A2

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Q1

A

First transform (u, v) to (x, y) where $x = v/u, y = u, v = xu$

Jacobian of this transformation is

$$\begin{bmatrix} rrr\frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{bmatrix}$$

absolute value of determinant of jacobian is $|0 \times \frac{\partial v}{\partial y} - 1 \times u| = u(\text{since } 0 \leq u)$

Then joint density of (u,x) is $f(x, y) = f(y, x) = f(u, x) = f(u, v)|J| = \frac{u}{|C_h|}$

Marginal density of X is

$f_X(x) = \int_0^{\sqrt{h(x)}} \frac{u}{|C_h|} du = \frac{h(x)}{2|C_h|}$

which is $\gamma h(x)$ where $\gamma = \frac{1}{2|C_h|}$

B

If $(u, v) \in C_h$, then as given above $0 \leq u \leq \sqrt{h(v/u)}$, also we have $X = V/U$, so $u \leq \sqrt{h(x)}$

therefore we have $u_+ = \max_x \sqrt{h(x)}$

$u \leq \sqrt{h(x)} \Rightarrow 1 \leq \frac{1}{u}\sqrt{h(x)}$

Then multiply both side by v, two cases here

Case 1. if v >0, then $\Rightarrow v \leq \frac{v}{u}\sqrt{h(x)} = x\sqrt{h(x)} \Rightarrow v \leq x\sqrt{h(x)}$

Therefore $v_+ = \max_x x\sqrt{h(x)}$

Case 2. if v <0, then $\Rightarrow v \geq \frac{v}{u}\sqrt{h(x)} = x\sqrt{h(x)} \Rightarrow v \geq x\sqrt{h(x)}$

Then $v_- = \min_x x\sqrt{h(x)}$

Case $v = 0$ is not dicussed since this case is not related to v_- nor v_+

C

```r
rnormal <- function(n) {
  x <- NULL
  upperv <- sqrt(2/exp(1))
  lowerv <- -upperv
  i <- 1

  while (i < n) {
    u <- runif(1) # generate Unif(0,1) rv
    v <- runif(1,lowerv,upperv) # generate unif rv with min lowerv and max upperv

    #rejection sampling test
    if (u <= exp(-(v/u)^2/4)) {
      x <- c(x, v/u)
      i <- i+1
      }
    }

  x
  }
```

D_h is the region rejection method workin on, and C_h is the region accepted. Therefore, probability of that proposals are accepted is

Area of D_h is $|D_h| = 2 \times \sqrt{\frac{2}{e}}$

The area of C_h is $|C_h| = \int_{-\infty}^{\infty} e^{\frac{-x^2}{2}} dx = \sqrt{\pi/2}$

D_h is the region rejection method workin on, and C_h is the region accepted.

Therefore, probability of that proposals are accepted is the ratio $\frac{|C_h|}{|D_h|} = 73.057\%$

Q2

A

Given $y_i = a \times i + b$. Also we have the objective function is non-negative since it's squared

To minimize the function, it is supposed to be as close to 0 as possible

The objective function for estimate$\{\theta_i\}$ is exactly 0 given $\theta_i = y_i$ since

$\theta_{i+1} - 2\theta_i + 2\theta_{i+1} = a(i+1) + b - 2[ai + b] + a(i-1) + b = 0$

and $y_i - \theta_i = 0$

Therefore, if y_i is exactly linear, $\hat{\theta}_i = y_i$ for all i.

B

Objective function is given above.

$$y^* - X\theta = \begin{bmatrix} y_1 - \theta_1 \\ y_2 - \theta_2 \\ y_n - \theta_n \\ \sqrt{\lambda}(\theta_3 - 2\theta_2 + \theta_1) \\ ... \\ \sqrt{\lambda}(\theta_n - 2\theta_{n-1} + \theta_{n-2}) \end{bmatrix}$$

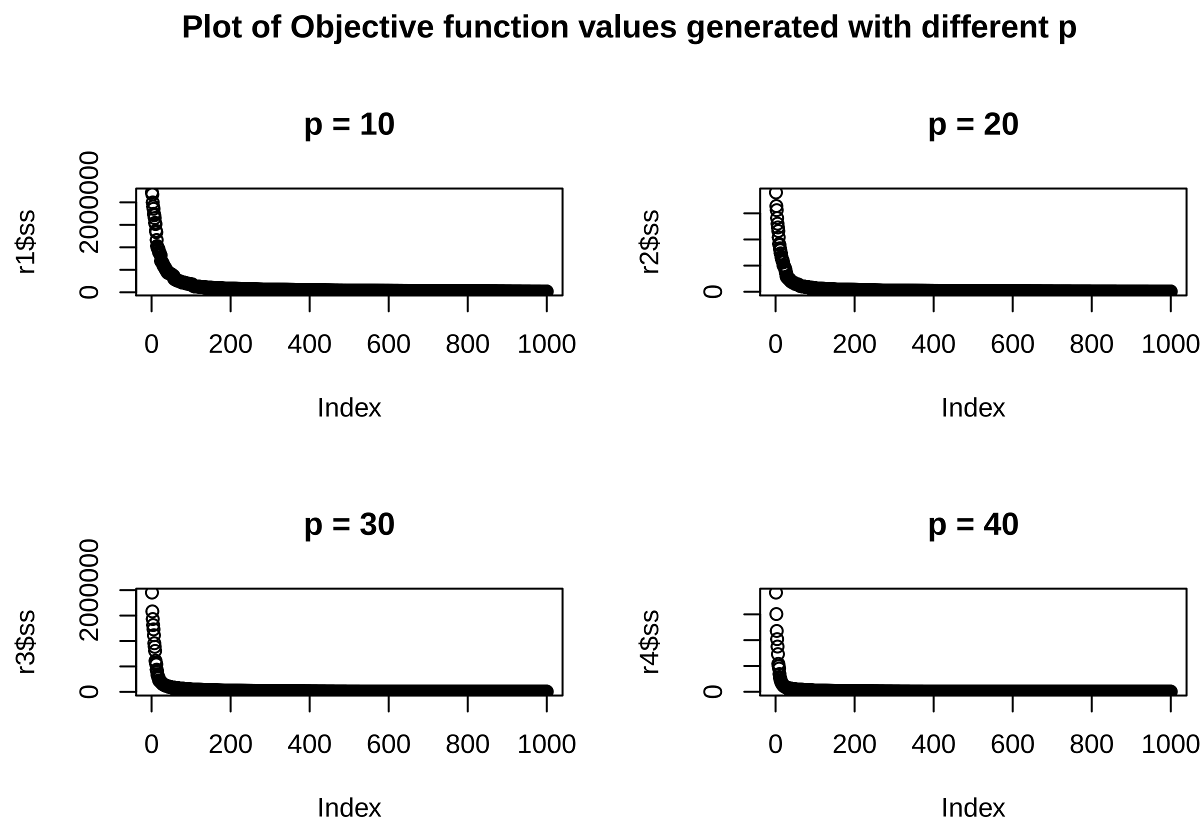
then $\|y^* - X\theta\|^2$ equals to the given objective function, and thus value minmizing the objective function will minimize the function for this question

$$y^* = \begin{bmatrix} y_1 \\ y_2 \\ ... \\ y_n \\ 0 \\ ... \\ 0 \end{bmatrix}$$

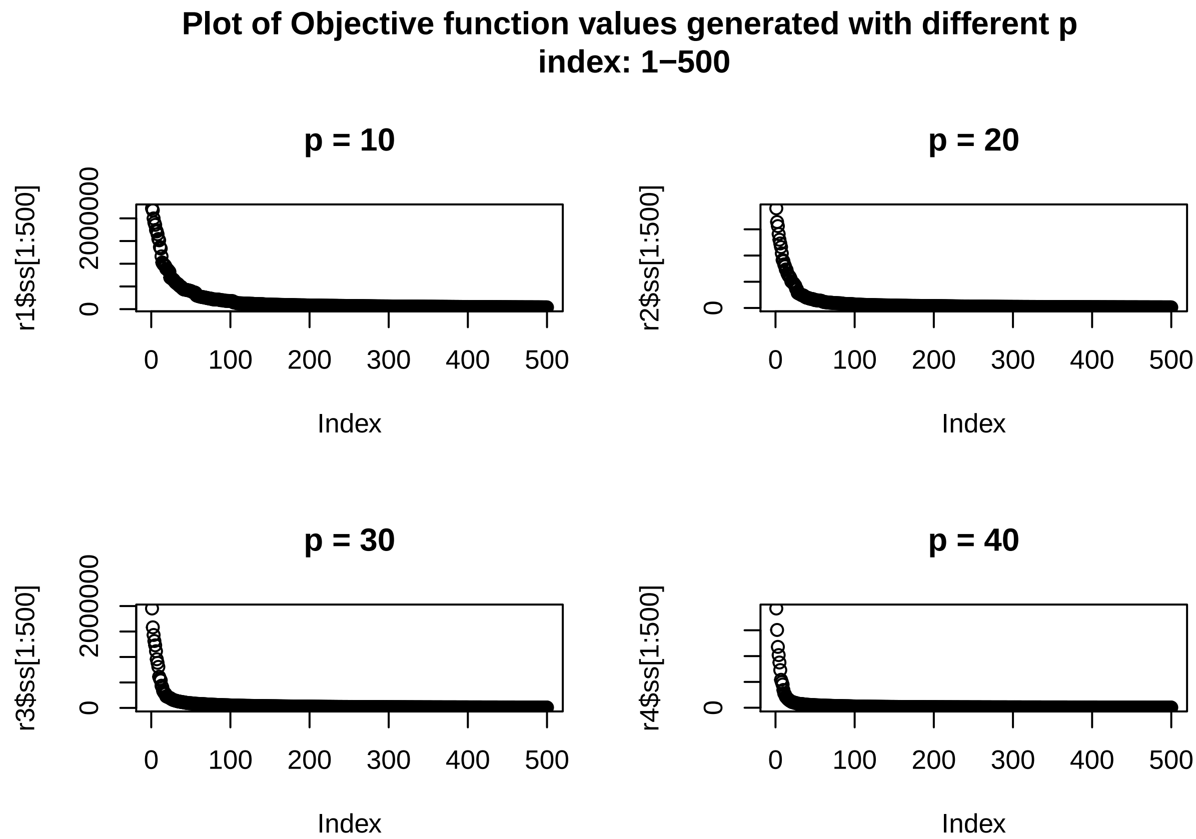
$$X = \begin{bmatrix} 1 & 0 & 0 & ... & 0 \\ 0 & 1 & 0 & ... & 0 \\ ... \\ \sqrt(\lambda) & 0 & 0 & ... & 1 \\ 0 & 0 & 0 & ... & 0 \\ ... \\ 0 & 0 & 0 & ... & 0 \end{bmatrix}$$

C

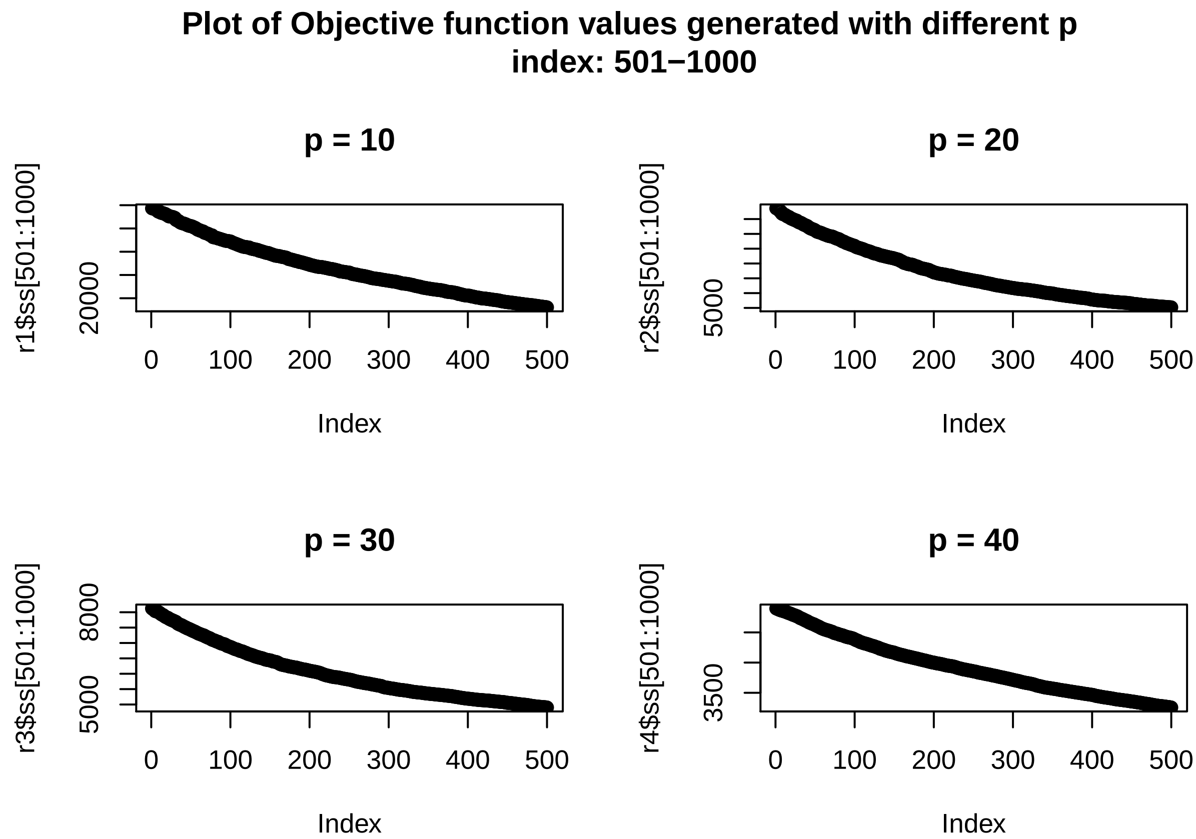
D




```
## [1] "Plot of Objective function values generated with different p"
```




```
## [1] "Plot of Objective function values generated with different p\n index: 1-500"
```




```
## [1] "Plot of Objective function values generated with different p\n index: 501-1000"
```

Refer to plot, as **p** increase, objective function decreases faster and ultimately reach a smaller value. More specifically, objective function with greater **p** decreases much faster each iteration at the begining, but the differences in rate of change is getting less apparent as the iteration continue.