STA457A2 Theory

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Theory

1.

A.

$E(Y|X=-1) = (-1) \times P(Y=-1|X=-1) + 0 \times P(Y=0|X=-1) + 1 \times P(Y=1|X=-1)$

$= (-1) \times \frac{0.05}{0.05+0.1+0.15} + 1 \times \frac{0.15}{0.05+0.1+0.15} = \frac{1}{3}$

$E(Y|X=0) = (-1) \times P(Y=-1|X=0) + 0 \times P(Y=0|X=0) + 1 \times P(Y=1|X=0)$

$= (-1) \times \frac{0.15}{0.15+0.15+0.1} + 1 \times \frac{0.1}{0.15+0.15+0.1} = -\frac{1}{8}$

$E(Y|X=1) = (-1) \times P(Y=-1|X=1) + 0 \times P(Y=0|X=1) + 1 \times P(Y=1|X=1)$

$= (-1) \times \frac{0.15}{0.15+0.15} + 1 \times \frac{0.15}{0.15+0.15} = 0$

Therefore, $E[(Y-g(X))^2] = E[E[Y-g(X)]^2|X]$

$= E[(Y-g(X=-1))^2|X=-1] \times P(X=-1) + E[(Y-g(X=0))^2|X=0] \times P(X=0) + E[(Y-g(X=1))^2|X=1] \times P(X=1)$

$= E[(Y-\frac{1}{3})^2] \times 0.3 + E[(Y-(-\frac{1}{8}))^2] \times 0.4 + E[(Y-0)^2] \times 0.3$

$E[Y^2 - \frac{2}{3}Y + \frac{1}{9}|X=-1] \times 0.3 + E[Y^2 + \frac{1}{4}Y + \frac{1}{64}|X=0] \times 0.4 + E[Y^2|X=1] \times 0.3$

where $E[Y] = -1 \times 0.35 + 0 \times 0.25 + 1 \times 0.4 = 0.05$,$E[Y^2|X=-1] = \frac{2}{3}$, $E[Y^2|X=0] = \frac{5}{8}$, $E[Y^2|X=1] = 1$

and $E[Y^2] = (-1)^2 \times 0.35 + 0^2 \times 0.25 + 1^2 \times 0.4 = 0.75$

Therefore, $E[(Y-g(X))^2] = 0.71$

b.

$E[(Y-\hat{Y})^2] = E[(Y-a-bX)^2]$

To find minimum value, take partial derivative of both a and b and set the result equals 0.

we have $E[(-2)(Y-a-bX)] = 0$

$E[(-2X)(Y-a-bX)] = 0$

Then we have $-2E[(Y-a-bX)] = 0 \rightarrow E[(Y-a-bX)] = 0 \rightarrow E[Y] - a - bE[X] = 0 \rightarrow a = E[Y] - bE[X]$

and $2E[(-X)(Y-a-bX)] = 0 \rightarrow E[(-X)(Y-a-bX)] = 0 \rightarrow E[XY] - aE[X] - bE[X^2] = 0$

subsititute a, we have $E[XY] - (E[Y] - bE[X])E[X] - bE[X^2] = 0 \rightarrow E[XY] - E[X]E[Y] + bE[X]^2 - bE[X^2] = 0 \rightarrow b = \frac{cov(X,Y)}{var[X]}$

From Part A, we know $E[Y] = 0.05$,$E[Y^2] = 0.75$

also, $E[X] = 0$,$E[X^2] = 0.6$

then $Var[X] = 0.6 - 0^2 = 0.6$, $cov(X,Y) = E[XY] = 0.05 \times (-1)^2 + 0.15 \times (+1)^2 + (-1)(+1) \times 0.15 + (+1)(-1)0.15 = -0.1$

Then $a = 0.05, b = -\frac{1}{6}$

$E[(Y - \hat{Y})^2] = E[Y^2 - 2Y\hat{Y} + \hat{Y}^2] = E[Y^2] - 2E[Y\hat{Y}] + E[\hat{Y}^2]$ where $E[\hat{Y}] = E[a + bX]$

then we have$E[(Y - \hat{Y})^2] = E[Y^2 - 2Y\hat{Y} + \hat{Y}^2] = E[Y^2] + 2E[Y\hat{Y}] + E[\hat{Y}^2]$

$= E[Y^2] + 2E[Y(a + bX)] + E[(a + bX)^2] = E[Y^2] - 2(aE[Y] + bE[XY]) + E[a^2] + 2abE[X] + b^2E[X^2]$

subsititute values of $E[Y^2], E[XY]$ and $E[X]$, we have $E[(Y - \hat{Y})^2] = 0.778$

2.

A.

$X^n_{n+1} = E[X_{n+1}|X_1...X_n] = E[\phi X_n + W_{n+1}|X_1...X_n] = \phi X_n$

$X^n_{n+2} = E[X_{n+2}|X_1...X_n] = E[\phi X_{n+1} + W_{n+1}|X_1...X_n] = E[\phi(\phi X_n + W_{n+1})|X_1...X_n] = \phi^2 X_n$

$Cov[(X_{n+1} - X^n_{n+1}), (X_{n+2} - X^n_{n+2})] = Cov[W_{n+1}, \phi W_{n+1} + W_{n+2}] = \phi Cov[W_{n+1}, W_{n+1}] = \phi\sigma^2_W$

b.

$X^{n-1}_n = E[X_n|X_1...X_{n-1}] = E[\phi X_{n-1} + W_n|X_1...X_{n-1}] = \phi X_{n-1}$

$Cov[(X_n - X^{n-1}_n), (X_{n+1} - X^n_{n+1})] = Cov[\phi X_{n-1} + W_n - (\phi X_{n-1}), \phi X_n + W_{n+1} - (\phi X_n)] = 0$