A4

Yuhan Hu

2019/11/21

Contents

Q1 **2**

a. 2

b. 2

c. 2

d. 2

Q2 **3**

a. 3

b. 3

c. 3

Q1

a.

$\theta_1 + \sum_{i=2}^{k} \phi_i = \theta_1 + (\theta_2 - \theta_1) + (\theta_3 - \theta_2) + ... + (\theta_k - \theta_{k-1})$

All terms cancel out

so $\theta_1 + \sum_{i=2}^{k} \phi_i = \theta_k$

b.

$\frac{\partial}{\partial \theta_1}(\sum_{i=1}^{n}(y_i - \theta_i)^2 + \lambda \sum_{i=2}^{n} |\phi_i|) = \frac{\partial}{\partial \theta_1} \sum_{i=1}^{n}(y_i - \theta_i)^2$

From a. we have $\theta_i = \theta_1 + \sum_{j=2}^{n} \phi_j$

$\frac{\partial}{\partial \theta_1} \sum_{i=1}^{n}(y_i - \theta_i)^2 = \frac{\partial}{\partial \theta_1}(\sum_{i=1}^{n}(y_i - \theta_1 - \sum_{j=2}^{n} \phi_j))^2 = -2\sum_{i=1}^{n}(y_i - \theta_1 - \sum_{j=2}^{n} \phi_j)$

(2) is minimized when the derivative reaches 0

$-2\sum_{i=1}^{n}(y_i - \hat{\theta}_1 - \sum_{j=2}^{n} \phi_j) = 0$

since $\hat{\theta}_1 - \sum_{j=2}^{n} \phi_j = \hat{\theta}_1 - \sum_{j=2}^{n} \hat{\theta}_j - \hat{\theta}_{i-1} = \hat{\theta}_i$

so (2) is minimized when $\sum_{i=1}^{n}(y_i - \hat{\theta}_i) = 0$

c.

$\partial\lambda \sum |\theta_i - \theta_{i-1}| = \partial\lambda \sum |\phi_i| = \lambda\partial \sum |\phi_i|$

$\partial|\phi_i| = 1, -1, [-1, 1]$given $\phi_i > 0, \phi_i < 0, \phi_i = 0$ respectively

so $\partial|\phi_i| \in [-\lambda, \lambda]$

In addition, $\lambda \sum_{i=2}^{n} |\phi_i|$ can be written as $\lambda|\theta_i - \theta_{i-1}| + \lambda|\theta_{i+1} - \theta_i|$ for each i

so $\partial\lambda \sum |\theta_i - \theta_{i-1}| = \partial(\lambda|\theta_i - \theta_{i-1}| + \lambda|\theta_{i+1} - \theta_i|) \in [-2\lambda, 2\lambda]$

therefore $\partial\lambda \sum |\theta_i - \theta_{i-1}| \in [-2\lambda, 2\lambda]^n$

d.

$\frac{\partial}{\partial \phi_j}(\sum_{i=1}^{n}(y_i - \theta_1 - \sum_{j=2}^{n} \phi_j)^2 + \lambda \sum_{i=2}^{n} |\phi_i|) = -2\sum_{i=1}^{n}(y_i - \theta_1 - \sum_{j=2}^{n} \phi_j) + \lambda\partial|\phi_i|$

$= -2\sum_{i=1}^{n}(y_i - \theta_1 - \sum_{j=2}^{n} \phi_j) + [-\lambda, \lambda]$

since we want the derivative to equal 0, we have

$-2\sum_{i=1}^{n}(y_i - \theta_1 - \sum_{j=2}^{n} \phi_j) \in [-\lambda, \lambda]$

$2\sum_{i=1}^{n}(y_i - \theta_1 - \sum_{j=2}^{n} \phi_j) \in [-\lambda, \lambda]$

Given $\hat{\theta}_1 = ... = \hat{\theta}_n = \bar{y}$

we have $\sum_{i=1}^{n} y_i - \theta_i = n\bar{y} - n\theta_1 = 0$

so$2\sum(y_i - \theta_1 - \sum \phi_j) = 2\sum_{i=1}^{n}\sum_{j=2}^{n} \phi_j \in [-\lambda, \lambda]$

therefore $|\sum\sum \phi_j| \leq \frac{\lambda}{2}$

Q2

a.

This given pdf follows negative binomial distribution with $k = m, r = n, p = 1 - \kappa_\lambda(r)$

mean of negative binomial distribution is $\frac{pr}{1-p}$, in this case it is $E_\lambda(M) = n(1 - \kappa_\lambda(r)/\kappa_\lambda(r)$

Negative binomial distribution mean proof:

$PDF = \binom{k+r-1}{k} \times (1-p)^r p^k$

$E(X) = \sum_{k=0}^{\infty} k \binom{k+r-1}{k} \times (1-p)^r p^k$

$= \sum_{k=1}^{\infty} \frac{(k+r-1)!}{(k-1)!(r-1)!} p^k (1-p)^r$

$= pr \sum_{k=1}^{\infty} \frac{(k+r-1)!}{(k-1)!r!} p^k (1-p)^r$

let $k + r - 1 = n$

then sum part of the above equation becomes$\sum_{n=r}^{\infty} \frac{n!}{(n-r)!r!} p^{n-r} (1-p)^r = \frac{1}{1-p}$

so $= pr \sum_{k=1}^{\infty} \frac{(k+r-1)!}{(k-1)!r!} p^k (1-p)^r = \frac{pr}{1-p}$

b.

$E_\lambda(\sum_{i=n+1}^{n+M} X_i | X_1 = x_1, ..., X_n = x_n) = E_\lambda[E_\lambda[\sum_{i=n+1}^{n+M} X_i | M = m]]$ by tower rule

$= E_\lambda[E_\lambda[(X_{n+1} + ... + X_{n+M})]] = E_\lambda[M E_\lambda(X)]$

$= E_\lambda[M] E_\lambda[X_i | X_i \leq r]$

c.

```
n = 1317+239+42+14+4+4+1
lambda = (1*1317+2*239+3*42+4*14+5*4+6*4+7*1)/n

for (i in 1:1000){
  k = 1-exp(-lambda)
  M = n*(1-k)/k
  lambda = (1*1317+2*239+3*42+4*14+5*4+6*4+7*1)/(n+M)
  i = i+1
}

M
```

```
## [1] 2730.148
```

```
lambda
```

```
## [1] 0.4660839
```

Since this method did converge, this estimate is reasonable.