1. Sobolev spaces

Motivation

Let $\lambda > 0$, $-\Delta u(x) + \lambda u(x) = f(x) \quad \forall x \in \mathbb{R}^d$ (PDE)

Given λ, f, find u ; PDE holds

This is similar to: given $v \in \mathbb{R}^d$, find $u \in \mathbb{R}^d$

$-Au + \lambda u = v$ (LE)

where $A \in M_d(\mathbb{R})$

Solving (LE): $(-A + \lambda I)$ is invertible!

if yes, $u = (-A + \lambda I)^{-1} v$

we work ; $(-\Delta + \lambda I)$ is invertible and $u = (-\Delta + \lambda I)^{-1} f$

$\forall f \in ??$ $\exists ! u \in ??$

($u \in C^2(\mathbb{R}^d)$ would be natural, then $\Delta u \in C(\mathbb{R}^d)$)

Answer: $\forall f \in L^2(\mathbb{R}^d)$ $\exists ! u \in H^1(\mathbb{R}^d) \subset L^2(\mathbb{R}^d)$

$C^1[-1,1] \subsetneq H^1(-1,1)$

Let $f \in C^1[-1,1]$, $\phi \in C_c^\infty(-1,1)$

$\int_{-1}^1 |f'(x)|^2 dx \le 2\|f'\|_\infty < \infty$ Thus $f' \in L^2(-1,1)$

$$\int_{-1}^1 f(x)\phi'(x)\,dx \overset{IBP}{=\!=} \underbrace{f(1)\phi(1)}_{0} - \underbrace{f(-1)\phi(-1)}_{0} - \int_{-1}^1 f'(x)\phi(x)\,dx = -\int_{-1}^1 f'(x)\phi(x)\,dx$$

since $\phi \in C_c[-1,1]$

Thus $f \in H^1(-1,1)$ 包含得证

Define $f(x)=|x|$ 不可微 $\forall x\in(-1,1)$

$$g(x)=\begin{cases}-1 & \forall x\in(-1,0)\\ 1 & \forall x\in(0,1)\\ 2020 & \text{if } x=0\end{cases}$$

$g\in L^2(-1,1)$ let $\phi\in C_c^\infty(-1,1)$

$$\int_{-1}^{1} f(x)\,\phi'(x)\,dx=\int_{-1}^{0} -x\,\phi'(x)\,dx+\int_0^1 x\,\phi'(x)\,dx$$

$$\underline{\underline{IBP}}\ -\underbrace{\phi(-1)}_{0}-\int_{-1}^{0}(-1)\phi(x)\,dx+\underbrace{\phi(1)}_{0}-\int_0^1\phi(x)\,dx$$

$$=\int_{-1}^{0}\phi(x)\,dx-\int_0^1\phi(x)\,dx$$

$$-\int_{-1}^{1} g(x)\phi(x)=-\int_{-1}^{0}(-1)\phi(x)-\int_0^1\phi(x)\,dx$$

Thus $\int_{-1}^{1} f(x)\,\phi'(x)\,dx=-\int_{-1}^{1} g(x)\phi(x)\,dx$

so $g=\frac{d}{dx}f$ weakly

分部积分法 (compact supported 上)

若在 sobolev space 定义中，令 $d=2$，则 i 可能为 1，可能为 2

$\int_\Omega f(x)\,\partial_i\phi(x)\,dx$ 有两种可能 $\begin{cases}\int_\Omega f(x)\frac{\partial\phi(x)}{\partial x}\,dx\\ \int_\Omega f(x)\frac{\partial\phi(x)}{\partial y}\,dy\end{cases}$

$i=1$ 时：$\int_\Omega f(x)\frac{\partial\phi(x)}{\partial x}\,dx=\underbrace{f(x)\int_\Omega\frac{\partial\phi(x)}{\partial x}\,dx\,dy}_{(*)}-\int_\Omega\frac{\partial f(x)}{\partial x}\phi(x)\,dx$

利用格林公式：$\int_{\partial D}P\,dx+Q\,dy=\iint_D\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dx\,dy$) (注意这里只有Q)

$(*)=f(x)\int_{\partial\Omega}\phi(x)\,dy=0$ (注意 $C_c^\infty(\Omega)$ 的定义)

$i=2$ 时：$\int_\Omega f(x)\frac{\partial\phi(x)}{dy}\,dx=\underbrace{f(x)\int_\Omega\frac{\partial\phi(x)}{\partial y}}_{(*)}\,dx\,dy-\int_\Omega\frac{\partial f(x)}{\partial y}\phi(x)\,dx$

利用格林公式，这里只有P

$(*)=f(x)\int_{\partial\Omega}\phi(x)\,dx=0$

综上 $\int_\Omega f(x)\partial_i\phi(x)=0-\int_\Omega g_i(x)\phi(x)\,dx$ ($\partial_i f=g_i$)

2. Projection on convex sets

定理2.1证明: 1. existence
2. unique
3. characterstion
4. subspace $\Rightarrow P_K$ linear

1. existence

let $f \in H$

$$\operatorname{dist}(f, K) = \inf\{\|f - v\| : v \in K\} = \lim_{n\to\infty} \|f - v_n\| \quad \text{for some } (v_n)_{n\in\mathbb{N}} \in K$$

$d := \operatorname{dist}(f, K)$

$d_n := \|f - v_n\| \quad \forall n \in \mathbb{N}$

let $m, n \in \mathbb{N}$

$$2d_n^2 + 2d_m^2 \overset{\substack{\text{parallellgen}\\ \text{identity}}}{=\!=} \|f - v_n - (f - v_m)\|^2 + \|2f - (\cancel{x}\, v_n + v_m)\|^2$$

$$\frac{1}{2}(d_n^2 + d_m^2) = \frac{1}{4}\|v_n - v_m\|^2 + \|f - \underbrace{\frac{v_n + v_m}{2}}_{\in K \text{ by convex}}\|^2 \geq \frac{1}{4}\|v_n - v_m\|^2 + d^2$$

$$\Rightarrow \|v_n - v_m\|^2 \leq 2(d_n^2 + d_m^2) - 4d^2 \xrightarrow[\substack{d_n, d_m \to d\\ (n, m\to\infty)}]{} 0$$

Therefore, $(v_n)_{n\in\mathbb{N}}$ is Cauchy sequence and hence convergent.
Let $u = \lim_{n\to\infty} v_n$, Then $u \in K$ since K is closed.

2. Characterision

① $\|f - u\| = \inf\{\|f - v\| ; v \in K\}$

② want to show $\forall w \quad \langle f - u, w - u\rangle \leq 0$

① $\Rightarrow$ ②

Let $w \in K \quad \forall t \in (0,1)$

$tw + (1-t)u = v \in K$ by convexity

Thus $\|f - u\| \leq \|f - v\|^2 = \|f - (tw + (1-t)u)\|^2 = \|f - u + t(\overset{u}{\cancel{w}} - \overset{w}{\cancel{u}})\|^2$

$$= \|f - u\|^2 + t^2\|u - w\|^2 + 2t\langle f - u, u - w\rangle$$

$\Rightarrow 2t\langle f - u, \overset{w-u}{\cancel{u-w}}\rangle \leq t^2\|u - w\|^2 \quad \forall t \in (0,1)$

$\Rightarrow 2\langle f - u, w - u\rangle \leq t\|v - w\|^2$

$\Rightarrow \langle f - u, w - u\rangle \leq 0 \quad \square$

② ⇒ ①

Assume $\exists u \in K$, such that $\langle f-u, w-u \rangle \le 0 \quad \forall w \in K$

$\|f-w\|^2 = \|f-u+u-w\|^2 = \|f-u\|^2 + \|u-w\|^2 + 2\langle f-u, u-w \rangle \ge \|f-u\|^2 + \|u-w\|^2 \ge \|f-u\|^2$

Thus $\|f-u\| = \inf\{\|f-v\| ; v \in K\}$

3. uniqueness

let $u_1, u_2 \in K$

$$\begin{cases} \langle f-u_1, w-u_1 \rangle \le 0 & \forall w \in K \\ \langle f-u_2, w-u_2 \rangle \le 0 & \forall w \in K \end{cases} \begin{matrix} \xrightarrow{\text{令 } w=u_2} \\ \xrightarrow{\text{令 } w=u_1} \end{matrix} \begin{cases} \langle f-u_1, u_2-u_1 \rangle \le 0 \\ \langle f-u_2, u_1-u_2 \rangle \le 0 \end{cases} \Rightarrow \begin{cases} \langle f-u_1, u_2-u_1 \rangle \le 0 & ① \\ \langle f-u_2, u_2-u_1 \rangle \ge 0 & ② \end{cases}$$

① − ② 得: $\langle u_2-u_1, u_2-u_1 \rangle \le 0 \qquad \|u_2-u_1\|^2 \le 0 \qquad u_2 = u_1$

4. subspace ⇒ Linear P_K

Assume that K is a subspace

let $f, g \in H$, and $\lambda \in \mathbb{R}^*$ (不为0)

$\langle f - P_K(f), v - P_K(f) \rangle \le 0 \quad \forall v \in K$

$\langle g - P_K(g), v - P_K(g) \rangle \le 0 \quad \forall v \in K$

since K is a subspace

$\langle f - P_K(f), \frac{1}{\lambda} v - P_K(f) \rangle \le 0 \quad \forall v \in K$

$\Rightarrow \langle \lambda f - \lambda P_K(f), v - \lambda P_K(f) \rangle \le 0 \quad \forall v \in K$

Thus $\lambda P_K(f) = P_K(\lambda f)$

$\forall v \in K \quad v - P_K(g), v - P_K(f) \in K$ since K is linear

$$\begin{cases} \langle f - P_K(f), v - (P_K(f) + P_K(g)) \rangle \le 0 & \forall v \in K \\ \langle g - P_K(g), v - (P_K(f) + P_K(g)) \rangle \le 0 & \forall v \in K \end{cases}$$

$\Rightarrow \langle f+g - (P_K(f) + P_K(g)), v - (P_K(f) + P_K(g)) \rangle \le 0 \quad \forall v \in K$

Thus $P_K(f+g) = P_K(f) + P_K(g)$

Therefore P_K is a linear map

3. Orthogonal complements, Lax-Milgram Theorem

H: Hilbert space

$E \subsetneq H$ subspace (closed)

1. $\exists\, u \in H \setminus E \quad \|u\| = 1 \;\Rightarrow\; dist(u, E) = 1$
2. $H = E \oplus E^{\perp}$ where $E^{\perp} = \{h \in H ;\ <h, v> = 0 \ \forall v \in E\}$

1. Let $v \in H \setminus E$

Define $u = \dfrac{v - P_E(v)}{\| v - P_E(v) \|}$

$$dist(u, E) = \| v - P_E(u) \| = \frac{\| v - P_E(v) - P_E(v - P_E(v)) \|}{\| v - P_E(v) \|}$$

$$= \frac{\| v - P_E(v) - P_E(v) + P_E(P_E v) \|}{\| v - P_E(v) \|}$$

$$= 1$$

2. $E \cap E^{\perp} = \{h \in E ;\ <h, v> \ \forall v \in E\} \subset \{h \in E ;\ \|h\|^2 = 0\} = \{0\}$

<u>uniqueness:</u> Given $h \in H$, write $h = \underbrace{P_E(h)}_{E} + h - P_E(h)$

WTS $\quad h - P_E(h) \in E^{\perp}$

let $v \in E$, $\varepsilon > 0$

$$\| h - P_E(h) \|^2 \leq \| h - \underbrace{P_E(h) - \varepsilon(v - P_E(h))}_{E} \| = \| h - P_E(h) \|^2 + \varepsilon^2 \| v - P_E(h) \|^2 - 2\varepsilon <h - P_E(h), v - P_E(h)>$$

$$0 \leq \varepsilon^2 \| v - P_E(h) \|^2 - 2\varepsilon <h - P_E(h), v - P_E(h)>$$

$\Rightarrow \; <h - P_E(h), v - P_E(h)> \ \leq \frac{\varepsilon}{2} \| v - P_E(h) \|$

$<h - P_E(h), v - P_E(h)> \ \leq 0 \quad \forall v \in E$

$\Rightarrow \; <h - P_E(h), w> \ \leq 0 \quad \forall w \in E$

$\Rightarrow \; <h - P_E(h), -w> \ \leq 0 \quad \forall w \in E$

$\Rightarrow \; <h - P_E(h), w> = 0 \quad$ i.e. $h - P_E(h) \in E^{\perp}$ $\diamond$

Stampacchia & Lax–Milgram Theorem

Remark

if $K = H$ $\exists!\, u \in H$ $a(u, v-u) \geq \phi(v-u)$ $\forall v \in H$

$$\Rightarrow a(u, w) \geq \phi(w) \quad \forall w \in H$$

$$\Rightarrow a(u, \epsilon w) \geq \phi(\epsilon w) \quad \forall w \in H \quad \forall \epsilon \in \mathbb{R}$$

Stampacchia $\Rightarrow$ Lax–Milgram

$$\begin{cases} a(u, w) \geq \phi(w) & \forall w \in H \\ a(u, -w) \geq \phi(-w) & \forall w \in H \end{cases} \Rightarrow a(u, w) = \phi(w) \quad \forall w \in H$$

Find $u \in K$: $a(u, v-u) \geq \phi(v-u)$ $\forall v \in K$

By Riesz representation Thm:

$\exists!\, f \in H$ s.t. $\forall v \in H$ $\phi(v) := \langle f, v \rangle$

~~$\forall u \in H$ $\exists!\, A(u) \in H$ st. $a(u, v)$,~~

$\forall u \in H$ $\exists!\, A(u) \in H$ s.t. $\forall v \in H$ $a(u, v) = \langle A(u), v \rangle$

Q: Find $u \in K$ s.t. $\forall v \in K$

$\langle A(u), v-u \rangle \geq \langle f, v-u \rangle$

$\Rightarrow$ $\forall v \in K$ $\langle A(u) - f, v-u \rangle \geq 0$

$\Rightarrow$ $\forall \rho > 0$ $\forall v \in K$ $\langle \rho A(u) - \rho f, v-u \rangle \geq 0$

$\Rightarrow$ $\forall \rho > 0$ $\forall v \in K$ $\langle \rho A(u) + u - \rho f - u, v-u \rangle \geq 0$

$\Rightarrow$ $\forall \rho > 0$ $\forall v \in K$ $\langle -\rho A(u) \underline{-u} + \rho f + u, v-u \rangle \leq 0$

$\Rightarrow$ $\forall \rho > 0$ $u = P_K(\rho f - \rho A(u) + u)$

Define $S_\rho : H \to H$

$$u \mapsto P_K(\rho f - \rho A(u) + u)$$

WTS $\exists!\, u \in H$ $S_\rho(u) = u$

Claim $\| P_K(u) - P_K(v) \| \leq \| u - v \| \quad \forall\, u, v \in H$

Indeed $\begin{cases} \langle u - P_K(u),\ v - P_K(u) \rangle \leq 0 & \forall\, v \in K \\ \langle v - P_K(v),\ w - P_K(v) \rangle \leq 0 & \forall\, w \in K \end{cases}$

$\Rightarrow \begin{cases} \langle u - P_K(u),\ P_K(v) - P_K(u) \rangle \leq 0 \\ \langle v - P_K(v),\ P_K(u) - P_K(v) \rangle \leq 0 \end{cases}$

$\Rightarrow \begin{cases} \langle u - P_K(u),\ P_K(v) - P_K(u) \rangle \leq 0 \\ \langle P_K(v) - v,\ P_K(v) - P_K(u) \rangle \leq 0 \end{cases}$

$\cancel{\Rightarrow \langle u - v,\ P_K(v) - P_K(u) \rangle \leq 0}$

$\Rightarrow \langle P_K(v) - P_K(u) + u - v,\ P_K(v) - P_K(u) \rangle \leq 0$

$\Rightarrow \| P_K(v) - P_K(u) \|^2 \leq \langle v - u,\ P_K(v) - P_K(u) \rangle$

$\xRightarrow{\text{Cauchy-Schwartz}} \| P_K(v) - P_K(u) \|^2 \leq \| v - u \| \, \| P_K(v) - P_K(u) \|$

Thus $\| P_K(u) - P_K(v) \| \leq \| u - v \|$

$S_\rho(u) = P_K(\rho f - \rho A(u) + u)$

$\forall\, u, v \in H \qquad \| S_\rho(u) - S_\rho(v) \|^2 \leq \| \rho A(u-v) - (u-v) \|^2$

$= \rho^2 \| A(u-v) \|^2 + \| u - v \|^2 - 2\rho \langle A(u-v), u-v \rangle$

$= \rho^2 \sup\limits_{\substack{w \in H \\ \|w\| = 1}} a(u-v, w)^2 + \| u - v \|^2 - 2\rho\, a(u-v, u-v)$

$\Downarrow$ continuity $\qquad\qquad \Downarrow$ coercivity

$\leq C^2 \rho^2 \| u - v \|^2 + \| u - v \|^2 - 2\rho\delta \| u - v \|^2$

$= (1 + C^2\rho^2 - 2\rho\delta) \| u - v \|^2$

Pick $\rho > 0 \qquad 1 + C^2\rho^2 - 2\rho\delta \leq 1 \iff C^2\rho^2 < 2\rho\delta \iff \rho < \frac{2\delta}{C^2}$

$\xRightarrow{\text{Banach 不动点定理}} \exists!\, u \in H \quad S_\rho(u) = u$

Assume a is symmetric

Then $\forall u \in H \quad \delta \|u\|^2 \leq a(u,u) \leq C\|u\|^2$

Define $\forall v, u \in H \quad \langle u, v \rangle_a = a(u,v)$

exercise: check that $(H, \langle \cdot, \cdot \rangle_a)$ is a Hilbert space with norm $\|\cdot\|_a^2 = \langle \cdot, \cdot \rangle_a$ equivalent to $\|\cdot\|$.

By Riesz representation Thm:

$\forall \phi \in H^* \quad \exists! g \in H \quad \forall v \in H \quad \phi(v) = \langle g, v \rangle = a(g, v)$

$\forall v \in K \quad a(u, v-u) \geq \phi(v-u)$

$\Longleftrightarrow \forall v \in K \quad \langle u, v-u \rangle_a \geq \langle g, v-u \rangle_a$

$\Longleftrightarrow \forall v \in K \quad \langle g-u, v-u \rangle_a \leq 0$

$\xRightarrow{\text{Thm 2.1}} \|g-u\|_a^2 = \inf\{\|g-v\|_a ; v \in K\}^2$

$\Longleftrightarrow a(g-u, g-u) = \inf\{a(v-g, v-g) ; v \in K\}$

$\Longleftrightarrow a(g,g) + a(u,u) - 2a(g,u) = \inf\{a(g,g) + a(v,v) - 2a(v,g) ; v \in K\}$

$\Longleftrightarrow \frac{1}{2}a(u,u) - a(g,u) = \inf\{\frac{1}{2}a(v,v) - a(v,g) ; v \in K\}$ ◇

4. Spectrum of compact self-adjoint operators

Thm 3.2 $T \in B(H)$ self-adjoint

$$m = \inf_{\|h\|=1} \langle T(h), h \rangle \qquad M = \sup_{\|h\|=1} \langle T(h), h \rangle$$

① $M < \infty$ and $m > -\infty$

$\forall h \in H \quad |\langle T(h), h \rangle| \underset{\text{Cauchy-Schwarz ineq}}{\leq} \|T(h)\| \|h\| \leq \|T\|_{B(H)} \|h\|_H^2 = \|T\|_{B(H)}$

Thus $M \leq \|T\|_{B(H)}$

Assume $m = -\infty$, then $\exists (h_n)_{n \in \mathbb{N}} \in H^{\mathbb{N}}$ $\begin{cases} \|h_n\| = 1 & \forall n \in \mathbb{N} \\ \langle T(h_n), h_n \rangle \underset{n \to +\infty}{\longrightarrow} -\infty \end{cases}$

This would imply $|\langle T(h_n), h_n \rangle| \underset{n \to +\infty}{\longrightarrow} +\infty$

but $|\langle T(h_n), h_n \rangle| \leq \|T\|_{B(H)} \quad \forall n \in \mathbb{N}$

Hence $m > -\infty$

② $\sigma(T) \subset [m, M]$

Let $\lambda > M$ WTS $\lambda \in \rho(T)$

Define $a(u,v) = \langle \lambda u - T(u), v \rangle \quad \forall u, v \in H$

$\forall u, v \in H \quad |a(u,v)| \leq |\lambda| |\langle u, v \rangle| + |\langle T(u), v \rangle|$

$$\underset{CS}{\leq} |\lambda| \|u\| \|v\| + \|T(u)\| \|v\|$$

$$\leq (|\lambda| + \|T\|) \|u\| \|v\|$$

$\Rightarrow$ a is continous

$\forall u \in H \quad a(u,u) = \langle \lambda u, u \rangle - \langle T(u), u \rangle$

$$= \lambda \|u\|^2 - \|u\|^2 \langle T(\tfrac{u}{\|u\|}), \tfrac{u}{\|u\|} \rangle$$

$$\geq \lambda \|u\|^2 - M \|u\|^2 = \underbrace{(\lambda - M)}_{>0} \|u\|^2$$

$\Rightarrow$ a is coercive

Thus by Lax-Milgram Thm

$\forall f \in H \quad \exists! u \in H \quad \forall v \in H \quad a(u,v) = \langle f, v \rangle$

$\forall f \in H \quad \exists! u \in H \quad \forall v \in H \quad \langle \lambda u - Tu, v \rangle = \langle f, v \rangle$

$\forall f \in H \quad \exists! u \in H \quad \lambda u - Tu = f$

$\Rightarrow \lambda T - T$ is a bijection

$\Rightarrow \lambda \in \rho(T)$

we have shown $\lambda > M \Rightarrow \lambda \in \rho(T)$

$\sigma(T) \subset (-\infty, M]$

In the same way : $\sigma(T) \subset [m, +\infty)$ (exercise)

③ $\mu := \max\{|m|, |M|\}$ WTS $\mu = \|T\|$

$\forall u \in H \quad \|u\| = 1 \quad |\langle T(u), u \rangle| \leq \|T\| \|u\|^2 = \|T\|$

let $(u_n)_{n \in \mathbb{N}} \in H^{\mathbb{N}}$: $\begin{cases} \|u_n\| = 1 \quad \forall n \in \mathbb{N} \\ |\langle T(u_n), u_n \rangle| \xrightarrow{n \to +\infty} \mu \end{cases}$

$\|T\|_{B(H)} \geq |\langle T(u_n), u_n \rangle| \xrightarrow{n \to +\infty} \mu$

so $\mu \leq \|T\|$

let $u, v \in H$

$\langle T(u+v), u+v \rangle = \langle T(u), u \rangle + \langle T(u), v \rangle + \langle T(v), u \rangle + \langle T(v), v \rangle$

$\xRightarrow{\text{self-adjoint}} \langle T(u), u \rangle + \langle T(v), v \rangle + 2 \langle T(u), v \rangle$ ①

$\langle T(u-v), u-v \rangle \xRightarrow{\text{self-adjoint}} \langle T(u), u \rangle + \langle T(v), v \rangle - 2 \langle T(u), v \rangle$ ②

① − ②: $\cancel{4 \langle T(u), u \rangle +}$ $4 \langle T(u), v \rangle = \langle T(u+v), u+v \rangle - \langle T(u-v), u-v \rangle$

i.e. $\forall u, v \in H$

$|\langle T(u), v \rangle| \leq \frac{1}{4} \mu (\|v+u\|^2 + \|u-v\|^2) \xlongequal[\text{identity}]{\text{parallegram}} \frac{\mu}{2} (\|u\|^2 + \|v\|^2)$

$\forall u, v \in H \quad \forall \alpha > 0 \quad |\langle T(\frac{u}{\alpha}), \alpha v \rangle| \leq \frac{\mu}{2} (\|\frac{u}{\alpha}\|^2 + \|\alpha v\|^2)$

Pick α : $\alpha^2 = \frac{\|u\|}{\|v\|}$. Then $|\langle T(u), v \rangle| \leq \frac{\mu}{2} (2\|u\|\|v\|) = \mu \|u\| \|v\| \quad \forall u, v \in H$

取 $\|v\| = 1$ Thus $\|T(u)\| \leq \mu \|u\| \quad \forall u \in H$

so $\|T\| \leq \mu$

④ $T \in K(H) \Rightarrow m, M \in EV(T)$

step 1:

show $\exists (u_n)_{u \in \mathbb{N}} \in H^{\mathbb{N}}$: $\begin{cases} \|u_n\| = 1 & \forall n \in \mathbb{N} \\ \|Mu_n - T(u_n)\| \xrightarrow[n \to +\infty]{} 0 \end{cases}$

We have that $\exists (U_n) \in H^{\mathbb{N}}$: $\begin{cases} \|U_n\| = 1 & \forall n \in \mathbb{N} \\ < MU_n - T(U_n), U_n > \xrightarrow{n \to +\infty} 0 \end{cases}$

Define $a(u,v) = < Mu - T(u), v > \quad \forall u, v \in H$ is symmetric (since T s.a. ~~...~~ and ~~bilinear~~ bilinear)

$a(u,u) \geq 0 \quad \forall u \in H$ (by def of M)

Thus $a(u,v)$ define an inner product

Thus we have Cauchy-Schwartz ineq:

$a(u,v) \leq a(u,u)^{\frac{1}{2}} a(v,v)^{\frac{1}{2}} \quad \forall u, v \in H$

and $|< Mu - T(u), v >| \leq < Mu - T(u), u >^{\frac{1}{2}} < Mu - T(v), v >^{\frac{1}{2}}$

$\leq \|MI - T\|^{\frac{1}{2}} \|v\|^{\frac{1}{2}} < Mu - T(u), u >^{\frac{1}{2}}$

Then $\|MU_n - T(U_n)\| = \sup\limits_{\|v\|=1} |< MU_n - T(U_n), v >| \leq \|MI - T\|^{\frac{1}{2}} < MU_n - T(U_n), U_n >^{\frac{1}{2}} \xrightarrow{n \to +\infty} 0$

step 2: since $T \in K(H)$ $\quad \exists v \in H \quad T(U_{n_k}) \xrightarrow{k \to \infty} v$

for some $n_k \nearrow$ $(k \in \mathbb{N})$

Therefore $MU_{n_k} \xrightarrow{k \to +\infty} v$ i.e. $U_{n_k} \xrightarrow{k \to +\infty} \frac{1}{M} v$

let $u = \frac{1}{M} v \quad \|Mu - T(U_{n_k})\| \xrightarrow{k \to +\infty} 0$

So $\|Mu - T(u)\| = 0$ i.e. $M \in EV(T)$

Exercise: m the same way. show that $m \in EV(T)$

5. Spectral Thm

Lemma: if $T \in K(H)$. Then $\sigma(T)\setminus\{0\} = EV(T)\setminus\{0\}$

proof let $\lambda \in \sigma(T)\setminus\{0\}$. Assume $\lambda \notin EV(T)$

Define $\forall n \in \mathbb{N}$. $E_n := R((\lambda I - T)^n)$

Claim $E_{n+1} \subsetneq E_n \subsetneq H \quad \forall n \in \mathbb{N}$

$N(\lambda I - T) = \{0\}$ since $\lambda \notin EV(T)$

$\lambda I - T$ is not bijection, since $\lambda \in \sigma(T)$

and because $\lambda I - T$ is injection

$\Rightarrow \lambda I - T$ is not surjection

$\Rightarrow E_1 \neq H$

let $n \in \mathbb{N}$, if $\forall u, v \in H$

$(\lambda I - T)^{n+1} u = (\lambda I - T)^n v$ (i.e. $E_{n+1} = E_n$)

Then $\forall u \in H \quad \exists v \in H \quad (\lambda I - T)^n((\lambda I - T)u - v) = 0$

Because $N(\lambda I - T) = \{0\}$

$\Rightarrow (\lambda I - T)^{n-1}((\lambda I - T)u - v) = 0$

Hence $\forall u \in H \quad \exists v \in H$

$(\lambda I - T)^n u = (\lambda I - T)^{n-1} v$

Now we have prove that $E_{n+1} = E_n \Rightarrow E_n = E_{n-1} \Rightarrow E_1 = H$

Contradiction!

So we have that $E_n \neq E_{n+1} \quad \forall n \in \mathbb{N}$

Moreover, $\forall u \in E_{n+1} \quad \exists v \in H: \ u = (\lambda I - T)^{n+1} v$

$u = (\lambda I - T)^n ((\lambda I - T) v) \in E_n$

This proves the Claim.

By Riesz Lemma: $\exists (u_n)_{n\in\mathbb{N}} \subset H$ $\begin{cases} \|u_n\| = 1 & \forall n \in \mathbb{N} \\ u_n \in E_n & \forall n \in \mathbb{N} \\ dist(u_n, E_{n+1}) = 1 & \forall n \in \mathbb{N} \end{cases}$

let $n, m \in \mathbb{N}$ $n > m$

$$T(u_n) - T(u_m) = \underbrace{T(u_n) - \lambda u_n}_{\in E_{n+1} \subset E_{m+1}} + \underbrace{\lambda u_n}_{\in E_n \subset E_{m+1}} - \lambda u_m + \underbrace{\lambda u_m - T(u_m)}_{\in E_{m+1}}$$

$$= -\lambda u_m + v_{m+1} \quad \text{for some } v_{m+1} \in E_{m+1}$$

$\| T(u_n) - T(u_m)\| = |\lambda| \, \| u_m - \frac{v_{m+1}}{\lambda} \| \geq |\lambda| \, dist(u_m, E_{m+1}) = |\lambda| \neq 0$ $(\forall m, n \in \mathbb{N} \ n > m)$

Thus $(T(u_n))_{n\in\mathbb{N}}$ dose not have a convergent subsequence

That is a contradiction since $T \in K(H)$ and $u_n \in \overline{B(0,1)}$ $\forall n \in \mathbb{N}$ ◇

Lemma: let $T \in K(H)$ and $(\lambda_n)_{n\in\mathbb{N}} \in (EV(T)\setminus\{0\})^{\mathbb{N}}$ with $\lambda_j \neq \lambda_k$ $\forall j \neq k$.
If $\exists \lambda \in \mathbb{R}$: $\lambda_n \xrightarrow[n\to\infty]{} \lambda$ then $\lambda = 0$

proof: pick $n \in \mathbb{N}$, $e_n \in H$ $T(e_n) = \lambda_n e_n$

Claim: $\boxed{(e_n)_{n\in\mathbb{N}} \text{ linearly independent}}$

By induction. Assume ~~not~~ true for $n \in \mathbb{N}$

By contradiction. Assume that $e_{n+1} = \sum_{j=1}^{n} \alpha_j e_j$ for some $(\alpha_j)_{j=1}^{n} \in \mathbb{R}^n$

Then $T(e_{n+1}) = \sum_{j=1}^{n} \alpha_j \lambda_j e_j = \lambda_{n+1} e_{n+1} = \lambda_{n+1} \sum_{j=1}^{n} \alpha_j e_j$

since $(e_j)_{j=1}^{n}$ linearly independent. We leave that $\alpha_j \underbrace{(\lambda_j - \lambda_{n+1})}_{\neq 0} = 0$ $\forall j = 1, 2 \cdots n$

Hence $\alpha_j = 0$ $\forall j = 1 \cdots n$ □

$E_n = span\{e_1, \dots e_n\}$ $\forall n \in \mathbb{N}$

$E_n \subsetneq E_{n+1}$

By Riesz Lemma $\exists (u_n)_{n\in\mathbb{N}}$: $\begin{cases} \|u_n\| = 1 & \forall n \in \mathbb{N} \\ u_n \in E_n & \forall n \in \mathbb{N} \\ d(u_n, E_{n-1}) = 1 & \forall n \in \mathbb{N}^+ \end{cases}$

let $n, m \in \mathbb{N}$ $n > m$

$$\frac{T(u_n)}{\lambda_n} - \frac{T(u_m)}{\lambda_m} = \underbrace{\frac{T(u_n) - \lambda_n u_n}{\lambda_n}}_{\in E_{n-1}} + u_n - \underbrace{\frac{T(u_m)}{\lambda_m}}_{\in E_m \subset E_{n-1}}$$

since $T(e_n) - \lambda_n e_n = 0$

so $T(\sum_{j=1}^{n} \alpha_j e_j) - \lambda_n(\sum_{j=1}^{n} \alpha_j e_j)$

$= \sum_{j=1}^{n-1} \alpha_j (\lambda_j - \lambda_n) e_j$

Thus $\| \frac{T(u_n)}{\lambda_n} - \frac{T(u_m)}{\lambda_m} \| \geq \operatorname{dist}(u_n, E_{n-1}) = 1$

Assume $\lambda_n \xrightarrow[n \to +\infty]{} \lambda \neq 0$

let $n_k \uparrow$ $(T(u_{n_k}))_{k \in \mathbb{N}}$ converges (using $T \in K(H)$)

Then $1 \leq \| \frac{T(u_{k+1})}{\lambda_{k+1}} - \frac{T(u_k)}{\lambda_k} \| \xrightarrow{n \to \infty} 0$ contradiction. □

Thm 3.5 H is a ~~sepra~~ separable Hilbert space

Let $T \in K(H)$ be self-adjoint

$\exists (e_n)_{n \in \mathbb{N}}$ orthonormal basis of H

$\exists (\lambda_n)_{n \in \mathbb{N}} \in \mathbb{R}$

$T(e_n) = \lambda_n e_n$ $\forall n \in \mathbb{N}$

proof

Remark that $EV(T) \cap [\frac{1}{n}, +\infty)$ is finite $\forall n \in \mathbb{N}$

Indeed if it were not, then

$EV(T) \cap [\frac{1}{n}, \infty) \subset [\frac{1}{n}, \|T\|]$

If $\exists (\lambda_j)_{j \in \mathbb{N}} \subset EV(T) \cap [\frac{1}{n}, +\infty)$ distinct

then $\exists (\lambda_{j_k})_{k \in \mathbb{N}} \subset EV(T) \cap [\frac{1}{n}, \infty)$ convergent

and $\lambda_{j_k} \xrightarrow{k \to \infty} 0$ by the lemma. Contradiction!

Let us define (write) $EV(T) \setminus \{0\} = \{\lambda_j \in \mathbb{R} \; ; \; j \in I\}$ for $I \subset \mathbb{N}$

Define $E_j := N(\lambda_j I - T) \quad \forall j \in I$

$E_0 := N(T)$

Claim: 1. $E_0 \perp E_j \quad \forall j \in I$

2. $E_j \perp E_k \quad \forall j, k \in I, \; j \neq k$

Indeed: 1. let $j \in I$

$\forall u \in E_0 \quad \forall v \in E_j$

$0 = \langle T(u), v \rangle \underset{\text{self-adjoint}}{\Longrightarrow} \langle u, T(v) \rangle = \lambda_j \langle u, v \rangle$

so $\langle u, v \rangle = 0$

2. let $j, k \in I \quad \forall u \in E_j \quad \forall v \in E_k$

$\langle u, v \rangle = 0$ (exercise)

Define $F = \left(\bigoplus_{j \in I} E_j \right) \oplus E_0$ WTS $F = H$ i.e. $F^{\perp} = \{0\}$

since $H = F \oplus F^{\perp}$

This would conclude the proof taking $(e_n^j)_{n \in K_j}$ an onb of E_j for all $j \in I \cup \{0\}$.

let $(e_n)_{n \in \mathbb{N}}$ be an onb of F

We have $\forall u \in F \quad \exists j \in I \quad T(u) = \lambda_j u$ or $T(u) = 0$

i.e. $\forall n \in \mathbb{N} \quad T(e_n) = \lambda_n e_n$ for some $\lambda_n \in EV(T)$

Rk $T(F^{\perp}) \subset F^{\perp}$ since

$\forall u \in F \quad \forall v \in F^{\perp} \quad \langle u, T(v) \rangle \xRightarrow{\text{self-adjoint}} \langle T(u), v \rangle = \lambda_n \langle u, v \rangle = 0$

Define $T_0 : F^{\perp} \to F^{\perp}$

$u \longmapsto T(u)$

$T_0 \in K(F^{\perp})$ and self-adjoint

so $\sigma(T_0) \setminus \{0\} = EV(T_0) \setminus \{0\}$

if $\lambda \in EV(T_0) \setminus \{0\} \subset EV(T) \setminus \{0\}$ we have

$\exists\, n \in \mathbb{N} \quad \lambda = \lambda_n$

$\exists\, u \in F^{\perp} \quad \lambda u = T(u) = \lambda_n u$

so $\quad u \in F \cap F^{\perp} = \{0\}$

Therefore $\quad EV(T_0) \setminus \{0\} = \phi$

$\inf\limits_{\substack{\|u\|=1 \\ u \in F^{\perp}}} \langle T_0 u, u \rangle = \sup\limits_{\substack{\|u\|=1 \\ u \in F^{\perp}}} \langle T_0 u, u \rangle = 0$

$\Rightarrow \|T_0\| = 0 \Rightarrow T_0 = 0$

~~Hence~~ $\forall u \in F^{\perp} \quad T(u) = T_0(u) = 0$

Hence $F^{\perp} \subset N(T) \subset F$

so $F^{\perp} \subset F \cap F^{\perp} = \{0\}$

Therefore $F = H \qquad \square$

6. Soblev embedding

$\forall f \in L^2(0,1) \quad \exists!\, v \in H^1(0,1) \quad v''(x) + v(x) = f(x) \quad x \in (0,1)$ weakly

i.e. $\int_0^1 v(y)\varphi''(y)\,dy + \int_0^1 v(y)\varphi(y)\,dy = \int_0^1 f(y)\varphi(y)\,dy \quad \forall \varphi \in C_0^\infty(0,1)$

Thm $H^1(0,1) \subset C[0,1]$ (Soblev embedding)

i.e. $\forall u \in H^1(0,1) \quad \exists!\, \underline{u} \in [0,1] \quad u = \underline{u}$ a.e.

Collary $\forall f \in C[0,1] \quad \exists u \in C^2(0,1) \quad u''(x) + u(x) = f(x) \quad \forall x \in (0,1)$

- existence
- uniqueness
- regularity

proof:

let $f \in C[0,1] \subset L^2(0,1)$

By Lax-Milgram Thm: $\exists!\, u \in H^1(0,1)$:

$\forall \varphi \in C_c^\infty(0,1) \quad \int_0^1 u(y)\varphi''(y)\,dy + \int_0^1 u(y)\varphi(y) = \int_0^1 f(y)\varphi(y)\,dy \quad (*)$

By Soblev embedding Thm: $\exists!\, \underline{u} \in C[0,1] \quad u = \underline{u}$ a.e.

By $(*)$ $\forall \varphi \in C_0^\infty(0,1) \quad \underbrace{\int_0^1 \underline{u}(y)\varphi''(y)\,dy}_{=\int_0^1 \underline{u}''(y)\varphi(y)\,dy} + \int_0^1 (\underline{u}(y) - f(y))\varphi(y)\,dy = 0$

i.e. $\underline{u}'' + \underline{u} - f = 0$ weakly, since $\int fg = 0 \quad \forall g \in C_0^\infty \Rightarrow f = 0$ a.e.

Thus $\underline{\underline{u}}'' = f - \underline{u}$ a.e. and $f - \underline{u} \in C[0,1]$

so $\exists!\, v \in C[0,1] \quad \underline{\underline{u}}'' = v$

Exercise: This implies that $\underline{u} \in C^2(0,1)$

Hint: $\underline{u}(x) = \int_0^x \underline{u}'(t)\,dt \qquad \lim_{\delta \to 0} \frac{\underline{u}(x) - \underline{u}(x+\delta)}{\delta} = \lim_{\delta \to 0} -\frac{1}{\delta}\int_t^{t+\delta} \underline{u}'(t)\,dt = -\underline{u}'(x)$

Moreover $0 = \underline{u}''(x) - f(x) + \underline{u}(x)$ a.e. $x \in (0,1)$

and $\underline{u}'' - f + \underline{u} \in C[0,1]$ therefore $\underline{u}''(x) = f(x) - \underline{u}(x) \quad \forall x \in (0,1)$ $\square$

proof of $H^1(0,1) \subset C[0,1]$

let $u \in H^1(0,1)$. Define $\underline{u}(x) = \int_0^x \underbrace{u'(t)}_{\in L^2 \subset L^1(0,1)} dt \quad \forall x \in [0,1]$

WTS (1) $\underline{u} \in C[0,1]$

(2) $\underline{u} = u$ a.e.

(1) let $x, y \in [0,1]$ $x \geq y$

$|\underline{u}(x) - \underline{u}(y)| = |\int_y^x u'(t)\,dt| \overset{C.S}{\leq} \|v'\|_{L^2} |y-x|^{\frac{1}{2}}$

Thus $\underline{u} \in C[0,1]$

(2) WTS $\forall \varphi \in C_c^\infty(0,1)$

$\int \underline{u}\varphi = \int u\varphi$ ($\Rightarrow \int(\underline{u}-u)g = 0 \quad \forall g \in L^2 \Rightarrow \underline{u} - u = 0$ in L^2 $\Rightarrow$ i.e. $\underline{u} = u$ a.e.)

We have that

$\forall \varphi \in C_c^\infty(0,1) \quad \int_0^1 \underline{u}(x)\varphi'(x)\,dx = \int_0^1 \int_0^x u'(t)\,dt\, \varphi'(x)\,dx \xRightarrow{\text{Fubini's Thm}} \int_0^1 (\int_t^1 \varphi'(x)\,dx) u'(t)\,dt$

$= \int_0^1 [\underbrace{\varphi(1)}_{0} - \varphi(t)]\, u'(t)\,dt$ (since $\varphi \in C_c(0,1)$)

$= -\int_0^1 \varphi(t) u'(t)\,dt = \int_0^1 u(t)\varphi'(t)\,dt$

Let $v \in C_c^\infty(0,1)$. Define $h = v - (\int_0^1 v)\psi$

where $\psi \in C_c^\infty(0,1)$ such that $\int_0^1 \psi(x)\,dx = 1$

We have that $h \in C_c^\infty(0,1)$ and $\int_0^1 h(x)\,dx = 0$

Define $\varphi(x) = \int_0^x h(t)\,dt \quad \forall x \in (0,1)$

Then $\varphi'(x) = h(x) \quad \forall x \in (0,1)$

$\varphi' \in C^\infty \Rightarrow \varphi \in C^\infty$

supp $h \subset [\alpha, \beta] \subset (0,1)$

$\forall x > \beta \quad \varphi(x) = \int_0^x h(t)\,dt = \int_0^\beta h(t)\,dt + \int_\beta^x h(t)\,dt = \int_0^\beta h(t)\,dt = \int_0^1 h(t)\,dt \overset{\text{By Def}}{=} 0$

$\forall x < \alpha \quad \varphi(x) = \int_0^x h(t)\,dt = 0$

so supp φ < supp h and thus $\varphi \in C_c^\infty(0,1)$

Now by the first step . we have

$$\int_0^1 \underline{u}(x)\varphi'(x)\,dx = \int_0^1 u(x)\varphi'(x)\,dx$$

i.e. $\int_0^1 \underline{u}(x)h(x)\,dx = \int_0^1 u(x)h(x)\,dx$

$$\int_0^1 [\underline{u}(x) - u(x)]v(x)\,dx = \int_0^1 [\underline{u}(x) - u(x)]\psi(x)\left[\int_0^1 v(y)\,dy\right]dx$$

$$= \int_0^1 \left(\int_0^1 [\underline{u}(x) - u(x)]\psi(x)\right) v(y)\,dy \qquad \forall\, v \in C_0^\infty(0,1)$$

i.e. $\underline{u}(x) - u(x) = \int_0^1 (\underline{u}(z) - u(z))\psi(z)\,dz$ a.e. $x \in (0,1)$

$$u(x) = \underline{u}(x) - \int_0^1 [\underline{u}(z) - u(z)]\psi(z)\,dz$$

7. Duality of Banach space

proof of Lemma 4.2

By continuity: $\exists \delta > 0 \quad \forall x \in X \quad \|x\| < \delta$ we would have $|\langle x^*, x\rangle - \langle x^*, 0\rangle| \leq 1$

$\forall x \in X \quad \|x\| \leq 1 \quad |\langle x^*, \delta x\rangle| \leq 1$

$\forall x \in X \quad \|x\| = 1 \quad |\langle x^*, x\rangle| \leq \frac{1}{\delta}$

proof of Thm 4.3

let $x^*, y^* \in X^*$

$\|x^* + y^*\| := \sup\{\langle x^* + y^*, x\rangle ; \|x\| = 1\} \leq \sup\{x^*(x); \|x\| = 1\} + \sup\{y^*(x); \|x\| = 1\}$

$= \|x^*\| + \|y^*\|$

let $\lambda \in \mathbb{R}^*$

$\|\lambda x^*\| = \sup\{\lambda x^*(x) ; \|x\| = 1\} = \sup\{|\lambda| x^*(x) ; \|x\| = 1\} = |\lambda| \sup\{x^*(x) ; \|x\| = 1\} = |\lambda| \|x^*\|$

if $\|x^*\| = 0$

$\forall x \in X \quad \|x\| = 1 \Rightarrow x^*(x) = 0$

$\Rightarrow \forall x \in X \setminus \{0\} \quad x^*\left(\frac{x}{\|x\|}\right) = 0 \xrightarrow{\text{linearity}} x^* = 0$

So $\underline{\|\cdot\|_{X^*}}$ is norm.

let $(x_n^*)_{n \in \mathbb{N}} \in (X^*)^{\mathbb{N}}$ be Cauchy. let $n, m \in \mathbb{N}$

$\forall x \in X \setminus \{0\} \quad |\langle x_n^*, \frac{x}{\|x\|}\rangle - \langle x_m^*, \frac{x}{\|x\|}\rangle| \leq \|x_n^* - x_m^*\|_{X^*}$

Thus $(\langle x_n^*, x\rangle)_{n \in \mathbb{N}}$ is Cauchy in $\mathbb{R}$ $\forall x \in X$

and $l(x) = \lim\limits_{n \to +\infty} \langle x_n^*, x\rangle$ exists $\forall x \in X$

l is linear. WTS (1) l is continous

(2) $\|x_n^* - l\|_{X^*} \xrightarrow{n \to +\infty} 0$

11)

let $x \in X$. $\|x\|=1$

Pick $n \in \mathbb{N}$. such that $|l_n(x) - \langle x_n^*, x\rangle| \le 1$

Then $|l(x)| \le 1 + |\langle x_n^*, x\rangle| \le 1 + \|x_n^*\|_{X^*} \le 1 + \sup_{n\in\mathbb{N}} \|x_n^*\| < \infty$

$\Downarrow$

So $l \in X^*$ since $(x_n^*)_{n\in\mathbb{N}}$ is Cauchy

12)

let $\varepsilon > 0$. let $N \in \mathbb{N}$ $\|x_n^* - x_N^*\| < \varepsilon$ $\forall n > N$

let $x \in X$ $\|x\|=1$ let $N(x) \in \mathbb{N}$ be such that $N(x) \ge N$. $|l(x) - x_n^*(x)| \le \varepsilon$ $\forall n \ge N(x)$

~~$\forall j \in \mathbb{N}$~~ $\forall j \ge N$ $\forall x \in X$ $\|x\|=1$ $|l(x) - x_j^*(x)| \le |l(x) - x_{N(x)}^*(x)| + |x_{N(x)}^*(x) - x_j^*(x)|$

$$\le \varepsilon + \|x_{N(x)}^* - x_j^*\| \le 2\varepsilon$$

so $\forall \varepsilon > 0$ $\exists N \in \mathbb{N}$ $\forall j \in N$ $\|l - x_j^*\|_{X^*} \le 2\varepsilon$ $\square$

<u>Note</u>: $\langle x, x^*\rangle := x^*(x)$

8. Hahn-Banach theorem extension form

proof of Thm 4.5

$V \neq X$

Step 1 : Pick $v_1 \in X \setminus V$ Define $V_1 = \text{span}\{V, \{v_1\}\}$

$\forall u, v \in V \quad l(u) + l(v) = l(u+v) \leq \| u+v \| \leq \| u - v_1 \| + \| v_1 + u \|$

$\forall u, v \in V \quad l(u) - \| u - v_1 \| \leq -l(v) + \| v_1 + u \|$

$$\sup_{u \in V} l(u) - \| u - v_1 \| \leq \inf_{v \in V} -l(v) + \| v_1 + u \|$$

Pick $R \in [\sup_{u \in V}(l(u) - \| u - v_1 \|), \inf_{v \in V}(-l(v) + \| v_1 + u \|)]$

Define $l^*(u_1) = R$

$l^* : V_1 \longmapsto \mathbb{R}$

$v + \alpha v_1 \longmapsto \alpha R + l(v) \quad \forall v \in V . \forall \alpha \in \mathbb{R}$

l^* is linear and ~~$l^*(u) =$~~ $l^*(v) = l(v) \quad \forall v \in V$

WTS $|l^*(v + \alpha v_1)| \leq \| v + \alpha v_1 \| \quad \forall v \in V \quad \forall \alpha \in \mathbb{R}$

let $v \in V$, $\alpha = 1$

$|l^*(v + v_1)| = |R + l(v)|$

If $R + l(v) > 0$ then $|l^*(v + v_1)| = l(v) + R \leq l(v) + (-l(v) + \| v + v_1 \|) = \| v + v_1 \|$

If $R + l(v) \leq 0$ then $|l^*(v + v_1)| = -l(v) - R \leq -(l(-v) - \| -v - v_1 \|) - l(v) = \| v + v_1 \|$

$\forall v \in V \quad \forall \alpha \in \mathbb{R}^* \quad |l^*(\alpha v_1 + v)| = |\alpha R + l(v)| = |\alpha| \, |R + l(\frac{v}{\alpha})| \leq |\alpha| \, \| \frac{v}{\alpha} + v_1 \| = \| v + \alpha v_1 \|$

Thus $|l^*(\alpha v_1 + v_0)| \leq \| \alpha v_1 + v \|$

If $V_1 = X$ then we're done

By induction, if X is separable, the Thm is proven.

Step 2:

For a pair (W, l_W) with $V \subset W \subset X$, W subspace

$l_W : W \to \mathbb{R}$ linear

and such that $\begin{cases} l_W(x) = l(x) & \forall x \in V \\ |l_W(y)| \leq \|y\| & \forall y \in W \end{cases}$

Define $(W, l_W) \preceq (W', l_{W'})$ if $\begin{cases} W \subset W' \\ W' \text{ subspace} \\ l_{W'}(y) = l_W(y) & \forall y \in W \end{cases}$

Let F be a totally ordered set of such pairs

Define $\tilde{W} = \bigcup_{(W, l_W) \in F} W$. $\forall x \in \tilde{W}$, $\exists (W_x, l_{W_x}) \in F : x \in W$

$l_{\tilde{W}} : x \mapsto l_{W_x}(x)$

Then $(\tilde{W}, l_{\tilde{W}})$ is an upper bound for F

By Zorn's lemma: $\exists (\underline{W}, l_{\underline{W}}) \in F$ maximal i.e. $\forall (W', l_{W'}) \in F$

we have $\begin{cases} W' \subset \underline{W} \\ \cancel{then}\ l_{\underline{W}}(y) = l_{W'}(y) & \forall y \in W' \end{cases}$

If $\underline{W} = X$, then take $l^* = l_{\underline{W}}$ does the job.

If $\underline{W} \neq X$, then $w_1 \in X \setminus \underline{W}$ and construct $(\underline{W}_1, l_{\underline{W}_1})$ by step 1.

Thus $(\underline{W}, l_{\underline{W}})$ is not maximal, contradiction. $\square$

Corollary 1. $\forall x \in X \quad \exists x^* \in X^* \quad \begin{cases} \|x^*\| = 1 \\ \langle x^*, x \rangle = \|x\| \end{cases}$

proof: Let $x \in X$. Define $l: \text{span}\{x\} \to \mathbb{R}$

$$\alpha x \mapsto \alpha\|x\|$$

l linear and ~~$\|l(x)\|$~~ $|l(x)| = |\alpha|\,\|x\| = \|\alpha x\|$

By HB $\exists x^* \in X^* : \begin{cases} \|x^*\| \leq 1 \\ \langle x^*, y \rangle = l(y) \end{cases} \quad \forall y \in \text{Span}\{x\}$

In paticular, $\langle x^*, x \rangle = l(x) = \|x\|$

$x^*\left(\frac{x}{\|x\|}\right) = 1 \qquad$ so $\quad \|x^*\| \geq 1 \qquad \square$

Corollary 2. $T \in B(X, Y) \quad \langle T^*(y^*), x \rangle := \langle y^*, T(x) \rangle \quad \begin{matrix} \forall x \in X \\ \forall y^* \in Y^* \end{matrix}$

Then $\|T^*\| = \|T\|$

proof: let $y^* \in Y^*$, $x^* \in X$

without loss of gerneralaity

$$|\langle T^*(y^*), x \rangle| = |\langle y^*, T(x) \rangle| = |\langle y^*, \frac{T(x)}{\|T(x)\|} \rangle|\, \|T(x)\| \leq \|y^*\|\,\|T(x)\| \leq \|y^*\|\,\|T\|_{B(X,Y)}\|x\|$$

so $\|T^*\|_{B(Y^*,X^*)} \leq \|T\|_{B(X,Y)}$

Let $\varepsilon > 0$. Pick $x \in X : \begin{cases} \|x\| = 1 \\ \|T(x)\| \geq \|T\| - \varepsilon \end{cases}$

By corollary 1, $\exists y^* \in Y^* : \begin{cases} \|y^*\| = 1 \\ \langle y^*, T(x) \rangle = \|T(x)\| \end{cases}$

$\|T(x)\| \geq \|T\| - \varepsilon$

$$\|T^*\|_{B(Y^*,X^*)} \geq \|T^*(y^*)\|_{X^*} \geq |\langle T^*(y^*), x \rangle| = \langle y^*, T(x) \rangle = \|T(x)\| \geq \|T\| - \varepsilon$$

since ε was arbitary. $\|T^*\|_{B(Y^*,X^*)} \geq \|T\|_{B(X,Y)} \qquad \square$

9. Hahn-Banach Theorem (geometric-form)

Lemma 5.3 (Minkowski gauge)

$A \neq \phi$ convex open

$p(x) = \inf \{ \alpha > 0 \ ; \ \alpha^{-1} x \in A \}$

(1) let $\alpha \in \mathbb{R}^*$, $\beta > 0$, $x \in X$

$\frac{1}{\alpha} x \in A \iff \frac{\beta}{\alpha\beta} x \in A$

$p(x) \leq \alpha \iff \cancel{\beta} \ p(\beta x) \leq \alpha \beta$

Thus $p(\beta x) = \beta p(x)$

(3) $0 \in A$ and A open. $\exists \delta > 0$ $B(0, \delta) \subset A$

let $x \in X \setminus \{0\}$ $\frac{x}{\|x\|} \frac{\delta}{2} \in B(0, \delta) \subset A$

Therefore $p(x) \leq \frac{\|x\|}{\frac{\delta}{2}} =: M\|x\|$

(4) WTS $A = \{ x \in X \ ; \ p(x) < 1 \}$

- let $x \in A$ $\exists \varepsilon$ $(1+\varepsilon) x \in A \Rightarrow \exists \varepsilon > 0$ $p(x) \leq \frac{1}{1+\varepsilon} < 1$

  so $A \subset \{ x \in X \ ; \ p(x) < 1 \}$

- let $x \in X$: $p(x) < 1$

  $\exists \alpha \in (0,1)$ $\frac{1}{\alpha} x \in A \overset{\{0 \in A,\ \text{convexity of } A}{\Longrightarrow} \exists \alpha \in (0,1)$ $(\frac{1}{\alpha} x) \cdot \alpha + (1-\alpha) 0 \in A$

  $\Rightarrow x \in A$

(2) let $x, y \in A$, $\varepsilon > 0$

$\frac{x}{p(x)+\varepsilon} \in A$ $\frac{y}{p(y)+\varepsilon} \in A$

By convexity $\forall t \in (0,1)$ $\frac{tx}{p(x)+\varepsilon} + \frac{(1-t)y}{p(y)+\varepsilon} \in A$

Pick $t = \frac{p(x)+\varepsilon}{p(x)+p(y)+2\varepsilon}$

Then $\frac{1}{p(x)+p(y)+2\varepsilon} \cdot x + \frac{y}{p(x)+p(y)+2\varepsilon} \in A$

i.e. $p(x+y) \leq p(x) + p(y) + 2\varepsilon$ $\forall \varepsilon > 0$

so $p(x+y) \leq p(x) + p(y)$ $\square$

lemma 5.4 $B \neq \phi$ convex open. $x_0 \notin B$. $A = \{x_0\}$ $\exists x^* \in X^*$, $\exists \alpha \in \mathbb{R}$

P_{α, x^*} separates A from B.

proof Define $p : x \mapsto \inf\{\alpha > 0 ;\ \alpha^{-1} x \in B\}$

(X, p) is a Banach space

Applying HB in (X, p) to extend $g : \operatorname{span}\{x_0\} \to \mathbb{R}$
$$t x_0 \mapsto t$$

This gives $\exists\, x^* \in (X, p)^*$: $\begin{cases} |x^*(x)| \leq p(x) & \forall x \in X \\ x^*(tx) = g(tx_0) \end{cases}$

$|x^*(x)| \leq p(x) \overset{\text{lemma}}{\leq} M\|x\|$ $\forall x \in X$ so $x^* \in X^*$

Take $\alpha = x^*(x_0) = 1$ since $x^*(x_0) = g(x_0) = 1$

$\begin{cases} \forall x \in B & x^*(x) \leq p(x) \leq 1 \quad \text{(By lemma)} \\ \forall x \in A & x^*(x) = x^*(x_0) = 1 \geq 1 \end{cases}$ $\square$

Hahn-Banach Thm (geometric form)

$A, B \subset X$ convex non-empty. $A \cap B = \phi$. B open $\Rightarrow A, B$ separated

proof

Define $Y := A - B = \{u - v \in X ;\ u \in A,\ v \in B\}$

Y is convex and non-empty

$Y = \bigcup_{u \in A} \{u - v ;\ v \in B\}$ is open as a union of open sets

$0 \notin Y$ since $A \cap B = \phi$

By the lemma : $\exists\, x^* \in X^*$: $P_{x^*(0), x^*}$ separates Y from $\{0\}$

i.e. $\forall u \in A$ $\forall v \in B$ $\langle x^*, u - v \rangle \leq x^*(0) = 0$

$\Rightarrow$ $\forall u \in A$ $\forall v \in B$ $\langle x^*, u \rangle \leq \langle x^*, v \rangle$

$\Rightarrow$ $\sup_{u \in A} \langle x^*, u \rangle \leq \inf_{v \in B} \langle x^*, v \rangle$

Pick $\alpha \in [\sup_{u \in A} \langle x^*, u \rangle,\ \inf_{v \in B} \langle x^*, v \rangle]$

$\forall u \in A$ $\forall v \in B$ $\begin{cases} \langle x^*, u \rangle \leq \alpha \\ \langle x^*, v \rangle \geq \alpha \end{cases}$ $\square$

Corollary 5.5

$A, B \subset X$ closed non-empty convex. $A \cap B = \phi$. B compact

$\Rightarrow$ A, B strictly separated

<u>proof</u>

let $Y := A - B$

<u>claim</u> : Y is closed

Indeed let $(u_n)_{n \in \mathbb{N}} \in A^{\mathbb{N}}$, $(v_n)_{n \in \mathbb{N}} \in B^{\mathbb{N}}$, $y \in X$

such that $u_n - v_n \xrightarrow{n \to \infty} y$

since B is compact.

$\exists$ $(v_{n_k})_{k \in \mathbb{N}} \in B^{\mathbb{N}}$ $\exists v \in B$ $v_{n_k} \xrightarrow{k \to \infty} v$

Thus $u_{n_k} - v_{n_k} \xrightarrow{k \to \infty} y$

and $u_{n_k} \xrightarrow{k \to \infty} y + v =: u \in A$

Therefore $y = \underbrace{u}_{\in A} - \underbrace{v}_{\in B} \in A - B$ □

$0 \notin Y$ since $A \cap B = \phi$

Y^c open. so $\exists \delta > 0$ $B(0, \delta) \cap Y = \phi$

By HB. $B(0, \delta)$ and Y are separated i.e.

$\exists \alpha \in \mathbb{R}$, $\exists x^* \in X^*$
$$\begin{cases} x^*(\delta w) \geq \alpha & \forall w \in B(0,1) \\ x^*(u - v) \leq \alpha & \forall u \in A \quad \forall v \in B \end{cases}$$

$\forall u \in A$
$\forall v \in B$ This gives $x^*(u-v) \leq x^*(\delta w)$ $\forall w \in B(0,1)$

$$\leq -\delta \|x^*\|$$

let $\varepsilon = \frac{\delta \|x^*\|}{2}$

We have that $\forall u \in A$ $\forall v \in B$ $x^*(u - v) \leq -2\varepsilon$

$\forall\, u \in A \qquad \forall v \in B$

$$x^*(u) + \varepsilon \leq x^*(v) - \varepsilon$$

$$\sup_{u\in A} (x^*(u) + \varepsilon) \leq \inf_{v \in B} (x^*(v) - \varepsilon)$$

Pick $\alpha \in [\sup_{u\in A} (x^*(u) + \varepsilon), \inf_{v\in B} (x^*(v) - \varepsilon)]$

Then $\begin{cases} x^*(u) + \varepsilon \leq \alpha & \forall\, u \in A \\ x^*(v) - \varepsilon \geq \alpha & \forall\, v \in B \end{cases}$ □

Thm 5.7

<u>Step 1</u> : $\overline{Y} \subset Y^{\perp\perp}$

let $y \in \overline{Y}$. Pick $(y_n)_{n\in\mathbb{N}} \in Y^{\mathbb{N}}$: $\|y_n - y\| \xrightarrow{n\to\infty} 0$

let $y^* \in Y^{\perp}$ $\quad \langle y^*, y_n \rangle = 0 \quad \forall n \in \mathbb{N}$ (since $y_n \in Y$)

so $\langle y^*, y \rangle = \lim_{n\to\infty} \langle y^*, y_n \rangle = 0$ i.e. $y \in Y^{\perp\perp}$

<u>Step 2</u> : $Y^{\perp\perp} \subset \overline{Y}$

By contradiction.

Assume $\exists y_0 \in Y^{\perp\perp} \setminus \overline{Y}$

Then $\overline{Y}$ and $\{y_0\}$ are strictly separated by the corollary

i.e. $\exists \alpha \in \mathbb{R}$ $\quad \exists y^* \in X^*$ $\quad \begin{cases} \langle y^*, y \rangle \leq \alpha - \varepsilon & \forall y \in \overline{Y} \\ \langle y^*, y_0 \rangle \geq \alpha + \varepsilon \end{cases}$

$\exists \varepsilon > 0$

<u>Remark</u> : $\langle y^*, y \rangle = 0 \quad \forall y \in \overline{Y}$

since for $y \in \overline{Y}$, such that $\langle y^*, y \rangle = \beta > 0$

$2|\alpha| = \langle y^*, \frac{y}{\beta} 2|\alpha| \rangle \leq \alpha$ contradiction

Thus $\langle y^*, y_0 \rangle \geq \alpha + \varepsilon > \langle y^*, y \rangle = 0 \quad \forall y \in \overline{Y}$

$\langle y^*, y \rangle = 0 \quad \forall\, y \in \overline{Y}$ so $y^* \in \overline{Y}^{\perp} \subset Y^{\perp}$

$\langle y^*, y_0 \rangle > 0 \Rightarrow y_0 \notin Y^{\perp\perp}$ contradiction

11. L^p spaces

Thm 6.3 (Hölder inequality)

$f \in L^p,\ g \in L^{p'} \Rightarrow fg \in L^1$ and $\|fg\|_1 \leq \|f\|_p \|g\|_{p'}$

proof

Claim $\forall a, b > 0 \quad ab \leq \frac{1}{p} a^p + \frac{1}{p'} b^{p'} \quad (1)$

$\ln$ concave $+ \quad \frac{1}{p} + \frac{1}{p'} = 1$

$\forall a, b > 0 \quad \frac{1}{p} \ln(a^p) + \frac{1}{p'} \ln(b^{p'}) \leq \ln(\frac{1}{p} a^p + \frac{1}{p'} b^{p'})$

$\forall a, b > 0 \quad \ln(ab) \leq \ln(\frac{1}{p} a^p + \frac{1}{p'} a^{p'})$

$\ln$ increasing function $\quad ab \leq \frac{1}{p} a^p + \frac{1}{p'} a^{p'}$

let $f \in L^p,\ g \in L^{p'}$

$\int |fg| \underset{(1)}{\leq} \int (\frac{1}{p} |f|^p + \frac{1}{p'} |g|^{p'}) = \frac{1}{p} \|f\|_p^p + \frac{1}{p'} \|g\|_{p'}^{p'}$

let $\lambda > 0$, replace f by λf

$\lambda \|fg\|_1 \leq \frac{\lambda^p}{p} \|f\|_p^p + \frac{1}{p'} \|g\|_{p'}^{p'}$

choose $\lambda = \frac{\|g\|_{p'}^{\frac{p'}{p}}}{\|f\|_p}$

$\frac{\|g\|_{p'}^{\frac{p'}{p}}}{\|f\|_p} \|fg\|_1 \leq \frac{1}{p} \frac{\|g\|_{p'}^{p'} \|f\|_p^p}{\|f\|_p^p} + \frac{1}{p'} \|g\|_{p'}^{p'} = \|g\|_{p'}^{p'}$

$\|fg\|_1 \leq \|f\|_p \|g\|_{p'}^{p'(1-\frac{1}{p})} = \|f\|_p \|g\|_{p'}$ $\square$

Thm $1 \le p \le \infty$ L^p is a Banach space

<u>proof</u>

let $f, g \in L^p$ $1 < p < \infty$

$$\|f+g\|_p^p \le \int |f+g|^{p-1}|f| + \int |f+g|^{p-1}|g| \overset{\text{Hölder}}{\le} \left(\int |f+g|^{(p-1)p'}\right)^{\frac{1}{p'}} \|f\|_p + \left(\int |f+g|^{(p-1)p'}\right)^{\frac{1}{p'}} \|g\|_{p'}$$

$= \|f+g\|_p^{p-1} (\|f\|_p + \|g_p\|)$ triangle ineq holds. $\|\cdot\|_p$ is a norm

(leave rest as exercise)

<u>Case 1</u> : $1 \le p < \infty$

let $(f_n)_{n \in \mathbb{N}} \in (L^p)^{\mathbb{N}}$ be a Cauchy sequence

Pick $(f_{n_k})_{k \in \mathbb{N}} \in (L^p)^{\mathbb{N}}$; $\|f_{n_{k+1}} - f_{n_k}\| \le 2^{-k}$ $\forall k \in \mathbb{N}$

Define $\forall x \in \mathbb{R}$: $g_m(x) = \sum_{k=0}^{m} |f_{n_{k+1}}(x) - f_{n_k}(x)|$

$\|g_m\|_p \le \sum_{k=0}^{m} \|f_{n_{k+1}} - f_{n_k}\|_p \le \sum_{k=0}^{m} 2^{-k} \le 2$ $\forall m \in \mathbb{N}$

By monotone covergence Thm $\exists g \in L^p$ $g_m \to g$ a.e.

$\forall l \in \mathbb{N}^*$
$\forall x \in \mathbb{R}$ $|f_{n_{k+l}}(x) - f_{n_k}(x)| = |\sum_{j=k}^{n+l-1} (f_{n_{j+1}}(x) - f_{n_j}(x))| \le |g_{k+l}(x) - g_k(x)| \le |g(x) - g_k(x)|$

since $g_k \to g$ a.e. $(f_{n_k}(x))_{k \in \mathbb{N}}$ is Cauchy for a.e. x

let $f(x) = \lim_{k \to \infty} f_{n_k}(x)$ a.e. x

$|f(x) - f_{n_k}(x)| \le |\sum_{j=k}^{\infty} f_{n_{j+1}}(x) - f_{n_j}(x)| \le \sum_{j=k}^{\infty} |f_{n_{j+1}}(x) - f_{n_j}(x)| = g(x) - g_k(x)$

$|f(x)| \le |g(x)|$ a. e. x

Thus $f \in L^p$ and $\|f\|_p \le \|g\|_p$

Thus $\|f_{n_k} - f\|_p \xrightarrow[n \to +\infty]{} 0$ by dominate convergence Thm

Since (f_n) Cauchy : $\|f_n - f\|_p \xrightarrow[n \to \infty]{} 0$

Case 2 : $p = +\infty$

Let $(f_n)_{n\in\mathbb{N}} \in (L^\infty)^{\mathbb{N}}$ be Cauchy

See Brezis' book

12. Dual of L^p

proof of Lemma 6.6:

let $l \in (L^p)^*$ and $f \in L^p$

If f is positive then $\bar{l}(f) := \sup\{l(g);\ g \in L^p : 0 \le g \le f\}$

If f is not positive, let us define

$f_+(x) = \max(f(x), 0) \ge 0 \qquad \forall x \in \mathbb{R}$

$f_-(x) = f_+(x) - f(x) \ge 0 \qquad \forall x \in \mathbb{R}$

Then define $\forall f \in L^p$

$l_+(f) := \bar{l}(f_+) - \bar{l}(f_-)$

$l_-(f) := l_+(f) - l(f)$

WTS $l_+, l_- \in (L^p)^*$ and l_+, l_- are positive

if f is positive: $l_+(f_+) = \bar{l}(f_+) \ge l(0) = 0$ so l_+ is positive

$l_-(f_+) = \bar{l}(f_+) - l(f_+) \ge l(f_+) - l(f_+) = 0$ so l_- is positive

It remains to show that $l_+ \in (L^p)^*$

Step 1

let $f, h \in L^p$. assume they are positive: $f = f_+ \quad h = h_+$

let ~~$\forall$~~ $g_1, g_2 \in L^p \quad \begin{cases} 0 \le g_1 \le f \\ 0 \le g_2 \le h \end{cases}$

Then $0 \le g_1 + g_2 \le f + h$

so $l_+(f+h) = \bar{l}(f+h) \ge l(g_1+g_2) = l(g_1) + l(g_2)$

$l_+(f+h) \ge \bar{l}(f) + \bar{l}(h) = l_+(f) + l_+(h)$

let $\varepsilon > 0$ let $0 \le g \le f+h$: $l_+(f+h) = \bar{l}(f+h) < f(g) + \varepsilon$

Define $g_1 = \min(g, f) \qquad 0 \le g_1 \le f$

$\qquad 0 \le g - g_1 \le h$

$g = g_1 + g - g_1$

$l_+(f+h) < l(g) + \varepsilon = l(g_1) + l(g - g_1) + \varepsilon \le \bar{l}(f) + \bar{l}(h) + \varepsilon = l_+(f) + l_+(h) + \varepsilon$

we have $l_+(f) + l_+(h) \le l_+(f+h) \le l_+(f) + l_+(h) + \varepsilon \quad \forall \varepsilon > 0$

Thus $l_+(f+h) = l_+(f) + l_+(h) \quad \forall f, h \in L^p : \begin{cases} f = f_+ \\ h = h_+ \end{cases}$

step 2:

let $f, h \in L^p$

$f + h = (f_+ - f_-) + (h_+ - h_-) = (f+h)_+ - (f+h)_-$

Thus $(f+h)_+ + (f_- + h_-) = (f+h)_- + f_+ + h_+$

and by step 1 $\bar{l}((f+h)_+) + \bar{l}(f_- + h_-) = \bar{l}(f+h)_- + \bar{l}(f_+) + \bar{l}(h_+)$

$l_+(f+h) = \bar{l}((f+h)_+) - \bar{l}((f+h)_-) = \bar{l}(f_+) + \bar{l}(h_+) - \bar{l}(f_-) - \bar{l}(h_-)$

$= l_+(f) + l_+(h)$

Exercise $l_+(\lambda f) = \lambda l_+(f) \quad \forall \lambda \in \mathbb{R}$ (Hint: do $\lambda > 0$ first)

$|l_+(f)| \le \bar{l}(f_+) + \bar{l}(f_-) \le \sup\{\|l\|_{(L^p)^*} \|g\|_{L^p} ; 0 \le g \le f_+\} + \sup\{\|l\|_{(L^p)^*} \|g\|_{L^p} ; 0 \le g \le f_-\}$

$\le \|l\|_{(L^p)^*} (\|f_+\|_{L^p} + \|f_-\|_{L^p}) \le 2 \|l\|_{(L^p)^*} \|f\|_{L^p}$

Thus $l_+ \in (L^p)^*$ $\square$

Thm. $(L^p)^* = L^{p'}$ $(1 \le p < \infty)$

meaning that $(L^p)^* = \{l_g : L^p \to \mathbb{R} ; g \in L^{p'}\}$, and $\|l_g\|_{(L^p)^*} = \|g\|_{L^{p'}}$

where $l_g : f \mapsto \int fg$

proof

step 1: $l_g \in (L^p)^*$ and $\|l_g\|_{(L^p)^*} = \|g\|_{L^{p'}} \quad \forall g \in L^{p'}$

$\forall g \in L^p \quad |l_g(f)| \le \int |fg| \overset{\text{Hölder}}{\le} \|f\|_p \|g\|_{p'}$

$\|l_g\|_{(L^p)^*} \le \|g\|_{p'}$

let $g \in L^{p'}$. Define $f(x) = \mathrm{sgn}(g(x)) \dfrac{|g(x)|^{p'-1}}{\|g\|_{p'}^{p'-1}}$ $\forall x \in \mathbb{R}$

where $\mathrm{sgn}(g(x)) = \begin{cases} 1 & \text{if } g(x) \geq 0 \\ -1 & \text{if } g(x) < 0 \end{cases}$

$$\|f\|_p^p = \int \frac{|g'(x)|^{(p'-1)p}}{\|g\|_{p'}^{(p'-1)p}} dx = \frac{\|g\|_{p'}^{p'}}{\|g\|_{p'}^{p'}} = 1$$

$$L_g(f) = \int \frac{|g(x)||g(x)|^{p'-1}}{\|g\|_{p'}^{p'-1}} dx = \frac{\|g\|_{p'}^{p'}}{\|g\|_{p'}^{p'-1}} = \|g\|_{p'}$$

Thus $\|g\|_{p'} \leq \|L_g\|_{(L^p)^*}$

<u>step 2:</u> let $L \in (L^p)^*$. By the lemma WLOE L is positive

Define Let $N \in \mathbb{N}$ $\nu_N(E) := L(1_{E \cap B(0,N)})$ $\forall E \subset \mathbb{R}$ Borel

<u>Claim:</u> ν_N is a measure abosolutely continous with Lebesgue measure

<u>Rk</u>: $\nu_N(E) \leq \|L\|_{(L^p)^*} \| 1_{E \cap B(0,N)} \|_{L^p} \leq \|L\|_{(L^p)^*} |E \cap B(0,N)|^{\frac{1}{p}}$ (1)

In particular. $|E| = 0 \Rightarrow \nu_N(E) = 0$

If $E = \bigcup_{n=0}^{\infty} E_n$ $(E_n)_{n \in \mathbb{N}}$ disjoint Borel sets

WTS $\nu_N(E) = \sum_{n=0}^{\infty} \nu_N(E_n)$

let $M \in \mathbb{N}$

$$\sum_{n=0}^{M} \nu_n(E_n) = L\left(\sum_{n=0}^{m} 1_{E_n \cap B(0,N)}\right) = L\left(1_{\bigcup_{n=0}^{m} E_n \cap B(0,N)}\right) = \underbrace{\nu_n(E) - L\left(1_{\bigcup_{n=M+1}^{\infty} E_n \cap B(0,N)}\right)}_{\leq \nu_n(E)}$$

$$\begin{aligned} \nu_N(E) &\leq \sum_{n=0}^{M} \nu_n(E_n) + \nu_N\left(\bigcup_{n=M+1}^{\infty} E_n \cap B(0,N)\right) \\ &\overset{(1)}{\leq} \sum_{n=0}^{M} \nu_N(E_n) + \|L\|_{(L^p)^*} \left|\bigcup_{n=M+1}^{\infty} E_n \cap B(0,N)\right|^{\frac{1}{p}} \\ &\leq \sum_{n=0}^{m} \nu_n(E_n) + \|L\|_{(L^p)^*} \Big(\underbrace{\sum_{n=M+1}^{\infty} |E_n \cap B(0,N)|}_{\leq |B(0,N)| = 2N}\Big)^{\frac{1}{p}} \xrightarrow{M \to +\infty} \sum_{n=0}^{\infty} \nu_N(E_n) \end{aligned}$$

which proves the claim

Therefore by the Raden-Nykedym differentiation Thm

$\forall N \in \mathbb{N} \quad \exists g_N \in L^1(\mathbb{R}) \quad \forall E \in \mathcal{S} \quad \nu_N(E) = \int_E g_N$

Rk: If $E \subset B(0,M)$ $M \in \mathbb{N}$

$\forall N \geq M \quad \nu_N(E) = \nu_M(E) := l(1_{E \cap B(0,M)})$

Thus if $x \in B(0,M)$ $\quad g_N(x) = g_M(x) \quad \forall N \geq M$

Define $g(x) = \lim_{n \to +\infty} g_N(x) \quad \forall x \in \mathbb{R}$

we have that $\nu_N(E) = \int_E g$ if $E \in B(0,N)$

$= \int 1_E g$

and for $f = \sum_{k=0}^{K} c_k 1_{E_k}$ with $(c_k)_{k=0}^{K} \in \mathbb{R}^{K+1}$

and $(E_k)_{k=0}^{K} \in \mathcal{B}^{K+1}$ with $E_k \subset B(0,N)$

we have that $l(f) = \sum_{k=0}^{K} c_k \nu(E_k) = \int fg = l_g(f)$

$|l_g(f)| \leq \|l\|_{(L^p)^*} \|f\|_{L^p} \quad \forall f \in L^p$ simple function compact supported

But the set of compact supported function is dense in L^p

By density l_g has a unique extension to an element $(L^p)^*$

Therefore $l_g \in (L^p)^*$ and $l_g = l$

By step 1 $\|l_g\|_{(L^p)^*} = \|g\|_{L^{p'}}$ $\square$

13. interpolation

weak type p-p definition

$T \in B(L^p) \Rightarrow T \in B(L^p, L^{p,\infty})$

$\alpha^p \mu(|Tf|>\alpha) \le \alpha^p \int_\Omega \frac{|Tf|^p}{\alpha^p} = \|Tf\|_p^p \le \|T\|^p \|f\|_p^p$

Thm 6.9 (Marcinkiewicz)

let $f \in S \in L^p$ (S dense)

$\|T(f)\|_p^p = \int_0^\infty p\alpha^{p-1} \mu(|Tf|>\alpha)\, d\alpha$

Given $\alpha>0$. Define $g_\alpha = f 1_{|f|\le\alpha}$, $h_\alpha = f 1_{|f|>\alpha}$, $f = g_\alpha + h_\alpha$

$\|T(f)\|_p^p = \int_0^\infty p\alpha^{p-1} \mu(|Tf|>\alpha) \le \int_0^\infty p\alpha^{p-1} \mu(|Tg_\alpha|>\frac{\alpha}{2})\, d\alpha + \int_0^\infty p\alpha^{p-1}\mu(|Th_\alpha|>\frac{\alpha}{2})\, d\alpha$

$\le \int_0^\infty p\alpha^{p-1} (\frac{\alpha}{2})^{-p_1} C_1 \|g_\alpha\|_{p_1}^{p_1}\, d\alpha + \int_0^\infty p\alpha^{p-1} (\frac{\alpha}{2})^{-p_0} C_0 \|h_\alpha\|_{p_0}^{p_0}\, d\alpha$

$= C_1 \int_0^\infty p\alpha^{p-1} (\frac{\alpha}{2})^{-p_1} \int_\Omega 1_{|f|\le\alpha}(w) |f(w)|^{p_1} d\mu(w)\, d\alpha + C_0 \int_0^\infty p\alpha^{p-1} (\frac{\alpha}{2})^{-p_0} \int_\Omega 1_{|f|>\alpha}(w) |f(w)|^{p_0} d\mu(w)\, d\alpha$

$\le \frac{C_1}{2^{p_1}} \int_\Omega |f(w)|^{p_1} \left(\int_{|f(w)|}^\infty p\alpha^{p-1} \alpha^{-p_1}\, d\alpha \right) d\mu(w) + \frac{C_0}{2^{p_0}} \int_\Omega |f(w)|^{p_0} \left(\int_0^{|f(w)|} p\alpha^{p-1}\, d\alpha \right) d\mu(w)$

$= \frac{C_1}{2^{p_1}} \int_\Omega |f(w)|^{p_1} \frac{p}{p_1-p_0} |f(w)|^{p-p_1} d\mu(w) + \frac{C_0}{2^{p_0}} \int_\Omega |f(w)|^{p_0} \frac{p}{p-p_0} |f(w)|^{p-p_0} d\mu(w)$

$\le \left(\frac{C_1}{2^{p_1}} \frac{p}{p_1-p} + \frac{C_0}{2^{p_0}} \frac{p}{p-p_0} \right) \|f\|_p^p$

14. Application to the Riesz projection

Thm 6.10 For all $p \in (1,\infty)$, $P \in B(L^p)$

proof Let $f = \sum_{n=-N}^{N} \langle f, e_n \rangle e_n$ for some $N \in \mathbb{N}^*$

WTS $\exists C > 0 \quad \| P(f) \| \leq C \|f\|$

without loss of generality ~~$\exists c$~~ $= 0$ #

$\langle f, e_0 \rangle = \int_0^1 f(t)\, dt = 0$

$P(f) = \sum_{n=1}^{N} \langle f, e_n \rangle e_n$

We note that $\int P(f)^M = \langle P(f)^M, e_0 \rangle = 0 \qquad \forall M \in \mathbb{N}$

since $P(f)^M = \sum_{n=1}^{\infty} \langle P(f)^M, e_n \rangle e_n$

because $e_n \cdot e_m = e_{n+m}$

$P = \frac{1}{2}(I + iT) \qquad$ where $T(f) = -i \sum_{n=-N}^{N} \langle f, e_n \rangle \operatorname{sgn}(n) e_n$

We have that $\forall k \in \mathbb{N}^* \qquad \int (f + iT(f))^{2k} = 0$

$\Rightarrow \forall k \in \mathbb{N}^* \qquad \sum_{l=0}^{2k} i^{2k-l} \binom{2k}{l} \int f^l T(f)^{2k-l} = 0 \qquad (1)$

since $f(t) \in \mathbb{R} \quad \forall t \in (0,1)$

$$\sum_{n=-N}^{N} \langle f, e_n \rangle (\cos(nt) + i \sin(nt)) = \sum_{n=1}^{N} \langle f, e_n + e_{-n} \rangle \cos(nt) + \underbrace{i \sum_{n=1}^{N} \langle f, e_n - e_{-n} \rangle \sin(nt)}_{\in \mathbb{R}}$$

$\Rightarrow Tf(t) \in \mathbb{R} \quad \forall t \in (0,1)$

Taking real part of (1)

$\Rightarrow \forall k \in \mathbb{N}^* \qquad \sum_{j=0}^{k} (-1)^{k-j} \binom{2k}{2j} \int f^{2j} T(f)^{2k-2j} = 0$

$\Rightarrow \forall k \in \mathbb{N}^* \qquad (-1)^k \| T(f) \|_{2k}^{2k} = - \sum_{j=1}^{k} (-1)^{k-j} \binom{2k}{2j} \underline{\int f^{2j} T(f)^{2k-2j}}$

use Hölder with $p = \frac{2k}{2k-2j} \qquad \frac{1}{p'} = 1 - \frac{1}{p} = \frac{2j}{2k}$

$\| T(f) \|_{2k}^{2k} \leq \sum_{j=1}^{k} \binom{2k}{2j} \| T(f) \|_{2k}^{2k-2j} \|f\|_{2k}^{2j}$

$\frac{\| T(f) \|_{2k}^{2k}}{\|f\|_{2k}^{2k}} \leq \sum_{j=1}^{k} \binom{2k}{2j} \| T(f) \|_{2k}^{2k-2j} \|f\|_{2k}^{2j-2k}$

Define $R = \dfrac{\|T(f)\|_{2k}}{\|f\|_{2k}}$

$$R^{2k} \leq \sum_{j=1}^{k} \binom{2k}{2j} R^{2k-2j} \leq (R+1)^{2k} - R^{2k} \leq 2k(R+1)^{2k-1} \quad (*)$$

If $R \leq 1$ then $\|T\|_{B(L^{2k})} \leq 1$

If $R \geq 1$ then by $(*)$ $R \leq 2k\left(1+\frac{1}{R}\right)^{2k-1} \leq 2k \cdot 2^{2k-1}$

Thus $\|T\|_{B(L^{2k})} \leq 2k \cdot 2^{k-1} \quad \forall k \in \mathbb{N}^*$

since $P = \frac{1}{2}(I + iT)$ so $P \in B(L^{2k}) \quad \forall k \in \mathbb{N}^*$

We have that $P \in B(L^p) \quad \forall p \in (2k, 2(k+1)) \quad \forall k \in \mathbb{R}^*$ by interpolation

Therefore $P \in B(L^p) \quad \forall p \in [2, \infty)$

let $p \in (1,2]$

let $f \in L^p \cap L^2$, $g \in L^{p'} \cap L^2$

$$\langle P(f), g\rangle = \sum_{n\in\mathbb{Z}} \sum_{m\in\mathbb{Z}} \langle P(f), e_n\rangle \langle g, e_m\rangle \langle e_n, e_m\rangle$$

$$= \sum_{n\in\mathbb{Z}} \langle P(f), e_n\rangle \langle g, e_n\rangle$$

$$= \sum_{n\in\mathbb{N}} \langle f, e_n\rangle \langle g, e_n\rangle$$

$$= \sum_{n\in\mathbb{Z}} \langle f, e_n\rangle \langle P(g), e_n\rangle$$

$$= \langle f, P(g)\rangle$$

$$|\langle P(f), g\rangle| \leq \int |f||P(g)| \overset{\text{Hölder}}{\leq} \|f\|_p \|P(g)\|_{p'} \leq \|f\|_p \|P\|_{B(L^{p'})} \|g\|_{p'}$$

since $L^p = (L^{p'})^*$

$$\|P(f)\|_{L^p} = \sup\{\langle P(f), g\rangle \ ; \ g \in L^{p'}, \|g\|_{L^{p'}} = 1\} \leq \|P\|_{B(L^{p'})} \|f\|_p \quad \square$$

15. Baire's Thm

Thm 7.1 (Baire)

let $(O_n)_{n\in\mathbb{N}}$ be dense. Assume $\overline{O_n} = X \quad \forall n \in \mathbb{N}$

let $G = \bigcap_{n\in\mathbb{N}} O_n$. WTS $\overline{G} = X$ i.e. $\forall \Omega \subset X$ open. $\Omega \cap G \neq \phi$

let Ω be open

Pick $x_0 \in \Omega$ and $R_0 > 0$: $B(x_0, R_0) \subset \Omega$

since $\overline{O_1} = X$, pick $x_1 \in O_1 \cap B(x_0, R_0)$

Choose $R_1 \in (0, \frac{R_0}{2})$: $\overline{B}(x_1, R_1) \subset O_1 \cap B(x_0, R_0)$

Inductively, we choose $(x_n) \in X^{\mathbb{N}}$ and $(R_n)_{n\in\mathbb{N}} \in (0, +\infty)^{\mathbb{N}}$

$$\begin{cases} \overline{B(x_n, R_n)} \subset O_n \cap B(x_{n-1}, R_{n-1}) & \forall n \in \mathbb{N}^* \\ R_n \leq \frac{R_{n-1}}{2} & \forall n \in \mathbb{N}^* \end{cases}$$

We have that $R_n \leq \frac{R_0}{2^n} \quad \forall n \in \mathbb{N}$ so $d(x_n, x_{n+1}) \leq \frac{R_0}{2^n} \quad \forall n \in \mathbb{N}$

Therefore $\forall n, m \in \mathbb{N}$ $n > m$: $d(x_n, x_m) \leq R_0 \sum_{k=m}^{n} 2^{-k}$

and thus $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, hence convergent

$\exists x \in X \quad d(x_n, x) \longrightarrow 0$

$\forall n \in \mathbb{N} \quad x \in \overline{B(x_n, R_n)} \subset O_n \cap B(x_0, R_0)$

$\forall n \in \mathbb{N} \quad x \in (\bigcap_{n\in\mathbb{N}} O_n) \cap \Omega = G \cap \Omega$ i.e. $G \cap \Omega \neq \phi$ □

16. Uniform boundedness principle

Thm 7.2 (UBP)

$(T_n)_{n\in\mathbb{N}} \in B(X,Y)^{\mathbb{N}}$: $\sup_{n\in\mathbb{N}} \|T_n(x)\| < \infty \quad \forall x \in X$

Define $\forall M \in \mathbb{N} \quad X_M := \{x\in X ;\ \forall n\in\mathbb{N} \ \|T_n(x)\| \leq M\}$

$\bigcup_{M\in\mathbb{N}} X_M = X$ Moreover X_M is closed

$\mathrm{Int} \bigcup_{M\in\mathbb{N}} X_M \neq \phi \xRightarrow[\text{Baire}]{\text{contropositive}} \exists N \in \mathbb{N} : \mathrm{Int}(X_M) \neq \phi$

$\exists M \in \mathbb{N} \quad \exists x_0 \in X_M \quad \exists R_0 > 0 \quad B(x_0, R_0) \subset \mathrm{Int}(X_M)$

$\exists M \in \mathbb{N} \quad \forall z \in B(0,1) \quad \forall n\in\mathbb{N} \quad \|T_n(x_0 + \frac{R_0}{2} z)\| \leq M$

$\forall z \in B(0,1) \quad \forall n \in \mathbb{N} \quad \|T_n(z)\| = \frac{2}{R_0}\|T_n(\frac{R_0}{2}z)\| \leq \frac{2}{R_0}(M + \|T_n(x_0)\|) \leq \frac{4M}{R_0}$

Thus $\sup_{n\in\mathbb{N}} \|T_n\| \leq \frac{4M}{R_0}$ □

Corollary 7.3

$B \subset X$ bounded $\iff \{\langle x^*, x\rangle ; x \in B\}$ are bounded $\forall x^* \in X^*$
(strong boundedness) (weak boundedness)

<u>proof</u>

$\Rightarrow \ |\langle x^*, x\rangle| \leq \|x^*\|\|x\| \leq \|x^*\| \sup_{x\in B} \|x\|$

$\Leftarrow \ \forall b \in B \quad T_b \in B(X^*, \mathbb{R}) : T_b(x^*) = \langle x^*, b\rangle$

By assumption $\forall x^* \in X^* \quad \exists C_{x^*} > 0 : \sup_{b\in B} |T_b(x^*)| \leq C_{x^*}$

By UBP : $\exists C > 0 \quad \sup_{b\in B} \|T_b\|_{B(X^*,\mathbb{R})} \leq C$

By HB : $\forall b \in B \quad \exists b^* \in X^* \begin{cases} \|b^*\| = 1 \\ \|b\| = \langle b^*, b\rangle = T_b(b^*) \end{cases}$

$\forall b \in B \quad \|b\| \leq \|T_b\| \leq C$

$\exists C > 0 \quad \forall b \in B \quad \|b\| \leq C$ □

17. Open mapping and closed graph

OMT: $T \in B(X,Y)$. surjective $\Longrightarrow \exists \delta > 0 \quad B(0,\delta) \subset T(B(0,\delta))$

proof

step 1: $Y_n = n\overline{T(B(0,1))} \quad \forall n \in \mathbb{N}$

$\bigcup_{n\in\mathbb{N}} Y_n = Y$ since T is surjective

$\mathrm{Int}(\bigcup_{n\in\mathbb{N}} Y_n) \neq \phi$ so by the Baire's theorem: $\exists n_0 \in \mathbb{N} \quad \mathrm{int}(Y_{n_0}) \neq \phi$

$\exists n_0 \in \mathbb{N} \quad \exists y_0 \in Y \quad \exists R_0 > 0 : B(y_0, R_0) \subset n_0\overline{T(B(0,1))}$

$B(\frac{y_0}{n_0}, \frac{R_0}{n_0}) \subset \overline{T(B(0,1))}$

$\forall z \in B(0, \frac{R_0}{n_0}) \quad \| \frac{y_0}{n_0} - (\frac{y_0}{n_0} + z) \| = \|z\| < \frac{R_0}{n_0}$

Thus $\frac{y_0}{n_0} + z \in B(\frac{y_0}{n_0}, \frac{R_0}{n_0}) \subset \overline{T(B(0,1))} \quad \forall z \in B(0, \frac{R_0}{n_0})$

in the same way $\frac{y_0}{n_0} - z_0 \in \overline{T(B(0,1))}$

subtract those two things, you'll get: $2z \in \overline{T(B(0,1))}$

$\exists (x_n)_{n\in\mathbb{N}} \in B(0,1)^{\mathbb{N}} \quad \exists (w_n)_{n\in\mathbb{N}} \in B(0,1)^{\mathbb{N}} : \begin{cases} T(x_n) \xrightarrow{n\to+\infty} \frac{y_0}{n_0} + z \\ T(w_n) \xrightarrow{n\to+\infty} \frac{y_0}{n_0} - z \end{cases}$

$T(\frac{x_n - w_n}{2}) \xrightarrow{n\to+\infty} z$

i.e. $z \in \overline{T(B(0,1))}$

we have shown $B(0, \frac{R_0}{n_0}) \subset \overline{T(B(0,1))}$

step 2: $C := \frac{R_0}{n_0}$

RK by step 1 $B(0, \frac{C}{2^j}) \subset \overline{T(B(0, \frac{1}{2^j}))} \quad j \in \mathbb{N} \quad (*)$

let $z \in B(0, \frac{C}{2})$ By $(*)$, $\exists x_0 \in B(0, \frac{1}{2}) : \|T(x_0) - z\| \leq \frac{C}{4}$

By $(*)$ again $\exists x_1 \in B(0, \frac{1}{4}) \quad \|T(x_0) - z + T(x_1)\| \leq \frac{C}{8}$

Inductively $\exists (x_n)_{n\in\mathbb{N}} \in B(0, \frac{1}{2})^{\mathbb{N}} \quad \begin{cases} \|T(\sum_{j=0}^{n} x_j) - z\| \leq \frac{C}{2^{j+1}} & \forall j \in \mathbb{N} \\ \|x_j\| \leq 2^{-j} & \forall j \in \mathbb{N} \end{cases}$

$\| \sum_{j=n}^{m} x_j \| \leq \sum_{j=n}^{m} 2^{-j} \quad \forall n, m \in \mathbb{N} \quad m \geq n$

Thus $\sum_{j=0}^{\infty} x_j$ converges to some $x \in \overline{B(0,\frac{1}{2})} \subset B(0,1)$

and $T(x) = z$ since $\| T(\sum_{j=0}^{n} x_j) - z\| \xrightarrow{n\to+\infty} 0$. $\square$

That is $\forall\ z \in B(0, \frac{c}{2})$ we can find an $x \in \overline{B(0,\frac{1}{2})} \subset B(0,1)$ such that $T(x) = z$

Corollary $T\in B(X,Y)$ bijective $\Rightarrow T^{-1}\in B(X,Y)$

<u>proof</u>:

OMT: $\exists c>0 \quad B(0,c) \subset T(B(0,1))$

$\forall y\in Y \quad \|y\|\le c \Rightarrow \exists x\in B(0,1) : \ y = T(x)$

$\forall y\in Y \quad \|y\|\le c \Rightarrow \|T^{-1}(y)\| \le 1$

$\forall y\in Y \quad \|T^{-1}(\frac{y}{\|y\|}c)\| \le 1$ i.e. $\|T^{-1}(z)\| \le \frac{1}{c}\ \forall z\in B(0,1)$ $\square$

<u>Rk</u>: $T\in B(X)$ $\rho(T) = \{\lambda\in\mathbb{R} ; (\lambda I - T) \text{ bijection}\} = \{\lambda\in\mathbb{R} ; (\lambda I - T) \text{ bijection and } (\lambda I - T)^{-1}\in B(X)\}$

CGT (Closed graph Thm)

$T: X\to Y$ linear. Consider $G(T) = \{(x, T(x)) ; x\in X\}$

$G(T)$ closed $\Leftrightarrow T\in B(X,Y)$

<u>proof</u>:

$\Leftarrow$ let $\begin{cases} x_n \xrightarrow{n\to+\infty} x \\ T(x_n) \xrightarrow{n\to+\infty} y \end{cases}$ Then $T(x_n) \xrightarrow{n\to+\infty} T(x)$ by $T\in B(X,Y)$
and $T(x) = y$

Thus $(x,y) = (x, T(x)) \in G(T)$

$\Rightarrow$ Define $\|x\|_T := \|x\|_X + \|T(x)\|_Y$

X is complete for $\|\cdot\|_T$ since

if x_n Cauchy for $\|\cdot\|_T$

$\Rightarrow \begin{cases} \exists x\in X & x_n \to x \\ \exists y\in Y & T(x_n)\to y \end{cases}$

since $G(T)$ is closed $y = T(x)$

$\| x_n - x\|_T = \|x_n - x\|_X + \|T(x_n - x)\|_Y \xrightarrow{n\to+\infty} 0$

Therefor X is complete on $\|\cdot\|_T$

consider the injection $j : (X, \|\cdot\|_T) \longrightarrow (X, \|\cdot\|_X)$

$$x \longmapsto x$$

j is bijection . $\|j(x)\| = \|x\| \leq \|x\|_T$

so $j \in B((X, \|\cdot\|_T), X)$

By OMT $j^{-1} \in B(X, (X, \|\cdot\|_T))$ i.e. $\|x\|_T \leq \|j^{-1}\| \|x\|_X$

i.e $\|x\|_X + \|T(x)\|_Y \leq \|j^{-1}\| \|x\|_X$

so $\|T(x)\|_Y \leq \|j^{-1}\| \|x\|_X \quad \forall x \in X \quad \square$

18. Weak Topology

Lemma 8.2

Let $\varepsilon > 0$. let $x_0 \in Y^c$. $d := \text{dist}(x_0, Y) = \inf\{\|x_0 - y\| ; y \in Y\}$

Pick $y_0 \in Y$: $0 < \|x_0 - y_0\| \leq \frac{d}{1-\varepsilon}$

Define $x = \frac{x_0 - y_0}{\|x_0 - y_0\|}$. Let $y \in Y$

$$\|x - y\| = \frac{1}{\|x_0 - y_0\|} \|x_0 - y_0 - y\|x_0 - y_0\|\| = \frac{1}{\|x_0 - y_0\|} \|x_0 - (y_0 + \|x_0 - y_0\| \cdot y)\| \geq \frac{1}{\|x_0 - y_0\|} d \geq \frac{d}{\frac{d}{1-\varepsilon}} = 1 - \varepsilon \quad \square$$

Thm 8.1

proof: Assume $\dim X = +\infty$.

Let $(X_n)_{n \in \mathbb{N}}$ be closed subspaces of X:

$X_n \subsetneq X_{n+1} \quad \forall n \in \mathbb{N}$

By Lemma 8.2. $\forall n \in \mathbb{N}$. $\exists x_{n+1} \in X_{n+1}$. $\|x_{n+1}\| = 1$: $d(x_{n+1}, X_n) \geq \frac{1}{2}$

Thus gives $\forall n, m \in \mathbb{N}$ $n > m$ $\|x_n - x_m\| \geq \frac{1}{2}$

So $(x_n)_{n \in \mathbb{N}}$ does not have a CV subsequence, which contradict with $\overline{B}(0,1)$ is compact.

Lemma 8.4

$\Rightarrow$ if $x_n \xrightarrow[n \to +\infty]{w} x_0$. Let $x^* \in X^*$

$x_n \xrightarrow[n \to +\infty]{w} x \Rightarrow \forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall n \geq N \quad x_n \in V_{\varepsilon, x, x^*}$

$\Leftrightarrow \forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall n \geq N \quad |\langle x_n - x, x^* \rangle| < \varepsilon$

$\Leftrightarrow \langle x_n - x, x^* \rangle \xrightarrow{n \to +\infty} 0$

$\Leftarrow$ Assume that $\langle x_n - x, x^* \rangle \to 0 \quad \forall x^* \in X^*$

let $\varepsilon > 0$, $N \in \mathbb{N}$, $x_1^*, \ldots, x_N^* \in X^*$

$\forall j \in 1, \cdots N \quad \langle x_n - x, x_j^* \rangle \xrightarrow{n \to +\infty} 0$

$\exists M \in \mathbb{N} \quad \forall n \geq M \quad \max_{j=1,\cdots N} |\langle x_n - x, x_j^* \rangle| < \varepsilon$

$\Rightarrow \exists M \in \mathbb{N} \quad \forall n \geq M \quad x_n \in V_{\varepsilon, x, x_1^*, \ldots, x_N^*} \quad \square$

Proposition 8.5 : T_w is Hausdorff

<u>proof</u>: Let $x, y \in X$. $x \neq y$

By Hahn-Banach. $\exists \alpha \in \mathbb{R}$ $\exists x^* \in X^*$

$\langle x^*, x \rangle < \alpha < \langle x^*, y \rangle$

Define $V = \{ z \in X ; \langle x^*, z \rangle < \alpha \}$

$x \in V$ $\quad y \notin V$

$\forall z \in V$ $\langle x^*, x \rangle < \alpha$ so $\exists \varepsilon > 0$: $\langle x^*, z \rangle = \alpha - \varepsilon$

Let $\tilde{z} \in V_{\frac{\varepsilon}{2}, z, x^*}$ $\quad \langle x^*, \tilde{z} \rangle < \langle x^*, z \rangle + \frac{\varepsilon}{2} \leq \alpha - \frac{\varepsilon}{2} < \alpha$

Thus $V_{\frac{\varepsilon}{2}, z, x^*} \subset V$ i.e. V is open for T_w

$\langle x^*, y \rangle = \alpha + \delta$ for some $\delta > 0$

$V_{\frac{\delta}{2}, y, x^*} \cap V = \phi$ $\square$

proposition 8.6: $x_n \xrightarrow[n \to +\infty]{w} x \Rightarrow \sup_{n \in \mathbb{N}} \|x_n\| < \infty$

<u>proof</u>: Define $T_n : X^* \longrightarrow \mathbb{R}$

$$x^* \longmapsto \langle x^*, x_n \rangle$$

$\forall x^* \in X^*$ $\quad \sup_{n \in \mathbb{N}} |T_n(x^*)| = \sup_{n \in \mathbb{N}} |\langle x^*, x_n \rangle| < \infty$ (since $x_n \xrightarrow{w} x$)

By UBP $\quad \sup_{n \in \mathbb{N}} \|T_n\|_{B(X^*, \mathbb{R})} < \infty$

$\|T_n\| := \sup \{ \langle x^*, x_n \rangle ; x^* \in S_{X^*} \}$

By HB $\forall n \in \mathbb{N}$ $\exists x^* \in S_{X^*}$, $\langle x^*, x_n \rangle = \|x_n\|$

Thus $\|T_n\| \geq \|x_n\|$ $\forall n \in \mathbb{N}$ and $\sup_{n \in \mathbb{N}} \|x_n\| < \infty$ $\square$

Thm (Mazur) $E \subset X$ convex. E weakly ~~bounded~~ closed $\Longleftrightarrow$ E closed

<u>proof</u> E^c weakly open $\Longleftrightarrow$ $\forall x \in E^c$ $\exists \delta > 0$ $\exists n \in \mathbb{N}$ $\exists x_1^*, \dots, x_n^* \in X^*$

$\forall y \in X$ $\quad |\langle x_j^*, x - y \rangle| < \delta$ $\quad \forall j = 1 \dots n \Rightarrow y \in E^c$

$\Rightarrow \forall x \in E^c$ $\quad \exists \delta' = \min_{j=1\dots n} \frac{\delta}{\|x_j^*\|}$ $\quad \forall y \in X$ $\|x - y\| < \delta'$

$|\langle x-y, x_j^* \rangle| < \delta' \|x_j^*\| \leq \delta$ $\quad \forall j = 1 \dots n \Rightarrow y \in E^c$

$\Longleftrightarrow \forall x \in E^c$ $\quad \underset{\|\cdot\|}{B}(x, \delta') \subset E^c$

$\Longleftrightarrow E^c$ open

if E is closed

E^c open. $x \in E^c$. By HB: $\exists \alpha \in \mathbb{R}$ $\exists x^* \in X^*$

$\langle x^*, x \rangle < \alpha < \langle x^*, y \rangle \quad \forall y \in E$

Thus $x \in V = \{ y \in X ; \langle x^*, y \rangle < \alpha \}$ and V is weakly open as shown in prop 8.5. Moreover $V \subset E^c$ $\square$

Corollary 8.8 $x_n \xrightarrow[n \to +\infty]{w} x$ $\quad \exists (y_n)_{n \in \mathbb{N}} \in \operatorname{conv}\{x_j, j \in \mathbb{N}\}$ $\quad y_n \xrightarrow{n \to +\infty} x$

proof $\quad E = \operatorname{conv}\{x_j, j \in \mathbb{N}\}$ $\quad x \in \overline{E}^w = \overline{E}$

19. Weak* Topology

Proposition 8.12 Let X be a separable Banach space.

$$(X^*, T_{w^*}) = (X^*, T_w) \iff X = X^{**}$$

proof Assume $(X^*, T_w) = (X^*, T_{w^*})$

Let $x^{**} \in (X^*)^*$

x^{**} is weakly continous

i.e $\exists \varepsilon > 0 \quad \exists x_0^* \in X^* \quad \exists x_1, \dots, x_n \in X$

$V_{\varepsilon, x_0^*, x_1, \dots, x_n} \subset (x^{**})^{-1}(-1, 1)$

Claim: $\bigcap_{j=1}^{n} \operatorname{Ker} x_j \subset \operatorname{Ker} x^{**}$

Indeed: let $y^* \in \bigcap_{j=1}^{n} \ker(x_j)$ i.e $\langle x_j, y^* \rangle = 0 \quad \forall j = 1 \dots n$

$\forall \varepsilon \in \mathbb{R} \qquad |\langle x_j, ty^* + x_0^* - x_0^* \rangle| = 0 < \varepsilon$

so $ty^* + x_0^* \in V_{\varepsilon, x_0^*, x_1, \dots, x_n} \subset (x^{**})^{-1}(-1, 1) \quad \forall \varepsilon \in \mathbb{R}$

$|x^{**}(ty^* + x_0^*)| < 1$

$|x^{**}(y^* + \frac{x_0^*}{t})| < \frac{1}{t} \quad \forall \varepsilon > 0$

$\Rightarrow x^{**}(y^*) = 0$ i.e $y^* \in \ker x^{**}$

Claim: $x^{**} \in \operatorname{Span}\{x_j ; j = 1 \dots n\}$

Define $F: X^* \to \mathbb{R}^{n+1}$

$x^* \mapsto (\langle x^*, x^{**} \rangle, \langle x^*, x_1 \rangle, \langle x^*, x_2 \rangle, \dots, \langle x^*, x_n \rangle)$

By HB in $\mathbb{R}^{n+1}$, one can separate $\{(1, 0, \dots 0)\}$ from $R(F)$

i.e $\exists \alpha \in \mathbb{R} \quad \exists \lambda = (\lambda_0, \lambda_1, \dots \lambda_n) \in \mathbb{R}^{n+1}$

$\langle \lambda, (1, 0, \dots 0) \rangle < \alpha < \langle \lambda, F(x^*) \rangle \qquad \forall x^* \in X^*$

$\Rightarrow \lambda_0 < \alpha < \lambda_0 \langle x^*, x^{**} \rangle + \sum_{j=1}^{n} \lambda_j \langle x^*, x_j \rangle \qquad \forall x^* \in X^*$

$\Rightarrow \lambda_0 < \alpha < \lambda_0 \langle x^{**}, tx^* \rangle + \sum_{j=1}^{n} \lambda_j \langle tx^*, x_j \rangle \qquad \forall t \in \mathbb{R}$

$\lambda_0 < \alpha < 0 = \lambda_0 \langle x^{**}, x^* \rangle + \sum_{j=1}^{n} \lambda_j \langle x^*, x_j \rangle \quad \forall x^* \in X^*$

$\langle x^{**}, x^* \rangle = -\sum_{j=1}^{n} \frac{\lambda_j}{\lambda_0} \langle x^*, x_j \rangle \quad \forall x^* \in X^*$

i.e $x^{**} \in \operatorname{Span}\{x_j, j = 1 \dots n\} \subset X \quad \square$

Thm 8.11 (B_{X^*}, T_{w^*}) metrisable $\iff X$ saparable

<u>proof</u>

$\Rightarrow$ d distance

$(B_{X^*}, T_{w^*}) \underset{homeo}{\sim} (B_{X^*}, d)$

Define $\forall n \in \mathbb{N}$. $V_n = \{x^* \in B_{X^*} : d(x^*, 0) < \frac{1}{n}\}$

$\forall n \in \mathbb{N}$ $\exists \varepsilon_n > 0$ $\exists x_n^* \in B_{X^*}$ $\exists N_n \in \mathbb{N}$ $\exists x_{n,1}, \cdots, x_{n,N_n} \in X$

$V_{\varepsilon, x_n^*, x_{n,1} \cdots x_{n,N_n}} \subset V_n$

Define $Y = \operatorname{span} \bigcup_{n \in \mathbb{N}} \{x_{n,j} : j = 1, \cdots, N_n\}$

It suffices to show that $Y^\perp = \{0\}$

(This implies $\overline{Y} = X$ hence X is separable)

let $y^* \in Y^\perp$.

$\langle y^*, y \rangle = 0$ $\forall y \in Y$

$\langle y^*, x_{n,j} \rangle = 0$ $\forall n \in \mathbb{N}$ $\forall j = 1, \cdots N_n$

$\rightarrow y^* \in V_{\varepsilon_N, x_n^*, x_{n,1}, \cdots x_{n,N}} \subset V_n$ $\forall n \in \mathbb{N}$

$\Rightarrow d(y^*, 0) < \frac{1}{n}$ $\forall n \in \mathbb{N}$

$\Rightarrow y^* = 0$ $\diamond$

$\Leftarrow$ Assume X separable.

Take $(x_j)_{j \in \mathbb{N}}$ dense in B_X

Define $|||x^*||| := \sum_{j=0}^{\infty} 2^{-j} |\langle x^*, x_j \rangle|$

$|||\cdot|||$ is a norm on X^*

$|||x^*||| \le \|x^*\|_{X^*}$ $\forall x^* \in X^*$

$\boxed{\text{One inclusion}}$

let $\varepsilon > 0$, $x_0^* \in B_{X^*}$, $y_1 \cdots, y_n \in B_X$

We need to find $\delta > 0$

$\forall x^* \in B_{X^*}$ $|||x^* - x_0^*||| < \delta \Rightarrow x^* \in V_{\varepsilon, x_0^*, y_1, \cdots, y_n}$

we have that $\forall x^* \in B_{X^*}$, $|||x^* - x_0^*||| < \delta$

$\forall j = 1, \cdots, n$ $|\langle x^* - x_0^*, x_j \rangle| < 2^j \delta$

By density of $(x_j)_{j \in \mathbb{N}}$, we can pick $j_1, \cdots, j_n : \|x_{j_k} - y_k\| < \frac{\varepsilon}{3}$

Choose $\delta > 0$: $2^{j_k}\delta < \frac{\varepsilon}{3}$ $\quad \forall k = 1 \dots n$

$$|\langle x^* - x_0^*, y_k\rangle| < |\langle x^* - x_0^*, x_{j_k}\rangle| + |\langle x^* - x_0^*, y_k - x_{j_k}\rangle|$$

$$< 2^{j_k}\delta + \|x^* - x_0^*\| \, \|x_{j_k} - y_k\|$$

$$< \frac{\varepsilon}{3} + 2\frac{\varepsilon}{3} = \varepsilon$$

$\boxed{\text{other inclusion}}$

Let $\delta > 0$, $x_0^* \in B_{X^*}$

Let $x^* \in B_{X^*}$

$$|||\, x^* - x_0^* \,||| = \sum_{j=1}^{\infty} 2^{-j} |\langle x^* - x_0^*, x_j\rangle| \leq \sum_{j=1}^{n} 2^{-j} |\langle x^* - x_0^*, x_j\rangle| + 2\sum_{j=n+1}^{\infty} 2^{-j} \leq \max_{j=1,\dots,n} |\langle x^* - x_0^*, x_j\rangle| + 2^{-n}$$

Pick $n \in \mathbb{N}$: $2^{-n} < \frac{\delta}{2}$ $\qquad$ Pick $\varepsilon = \frac{\delta}{2}$

$\forall x^* \in V_{\varepsilon = \frac{\delta}{2}, x_0^*, x_1, \dots, x_n}$ $\qquad |||\, x^* - x_0^* \,||| < \delta$ $\quad \square$

20. Banach-Alaoglu's compactness Thm

Thm (Banach-Alaoglu's) : (B_{X^*}, T_{w^*}) is compact

proof :

$$\phi : \cancel{B(X)}\ (B_{X^*}, T_{w^*}) \longrightarrow \mathbb{R}^X$$

$$x^* \longmapsto (\langle x^*, x\rangle)_{x\in X}$$

$A := R(\phi) \subset \prod_{x\in X} [-\|x\|, \|x\|]$ since $\|x^*\| \le 1$

By Tychenoff Thm $\prod_{x\in X} [-\|x\|, \|x\|]$ is compact

we show (1) A is closed

(2) ϕ is a homeomorphism between (B_{X^*}, T_{w^*}) and A

(1)

let $\lambda = (\lambda_x)_{x\in X} \in \overline{A}$

WTS $\lambda \in A$ i.e (a) $x \mapsto \lambda_x$ is linear

(b) $\|\lambda_x\| \le \|x\|$ $\forall x\in X$

This gives that $\lambda = \phi(x \mapsto \lambda_x)$

For $\varepsilon > 0$, $x_1, \dots, x_n \in X$. Define

$U_{\varepsilon, x_1, \dots x_n} := \{ (\mu_x)_{x\in X} \in \mathbb{R}^X \ ; \ |\mu_{x_j} - \lambda_{x_j}| < \varepsilon \ \ \forall j = 1 \dots n \}$

$U_{\varepsilon, x_1 \dots x_n}$ is open in $\mathbb{R}^X$

Thus $U_{\varepsilon, x_1 \dots x_n} \cap A \neq \phi$ since $\lambda \in \overline{A}$

(a)

let $x_0, y_0 \in X$ $\alpha \in \mathbb{R}^*$, $\varepsilon > 0$

$U_{\min(\varepsilon, \frac{\varepsilon}{|\alpha|})\, x_0, \alpha x_0} \cap A \neq \phi$

let $x^* \in B_{X^*}$: $\begin{cases} |\langle x^*, x_0\rangle - \lambda_{x_0}| < \frac{\varepsilon}{|\alpha|} \\ |\langle x^*, \alpha x_0\rangle - \lambda_{\alpha x_0}| < \varepsilon \end{cases}$

$|\alpha \lambda_{x_0} - \langle x^*, \alpha x_0\rangle| < \varepsilon$

$\forall \varepsilon > 0$ $|\alpha\lambda_{x_0} - \lambda_{\alpha x_0}| < 2\varepsilon$

Therefore $\lambda_{\alpha x_0} = \alpha \lambda_{x_0} \quad \forall \alpha \in \mathbb{R}^*$

$U_{\varepsilon, 0} \cap A \neq \phi$

$\forall \varepsilon > 0 \quad | \langle x^*, 0 \rangle - \lambda_0 | < \varepsilon \Rightarrow \lambda_0 = 0$

so $\lambda_{\beta x_0} = \beta \lambda_{x_0} \quad \forall \beta \in \mathbb{R}$

$U_{\varepsilon, x_0, y_0, x_0 + y_0} \cap A \neq \phi$, since $\lambda \in \bar{A}$

$\exists x^* \in B_{X^*}$ $\begin{cases} |\langle x^*, x_0 \rangle - \lambda_{x_0}| < \varepsilon \\ |\langle x^*, y_0 \rangle - \lambda_{y_0}| < \varepsilon \\ |\langle x^*, x_0 + y_0 \rangle - \lambda_{x_0 + y_0}| < \varepsilon \end{cases}$

$| \lambda_{x_0 + y_0} - (\lambda_{x_0} + \lambda_{y_0}) | < 3\varepsilon \quad \forall \varepsilon > 0$

This proves (a) i.e $x \to \lambda_x$ is linear.

(b) $\exists x^* \in B_{X^*} \quad | \langle x^*, x \rangle - \lambda x | < \varepsilon$

$|\lambda x| \leq \|x\| + \varepsilon \quad \forall \varepsilon > 0$

Thus $|\lambda x| \leq \|x\|$

We have proven that $A = \bar{A}$, and thus A is compact.

(2)

let $x^*, y^* \in B_{X^*}$

$\phi(x^*) = \phi(y^*) \iff \langle x^* - y^*, x \rangle = 0 \quad \forall x \in X \iff x^* = y^*$

so ϕ is a bijection and $B_{X^*} = \phi^{-1}(A)$

WTS ϕ^{-1} is continous from A to (B_{X^*}, T_{w^*})

i.e $\forall \mathcal{O} \subset B_{X^*}$ open , $\phi(\mathcal{O})$ is open.

WTS $\forall \varepsilon > 0 \quad \forall n \in \mathbb{N} \quad \forall x_0^* \in B_{X^*} \quad \forall x_1, \cdots, x_n \in X$

$\phi(V_{\varepsilon, x_0^*, x_1, \cdots, x_n} \cap B_{X^*})$ is open in A

so let $\lambda \in \phi(V_{\varepsilon, x_0^*, x_1, \dots, x_n} \cap B_{X^*})$

$\lambda = \phi(y_0^*)$ for some $y_0^* \in B_{X^*} \cap V_{\varepsilon, x_0^*, x_1, \dots x_n}$

Pick $\delta \in (0,1) \quad |\langle y_0^* - x_0^*, x_j \rangle| \leq (1-\delta)\varepsilon \quad \forall j = 1, \dots, n$

Consider $U_{\frac{\varepsilon\delta}{2}, x_1 \dots x_n} = \{ (\mu_x)_{x \in X} \in \mathbb{R}^X ; |\mu_{x_j} - \lambda_{x_j}| < \frac{\varepsilon\delta}{2} \quad \forall j = 1 \dots n \}$

$U_{\frac{\varepsilon\delta}{2}, x_1 \dots x_n} \cap A$ is open in A

For $\mu = (\mu_x)_{x \in X} \in U_{\varepsilon, x_1, \dots, x_n} \cap A$

$\exists y_0^* \in B_{X^*} \; : \mu = \phi(y_0^*)$

WTS $\quad y_0^* \in V_{\frac{\varepsilon}{2}, x_0^*, x_1 \dots x_n}$

let $\quad j \in \{1, \dots n\}$

$$|\langle y_0^* - x_0^*, x_j \rangle| = |\mu_{x_j} - \lambda_{x_j} + \lambda_{x_j} - \langle x_0^*, x_j \rangle| < \frac{\varepsilon\delta}{2} + |\lambda_{x_j} - \langle x_0^* - x_j \rangle| < \frac{\varepsilon\delta}{2} + (1-\delta)\varepsilon = (1 - \frac{\delta}{2})\varepsilon < \varepsilon$$

$\square$

21. Reflexive spaces

Proposition 9.2

Proof : X reflexive $\Rightarrow (B_{X^{**}}, T_{w^*}) = (B_X, T_w)$

By Banach - Alaoglu ~~Thm~~ $(B_{X^{**}}, T_{w^*})$ is compact.

so (B_X, T_w) is compact

Therefore $\forall R>0$ $R B_X$ is weakly ~~to~~ compact

since K is bounded . $\exists R>0$ $K \subset R B_X$

since K is closed and convex, K is weakly closed by Mazur's Thm.

Therefore K is weakly-compact since $R B_X$ is weakly compact □

Lemma 9.4 (w^* HB)

$x_0^* \in B^c$ and B^c w^* open

$\exists \varepsilon>0$ $\exists n \in N$ $\exists x_1 \dots x_n \in X$

$V := V_{\varepsilon, x_0^*, x_1, \dots x_n} \subset B^c$

B is convex

By HB $\exists x_0^{**} \in X^{**}$ $\exists \alpha \in \mathbb{R}$

$\langle x_0^{**}, x_0^* \rangle < \alpha < \langle x_0^{**}, y^* \rangle$ $\forall x^* \in V$ $\forall y^* \in B$

Recall $\bigcap_{j=1}^{n} \ker x_j \subset \ker x_0^{**} \Longrightarrow x_0^{**} \in \operatorname{Span}\{x_j, j=1 \cdots n\} \subset X$

Let $x^* \in \bigcap_{j=1}^{n} \ker x_j$ and $t \in \mathbb{R}$

$\forall t \in \mathbb{R}$ $\forall j=1 \cdots n$ $\langle x_0^* + t x^*, x_j \rangle = \langle x_0^*, x_j \rangle$

$x_0^* + t x^* \in V$

$\langle x_0^{**}, x_0^* + t x^* \rangle < \alpha$ $\forall t \in \mathbb{R}$

$\Rightarrow \langle x^*, x_0^{**} \rangle = 0 \iff x_0^* \in \ker x_0^{**}$

Lemma 9.5 (Goldstine)

By contradiction.

$B_{X^{**}} \setminus \overline{i(B_X)}^{w} \neq \phi$

$\exists\, x_0^{**} \in B_{X^{**}} \setminus \overline{i(B_X)}^{w^*} \neq \phi$

By Lemma 9.4 : $\exists\, x_0^* \in X^* \quad \exists \alpha \in \mathbb{R}$

$\langle i(x), x_0^* \rangle < \alpha < \langle x_0^*, x_0^{**} \rangle \quad \forall x \in B_X$

$\forall x \in X \quad \langle x, x_0^* \rangle < \alpha < \|x_0^*\| \, \|x_0^{**}\| < \|x_0^*\|$

$\|x_0^*\| = \sup_{x \in B_X} \langle x, x_0^* \rangle \leq \alpha < \|x_0^*\|$ contradiction

Thm 9.3

$\Rightarrow$

X reflexive $\xRightarrow{\text{Banach-Alaoglu}}$ (B_{X^*}, T_{w^*}) is compact $\xRightarrow{\text{prop 8.12}}$ (B_{X^*}, T_w) compact

$\Leftarrow$

(B_X, T_w) is compact $\Rightarrow (i(B_X), T_w)$ is compact

$i(B_X)$ is weakly dense in $B_{X^{**}}$ by Lemma 9.5

Thus $i(B_X) = B_{X^{**}}$ and $\forall R > 0 \quad i(RB_X) = R\, \cancel{B_X}\, B_{X^{**}}$

Therefore i is bijection $\square$

Corollary 9.6

- Assume X is reflexive

  $(B_{X^*}, T_w) = (B_{X^*}, T_{w^*}) \xrightarrow{\text{Banach-Alaoglu}} (B_{X^*}, T_w)$ compact $\xrightarrow{\text{Thm 9.3}} X^*$ reflexive

- Assume X^* is reflexive

  X^* reflexive $\Rightarrow X^{**}$ reflexive $\xRightarrow{\text{Thm 9.3}} (B_{X^{**}}, T_w)$ is compact

  $i(B_X)$ is closed and convex, so by Mazur's Thm it is weakly closed

  Thus $(i(B_X), T_w)$ is compact

  $(i(B_X), T_w) = (i(B_X), T_{w^*})$ is compact $\xRightarrow{\text{Thm 9.3}}$ X reflexive $\square$
  $= (B_X, T_w)$ ~~#####~~

Corollary 9.7

- X reflexive $\xRightarrow{\text{Thm } 9.3}$ (B_X, T_w) is compact

$B_{X_0} \subset B_X$ is closed and convex

By Mazur's Thm B_{X_0} is weakly closed.

i.e B_{X_0} can be covered by a finite union of sets $V_{\varepsilon, x, x_1^*, \dots, x_n^*}$

where $x \in B_{X_0}$ and $x_1^*, \dots, x_n^* \in X^* \subset X_0^*$

i.e B_{X_0} is weakly compact in the duality of X_0, which is X_0^*

Thus by Thm 9.3, X_0 is reflexive.

- $T: X_0 \longrightarrow Y$ isomorphism

let $V := V_{\varepsilon, T(x_0), y_1^*, \dots, y_n^*} \subset Y$

$$T^{-1}(V) = \{ x \in X ;\ |\langle T(x) - T(x_0), y_j^* \rangle| < \varepsilon \quad \forall j = 1 \dots n \}$$

$$= \{ x \in X ;\ |\langle x - x_0, T^*(y_j^*) \rangle| < \varepsilon \quad \forall j = 1 \dots n \}$$

$$= V_{\varepsilon, x_0, T^*(y_1^*) \dots T^*(y_n^*)}$$

so $T: (X_0, T_w) \longrightarrow (Y, T_w)$ is continous

$(B_Y, T_w) = (T(T^{-1}(B_Y)), T_w)$

$T^{-1}(B_Y)$ is w-closed

(B_{X_0}, T_w) is compact since X_0 is reflexive

Thus $T^{-1}(B_Y)$ is weakly compact

and $(T(T^{-1}(B_Y)), T_w)$ is weakly compact since T is continous

Thm 9.3 $\Longrightarrow$ Y reflexive $\square$

22. Uniformly convex spaces

prop 9.8

Let $(x_n)_{n\in\mathbb{N}}$ be bounded

$Y = \overline{\text{span}\{x_n, n\in\mathbb{N}\}} \subset X$

Y is a closed, separable subspace of X

Thus Y is a separable and reflexive Banach space

If Y^* is separable, then $\underbrace{(B_{Y^{**}}, T_{w^*})}_{= (B_Y, T_w)}$ is metrisable

Therefore we have the unit ball B_Y, B_Y is weakly compact by ~~prop~~ Thm 9.3.

Hence we have a weakly compact set B_Y for metrisable topology.

and a compact set in a metrisable space is sequentially compact

and a sequentially compact in a metrisable space is compact

which implies the result.

It suffices to show that Y^* is separable

Pick $(y_n)_{n\in\mathbb{N}} \in Y^{\mathbb{N}}$ dense in $Y = Y^{**}$

Pick $(y_n^*)_{n\in\mathbb{N}} \in (Y^*)^{\mathbb{N}}$: $\begin{cases} \langle y_n^*, y_n\rangle \geq \frac{1}{2}\|y_n\| \\ y_n^* \in S_{Y^*} \quad \forall n\in\mathbb{N} \end{cases}$

It suffices to show that $\overline{\text{Span}\{y_n^*, n\in\mathbb{N}\}} = Y^*$

Define $Z = \overline{\text{span}\{y_n^*; n\in\mathbb{N}\}}$

WTS $Z^\perp = \{0\}$

Let $z \in Z^\perp$. We have that $\langle z, y_n^*\rangle = 0 \quad \forall n\in\mathbb{N}$

Given $\varepsilon > 0$, pick $n\in\mathbb{N}$ $\|z - y_n\| < \varepsilon$

$$\frac{1}{2}\|y_n\| \leq \langle y_n^*, y_n\rangle = \langle y_n^*, y_n - z\rangle + \langle y_n^*, z\rangle = \langle y_n^*, y_n - z\rangle$$

$$\frac{1}{2}\|y_n\| \leq \|y_n^*\|\,\|y_n - z\| < \varepsilon$$

$\|z\| < 3\varepsilon \quad \forall \varepsilon > 0 \Rightarrow \varepsilon = 0$ □

Corollary 9.9 $\forall x \in X \quad \exists y \in K \quad d(x,K) = \|x-y\|$

Let $(y_n) \in K^{\mathbb{N}}$. $\|x - y_n\| \xrightarrow{n\to+\infty} d(x,K) = d$

By Prop 9.8 $\exists\, y_{n_k} \in X \quad y_{n_k} \xrightarrow{w} y$

K is weakly-compact, so $y \in K$

$\|x-y\| \geq d$ since $y \in K$

$\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall k \geq N \quad \|x - y_{n_k}\| \leq (1+\varepsilon) d$

$\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall k \geq N \quad x - y_{n_k} \in \underbrace{(1+\varepsilon) d B_X}_{\text{weakly-closed}}$

Thus $x - y \in (1+\varepsilon) d B_X \quad \forall \varepsilon > 0$

i.e $\|x-y\| \leq (1+\varepsilon) d \quad \forall \varepsilon > 0$

so $\|x-y\| = d$ □

Thm 9.11

By ~~contrapositive~~ contradiction

if X is not reflexive, that is $\exists\, x^{**} \in S_{X^{**}} \quad \exists \varepsilon > 0 \quad \forall x \in B_X \quad \|i(x) - x^{**}\| > \varepsilon$

By Uniformly convex

pick $\delta > 0 \quad \forall x, y \in X \quad \|x-y\| \geq \varepsilon \Rightarrow \|\frac{x+y}{2}\| \leq 1 - \delta$

Pick $x^* \in S_{X^*} \quad \langle x^{**}, x^* \rangle > 1 - \frac{\delta}{2}$

Define $V = \{ y^{**} \in X^{**} \mid |\langle y^{**} - x^{**}, x^* \rangle| < \frac{\delta}{2} \}$

V is weak * open set in X^{**}

By Goldstine Lemma $i(B_X) \cap V \neq \phi$

Pick $x \in B_X$: $i(x) \in V$

Recall that $\| i(x) - x^{**} \| > \varepsilon$

Define $W = (i(x) + \varepsilon B_{X^{**}})^c$ is w^* open in X^{**}

By Goldstine Lemma $i(B_X) \cap V \cap W \neq \phi$

Pick $y \in B_X \quad i(y) \in V \cap W$

Then
$$\begin{cases} |\langle x^*, i(x) - x^{**}\rangle| < \frac{\delta}{2} & \text{since } i(x) \in V \\ |\langle x^*, i(y) - x^{**}| < \frac{\delta}{2} & \text{since } i(y) \in V \\ \| i(x) - i(y)\|_{X^{**}} > \varepsilon \\ \| \frac{x+y}{2} \| \le 1-\delta \end{cases}$$

$$|\langle x^*, i(x+y) - 2x^{**}\rangle| < \delta$$

$$2\langle x^*, x^{**}\rangle < \delta + \langle x^*, i(x+y)\rangle$$

$$\Rightarrow \quad 1 - \frac{\delta}{2} < \frac{\delta}{2} + \|x^*\| \, \| \frac{x+y}{2} \|$$

$$\Rightarrow \quad 1-\delta < 1-\delta \qquad \text{contradiction } \square$$

Thm 9.12

proof

By Thm 9.11 and Corollary 9.9

$\forall x \in X \quad \exists y \in K \quad \|x-y\| = \operatorname{dist}(x, K) =: d$

Assume $\exists x \in K \quad \exists y_1, y_2 \in k \quad \exists \varepsilon > 0$

$$\begin{cases} \|x-y_1\| = \|x-y_2\| = d \\ \|y_1 - y_2\| > \varepsilon \end{cases}$$

$$\| \frac{x-y_1}{\|x-y_1\|} + \frac{x-y_2}{\|x-y_2\|} \| = \frac{1}{d}(2x - (y_1+y_2))$$

$$\| \frac{x-y_1}{\|x-y_1\|} - \frac{x-y_2}{\|x-y_2\|} \| = \frac{1}{d}\|y_1 - y_2\| > \frac{\varepsilon}{d}$$

By uniformly convexity, $\exists \delta > 0$

$$\frac{1}{2} \| \frac{x-y_1}{\|x-y_1\|} + \frac{x-y_2}{\|x-y_2\|} \| < 1-\delta$$

$$\exists \delta > 0 \quad \frac{1}{d} \|x - \frac{y_1+y_2}{d}\| < (1-\delta)$$

$$\Rightarrow \quad \|x - \frac{y_1+y_2}{2}\| < \operatorname{dist}(x, K)$$

This is a contradiction since $\frac{y_1+y_2}{2} \in K$ by convexity $\square$

24. Schauder bases

Prop 10.2

(1) want to show S is complete

let $((a_n^{(j)})_{n\in\mathbb{N}})_{j\in\mathbb{N}} \in S^{\mathbb{N}}$ be a Cauchy sequence

$\forall \varepsilon>0 \quad \exists J\in\mathbb{N} \quad \forall j\geq k\geq J \quad \forall N\in\mathbb{N}$

$\|\sum_{n=0}^{N}(a_n^{(j)}-a_n^{(k)})e_n\|<\varepsilon$

In particular

$|a_0^{(j)}-a_0^{(k)}|\,\|e_0\|<\varepsilon$

$|a_1^{(j)}-a_1^{(k)}|<\frac{2\varepsilon}{\|e_1\|}$

Inductively: $|a_n^{(k)}-a_n^{(j)}|<\frac{(n+1)\varepsilon}{\|e_n\|} \quad \forall n\in\mathbb{N}$

Define $b_n := \lim_{j\to+\infty} a_n^{(j)} \quad \forall n\in\mathbb{N}$

$\forall\varepsilon>0 \quad \exists J\in\mathbb{N} \quad \forall j\geq J \quad \sup_{N\in\mathbb{N}}\|\sum_{n=0}^{N}(a_n^{(j)}-b_n)e_n\|<\varepsilon \quad (2)$

$\forall\varepsilon>0 \quad \exists J\in\mathbb{N} \quad \forall N,M\in\mathbb{N} \quad N>M$

$\|\sum_{n=M}^{N}b_ne_n\| \leq \|\sum_{n=M}^{N}(b_n-a_n^{(J)})e_n\| + \|\sum_{n=M}^{N}a_n^{(J)}e_n\| \overset{(2)}{\leq} 2\varepsilon + \|\sum_{n=M}^{N}a_n^{(j)}e_n\|$

Thus $(\sum_{n=0}^{N}b_ne_n)_{N\in\mathbb{N}}$ is Cauchy in X hence convergent.

i.e $b_n\in S$

(2) $T: S \longrightarrow X$

$(a_n)_{n\in\mathbb{N}} \longmapsto \sum_{n=0}^{\infty} a_n e_n$

T is a bijection since $(e_n)_{n\in\mathbb{N}}$ is a basis

$\| T((a_n)_{n\in\mathbb{N}})\| = \| \sum_{n=0}^{\infty} a_n e_n \| \le \sup_{N\in\mathbb{N}} \| \sum_{n=0}^{N} a_n e_n \| = \| (a_n)_{n\in\mathbb{N}} \|_S$ 等距的

$T^{-1} \in B(X, S)$ by open mapping Thm

Therefore T is an isomorphism

(3) Define $\forall n \in \mathbb{N}$

$\varphi_n : S \longrightarrow \mathbb{R}$

$(a_j)_{j\in\mathbb{N}} \longmapsto a_n$

$e_n^*(x) = \varphi_n(T^{-1}(x)) \quad \forall n\in\mathbb{N} \quad \forall x \in X$

$\forall n\in\mathbb{N}^*$
$\forall (a_j)_{j\in\mathbb{N}} \in S$ $\quad |\varphi_n((a_j)_{j\in\mathbb{N}})| = |a_n| = \frac{1}{\|e_n\|} \|a_n e_n\| = \frac{1}{\|e_n\|} \| \sum_{j=0}^{n} a_j e_j - \sum_{j=0}^{n-1} a_j e_j \| \le \frac{2}{\|e_n\|} \| (a_j)_{j\in\mathbb{N}} \|_S$

$\forall n\in\mathbb{N}$
$\forall x\in X$ $\quad |e_n^*(x)| \le \frac{2}{\|e_n\|} \|T^{-1}(x)\|_S \le \frac{2}{\|e_n\|} \|T^{-1}\|_{B(X,S)} \|x\|_X$

Therefore $e_n^* \in X^* \quad \forall n\in\mathbb{N}$

(4)

$\forall n\in\mathbb{N} \quad \|e_n^*\| \|e_n\| \le \frac{2}{\|e_n\|} \|T^{-1}\|_{B(X,S)} \|e_n\| = 2\|T^{-1}\|$

(5)

$\forall n, m\in\mathbb{N} \quad \langle e_n^*, e_m \rangle = \langle e_n^*, \sum_{j=0}^{\infty} \delta_{j,m} e_j \rangle = \delta_{n,m}$

(6)

Let $x\in X$ $\quad \sup_{N\in\mathbb{N}} \|P_N(x)\| = \|T^{-1}(x)\|_S \le \|T^{-1}\|_{B(X,S)} \|x\|_X$

By UBP $\quad \sup_{N\in\mathbb{N}} \|P_N\|_{B(X)} < \infty$

Prop 10.3

basis $\Longrightarrow$ (1)(2)(3) (exercise)

Assume (1), (2) and (3)

Define $E = \{x \in X \ ; \ \exists (a_n)_{n \in \mathbb{N}} : x = \sum_{n=0}^{\infty} a_n e_n\}$

$\overline{E} = X$ by (2)

WTS E is closed. Let $(x^{(j)})_{j \in \mathbb{N}} \in E^{\mathbb{N}}$ converge to some $x \in X$

$x^{(j)} = \sum_{n=0}^{\infty} a_n^{(j)} e_n$

$\forall n \neq 0 \in \mathbb{N}$

$\forall j, k \in \mathbb{N}$ $\quad |a_n^{(j)} - a_n^{(k)}| \overset{(1)}{=} \frac{1}{\|e_n\|} \| \sum_{m=0}^{n} (a_m^{(j)} - a_m^{(k)}) e_m - \sum_{m=0}^{n-1} (a_m^{(j)} - a_m^{(k)}) e_m \|$

$$\overset{(3)}{\leq} \frac{2C}{\|e_n\|} \| \sum_{m=0}^{M} (a_m^{(j)} - a_m^{(k)}) e_m \| \quad \forall M \geq n$$

$$= \frac{2C}{\|e_n\|} \| x^{(j)} - x^{(k)} \|$$

Thus $\forall n \in \mathbb{N}^*$ $(a_n^{(j)})_{j \in \mathbb{N}}$ is Cauchy in $\mathbb{R}$

$b_n = \lim_{j \to +\infty} a_n^{(j)} \quad \forall n \in \mathbb{N}$

let $\varepsilon > 0$ Pick $j \in \mathbb{N}$: $\|x^{(j)} - x\| \leq \varepsilon$

$\forall N \in \mathbb{N}$ $\quad \| \sum_{n=0}^{N} b_n e_n - x \| \leq \|x^{(j)} - x\| + \| \sum_{n=0}^{N} b_n e_n - \sum_{n=0}^{N} a_n^{(j)} e_n \| + \| \sum_{n=N+1}^{\infty} a_n^{(j)} e_n \|$

$$\leq \varepsilon + \lim_{k \to +\infty} \| \sum_{n=0}^{N} a_n^{(k)} e_n - \sum_{n=0}^{N} a_n^{(j)} e_n \| + \| \sum_{n=N+1}^{\infty} a_n^{(j)} e_n \|$$

$$\overset{(3)}{\leq} \varepsilon + \lim_{k \to +\infty} C \| x^{(k)} - x^{(j)} \| + \| \sum_{n=N+1}^{\infty} a_n^{(j)} e_n \|$$

$$\leq (1+C)\varepsilon + \| \sum_{n=N+1}^{\infty} a_n^{(j)} e_n \|$$

$\forall \varepsilon > 0 \quad \exists j \in \mathbb{N} \quad \forall N \in \mathbb{N} \quad \| \sum_{n=0}^{N} b_n e_n - x \| \leq (1+C)\varepsilon + \underbrace{\| \sum_{n=N+1}^{\infty} a_n^{(j)} e_n \|}_{\leq \varepsilon}$

$\forall \varepsilon > 0 \quad \exists j \in \mathbb{N} \quad \exists \bar{N} \in \mathbb{N} \quad \forall N \geq \bar{N} \quad \| \sum_{n=0}^{N} b_n e_n - x \| \leq (2+C)\varepsilon$

i.e $\sum_{n=0}^{\infty} b_n e_n = x$

Thus $x \in E$ and $E = X$

So $\forall x \in X$ $\exists (a_n)_{n \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$: $x = \sum_{n=0}^{\infty} a_n e_n$

Assume $\exists (b_n) \in \mathbb{R}^{\mathbb{N}}$: $x = \sum_{n=0}^{\infty} b_n e_n$

$\forall N \in \mathbb{N}$ $\quad \| \sum_{n=0}^{N} (a_n - b_n) e_n \| \underset{(3)}{\leq} C \| \sum_{n=0}^{\infty} (a_n - b_n) e_n \| = 0$

Thus $(a_n - b_n) e_n = 0$ $\quad \forall n \in \mathbb{N}$

Thus $a_n = b_n$ $\forall n \in \mathbb{N}$ by (1) $\square$

25. Basic sequences

Lemma 10.6

let $\varepsilon > 0$ since $\dim Y < \infty$, S_Y is compact

$\exists (y_j)_{j=1}^N \in S_Y^N$ let $(y_j)_{j=1}^N$ be a cover to S_Y^N

Then we have $\forall y \in S_Y$ $\exists j \in \{1, \dots, N\}$ $\|y - y_j\| < \frac{\varepsilon}{2}$

By HB $\exists (y_j^*)_{j=1}^N \in S_{X^*}^N$ $\langle y_j^*, y_j \rangle = 1$ $\forall j = 1, \dots, N$

$\overline{\text{span}} \{ y_j^* ; j=1 \dots N \} \neq X^*$

Rk: If $\bigcap_{i=1}^N \ker y_j^* = \{0\}$, then $\bigcap_{j=1}^N \ker y_j^* \subset \ker y^*$ $\forall y \in S_{X^*}$

Then by HB, Thm 5.7 $y^* \in \overline{\text{span}} \{ y_j^* : j = 1 \dots N\}$ $\forall y^* \in S_{X^*}$ which gives contradiction.

Therefore $\exists x \in (\bigcap_{j=1}^N \ker y_j^*) \cap S_X$

let $y \in S_Y$, $\lambda \in \mathbb{R}$, Pick $j \in \{1, \dots, N\}$ $\|y - y_j\| < \frac{\varepsilon}{2}$

$\|\lambda x + y\| \geq \|\lambda x + y_j\| - \|y - y_j\| \geq \langle \lambda x + y_j, y_j^* \rangle - \frac{\varepsilon}{2} = \langle y, y_j^* \rangle - \frac{\varepsilon}{2} = 1 - \frac{\varepsilon}{2} \geq \frac{1}{1+\varepsilon} \|y\|$

$\forall y \in Y$ $\forall \lambda \in \mathbb{R}$

$(1+\varepsilon) \|\lambda x + \frac{y}{\|y\|}\| \geq 1$

$(1+\varepsilon) \| \lambda \|y\| x + y \| \geq \|y\|$ □

Thm 10.5

Proof: Pick $e_1 \in S_X$

let $\varepsilon_1 > 0$, by lemma 10.6 $\exists e_2 \in S_X$

$\forall \lambda_1, \lambda_2 \in \mathbb{R}$

$\| \lambda_1 e_1 \| \leq (1 + \varepsilon_1) \| \lambda_1 e_1 + \lambda_2 e_2 \|$

let $(\varepsilon_n)_{n \in \mathbb{N}} \in \mathbb{R}_+^{\mathbb{N}}$, Pick $(e_n)_{n=1}^N \in S^N$ $\forall (\lambda_j)_{j=1}^N$, Iterately, we have

$\| \sum_{j=1}^{N-1} \lambda_j e_j \| \leq (1 + \varepsilon_N) \| \sum_{j=1}^N \lambda_j e_j \|$ (*)

Then $\forall N, M \in \mathbb{N}$, $N > M$

$$\left\| \sum_{j=1}^{M} \lambda_j e_j \right\| \leq \prod_{j=M}^{N} (1+\varepsilon_j) \left\| \sum_{j=1}^{N} \lambda_j e_j \right\|$$

choosing $(\varepsilon_n)_{n \in \mathbb{N}}$ such that $\prod_{j=1}^{\infty} (1+\varepsilon_j) \leq 2$

Then we have $\|e_n\| = 1$ $\forall n \in \mathbb{N}$

$\forall N, M \in \mathbb{N}$ $N > M$

$\forall (\lambda_j)_{j=1}^{N} \in \mathbb{R}^N$ $\quad \left\| \sum_{j=1}^{M} \lambda_j e_j \right\| \leq 2 \left\| \sum_{j=1}^{N} \lambda_j e_j \right\|$ $\square$

26. Shrinking bases

Prop. 10.7 (e_n) basis of $X \Rightarrow (e_n^*)_{n\in\mathbb{N}}$ is a basic sequence in X

proof

Remark : let $x \in X$, $x^* \in X^*$ $N \in \mathbb{N}$

$$\langle P_N(x), x^* \rangle = \sum_{n=0}^{N} \langle x, e_n \rangle \langle e_n, x^* \rangle$$

$$= \langle x, \sum_{n=0}^{N} \langle e_n, x^* \rangle e_n^* \rangle$$

$$= \langle x, P_N^*(x^*) \rangle$$

i.e $P_N^*(x^*) = \sum_{n=0}^{N} \langle e_n, x^* \rangle e_n^*$

let $Y^* = \overline{\text{span}}\{e_j^* ; j \in \mathbb{N}\} \subset X^*$

$e_n^* \neq 0$ $\forall n \in \mathbb{N}$ $e_n^*(e_m) = \delta_{n,m}$ $\forall n,m \in \mathbb{N}$

let $x^* \in Y^*$. $\exists (x_k^*)_{k\in\mathbb{N}} \in \text{span}\{e_j^* ; j \in \mathbb{N}\}$

$\|x^* - x_k^*\| \xrightarrow{k \to +\infty} 0$

$\forall k \in \mathbb{N}$ $\exists N_k \in \mathbb{N}$ $\exists (\lambda_{k,j})_{j=0}^{N_k}$: $x_k^* = \sum_{j=0}^{N_k} \lambda_{k,j} e_j^*$

$\forall n \in \mathbb{N}$ $\langle x_n^*, e_n \rangle = \sum_{j=0}^{N_k} \lambda_{k,j} \langle e_j^*, e_n \rangle = \begin{cases} \lambda_{k,n} & \text{if } n \leq N_k \\ 0 & \text{otherwise} \end{cases}$

$$x_k^* = \sum_{n=0}^{N_k} \langle x_k^*, e_n \rangle e_n^* \quad \forall k \in \mathbb{N}$$

$$= P_{N_k}^*(x_k^*)$$

$$= P_N^*(x_k^*) \quad \forall N \geq N_k$$

let $k \in \mathbb{N}$
let $N > N_k$ $\|x^* - \sum_{n=0}^{N} \langle x^*, e_n \rangle e_n^*\| \leq \|x^* - x_k^*\| + \|x_k^* - P_N^*(x^*)\|$

$$= \|x^* - x_k^*\| + \|P_N^*(x_k^* - x^*)\|$$

$$\leq (1 + \|P_N^*\|)\|x_k^* - x^*\|$$

$$\leq (1 + \sup_{N\in\mathbb{N}} \|P_N\|)\|x_k^* - x\|$$

$\forall \varepsilon > 0 \quad \exists k \in \mathbb{N} \quad \forall N > N_k \quad \| x^* - \sum_{n=0}^{N} \langle x^*, e_n \rangle e_n^* \| < \varepsilon$

i.e $\quad x^* = \sum_{n=0}^{\infty} \langle x^*, e_n \rangle e_n^*$

Thus $(e_n^*)_{n\in\mathbb{N}}$ is a basis of Y^* $\square$

Prop 10.8

Let $(e_n)_{n\in\mathbb{N}}$ be a basis of X

$(e_n^*)_{n\in\mathbb{N}}$ is a basis of X^* $\iff \forall x^* \in X^* \quad \|x^*|_{\overline{\text{span}}\{e_j ; j\geq n\}}\| \xrightarrow{n\to+\infty} 0$

<u>proof</u>

$\Rightarrow$ let $x^* \in Y^*$, $n \in \mathbb{N}$

$\| x^*|_{\overline{\text{span}}\{e_j ; j\geq n\}} \| = \| x^* - P_{n-1}^*(x^*) \|$

since $\langle P_{n-1}^*(x^*), \sum_{k=n}^{\infty} \langle e_k^*, y\rangle e_k \rangle = \langle x^*, P_{n-1}(\sum_{k=n}^{\infty} \langle e_k^*, y\rangle e_k) \rangle = 0$

$\| x^*|_{\overline{\text{span}}\{e_j ; j\geq n\}} \| \leq \| x^* - P_{n-1}^*(x^*) \| \xrightarrow{n\to+\infty} 0$

since $(e_n^*)_{n\in\mathbb{N}}$ is a basis

$\Leftarrow$ let $n \in \mathbb{N}^*$

$\|x^* - P_{n-1}^*(x^*)\| = \| \underbrace{(x^* - P_{n-1}^*(x^*))|_{\overline{\text{span}}\{e_j ; j\leq n-1\}}}_{0} + (x^* - P_{n-1}^*(x^*))|_{\overline{\text{span}}\{e_j ; j\geq n\}} \|$

$= \| (x^* - P_{n-1}^*(x^*))|_{\overline{\text{span}}\{e_j ; j\geq n\}} \|$

$= \| x^* |_{\overline{\text{span}}\{e_j ; j\geq n\}} \| \longrightarrow 0$ $\square$

Prop 10.10

<u>proof</u> let $x^{**} \in X^{**}$ $\quad x^* \in X^*$

$\langle P_N^{**}(x^{**}), x^* \rangle = \langle x^{**}, P_N^*(x^*) \rangle = \langle x^{**}, \sum_{j=0}^{N} \langle x^*, e_j \rangle e_j^* \rangle = \sum_{j=0}^{N} \langle x^*, e_j \rangle \langle x^{**}, e_j^* \rangle$

$= \langle x^*, \sum_{j=0}^{N} \langle x^{**}, e_j^* \rangle e_j \rangle$

Therefore $P_N^{**}(x^{**}) = \sum_{j=0}^{N} \langle x^{**}, e_j^* \rangle e_j \quad \forall N \in \mathbb{N} \quad \forall x^{**} \in X^{**}$

$\| (\langle x^{**}, e_j^* \rangle)_{j\in\mathbb{N}} \|_S = \sup_{N\in\mathbb{N}} \| \sum_{j=0}^{N} \langle x^{**}, e_j^* \rangle e_j \| = \sup_{N\in\mathbb{N}} \| P_N^{**}(x^{**}) \|_{X^*} \leq \sup_{N\in\mathbb{N}} \|P_N\|_{B(X)} \|x^{**}\|_{X^{**}} < \infty$ (by prop 10.2(6))

Therefore $T : x^{**} \longmapsto (\langle x^{**}, e_j^* \rangle)_{j\in\mathbb{N}} \in B(X^{**}, S)$

let $x^{**}, y^{**} \in X^{**}$

$T(x^{**}) = T(y^{**})$

$\langle x^{**}, e_j^* \rangle = \langle y^{**}, e_j^* \rangle \quad \forall j \in \mathbb{N}$

$x^{**} = y^{**}$ since $(e_j^*)_{j\in\mathbb{N}}$ is a basis of X^*

let $(a_j)_{j\in\mathbb{N}} \in S$. $\sup_{N\in\mathbb{N}} \| \sum_{j=0}^{N} a_j e_j \|_X < \infty$

Define $x_N := \sum_{j=0}^{N} a_j e_j \quad \forall N \in \mathbb{N}$

$(x_N)_{N\in\mathbb{N}}$ is bounded so by the Banach Aloglu Thm.

$\exists (x_{N_k})_{k\in\mathbb{N}} \quad \exists x^{**} \in X^{**} \quad : x_{N_k} \xrightarrow{w^*} x^{**}$

$$\langle x_{N_k}, e_j^* \rangle = \sum_{n=0}^{N_k} \langle a_n e_n, e_j^* \rangle = \begin{cases} a_j & \text{if } j \le N_k \\ 0 & \text{if } j > N_k \end{cases}$$

$\langle x_{N_k}, e_j^* \rangle = a_j \quad \forall j \le N_k$

$\langle x^{**}, e_j^* \rangle = a_j \quad \forall j \in \mathbb{N}$

Therefore $T(x^{**}) = (e_j)_{j\in\mathbb{N}}$ hence T is surjective

So by OMT T is an isomorphism $\square$

27. B-complete Bases

Prop 10.12

Let $(a_j)_{j\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$.

Assume $\sup_{N\in\mathbb{N}} \| \sum_{j=0}^{N} a_j e_j^* \| < \infty$

Define $x_N^* = \sum_{j=0}^{N} a_j e_j^*$

x_N^* is bounded

By Banach-Alaoglu Thm, $\exists (x_{N_k}^*)_{k\in\mathbb{N}} \in (X^*)^{\mathbb{N}}$. $\exists x^* \in X^*$. $x_{N_k}^* \xrightarrow{w^*} x^*$

$\langle x_{N_k}^*, e_j \rangle \xrightarrow{k\to+\infty} \langle x^*, e_j\rangle \quad \forall j$

If $j \le N_k$ $\langle x_{N_k}^*, e_j \rangle = \langle \sum_{n=0}^{N_k} a_n e_n^*, e_j \rangle = a_j$

$x^* = \sum_{j=0}^{\infty} b_j e_j^* \qquad \langle x^*, e_j\rangle = \sum_{k=0}^{\infty} b_k \langle e_k^*, e_j\rangle = b_j \quad \forall j\in\mathbb{N}$

$x^* = \sum_{j=0}^{\infty} \langle x^*, e_j\rangle e_j^*$

Thus $\langle x^*, e_j\rangle = a_j \quad \forall j\in\mathbb{N}$

Therefore $x^* = \sum_{j=0}^{\infty} a_j e_j^*$ □

Thm 10.13

$\Rightarrow$: X reflexive. $(e_n)_{n\in\mathbb{N}}$ a basis of X

Define $Y_* = \text{span}\{e_j^* ; j\in\mathbb{N}\}$

WTS $Y_*^{\perp} = \{0\}$

Let $y \in Y_*^{\perp} \subset X^{**} = X$

$$\begin{cases} \langle y, e_n^* \rangle = 0 \\ y = \sum_{n=0}^{\infty} \langle y, e_n^*\rangle e_n \end{cases} \Rightarrow y = 0$$

Thus $(e_n)_{n\in\mathbb{N}}$ is shrinking let $(a_j)_{j\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$. Assume $\sup_N \|\sum_{i=0}^{N} a_j e_j\| < \infty$

let $x_N := \sum_{j=0}^{N} a_j e_j$ $\quad (x_N)_{N\in\mathbb{N}}$ is bounded

By Banach – Alaoglu + Reflexive

$\exists\, (x_{N_k})_{k\in\mathbb{N}} \in X^{\mathbb{N}}$, $\exists\, x \in X$: $\quad x_{N_k} \xrightarrow[k\to+\infty]{w} x$

$\langle x, e_j^* \rangle = a_j \quad \forall j \in \mathbb{N}$

$$x = \sum_{j=0}^{\infty} \langle x, e_j^* \rangle e_j = \sum_{j=0}^{\infty} a_j e_j$$

We have proven $(e_j)_{j\in\mathbb{N}}$ is b-complete

$\Leftarrow$ let $x^{**} \in X^{**}$. Since (e_j) is shrinking

$T: X^{**} \longrightarrow S$ is an isomorphism:

$y^{**} \longmapsto (\langle y^{**}, e_j^* \rangle)_{j\in\mathbb{N}}$

$T(x^{**}) \in S$ i.e $\sup_{N\in\mathbb{N}} \|\sum_{j=0}^{N} \langle x^{**}, e_j^* \rangle e_j\| < \infty$

since (e_j) is b-complete:

$\sum_{j=0}^{\infty} \langle x^{**}, e_j^* \rangle e_j =: x \in X$

$\forall n \in \mathbb{N} \quad \langle x, e_n^* \rangle = \langle x^{**}, e_n^* \rangle$

since $(e_n^*)_{n\in\mathbb{N}}$ is a basis of X^*: $\langle x, x^* \rangle = \langle x^{**}, x^* \rangle \quad \forall x^* \in X^*$

Thus $x = x^{**}$ □

28. Unconditional convergence

Prop 11.2

(1) $\Rightarrow$ (2)

Let $(x_{n_k})_{k\in\mathbb{N}}$ be a subsequence of $(x_n)_{n\in\mathbb{N}}$ and $\sum x_n$ CV unconditionally.

$\forall n\in\mathbb{N}$ $\quad \varepsilon_n := \begin{cases} 1 & \text{if } n\in\{n_k\,;\,k\in\mathbb{N}\} \\ -1 & \text{otherwise} \end{cases}$

Then $\quad \frac{1+\varepsilon_n}{2} = \begin{cases} 1 & \text{if } n\in\{n_k\,;\,k\in\mathbb{N}\} \\ 0 & \text{otherwise} \end{cases}$

$\forall N\in\mathbb{N}$ $\quad \exists M\in\mathbb{N}$ $\quad \sum_{k=0}^{N} x_{n_k} = \sum_{n=0}^{M}\left(\frac{1+\varepsilon_n}{2}\right)x_n$

$$\begin{cases} \sum_{n=0}^{\infty}\varepsilon_n x_n & \text{CV} \\ \sum_{n=0}^{\infty} x_n & \text{CV} \end{cases} \Rightarrow \sum_{k=0}^{\infty} x_{n_k} \text{ CV}$$

(2) $\Rightarrow$ (1)

Let $(\varepsilon_n)_{n\in\mathbb{N}}\in\{-1,1\}^{\mathbb{N}}$

Define $A=\{n\in\mathbb{N}\ ;\ \varepsilon_n=1\}$

If A is finite, $\sum_{n=0}^{\infty}\varepsilon_n x_n$ converges.

If A is infinite, $\quad A=\{n_k\,;\,k\in\mathbb{N}\}$ with $(n_k)_{k\in\mathbb{N}}$ increasing.

$\sum_{k=0}^{\infty}x_{n_k}$ CV and $\sum_{k=0}^{\infty}x_{n_k} = \sum_{n=0}^{\infty}\left(\frac{1+\varepsilon_n}{2}\right)x_n$

So $\sum_{n=0}^{\infty}\left(\frac{1+\varepsilon_n}{2}\right)x_n$ CV, since $\sum_{n=0}^{\infty}x_n$ CV

we have that $\sum_{n=0}^{\infty}\varepsilon_n x_n$ CV

$(2) \Rightarrow (3)$

By contradiction, assume

$\exists \varepsilon > 0 \quad \forall N \in \mathbb{N} \quad \exists F_N \subset \mathbb{N} \setminus \{1, \dots, N\}$ finite $\|\sum_{n \in F_N} x_n\| > \varepsilon$

Pick $N_1 = 1 \quad N_{k+1} := \max F_{N_k} + 1 \quad \forall k \in \mathbb{N}$

$\forall j, k \in \mathbb{N} \quad j \neq k \Rightarrow F_{N_j} \cap F_{N_k} = \phi$

$\bigcup_{j=1}^{\infty} F_{N_j} = \{n_k ; k \in \mathbb{N}\}$ for some increasing $(n_k)_{k \in \mathbb{N}}$

$\forall M \in \mathbb{N} \quad \exists k_1 \geq k_0 \geq M \quad \exists j \in \mathbb{N} \quad \|\sum_{k=k_0}^{k_1} x_{n_k}\| = \|\sum_{n \in F_{N_j}} x_n\| > \varepsilon$

Thus $\sum_{k=0}^{\infty} x_{n_k}$ does not converge.

$(3) \Rightarrow (2)$

Let $(x_{n_k})_{k \in \mathbb{N}}$ be a subsequence

We have that $\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall F_N \subset \mathbb{N} \setminus \{1, \dots, N\}$ finite $\|\sum_{n \in F_N} x_n\| < \varepsilon$

Given $N \in \mathbb{N}$, and $k_1 \geq k_0 \geq N$, define $F_{k_1, k_0, N} = \{n_j ; j = k_0, \dots, k_1\}$

$\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall k_1 \geq k_0 \geq N \quad \|\sum_{j=k_0}^{k_1} x_{n_j}\| < \varepsilon$ (partifal sum, Cauchy seq)

Thus $\sum x_{n_k}$ converge

$(3) \Rightarrow (4)$

Let $\sigma : \mathbb{N} \to \mathbb{N}$ be a bijection

Give $N \in \mathbb{N}$, define $F_{N, \sigma, k_0, k_1} := \{\sigma(j) \mid j = k_0, \dots, k_1\}$

$\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall k_1 \geq k_0 \geq N \quad \|\sum_{j=k_0}^{k_1} x_{\sigma(j)}\| < \varepsilon$ (partial sum, Cauchy seq)

Thus $\sum_{j=0}^{\infty} x_{\sigma(j)}$ converges.

$(4) \Rightarrow (3)$

By contradiction, assume

$\exists \varepsilon > 0 \quad \exists (F_{N_j})_{j \in \mathbb{N}} \begin{cases} F_{N_j} \subset \mathbb{N} \\ \min F_{N_j} \geq N_j \\ N_j \xrightarrow{j \to +\infty} +\infty \end{cases} \quad \forall j \in \mathbb{N}$ finite : $\|\sum_{n \in F_{N_j}} x_n\| > \varepsilon \quad \forall j \in \mathbb{N}$

Pick $\sigma : \mathbb{N} \to \mathbb{N}$ bijection, such that

$\forall j \in \mathbb{N} \quad \sigma^{-1}(F_{N_j}) = \{N_j, \dots, N_j + M_j\}$ for some $M_i \in \mathbb{N}$

Then $\|\sum_{n=N_j}^{N_j+M_j} x_{\sigma(n)}\| = \|\sum_{n \in F_j} x_n\| > \varepsilon$

Thus $\sum x_{\sigma(n)}$ does not converge $\square$

29. Unconditional bases

Lemma 11.4

By contradiction, assume $\sup\limits_{J \text{ finite}} \|P_J\|_{B(X)} = +\infty$

Pick $J_1 \subset \mathbb{N}$ finite: $\|P_{J_1}\| \geq 2$

<u>Claim</u>: $\sup\limits_{\substack{J \supseteq J_1 \\ \text{finite}}} \|P_J\|_{B(X)} = +\infty$

Indeed. If $\sup\limits_{J \supseteq J_1} \|B_J\|_{B(X)} < \infty$

then we have the following

let $J \subset \mathbb{N}$ be finite. $\forall x \in X$

$$\|P_J(x)\| = \|P_{J \cup J_1}(x) - P_{J_1 \cap J^c}(x)\| \leq \|P_{J \cup J_1}(x)\| + \sum_{j \in J_1} |\langle e_j^*, x\rangle| \, \|e_j\|$$

$$\leq \Big(\sup_{\substack{K \supseteq J_1 \\ K \text{ finite}}} \|P_K\| + \sum_{j \in J_1} \|e_j^*\| \|e_j\|\Big) \|x\|$$

$\forall x \in X$ $\quad \sup\limits_{J \text{ finite}} \|P_J(x)\| < \infty$ $\quad$ By UBP $\quad \sup\limits_{J \text{ finite}} \|P_J\|_{B(X)} < \infty$ $\quad$ Contradiction

Inductively $\exists (J_k)_{k \in \mathbb{N}}$ finite subsets of $\mathbb{N}$: $J_k \subset J_{k+1}$ $\forall k \in \mathbb{N}$

and $\|P_{J_k}\| \geq 2^k$ $\forall k \in \mathbb{N}$

$\bigcup\limits_{k=0}^{\infty} J_k = \{n_j \mid j \in \mathbb{N}\}$ for some $(n_j)_{j \in \mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$ increasing.

By unconditionality : $\sum\limits_{j=0}^{\infty} \langle x, e_{n_j}^* \rangle e_{n_j}$ converges. $\forall x \in X$

$\lim\limits_{k \to +\infty} P_{J_k}(x) = \sum\limits_{j=0}^{\infty} \langle x, e_{n_j}^* \rangle e_j$

$\sup\limits_{k \in \mathbb{N}} \|P_{J_k}(x)\| < \infty$ $\quad \forall x \in X$

By UBP. $\quad \sup\limits_{k \in \mathbb{N}} \|P_{J_k}\|_{B(X)} < \infty$ $\quad \square$

Thm 11.5

$\lambda \in l_\infty$ WLOG $\|\lambda\|_\infty \leq 1$

let $\varepsilon > 0$. Pick $d \in \mathbb{N}$ $(a_l)_{l=1}^{d}$: $\begin{cases} a_1 = -1 \\ a_{d+1} = 1 \\ |a_{l+1} - a_l| < \varepsilon \quad \forall l = 1, \dots, d \end{cases}$

$\forall l = 1, \dots, d-1$ $A_l = \{n \in \mathbb{N} \,;\, \lambda_n \in [a_l, a_{l+1})\}$

$A_d = \{n \in \mathbb{N} \,;\, \lambda_n \in [a_d, 1]\}$

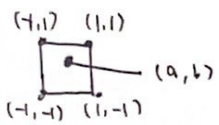
$\bigcup_{l=1}^{d} A_l = \mathbb{N}$ $\quad \forall n \in \mathbb{N} \quad \exists! l \in \{1, \dots, d\} \quad n \in A_l$

Define $\mu_n := \sum_{l=1}^{d} \mathbb{1}_{A_l}(n)\, a_l \quad \forall n \in \mathbb{N}$

Rk $\quad |\mu_n - \lambda_n| < \varepsilon \quad \forall n \in \mathbb{N}$

Rk $\quad a \in [-1, 1] \quad a = \left(\frac{1+a}{2}\right) \cdot 1 + \left(\frac{1-a}{2}\right)(-1)$

$(a,b) \in [-1, 1]^2 \quad (a,b) = \left(\frac{1+a}{2}\right)(1, b) + \left(\frac{1-a}{2}\right)(-1, b) = \left(\frac{1+a}{2}\right)\left(\frac{1+b}{2}\right)(1,1) + \left(\frac{1+a}{2}\right)\left(\frac{1-b}{2}\right)(1,-1) + \cdots (-1,1) + \cdots + (-1,-1)$



$\forall (a_1, \dots, a_d) \in [-1, 1]^d \quad \exists (t_j)_{j=1}^{2^d} \in [0,1]^{2^d}$

$\sum_{j=1}^{2^d} t_j = 1 \quad (a_1, \dots, a_d) = \sum_{j=1}^{2^d} t_j \left(\theta_1^{(j)}, \dots, \theta_d^{(j)}\right)$

$\left\{ \left(\theta_1^{(j)}, \dots, \theta_d^{(j)}\right) \, j = 1, \dots, 2^d \right\} = \{1, -1\}^d$

$\forall n \in \mathbb{N} \quad \mu_n = \sum_{l=1}^{d} \mathbb{1}_{A_l}(n) \sum_{j=1}^{2^d} t_j\, \theta_l^{(j)}$

let $N \in \mathbb{N}$ and $x \in X$

$$\left\| \sum_{n=0}^{N} \mu_n \langle x, e_n^* \rangle e_n \right\| = \left\| \sum_{n=0}^{N} \sum_{l=1}^{d} \mathbb{1}_{A_l}(n) \sum_{j=1}^{2^d} t_j\, \theta_l^{(j)} \langle x, e_n^* \rangle e_n \right\|$$

$$\leq \sum_{j=1}^{2^d} t_j \left\| \sum_{n=0}^{N} \Big[\underbrace{\sum_{l=1}^{d} \mathbb{1}_{A_l}(n)\, \theta_l^{(j)}}_{\in \{-1,1\}} \Big] \langle x, e_n^* \rangle e_n \right\|$$

$$\overset{\text{lemma 11.4}}{\leq} C \sum_{j=1}^{2^d} t_j \left\| \sum_{n=0}^{N} \langle x, e_n^* \rangle e_n \right\| \qquad \Big(C = 1 + \sup_{\substack{J \subset \mathbb{N} \\ \text{finite}}} \|P_J\|_{B(X)}\Big)$$

for some C indep of N, x and d

$$= C \left\| \sum_{n=0}^{N} \langle x, e_n^* \rangle e_n \right\|$$

$$\left\|\sum_{n=0}^{N}\lambda_n \langle x, e_n^*\rangle e_n\right\| \leq c\|x\| + \left\|\sum_{n=0}^{N}(\lambda_n-\mu_n)\langle x, e_n^*\rangle e_n\right\|$$

$$\leq c\|x\| + \sup_{x^*\in S_{X^*}}\left|\sum_{n=0}^{N}(\lambda_n-\mu_n)\langle x, e_n^*\rangle\langle e_n, x^*\rangle\right|$$

$$\leq c\|x\| + \varepsilon \sup_{x^*\in S_{X^*}}\sum_{n=0}^{N}|\langle x, e_n^*\rangle||\langle e_n, x^*\rangle|$$

$$= c\|x\| + \varepsilon \sup_{x^*\in S_{X^*}}\sum_{n=0}^{N}\langle x, e_n^*\rangle\langle e_n, x^*\rangle \operatorname{sgn}[\langle x, e_n^*\rangle\langle e_n, x^*\rangle]$$

$$= c\|x\| + \varepsilon \sup_{x^*\in S_{X^*}}\left\langle \sum_{n=0}^{N}\operatorname{sgn}(\langle x, e_n^*\rangle\langle e_n, x^*\rangle)\langle x, e_n^*\rangle e_n, x^*\right\rangle$$

$$\leq c\|x\| + \varepsilon \sup_{x^*\in S_{X^*}}\left\|\sum_{n=0}^{N}\operatorname{sgn}(\langle x, e_n^*\rangle\langle e_n, x^*\rangle)\langle x, e_n^*\rangle e_n\right\|$$

$$\leq c\|x\| + c\,\varepsilon \sup_{x^*\in S_{X^*}}\left\|\sum_{n=0}^{N}\langle x, e_n^*\rangle e_n\right\| \leq c(1+\varepsilon)\|x\| \quad \square$$

30. Haar basis

Thm 11.7

- $e_n \neq 0 \quad \forall n \in \mathbb{N}$ # since $\|e_n\|_p = 1$

let $k, l \in \mathbb{N}$

$j \in \{0, \dots, 2^l - 1\}$

If $l > k$

$\forall t \in (0,1)$

$$\mathcal{E}_{2^k}(e_{2^l+j})(t) = \sum_{m=0}^{2^k-1} \mathbb{1}_{(m2^{-k}, (m+1)2^{-k})}(t) \cdot 2^k \int_{m2^{-k}}^{(m+1)2^{-k}} e_{2^l+j}$$

$$= \sum_{m=0}^{2^k-1} \mathbb{1}_{(m2^{-k}, (m+1)2^{-k})}(t) \cdot 2^k \int_{m2^{-k}}^{(m+1)2^{-k}} 2^{\frac{l}{p}} \left(\mathbb{1}_{(j2^{-l}, (j+\frac{1}{2})2^{-l})}(s) - \mathbb{1}_{((j+\frac{1}{2})2^{-l}, (j+1)2^{-l})}(s) \right) ds$$

Two possibilities: $(j2^{-l}, (j+1)2^{-l}) \cap (m2^{-k}, (m+1)2^{-k}) = \phi$

or $(j2^{-l}, (j+1)2^{-l}) \subset (m2^{-k}, (m+1)2^{-k})$

$$\mathcal{E}_{2^k}(e_{2^l+j}) = 0 \quad \text{if } l > k$$

If $k > l$

$$\mathcal{E}_{2^k}(e_{2^l+j})(t) = \sum_{m=0}^{2^k-1} \mathbb{1}_{(m2^{-k}, (m+1)2^{-k})}(t) \; \mathbb{1}_{(j2^{-l}, (j+\frac{1}{2})2^{-l})}(t) \cdot 2^{\frac{l}{p}}$$

$$+ \sum_{m=0}^{2^k-1} \mathbb{1}_{(m2^{-k}, (m+1)2^{-k})}(t) \; \mathbb{1}_{((j+\frac{1}{2})2^{-l}, (j+1)2^{-l})}(t) \; (-2^{\frac{l}{p}})$$

$$= e_{2^l+j}(t)$$

If $k = l$

$$\mathcal{E}_{2^k}(e_{2^k+j})(t) = \mathbb{1}_{(j2^{-k}, (j+1)2^{-k})}(t) \; 2^{-k} \int_{j2^{-k}}^{(j+1)2^{-k}} e_{2^k+j}(s)\, ds = 0$$

$$\mathcal{E}_{2^k}(e_{2^l+j}) = \begin{cases} e_{2^l+j} & \text{if } l < k \\ 0 & \text{otherwise} \end{cases}$$

Define $P_{2^k+j}(f)(t) = \begin{cases} \mathcal{E}_{2^{k+1}}(f)(t) & \text{if } t < (j+1)2^{-k} \\ \mathcal{E}_{2^k}(f)(t) & \text{if } t \geq (j+1)2^{-k} \end{cases}$ $\quad \forall k \in \mathbb{N}$ $\quad \forall j \in \{0, \dots 2^k-1\}$ $\quad \forall t \in (0,1)$

$$P_{2^k+j}(e_{2^k+m}) = \begin{cases} e_{2^k+m} & \text{if } m \leq j \\ 0 & \text{if } m > j \end{cases} \qquad \forall m \in \{0, \dots 2^k-1\}$$

$$P_{2^k+j}(e_{2^l+m}) = \begin{cases} e_{2^l+m} & \text{if } l < k \\ 0 & \text{if } l > k \end{cases} \qquad \forall l \in \mathbb{N}$$

$\forall x \in X \quad \forall N, M \in \mathbb{N} \quad N \geq M \quad \| \sum_{n=0}^{M} \langle x, e_n \rangle e_n \|_{L^p} = \| P_M (\sum_{n=0}^{N} \langle x, e_n \rangle e_n) \|_{L^p} \quad (*)$

$P_M(f)(t) \leq \|f\|_\infty \quad \forall t \in (0, \infty) \quad \forall f \in L^\infty(0,1) \quad P_M \in B(L^\infty) \quad \|P_M\|_{B(L^\infty)} \leq 1$

$$\begin{matrix} \|P_M\|_{B(L^\infty)} \leq 1 \\ \|P_M\|_{B(L^1)} \leq 1 \end{matrix} \xrightarrow[\text{Interpolation Thm}]{\text{Thm 6.9}} \|P_M\|_{B(L^p)} \leq 1$$

plug in $(*)$ Then $(e_n)_{n \in \mathbb{N}}$ is a basis sequence in $L^p(0,1)$

let $f \in C[0,1]$. let $k \in \mathbb{N}$ $\forall t \in (0,1)$

$$| \mathcal{E}_{2^k}(f)(t) - f(t) | \leq \sum_{j=0}^{2^k-1} 1_{(j2^{-k}, (j+1)2^{-k})}(t) \cdot 2^k \int_{j2^{-k}}^{(j+1)2^{-k}} |f(s) - f(t)| \, ds$$

Thus $\forall \varepsilon > 0 \quad \exists K \in \mathbb{N} \quad \forall k \geq K \quad \forall t \in (0,1)$

$$| \mathcal{E}_{2^k}(f)(t) - f(t) | \leq \sum_{j=1}^{2^k-1} 1_{(j2^{-k}, (j+1)2^{-k})}(t) \, \varepsilon = \varepsilon \quad \text{(uniformly continuity)}$$

Therefore $\forall f \in C[0,1] \quad \forall \varepsilon > 0 \quad \exists K \in \mathbb{N} \quad \forall k \geq K$

$\| \mathcal{E}_{2^k}(f) - f \|_p \leq \varepsilon$

since $C[0,1]$ is dense in L^p

Let $\forall f \in L^p$, let $\varepsilon > 0$, Pick $g \in C[0,1]$: $\|f - g\|_p < \varepsilon$

$\forall f \in L^p \quad \forall \varepsilon > 0 \quad \exists K \in \mathbb{N} \quad \forall k \geq K$

$$\| \mathcal{E}_{2^k}(f) - f \|_p \leq \| \mathcal{E}_{2^k}(f-g) \|_p + \|f-g\|_p + \| \mathcal{E}_{2^k}(g) - g \|_p \leq 2\varepsilon + \| \mathcal{E}_{2^k}(g) - g \|_p \leq 3\varepsilon$$

Therefore $\overline{\text{span}\{e_j \mid j \in \mathbb{N}\}} = L^1(0,1)$

Thus $(e_n)_{n \in \mathbb{N}}$ is a basis of $L^p(0,1)$ ∎

$\forall N \in \mathbb{N} \quad \forall (a_k)_{k=0}^N \in \mathbb{R}^{N+1} \qquad \forall (\varepsilon_k)_{k=0}^N \in \{-1, +1\}^{N+1}$

$$\|\sum_{k=0}^N \varepsilon_k a_k e_k\|_{L^p} \leq (\max(p,p') - 1) \|\sum_{k=0}^N a_k e_k\|_{L^p} \qquad (U_p)$$

<u>Remark 1</u>: $(U_q)\ \forall q \geq 2 \implies (U_p)\ \forall p \leq 2$

let $p \leq 2$. Define $g_N := \sum_{k=0}^N \varepsilon_k a_k e_k \qquad f_N := \sum_{k=0}^N a_k e_k$

Pick $\tilde{g}_N \in L^{p'}$: $\int g_N \tilde{g}_N = \|g_N\|_p \quad \|\tilde{g}_N\|_{p'} = 1$ (Hahn–Banach exists a norming functional)

$\exists (\tilde{a}_k)_{k=0}^N \in \mathbb{R}^{N+1} \quad : \quad \tilde{g}_N = \sum_{k=0}^N \tilde{a}_k e_k$

$$\|g_N\|_p = \int_0^1 g_N \tilde{g}_N = \sum_{k=0}^N \varepsilon_k a_k \tilde{a}_k \int_0^1 e_k^2(s)\, ds$$

$$= \int (\sum_{k=0}^N a_k e_k)(\sum_{j=0}^N \varepsilon_j \tilde{a}_j e_j)$$

$$\overset{\text{Hölder}}{\leq} \|\sum_{k=0}^N a_k e_k\|_p \|\sum_{j=0}^N \varepsilon_j \tilde{a}_j e_j\|_{p'}$$

$$\leq \|\sum_{k=0}^N a_k e_k\|_p (\max(p,p') - 1) \|\tilde{g}_N\|_{p'} \quad \diamond$$

WLOG $p \geq 2$

let $p \geq 2$

WTS $\forall N \in \mathbb{N} \quad \int_0^1 \phi_p(f_N(s), g_N(s))\, ds \geq 0$ where $\phi_p(x,y) = (p-1)^p |x|^p - |y|^p \quad \forall x,y \in \mathbb{R}$

<u>Buekhddek 1980's</u> (Bellnan function argument)

<u>Rk\ idea</u>: $\exists \lambda > 0 \quad$ ~~$\exists \lambda \varphi \in C$~~ $\quad \exists \varphi \in C^2(\mathbb{R}^2)$

$$\begin{cases} \phi_p(x,y) \geq \lambda \varphi(x,y) \quad \forall x,y \in \mathbb{R} & (\mathrm{Dom}) \\ \forall N \in \mathbb{N} \quad \int_0^1 \varphi(f_N(s), g_N(s))\, ds \geq 0 & (B_N(\varphi)) \end{cases}$$

would imply the result

$B_1(\varphi) \iff \int_0^1 \varphi(\varepsilon_1 a_1 e_1(s), a_1 e_1(s))\, ds \geq 0$

$\varphi(x,y) = \varphi(|x|, |y|)$

$\varphi(x,x) \geq 0 \quad \forall x \in \mathbb{R}$ (Pos)

WTS $\quad B_{N-1}(\varphi) \Longrightarrow B_N(\varphi) \quad \forall N \in \mathbb{N}^*$

Define $\quad \forall t \in [0,1] \quad F(t) := \int_0^1 \varphi((1-t) g_{N-1}(s) + t g_N(s),\ (1-t) f_{N-1}(s) + t f_N(s)) ds$

$B_{N-1}(\varphi) \iff F(0) \geq 0$

$B_N(\varphi) \iff F(1) \geq 0$

It suffices to show that $\quad F'(t) \geq 0 \quad \forall t \in [0,1]$

$$F'(t) = \int_0^1 (-g_{N-1}(s) + g_N(s)) \partial_1 \varphi((1-t) g_{N-1}(s) + t g_N(s),\ (1-t) f_{N-1}(s) + t f_N(s))$$

$$+ (f_N(s) - f_{N-1}(s)) \partial_2 \varphi((1-t) g_{N-1}(s) + t g_N(s),\ (1-t) f_{N-1}(s) + t f_N(s))$$

$$F'(0) = a_N \int_0^1 e_N(s) \underbrace{(\varepsilon_N \partial_1 \varphi(\_\_) + \partial_2 \varphi(\_\_))}_{\text{constant in } s \text{ on the support of } e_N} ds = 0$$

Thus if $F''(t) \geq 0 \quad \forall t \in [0,1]$ then $F'(t) \geq 0 \quad \forall t \geq 0$, and the result will ~~prove~~. follow.

$$F''(t) = a_N^2 \int_0^1 \varepsilon_N^2 e_N(s)^2 \partial_1^2 \varphi(-) + e_N(s)^2 \partial_2^2 \varphi(-) + 2 e_N(s)^2 \varepsilon_N \partial_1 \partial_2 \varphi(-)\, ds$$

(DI) $\quad \partial_1^2 \varphi(x,y) + \partial_2^2 \varphi(x,y) \geq 2 |\partial_1 \partial_2 \varphi(x,y)| \quad \forall x,y \in \mathbb{R}$

If $\quad \exists \varphi \in C^2 :$ (Pom) + (Pos) + (DI) then the proof is complete.

$\varphi(x,y) := (|x| + |y|)^{p-1} ((p-1)|x| - |y|) \quad \forall x,y \quad$ does satisfy (Pom) + (Pos) + (DI) $\square$

31. Compact operators

Prop 12.2

let $n \in \mathbb{N}$ $\quad \overline{T_n(B_X)} \subseteq \overline{R(T_n)} = R(T_n)$ (since finite dimision space is always close)

In finite dim, closed bounded sets are indeed compact.

Thus $\overline{T_n(B_X)}$ is compact since it is a closed and bounded subset of a finite dimisional Banach space.

let $\varepsilon > 0$. Pick $n \in \mathbb{N}$ $\quad \|T - T_n\|_{B(X)} < \varepsilon$

$\exists N \in \mathbb{N}$ $\quad \exists (x_j)_{j=1}^N \in B_X^N$ $\quad \overline{T_n(B_X)} \subset \bigcup_{j=1}^N B(T_n(x_j), \varepsilon)$, since T_n compact, and X complete, Therefore totally bounded

let $z \in \overline{T(B_X)}$. Pick $x \in B_X$. $\|z - T(x)\| < \varepsilon$

Pick $j \in \{1, \dots, N\}$ $\quad \|T_n(x) - T_n(x_j)\| < \varepsilon$

$$\|z - T(x_j)\| \leq \|z - T(x)\| + \|T(x) - T_n(x)\| + \|T_n(x) - T_n(x_j)\| + \|T_n(x_j) - T(x_j)\|$$

$$\leq \varepsilon + \varepsilon + \varepsilon + \varepsilon = 4\varepsilon$$

i.e. $\overline{T(B_X)} \subset \bigcup_{j=1}^N B(T(x_j), 4\varepsilon)$ $\quad$ totally bounded in complete space

Therefore T compact $\square$

Example 1

(e_n) unconditional basis. $\lambda \in c_0$ (space of sequences tends to 0 at infinity)

$T_\lambda(x) := \sum_{n=0}^{\infty} \lambda_n \langle x, e_n^* \rangle e_n \quad \forall x \in X$

Define $\quad P_N(x) := \sum_{n=0}^{N} \langle x, e_n^* \rangle e_n$

$\exists C > 0$
$\forall x \in X$
$\forall N \in \mathbb{N}$

$$\|T_\lambda(x) - P_N T_\lambda(x)\| = \left\| \sum_{n=N+1}^{\infty} \lambda_n \langle x, e_n^* \rangle e_n \right\| \leq C \sup_{n \geq N+1} |\lambda_n| \left\| \sum_{n=0}^{\infty} \langle x, e_n^* \rangle e_n \right\|$$ just by unconditionality

$$\|T_\lambda - P_N T_\lambda\|_{B(X)} \leq C \sup_{n \geq N+1} |\lambda_n| \|\cdot\| \xrightarrow[N \to +\infty]{} 0 \quad \text{since } \lambda \in c_0$$

$\dim R(P_N T_\lambda) = N+1 \quad \forall N \in \mathbb{N}$

Thus by Prop 12.2 $\quad T_\lambda \in K(X)$

Example 2: let $f \in L^2$, let u be the unique weak solution of $-u'' + u = f$ in $L^2(0,1)$

~~let u~~ write $u = T(f)$

We have seen $T \in B(L^2)$ and $T \in K(L^2)$ by Sobolev embedding $i: H^1 \to L^2$ is compact

Idea: $T = \left(-\frac{d}{dx^2} + I\right)^{-1}$

prop. 12.5

$$\|T - P_n T\|_{B(X)} = \sup_{x \in B_X} \|T(x) - P_n(T(x))\|_X = \sup_{y \in T(B_X)} \|y - P_n(y)\|_X$$

$\forall y \in X$ $\quad \|y - P_n(y)\| \xrightarrow{n \to +\infty} 0$ since $(e_n)_{n \in \mathbb{N}}$ is a basis.

$\overline{T(B_X)}$ is compact so $\sup_{y \in \overline{T(B_X)}} \|y - P_n(y)\|_X \xrightarrow{n \to +\infty} 0$

Thus $\|T - P_n T\|_{B(X)} \xrightarrow{n \to +\infty} 0$ □

32. Compact operators and weak topologies

Prop 12.6

$\Rightarrow$ let $T \in K(X)$, $(x_n)_{n \in \mathbb{N}} \in X^{\mathbb{N}}$, $x \in X$, $x_n \xrightarrow[n \to +\infty]{w} x$

assume $\exists \varepsilon > 0$ $\exists (x_{n_k})_{k \in \mathbb{N}}$ $\| T(x_{n_k}) - T(x) \| > \varepsilon$ $\forall k \in \mathbb{N}$

$(T(x_{n_k}))_{k \in \mathbb{N}}$ is bounded since $x_{n_k} \xrightarrow{w} x$ so is bounded

$T \in K(X) \Rightarrow \exists (x_{n_{k_l}})_{l \in \mathbb{N}}$ $\exists y \in X$ $\| T(x_{n_{k_l}}) - y \| \xrightarrow{l \to +\infty} 0$

$\forall x^* \in X^*$ $\langle T(x_{n_{k_l}}), x^* \rangle \xrightarrow{l \to +\infty} \langle y, x^* \rangle$

$\forall x^* \in X^*$ $\langle x_{n_{k_l}}, T^*(x^*) \rangle \xrightarrow{l \to +\infty} \langle y, x^* \rangle$

so $y = T(x)$ $\| T(x_{n_{k_l}}) - T(x) \| \xrightarrow{l \to +\infty} 0$

This contradicts $\| T(x_{n_{k_l}}) - T(x) \| > \varepsilon$ $\forall l \in \mathbb{N}$

$\Leftarrow$

let $(y_n)_{n \in \mathbb{N}} \in \overline{T(B_X)}$. let $\varepsilon > 0$. Pick $\forall n \in \mathbb{N}$, $x_n \in B_X$: $\| T(x_n) - y_n \| < \varepsilon$

By (BA) + reflexivity, Pick $(x_{n_k})_{k \in \mathbb{N}}$, $x \in X$: $x_{n_k} \xrightarrow[k \to \infty]{w} x$

$\langle T(x_{n_k}), x^* \rangle \xrightarrow{k \to +\infty} \langle T(x), x^* \rangle$

By assumption: $\| T(x_{n_k}) - T(x) \| \xrightarrow{k \to +\infty} 0$

$\forall \varepsilon > 0$ $\exists K \in \mathbb{N}$ $\forall k \geq K$

$\| y_{n_k} - T(x) \| \leq \| T(x_{n_k}) - y_{n_k} \| + \| T(x_{n_k}) - T(x) \| \leq 2\varepsilon$

Therefore for any $\underset{\overline{T(B_X)}}{\underset{\uparrow}{y_n}}$ we have a subsequence y_{n_k} converging to $T(x)$

Thus $\overline{T(B_X)}$ is compact $\square$

Prop 12.7 X separable. $T \in K(X)$

let $(x_n^*)_{n\in\mathbb{N}} \in B_{X^*}^{\mathbb{N}}$

By (BA) $\exists (x_{n_k}^*)_{k\in\mathbb{N}}$ $\exists x^* \in X^*$ $x_{n_k}^* \xrightarrow[k\to+\infty]{w^*} x^*$

$$\| T^*(x_{n_k}^*) - T^*(x^*) \|_{X^*} = \sup_{x\in B_X} \langle x_{n_k}^* - x^*, T(x) \rangle \leq \sup_{y\in \overline{T(B_X)}} \langle x_{n_k}^* - x^*, y \rangle$$

$$\langle x_{n_k}^* - x^*, y \rangle \xrightarrow{k\to+\infty} 0 \quad \forall y \in X$$

$$\underset{\text{compactness of } \overline{T(B_X)}}{\Longrightarrow} \sup_{y\in\overline{T(B_X)}} \langle x_{n_k}^* - x^*, y \rangle \xrightarrow[k\to+\infty]{} 0$$

i.e. $T^*(x_{n_k}^*) \longrightarrow T^*(x^*)$

This gives $T^* \in K(X^*)$ by prop 12.6 □

34. Fredholm theory

Thm 13.3

WLOG $\lambda = 1$

Step 1 : $\dim N(I-T) < \infty$

$N(I-T)$ is a Banach space

$\forall x \in N(I-T) \quad \|x\| \leq 1 \quad : x = T(x) \in \overline{T(B_X)}$

Thus $B_{N(I-T)}$ is closed in $\overline{T(B_X)}$ which is compact

Therefore $B_{N(I-T)}$ is compact, and thus $N(I-T)$ is finite dimional

Step 2 : $\overline{R(I-T)} = R(I-T)$

Let $(u_n)_{n\in\mathbb{N}} \in X^{\mathbb{N}}$, $f \in X$: $u_n - Tu_n \xrightarrow{n\to+\infty} f$

Define $d_n := \operatorname{dist}(u_n, N(I-T)) \quad \forall n \in \mathbb{N}$

pick $v_n \in N(I-T) \quad \|u_n - v_n\| \leq d_n + \frac{1}{n} \quad \forall n \in \mathbb{N}^*$

- assume $(\|u_n - v_n\|)_{n\in\mathbb{N}}$ is bounded

  By compactness $\exists (u_{n_j} - v_{n_j})_{j\in\mathbb{N}}$: $(T(u_{n_j} - v_{n_j}))_{n\in\mathbb{N}}$ converges.

  $\exists g \in X$: $T(u_{n_j}) - T(v_{n_j}) \xrightarrow{j\to+\infty} g$

  Then $u_{n_j} - v_{n_j} = u_{n_j} - T(v_{n_j}) = u_{n_j} - T(v_{n_j}) + T(u_{n_j} - v_{n_j}) \xrightarrow{j\to+\infty} f+g$

  $(I-T)(f+g) = \lim\limits_{j\to\infty} (I-T)(u_{n_j} - v_{n_j}) = \lim\limits_{j\to+\infty} (I-T)u_{n_j} = f$

  Thus $f = (I-T)(f+g) \in R(I-T)$

- assume $(\|u_n - v_n\|)_{n\in\mathbb{N}}$ is not bounded

  Let $(u_{n_k} - v_{n_k})_{k\in\mathbb{N}}$: $\begin{cases} \|u_{n_k} - v_{n_k}\| \xrightarrow{k\to+\infty} +\infty \\ \exists y \in X \quad T\left(\dfrac{u_{n_k} - v_{n_k}}{\|u_{n_k} - v_{n_k}\|}\right) \xrightarrow{k\to+\infty} y \end{cases}$

  Define $w_k := \dfrac{u_{n_k} - v_{n_k}}{\|u_{n_k} - v_{n_k}\|}$

  $\|(I-T)w_k\| = \dfrac{\|(I-T)(u_{n_k})\|}{\|u_{n_k} - v_{n_k}\|} \xrightarrow{k\to+\infty} 0$

  $d(w_k, N(I-T)) = d\left(\dfrac{u_{n_k}}{\|u_{n_k} - v_{n_k}\|}, N(I-T)\right) = \dfrac{d_{n_k}}{\|u_{n_k} - v_{n_k}\|} \geq \dfrac{d_{n_k}}{d_{n_k} + \frac{1}{n_k}} \xrightarrow{k\to+\infty} 1$

$(I-T)y = \lim_{k\to\infty} T((I-T)(W_k)) = 0$ so $y \in N(I-T)$

$0 = \text{dist}(y, N(I-T)) = \lim_{k\to+\infty} \text{dist}(T(w_k), N(I-T)) = \lim_{n\to\infty} d(T(w_k) - w_k + w_k, N(I-T))$

$= \lim_{n\to\infty} d(w_k, N(I-T)) \geq 1$

contradiction.

Step 3 (main idea)

Assume (S) $\exists n_0 \in \mathbb{N}$ $X = N((I-T)^{n_0}) \oplus R((I-T)^{n_0})$

then $(I-T)^{n_0} = I - S_{n_0}$ where $S_{n_0} \in K(X)$

Step 1. $\dim(I - S_{n_0}) < \infty$

so $\dim N((I-T)^{n_0}) < \infty$

Thus codim $R((I-T)^{n_0}) < \infty$

$R((I-T)^{n_0}) \subset R(I-T)$ so codimension $R(I-T) < \infty$

Define $K_n := N((I-T)^n)$

$M_n := R((I-T)^n)$ $\forall n \in \mathbb{N}$

Rk: $M_{n+1} \subset M_n$ $K_n \subset K_{n+1}$ $\forall n \in \mathbb{N}$

(S1) $\exists n_0 \in \mathbb{N}$ $\forall n \geq n_0$ $K_n = K_{n_0}$

(S2) $\exists n_0 \in \mathbb{N}$ $\forall n \geq n_0$ $M_n = M_{n_0}$

Step 4 : (S1) + (S2) $\Rightarrow$ (S)

$\exists n_0 \in \mathbb{N}$: $\begin{cases} K_n = K_{n_0} \\ M_n = M_{n_0} \end{cases}$ $\forall n \geq n_0$

let $x \in K_{n_0} \cap M_{n_0}$

$\exists y \in X$: $x = (I-T)^{n_0} y$,

$(I-T)^{n_0} x = 0$

so $(I-T)^{2n_0} y = 0$ and $y \in K_{2n_0} = K_{n_0}$

Therefore $x = 0$ thus $K_{n_0} \cap M_{n_0} = \{0\}$

let $x \in X$. $\exists y \in X$. $(I-T)^{n_0} x = (I-T)^{2n_0} y$

$(I-T)^{n_0} (x - (I-T)^{n_0} y) = 0$

so $x = \underbrace{(I-T)^{n_0} y}_{\in M_{n_0}} + \underbrace{(x - (I-T)^{n_0} y)}_{\in K_{n_0}}$

step 5: prove (S1)

Assume (S1) does not hold

$\forall n \in \mathbb{N}$ $K_n \subsetneq K_{n+1}$

$\forall n \in \mathbb{N}$ $T(K_n) \subset T(K_{n+1})$

let $x_n \in K_{n+1}$ $\exists y_{n+1} \in K_n$: $T(x_n) = T(y_{n+1})$

$x_n = \underbrace{x_n - T(x_n)}_{\in (I-T)(K_{n+1}) \subset K_n} + \underbrace{T(y_{n+1})}_{\in K_n}$ Thus $K_n = K_{n+1}$

Thus $T(K_n) \subsetneq T(K_{n+1})$ $\forall n \in \mathbb{N}$

By Riesz Lemma : $\exists (x_n) \in S_X^{\mathbb{N}}$. $\begin{cases} x_n \in K_{n+1} & \forall n \in \mathbb{N} \\ d(T(x_n), T(K_n)) \geq \frac{1}{2} & \forall n \in \mathbb{N} \end{cases}$

let $n, m \in \mathbb{N}$ $n > m$ $T(x_m) \in T(K_n)$

so $\| T(x_n) - T(x_m) \| \geq \frac{1}{2}$

Thus $(T(x_n))_{n \in \mathbb{N}}$ does not have a convergent subsequence

and hence $T \notin K(X)$ contradiction.

step 6: prove (S2)

Assume (S2) does not hold

$M_{n+1} \subsetneq M_n$ $\forall n \in \mathbb{N}$

By Riesz Lemma : $\exists (x_n) \in S_X^{\mathbb{N}}$: $\begin{cases} x_n \in M_n & \forall n \in \mathbb{N} \\ \operatorname{dist}(x_n, M_{n+1}) \geq \frac{1}{2} & \forall n \in \mathbb{N} \end{cases}$

let $n, m \in \mathbb{N}$ $n > m$

$\| T(x_n) - T(x_m) \| = \| T(x_m) - \underbrace{x_n}_{\in M_n \subset M_{m+1}} + \underbrace{x_n - T(x_n)}_{\in M_{n+1} \subset M_{m+1}} \| \geq \operatorname{dist}(T(x_m), M_{m+1}) = \operatorname{dist}(\overbrace{T(x_m) - x_m + x_m}^{\in M_{m+1}}, M_{m+1})$

$= \operatorname{dist}(x_m, M_{m+1}) \geq \frac{1}{2}$

This contradicts $T \in K(X)$ □

Corollary 13.4

WLOG $\lambda = 1$

WTS (1) $N(I-T) = \{0\} \Rightarrow R(I-T) = X$

(2) $R(I-T) = X \Rightarrow N(I-T) = \{0\}$

proof of (1)

By the proof of Thm 13.3, we have that

$\exists n_0 \in \mathbb{N} \quad X = N((I-T)^{n_0}) \oplus R((I-T)^{n_0})$

let $x \in N((I-T)^{n_0})$ then $(I-T)(I-T)^{n_0-1}x = 0$

so $(I-T)^{n_0-1}x = 0$

Inductively $(I-T)x = 0$ and thus $x = 0$

So we have $X = R((I-T)^{n_0})$

let $x \in X$. $\exists y \in X \quad x = (I-T)^{n_0}y$

$\exists z \in X \quad (I-T)x = (I-T)^{n_0}z$

$(I-T)^{n_0+1}y = (I-T)^{n_0}z$

and thus $(I-T)((I-T)^{n_0}y - (I-T)^{n_0-1}z) = 0$

Therefore $x = (I-T)^{n_0}y = (I-T)^{n_0-1}z \in R((I-T)^{n_0-1})$

Inductively $x \in R(I-T)$

i.e $R(I-T) = X$

proof of (2)

Assume $R(I-T) = X$

let $x^* \in N(I-T^*)$, $x \in X$

$\exists y \in X \therefore x = (I-T)y$

$\langle x, x^* \rangle = \langle (I-T)y, x^* \rangle = \langle y, (I-T^*)x^* \rangle = 0$

Thus $N(I-T^*) = \{0\}$

By (1) in X^*, $R(I-T^*) = X^*$

let $x \in N(I-T)$, $x^* \in X^*$

$\exists y^* \in X^* \quad x^* = (I - T^*) y^*$

$\langle x, x^* \rangle = \langle x, (I-T^*) y^* \rangle = \langle (I-T) x, y \rangle = 0$

so $x = 0$ and $N(I-T) = \{0\}$ □

Corollary 13.5

let $\varepsilon > 0$. Assume $\exists (\lambda_n)_{n\in\mathbb{N}} \in \sigma(T)^{\mathbb{N}} \cap B(0,\varepsilon)^c : \lambda_n = \lambda_m \Leftrightarrow n = m$

$\lambda_n \in \sigma(T) \xRightarrow{\text{Thm 13.4}} N(\lambda_n I - T) \neq \{0\}$

$\exists (x_n)_{n\in\mathbb{N}} \in S_X^{\mathbb{N}} : \lambda_n x_n = T(x_n) \quad \forall n \in \mathbb{N}$

Claim: $(x_n)_{n\in\mathbb{N}}$ are linearly independent. By induction.

Assume that $\{x_j, j=0,\dots n\}$ are linearly independent.

$x_{n+1} = \sum_{j=0}^{n} \alpha_j x_j$ for some $(\alpha_j)_{j=0}^{n} \in \mathbb{R}^{n+1}$

Then $\lambda_{n+1} x_{n+1} = T(x_{n+1}) = \sum_{j=0}^{n} \alpha_j (T x_j) = \sum_{j=0}^{n} \alpha_j \lambda_j x_j$

i.e $\sum_{j=0}^{n} \lambda_{n+1} \alpha_j x_j = \sum_{j=0}^{n} \lambda_j \alpha_j x_j$

By linear independent of $\{x_j ; j=0,\dots,n\}$: $\lambda_{n+1}\alpha_j = \lambda_j \alpha_j \quad \forall j = 0\dots n.$

which is a contradiction. ◇

Define $\cancel{y_n} Y_n := \operatorname{span}\{x_0, \dots, x_n\} \quad \forall n \in \mathbb{N}$

We have that $\cancel{y_n \subsetneq y} \; Y_n \subsetneq Y_{n+1} \quad \forall n \in \mathbb{N}$

By Riesz Lemma: $\exists (y_n)_{n\in\mathbb{N}} \in S_X^{\mathbb{N}} : \begin{cases} y_n \in Y_n & \forall n \in \mathbb{N} \\ d(y_n, Y_{n-1}) \geq \frac{1}{2} & \forall n \in \mathbb{N} \end{cases}$

$\forall n \in \mathbb{N} \quad \cancel{\exists (\alpha_j^{(n)})_{j=0}^{(n)} \in \mathbb{R}} \quad \exists (\alpha_j^{(n)})_{j=0}^{n} \in \mathbb{R}^{n+1} : y_n = \sum_{j=0}^{n} \alpha_j^{(n)} x_j$

$\forall n, m \in \mathbb{N} \quad n > m$

$\| T(y_n) - T(y_m) \| = \| \sum_{j=0}^{n} \alpha_j^{(n)} \lambda_j x_j - \sum_{j=0}^{m} \alpha_j^{(m)} \lambda_j x_j \| = \| \lambda_n y_n - \underbrace{\sum_{j=0}^{n-1} \alpha_j^{(n)} (\lambda_j - \lambda_n) x_j - \sum_{j=0}^{m} \alpha_j^{(m)} \lambda_j x_j}_{\in Y_{n-1}} \|$

$= |\lambda_n| \, \| y_n - z_n \|$ for some $z_n \in Y_{n-1}$

$\geq \frac{\varepsilon}{2}$ This contradicts the compactness of $T \in K(X)$

Corollary 13.6

like the proof in Corollary 13.5

if $\exists (\lambda_n)_{n \in \mathbb{N}} \in (EV(T) \setminus \{0\})^{\mathbb{N}}$

by Riesz Lemma: $\left\| \frac{T(y_n)}{\lambda_n} - \frac{T(y_m)}{\lambda_m} \right\| \geq \frac{1}{2} \quad \forall n, m$

If $\lambda_n \xrightarrow{n \to +\infty} \lambda \neq 0$

Pick $(y_{n_k})_{n \in \mathbb{N}}$, $(\lambda_{n_k})_{n \in \mathbb{N}}$ $\therefore$ $T(y_{n_k})_{n \in \mathbb{N}}$ cv

by compactness then $\left\| \frac{T(y_{n_k})}{\lambda_{n_k}} - \frac{T(y_{n_k})}{\lambda_{n_k}} \right\| \xrightarrow{n \to +\infty} 0$

34. Von Neumann mean ergodic Theorem

Ergodic theory

$T \in B(X)$ $\quad \|T\|_{B(X)} \leq 1$ $\quad$ Q: $(T^n)_{n\in\mathbb{N}}$ converges ?

<u>Motivate:</u>

(1) $X = C(\Omega)$ or $L^p(\Omega)$

$\forall f \in X$ $\quad Tf(w) = f(\varphi(w))$ for some $\varphi: \Omega \to \Omega$ continous or measurable

<u>Typical result:</u>

$$\frac{1}{n}\sum_{j=1}^{n} T^j f(w') \xrightarrow{n\to+\infty} \frac{1}{\mu'(\Omega)} \int_\Omega f(w)\, d\mu(w) \qquad w' \in \Omega$$

ergordic hypothsis: $T^j f(w) = f(\varphi^{(j)}(w))$

$$\varphi^{(j)}(w) = \varphi^{(j-1)}(\varphi(w)) \quad \forall w \in \Omega$$

(2) $T = I - S$ $\quad S \in K(X)$

" $\frac{1}{n}\sum_{j=1}^{n} T^j \xrightarrow[n\to+\infty]{} P$ " $\qquad TP(x) = P(x) \quad \forall x \in X$

$P = \text{proj}$ onto $N(I-T) = N(S)$

Lemma 14.2

$\Rightarrow$ $\forall h \in N(I-T)$ $\quad \langle h, T(h)\rangle = \langle h, h\rangle = \|h\|^2$

$\Leftarrow$ let $h \in H$ $\quad$ Assume $\|h\|^2 = \langle h, T(h)\rangle$

$\|h - T(h)\|^2 = \|T(h)\|^2 + \|h\|^2 - 2\langle h, T(h)\rangle = \|T(h)\|^2 - \|h\|^2 \leq (\|T\|^2 - 1)\|h\|^2 \leq 0$

Thus $h = T(h)$ $\square$

Lemma 14.3

WTS $N(I-T)^\perp = \overline{R(I-T)}$ $\quad$ (Since $H = N(I-T) \oplus N(I-T)^\perp$

$\boxed{\supset}$ let $f \in N(I-T)$, $g \in H$ $\quad h = (I-T)g$

$\langle f, h\rangle = \langle f, (I-T)g\rangle = \langle (I-T^*)f, g\rangle \overset{\text{Lemma 14.2}}{=} 0$

Thus $h \in N(I-T)^\perp$

So $\overline{R(I-T)} \subset N(I-T)^\perp$

$\boxed{\subset}$ $f \in \overline{R(I-T)}^\perp$ $\quad g \in H$

$0 = \langle f, (I-T)g\rangle \Rightarrow \langle (I-T^*)f, g\rangle = 0$

$\overline{R(I-T)}^\perp \subset N(I-T^*) \overset{\text{Thm 14.2}}{=\!=} N(I-T)$

Thus $N(I-T)^\perp \subset \overline{R(I-T)}$ $\square$

Thm 14.1

$H = N(I-T) \oplus \overline{R(I-T)}$

- $\forall h \in N(I-T)$

  $\forall n \in \mathbb{N}^*$ $\quad \frac{1}{n} \sum_{j=1}^{n} T^j(h) = h = P(h)$

- $\forall g \in R(I-T)$

  let $h \in H$ $\quad g = (I-T)h$

  $\forall n \in \mathbb{N}^*$ $\quad \| \frac{1}{n} \sum_{j=1}^{n} T^j(g) \| = \| \frac{1}{n} \sum_{j=1}^{n} (T^j - T^{j+1})(h) \| = \frac{1}{n} \| h - T^{n+1}(h) \| \leq \frac{2\|h\|}{n} \xrightarrow{n \to +\infty} 0 = P(g)$

- let $h \in \overline{R(I-T)}$

  $\varepsilon > 0$ $\quad g \in R(I-T) : \quad \|g-h\| < \varepsilon$

  $\forall n \in \mathbb{N}^*$

  $\| \frac{1}{n} \sum_{j=1}^{n} T^j(h) \| \leq \| \frac{1}{n} \sum_{j=1}^{n} T^j(h-g) \| + \| \frac{1}{n} \sum_{j=1}^{n} T^j(g) \| \leq \frac{2}{n} \|g\| + \frac{1}{n} \sum_{j=1}^{n} \| T^j(h-g) \| \leq \frac{2}{n} \|g\| + \varepsilon \xrightarrow{n \to +\infty} \varepsilon$

  so $\frac{1}{n} \sum_{j=1}^{n} T^j(h) \xrightarrow{n \to +\infty} 0 = P(h)$ □

36. Von Neumann–Yosida mean ergodic theorem

Thm 14.4

(1) $\Rightarrow$ (2)

Let $y \in N(I-T) \cap \overline{\mathrm{conv}}\{T^n(x) \,;\, n \in \mathbb{N}\}$

let $\varepsilon > 0$ Pick $z \in \mathrm{conv}\{T^n(x) \,;\, n \in \mathbb{N}\} : \|z-y\| < \varepsilon$

$\exists N \in \mathbb{N}$ $\exists (C_k)_{k=1}^N \in [0,1]^N :$ $\begin{cases} \sum_{k=1}^N C_k = 1 \\ \sum_{k=1}^N C_k T^k(x) = z \end{cases}$

Let $n \in \mathbb{N}^*$

$$\|y - \frac{1}{n}\sum_{j=1}^n T^j(x)\| = \|\frac{1}{n}\sum_{j=1}^n (T^j(y) - T^j(x))\| \le \|\frac{1}{n}\sum_{j=1}^n T^j(y-z)\| + \|\frac{1}{n}\sum_{j=1}^n (T^j(z) - T^j(x))\|$$

$$\le \|y-z\| + \|\sum_{k=1}^N C_k \frac{1}{n}\sum_{j=1}^n (T^j(T^k(x)) - T^j(x))$$

$$\le \varepsilon + \sum_{k=1}^N \frac{C_k}{n} \|\sum_{j=k+1}^{n+k} T^j(x) - \sum_{j=1}^k T^j(x)\|$$

$$\le \varepsilon + \sum_{k=1}^N \frac{2kC_k}{n}\|x\| \le \varepsilon + \frac{2N}{n}\|x\| \sum_{k=1}^N C_k = \varepsilon + \frac{2N}{n}\|x\|$$

$$\xrightarrow{n \to +\infty} \varepsilon$$

(4) $\Rightarrow$ (1)

$$y = w - \lim_{k \to +\infty} \frac{1}{n_k}\sum_{j=1}^{n_k} T^j(x)$$

so $y \in \overline{\mathrm{conv}}^w\{T^n(x) \,;\, n \in \mathbb{N}\}$

By Mazur's Thm $y \in \overline{\mathrm{conv}}\{T^n(x) \,;\, n \in \mathbb{N}\}$

let $\varepsilon > 0$
let $x^* \in X^*$. Pick $K \in \mathbb{N}$: $\forall k \ge K$ $\begin{cases} \langle y - \frac{1}{n_k}\sum_{j=1}^{n_k} T^j(x), x^* \rangle < \varepsilon \\ \langle y - \frac{1}{n_k}\sum_{j=1}^{n_k} T^j(x), T^*(x^*) \rangle < \varepsilon \end{cases}$

$\forall k \ge K$

$$|\langle y - T(y), x^* \rangle| \le |\langle y - \frac{1}{n_k}\sum_{j=1}^{n_k} T^j(x), x^* \rangle| + |\langle \frac{1}{n_k}\sum_{j=1}^{n_k} T^j(x) - \frac{1}{n_k}\sum_{j=1}^{n_k} T^{j+1}(x), x^* \rangle| + |\langle \frac{1}{n_k}\sum_{j=1}^{n_k} T^{j+1}(x) - T(y), x^* \rangle|$$

$$\le 2\varepsilon + |\langle \frac{1}{n_k}T(x) - \frac{1}{n_k}T^{n_k+1}(x), x^* \rangle| \le 2\varepsilon + \frac{2\|x^*\|\|x\|}{n_k} \xrightarrow{k \to \infty} 2\varepsilon$$

$\langle y - T(y), x^* \rangle = 0$ $\forall x^* \in X$ so $y \in N(I-T)$ $\square$

Thm 14.5

let $x \in X$, $\| \frac{1}{n} \sum_{j=1}^{n} T^j(x) \| \leq \|x\| \quad \forall n \in \mathbb{N}^*$

By Reflexivity + (BA)

we can extract a weakly convergent subseq from this bounded seq

Therefore (4) from Thm 14.4 holds

Thus (1) and (2) hold so $y = \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} T^j(x)$ exists and $y \in N(I-T)$

let $P(x) = \lim_{n\to+\infty} \frac{1}{n} \sum_{j=1}^{n} T^j(x) \quad \forall x \in X$

Rk: $P(x) \in N(I-T) \quad \forall x \in X$ (since $y \in N(I-T)$)

$R(P) = N(I-T)$ (since $\forall x \in N(I-T) \quad x = \frac{1}{n} \sum_{j=1}^{n} T^j(x) \quad \forall n \in \mathbb{N}$)

$P^2 = P$ since $\frac{1}{n} \sum_{j=1}^{n} T^j(P(x)) = P(x) \quad \forall n \in \mathbb{N} \quad \forall x \in X$

$R(I-P) \cap R(P) = \{0\}$ (since $P(x) = (I-P)y \Rightarrow P(x) = (P-P^2)(y) = 0$)

$\forall x \in X \quad Px + (I-P)x = x$ so $X = R(P) \oplus \overline{R(I-P)}$

Thus we have $X = N(I-T) \oplus \overline{R(I-P)}$

WTS $\overline{R(I-P)} = \overline{R(I-T)}$

$\boxed{\supset}$ let $x \in X$

$\| P(I-T)x) = \lim_{n\to+\infty} \| \frac{1}{n} \sum_{j=1}^{n} (T^j - T^{j+1})(x) \| = \lim_{n\to+\infty} \frac{1}{n} \| T(x) - T^{n+1}(x) \| \leq \lim_{n\to\infty} \frac{2\|x\|}{n} = 0$

so $(I-T)x = (I-P)(I-T)x + \underbrace{P(I-T)(x)}_{0}$

$\overline{R(I-T)} \subset \overline{R(I-P)}$

$\boxed{\subset}$ Define $N_* = N(I-T^*) \subset X^*$

WTS $\overline{R(I-P)} \subset N_*^{\perp} \subset \overline{R(I-P)}$

let $x \in X$, $x^* \in N_*$

$\langle (I-P)(x), x^* \rangle = \langle x, x^* \rangle - \lim_{n\to+\infty} \langle (\frac{1}{n} \sum_{j=1}^{n} T^j)(x), x^* \rangle = \langle x, x^* \rangle - \lim_{n\to+\infty} \langle x, \frac{1}{n} \sum_{j=1}^{n} (T^*)^j (x^*) \rangle$

$= \langle x, x^* \rangle - \langle x, x^* \rangle = 0$

Thus $\overline{R(I-P)} \subset N_*^{\perp}$

Assume $N_*^\perp \not\subset \overline{R(I-T)}$

$\exists \; x \in N_*^\perp \subset X = X^{**}$

By (HB) $\quad \exists \, y^* \in X^* : \begin{cases} \langle y, y^* \rangle = 0 & \forall y \in \overline{R(I-T)} \\ \langle y^*, x \rangle \neq 0 \end{cases}$

$\forall z \in X \quad 0 = \langle (I-T)z, y^* \rangle = \langle z, (I-T^*) y^* \rangle$

so $\quad y^* \in N_*$ and thus $\langle y^*, x \rangle = 0$ contradiction!

Thus $\quad X = N(I-T) \oplus \overline{R(I-T)}$

and $\quad P(x) = \lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} T^j(x) \quad \forall x \in X$ defines the projection onto $N(I-T)$ $\square$

37. Birkhoff pointwise ergodic theorem

Lemma 14.7

Let $\lambda > 0$, $f \in L^p$

Define $M_k^\lambda := \sum_{j=1}^{k} T^j |f| - k\lambda \quad \forall k \in \mathbb{N}^*$

Rk: $\exists c > 0 \quad \forall N \in \mathbb{N}^* \quad \forall f \in L^p \quad \forall \lambda > 0$

$$\mu(\{\omega \in \Omega ; \max_{k \leq N} M_k^\lambda \geq 0\}) \leq \frac{c}{\lambda^p} \|f\|_p^p$$

implies the result (by Fatou's lemma)

$$M_k^\lambda = T|f| - \lambda 1_\Omega + \sum_{j=2}^{k} T^j |f| - (k-1)\lambda 1_\Omega$$

since $\|T(1_\Omega)\|_\infty \leq 1$

$$= T|f| - \lambda + T\Big(\sum_{j=1}^{k-1} T^j |f| - (k-1)\lambda 1_\Omega\Big) + (k-1)\lambda \underbrace{(T(1_\Omega) - 1_\Omega)}_{\leq 0}$$

$$\leq T|f| - \lambda + T(M_{k-1}^\lambda)$$

$$M_k^\lambda \leq T|f| - \lambda + T(M_{k-1}^\lambda)$$

$$\max_{k \leq N} M_k^\lambda \leq T|f| - \lambda + T(\max_{k \leq N} M_{k-1}^\lambda)$$

$$\leq T|f| - \lambda + T(1_{\max_{k \leq N} M_k^\lambda \geq 0} \cdot \max M_k^\lambda)$$

$$\int_{\max_{k \leq N} M_k^\lambda \geq 0} \max M_k^\lambda \leq \int_{\max M_k^\lambda \geq 0} (T|f| - \lambda) + \int_\Omega T(1_{\max M_k^\lambda \geq 0} \cdot \max M_k^\lambda)$$

$$\leq \int_{\max M_k^\lambda \geq 0} (T|f| - \lambda) + \int_{\max M_k^\lambda > 0} \max M_k^\lambda$$

$$\int_{\max_{k \leq N} M_k^\lambda \geq 0} T|f| \geq \lambda \mu(\max M_k^\lambda \geq 0)$$

$$\lambda \mu(\max_{k \leq N} M_k^\lambda \geq 0) \overset{\text{Hölder}}{\leq} \|T(f)\|_p \, \mu(\max M_k^\lambda > 0)^{\frac{1}{p'}}$$

$$\lambda^p \mu(\max_{k \leq N} M_k^\lambda \geq 0) \leq \|T(f)\|_p^p \leq \|f\|_p^p \quad \square$$

Thm 14.6

By Von Neuman-Yosiada : $X = N(I-T) \oplus \overline{R(I-T)}$

let $f \in N(I-T)$: $\forall n \in \mathbb{N}$ $\quad \frac{1}{n}\sum_{j=1}^{n} T^{(j)} f(w) = f(w)$ $\quad$ a.e $w \in \Omega$

let $g \in L^\infty$ $\quad \forall n \in \mathbb{N}$ $\quad \forall w \in \Omega$

$$\left|\frac{1}{n}\sum_{j=1}^{n} T^{j}(I-T)g(w)\right| = \frac{1}{n}\left|(T^{n+1} - T)g(w)\right| \leq \frac{1}{n}\|(T^{n+1}-T)g\|_\infty \leq \frac{2}{n}\|g\|_\infty \xrightarrow[n\to+\infty]{} 0$$

let $f \in \overline{R(I-T)}$. since $\overline{L^p \cap L^\infty}^{L^p} = L^p$, we have that

$\forall \varepsilon > 0 \quad \exists g \in L^\infty \quad \|f - (I-T)g\|_p < \varepsilon$

$h = (I-T)g \quad n, m \in \mathbb{N} \quad \lambda > 0$

$$\left|\frac{1}{n}\sum_{j=1}^{n} T^{j}(f) - \frac{1}{m}\sum_{j=1}^{m} T^{j}(f)\right| \leq 2\sup_{n\in\mathbb{N}}\left|\frac{1}{n}\sum_{j=1}^{n} T^{j}(f-h)\right| + \left|\frac{1}{n}\sum_{j=1}^{n} T^{j}(h) - \frac{1}{m}\sum_{j=1}^{m} T^{j}(h)\right|$$

$$\mu\left(\limsup_{n,m\to\infty}\left|\frac{1}{n}\sum_{j=1}^{n} T^{j}(f) - \frac{1}{m}\sum_{j=1}^{m} T^{j}(f)\right| > \lambda\right) \leq \mu\left(\sup_{n\in\mathbb{N}} \frac{1}{n}\sum_{j=1}^{n} T^{j}|f-h| > \frac{\lambda}{2}\right)$$

$$\underset{14.7}{\leq} \left(\frac{2}{\lambda}\right)^p \|f-h\|_p^p = \left(\frac{2}{\lambda}\right)^p \varepsilon^p \quad \forall \varepsilon > 0$$

so $\mu\left(\lim_{n,m\to\infty}\left|\frac{1}{n}\sum_{j=1}^{n} T^{j}(f) - \frac{1}{m}\sum_{j=1}^{n} T^{j}(f)\right| > \lambda\right) = 0$

Thus $\frac{1}{n}\sum_{j=1}^{n} T^{j}(f)$ cv. a.e $\square$

38. Blum-Hanson weak mixing theorem

Thm 14.8

(2) $\Rightarrow$ (1)

Assume $\neg(1)$ i.e $T^n h \xrightarrow[n\to+\infty]{w} 0$ (not)

$\exists \varepsilon > 0$ $\exists g \in S_H$ $\exists (n_j)_{j\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$ increasing

$\langle T^{n_j}(h), g\rangle \geq \varepsilon$ $\forall j \in \mathbb{N}$

$\forall N \in \mathbb{N}^*$ $\|\frac{1}{N}\sum_{j=1}^{N} T^{n_j}(h)\| \geq \langle \frac{1}{N}\sum_{j=1}^{N} T^{n_j}(h), g\rangle \geq \frac{1}{N}\sum_{j=1}^{N}\varepsilon = \varepsilon$

(1) $\Rightarrow$ (2)

$\alpha := \lim_{n\to+\infty} \|T^n(h)\| = \inf_{n\to+\infty} \|T^n(h)\|$

If $\alpha = 0$ and $(n_j)_{j\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$ increases, then

$\forall \varepsilon > 0$ $\exists J \in \mathbb{N}$ $\forall j \in J$ $\|T^{n_j}(h)\| < \varepsilon$

$\forall \varepsilon > 0$ $\exists J$ $\forall N \geq J$ $\|\frac{1}{N}\sum_{j=1}^{N} T^{n_j}h\| \leq \|\frac{1}{N}\sum_{j=1}^{J} T^{n_j}h\| + \|\frac{1}{N}\sum_{j=J+1}^{N} T^{n_j}h\| \leq \frac{J}{N}\|h\| + \varepsilon$

$\forall \varepsilon > 0$ $\lim_{n\to\infty} \|\frac{1}{N}\sum_{j=1}^{N} T^{n_j}h\| < \varepsilon$ Thus $\frac{1}{N}\sum_{n_j=1}^{N} T^{n_j}h \to 0$

If $\alpha \neq 0$

then WLOG $\alpha = 1$ (if not replace h by $\frac{h}{\alpha}$)

we have $\|T^n h\| \xrightarrow{n\to+\infty} 1$ and $T^n h \xrightarrow[n\to+\infty]{w} 0$

let $\varepsilon > 0$ Pick $M \in \mathbb{N}$ $\forall m \geq M$ $\|T^m(h)\|^2 < (1+\varepsilon)$ (1)

Pick $K \in \mathbb{N}$ $\forall k \geq K$ $|\langle T^{M+k}(h), T^M(h)\rangle| < \varepsilon$ (2)

let $n, m \in \mathbb{N}$ $m \geq M$ $n \geq m+K$

$2\langle T^n(h), T^m(h)\rangle = \|T^n(h) + T^m(h)\|^2 - \|T^n(h)\|^2 - \|T^m(h)\|^2 \leq \|T^n(h) + T^m(h)\|^2 - 2$ (since $\inf_{n\in\mathbb{N}} \|T^n(h)\| = 1$)

$= \|T^{m-M}(T^M(h) + T^{n-m+M}(h))\|^2 - 2 \leq \|T^M(h) + T^{n-m+M}(h)\|^2 - 2$

$= \|T^M(h)\|^2 + \|T^{(n-m)+M}(h)\|^2 + 2\langle T^M(h), T^{n-m+M}(h)\rangle - 2 \overset{(1)+(2)}{\leq} 2(1+\varepsilon) - 2 + 2\varepsilon = 4\varepsilon$

Let $(n_j)_{j\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$ be increasing. Let $N \in \mathbb{N}$

$$\left\| \frac{1}{N} \sum_{j=1}^{N} T^{n_j}(h) \right\|^2 = \frac{1}{N^2} \sum_{j,k=1}^{N} \langle T^{n_j}(h), T^{n_k}(h) \rangle$$

$$= \frac{1}{N^2} \sum_{j=1}^{N} \| T^{n_j}(h) \|^2 + \frac{2}{N^2} \sum_{j=1}^{N-1} \sum_{k=j+1}^{N} \langle T^{n_j}(h), T^{n_k}(h) \rangle$$

$$\leq \frac{1}{N} \|h\|^2 + \frac{2}{N^2} \sum_{j=1}^{N-1} \sum_{k=j+1}^{N} \langle T^{n_j}(h), T^{n_k}(h) \rangle$$

- $\frac{1}{N^2} \sum_{j=1}^{N-1} \sum_{k=j+1}^{N} 1_{[0,M]}(n_j) \left| \langle T^{n_j}(h), T^{n_k}(h) \rangle \right| \leq \frac{M}{N^2} \cdot N \|h\|^2 \xrightarrow{N\to+\infty} 0$

写重了....

- $\frac{1}{N^2} \sum_{j=1}^{N-1} \sum_{k=j+1}^{N} 1_{[0,M]}(n_j) \left| \langle T^{n_j}(h), T^{n_k}(h) \rangle \right| \leq \frac{M}{N^2} \|h\|^2 \xrightarrow{N\to+\infty} 0$

- $\frac{1}{N^2} \sum_{j=1}^{N-1} \sum_{k=j+1}^{N} 1_{(M,+\infty)}(n_j)\, 1_{[n_j+K,+\infty)}(n_k) \left| \langle T^{n_j}(h), T^{n_k}(h) \rangle \right| \leq \frac{2\varepsilon}{N^2} N^2 = 2\varepsilon \quad \forall N \in \mathbb{N}$

- $\frac{1}{N^2} \sum_{j=1}^{N-1} \sum_{k=j+1}^{N} 1_{(M,+\infty)}(n_j)\, 1_{[n_j+1,n_j+K)}(n_k) \left| \langle T^{n_j}(h), T^{n_k}(h) \rangle \right| \leq \frac{1}{N^2} \cdot N \cdot K \|h\|^2 = \frac{K\|h\|^2}{N} \xrightarrow{N\to+\infty} 0$

We have shown that

$$\lim_{N\to+\infty} \left\| \frac{1}{N} \sum_{j=1}^{N} T^{n_j}(h) \right\| < 2\varepsilon \quad \forall \varepsilon > 0$$

Thus $\frac{1}{N} \sum_{j=1}^{N} T^{n_j}(h) \xrightarrow{N\to+\infty} 0$ $\square$

39. Mueller-Tomilov weak mixing theorem

Thm 14.9

(2) $\Rightarrow$ (1) as in Blum-Hanson

(1) $\Rightarrow$ (2)

$$\alpha := \lim_{n\to+\infty} \|T^n(x)\| = \inf_{n\to+\infty} \|T^n(x)\|$$

WLOG $\alpha = 1$

Let $\varepsilon>0$. Pick $M\in\mathbb{N}$. $\forall m\geq M$. $\|T^m(x)\|<1+\varepsilon$ (1)

Denote the canonical basis of ℓ_p by $(e_n)_{n\in\mathbb{N}}$, and $(P_N)_{N\in\mathbb{N}}$ the converging partial sum projection, i.e $\forall N\in\mathbb{N}$, $P_N((y_j)_{j\in\mathbb{N}}) = (z_j^N)_{j\in\mathbb{N}}$ with $\begin{cases} z_j = y_j & \text{if } j\leq N \\ z_j = 0 & \text{if } j>N \end{cases}$

Pick $N\in\mathbb{N}$: $\|(I-P_N)T^M(x)\|<\varepsilon$ (2) (since $T(x)$ is in ℓ^p)

Pick $K\in\mathbb{N}$ $\forall k\geq K$ $\|P_N(T^{M+k}(x))\|_{\ell_p}^p = \sum_{l=0}^{N} |\langle T^{M+k}(x), e_l^*\rangle|^p<\varepsilon$ (3)

(since $\langle T^{M+k}(x), e_l^*\rangle \xrightarrow{k\to+\infty} 0$ $\forall l=0\dots N$ by assumption)

$$\boxed{\begin{array}{l} \text{Let } \delta>0 \quad \text{Pick } t\in\mathbb{N} \quad t^{-\frac{1}{p}}\leq\delta \\ \text{Chose } \varepsilon\in(0,1): \quad \forall s\in\{0,\dots t\} \\ ((1+\varepsilon)^p + 2^p s)^{\frac{1}{p}} + (s+1)\varepsilon \leq 2(s+1)^{\frac{1}{p}} \end{array}} \quad (\text{Box})$$

Claim: $\forall s\in\{1,\dots,t-1\}$

(P_s) $\forall (m_j)\nearrow_{j\in\mathbb{N}}$ $(m_1\geq M$ and $m_{j+1}-m_j\geq k$ $\forall j=1,\dots,s) \Rightarrow \|\sum_{j=1}^{s} T^{m_j}(x)\|\leq 2s^{\frac{1}{p}}$

Proof of the claim by induction

(P_1) $\|T^{m_1}(x)\|<1+\varepsilon<2<2s^{\frac{1}{p}}$ since $m_1\geq M$ and $s=1$

Assume (P_s). Let $(m_j)_{j\in\mathbb{N}}\nearrow$: $m_1\geq M$, $m_{j+1}-m_j\geq k$ $\forall j=1\dots s+1$

$$\|\sum_{j=1}^{s+1} T^{m_j}(x)\| = \|T^{-M+m_1}\sum_{j=1}^{s+1} T^{m_j-m_1+M}(x)\| \leq \|T^M(x) + \sum_{j=2}^{s+1} T^{(m_j-m_1)+M}(x)\|$$

by triangle ineq

$$\leq \|P_N T^M(x) + (I-P_N)\sum_{j=2}^{s+1} T^{(m_j-m_1)+M}(x)\| + \|(I-P_N)T^M(x)\| + \|P_N(\sum_{j=2}^{s+1} T^{(m_j-m_1)+M}(x))\|$$

$$\overset{(2)+(3)}{\leq} \|P_N T^M(x) + (I-P_N)\sum_{j=2}^{s+1} T^{(m_j-m_1)+M}(x)\| + \varepsilon + s\varepsilon$$

$$\overset{\text{disjoint support in } \ell^p}{=} (\|P_N T^M(x)\|^p + \|(I-P_N)(\sum_{j=2}^{s+1} T^{m_j-m_1+M}(x))\|^p)^{\frac{1}{p}} + (s+1)\varepsilon \overset{(1)}{\leq} ((1+\varepsilon)^p + (2s^{\frac{1}{p}})^p)^{\frac{1}{p}} + (s+1)\varepsilon$$

$$\overset{(\text{Box})}{\leq} 2(s+1)^{\frac{1}{p}} \quad \diamond$$

Let $(n_j)_{j\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$ be increasing

Given $\delta > 0$, picking ε, M, N, t, k as above

Pick $N \in \mathbb{N}$. $N' > M + kt$

Let us write $N' = M + mt + R$ for some $m \geq k$ and $R \in \{0, \dots, t-1\}$

Notice that $\forall j \in \{M+R, \dots, N'\}$ $\exists l \in \{0, \dots, m\}$ $\exists l' \in \{0, \dots t-1\}$

$\delta = M + R + l'm + l$

$$\left\| \frac{1}{N'} \sum_{j=1}^{N} T^{n_j}(x) \right\| \leq \frac{M+R}{N'} \| T^{n_j}(x) \| + \frac{1}{N'} \sum_{l=0}^{M} \left\| \sum_{l'=0}^{t-1} T^{n_{M+R+l'm+l}}(x) \right\|$$

$$\leq \frac{M+R}{N'} \|x\| + \frac{(m+1)}{N'} 2t^{\frac{1}{p}}$$

$$\leq \frac{M+R}{N'} \|x\| + 4t^{\frac{1}{p}-1} \leq \frac{M+R}{N'} \|x\| + 4\delta \xrightarrow{N' \to +\infty} 4\delta \quad \square$$