CM1020: Discrete Mathematics Summary

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1 Sets

Learning Outcomes

- ✓ Understand sets and powersets
- ✓ Learn the listing methods and rules of inclusion methods
- ✓ Learn to manipulate set operations
- ✓ Represent a set using Venn diagrams
- ✓ Understand and apply De Morgan's law
- ✓ Understand, and apply commutative, associative and distributive laws

1.1 Definition of a Set

A set is an *unordered collection* of *distinct objects*, called *elements*. We write $a \in A$ to denote that a is an element of the set A, and $a \notin A$ to denote that a is **not** an element of A.

We can describe a set using **roster method** as $A = \{a, b, c, d.\}$ or **set builder notation**:

$$A = \{x \in \mathbb{Z} \mid x \text{ is odd}\}$$

Two sets are considered equal if they have the same elements.

$$\forall x (x \in A \leftrightarrow x \in B)$$

An empty set is denoted as $\{\}$ or $\emptyset$. Do not confuse $\emptyset$ with $\{\emptyset\}$ (the set containing the empty set).

1.2 Subsets

A is a subset of B if and only if all elements of A are in B. B is a superset of A.

$$A \subseteq B$$
$$B \supseteq A$$

Further, we can show **proper subsets**, which imply that A $\neq$ B

$$A \subset B$$
$$B \supset A$$

Show that $A \subseteq B$ Show that if x belongs to A, then x belongs to B.

Show that $A \not\subseteq B$ Find a single $x \in A$ such that $x \notin B$

Show that $A = B$ Show that $A \subseteq B$ and $B \subseteq A$

Every nonempty set S has at least two subsets: the empty set $\emptyset$ and the set S itself.

1.3 Cardinality

The *cardinality* of a *finite* set S is denoted as $|S|$ and describes the number of distinct elements in a set. Because the empty set has no elements, it follows that $|\emptyset| = 0$.

1.4 Power Sets

The power set $\mathcal{P}(S)$ of S is the set of all subsets of the set S. For nonempty sets, the empty set and the set itself are members of the power set.

The empty set has exactly one subset, itself.

$$\mathcal{P}(\emptyset) = \{\emptyset\}$$

The set $\{\emptyset\}$ has two subsets.

$$\mathcal{P}(\{\emptyset\}) = \{\{\emptyset\}, \emptyset\}$$

If two sets have the same powerset, the sets themselves are equivalent. That is:

$$\mathcal{P}(A) = \mathcal{P}(B) \leftrightarrow A = B$$

The cardinality of a powerset is given as

$$|\mathcal{P}(S)| = 2^n \text{where n is the cardinality of } S$$

1.5 Set Operations & Identities

The Universal Set is denoted as U and contains everything.

1.5.1 Union

The union of A and B is denoted as $A \cup B$ and contains all elements that are in A or B, or both. This is shown in Figure 1.

$$A \cup B = \{x \mid x \in A \lor x \in B\}$$

The union of sets is **commutative**, **associative** and **distributive** over the interesection.

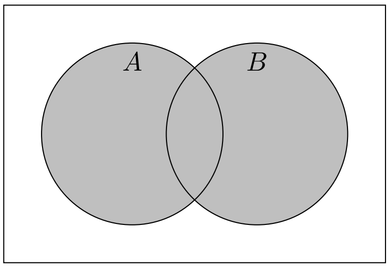

Figure 1. *Union of two sets*

$$A \cup B = B \cap A \quad \text{(Commutative Property)}$$
$$(A \cup B) \cup C = A \cup (B \cup C) \quad \text{(Associative Property)}$$
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \quad \text{(Distributive Property)}$$

1.5.2 Intersection

The intersection of A and B is denoted as $A \cap B$ and contains all elements that are in A and B. This is shown in Figure 2.

$$A \cap B = \{x \mid x \in A \land x \in B\}$$

The intersection of sets is **commutative**, **associative** and **distributive** over the union.

$$A \cap B = B \cap A \quad \text{(Commutative Property)}$$
$$(A \cap B) \cap C = A \cap (B \cap C) \quad \text{(Associative Property)}$$
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \quad \text{(Distributive Property)}$$

Two sets are called *disjoint* if $A \cap B = \emptyset$.

Inclusion-Exclusion Principle The cardinality of the union of two sets is given by adding the cardinalities of the two sets, and subtracting the cardinality of the intersection (as to not dobule count these elements). This is calles the principle of **inclusion-exclusion**

$$|A \cup B| = |A| + |B| - |A \cap B| \qquad \text{(Principle of inclusion-exclusion)}$$

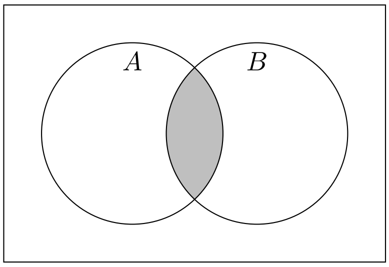

Figure 2. *Intersection of two sets*

1.5.3 Set Difference

The difference of A and B is denoted as $A - B$ and contains all elements that are in A but not B. This is shown in Figure 3.

$$A - B = \{x \mid x \in A \land x \notin B\}$$

The set difference of sets is **not commutative** and **not associative**.

$$A - B \neq B \cap B - A$$
$$(A - B) \cap C \neq B - A$$

An important identify for the set difference is

$$A - B = A \cap \overline{B}$$

Important: The difference of two sets is not treated like an arithmetic subtraction. It only means taking all elements of a set, and removing any elements that overlap with the second set.

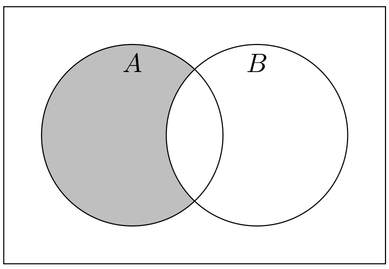

Figure 3. *Set difference of two sets*

1.5.4 Complement

$U$is the universal set comprising everything. The complement of A with respect to U is denoted as $\bar{A}$ and contains all elements that are in U but not A. This is equivalent to $U - A$ and is shown in Figure 4.

$$\bar{A} = \{x \mid x \in U \land x \notin A\}$$

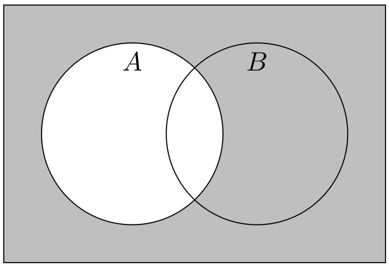

Figure 4. *Complement of a set*

1.5.5 Symmetric Difference

The symmetric difference of A and B is denoted as $A \oplus B$ and contains all elements that are in A or in B, but **not** in both. This is shown in Figure 5.

$$A \oplus B = \{x \mid (x \in A \lor x \notin B) \text{ and } x \notin A \cap B\}$$

In other words, $A \oplus B = A \cup B - A \cap B$.

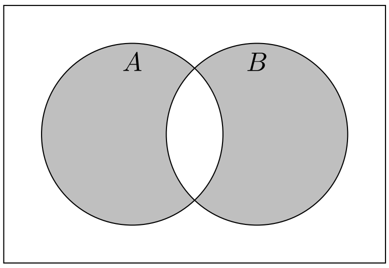

Figure 5. *Symmetric difference of two sets*

The symmetric difference of sets is **commutative** and **associative**. The intersection is **distributive** over the symmetric difference.

$$A \oplus B = B \oplus A \qquad \text{(Commutative Property)}$$

$$(A \oplus B) \oplus C = A \oplus (B \oplus C) \qquad \text{(Associative Property)}$$

$$A \cap (B \oplus C) = (A \cap C) \oplus (B \cap A) \qquad \text{(Distributive Property)}$$

1.5.6 Cartesian Products

The *ordered n-tuple* $(a_1, a_2, \ldots, a_n)$ is the ordered collection that has a_1 as its first element, a_2 as its second element, ..., and a_n as it nth element.

Let A and B be sets. The *Cartesian Product* of A and B, denoted by $A \times B$, is the set of all ordered pairs (a, b) where $a \in A$ and $b \in B$. Hence,

$$A \times B = (a, b) \mid a \in A \wedge b \in B$$

1.5.7 De Morgan's Laws

De Morgan's laws describe how mathematical statements and concepts are related through their opposites.

First Law The complement of the union of two sets is equal to the interesction of their complements.

$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$

Second Law The complement of the intersection of two sets is equal to the union of their complements.

$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$

1.6 Summary of Identity Laws

A compiled overview of identity laws of sets are shown in Table 1.

Name	Identity
Identity Laws	$A \cap U = A$ $A \cup \emptyset = A$
Domination Laws	$A \cup U = U$ $A \cap \emptyset = \emptyset$
Idempotent Laws	$A \cup A = A$ $A \cap A = A$
Complementation Law	$\overline{\overline{A}} = A$
Commutative Laws	$A \cup B = B \cup A$ $A \cap B = B \cap A$
Associative Laws	$(A \cup B) \cup C = A \cup (B \cup C)$ $(A \cap B) \cap C = A \cap (B \cap C)$
Distributive Laws	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $AA \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ $A \cap (B \oplus C) = (A \cap C) \oplus (B \cap A)$
De Morgan's Laws	$\overline{A \cup B} = \overline{A} \cap \overline{B}$ $\overline{A \cap B} = \overline{A} \cup \overline{B}$
Absorption Laws	$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$
Complement Laws	$A \cup \overline{A} = U$ $A \cap \overline{A} = \emptyset$

Table 1. *Summary of Identities in Set Theory*

1.7 Partitions

A **partition** of a set S is a a set of *disjointed, non-empty* subsets of S that have S as their union. This is shown in Figure 6.

In other words, the set of subsets A_i is a partition of S if and only if

$$A_i \neq \emptyset$$
$$A_i \cap A_j = \emptyset \text{when} i \neq j$$
$$A_1 \cup A_2 \cup A_3 \ldots \cup A_i = S$$

2 Functions

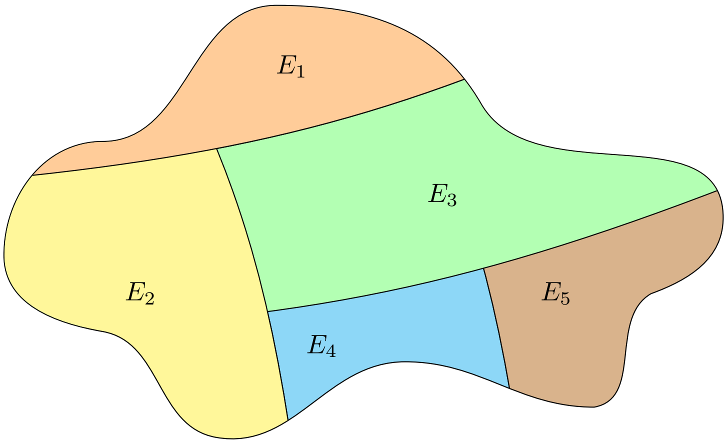

Figure 6. *Partitions of a set*

Learning Outcomes

- ✓ Understand what is a function and define its domain, and co-domain,
- ✓ Define the range of a function and the pre- image of an element
- ✓ Recognise linear, quadratic and exponential functions
- ✓ Define and learn examples of injective and surjective functions.
- ✓ Build functions compositions and a function's inverse.
- ✓ Recognise and define some particular functions such as absolute value, floor, ceiling

2.1 Definition of a function

Let A and B be nonempty sets. A *function* from A to B is an assignment of exactly one element of B to every element of A.

$$f\colon A \to B$$

A function is a subset of the cartesian product (see section 1.5.6) $A \times B$ that has only one ordered pair (a, b) for each $a \in A$.

2.2 Domain, Co-Domain and Range

In a function $f\colon A \to B$, A is the **domain** D_f, and B is the **co-domain** $Co\text{-}D_f$. When noted as $f(a) = b$, a is the *pre-image* and b is the *image*. The **range** R_f is the set of all images of A in B.

Two functions are *equal* if and only if they have the same domain, co-domain and map each element of the domain the same element in the co-domain.

A function is called *real-valued* if $D_f = \mathbb{R}$ and *integer-valued* if $D_f = \mathbb{Z}$.

2.3 Injective & Surjective Functions

2.3.1 Injective Functions

A function is **injective** (or one-to-one) if $f(a) = f(b) \to a = b$. In other words, if two images are the same, their pre-images are the same. **An injective funtion only assigns one image for each pre-image**. This is clarified by the contrapositive of the above implication: $f(a) \neq f(b) \to a \neq b$. In general:

$$\forall a \forall b(a = b \to f(a) = f(b))$$

If a function is **strictly increasing** or **decreasing** (i.e. $\forall a \forall b(a < b \to f(a) \leq f(b))$ or $\forall a \forall b(a > b \to f(a) \geq f(b))$), the function is injective.

2.3.2 Surjective Functions

A function is **surjective** (or onto) if $\forall y \exists x(f(x) = y)$. That is, for every image in $Co\text{-}D_f$ there exists a pre-image in D_f, thus $Co\text{-}D_f = R_f$.