CM1020: Discrete Mathematics Peer Graded Assignment 1.209

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1 Inclusion-Exclusion Principle for 3 Sets

We want to show that

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

1.1 Using set operations

We can apply the laws of set theory to prove this.

$$\begin{aligned}
|A \cup B \cup C| &= |(A \cup B) \cup C| && \text{(Associativity)} \\
&= |(A \cup B)| + |C| - |(A \cup B) \cap C| && \text{(Inclusion-Exclusion)} \\
&= |A| + |B| - |(A \cap B)| + |C| - |(A \cup B) \cap C| && \text{(Inclusion-Exclusion)} \\
&= |A| + |B| - |(A \cap B)| + |C| - |(A \cap B) \cup (B \cap C)| && \text{(Distributivity)} \\
&= |A| + |B| - |(A \cap B)| + |C| - (|(A \cap C)| + |(B \cap C)| + |A \cap C \cap B \cap C|) \\
& && \text{(Inclusion-Exclusion)} \\
&= |A| + |B| + |C| - |(A \cap B) - |(A \cap C)| - |(B \cap C)| + |A \cap B \cap C| && \text{(Commutativity)}
\end{aligned}$$

1.2 Using Venn diagrams

We will construct this by applying the principle of inclusion-exclusion, which starts by including all elements of given sets and then removing overcounts and returning undercounts. We know that if the sets A, B, C were disjoint, the cardinality of these sets would be stated as

$$|A \cup B \cup C| = |A| + |B| + |C|$$

In this case, the elements of each set are counted only once. However, let us assume that all three sets intersect as shown in Figure 1.

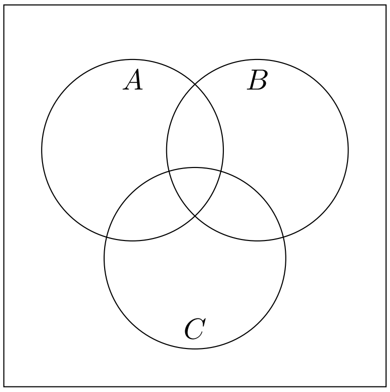

Figure 1: 3 sets intersecting

If we apply the above formula, we would count elements multiple times. Assuming each set has only 1 element, the total count is shown in Figure 2.

Therefore, we need to remove the overcounted elements that occur at the intersections of the sets. We construct the following set difference.

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C|$$

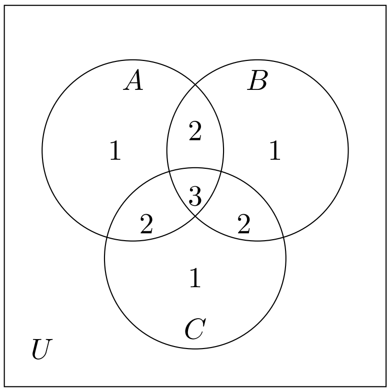

Figure 2: Overcount of intersecting elements

This results in removal of double-counted elements. It also however causes the elements of $|A \cap B \cap C|$ to be completely removed as shown in Figure 3

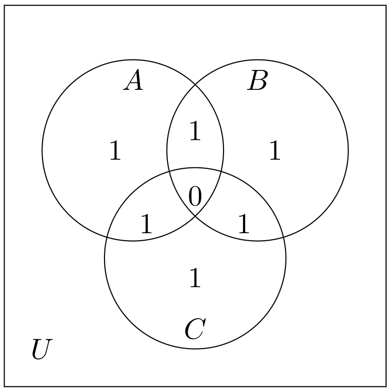

Figure 3: Undercount of $|A \cap B \cap C|$

We therefore need to ensure that the elements of the intersection $|A \cap B \cap C|$ are counted and added back.

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

The venn diagram shows that all elements are thus counted exactly once. This is shown in Figure 4

2 Listing Method

Let A and B two subsets of the universal set $U = \{x \mid x \in \mathbb{Z} \text{ and } 0 \leq x < 20\}$. A is the set of even numbers in U and B is the set of odd numbers in U. Therefore we know:

$$U = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19\}$$
$$A = \{0, 2, 4, 6, 8, 10, 12, 14, 16, 18\}$$
$$B = \{1, 3, 5, 7, 9, 11, 13, 15, 17, 19\}$$
$$A \cup B = U$$
$$A \cap B = \emptyset$$

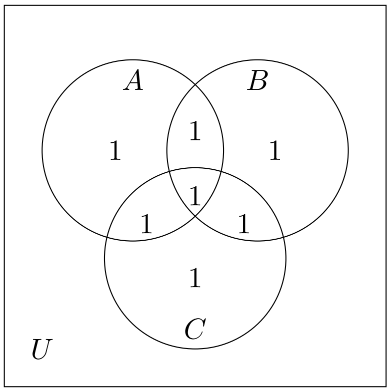

Figure 4: Corrected count of all elements

We now show the elements of the following sets using the listing method: $A \cap \overline{B}$, $\overline{A \cap B}$, $\overline{A \cup B}$, $\overline{A \oplus B}$.

2.1 $A \cap \overline{B}$

The intersection of two sets contains all elements that are in both sets.

$$A = \{0, 2, 4, 6, 8, 10, 12, 14, 16, 18\}$$
$$\overline{B} = U - B = A$$
$$A \cap \overline{B} = A = \{0, 2, 4, 6, 8, 10, 12, 14, 16, 18\}$$

2.2 $\overline{A \cap B}$

The complement of the intersection of two sets is equal to the union of their complements. The union of two sets contains all the elements in both sets.

$$\begin{aligned} \overline{A \cap B} &= \overline{A} \cup \overline{B} && \text{(De Morgan's Second Law)} \\ \overline{A} &= B \\ \overline{B} &= A \\ \overline{A} \cup \overline{B} &= B \cup A = U \\ U &= \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19\} \end{aligned}$$

2.3 $\overline{A \cup B}$

The complement of the union of two sets is equal to the intersection of their complements. The intersection of two sets contains all elements that are in both sets.

$$\begin{aligned} \overline{A \cup B} &= \overline{A} \cap \overline{B} && \text{(De Morgan's First Law)} \\ \overline{A} &= B \\ \overline{B} &= A \\ \overline{A} \cap \overline{B} &= B \cap A = \emptyset \end{aligned}$$

2.4 $\overline{A \oplus B}$

The symmetric difference of two sets contains all elements that are in either of the sets, but **not** in both.

$$
\begin{aligned}
A \oplus B &= U \\
\overline{A \oplus B} &= \overline{U} = \emptyset
\end{aligned}
$$