

Heat transfer – Analytical solutions

Introduction to Lithospheric Geodynamic Modelling



Contents & aims

- Solving the heat transfer equation
 - Simple one-dimensional cases
 - Basic behaviour of the geotherm in steady-state
 - Demonstrate the mathematical basis of *boundary conditions*
- Plotting solutions with *Python*
 - Get to know python scripts
 - Varying the parameters
 - Understand how the solutions react to changes in physical parameters



Heat transfer equation

$$\rho C_p \left(\frac{\partial T}{\partial t} + u_z \frac{\partial T}{\partial z} \right) = \frac{\partial}{\partial z} \left(k \frac{\partial T}{\partial z} \right) + A$$

- How do material properties affect the lithospheric geotherm?
- Analytical solutions → to get an idea how the geotherm should look like before we even consider running any numerical models



Heat transfer equation simplified

$$\rho C_p \left(\frac{\partial T}{\partial t} + u_z \frac{\partial T}{\partial z} \right) = \frac{\partial}{\partial z} \left(k \frac{\partial T}{\partial z} \right) + A$$

→ Assume no advection ($u_z = 0$)

$$\rho C_p \frac{\partial T}{\partial t} = \frac{\partial}{\partial z} \left(k \frac{\partial T}{\partial z} \right) + A$$

→ Assume no internal heat production ($A = 0$)

$$\rho C_p \frac{\partial T}{\partial t} = \frac{\partial}{\partial z} \left(k \frac{\partial T}{\partial z} \right)$$

→ Assume constant heat conductivity ($dk/dz = 0$)

$$\rho C_p \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial z^2}$$



Heat transfer equation *really* simplified

$$\rho C_p \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial z^2}$$



Heat transfer equation *really* simplified

$$\rho C_p \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial z^2}$$

→ Assume *steady-state* ($dT/dt = 0$)

$$\rho C_p \frac{\partial T}{\partial t} = \rho C_p 0 = 0 = k \frac{\partial^2 T}{\partial z^2}$$



Heat transfer equation *really* simplified

$$\rho C_p \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial z^2}$$

→ Assume *steady-state* ($dT/dt = 0$)

$$\rho C_p \frac{\partial T}{\partial t} = \rho C_p 0 = 0 = k \frac{\partial^2 T}{\partial z^2}$$



Steady-state geotherm

$$0 = k \frac{\partial^2 T}{\partial z^2}$$

$$\begin{aligned} \int \int 0 dz dz &= \int \int k \frac{\partial^2 T}{\partial z^2} dz dz \\ \int C_1 dz &= \int \left(k \frac{\partial T}{\partial z} + C_2 \right) dz \\ C_1 z + C_3 &= kT + C_2 z + C_4 \end{aligned}$$

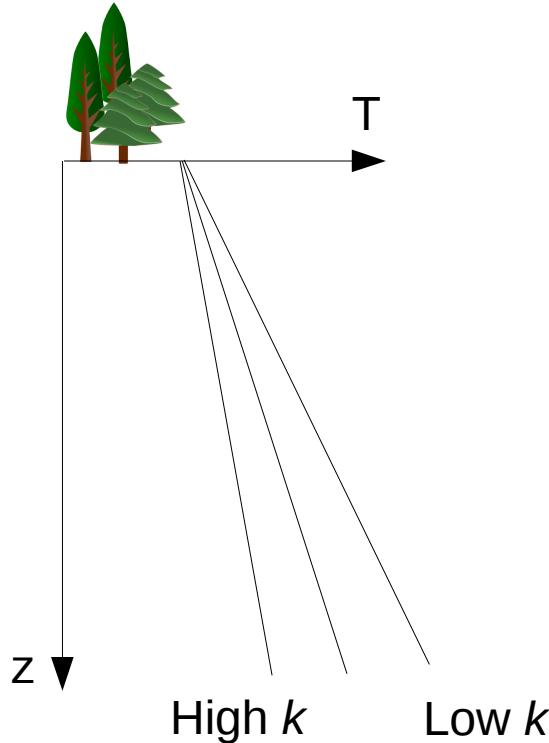
$$T = ((C_1 - C_2)z + (C_3 - C_4)) / k$$

$$T = (C_a z + C_b) / k$$



Steady-state geotherm

$$T = (C_a z + C_b)/k$$

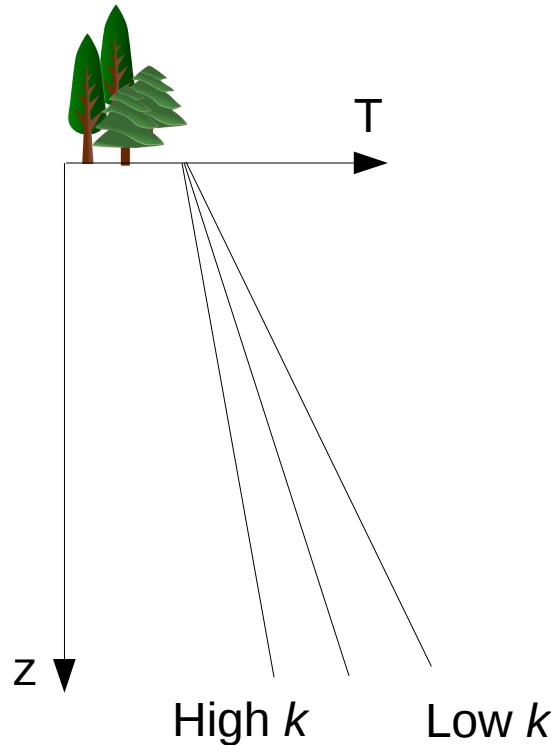




Steady-state geotherm

$$T = (C_a z + C_b)/k$$

Equation of a
straight line

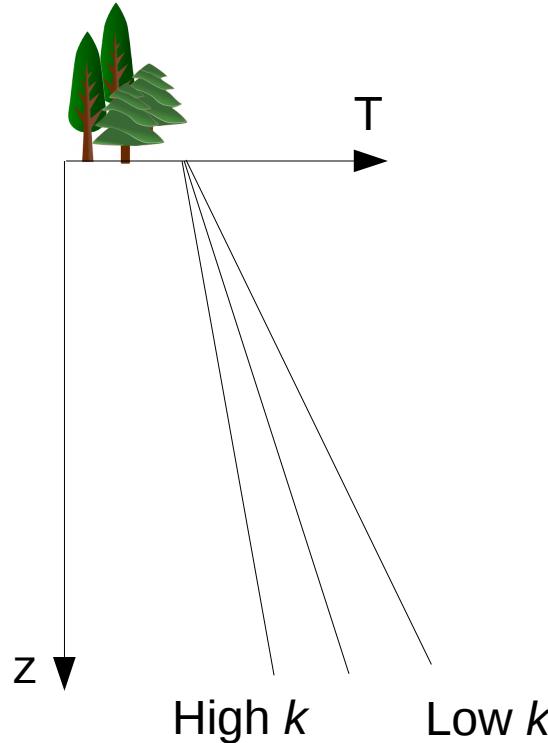




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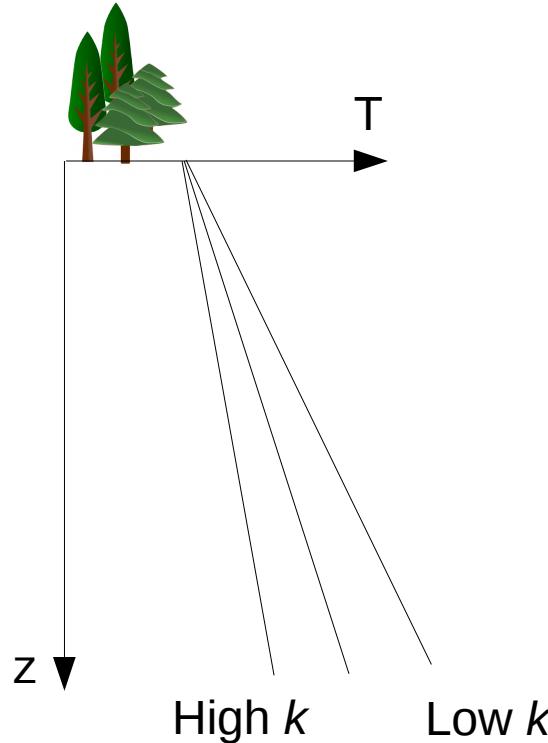
→ Heat conductivity k affects the steepness of the geotherm



Steady-state geotherm

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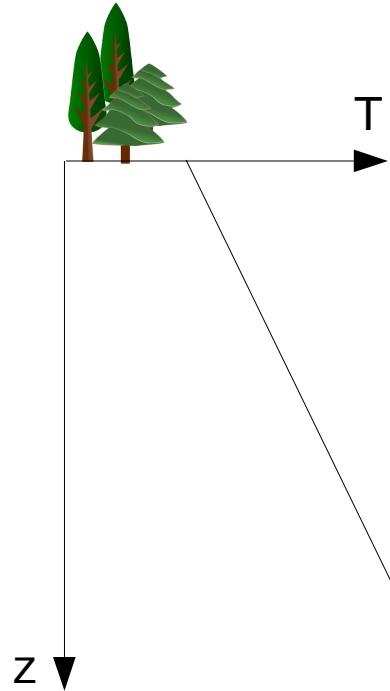


- Heat conductivity k affects the steepness of the geotherm
- What are the values of the constants?



Steady-state geotherm

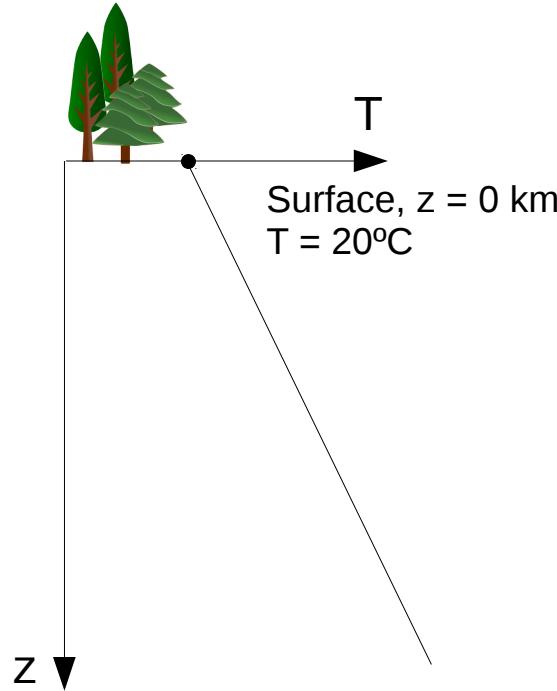
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Steady-state geotherm

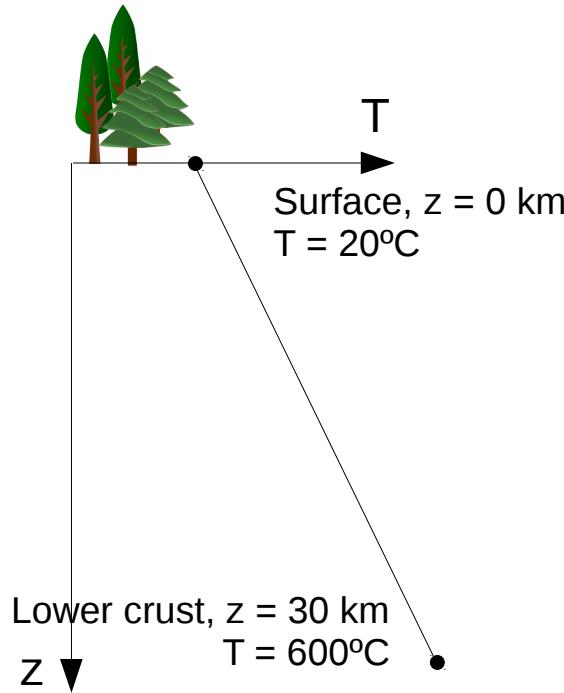
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Steady-state geotherm

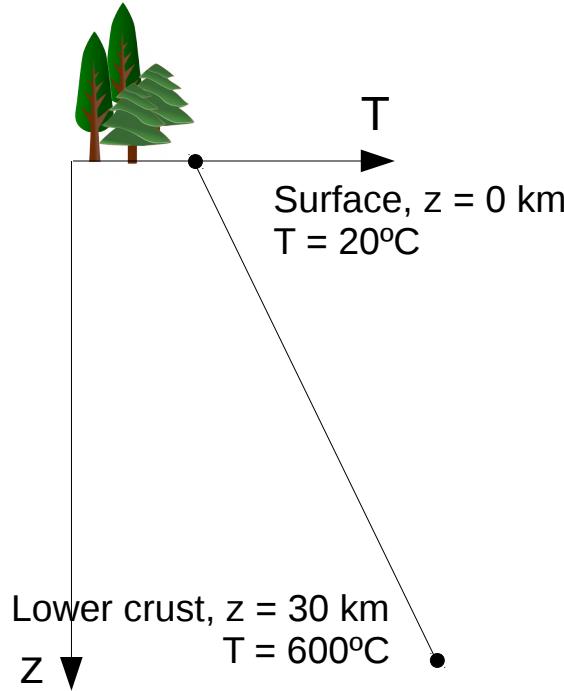
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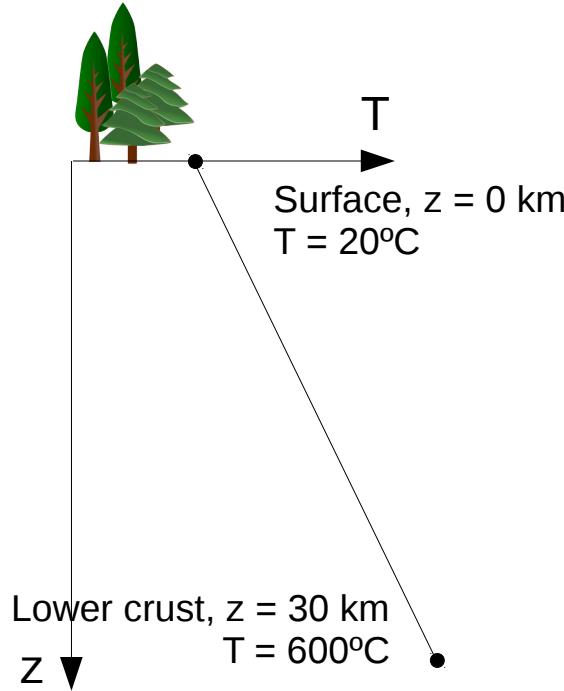


$$20 = (C_a \times 0 + C_b)/k$$



Steady-state geotherm

$$T = (C_a z + C_b)/k$$

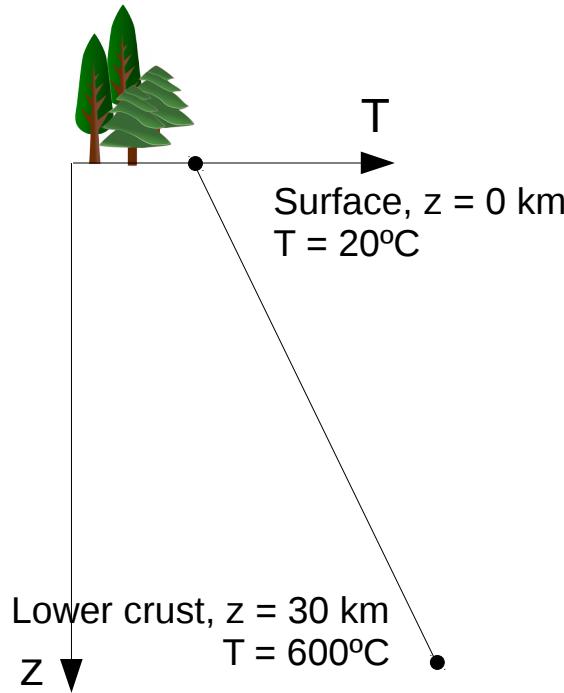


$$\begin{aligned} 20 &= (C_a \times 0 + C_b)/k \\ \Rightarrow 20k &= C_b \end{aligned}$$



Steady-state geotherm

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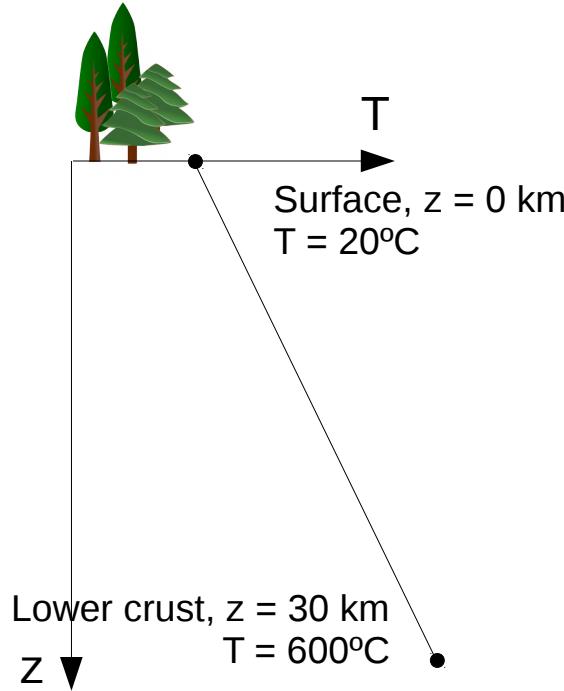
$$\begin{aligned} 20 &= (C_a \times 0 + C_b)/k \\ \Rightarrow 20k &= C_b \end{aligned}$$

$$\begin{aligned} 600 &= (C_a \times 30000 + C_b)/k \\ 600k &= 30000C_a + C_b \\ \Rightarrow C_a &= \frac{58}{3000}k \end{aligned}$$



Steady-state geotherm

$$T = (C_a z + C_b)/k$$



$$\begin{aligned} 20 &= (C_a \times 0 + C_b)/k \\ \Rightarrow 20k &= C_b \end{aligned}$$

$$\begin{aligned} 600 &= (C_a \times 30000 + C_b)/k \\ 600k &= 30000C_a + C_b \\ \Rightarrow C_a &= \frac{58}{3000}k \end{aligned}$$

→ T can be evaluated at any depth z once k is chosen



Steady-state geotherm

Heat transfer equation has an analytical solution ...

$$T = (C_a z + C_b)/k$$

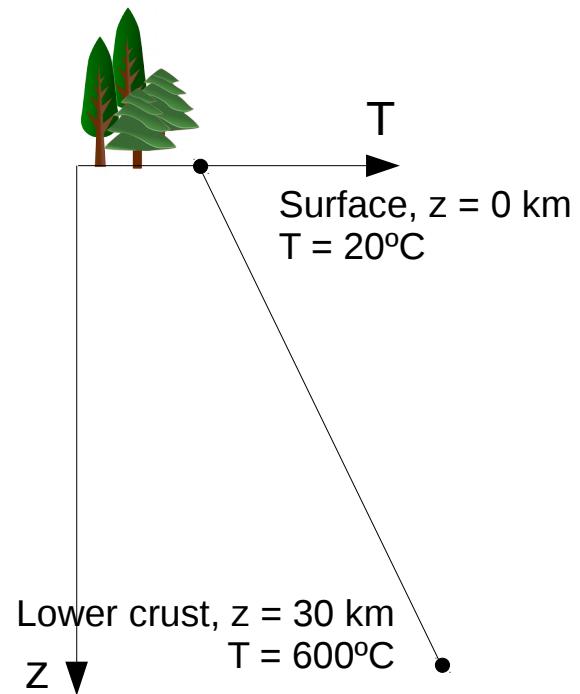
$$T = ((58/3000)kz + 20k)/k$$

$$T = (58/3000)z + 20$$

*... given the two **boundary conditions***

$$T = 20 \quad \text{at} \quad z = 0$$

$$T = 600 \quad \text{at} \quad z = 30000$$





Steady-state geotherm

Heat transfer equation has an analytical solution ...

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$$T = ((58/3000)kz + 20k)/k$$

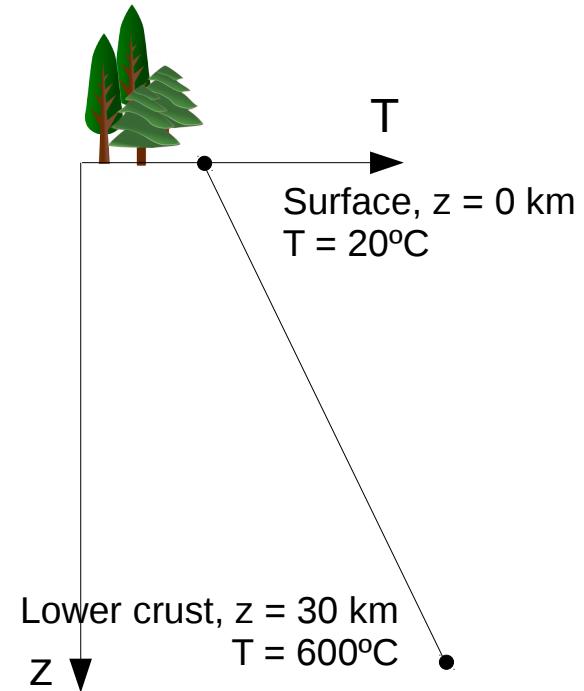
$$T = (58/3000)z + 20$$

NB! No k here

... given the two **boundary conditions**

$$T = 20 \quad \text{at} \quad z = 0$$

$$T = 600 \quad \text{at} \quad z = 30000$$





Steady-state geotherm with radiogenic heating

$$0 = k \frac{\partial^2 T}{\partial z^2} + A$$



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... integrate twice ...



Steady-state geotherm with radiogenic heating

$$0 = k \frac{\partial^2 T}{\partial z^2} + A$$

... integrate twice ...

$$T = \left(\frac{1}{2} A z^2 + C_a z + C_b \right) / k$$

*Not a
straight line*



Steady-state geotherm with radiogenic heating

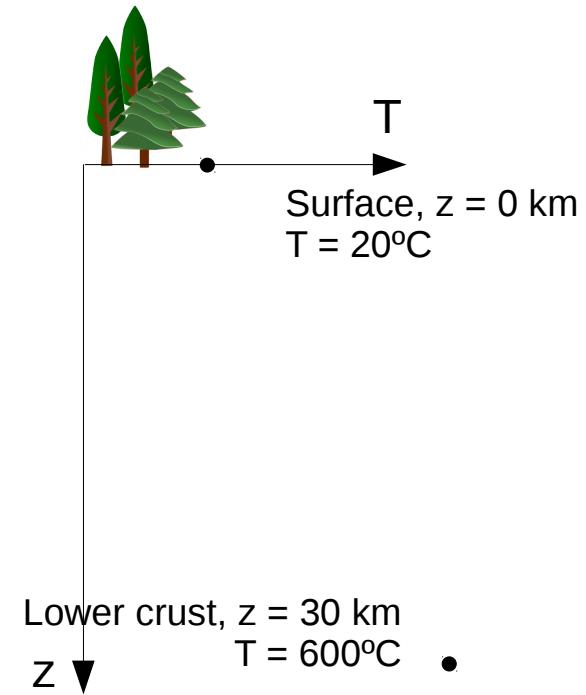
$$0 = k \frac{\partial^2 T}{\partial z^2} + A$$

... integrate twice ...

$$T = \left(\frac{1}{2} Az^2 + C_a z + C_b \right) / k$$

Not a straight line

... apply same boundary conditions ...





Steady-state geotherm with radiogenic heating

$$0 = k \frac{\partial^2 T}{\partial z^2} + A$$

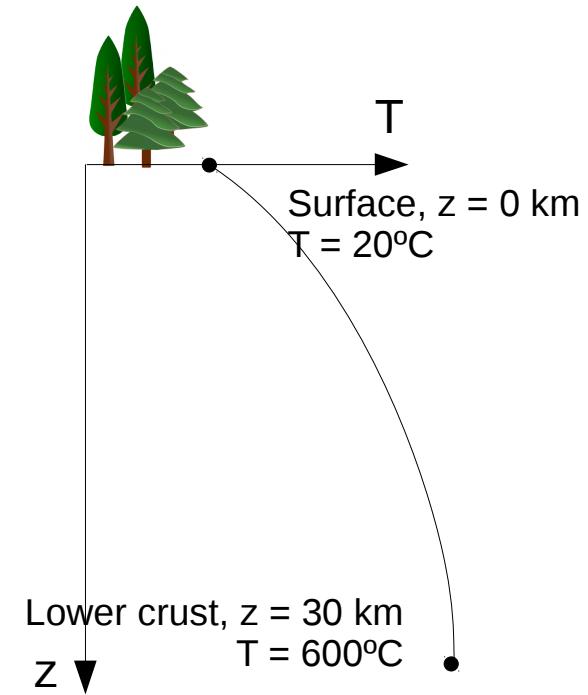
... integrate twice ...

$$T = \left(\frac{1}{2} Az^2 + C_a z + C_b \right) / k$$

Not a straight line

... apply same boundary conditions ...

$$T = -\frac{1}{2k} Az^2 + \frac{58}{3000} z + \frac{1}{2k} 30000 A z + 20$$





Mid-lesson wrap-up & play time

- Steady-state geotherm does not depend on density or heat capacity
- Form is dictated by heat conductivity and internal heat production and/or boundary conditions only (in absence of advection...)
 - Geotherm with $A=0$ – linear, straight line
 - Geotherm with $A>0$ – curved, parabolic
- Two boundary conditions are needed (for two integration constants)



Heat equation, analytical solutions in python

- Download the course package (ZIP) at
<https://github.com/HUGG/NGWM2016-modelling-course.git>
- Extract it in your home folder
- Change to folder NGWM2016-modelling-course / Lessons / 03-Analytical-solutions-heat-transfer /scripts
- Run `plot_steady_state_heat_eq_with_A.py` (double click or run at command prompt)
- Right click > Choose “Edit with IDLE”
- Change values of k and A and re-run
 - What happens with small / large values of A ?
 - What happens with small / large values of k ?



About boundary conditions

- We defined boundary conditions as a constant temperature at given depth
 - This is known as *Dirichlet* boundary condition
$$T = [\text{somevalue}]$$
- Instead of temperature, we can also define *heat flow*, the derivative of T , at either (or both) boundary
 - This is known as *von Neumann* boundary condition
$$q = k \frac{dT}{dz} = [\text{somevalue}] \text{ (mW/m}^2\text{)}$$



Heat flow boundary condition

The heat transfer equation has an analytical solution

$$T = (q_b z + 30000 A z + 20k - \frac{1}{2} A z^2)/k$$

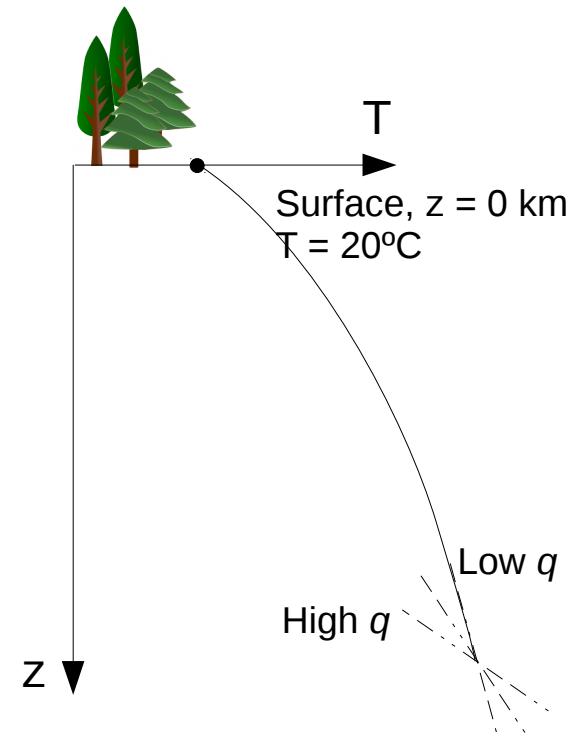
... given the two boundary conditions

$$T = 20 \quad \text{at} \quad z = 0$$

$$q = k \frac{dT}{dz} = q_b \quad \text{at} \quad z = 30000$$

What happens to the geotherm if ...

- › A is increased/decreased?
- › q_b is increased/decreased?
- › -> `plot_steady_state_heat_eq_vonNeumann.py`





Steady-state heat equation, internal boundary conditions

Despite the name, boundary conditions need not to be applied at the boundary. Consider the general solution to the steady-state heat equation.

$$T = \left(-\frac{1}{2}Az^2 + C_a z + C_b \right) / k$$

where

$$C_a = q_0 + Az_1$$

$$C_b = -q_0 z_2 - Az_1 z_2 + kT_0 + \frac{1}{2}Az_2^2$$

with boundary conditions

$$q = q_0 \text{ at } z = z_1$$

$$T = T_0 \text{ at } z = z_2$$



Steady-state heat equation, internal boundary conditions

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with boundary conditions

$$q = q_0 \text{ at } z = z_1$$

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For example, we can use measured values of heat flow and temperature at surface ($z_1 = 0; z_2 = 0$)

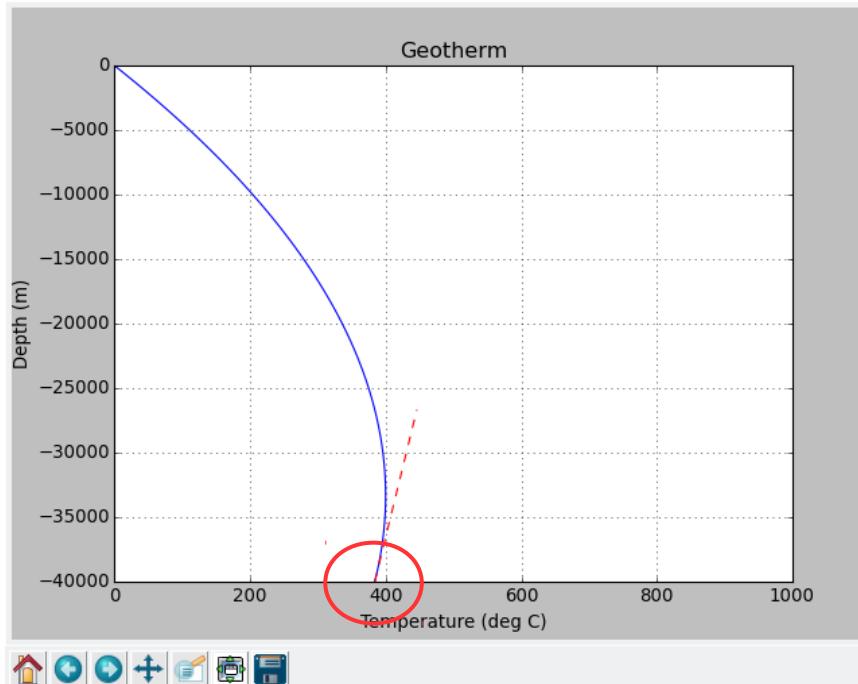


Steady-state heat equation, internal boundary conditions

- Open script `plot_steady_state_heat_eq_innerbnd.py`.
- By default this uses boundary conditions $T=20$ at surface ($z=0$) and $q=15\text{mW/m}^2$ at the bottom of the crust ($z=40\text{ km}$). These can be varied to different values at different depths.
- Using the script, find out the approximate value for internal heat generation within the crust, if we know that
 - $T=0$ at surface (measured)
 - $q=60\text{mW/m}^2$ at surface (measured)
 - $T\sim 600$ at the bottom of the crust (40 km) (estimate / literature)



Steady-state heat equation, internal boundary conditions

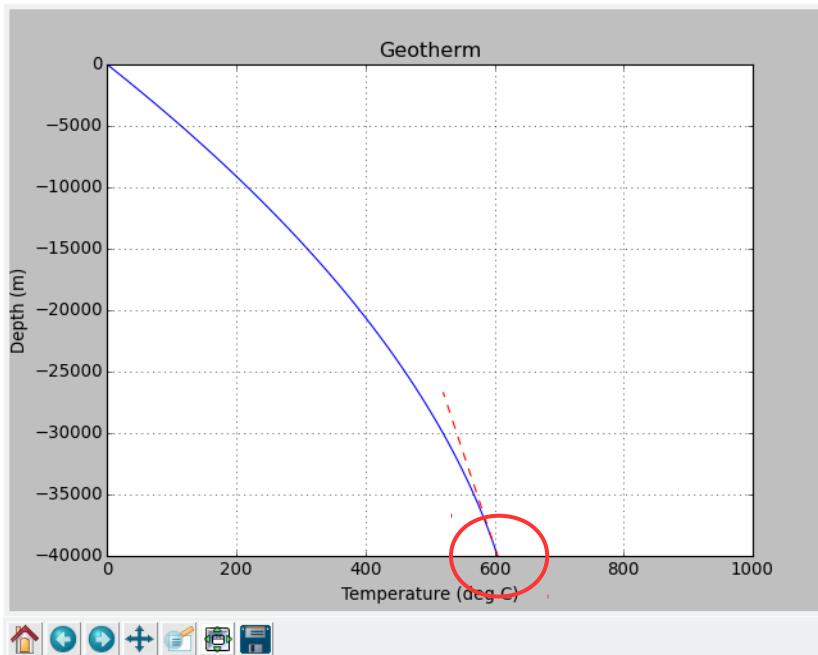


- $T=0$ at surface
 - $q=60\text{mW/m}^2$ at surface
 - A unchanged ($1.8\mu\text{W/m}^3$)
- Do we need to increase or decrease the value of A ?



Steady-state heat equation, internal boundary conditions

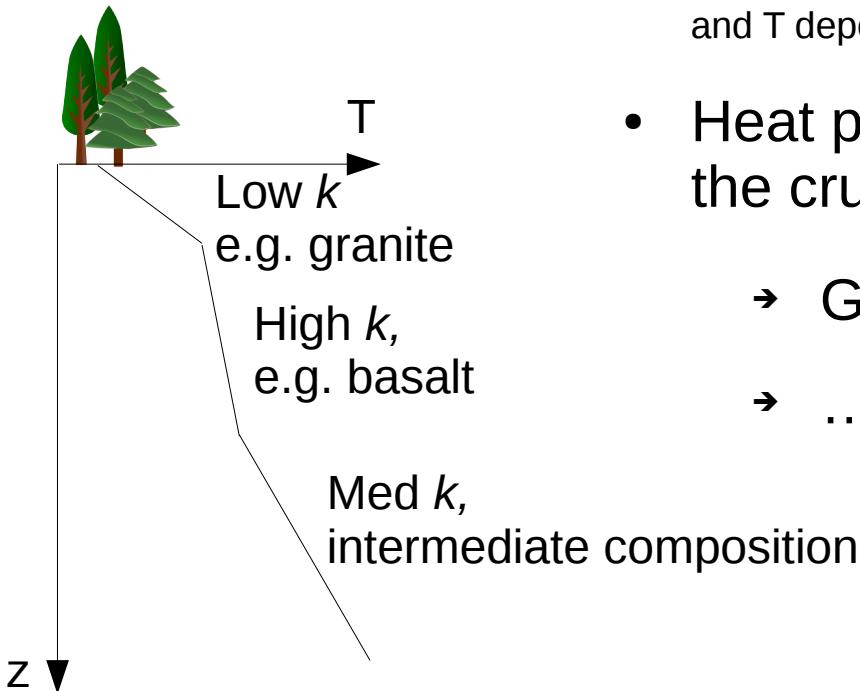
$$q_{surf} = q_{bot} + [\text{heat generated in crust}]$$



→ To keep the surface heat flow at 60mW/m² and to increase the value of bottom heat flow, heat generation value needs to be *decreased*



Complications that were not considered



- Heat conductivity is not constant throughout the crust (lithological variation and T dependence)
- Heat production is not constant throughout the crust (and is very low in mantle)
 - Geotherm can be defined piecewise
 - ... or we could use numerical models

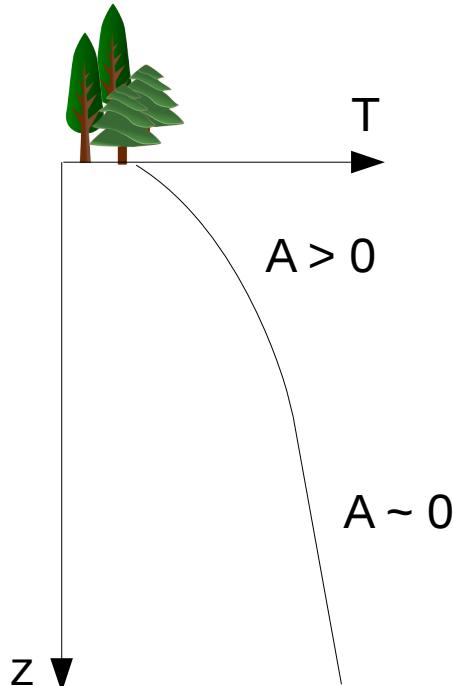


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Non-steady-state heat equation

$$\rho C_p \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial z^2}$$

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial z^2} \quad \kappa = \frac{k}{\rho C_p}$$

- Need two integrations in respect to z , one integration in respect to t
 - Three boundary condition values needed
 - Boundary condition in time = initial condition



Non-steady-state heat equation

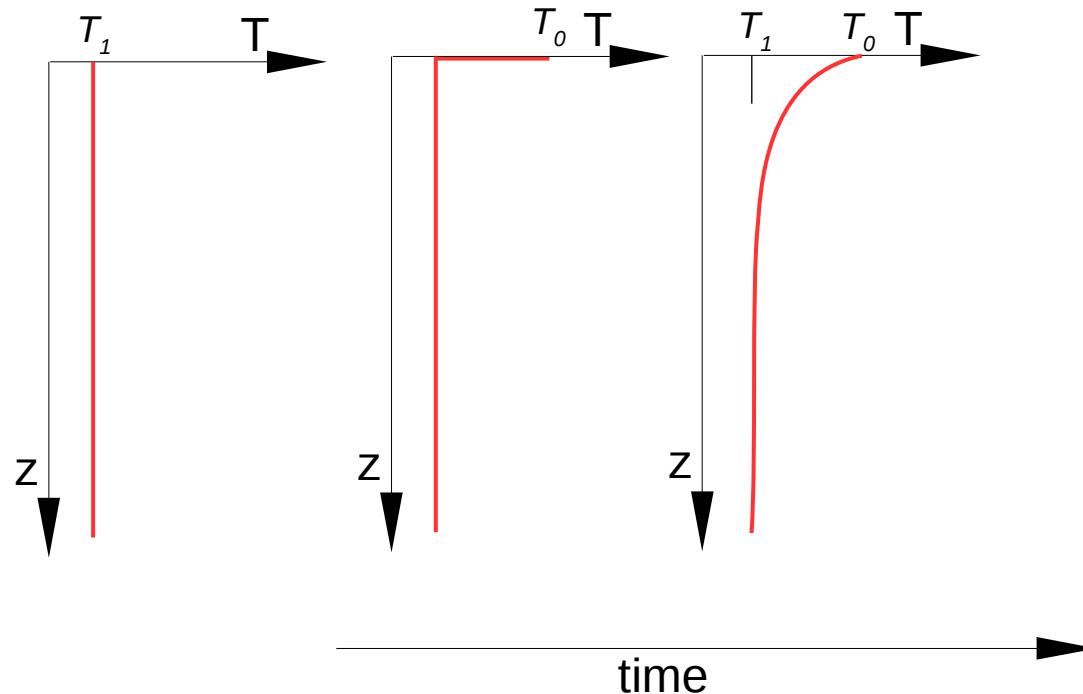
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- Need two integrations in respect to z , one integration in respect to t
 - Not that easy ...
- Let's have a look at a *semi-infinite half-space* case
 - Only one (spatial) boundary condition and one (temporal) initial condition



Heat equation in semi-infinite half-space

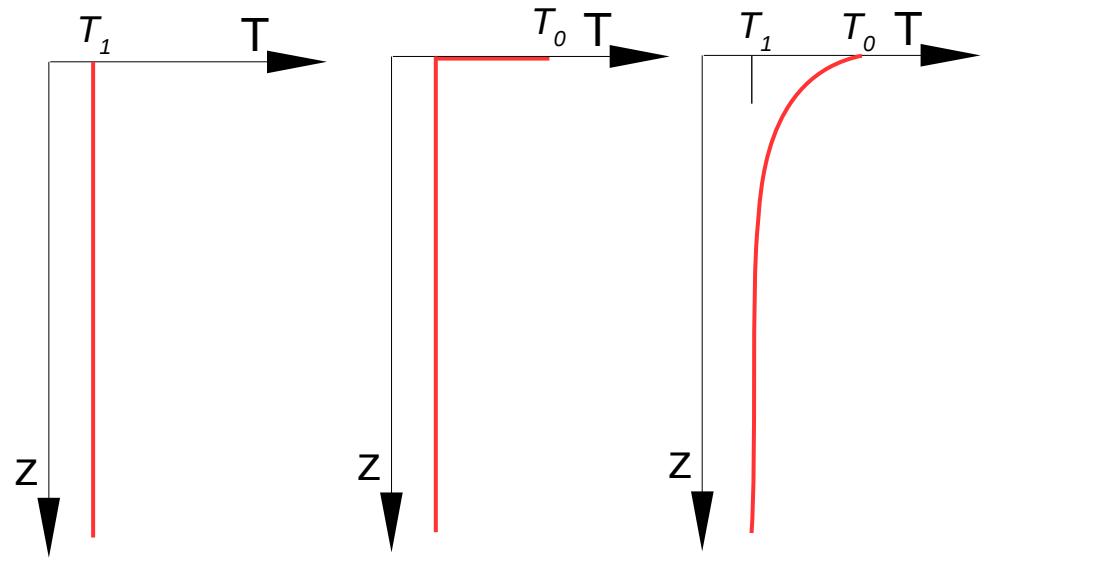


$T = T_1$ at $z = \infty$ at all times

$T = T_0$ at $z = 0$ at $t > 0$



Heat equation in semi-infinite half-space

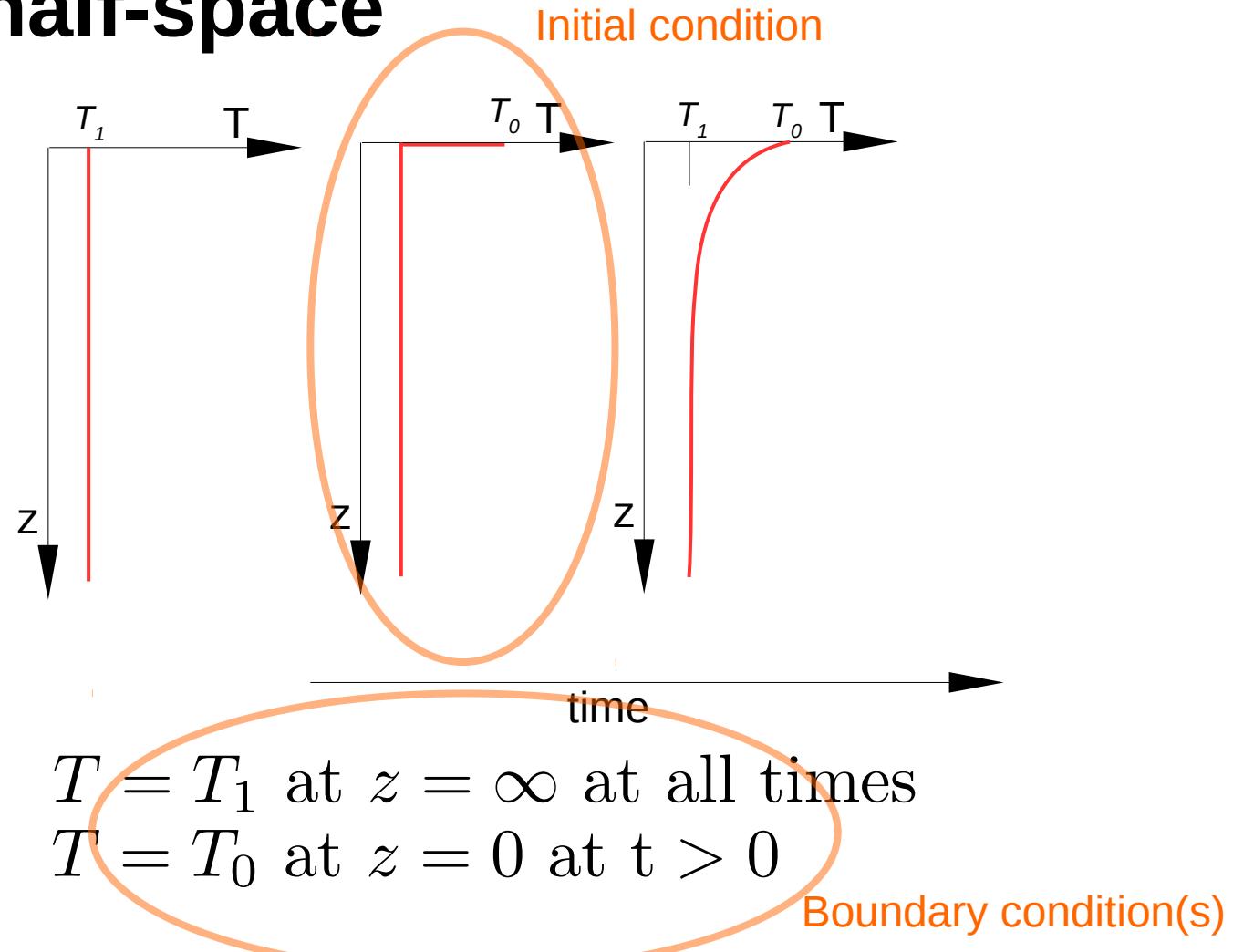


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Boundary condition(s)



Heat equation in semi-infinite half-space





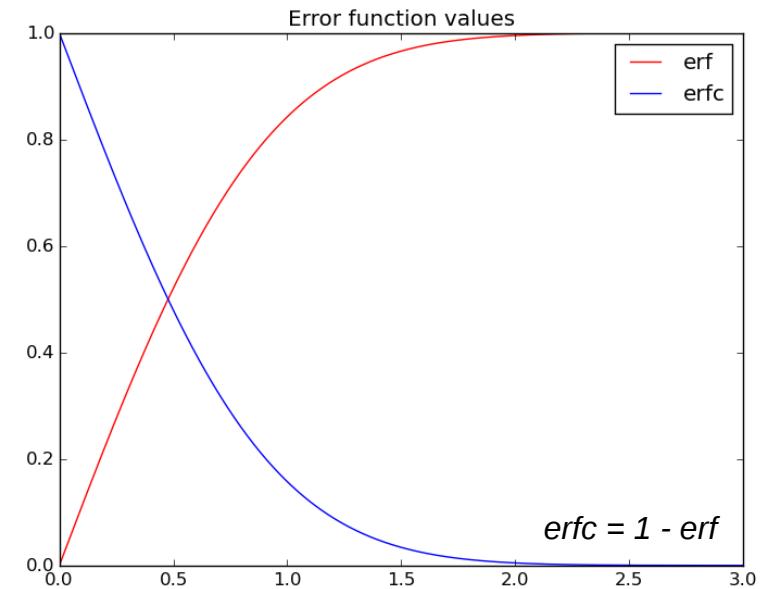
Heat equation in semi-infinite half-space

Solution:

$$\frac{T - T_1}{T_0 - T_1} = \operatorname{erfc} \frac{z}{2\sqrt{\kappa t}}$$

More about transient heat equations
later today...

Characteristic thermal diffusion distance
→ how far the heat diffuses in time t





Heat equation in semi-infinite half-space

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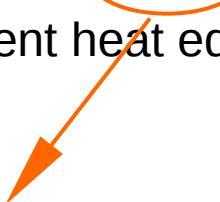


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Characteristic thermal diffusion distance
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“Back of the envelope” calculations:
E.g. is thermal conduction important process in crustal scale in a numerical model that runs for 100 kyr?



Heat equation in semi-infinite half-space

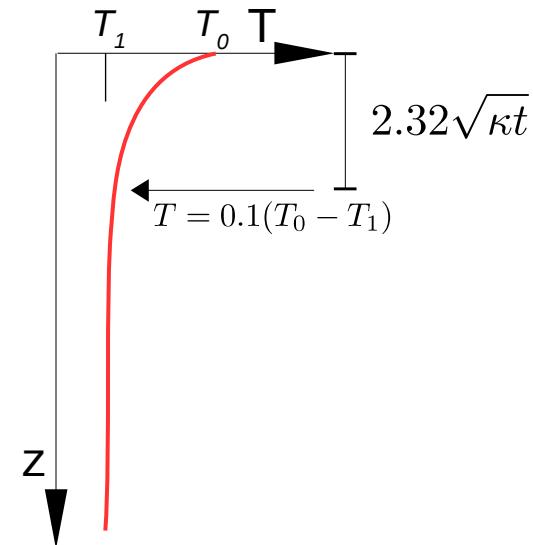
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Characteristic thermal diffusion distance
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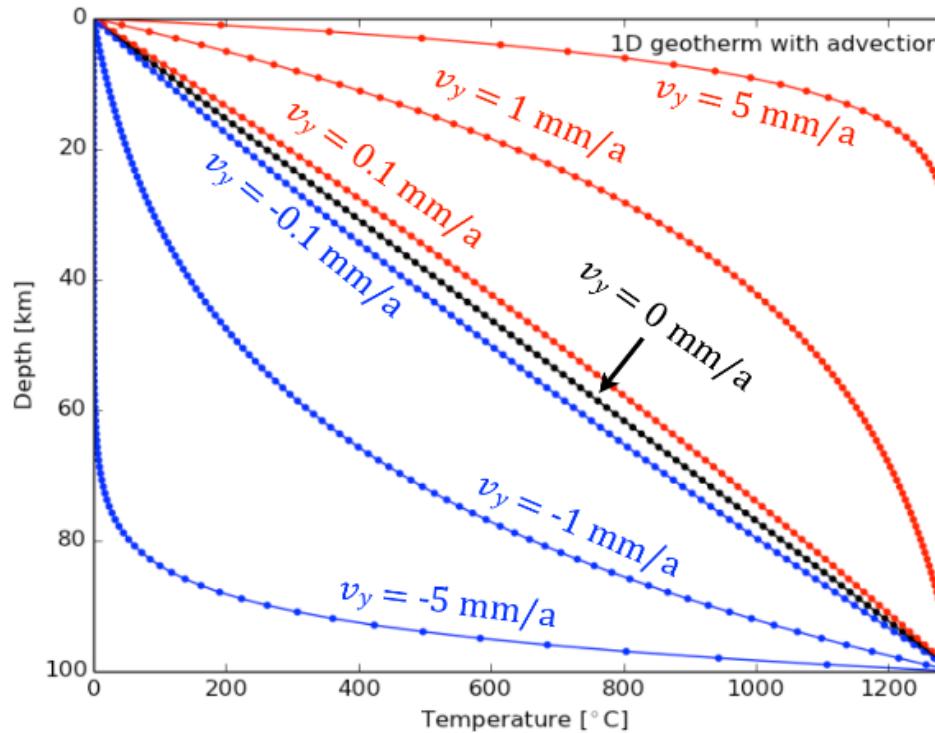


“Back of the envelope” calculations:
E.g. is thermal conduction important process in crustal scale in a numerical model that runs for 100 kyr?

$$2\sqrt{\kappa t} = 2\sqrt{10^{-6} \times 10^{7.5} \times 10^5} \approx 3.5 \text{ km}$$



Steady-state heat equation with advection





Steady-state heat equation with advection

$$\rho C_p \left(\cancel{\frac{\partial T}{\partial t}} + u_z \frac{\partial T}{\partial z} \right) = k \frac{\partial^2 T}{\partial z^2} \quad \kappa = \frac{k}{\rho C_p}$$

$$u_z \frac{\partial T}{\partial z} = \kappa \frac{\partial^2 T}{\partial z^2} \Rightarrow \frac{d}{dz} \left(\frac{dT}{dz} \right) - \frac{u_z}{\kappa} \frac{dT}{dz} = 0 \quad y = \frac{dT}{dz}$$

$$y' + \left(\frac{-u_z}{\kappa} \right) y = 0$$

$$y' + \alpha y = 0 \Rightarrow y = C e^{-\alpha x}$$

$$y = C_1 e^{\frac{u_z}{\kappa} z}$$

$$\frac{dT}{dz} = C_1 e^{\frac{u_z}{\kappa} z}$$



Steady-state heat equation with advection

$$\int \frac{dT}{dz} dz = \int C_1 e^{\frac{u_z}{\kappa} z} dz$$
$$T = \frac{C_1 \kappa}{u_z} e^{\frac{u_z}{\kappa} z} + C_2$$

$$T = 0 \text{ at } z = 0 \longrightarrow C_2 = -\frac{C_1 \kappa}{u_z}$$

$$T = T_{\text{bott}} \text{ at } z = L \longrightarrow C_1 = \frac{T_{\text{bott}} u_z}{\kappa} \Big/ \left(e^{-\frac{u_z}{\kappa} L} - 1 \right)$$

$$T = T_{\text{bott}} \frac{e^{\frac{u_z}{\kappa} z} - 1}{e^{\frac{u_z}{\kappa} L} - 1}$$



Steady-state heat equation with advection

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-> `plot_steady_state_heat_eq_advection.py`



Steady-state heat equation with advection

`plot_steady_state_heat_eq_advection.py`

Default case: $u_z = 5 \text{ mm/a}$ (sedimentation)

If sedimentation rate decreases to 2 mm/a, does heat conductivity need to be

- a) increased or
- b) decreased

to maintain the same geotherm?



Steady-state heat equation with advection

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$$\frac{dT}{dz} = C_1 e^{\frac{u_z}{\kappa} z}$$

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Peclet number

- Non-dimensional number describing the relevance of diffusive and advective heat (or mass) transfer processes

$$\text{Pe} = \frac{Lu}{\kappa}$$



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-> `plot_steady_state_heat_eq_advection.py`

- What's the effect of u_z if k has a really high value?