Heat transfer – Analytical solutions

Introduction to Lithospheric Geodynamic Modelling

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Contents & aims

- Solving the heat transfer equation
  - Simple one-dimensional cases
  - Basic behaviour of the geotherm in steady-state
  - Demonstrate the mathematical basis of *boundary conditions*
- Plotting solutions with *Python*
  - Get to know python scripts
  - Varying the parameters
  - Understand how the solutions react to changes in physical parameters

Heat transfer equation

$$\rho C_p \left(\frac{\partial T}{\partial t} + u_z \frac{\partial T}{\partial z} \right) = \frac{\partial}{\partial z} \left(k \frac{\partial T}{\partial z} \right) + A$$

- How do material properties affect the lithospheric geotherm?

- Analytical solutions → to get an idea how the geotherm should look like before we even consider running any numerical models

Heat transfer equation simplified

$$\rho C_p \left(\frac{\partial T}{\partial t} + u_z \frac{\partial T}{\partial z} \right) = \frac{\partial}{\partial z} \left(k \frac{\partial T}{\partial z} \right) + A$$

- Assume no advection ($u_z = 0$)

$$\rho C_p \frac{\partial T}{\partial t} = \frac{\partial}{\partial z} \left(k \frac{\partial T}{\partial z} \right) + A$$

  - Assume no internal heat production ($A = 0$)

$$\rho C_p \frac{\partial T}{\partial t} = \frac{\partial}{\partial z} \left(k \frac{\partial T}{\partial z} \right)$$

    - Assume constant heat conductivity ($dk/dz = 0$)

$$\rho C_p \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial z^2}$$

Heat transfer equation *really* simplified

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Heat transfer equation *really* simplified

$$\rho C_p \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial z^2}$$

- Assume *steady-state* (d*T*/dt = 0)

$$\rho C_p \frac{\partial T}{\partial t} = \rho C_p 0 = \boxed{0 = k \frac{\partial^2 T}{\partial z^2}}$$

Heat transfer equation *really* simplified

$$\rho C_p \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial z^2}$$

- Assume *steady-state* (d*T*/dt = 0)

$$\rho C_p \frac{\partial T}{\partial t} = \rho C_p 0 = \boxed{0 = k \frac{\partial^2 T}{\partial z^2}}$$

Steady-state geotherm

$$0 = k\frac{\partial^2 T}{\partial z^2}$$

$$\int\int 0 \mathrm{d}z\mathrm{d}z = \int\int k\frac{\partial^2 T}{\partial z^2}\mathrm{d}z\mathrm{d}z$$

$$\int C_1 \mathrm{d}z = \int \left(k\frac{\partial T}{\partial z} + C_2\right)\mathrm{d}z$$

$$C_1 z + C_3 = kT + C_2 z + C_4$$

$$T = \left((C_1 - C_2)z + (C_3 - C_4)\right)/k$$

$$T = (C_a z + C_b)/k$$

Steady-state geotherm

$$T = (C_a z + C_b)/k$$

Steady-state geotherm

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Equation of a straight line

Steady-state geotherm

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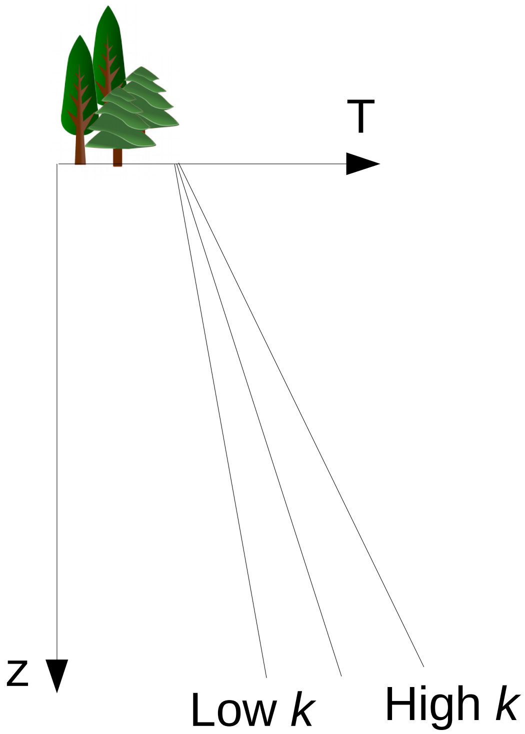
Equation of a straight line



Steady-state geotherm

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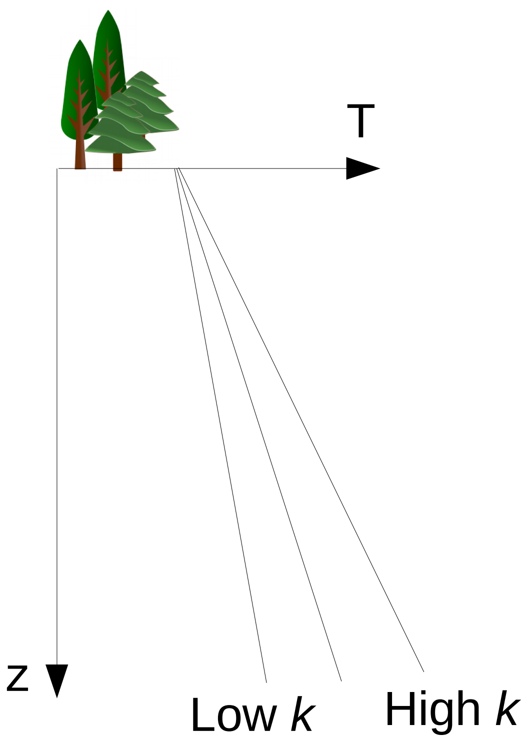

- Heat conductivity *k* affects the steepness of the geotherm

Steady-state geotherm

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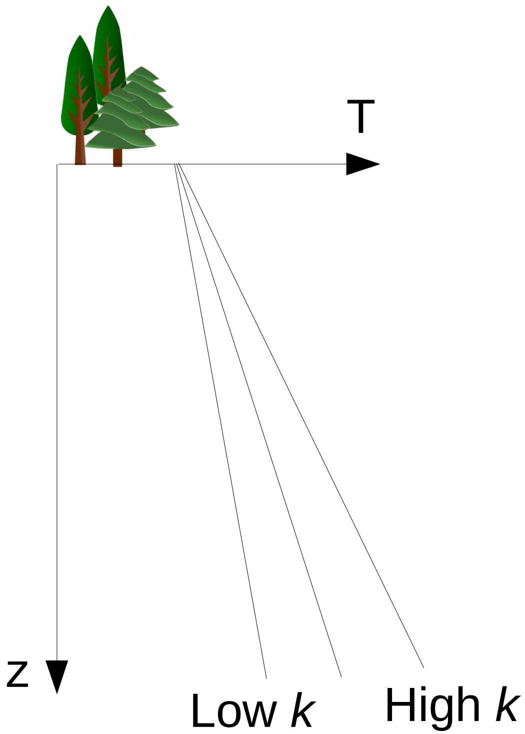
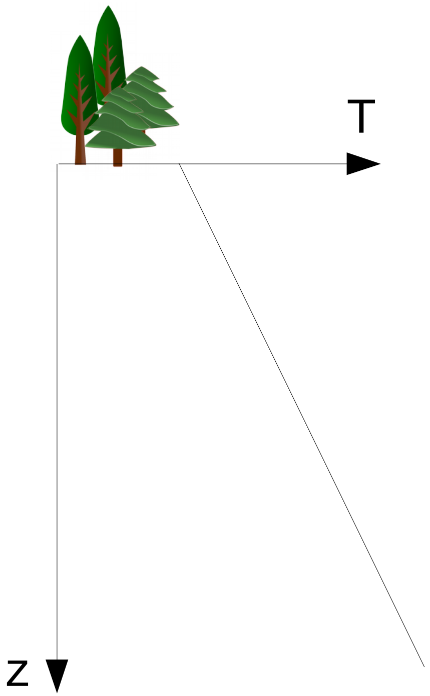
Equation of a straight line



- Heat conductivity *k* affects the steepness of the geotherm
- What are the values of the constants?

Steady-state geotherm

$$T = (C_a z + C_b)/k$$



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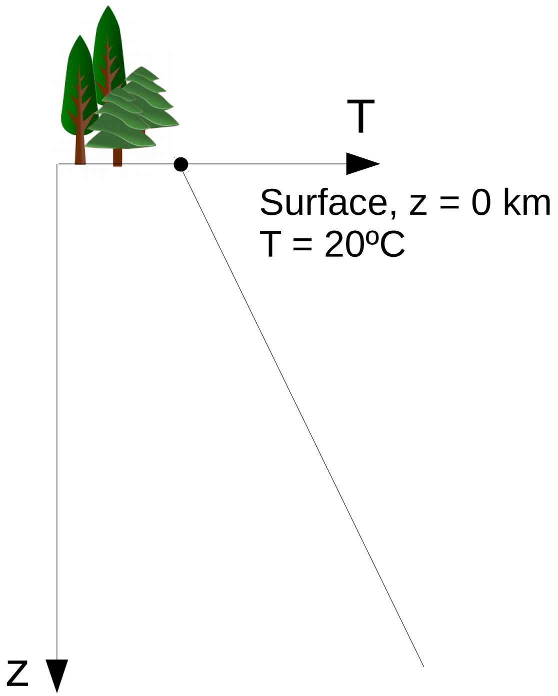
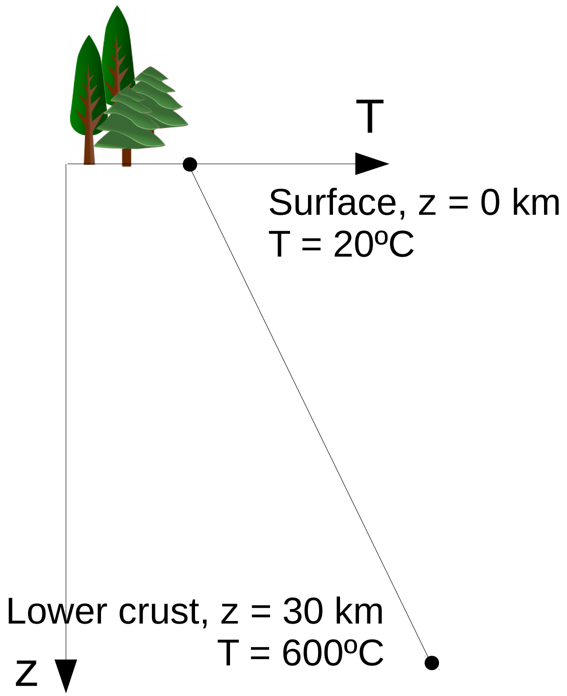


Steady-state geotherm

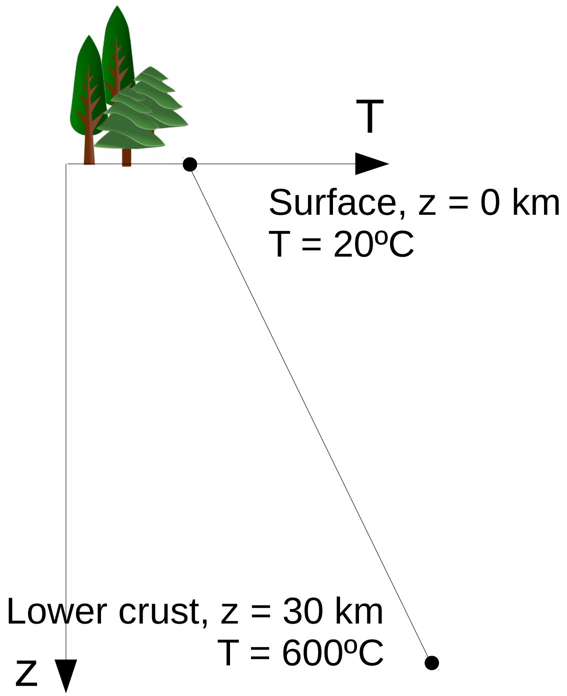
$$\boxed{T = (C_a z + C_b)/k}$$



Steady-state geotherm

$$\boxed{T = (C_a z + C_b)/k}$$

$$20 = (C_a \times 0 + C_b)/k$$



Steady-state geotherm

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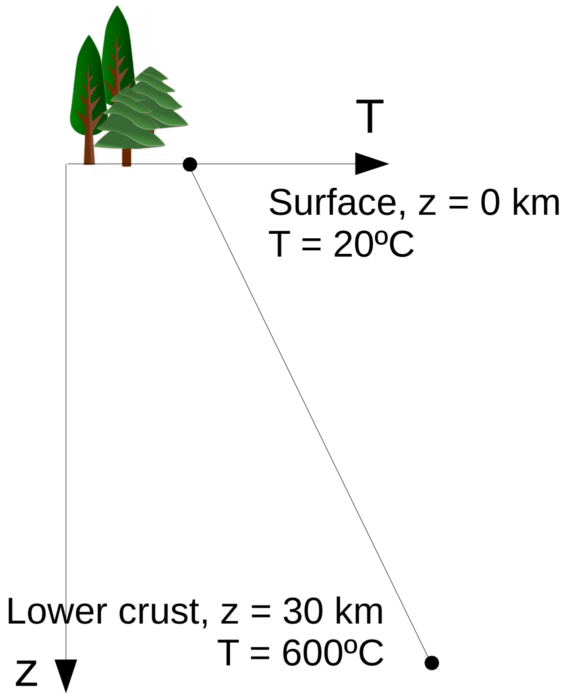

$$20 = (C_a \times 0 + C_b)/k$$
$$\Rightarrow 20k = C_b$$

Steady-state geotherm

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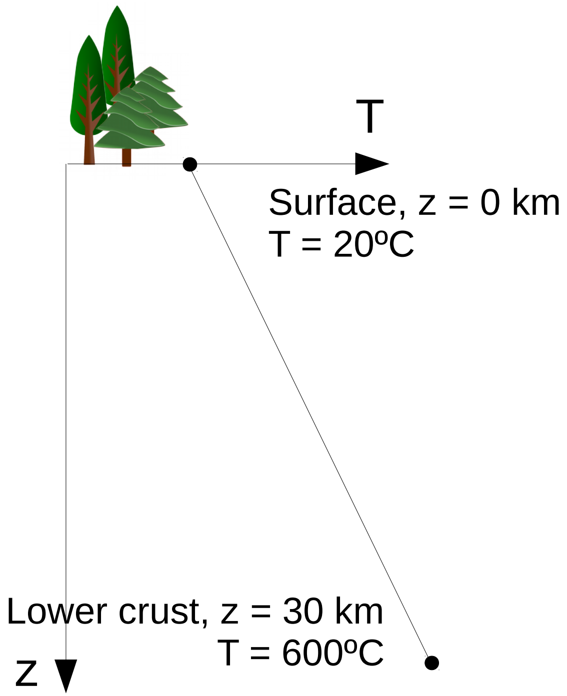
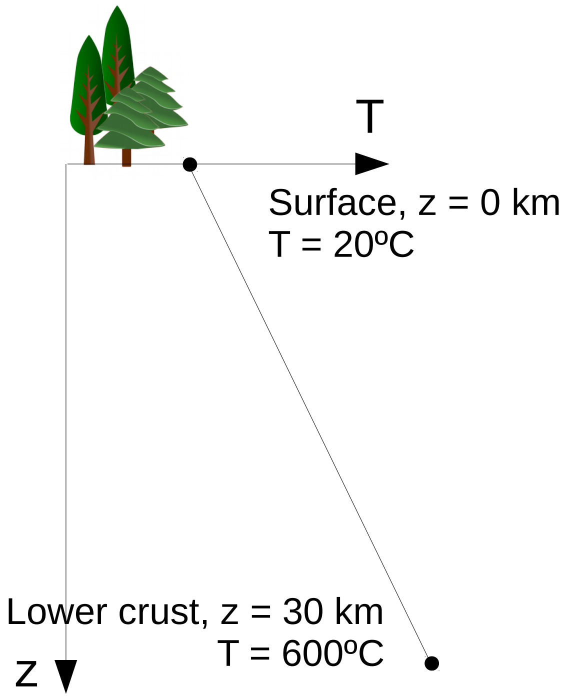


$$20 = (C_a \times 0 + C_b)/k$$
$$\Rightarrow 20k = C_b$$

$$600 = (C_a \times 30000 + C_b)/k$$
$$600k = 30000C_a + C_b$$
$$\Rightarrow C_a = \frac{58}{3000}k$$

Steady-state geotherm

$$\boxed{T = (C_a z + C_b)/k}$$



$$20 = (C_a \times 0 + C_b)/k$$
$$\Rightarrow 20k = C_b$$

$$600 = (C_a \times 30000 + C_b)/k$$
$$600k = 30000C_a + C_b$$
$$\Rightarrow C_a = \frac{58}{3000}k$$

–> *T* can be evaluated at any depth *z* once *k* is chosen

Steady-state geotherm

Heat transfer equation has an analytical solution ...

$$T = (C_a z + C_b)/k$$

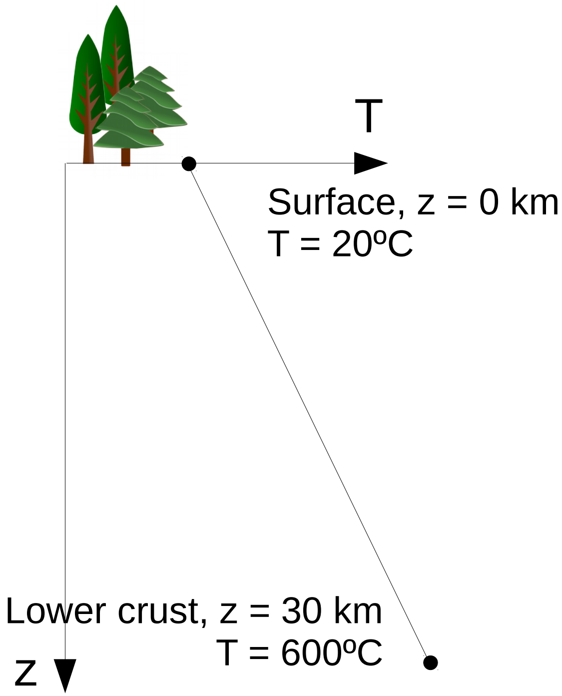
$$T = ((58/3000)kz + 20k)/k$$

$$T = (58/3000)z + 20$$

… given the two ***boundary conditions***

$T = 20$ at $z = 0$

$T = 600$ at $z = 30000$



Steady-state geotherm

Heat transfer equation has an analytical solution ...

$$T = (C_a z + C_b)/k$$

$$T = ((58/3000)kz + 20k)/k$$

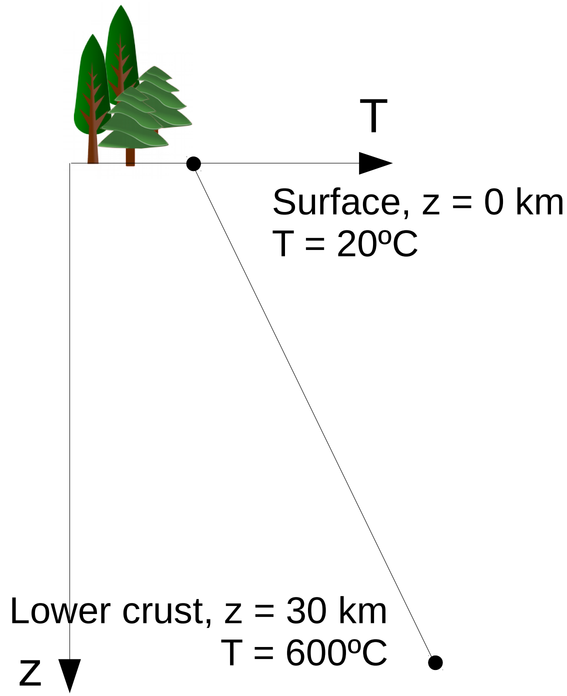
$$T = (58/3000)z + 20$$

NB! No k here

*… given the two **boundary conditions***

$$T = 20 \quad \text{at} \quad z = 0$$

$$T = 600 \quad \text{at} \quad z = 30000$$



Steady-state geotherm with radiogenic heating

$$0 = k\frac{\partial^2 T}{\partial z^2} + A$$

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… integrate twice …

Steady-state geotherm with radiogenic heating

$$0 = k\frac{\partial^2 T}{\partial z^2} + A$$

… integrate twice …

$$T = \left(\frac{1}{2}Az^2 + C_a z + C_b\right)/k$$

Not a straight line

Steady-state geotherm with radiogenic heating

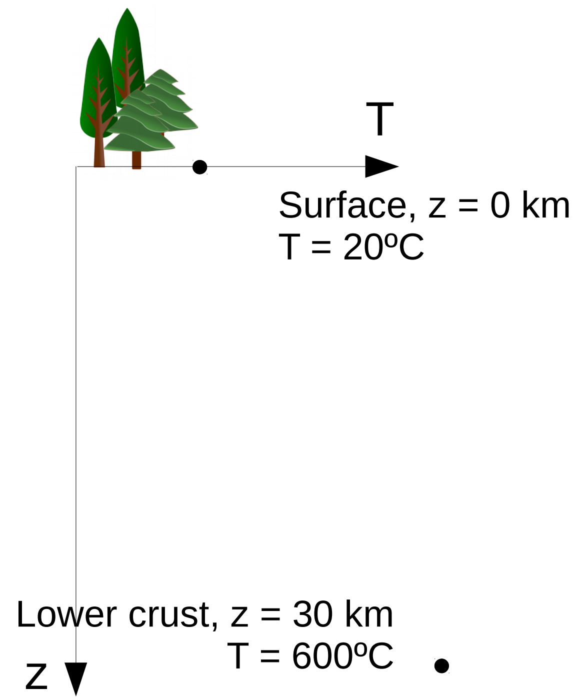
$$0 = k\frac{\partial^2 T}{\partial z^2} + A$$

… integrate twice …

$$T = \left(\frac{1}{2}Az^2 + C_a z + C_b\right)/k$$

Not a straight line

… apply same boundary conditions …



Steady-state geotherm with radiogenic heating

$$0 = k\frac{\partial^2 T}{\partial z^2} + A$$

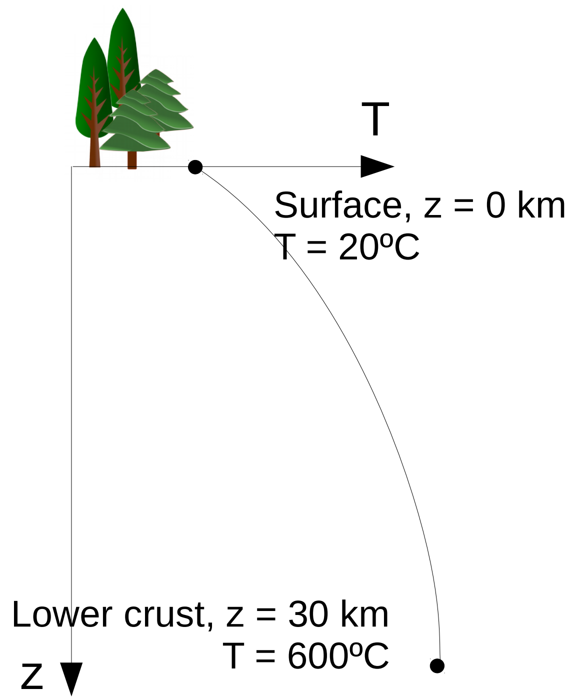
… integrate twice …

$$T = \left(\frac{1}{2}Az^2 + C_a z + C_b\right)/k$$

Not a straight line

… apply same boundary conditions …

$$T = -\frac{1}{2k}Az^2 + \frac{58}{3000}z + \frac{1}{2k}30000Az + 20$$



Mid-lesson wrap-up & play time

- Steady-state geotherm does not depend on density or heat capacity
- Form is dictated by heat conductivity and internal heat production and/or boundary conditions only (in absence of advection…)
  - Geotherm with A=0 – linear, straight line
  - Geotherm with A>0 – curved, parabolic
- Two boundary conditions are needed (for two integration constants)

Heat equation, analytical solutions in python

- Download the course package (ZIP) at https://github.com/HUGG/NGWM2016-modelling-course.git
- Extract it in your home folder
- Change to folder `NGWM2016-modelling-course / Lessons / 03-Analytical-solutions-heat-transfer /scripts`
- Run `plot_steady_state_heat_eq_with_A.py` (double click or run at command prompt)
- Right click > Choose “Edit with IDLE”
- Change values of *k* and *A* and re-run
  - What happens with small / large values of *A*?
  - What happens with small / large values of *k*?

About boundary conditions

- We defined boundary conditions as a constant temperature at given depth
  - This is known as *Dirichlet* boundary condition

$$T = [\mathrm{somevalue}]$$

- Instead of temperature, we can also define *heat flow*, the derivative of *T*, at either (or both) boundary
  - This is known as *von Neumann* boundary condition

$$q = k\frac{\mathrm{d}T}{\mathrm{d}z} = [\mathrm{somevalue}]\ (\mathrm{mW/m^2})$$

Heat flow boundary condition

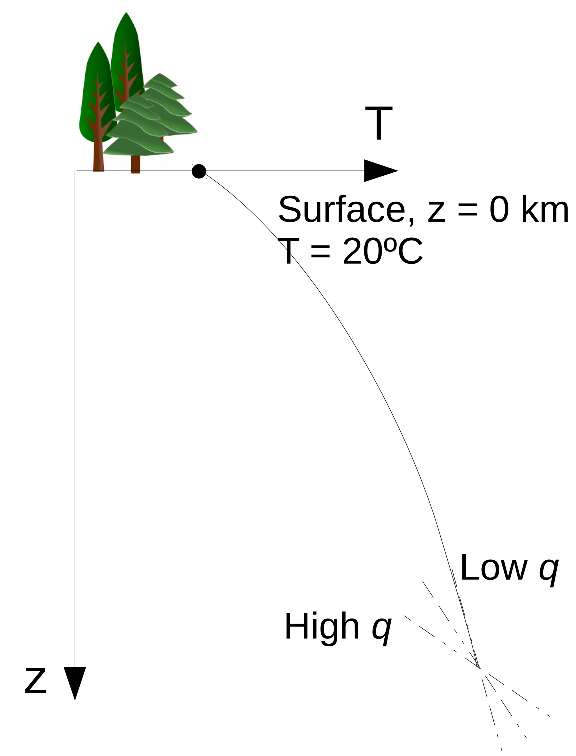
The heat transfer equation has an analytical solution

$$T = (q_b z + 30000Az + 20k - \frac{1}{2}Az^2)/k$$

… given the two boundary conditions

$$T = 20 \quad \text{at} \quad z = 0$$

$$q = k\frac{\mathrm{d}T}{\mathrm{d}z} = q_b \quad \text{at} \quad z = 30000$$



What happens to the geotherm if …

- *A* is increased/decreased?
- q_b is increased/decreased?
- `-> plot_steady_state_heat_eq_vonNeumann.py`

Steady-state heat equation, internal boundary conditions

Despite the name, boundary conditions need not to be applied at the boundary. Consider the general solution to the steady-state heat equation.

$$T = (-\frac{1}{2}Az^2 + C_a z + C_b)/k$$

where

$$C_a = q_0 + Az_1$$
$$C_b = -q_0 z_2 - Az_1 z_2 + kT_0 + \frac{1}{2}Az_2^2$$

with boundary conditions

$$q = q_0 \text{ at } z = z_1$$
$$T = T_0 \text{ at } z = z_2$$

Steady-state heat equation, internal boundary conditions

$$T = (-\frac{1}{2}Az^2 + C_a z + C_b)/k$$

where

$$C_a = q_0 + Az_1$$
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with boundary conditions

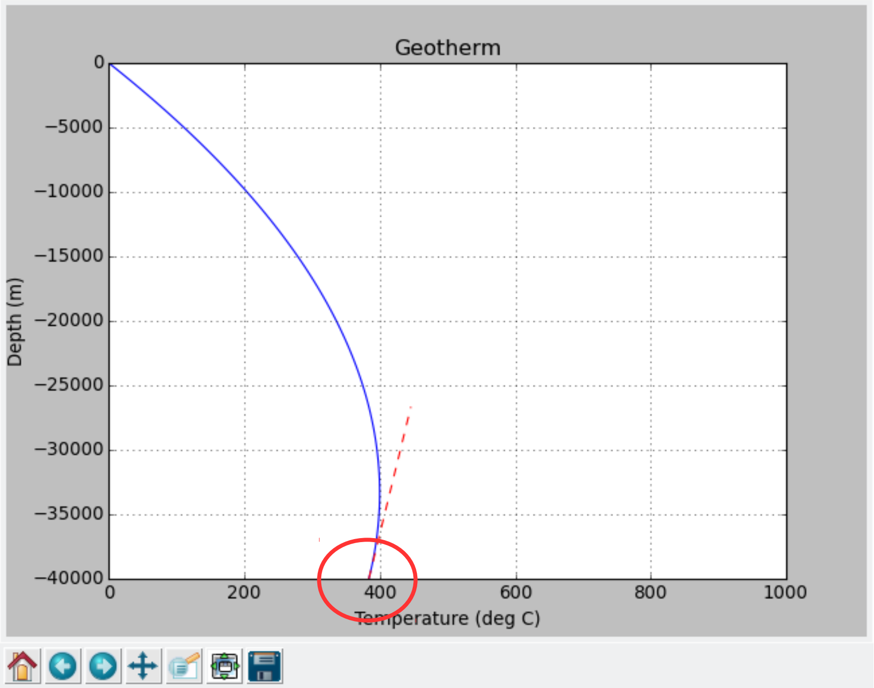
$$q = q_0 \text{ at } z = z_1$$
$$T = T_0 \text{ at } z = z_2$$

For example, we can use measured values of heat flow and temperature at surface (z_1 = 0; z_2 = 0)

Steady-state heat equation, internal boundary conditions

- Open script `plot_steady_state_heat_eq_innerbnd.py`.
- By default this uses boundary conditions T=20 at surface (z=0) and q=15mW/m^2 at the bottom of the crust (z=40 km). These can be varied to different values at different depths.
- Using the script, find out the approximate value for internal heat generation within the crust, if we know that
  - T=0 at surface (measured)
  - q=60mW/m^2 at surface (measured)
  - T~600 at the bottom of the crust (40 km) (estimate / literature)

Steady-state heat equation, internal boundary conditions

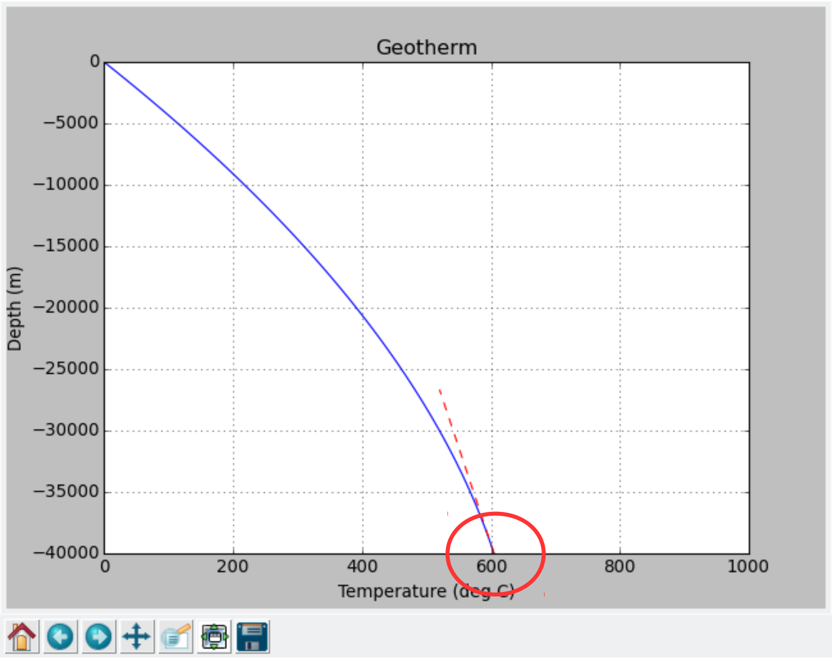


- T=0 at surface
- q=60mW/m² at surface
- A unchanged (1.8µW/m³)

> Do we need to increase or decrease the value of A?

Steady-state heat equation, internal boundary conditions

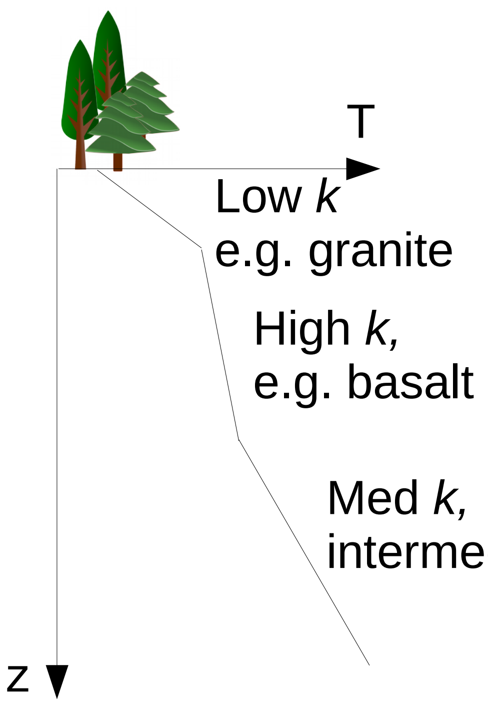
$$q_{surf} = q_{bot} + \text{[heat generated in crust]}$$



➔ To keep the surface heat flow at 60mW/m² and to increase the value of bottom heat flow, heat generation value needs to be *decreased*

Complications that were not considered

- Heat conductivity is not constant throughout the crust (lithological variation and T dependence)
- Heat production is not constant throughout the crust (and is very low in mantle)
  - ➔ Geotherm can be defined piecewise
  - ➔ … or we could use numerical models



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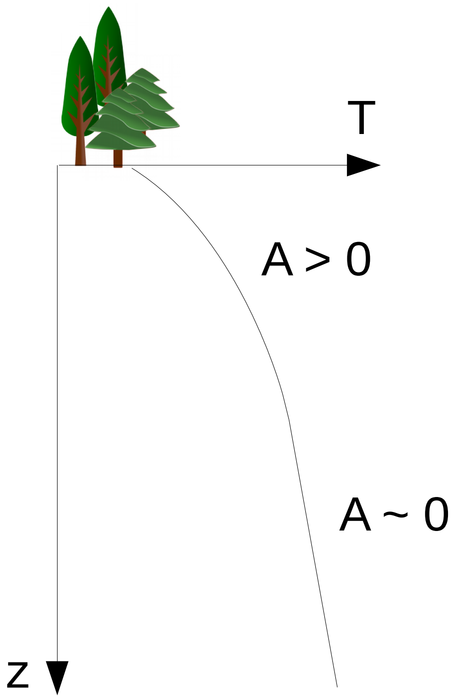

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Non-steady-state heat equation

$$\rho C_p \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial z^2}$$

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial z^2} \qquad \kappa = \frac{k}{\rho C_p}$$

- Need two integrations in respect to *z*, one integration in respect to *t*
  - Three boundary condition values needed
  - Boundary condition in time = initial condition

Non-steady-state heat equation

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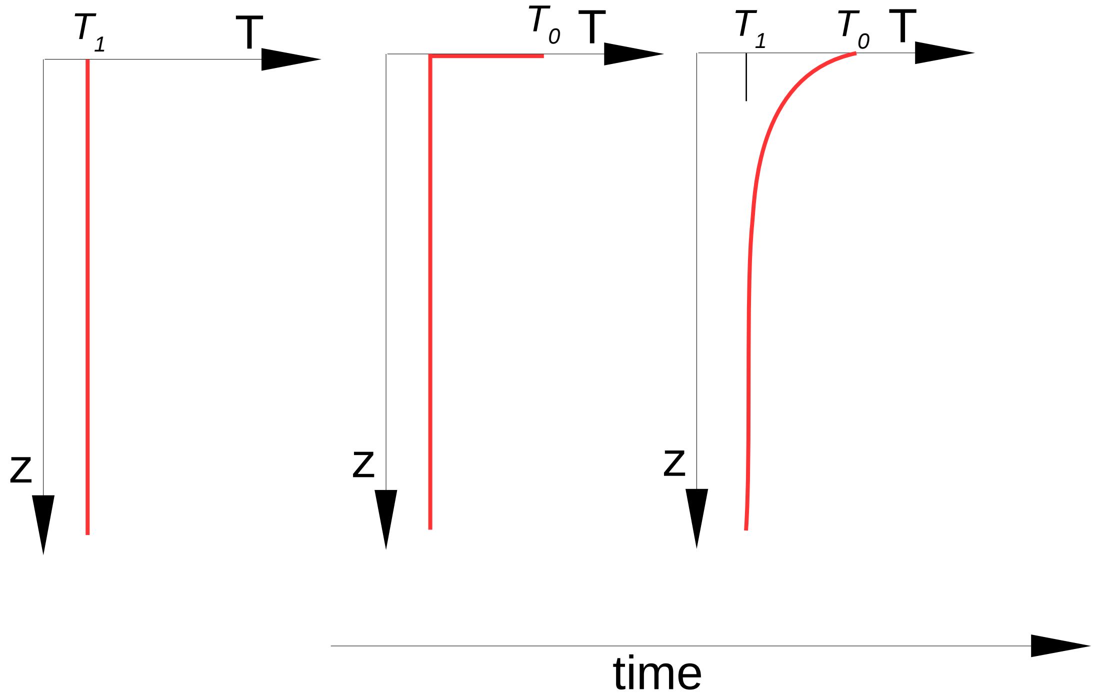
$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial z^2} \qquad \kappa = \frac{k}{\rho C_p}$$

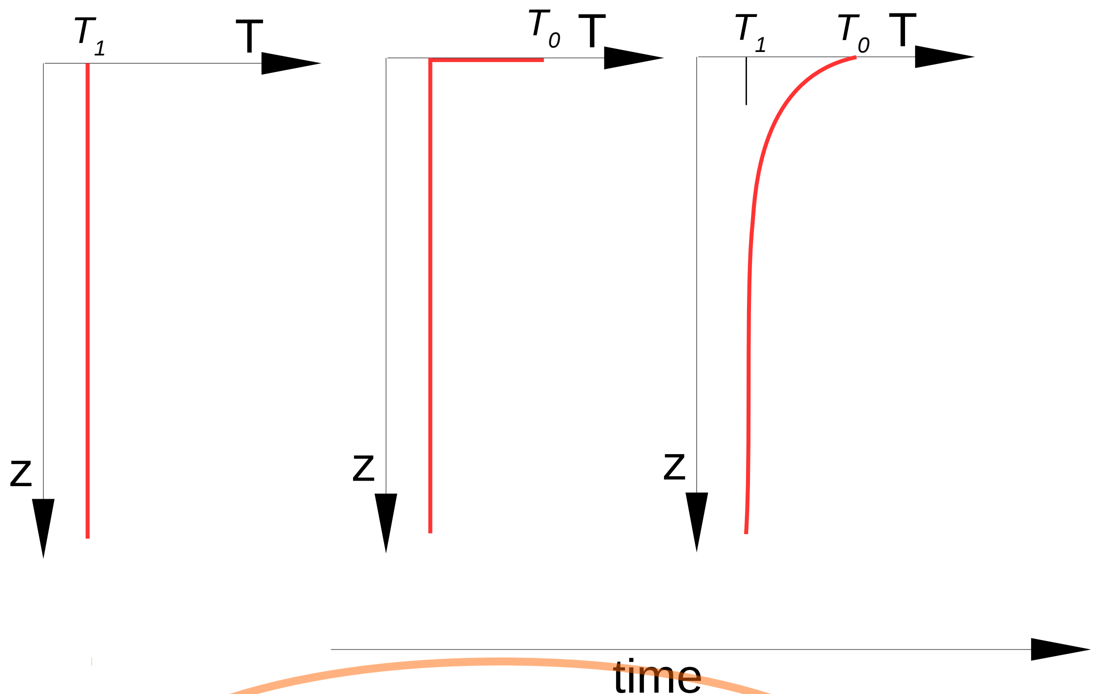
- Need two integrations in respect to *z*, one integration in respect to *t*
  - Not that easy …
- Let's have a look at a *semi-infinite half-space* case
  - Only one (spatial) boundary condition and one (temporal) initial condition

Heat equation in semi-infinite half-space



$$T = T_1 \text{ at } z = \infty \text{ at all times}$$
$$T = T_0 \text{ at } z = 0 \text{ at t} > 0$$

Heat equation in semi-infinite half-space



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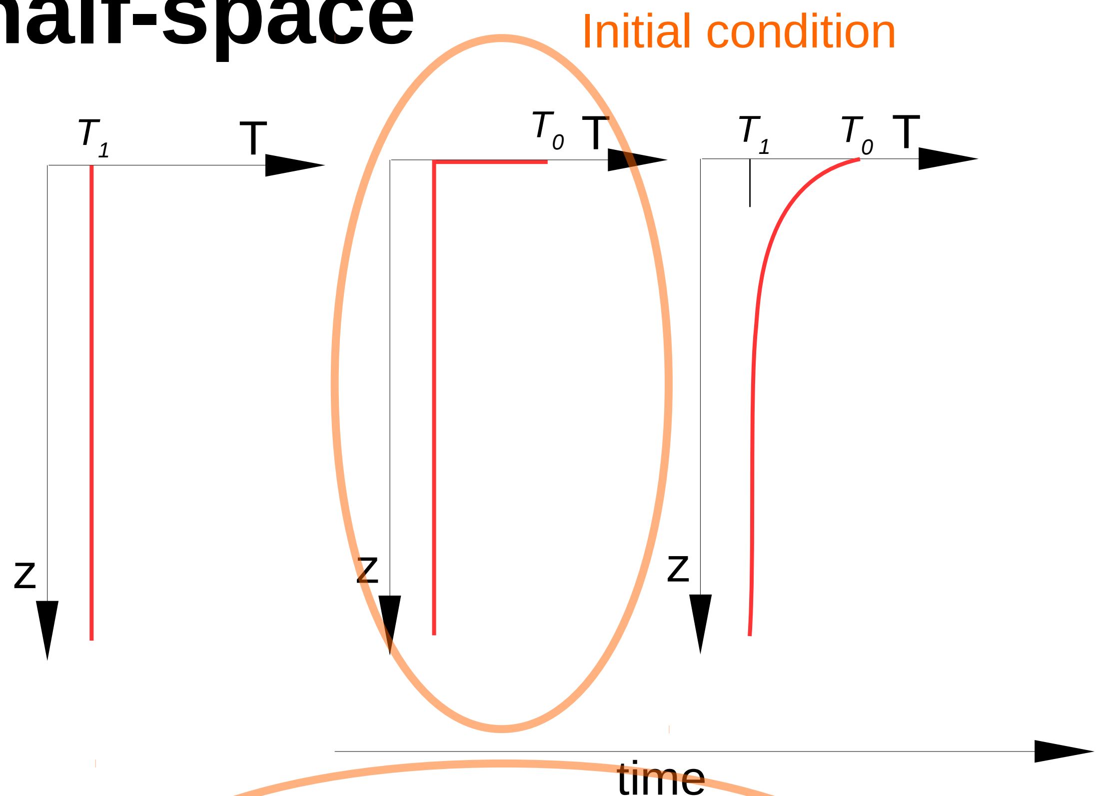
Boundary condition(s)

Heat equation in semi-infinite half-space



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Boundary condition(s)

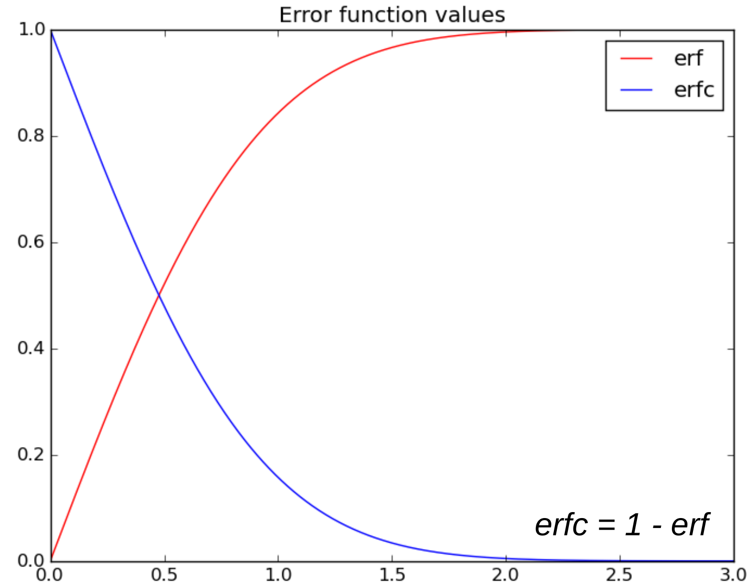
Heat equation in semi-infinite half-space

Solution:

$$\frac{T - T_1}{T_0 - T_1} = \operatorname{erfc} \frac{z}{2\sqrt{\kappa t}}$$

More about transient heat equations later today…

Characteristic thermal diffusion distance
–> how far the heat diffuses in time *t*



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Characteristic thermal diffusion distance
–> how far the heat diffuses in time *t*

“Back of the envelope” calculations:
E.g. is thermal conduction important process in crustal scale in a numerical model that runs for 100 kyrs?

Heat equation in semi-infinite half-space

Solution:

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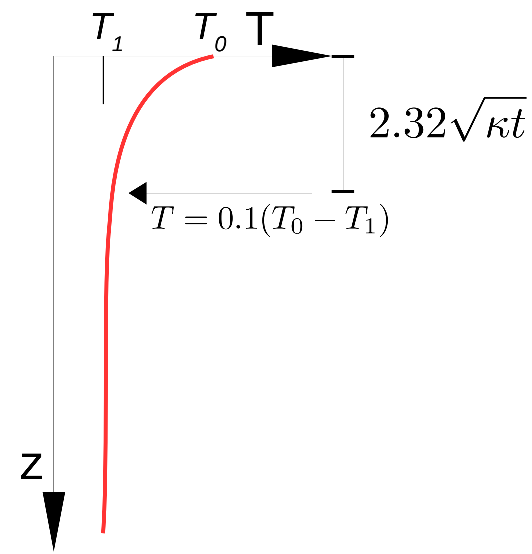
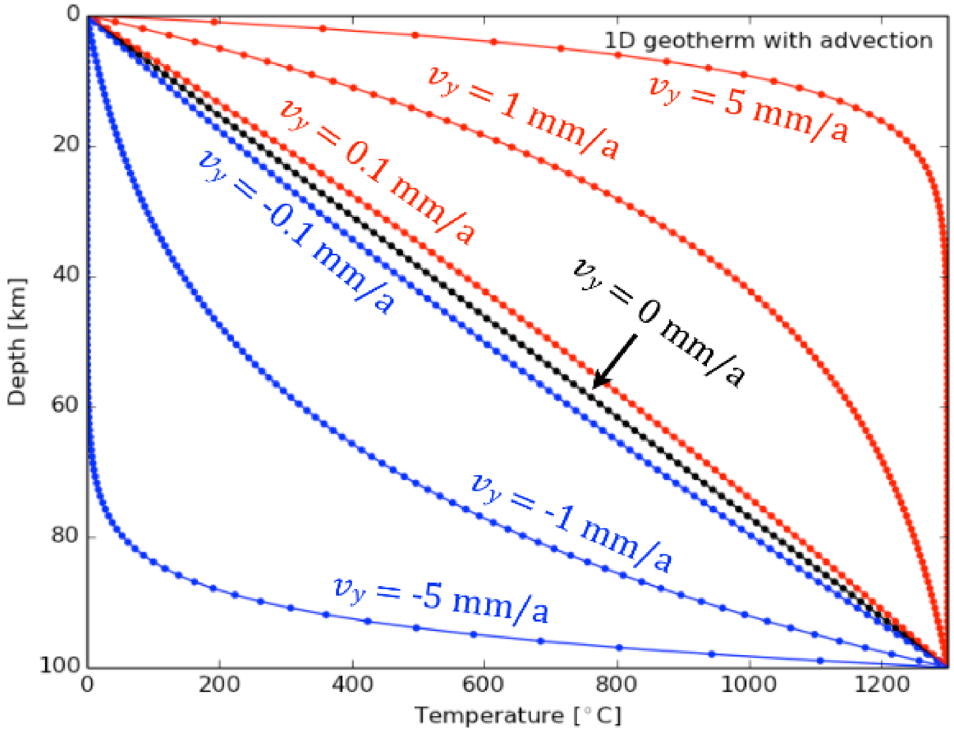


Characteristic thermal diffusion distance
–> how far the heat diffuses in time *t*

"Back of the envelope" calculations:
E.g. is thermal conduction important process in crustal scale in a numerical model that runs for 100 kyrs?

$$2\sqrt{\kappa t} = 2\sqrt{10^{-6} \times 10^{7.5} \times 10^{5}} \approx 3.5 \text{ km}$$

Steady-state heat equation with advection



Steady-state heat equation with advection

$$\rho C_p \left(\cancel{\frac{\partial T}{\partial t}} + u_z \frac{\partial T}{\partial z} \right) = k \frac{\partial^2 T}{\partial z^2} \qquad \kappa = \frac{k}{\rho C_p}$$

$$u_z \frac{\partial T}{\partial z} = \kappa \frac{\partial^2 T}{\partial z^2} \Rightarrow \frac{\mathrm{d}}{\mathrm{d}z} \left(\frac{\mathrm{d}T}{\mathrm{d}z} \right) - \frac{u_z}{\kappa} \frac{\mathrm{d}T}{\mathrm{d}z} = 0 \qquad y = \frac{\mathrm{d}T}{\mathrm{d}z}$$

$$y' + \left(\frac{-u_z}{\kappa} \right) y = 0$$

$$y' + \alpha y = 0 \Rightarrow y = C \mathrm{e}^{-\alpha x} \qquad y = C_1 \mathrm{e}^{\frac{u_z}{\kappa} z}$$

$$\frac{\mathrm{d}T}{\mathrm{d}z} = C_1 \mathrm{e}^{\frac{u_z}{\kappa} z}$$

Steady-state heat equation with advection

$$\int \frac{\mathrm{d}T}{\mathrm{d}z}\mathrm{d}z = \int C_1 \mathrm{e}^{\frac{u_z}{\kappa}z}\mathrm{d}z$$

$$T = \frac{C_1\kappa}{u_z}\mathrm{e}^{\frac{u_z}{\kappa}z} + C_2$$

$T = 0$ at $z = 0$ $\longrightarrow$ $C_2 = -\dfrac{C_1\kappa}{u_z}$

$T = T_{\mathrm{bott}}$ at $z = L$ $\longrightarrow$ $C_1 = \dfrac{T_{\mathrm{bott}}u_z}{\kappa}\Big/\left(\mathrm{e}^{-\frac{u_z}{\kappa}L} - 1\right)$

$$T = T_{\mathrm{bott}}\frac{\mathrm{e}^{\frac{u_z}{\kappa}z} - 1}{\mathrm{e}^{\frac{u_z}{\kappa}L} - 1}$$

Steady-state heat equation with advection

$$\int \frac{\mathrm{d}T}{\mathrm{d}z}\mathrm{d}z = \int C_1 \mathrm{e}^{\frac{u_z}{\kappa}z}\mathrm{d}z$$

$$T = \frac{C_1\kappa}{u_z}\mathrm{e}^{\frac{u_z}{\kappa}z} + C_2$$

$T = 0$ at $z = 0$ ⟶ $C_2 = -\frac{C_1\kappa}{u_z}$

$T = T_{\mathrm{bott}}$ at $z = L$ ⟶ $C_1 = \frac{T_{\mathrm{bott}} u_z}{\kappa} \Big/ \left(\mathrm{e}^{-\frac{u_z}{\kappa}L} - 1\right)$

$$T = T_{\mathrm{bott}} \frac{\mathrm{e}^{\frac{u_z}{\kappa}z} - 1}{\mathrm{e}^{\frac{u_z}{\kappa}L} - 1} \qquad \kappa = \frac{k}{\rho C_p}$$

-> plot_steady_state_heat_eq_advection.py

Steady-state heat equation with advection

`plot_steady_state_heat_eq_advection.py`

Default case: u_z = 5 mm/a (sedimentation)

If sedimentation rate decreases to 2 mm/a, does heat conductivity need to be
  a) increased or
  b) decreased
to maintain the same geotherm?

Steady-state heat equation with advection

```
plot_steady_state_heat_eq_advection.py
```

Default case: u_z = 5 mm/a (sedimentation)

If sedimentation rate decreases to 2 mm/a, does heat conductivity need to be
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$$T = T_{\mathrm{bott}} \frac{\mathrm{e}^{\frac{u_z}{\kappa} z} - 1}{\mathrm{e}^{\frac{u_z}{\kappa} L} - 1} \qquad \kappa = \frac{k}{\rho C_p}$$

Steady-state heat equation with advection

`plot_steady_state_heat_eq_advection.py`

Default case: u_z = 5 mm/a (sedimentation)

If sedimentation rate decreases to 2 mm/a, does heat conductivity need to be
a) increased or
b) decreased
to maintain the same geotherm?

$$\frac{\mathrm{d}T}{\mathrm{d}z} = C_1 \mathrm{e}^{\frac{u_z}{\kappa} z}$$

$$T = T_{\mathrm{bott}} \frac{\mathrm{e}^{\frac{u_z}{\kappa} z} - 1}{\mathrm{e}^{\frac{u_z}{\kappa} L} - 1}$$

$$\kappa = \frac{k}{\rho C_p}$$

Peclet number

- Non-dimensional number describing the relevance of diffusive and advective heat (or mass) transfer processes

$$\mathrm{Pe} = \frac{Lu}{\kappa}$$

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- Non-dimensional number describing the relevance of diffusive and advective heat (or mass) transfer processes

$$\mathrm{Pe} = \frac{Lu}{\kappa}$$

$$T = T_{\mathrm{bott}} \frac{\mathrm{e}^{\frac{u_z}{\kappa} z} - 1}{\mathrm{e}^{\frac{u_z}{\kappa} L} - 1}$$

```
-> plot_steady_state_heat_eq_advection.py
```

- What's the effect of u_z if k has a really high value?