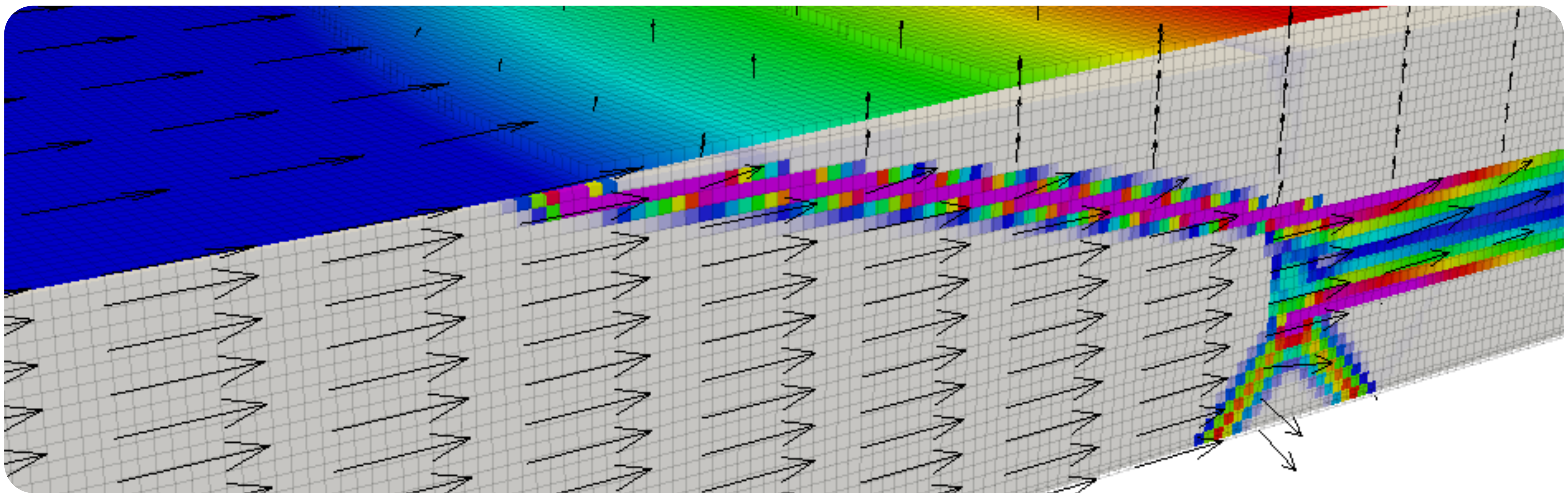
Introduction to lithospheric geodynamic modelling

Basic fluid mechanics

David Whipp and Lars Kaislaniemi
Department of Geosciences and Geography, Univ. Helsinki

Nordic Geological Winter Meeting 2016

Why fluid mechanics?



Velocities and strain rates in a lithospheric geodynamic model

- Most geodynamic models treat the Earth as a **continuum** such that there are no material gaps or voids at the macroscopic scale
  - Field variables such as pressure, velocity or stress are thus fully continuous
- In this context the Earth is a fluid with a very high viscosity (typically 10^{18} - 10^{23} Pa s)

Fluids and the Earth

- **Fluid**: Any material that flows in response to an applied stress

- Differences between **solids** and **fluids**

Solids	Fluids
Strain from being stressed	**Continuous** deformation under applied forces
Stresses related to strains	Stresses related to **rates of strain**
Strain result of displacement gradients	Strain result of **velocity gradients**

- **Rheological** (or **constitutive**) **law**: An equation relating stress to strain rates in a fluid

Fluid mechanics

- **Fluid mechanics** is the science of fluid motion

Fluid mechanics

- **Fluid mechanics** is the science of fluid motion

- Based on conservation of three basic physical property and their corresponding mathematical representations
  - **Conservation of mass** - The continuity equation
  - **Conservation of momentum** - The momentum equation
  - **Conservation of energy** - The heat transfer equation

Fluid mechanics

- **Fluid mechanics** is the science of fluid motion

- Based on conservation of three basic physical property and their corresponding mathematical representations
  - **Conservation of mass** - The continuity equation
  - **Conservation of momentum** - The momentum equation
  - **Conservation of energy** - The heat transfer equation

- Conservation of mass, momentum and energy are combined with rheological laws to <u>describe fluid movement under an applied force</u>

Roadmap

- **Fundamental equations** governing fluid flow
- Calculation of fluid flow velocities/patterns for **linear viscous materials**

Fluid mechanics

- **Fluid mechanics** is the science of fluid motion

- Based on conservation of three basic physical property and their corresponding mathematical representations
  - **Conservation of mass** - The continuity equation
  - **Conservation of momentum** - The momentum equation
  - **Conservation of energy** - The heat transfer equation

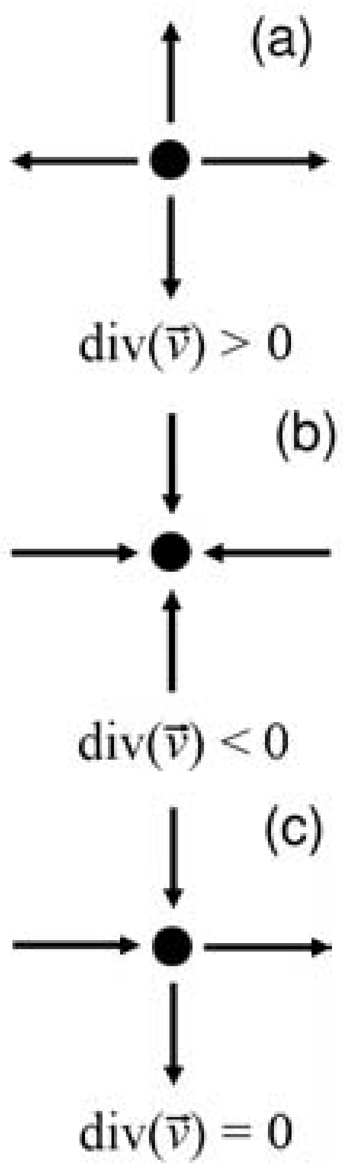
Fluid mechanics

- **Fluid mechanics** is the science of fluid motion

- Based on conservation of three basic physical property and their corresponding mathematical representations
  - **Conservation of mass** - The continuity equation
  - **Conservation of momentum** - The momentum equation

Covered in lectures 2-3 →
  - ~~Conservation of energy - The heat transfer equation~~

Conservation of mass - Continuity equation



Gerya, 2010

- Calculations in the continuum are performed by considering an infinitesimal volume of the material, the local volume
- The general form of conservation of mass for a local volume of a continuum in an Eulerian reference frame is

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{V}) = 0$$

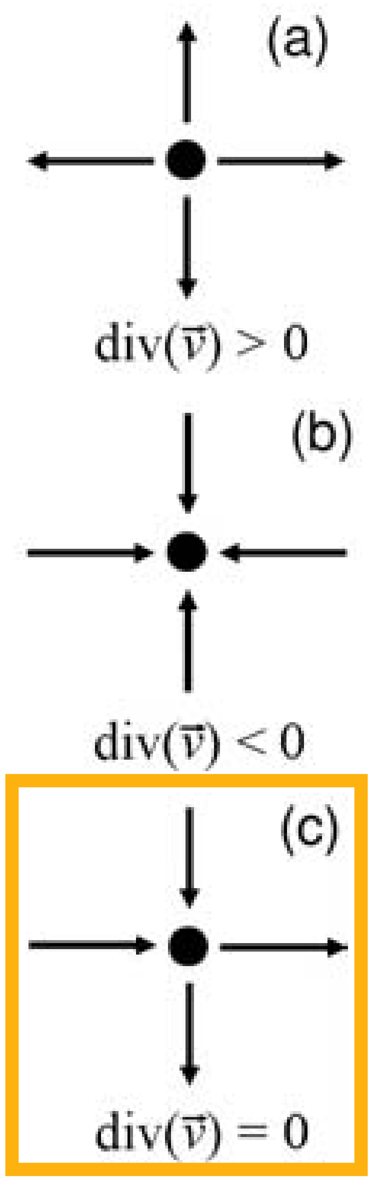
Change in local density

Mass or volume flux (divergence of velocity)

where ρ is the local density, t is time and $\boldsymbol{V}$ is the local velocity

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \boldsymbol{V}) \quad \text{Alternative form}$$

Conservation of mass - Continuity equation



Gerya, 2010

- It is common in geodynamic numerical models, particularly in the crust or lithosphere, to assume the material is incompressible
- In this case, the continuity equation simplifies to

$$\nabla \cdot \boldsymbol{V} = 0$$

stating simply that there is no divergence in the velocity field of the continuum

- In many numerical models, this condition is not strictly obeyed, allowing a very small amount of compressibility in the materials

Fluid mechanics

- Fluid mechanics is the science of fluid motion
- Based on conservation of three basic physical property and their corresponding mathematical representations
  - Conservation of mass - The continuity equation
  - Conservation of momentum - The momentum equation
  - Conservation of energy - The heat transfer equation
- Conservation of mass, momentum and energy are combined with rheological laws to describe fluid movement under an applied force

Fluid mechanics

- Fluid mechanics is the science of fluid motion

- Based on conservation of three basic physical property and their corresponding mathematical representations
  - Conservation of mass - The continuity equation
  - Conservation of momentum - The momentum equation
  - Conservation of energy - The heat transfer equation

- Conservation of mass, momentum and energy are combined with rheological laws to describe fluid movement under an applied force

What kind of forces might we expect to have acting on a fluid?

Conservation of momentum - Momentum eq.

Sir George Stokes

- The basic relationship that thus determines the dynamics of material in the continuum is conservation of momentum, the balance of internal and external forces acting on the material
- The **conservation of momentum** for a fluid subject to gravity is the Navier-Stokes equation

$$\nabla \cdot \eta(\nabla \boldsymbol{V} + \nabla \boldsymbol{V}^{\mathrm{T}}) - \nabla P - \rho g = \rho \dot{\boldsymbol{V}}$$

Fluid velocity   Fluid pressure   Body forces   Acceleration

where η is the fluid shear viscosity, P is pressure, g is the acceleration due to gravity, and $\dot{V}$ is the material time derivative of the fluid velocity (acceleration)

Conservation of momentum - Momentum eq.

Sir George Stokes

- For highly viscous fluids with a very small Reynolds number the acceleration term of the Navier-Stokes equation can be ignored reducing to the equation of **Stokes flow** (and simplifying the solutions)

$$\nabla \cdot \eta(\nabla \boldsymbol{V} + \nabla \boldsymbol{V}^{\mathrm{T}}) - \nabla P = \rho g$$

Fluid velocity Fluid pressure Body forces

- It is trivial to demonstrate that the Reynolds number of most geodynamic flows is extremely low (~10^{-20})

$$\mathbf{Re} = \frac{\rho \boldsymbol{V} L}{\eta} \qquad \frac{\text{Inertial forces}}{\text{Viscous forces}}$$

The Reynolds number

Roadmap

- **Fundamental equations** governing fluid flow
- Calculation of fluid flow velocities/patterns for **linear viscous materials**

Viscous flow - Newtonian (or linear) fluid

- A **Newtonian fluid** is a fluid in which there is a linear relationship between the rate of strain and the applied stress
    - **What would this relationship look like as an equation?**

Viscous flow - Newtonian (or linear) fluid

Material	Approximate Viscosity [Pa s]
Air	1×10^{-5}
Water	1×10^{-3}
Ice	1×10^{16}
Rock Salt	1×10^{17}
Granite	1×10^{20}

- A **Newtonian fluid** is a fluid in which there is a linear relationship between the rate of strain and the applied stress
  - What would this relationship look like as an equation?

$$\sigma \propto \dot{\varepsilon} \quad \text{or} \quad \sigma = \eta \dot{\varepsilon}$$

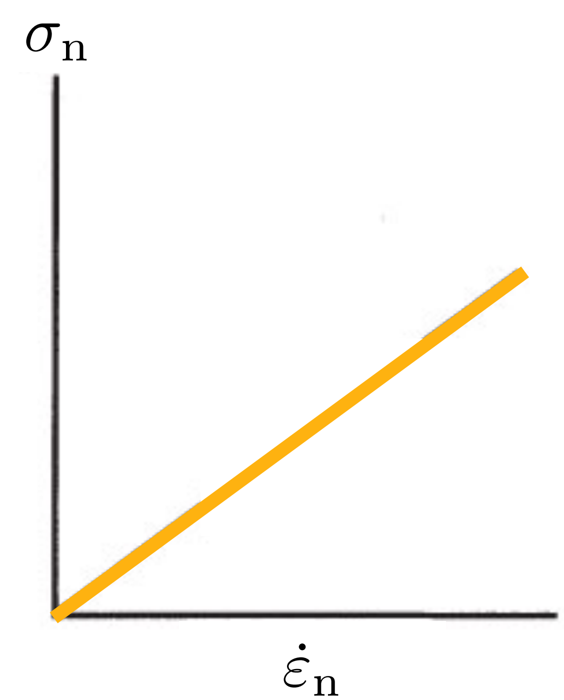
- The proportionality constant η is known as the **dynamic** (or **shear**) **viscosity**
  - Dynamic viscosity has units of [Pa s]

Viscous flow - (Linear) Viscous deformation

- In simple shear,

  $\tau_s = \eta \dot{\varepsilon}_s$ $\quad\quad$ η Dynamic viscosity

  Shear stress proportional to **shear strain rate**



A.

Twiss and Moores, 2007

Viscous flow - (Linear) Viscous deformation

- In simple shear,

  $\tau_s = \eta \dot{\varepsilon}_s$ η Dynamic viscosity

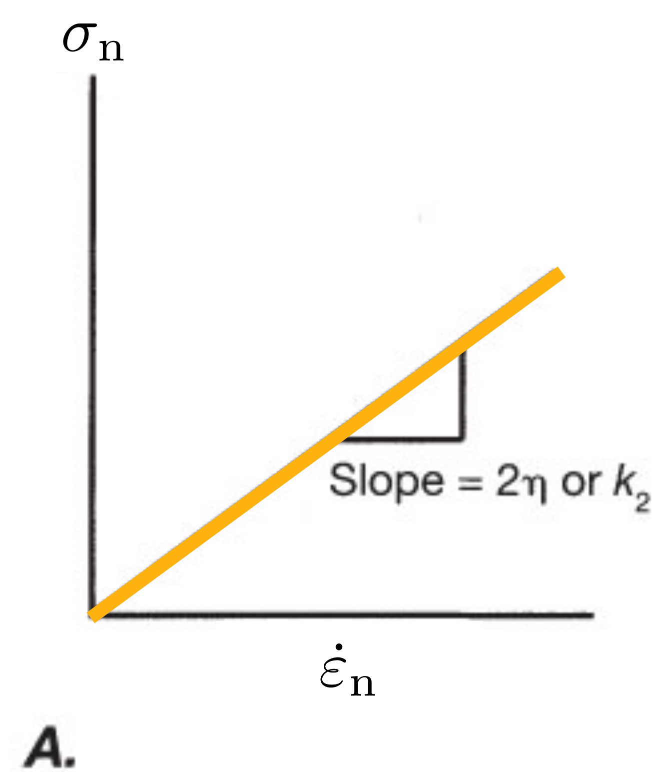
  Shear stress proportional to **shear strain rate**

- In general,

  $\sigma' = 2\eta\dot{\varepsilon}$

  deviatoric stress is proportional to **strain rate**

- For linear viscous (Newtonian) materials, η is constant



Twiss and Moores, 2007

Viscous flow - (Linear) Viscous deformation

- In simple shear,

  $$\tau_s = \eta \dot{\varepsilon}_s \qquad \eta \text{ Dynamic viscosity}$$

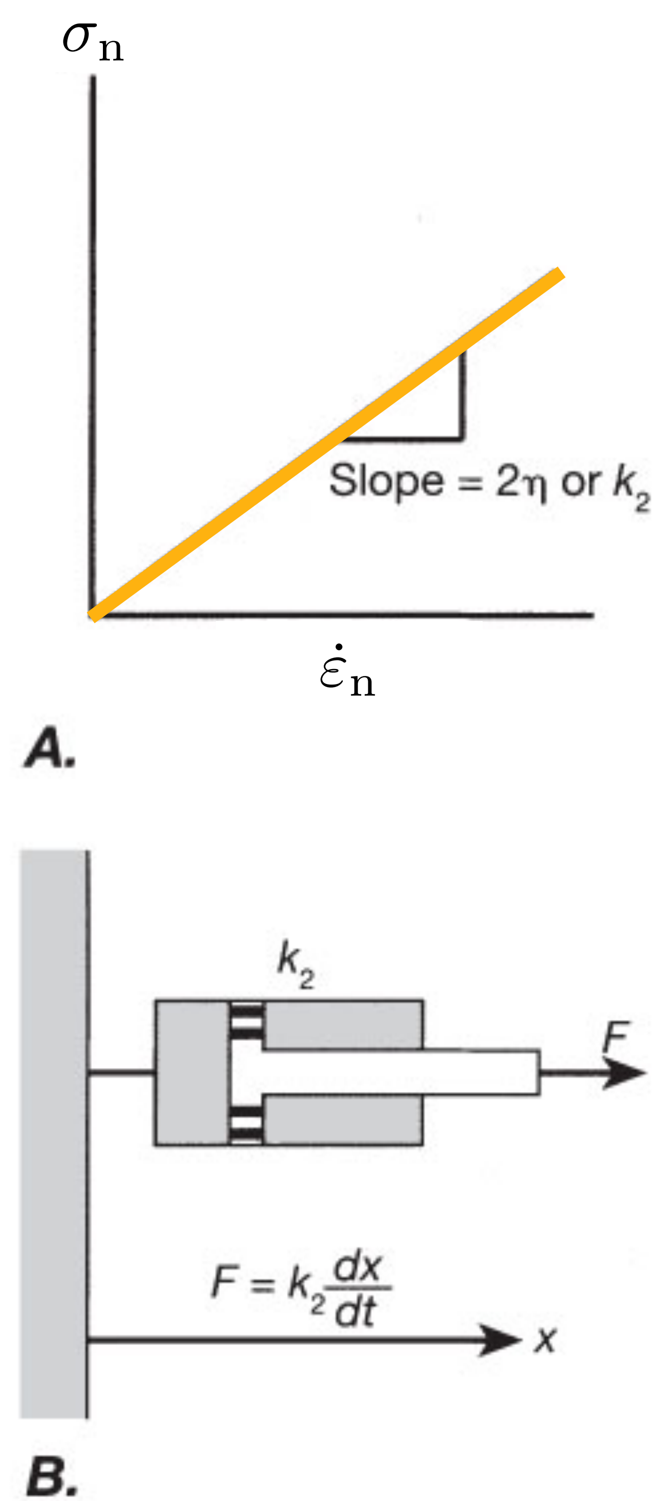
  Shear stress proportional to **shear strain rate**

- In general,

  $$\sigma' = 2\eta\dot{\varepsilon}$$

  deviatoric stress is proportional to **strain rate**

- For linear viscous (Newtonian) materials, η is constant



Twiss and Moores, 2007

Viscous flow - (Linear) Viscous deformation

- In simple shear,

  $\tau_s = \eta \dot{\varepsilon}_s$ η Dynamic viscosity

  Shear stress proportional to **shear strain rate**

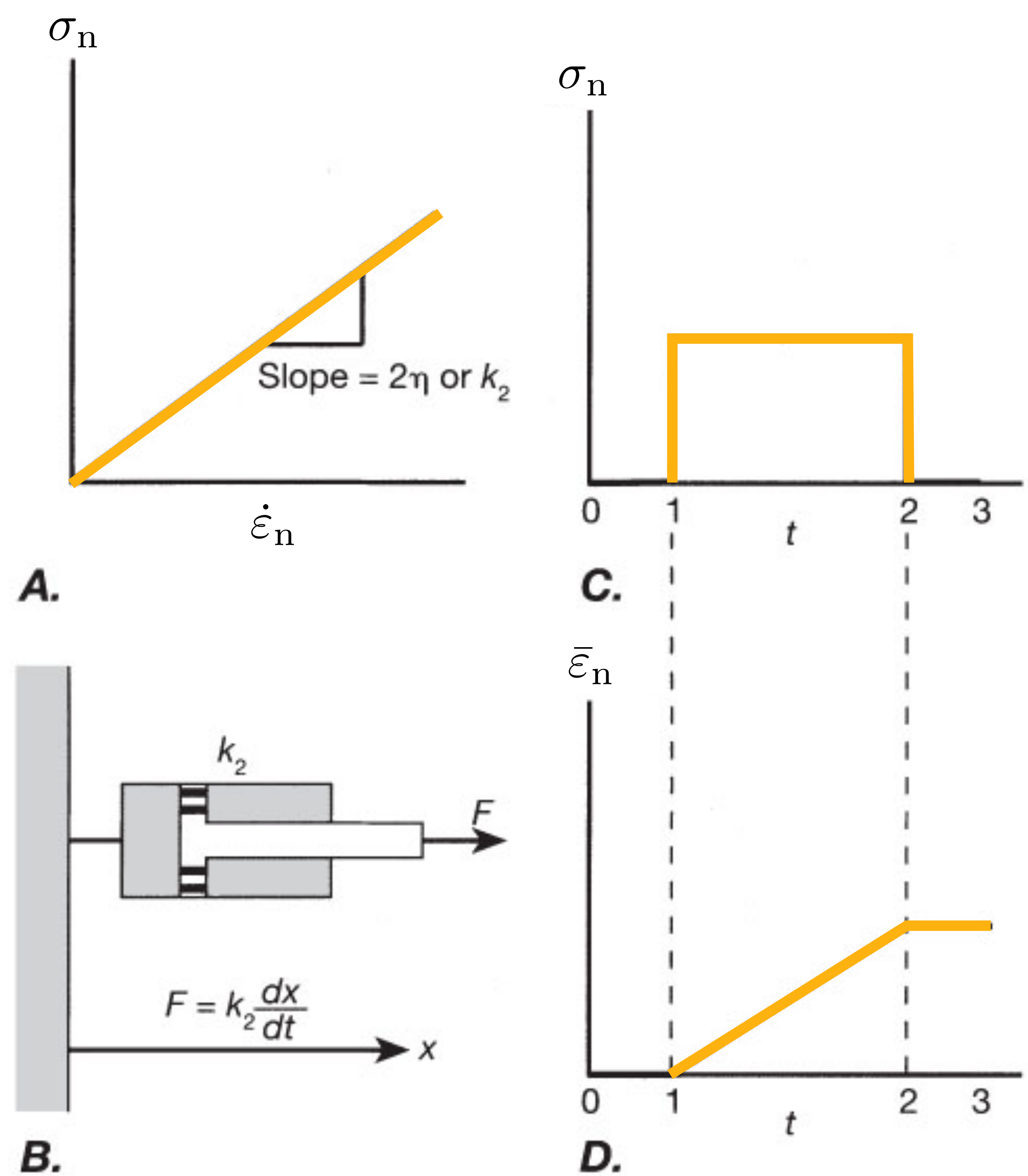
- In general,

  $\sigma' = 2\eta\dot{\varepsilon}$

  deviatoric stress is proportional to **strain rate**

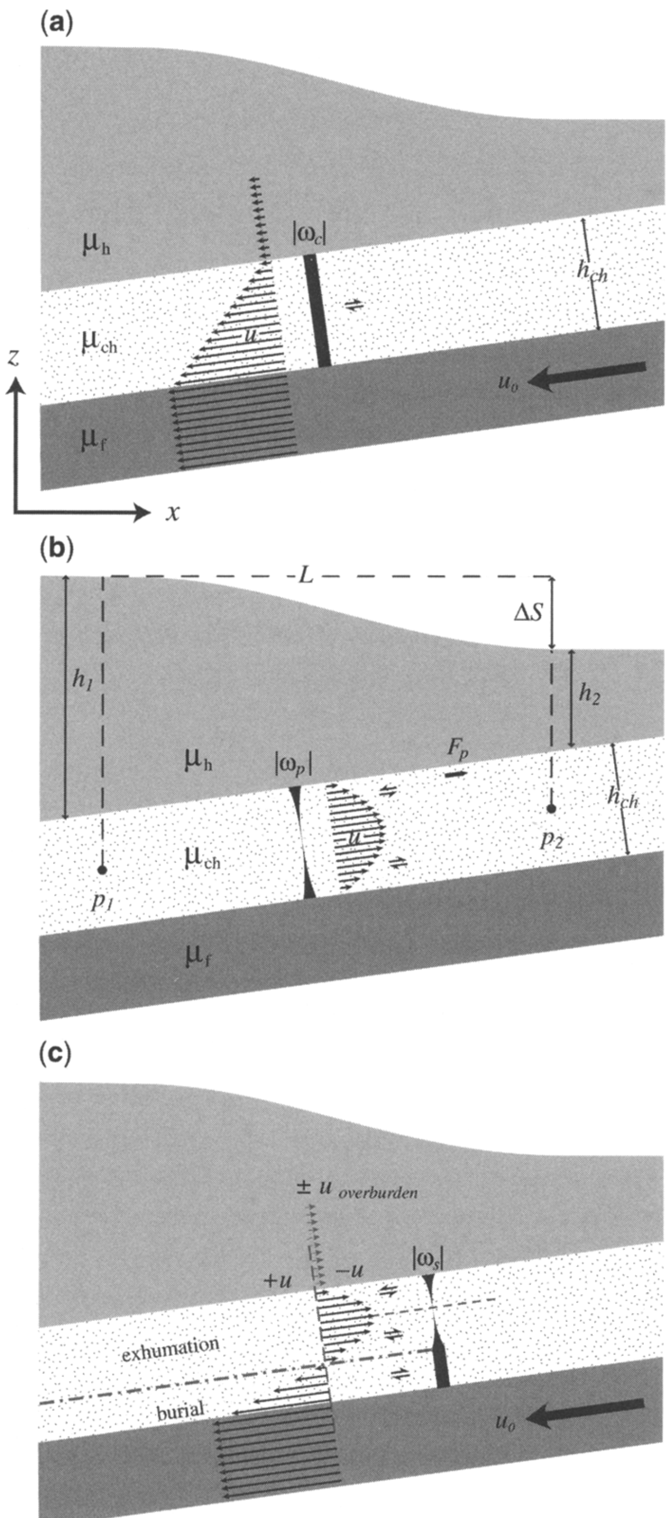
- For linear viscous (Newtonian) materials, η is constant

- Nonrecoverable



Twiss and Moores, 2007

Channel flows in the Earth

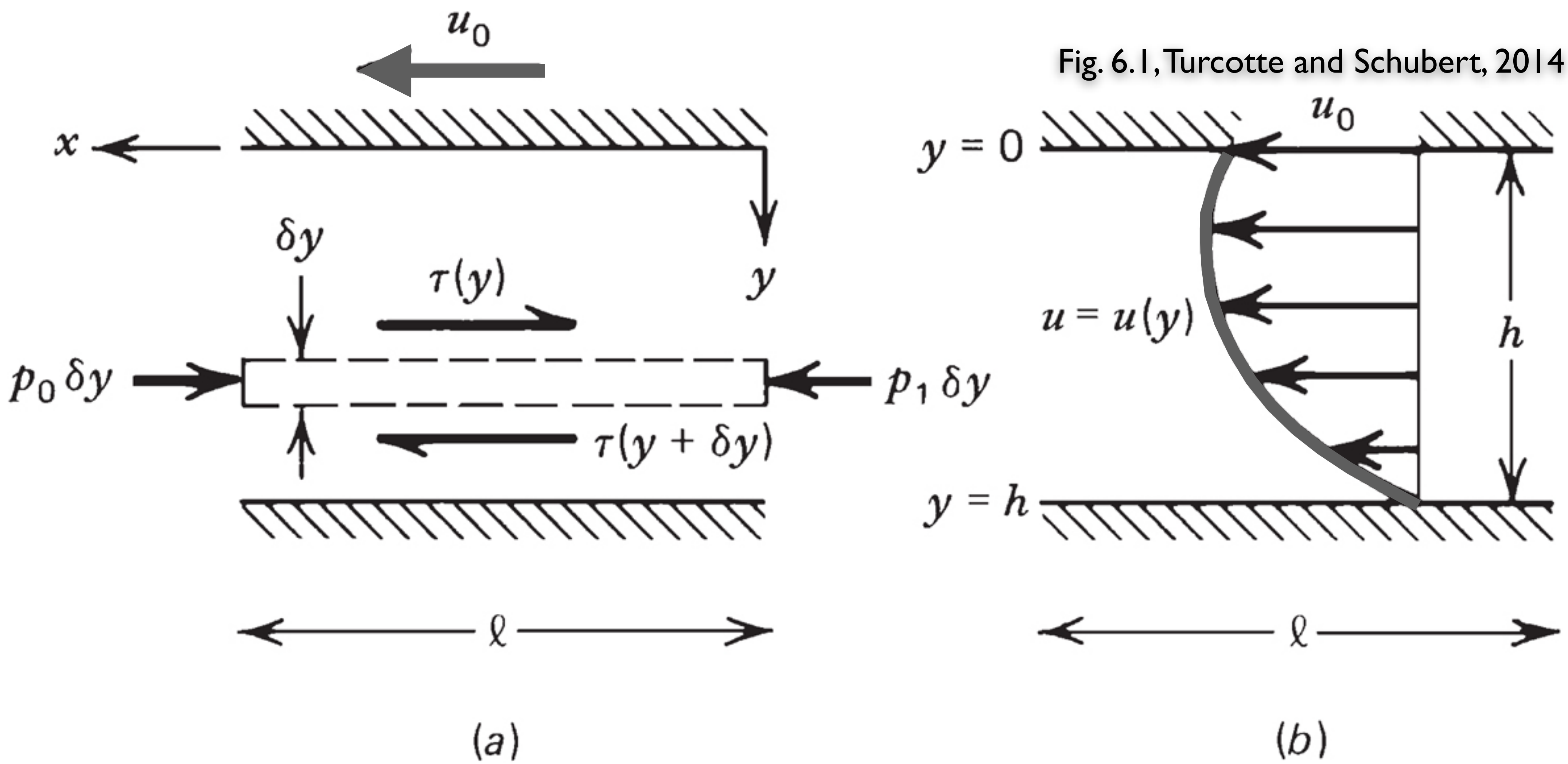


Grujic, 2006

- **Channel flows** in the Earth occur when a fluid flows within a channel, between two solid "walls"
- Such channels can be found in a number of geological settings:
    - Counterflow in the asthenosphere
    - Lower crustal flow
    - Intra-crustal channels (figure on left)
    - Subduction channels
    - Salt tectonic channels

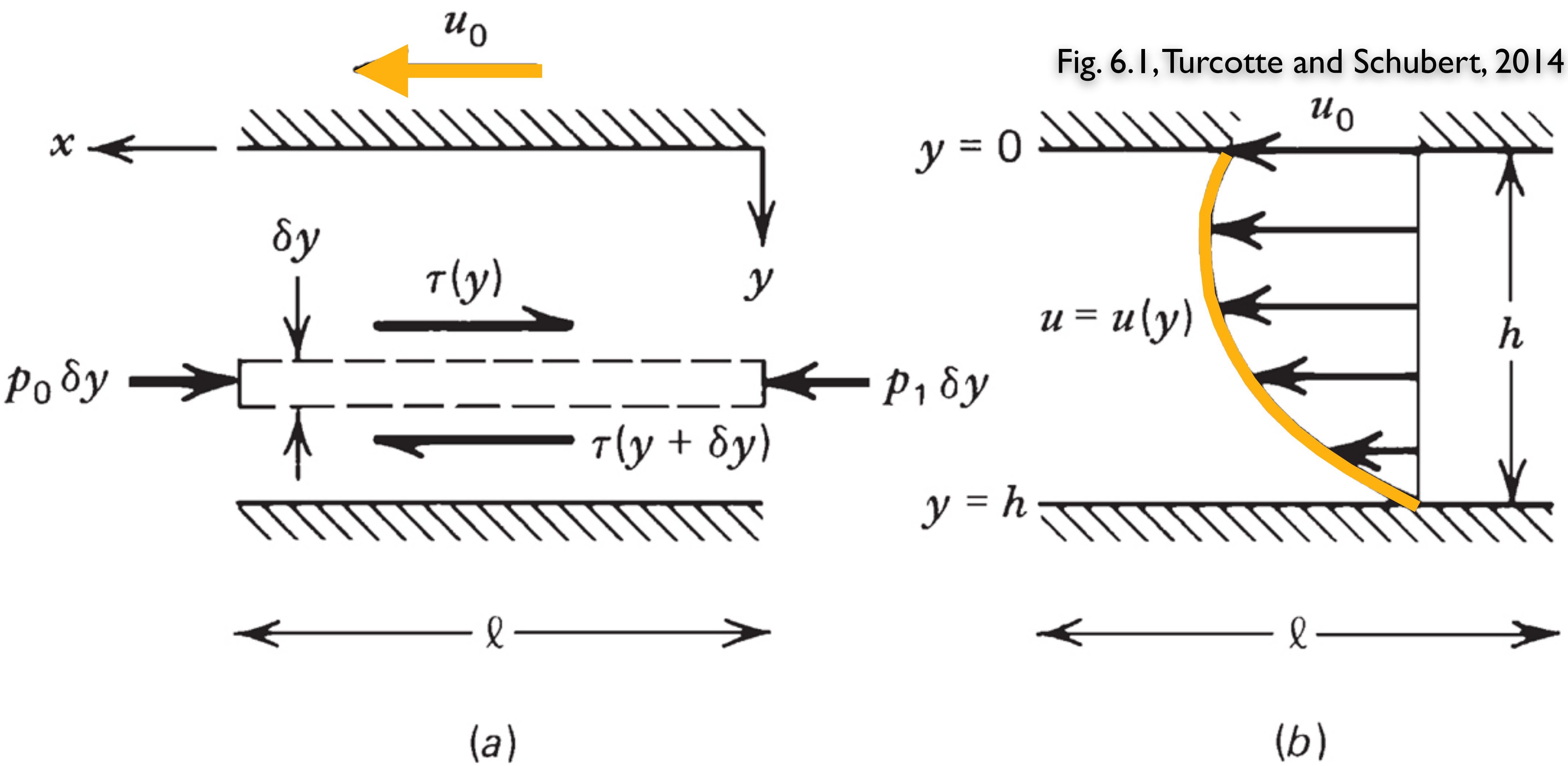
1D channel flows

Fig. 6.1, Turcotte and Schubert, 2014



- The most simple fluid flow we can consider is <u>flow of a fluid in one direction within a channel of fixed width</u>

1D channel flows



- Fluid is flowing with velocity u in the x direction, and the flow velocity u is a function of distance across the channel y
- Flow results from
  - a **pressure gradient** $(p_0 - p_1)/l$, and/or
  - **motion of the side wall** of the channel u_0

1D channel flows

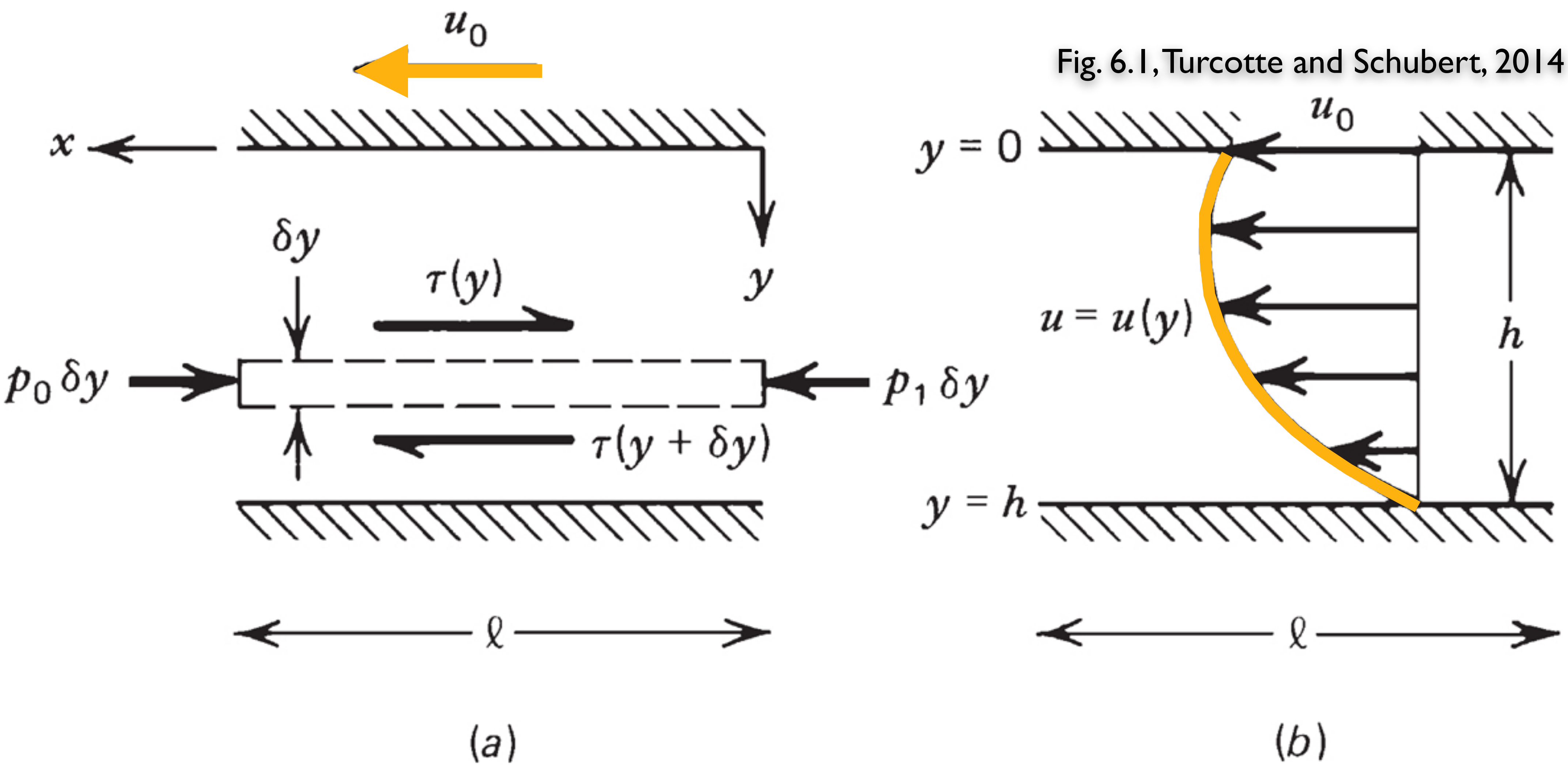
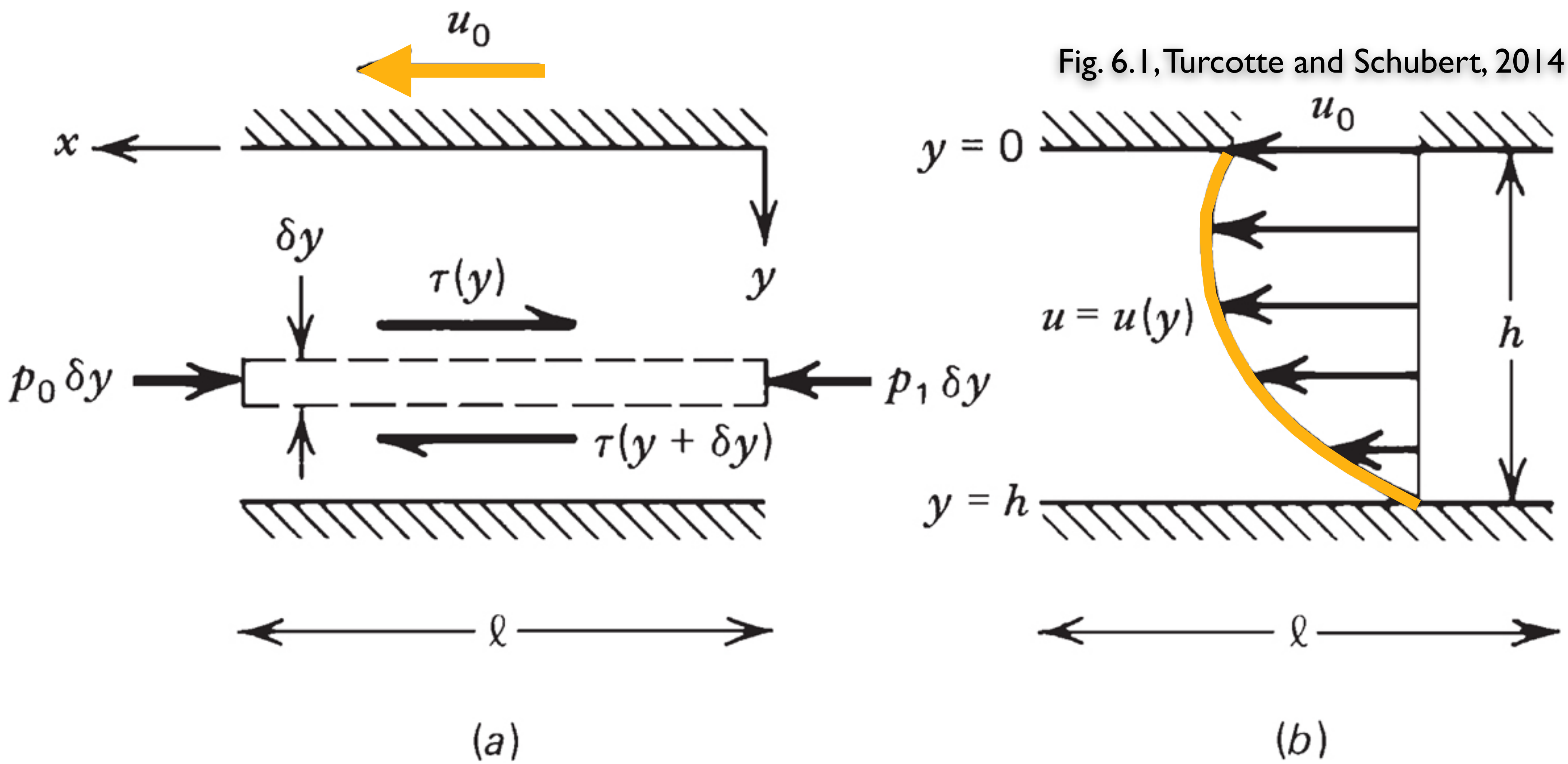


Fig. 6.1, Turcotte and Schubert, 2014

- **Shear**, or a gradient in the velocity, in the channel results in a **shear stress** τ that is exerted on horizontal planes in the fluid
- For a Newtonian fluid with a constant **dynamic viscosity** η we can state

$$\tau = \eta \frac{du}{dy}$$

1D channel flows



- **Shear**, or a gradient in the velocity, in the channel results in a **shear stress** τ that is exerted on horizontal planes in the fluid
- For a Newtonian fluid with a constant **dynamic viscosity** η we can state

$$\tau = \eta \frac{du}{dy}$$

stress

strain rate

1D channel flows

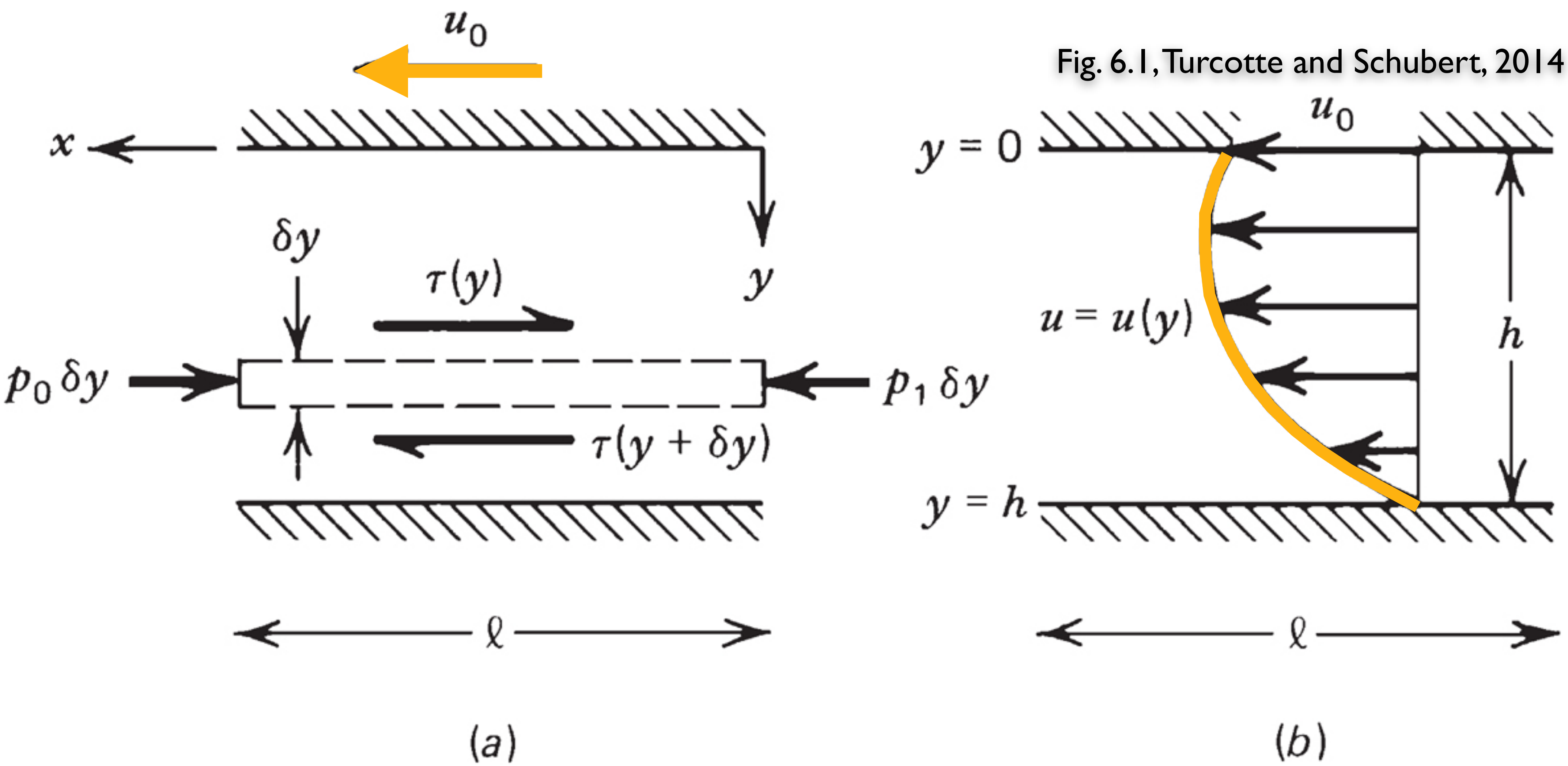


Fig. 6.1, Turcotte and Schubert, 2014

- We can now determine the flow in the channel using the **equation of motion**, based on the force balance on a layer of fluid of thickness δy and length l
- The net **pressure force** on the element in the x direction is

$$(p_1 - p_0)\delta y$$

1D channel flows

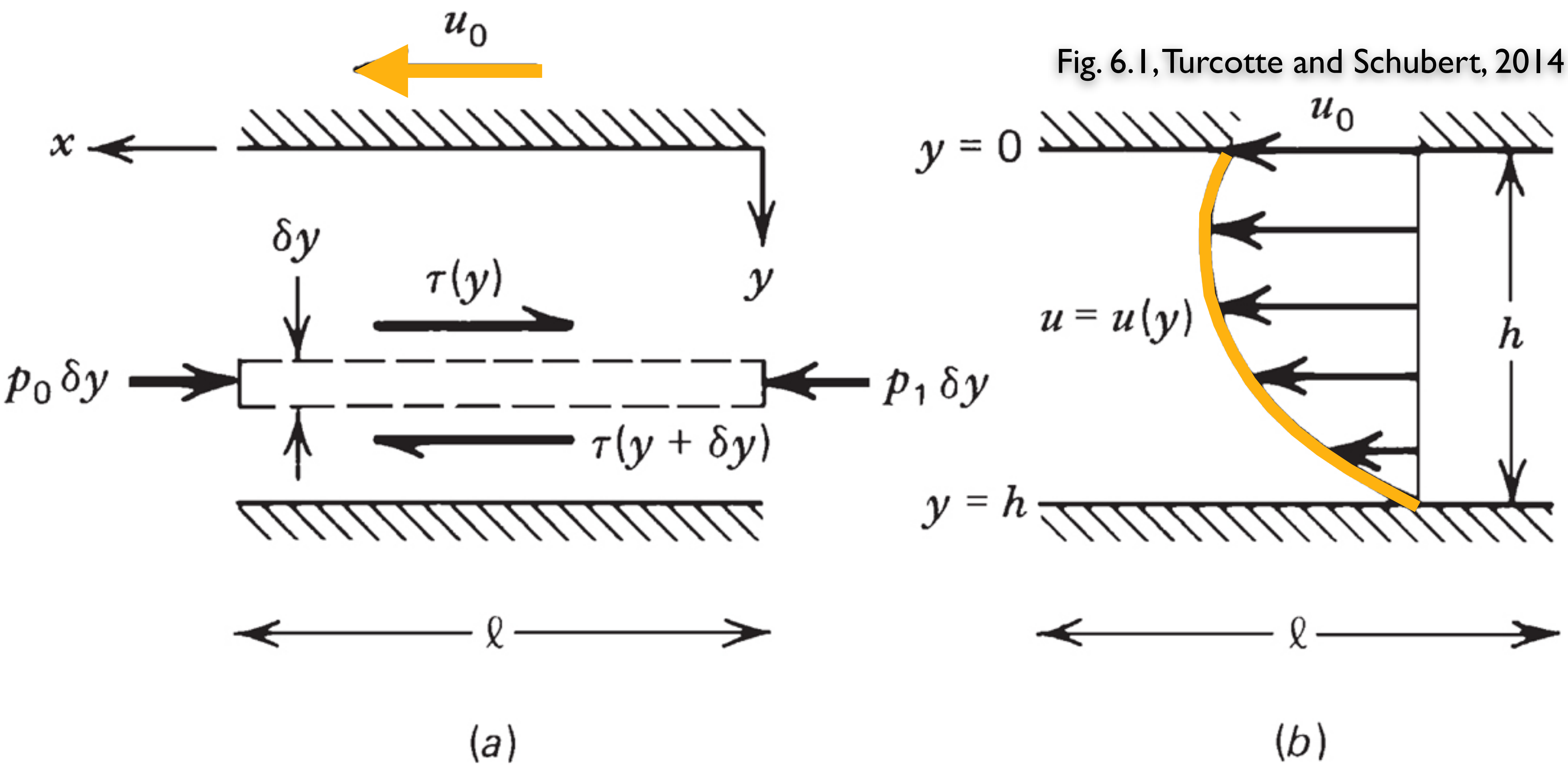


Fig. 6.1, Turcotte and Schubert, 2014

- Because the shear stress τ and velocity u are both only a function of distance y, the **shear force** on the upper boundary of the element is

$$-\tau(y)l$$

- The equivalent **shear force** on the lower boundary is

$$\tau(y + \delta y)l = \left(\tau(y) + \frac{d\tau}{dy}\delta y\right) l$$

1D channel flows

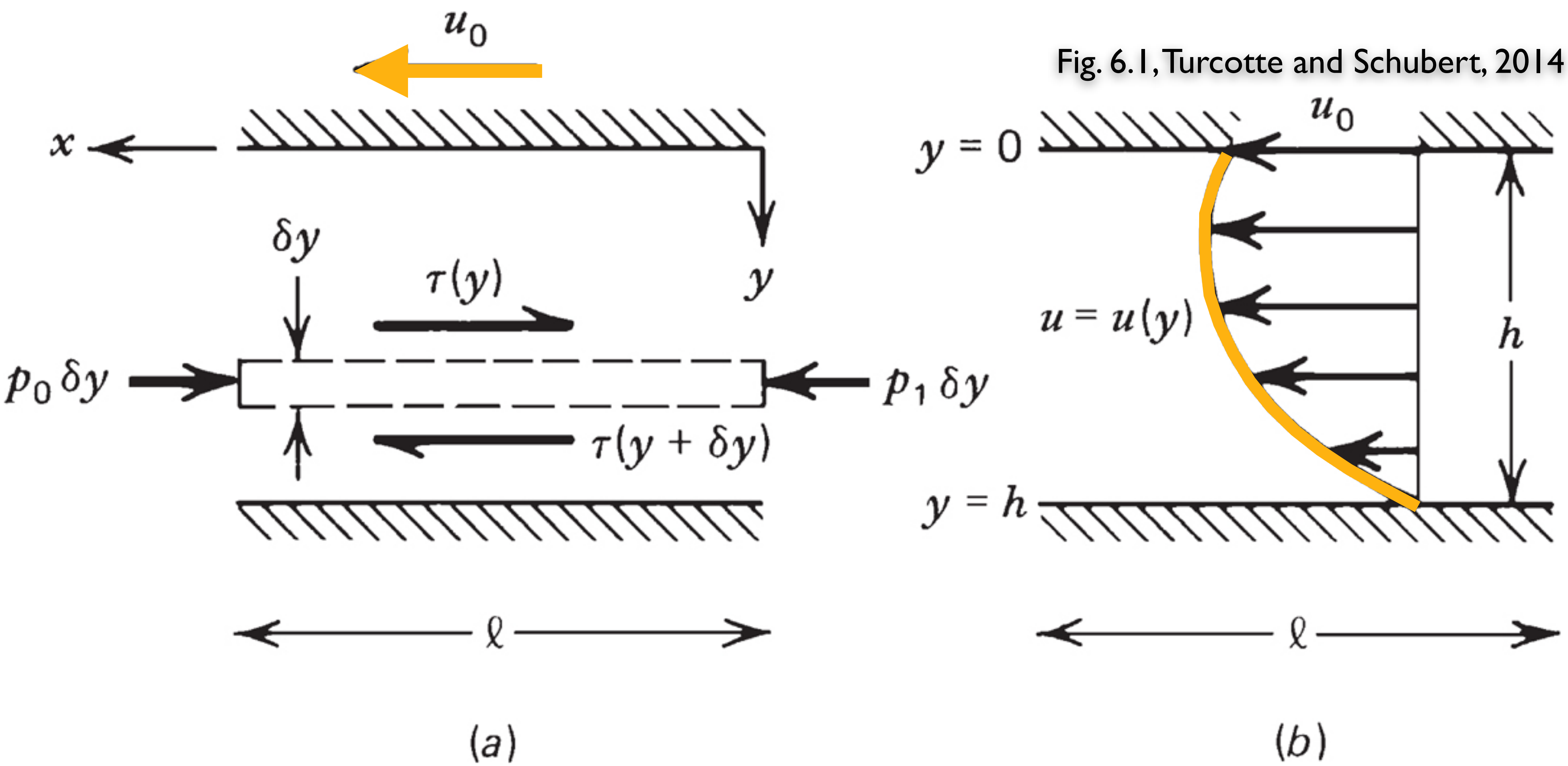


Fig. 6.1, Turcotte and Schubert, 2014

- The **net force** (or sum of the forces) must be equal to zero, or

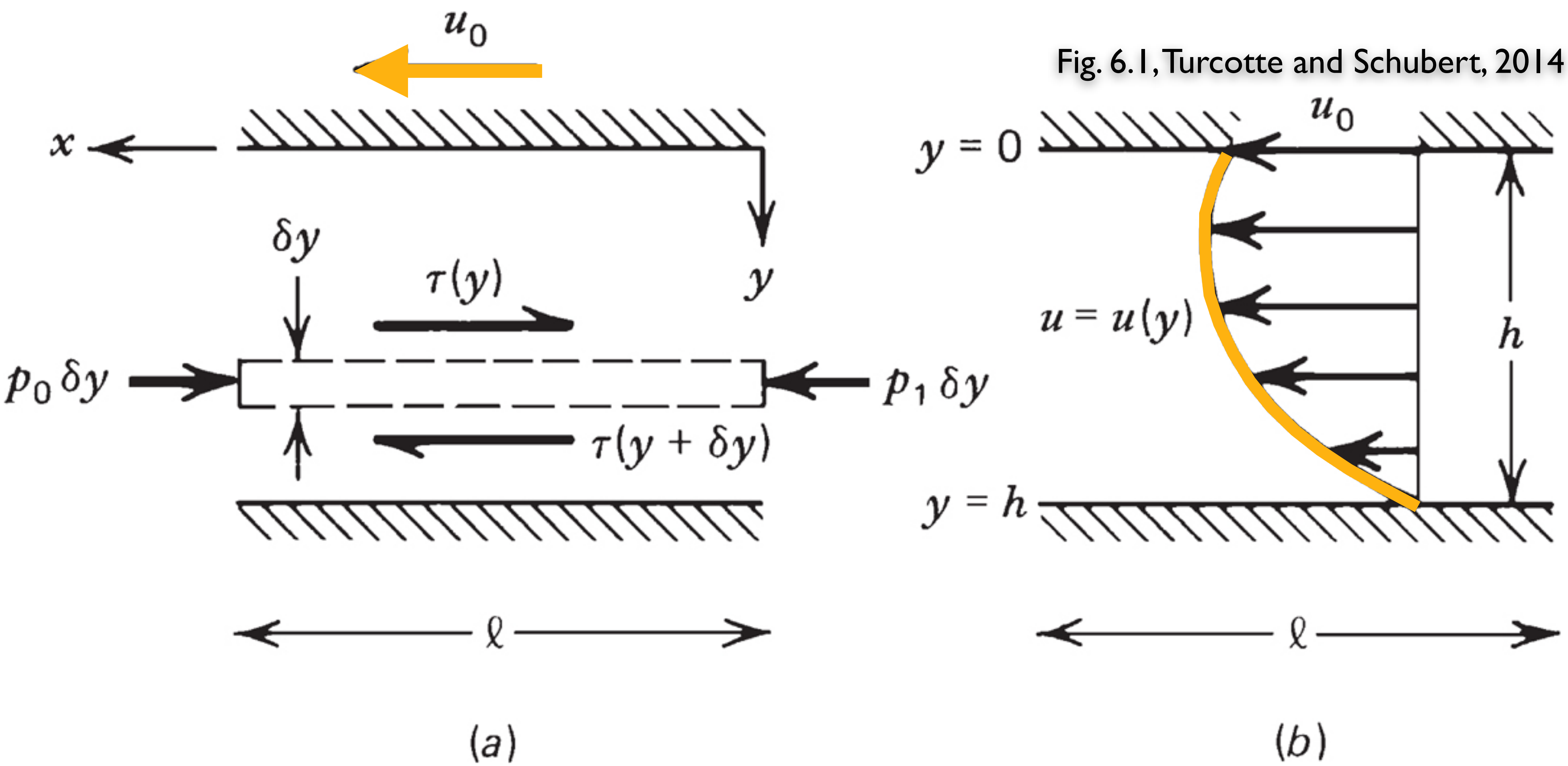
$$(p_1 - p_0)\delta y + \left[\tau(y) + \frac{d\tau}{dy}\delta y\right] l - \tau(y) l = 0$$

- As $\delta y \rightarrow 0$, the relationship above becomes

$$\frac{d\tau}{dy} = -\frac{(p_1 - p_0)}{l}$$

1D channel flows

Fig. 6.1, Turcotte and Schubert, 2014



- The right side of the previous equation is the **horizontal pressure gradient** in the channel

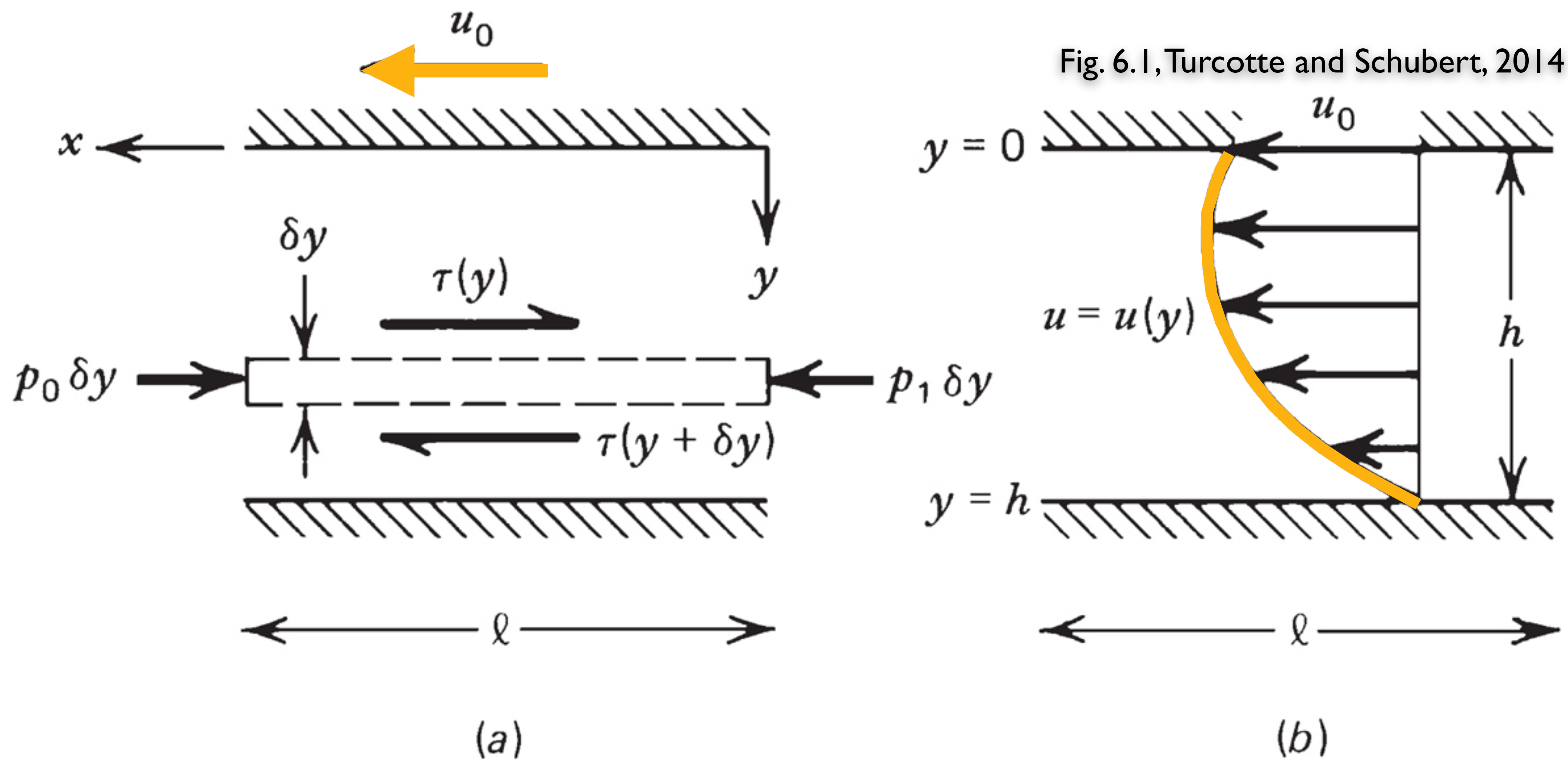
$$\frac{dp}{dx} = -\frac{(p_1 - p_0)}{l}$$

- From which the **equation of motion** can be written

$$\frac{d\tau}{dy} = \frac{dp}{dx}$$

1D channel flows

Fig. 6.1, Turcotte and Schubert, 2014



Newtonian fluid

$$\tau = \eta \frac{du}{dy}$$

- Velocity in the channel is found by substituting the rheological law for a Newtonian fluid into the equation of motion

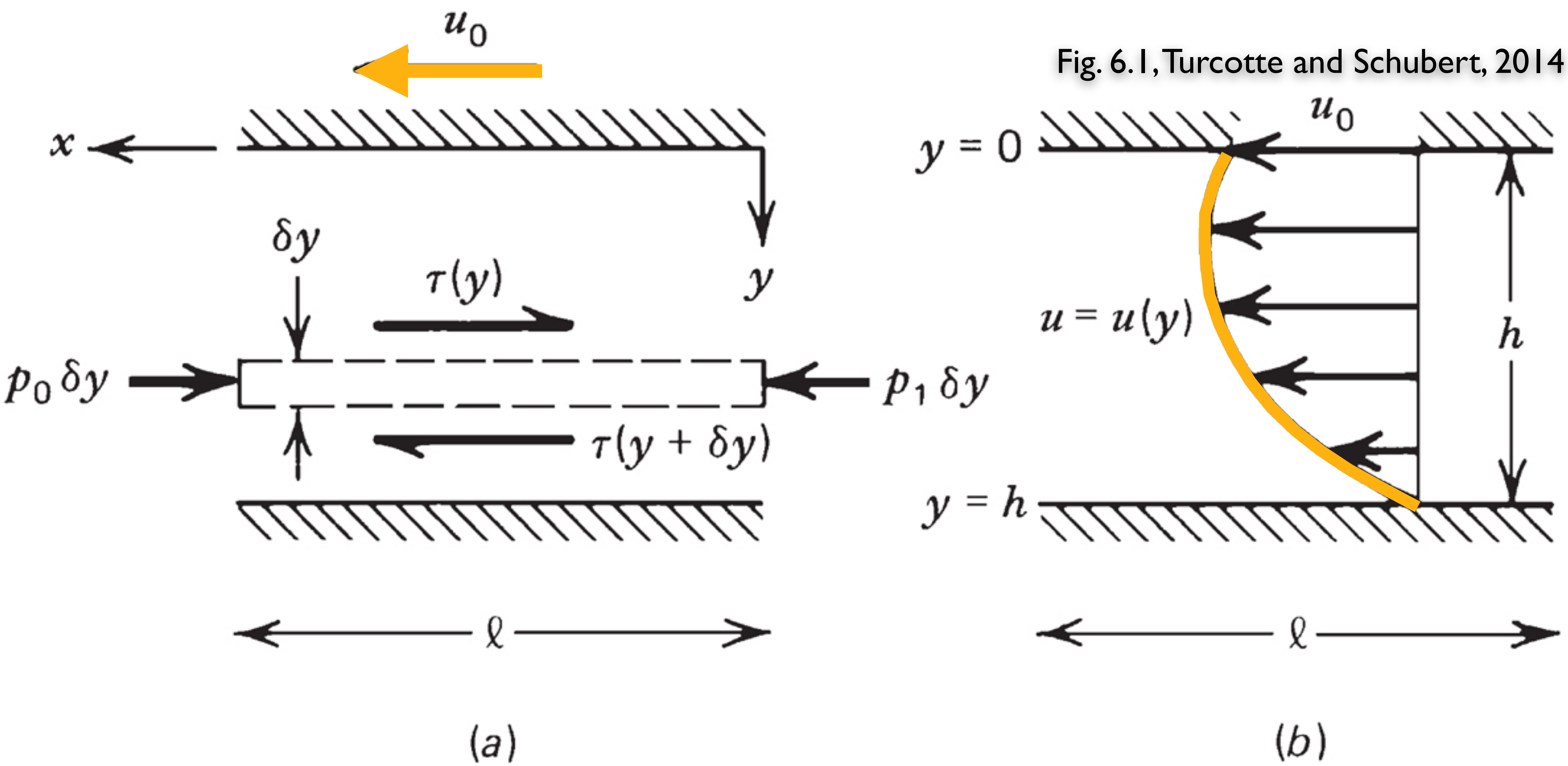
$$\frac{d\tau}{dy} = \frac{d}{dy}\eta\frac{du}{dy} = \eta\frac{d^2u}{dy^2} = \frac{dp}{dx}$$

- Integrating the equation above twice yields

$$\boxed{u = \frac{1}{2\eta}\frac{dp}{dx}y^2 + c_1 y + c_2}$$

1D channel flows

Fig. 6.1, Turcotte and Schubert, 2014



- The constants c_1 and c_2 can be found by applying the boundary conditions that $u = 0$ at $y = h$, and $u = u_0$ at $y = 0$ (no slip)

$$\boxed{u = \frac{1}{2\eta}\frac{dp}{dx}(y^2 - hy) - \frac{u_0 y}{h} + u_0}$$

1D channel flows

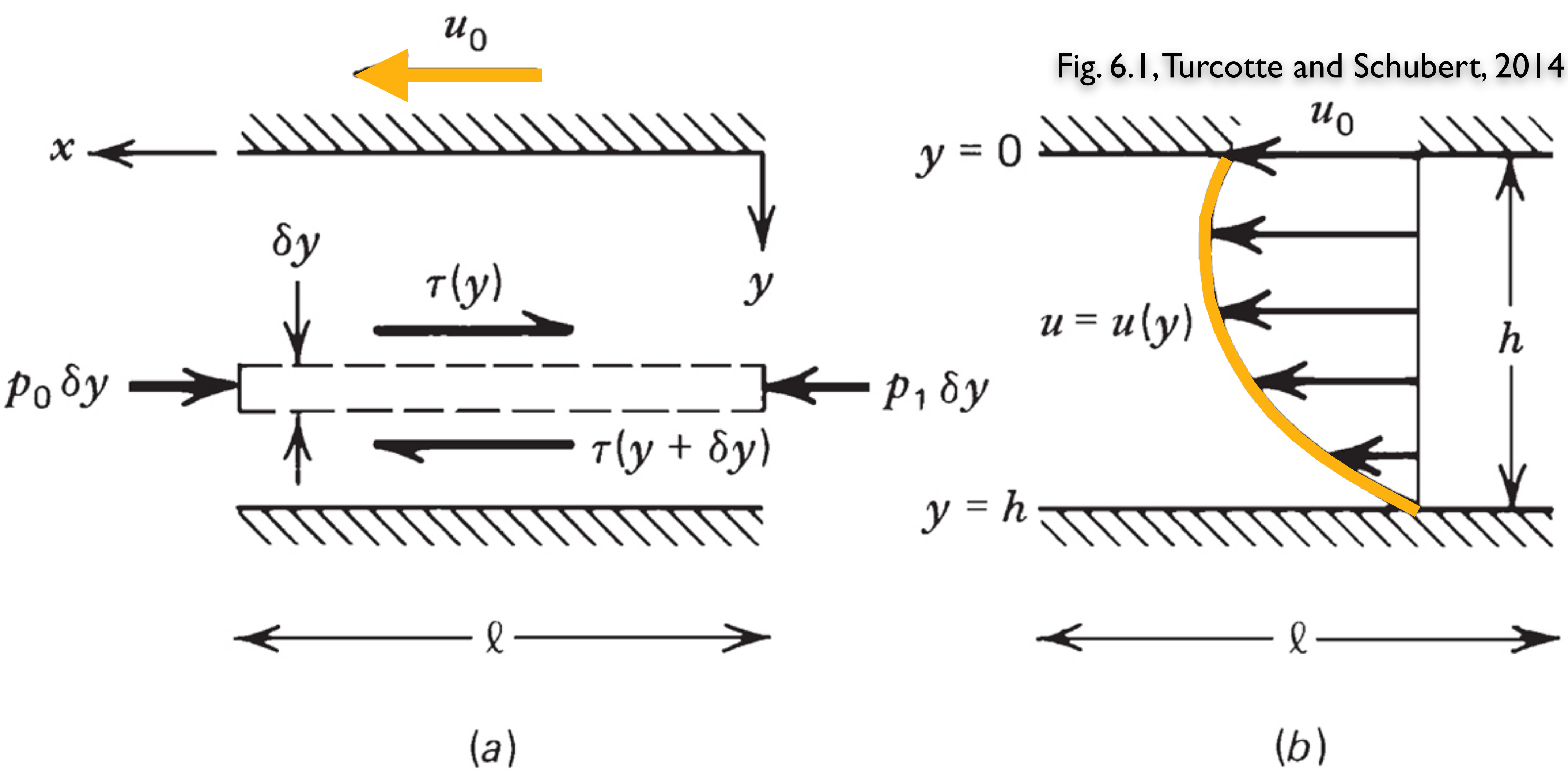


Fig. 6.1, Turcotte and Schubert, 2014

- The constants c_1 and c_2 can be found by applying the boundary conditions that $u = 0$ at $y = h$, and $u = u_0$ at $y = 0$ (no slip)

$$\boxed{u = \frac{1}{2\eta}\frac{dp}{dx}(y^2 - hy) - \frac{u_0 y}{h} + u_0}$$

pressure gradient

wall velocity

Channel flow challenge #1

- Start by navigating to the directory `NGWM2016-modelling-course/Lessons/04-Basic-fluid-mechanics/scripts`
- Right-click on the Python script called `1D-channel-flow.py` and choose "Edit with IDLE"
- The script cannot currently be run because it is missing the equation for the velocity in a 1D channel

$$u = \frac{1}{2\eta}\frac{dp}{dx}(y^2 - hy) - \frac{u_0 y}{h} + u_0$$

your task is to input this equation into the script and then run the script and save plots for the following scenarios:

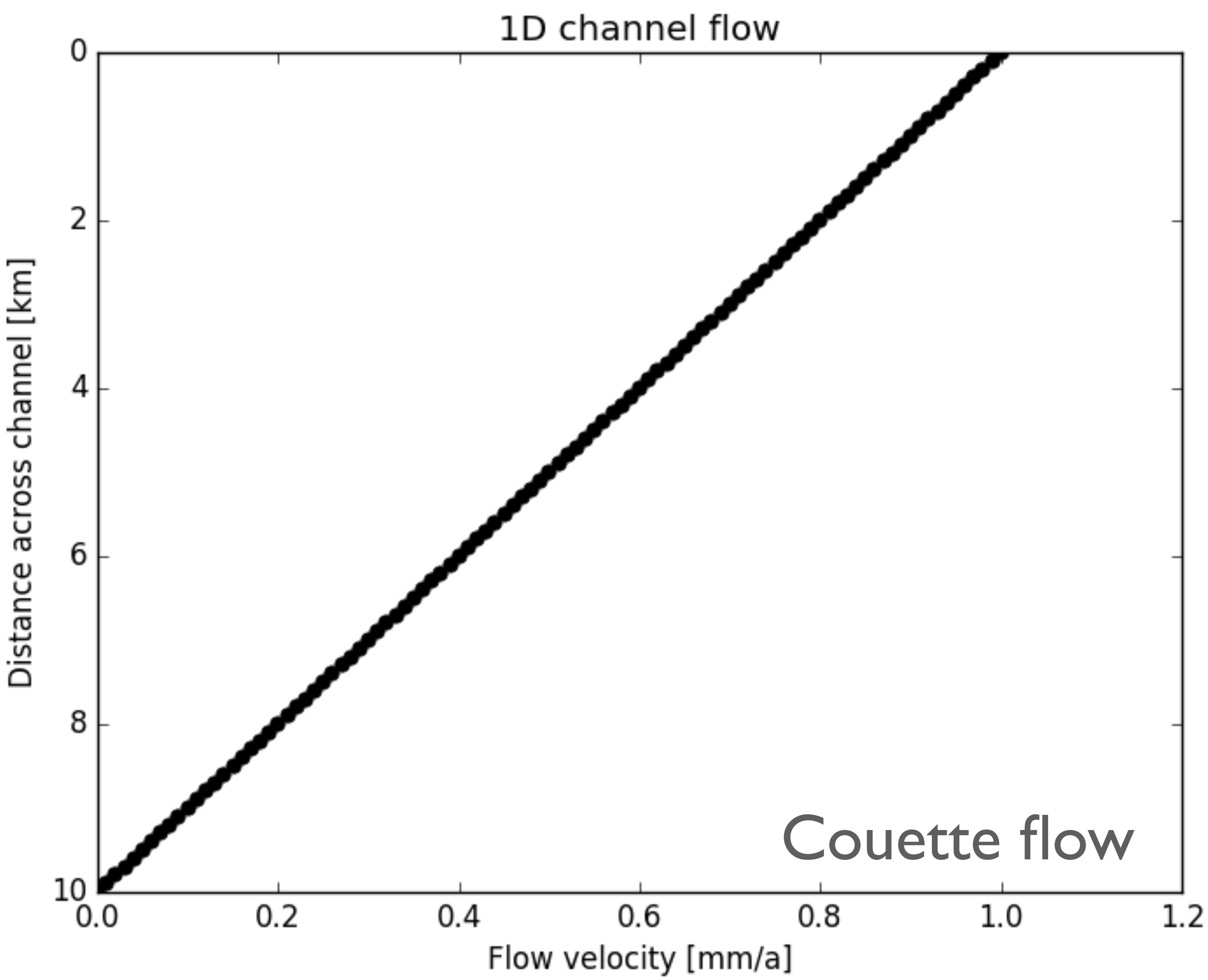
  - No pressure gradient: u_0 = 1.0 mm/a; dp/dx = 0.0 Pa
  - No wall velocity: u_0 = 0.0 mm/a; dp/dx = -2000.0 Pa
  - Other cases: No pressure gradient/wall velocity, both, etc.

What does this equation tell us?

$$u = \frac{1}{2\eta}\frac{dp}{dx}(y^2 - hy) - \frac{u_0 y}{h} + u_0$$

- We've now look at several simple fluid flow behaviors, including two important end members
  - Zero pressure gradient ($dp/dx = 0$)
  - Zero boundary velocity ($u_0 = 0$)

Couette flow

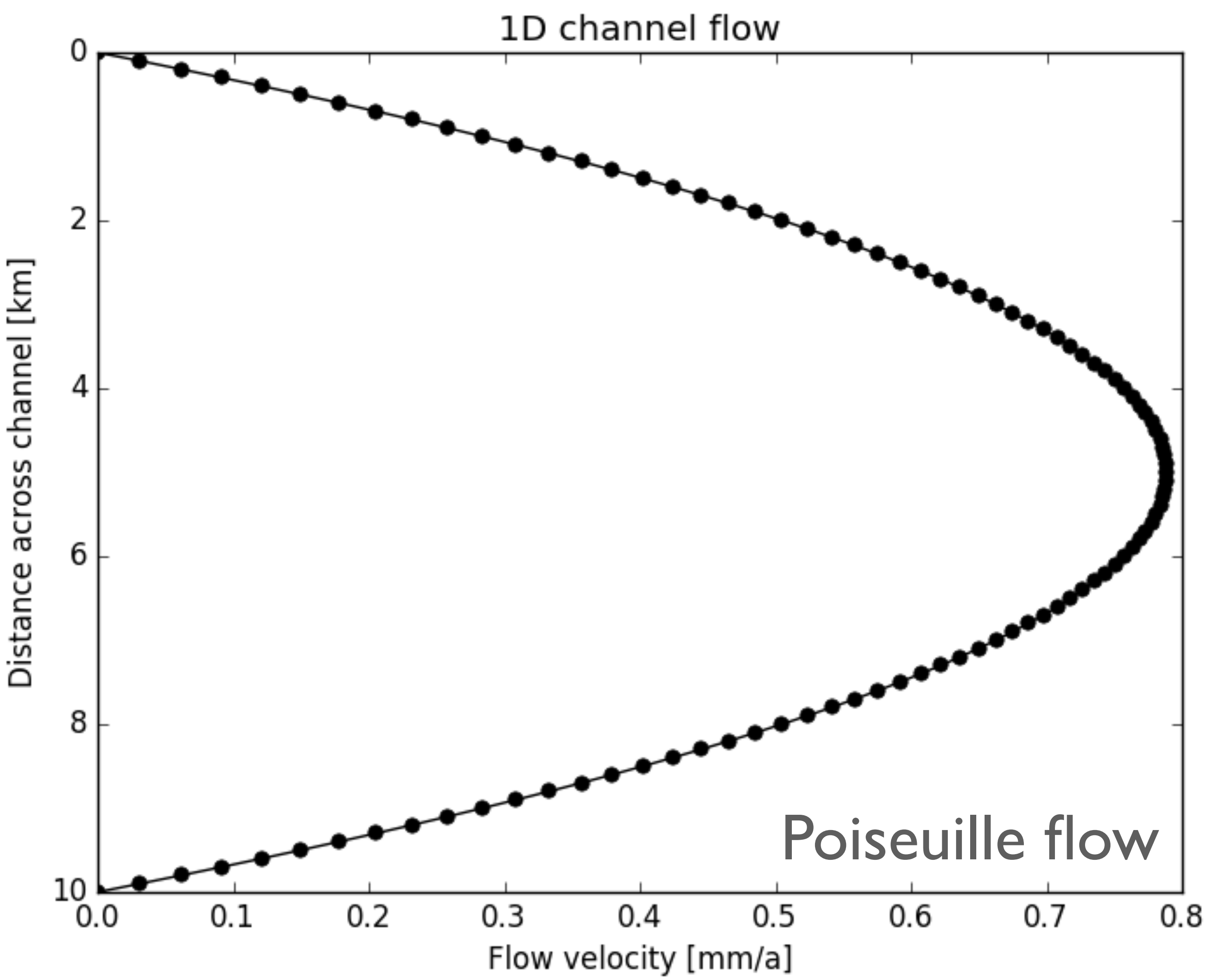


- A **Couette flow** has no pressure gradient, or $dp/dx = 0$, reducing the 1D equation for velocity in the channel down to

$$u = u_0 \left(1 - \frac{y}{h}\right)$$

- Clearly, this predicts a linear increase in velocity from $y = h$ to $y = 0$

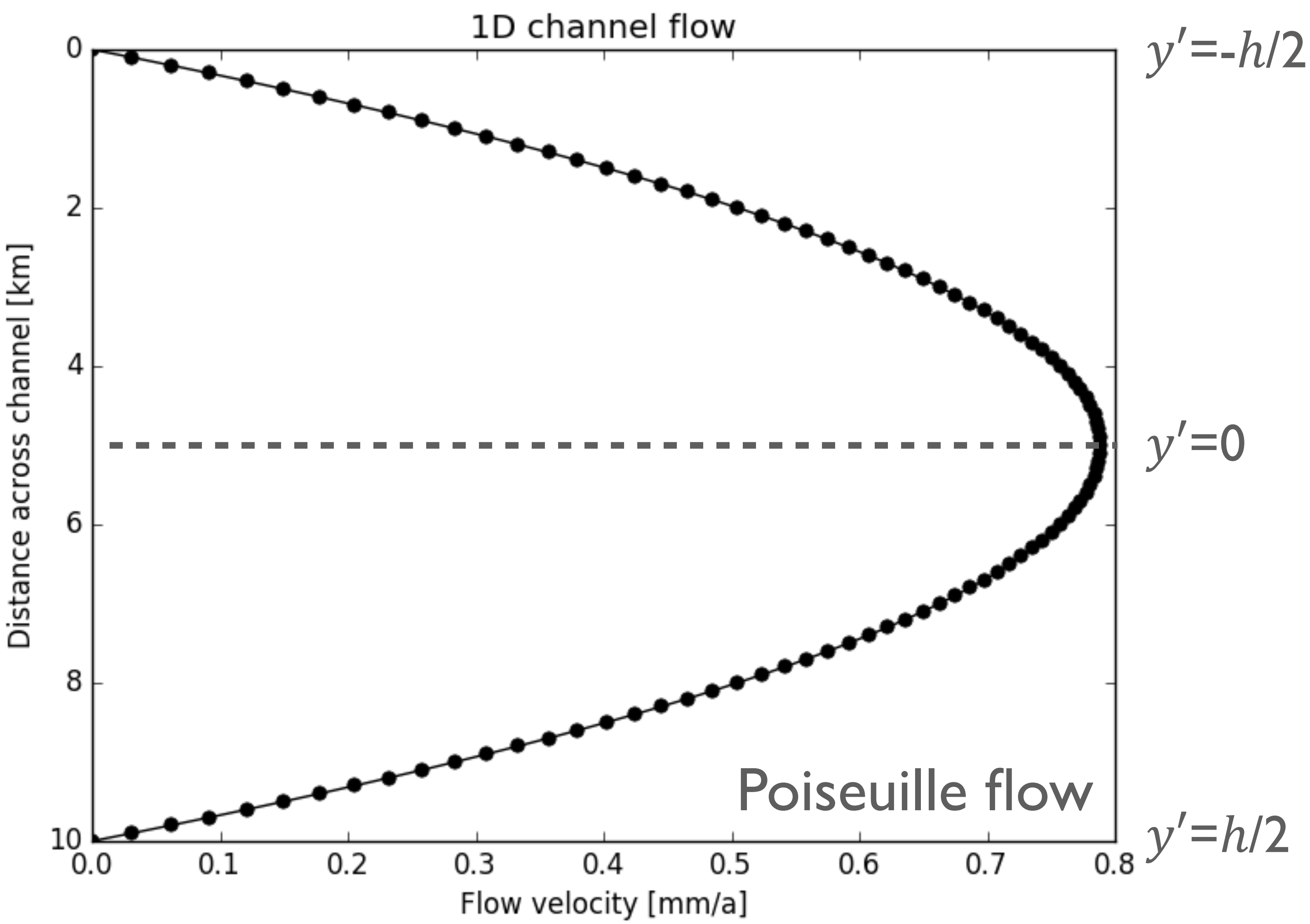
Poiseuille flow



- **Poiseuille flow** is driven only by a pressure gradient in the channel with zero boundary velocities ($u_0 = 0$), thus

$$u = \frac{1}{2\eta}\frac{dp}{dx}(y^2 - hy)$$

Poiseuille flow



- **Poiseuille flow** is driven only by a pressure gradient in the channel with zero boundary velocities ($u_0 = 0$), thus

$$u = \frac{1}{2\eta}\frac{dp}{dx}(y^2 - hy)$$

- In a coordinate system with y' at the middle of the channel we can say $y' = y - h/2$, which results in the relationship

$$u = \frac{1}{2\eta}\frac{dp}{dx}\left(y'^2 - \frac{h^2}{4}\right)$$

Asthenospheric counterflow

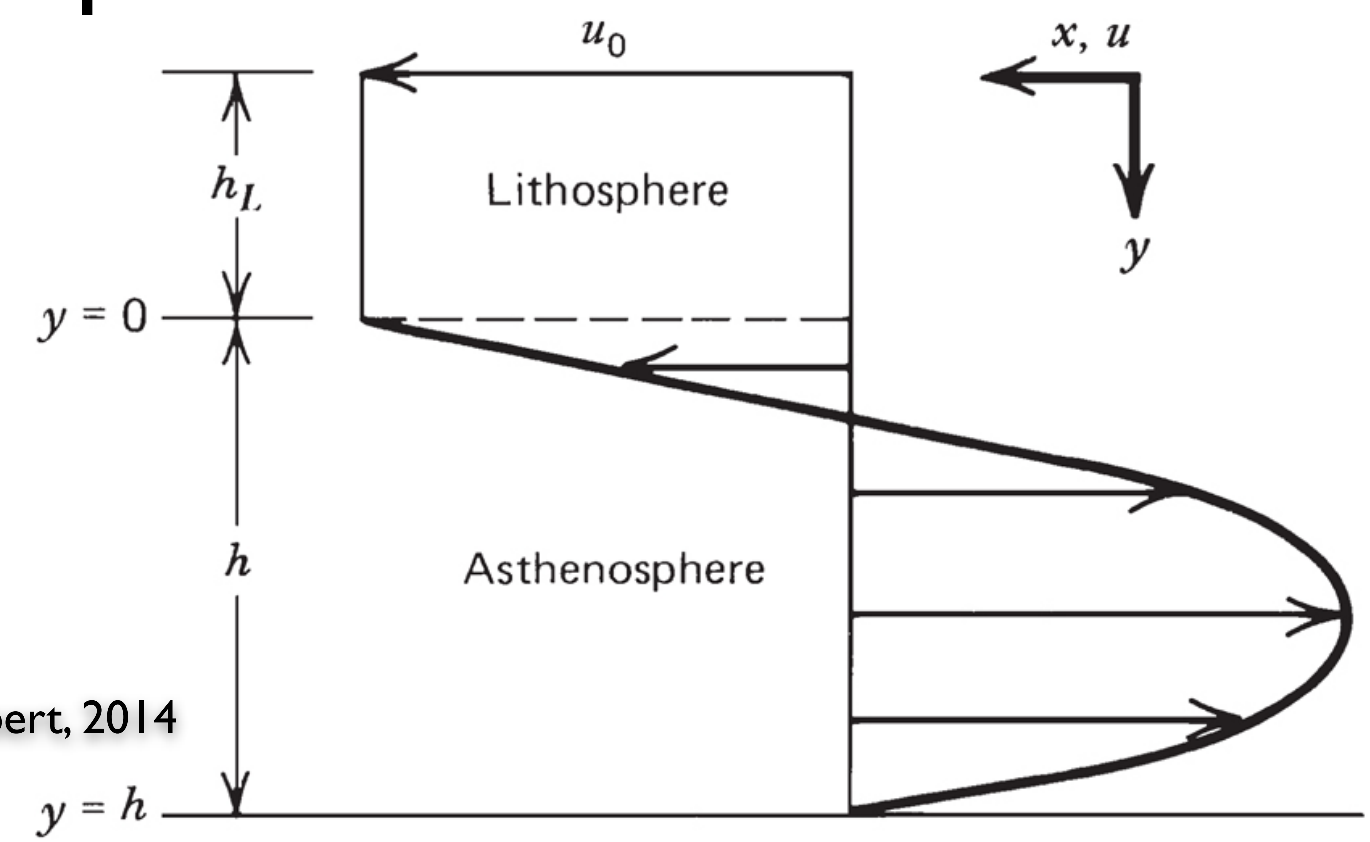


Fig. 6.4, Turcotte and Schubert, 2014

- One model for mantle flow is that the motion of lithospheric plates on the Earth's surface produces a **counterflow** in the uppermost asthenosphere (upper ~100-200 km)

- If we assume the plate is rigid and moving at velocity u_0, and that the velocity at some depth $y = h$ must be zero, it is clear that the counterflow is opposite in direction to the plate motion in order to **conserve mass**

Asthenospheric counterflow

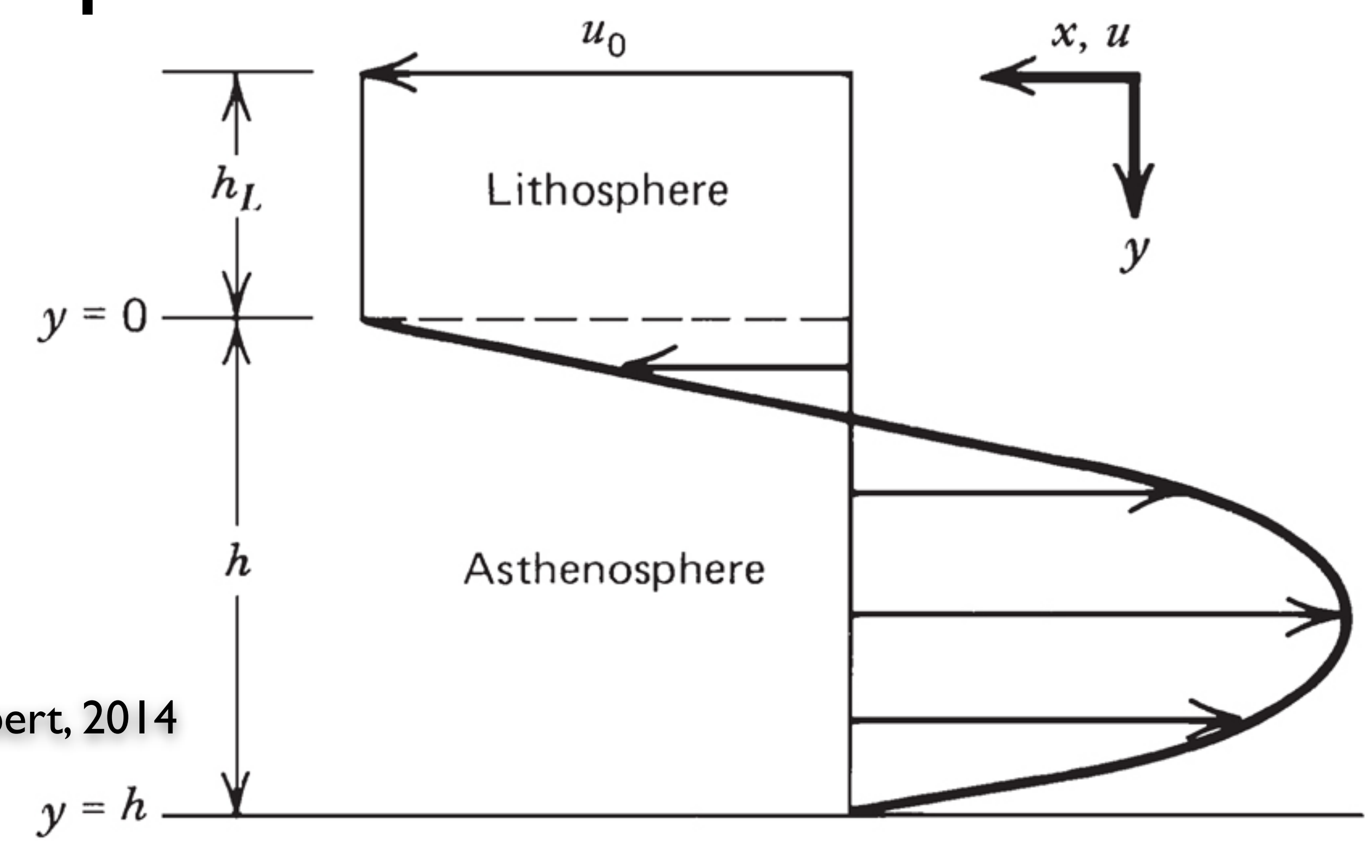


Fig. 6.4, Turcotte and Schubert, 2014

- Mathematically, we can state that as

$$u_0 h_L + \int_0^h u \; dy = 0$$

where h_L is the thickness of the lithosphere and h is the thickness of the asthenosphere involved in counterflow

- If we insert our equation for 1D channel flow in the second term, we get

$$u_0 h_L + \frac{h^3}{12\eta} \frac{dp}{dx} + \frac{u_0 h}{2} = 0$$

Asthenospheric counterflow

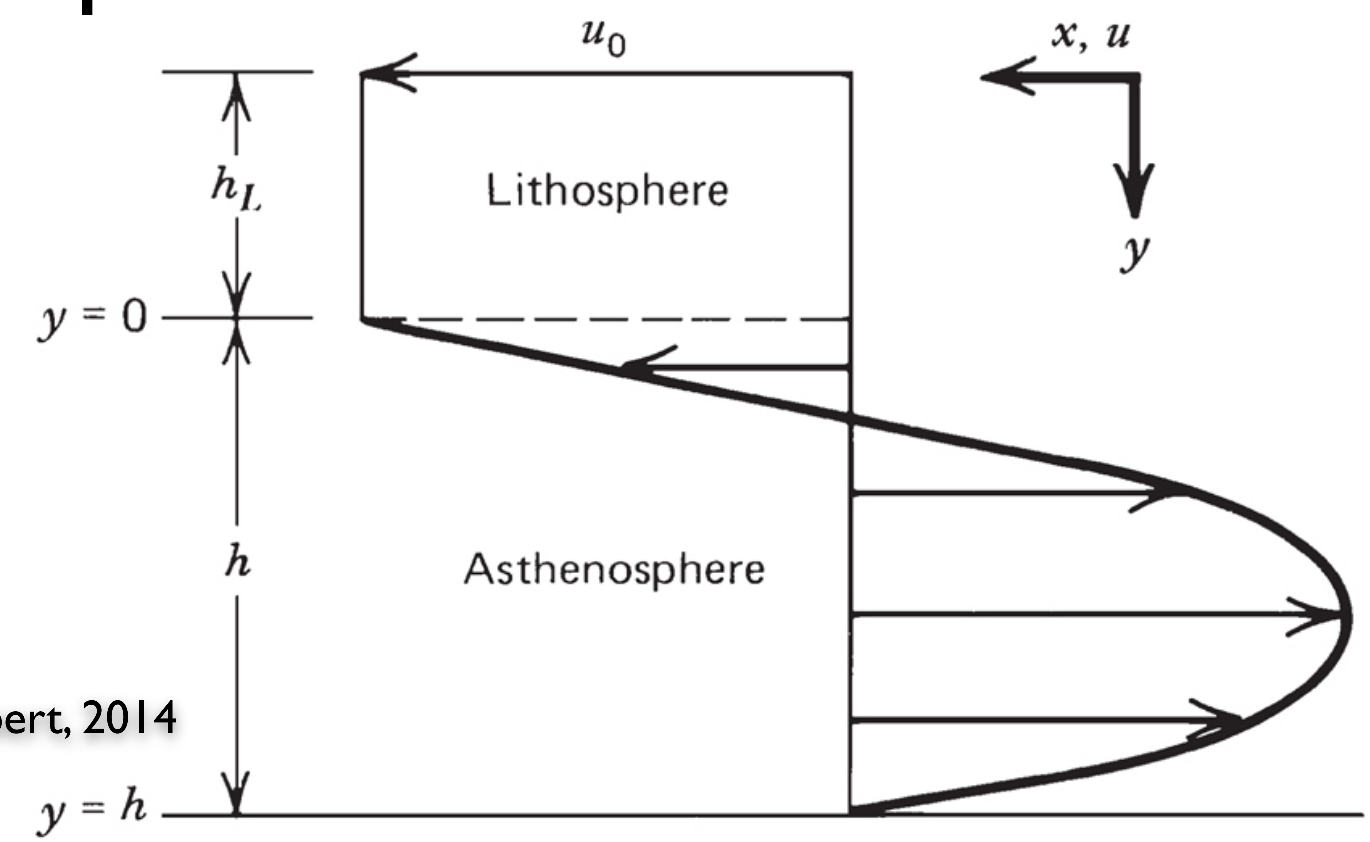


Fig. 6.4, Turcotte and Schubert, 2014

- If we now solve for the **pressure gradient**, we find

$$\frac{dp}{dx} = \frac{12\eta u_0}{h^2}\left(\frac{h_L}{h} + \frac{1}{2}\right)$$

- And this can be inserted into the equation for 1D channel flow to get the predicted velocity profile for a counterflow

$$\boxed{u = u_0\left[1 - \frac{y}{h} + 6\left(\frac{h_L}{h} + \frac{1}{2}\right)\left(\frac{y^2}{h^2} - \frac{y}{h}\right)\right]}$$

Asthenospheric counterflow

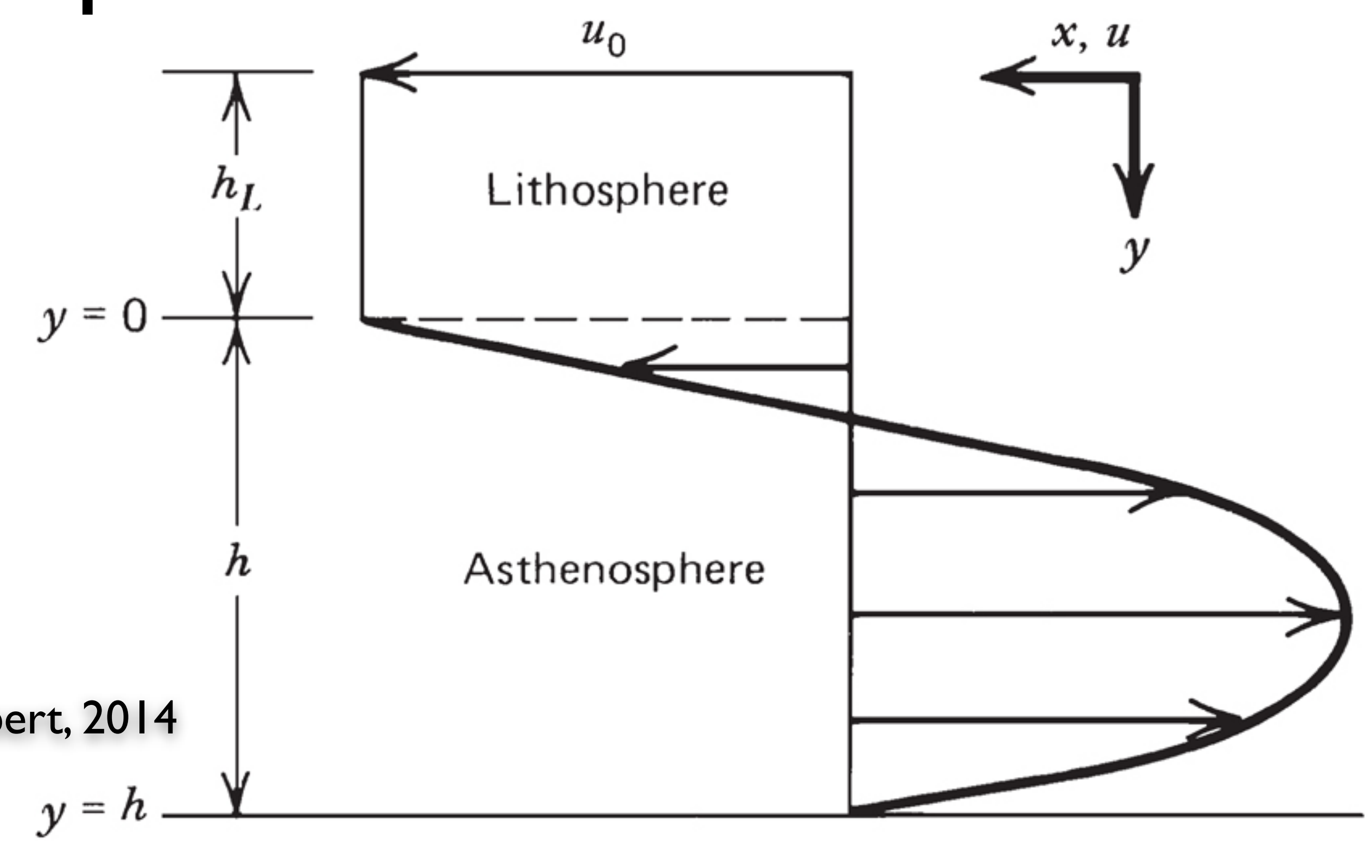


Fig. 6.4, Turcotte and Schubert, 2014

$$u = u_0 \left[1 - \frac{y}{h} + 6 \left(\frac{h_L}{h} + \frac{1}{2} \right) \left(\frac{y^2}{h^2} - \frac{y}{h} \right) \right]$$

- **Looking at this equation for a moment, is there anything missing that you might expect to see?**

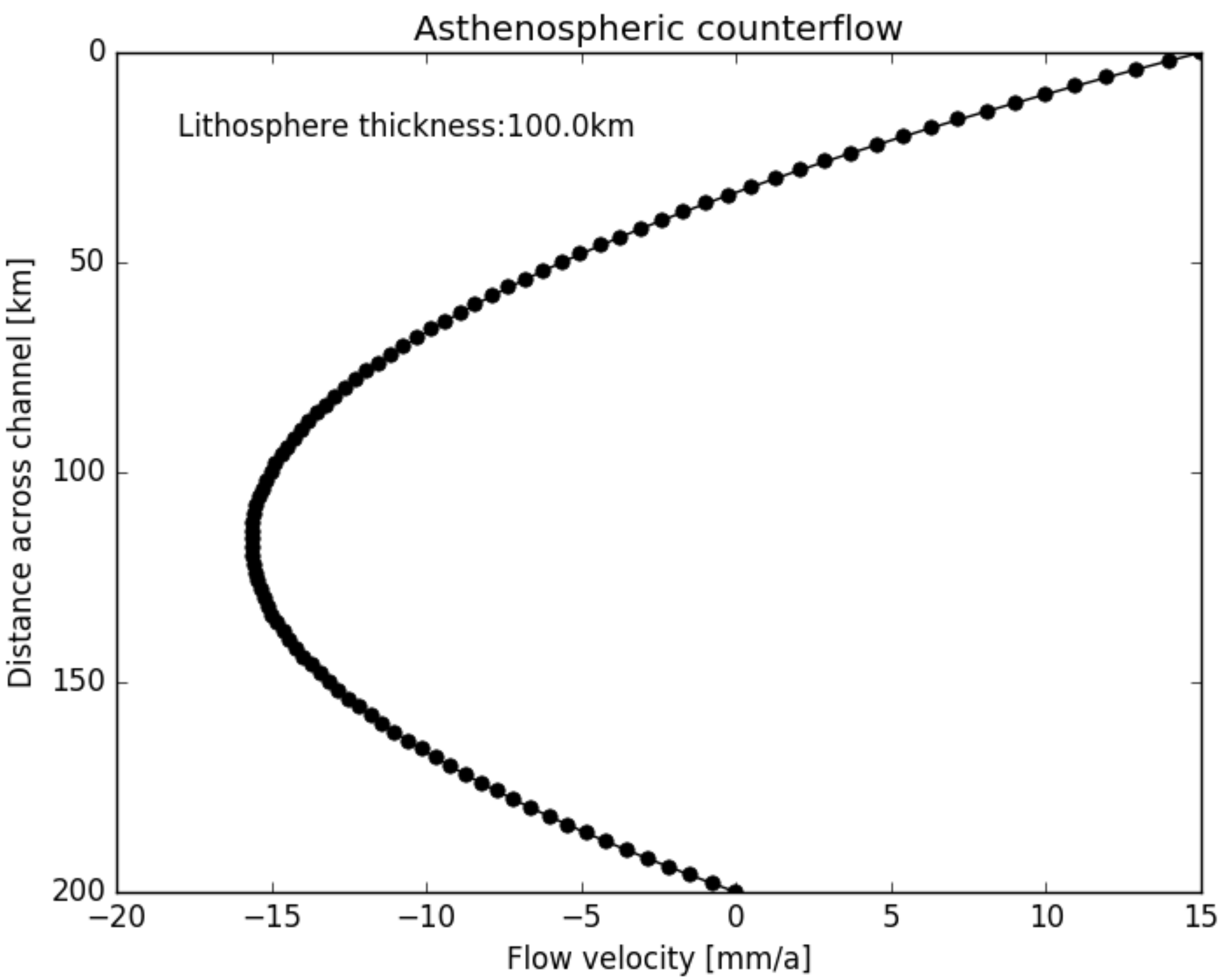
Channel flow challenge #2

- In the same directory as before, right-click on the script `1D-asthenospheric-counterflow.py` and choose "Edit with IDLE"
- Again, this script cannot be run because it is missing the equation for asthenospheric counterflow

$$u = u_0 \left[1 - \frac{y}{h} + 6\left(\frac{h_L}{h} + \frac{1}{2}\right)\left(\frac{y^2}{h^2} - \frac{y}{h}\right)\right]$$

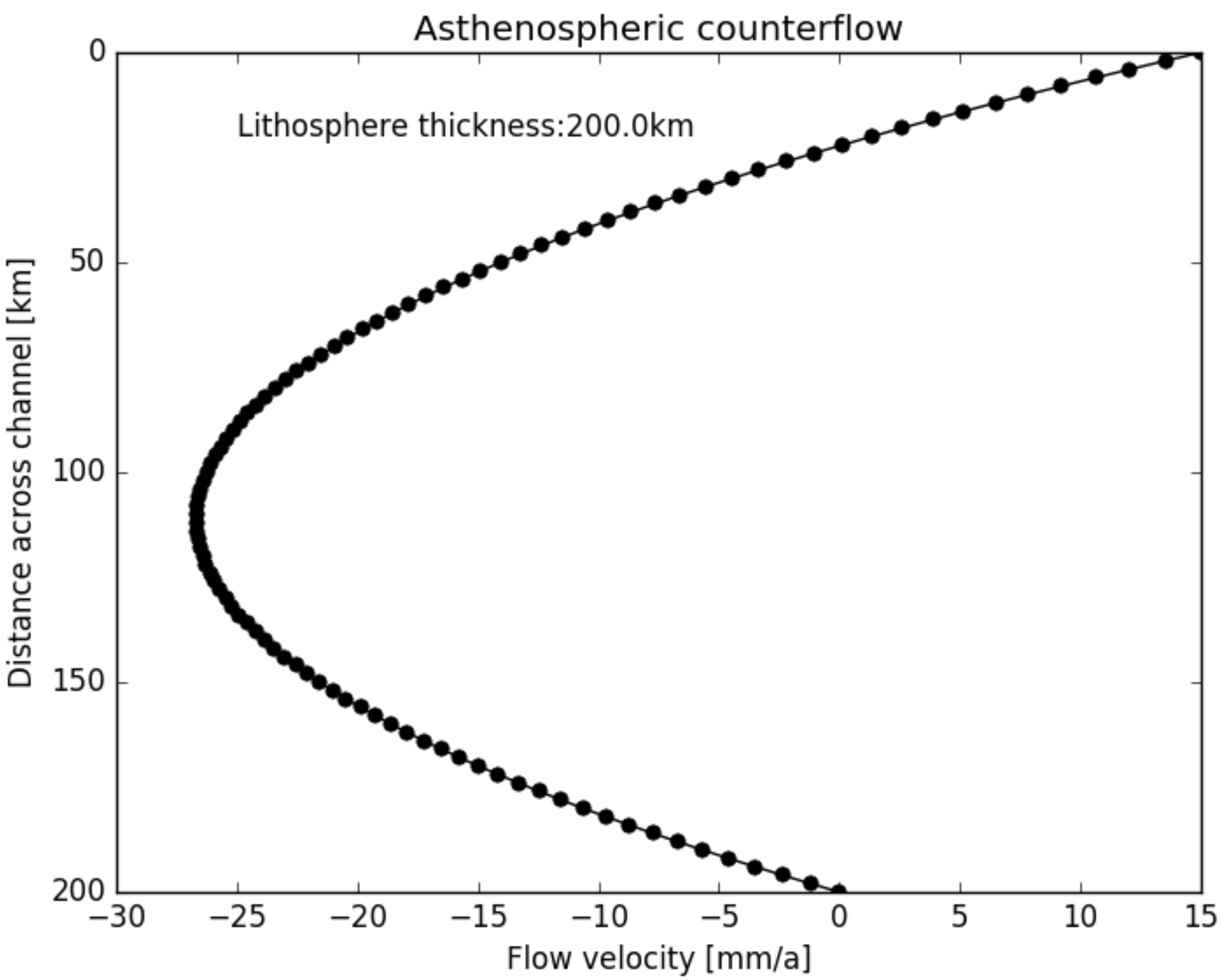
- In addition, you will need to add a text label to display the thickness of the lithospheric plate in [km] on the plot
  - You can find how to use the `plt.text()` function at `http://matplotlib.org/api/pyplot_api.html#matplotlib.pyplot.text`
- After making the changes, run the script with the default values and save the plot
- How do the flow velocities change when you vary h and h_l?

Asthenospheric counterflow



- As expected, movement of a 100-km-thick lithosphere at u_0 = 15 cm/a results in counter flow in the 200-km-thick asthenosphere
  - The maximum counterflow velocity is roughly the same as the plate velocity in this case

Asthenospheric counterflow



- When the plate thickness is doubled, the counterflow velocity must also increase in order to conserve mass, as required by our counterflow equation
    - The maximum counterflow velocity here is about 1.8 times the plate velocity

Limitations to our channel flow models

- Though we can predict the velocities in 1D channels for a number of different scenarios, we're not able to handle a few things of geological importance
  - Nonlinear viscosity in the channel
    - As you'll see tomorrow morning, viscous flow in rocks is generally not linear
  - Spatial variations in the channel material
    - Perhaps we have a channel with two different fluids
  - Changes in the boundary conditions with time
    - What if the channel wall velocity varies with time?

Summary

- **Fluid mechanics** is the science of fluid motion

- Fluid motions are caused by **internal and external forces**, and modelled using simple formulations of the **conservation of mass, momentum and energy**

- For geological applications, we treat the **Earth as a fluid with a high viscosity** and model flow using the **Stokes equation**

- Analytical solutions for 1D channel flows can provide insight into fluid flow in the Earth, but have <u>significant limitations</u>

References

Gerya, T. (2009). *Introduction to numerical geodynamic modelling*. Cambridge University Press.

Grujic, D. (2006). Channel flow and continental collision tectonics: an overview. *Geological Society, London, Special Publications*, *268*(1), 25-37.

Turcotte, D. L., & Schubert, G. (2014). *Geodynamics*. Cambridge University Press.

Twiss, R. J., & Moores, E. M. (2007). *Structural Geology*, 2nd Edition. W.H. Freeman Co.