MATH FINANCE LAB 4: Heston Model

Math Finance Lab4
In this lab you will:
simulate stock prices using the Heston model
price a call option using MOnte Carlo

Simulation of Stock Prices

Let $v(t,\omega)$ be the instantaneous variance in the model. The model for the stock price $S(t)$ is:

$$\begin{aligned}\frac{dS(t,\omega)}{S(t,\omega)} &= rdt + \sqrt{v(t,\omega)}d\tilde{W}_1(t,\omega) \\ S(0,\omega) &= S_0\end{aligned}$$

where $\tilde{W}_1$ is Brownian motion in the risk-neutral measure. In a *local volatility* model, $v(t,\omega)$ is a function of the stock price, i.e., $v(t,\omega) = \sigma^2(t, S(t,\omega))$. In a general stochastic volatility model, $v(t,\omega)$ is an Ito process. There is correlation between returns and volatilities:

$$Corr[\frac{dS}{S}, dv] = \rho dt$$

To achieve that, we use a second Brownian motion $\tilde{W}_2$, independent from $\tilde{W}_1$. The variance process is a positive process, given by:

$$\begin{aligned}dv(t,\omega) &= -\lambda(v(t,\omega)-\bar{v})dt + \eta\sqrt{v(t,\omega)}d\tilde{W}_2(t,\omega) \\ &= -\lambda(v(t,\omega)-\bar{v})dt + \eta\sqrt{v(t,\omega)}(\rho d\tilde{W}_1(t,\omega) + \sqrt{1-\rho^2}dZ_2(t,\omega)) \\ v(0,\omega) &= v_0\end{aligned}$$

In the lab, you will use the Euler scheme (see below) to simulate a discrete version $S^d(i\Delta,\omega)$ of the stock price, and a discrete version $v^d(i\Delta,\omega)$ of the instantaneous variance. The parameters are:

Name	Symbol	Value
Stock price at zero	S_0	100
Variance at zero	v_0	0.04
Speed of mean reversion	λ	1
Long run value of the variance	$\bar{v}$	0.04
Volatility of volatility	η	0.1
Correlation	ρ	0.5
Interest rate	r	0

You should estimate via Monte Carlo simulation the following values: $\tilde{E}[S^d(N\Delta)]$, $\widetilde{Var}[S^d(N\Delta)]$, $\tilde{E}[v^d(N\Delta)]$, as well as $\widetilde{Var}[v^d(N\Delta)]$, for $N = 20$. Choose $\Delta = 0.05$ and the numebr of scenarios Ω=1000.

Pricing a Call Option

Suppose you have a number Ω of scenarios. For any function $f(S^d(T))$, an approximation of its expected value is its average:

$$E[f(S(T))] \cong \frac{1}{\Omega}\sum_{\omega=1}^{\Omega} f(S^d(N\Delta,\omega))$$

Use Monte Carlo simulation to price a call option with strike $K = 99$ and expiration $T = 1$.

Euler scheme in Monte Carlo simulation

Let a continuous-time stochastic process $X(t)$ be given by the SDE:

$$dX(t,\omega) = \mu_X(t, X(t,\omega))dt + \sigma_X(t, X(t,\omega))d\tilde{W}(t,\omega)$$

where $\tilde{W}$ is Brownian motion. In the Euler scheme, a discrete-time approximation $X^d(i\Delta,\omega)$ of $X(i\Delta,\omega)$ can be obtained by:

$$X^d((i+1)\Delta,\omega) - X^d(i\Delta,\omega) = \mu_X(i\Delta, X^d(i\Delta,\omega))\Delta + \sigma_X(t, X^d(i\Delta,\omega))z_i(\omega)\sqrt{\Delta}$$

where $\{z_i\}$ is a collection of IID standard normal random variables.