Module 5 – Graphs I

Modified/Inspired from Stanford's CS161 by Nayyar Zaidi

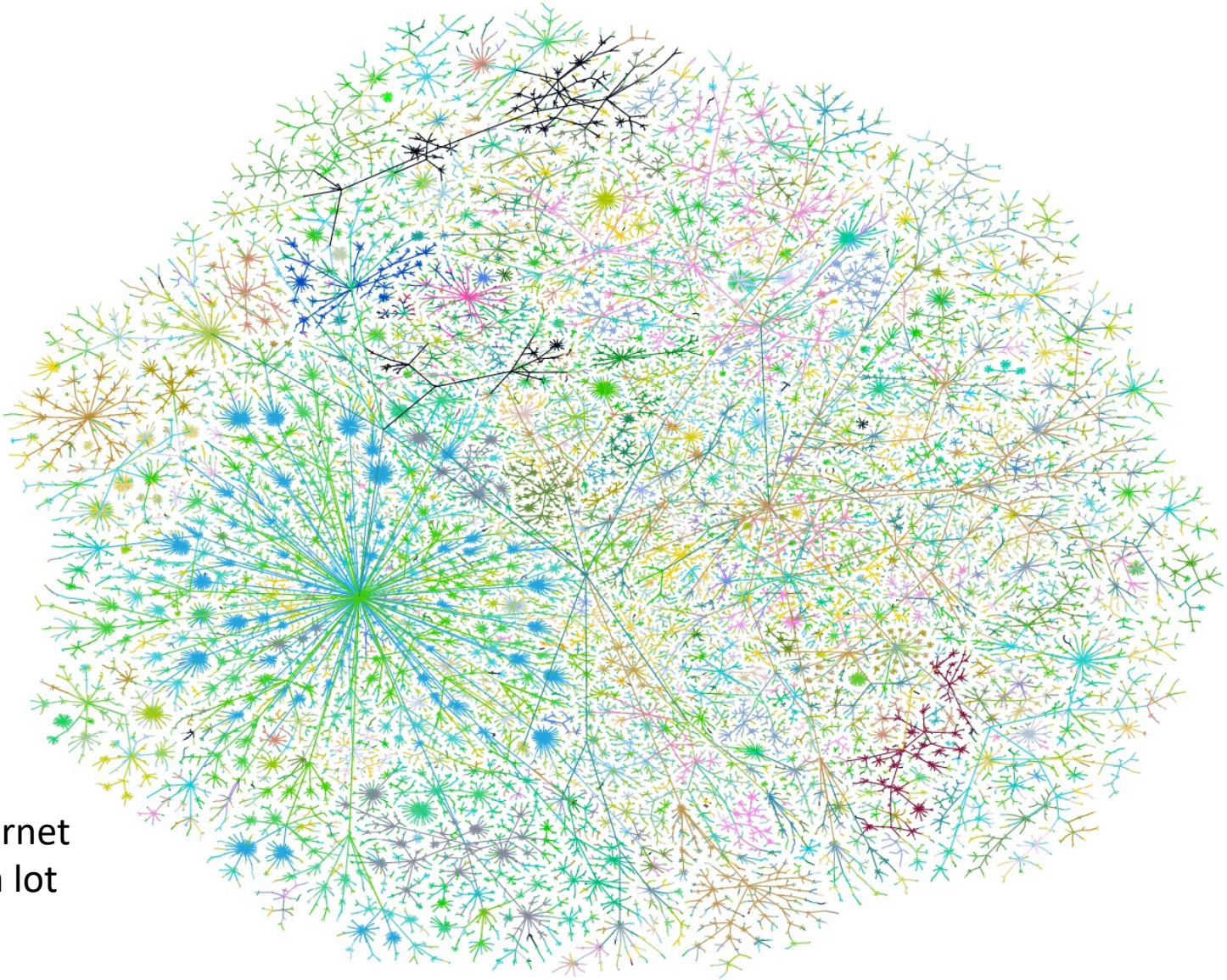
Outline

- Part 0: Graphs and terminology

- Part 1: Depth-first search
  - Application: topological sorting
  - Application: in-order traversal of BSTs
- Part 2: Breadth-first search
  - Application: shortest paths
  - Application (if time): is a graph bipartite?

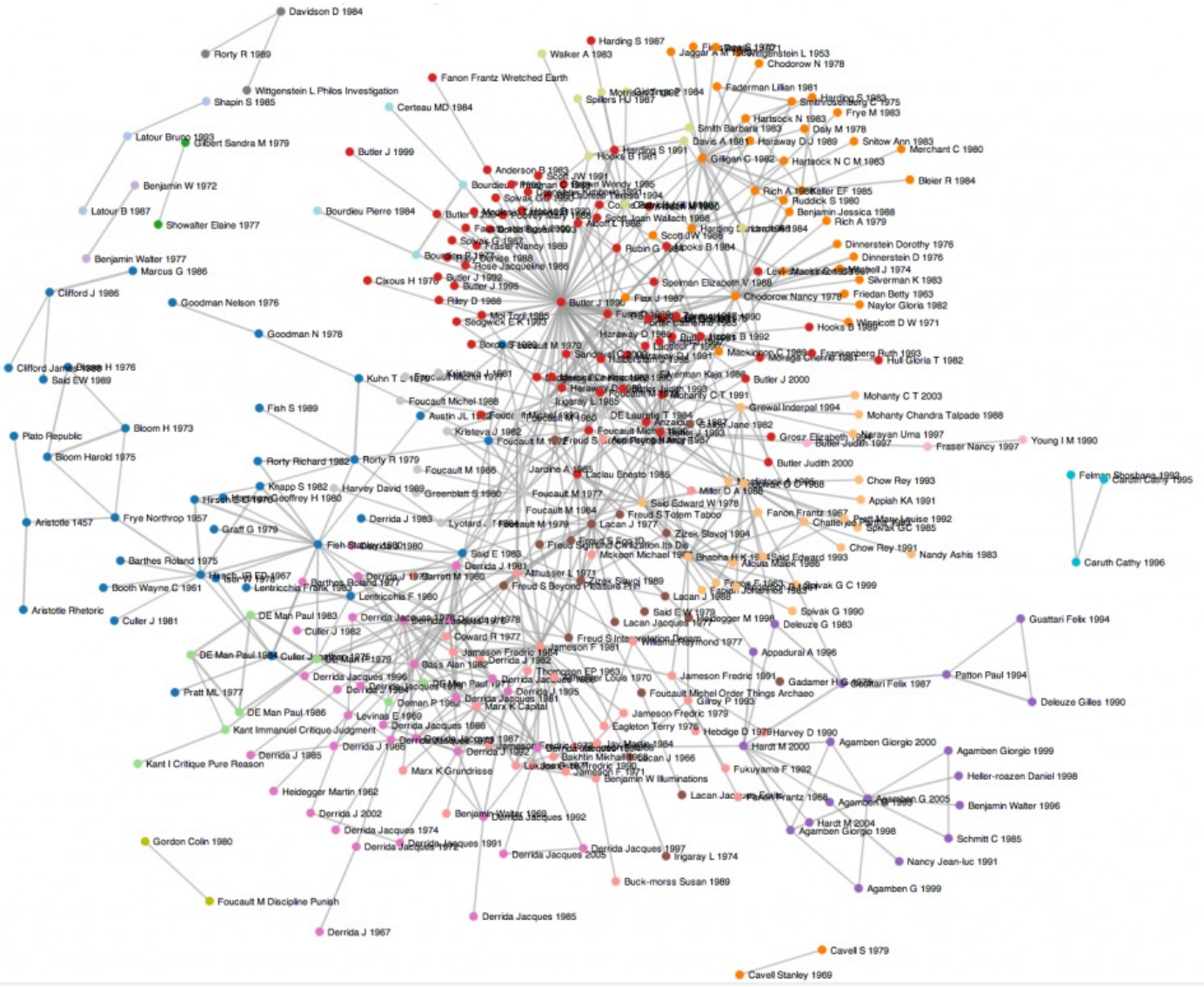
Part 0: Graphs

Graphs



Graph of the internet (circa 1999…it's a lot bigger now…)

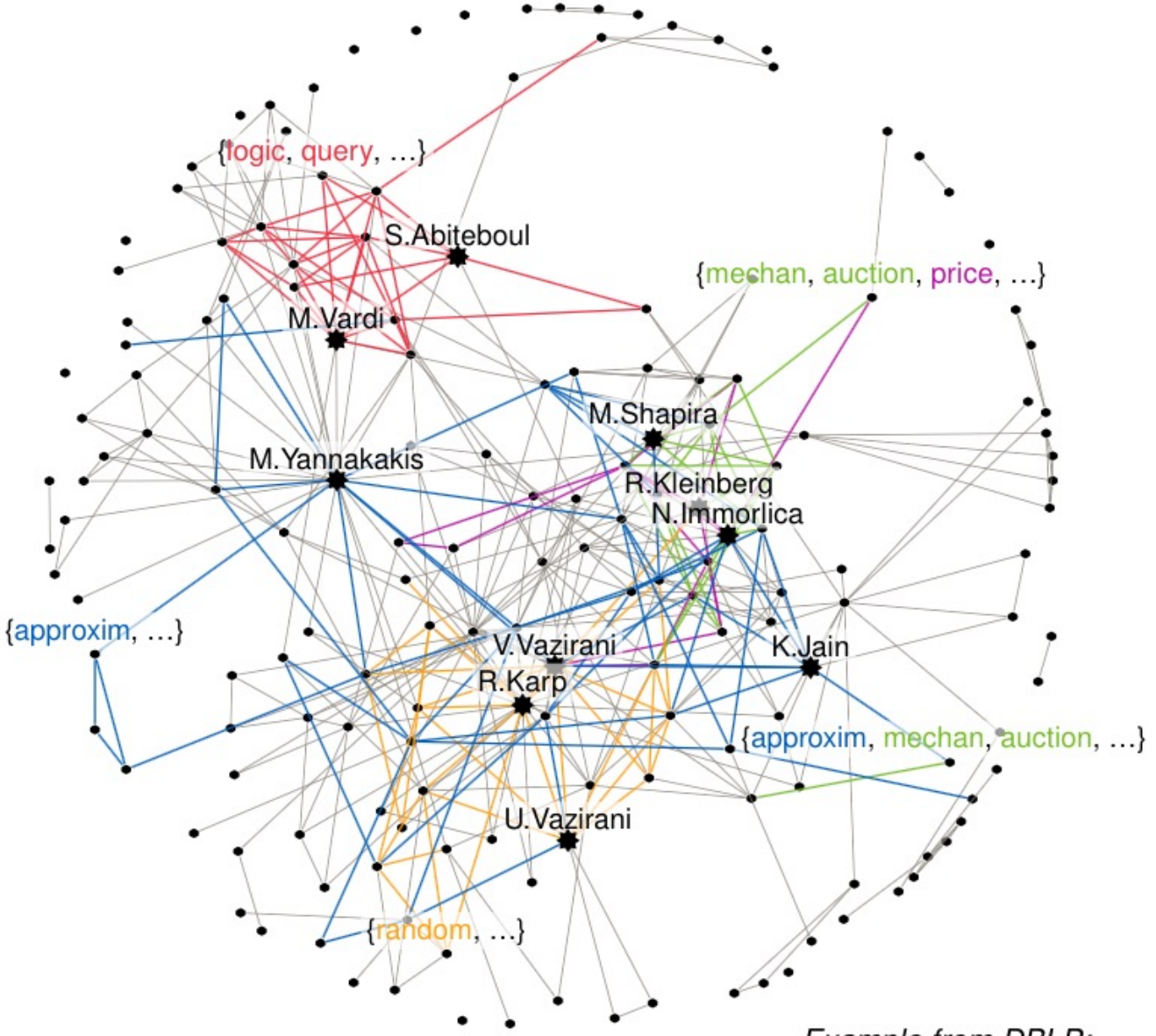
Graphs



Citation graph of literary theory academic papers

Graphs

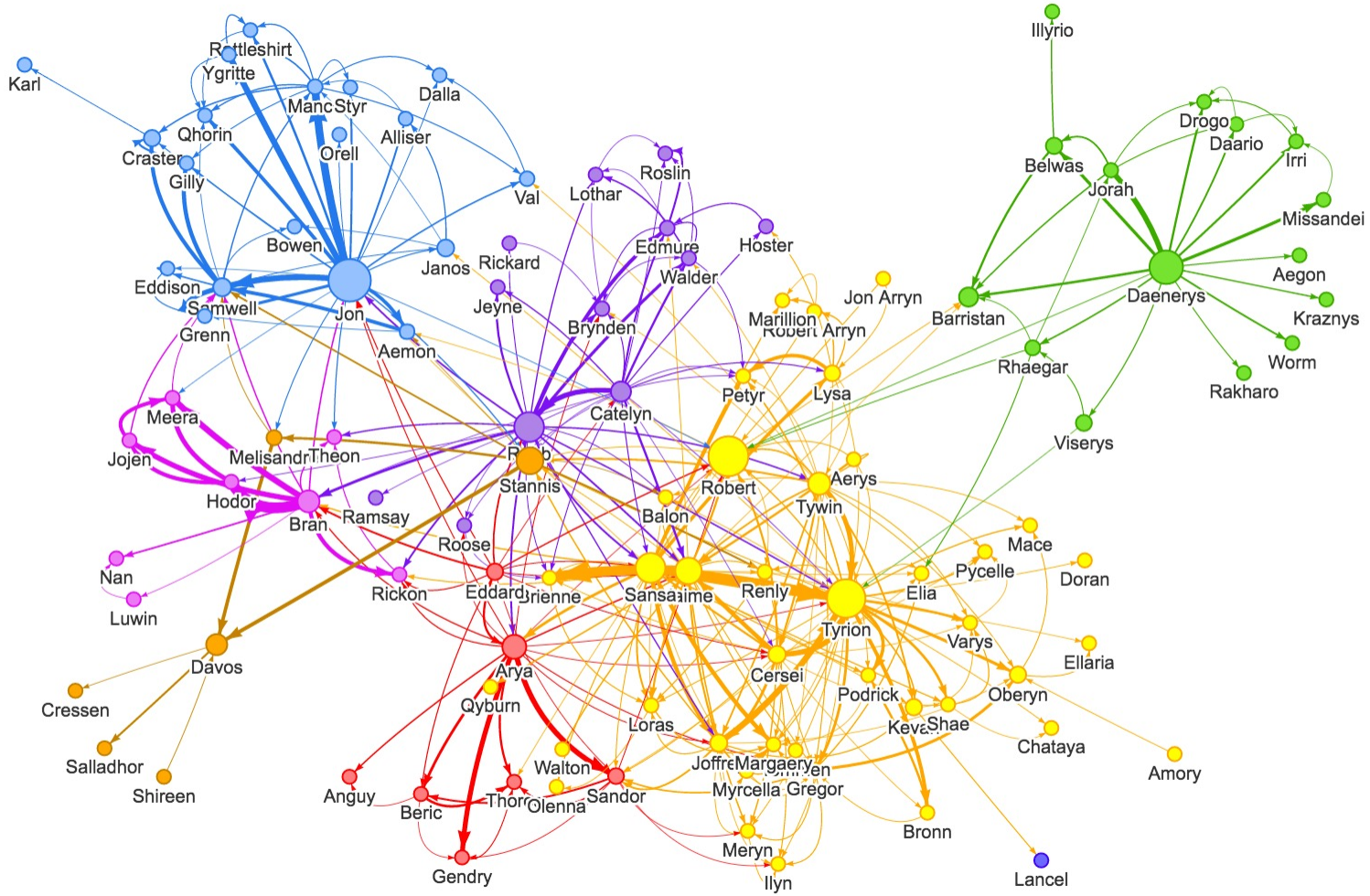
Theoretical Computer Science academic communities



Example from DBLP:
Communities within the co-authors of Christos H. Papadimitriou

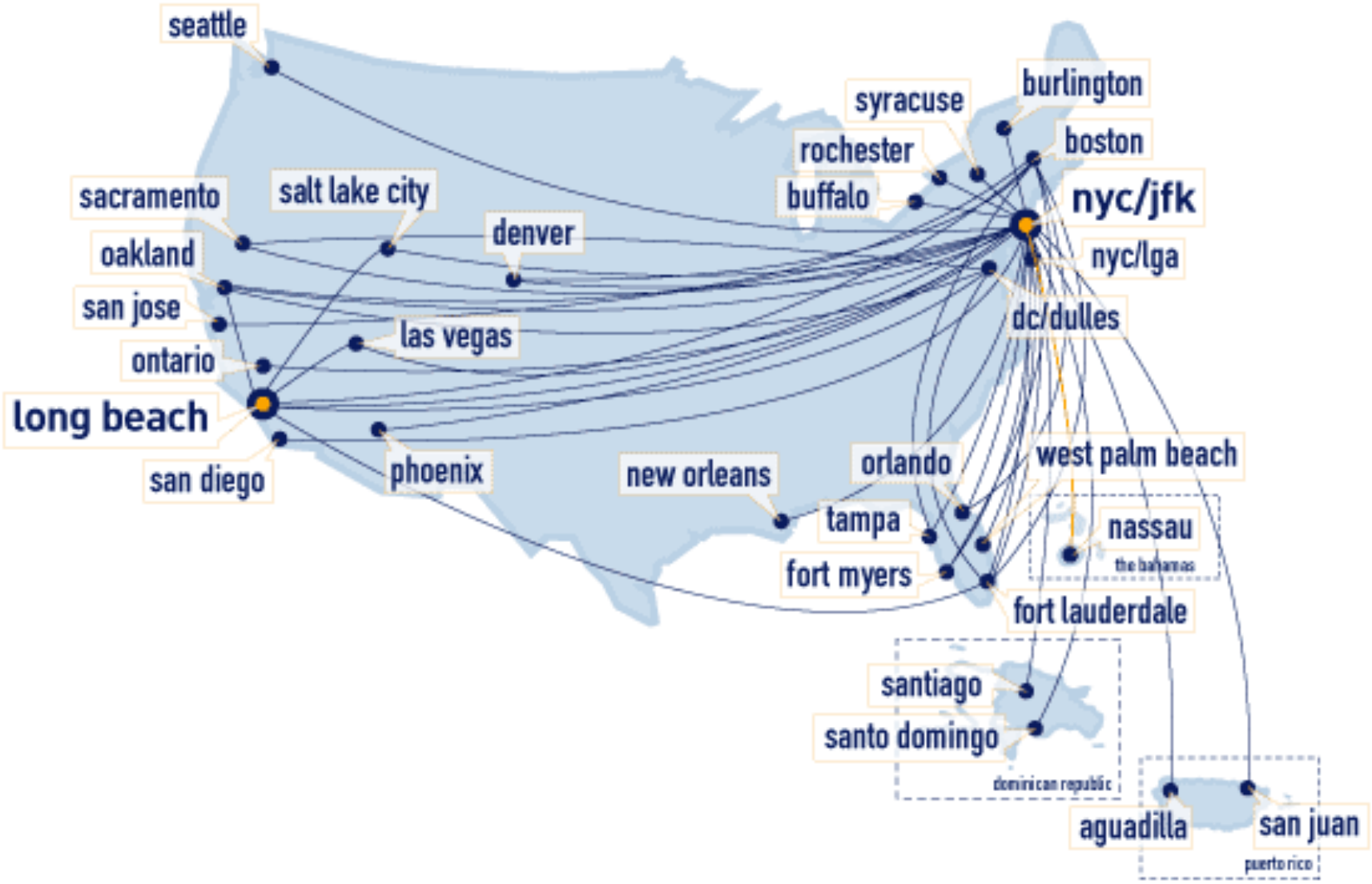
Graphs

Game of Thrones Character Interaction Network



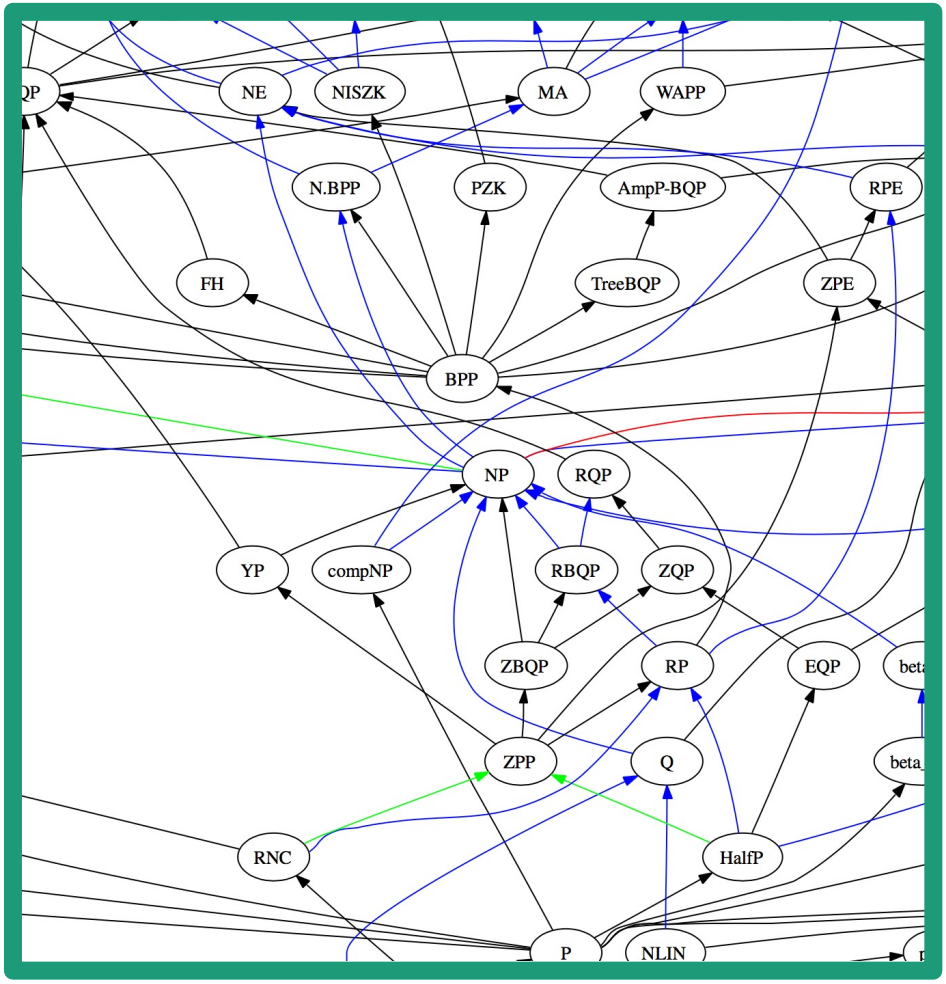
Graphs

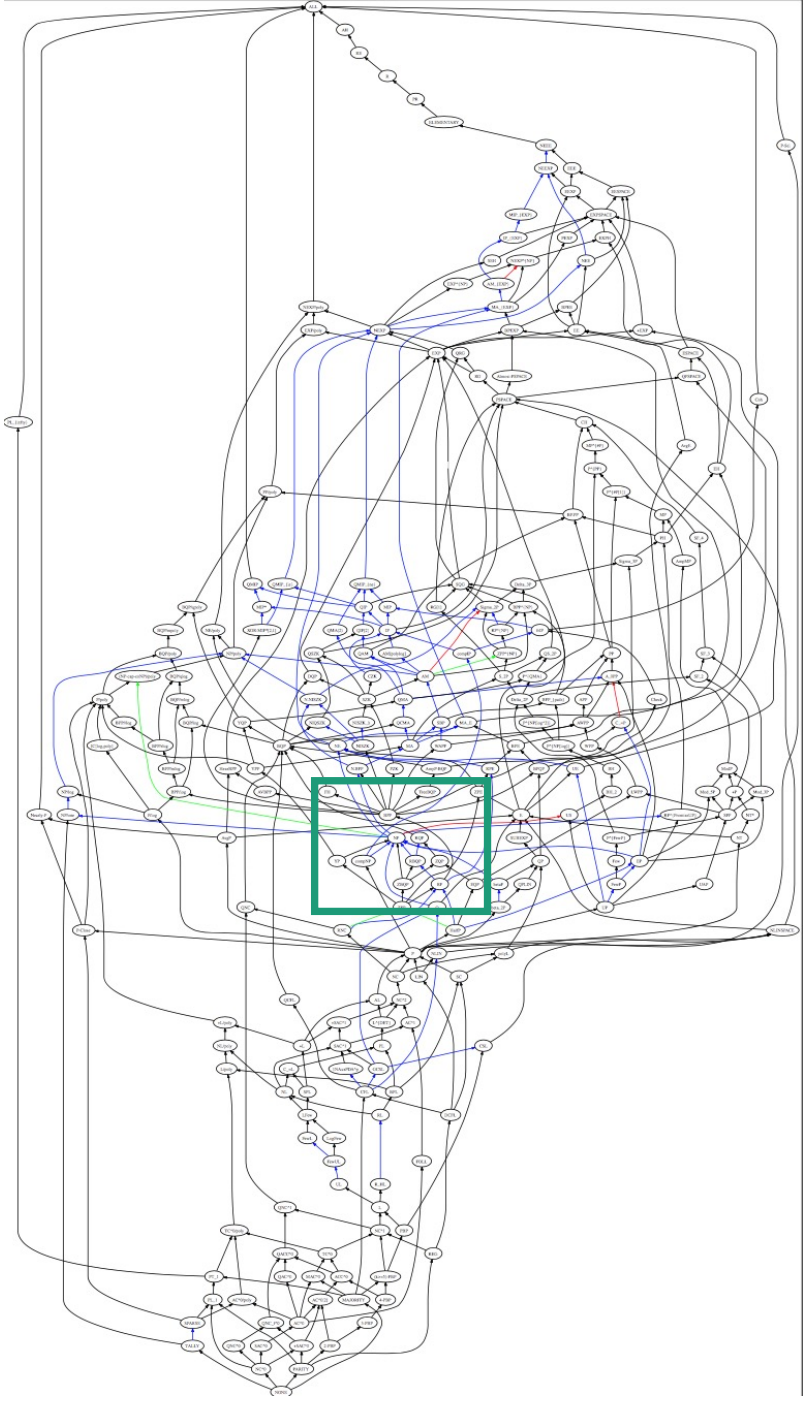
jetblue flights



Graphs

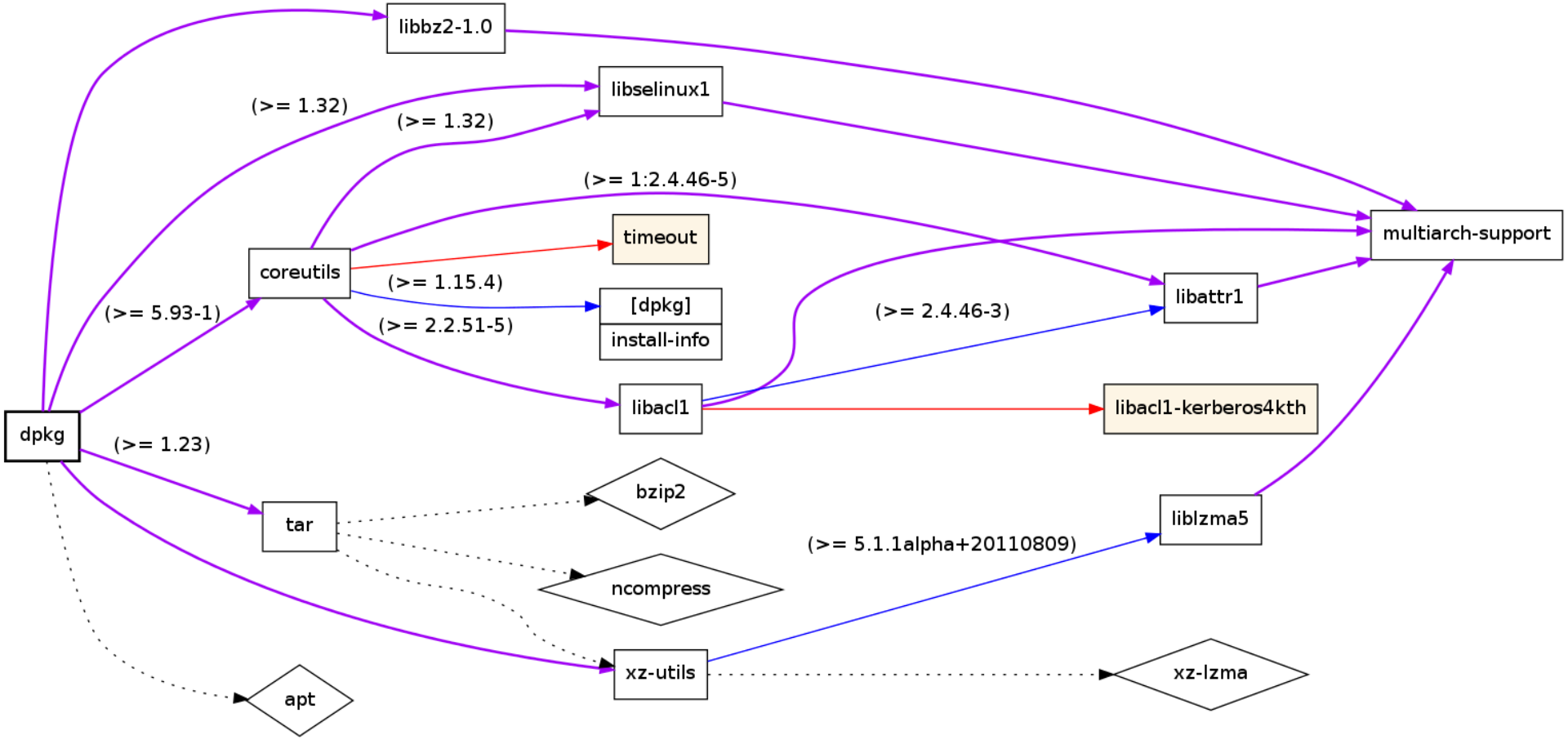
Complexity Zoo
containment graph





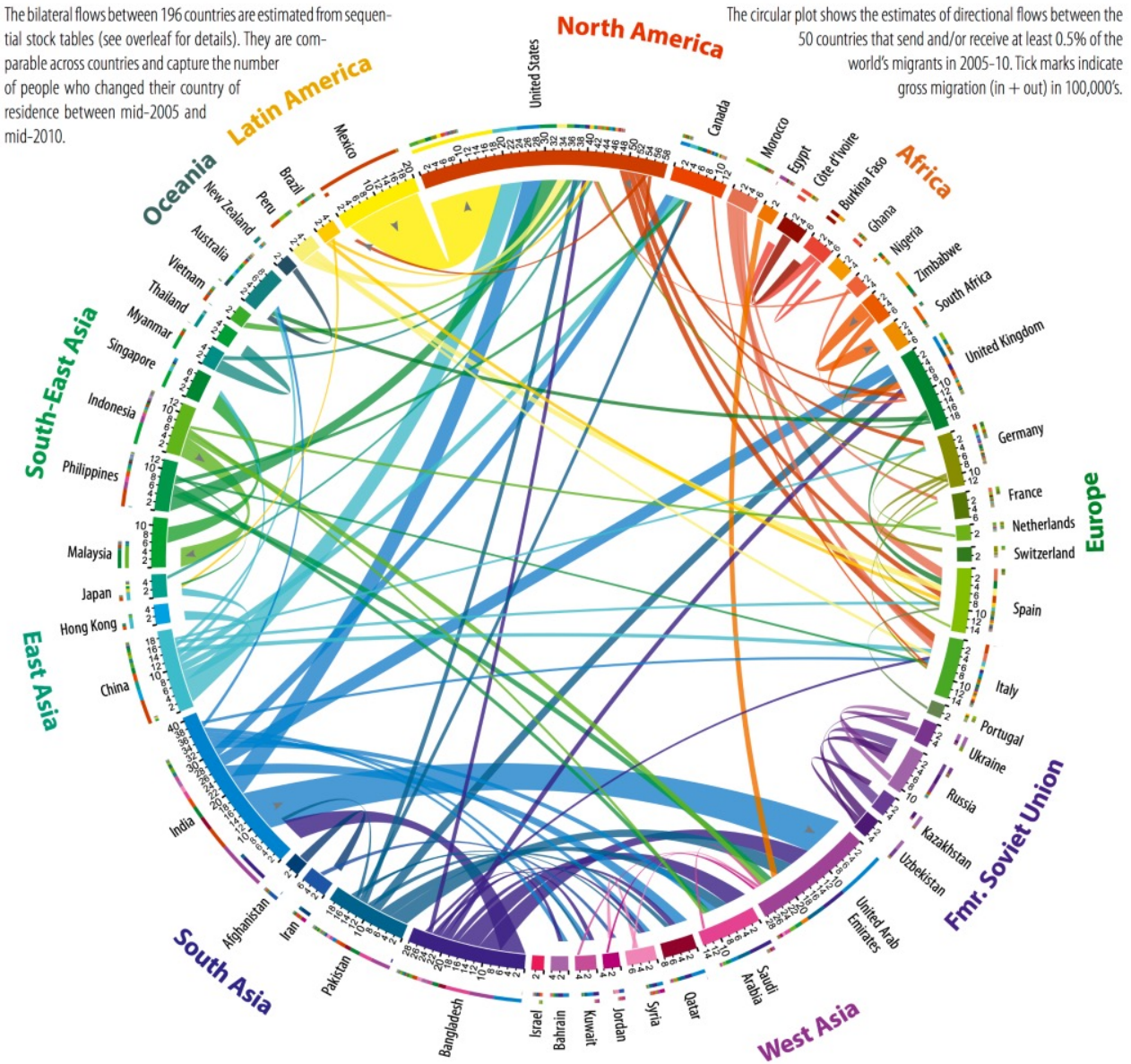
Graphs

debian dependency (sub)graph



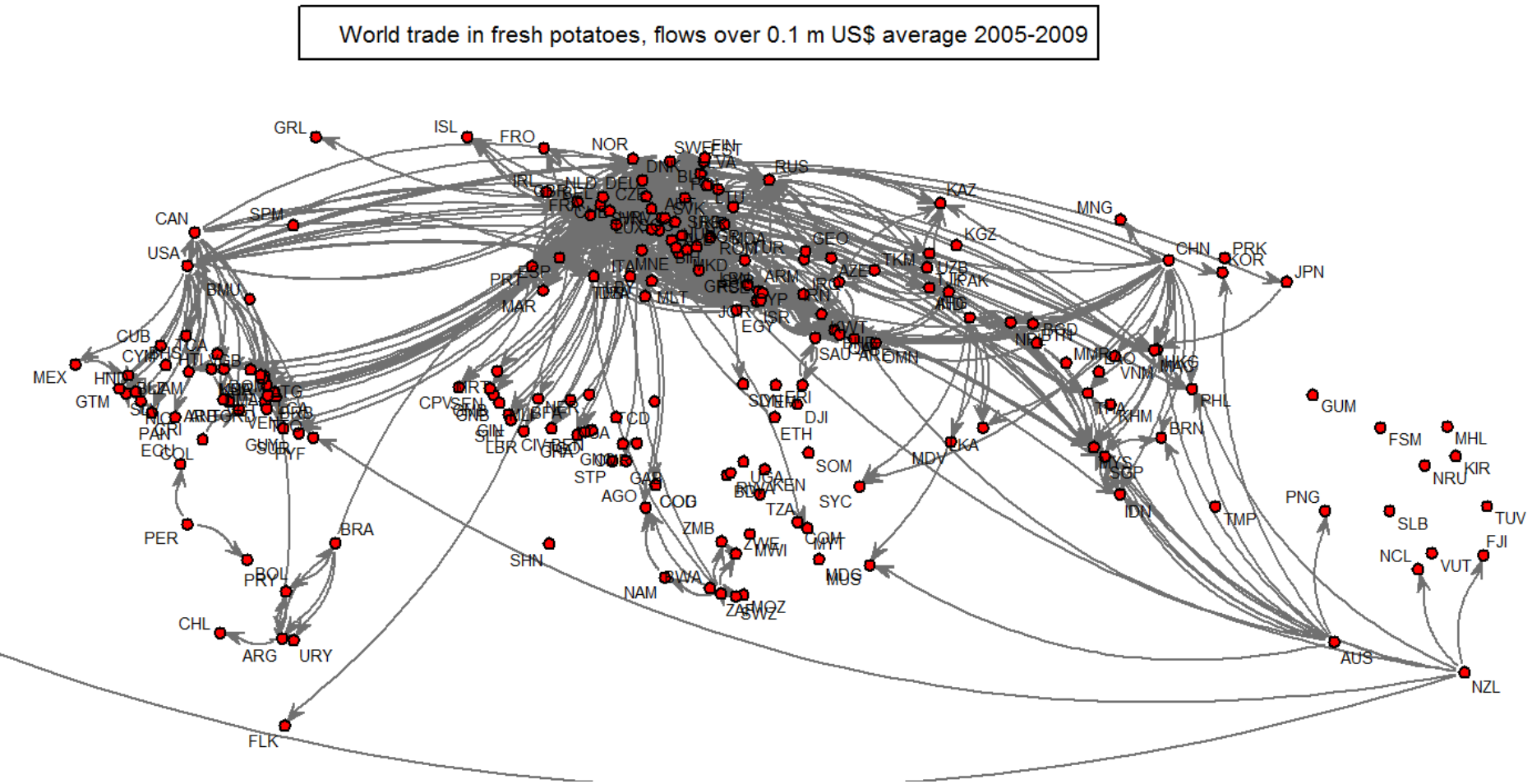
Graphs

Immigration flows



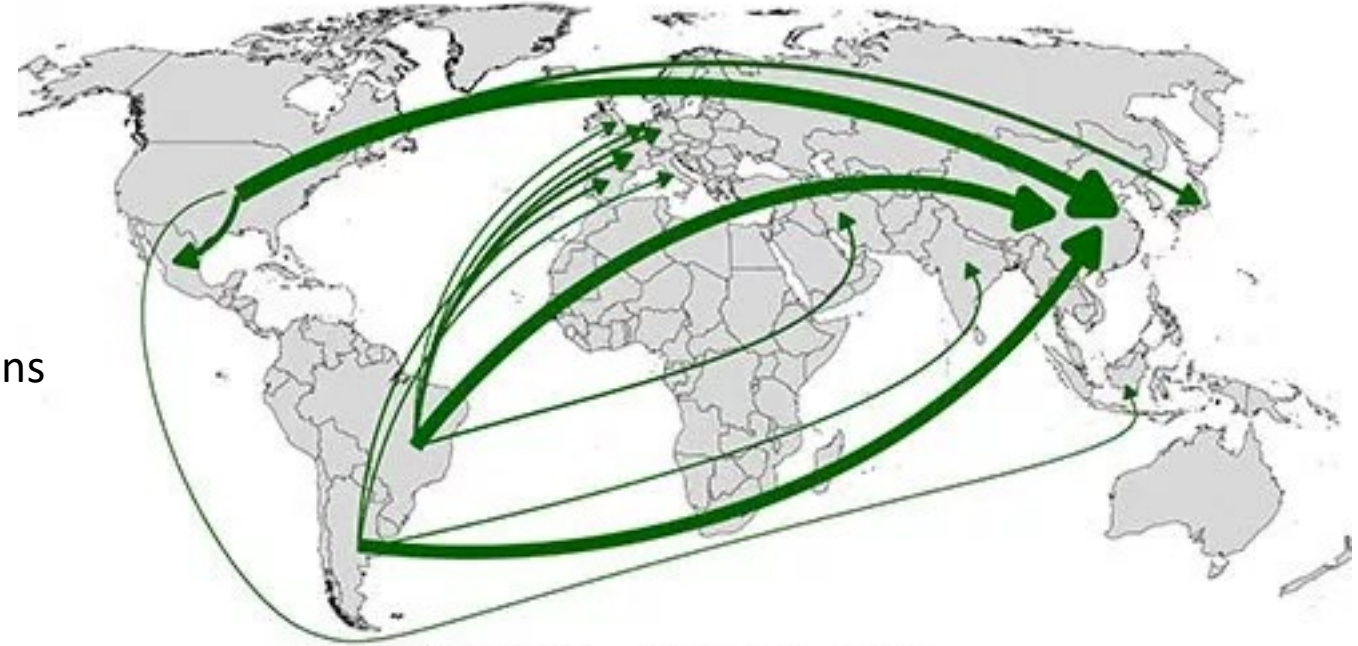
Graphs

Potato trade

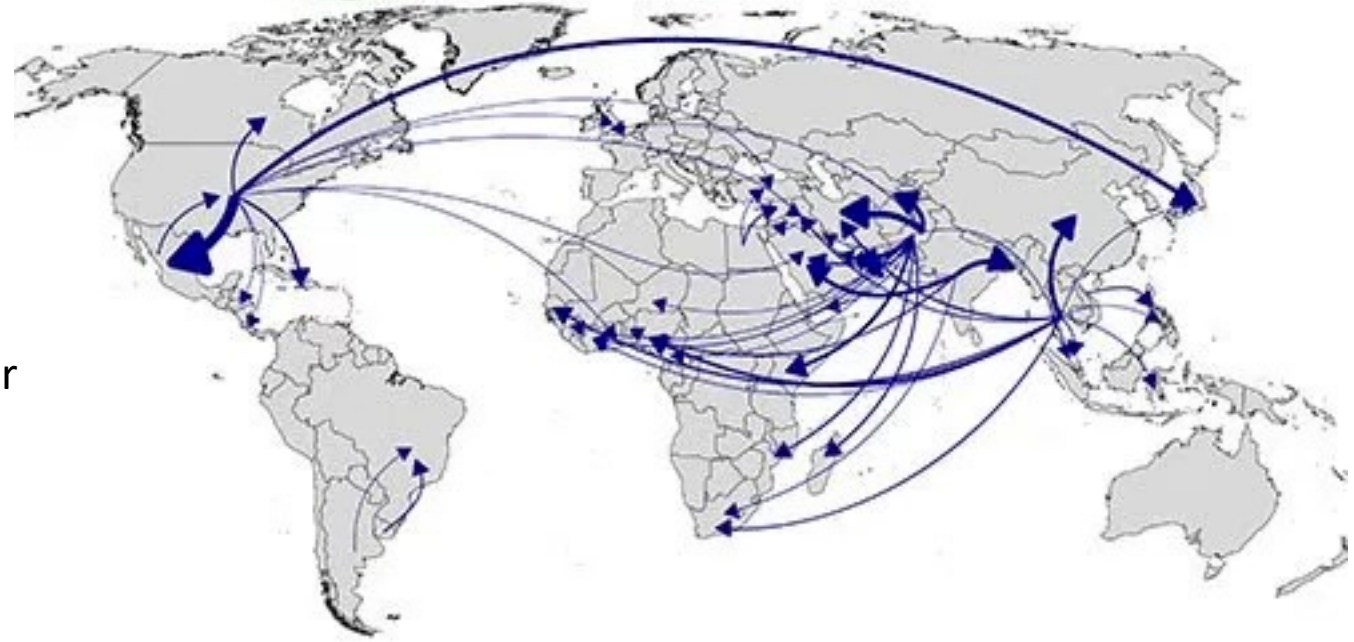


Graphs

Soybeans

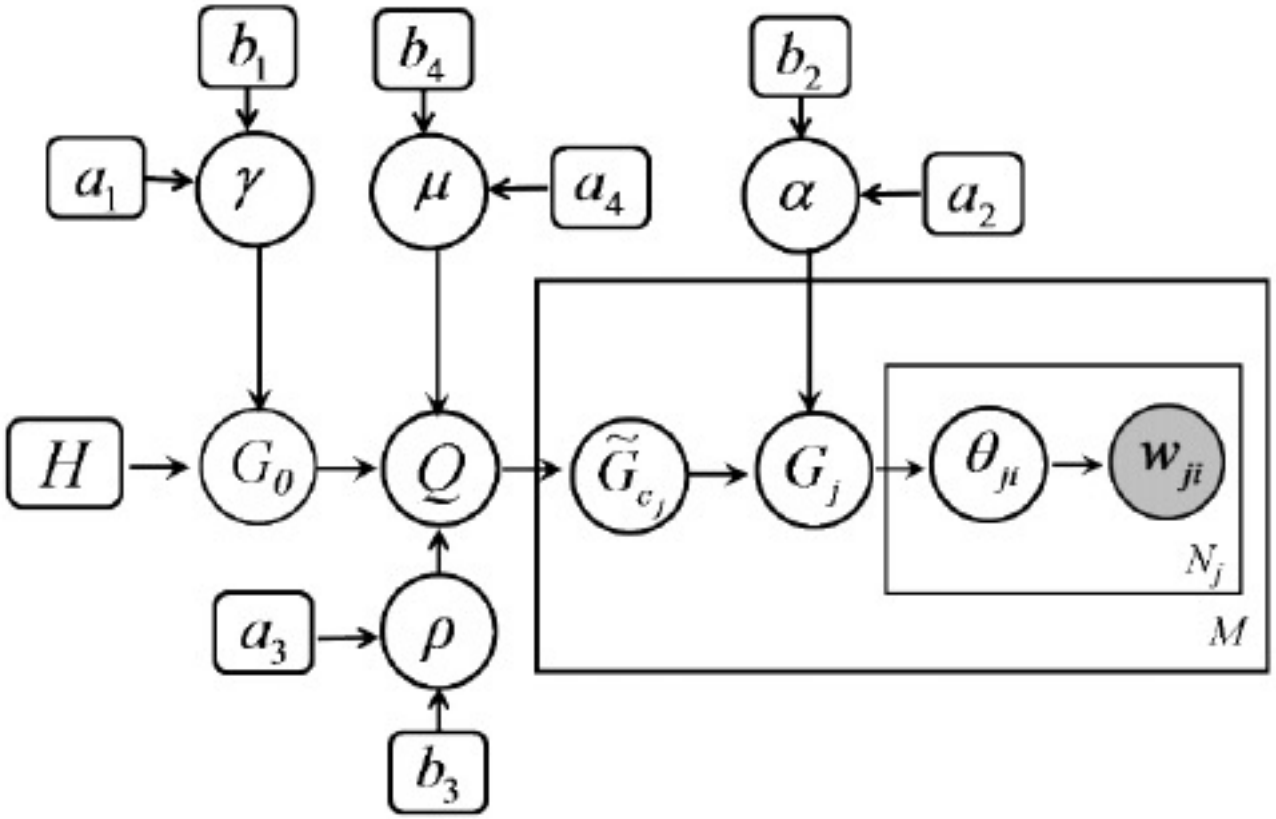


Water



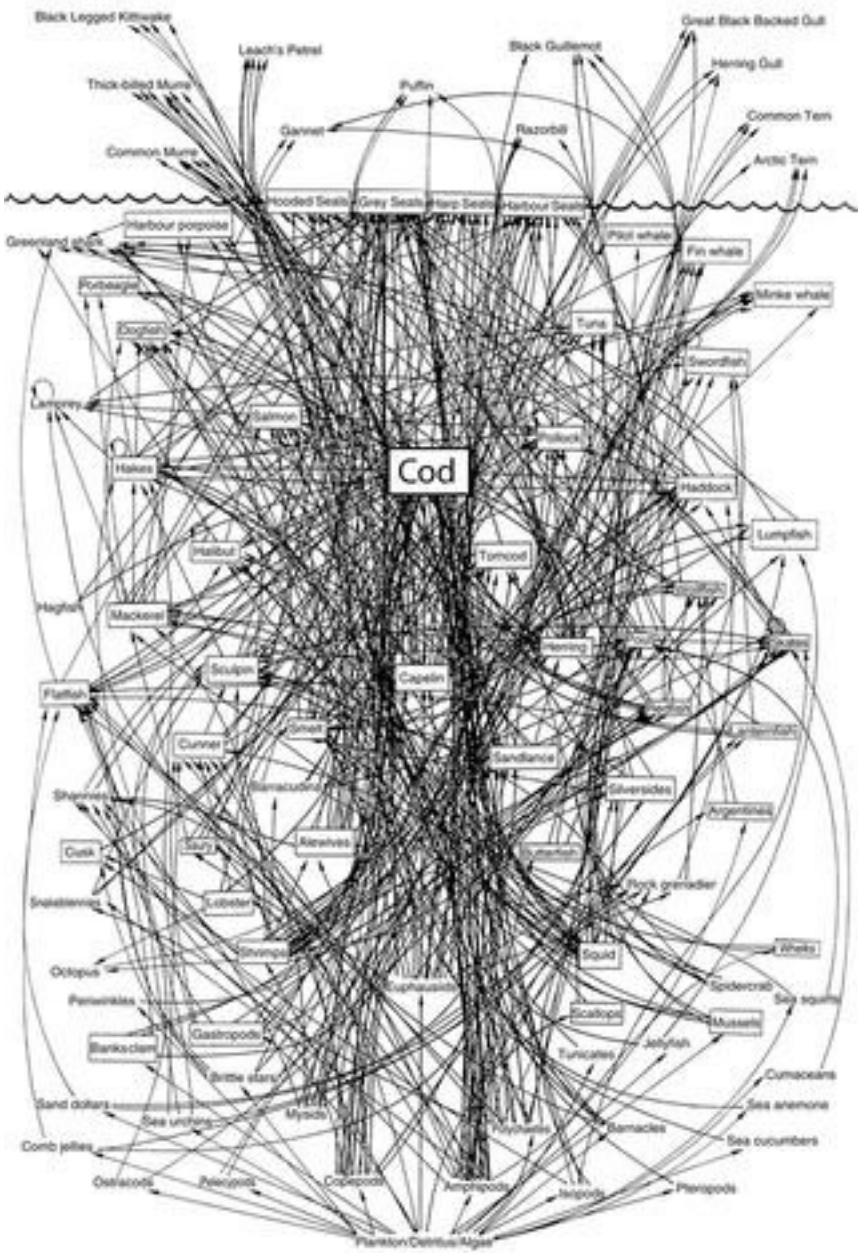
Graphs

Graphical models



Graphs

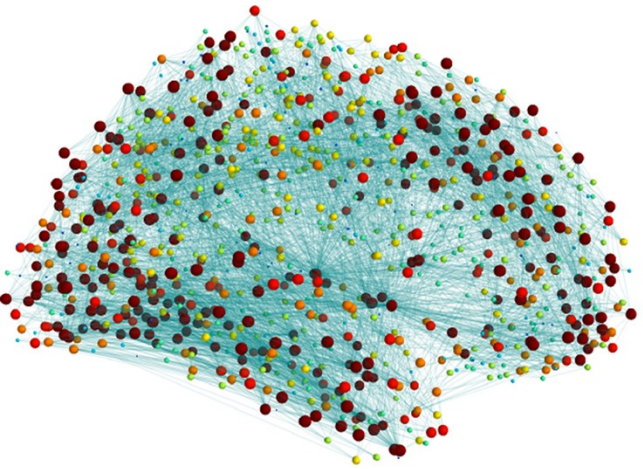
What eats what in the Atlantic ocean?

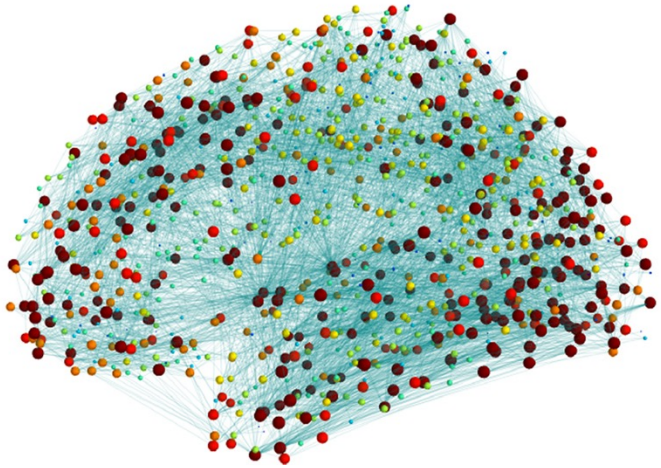


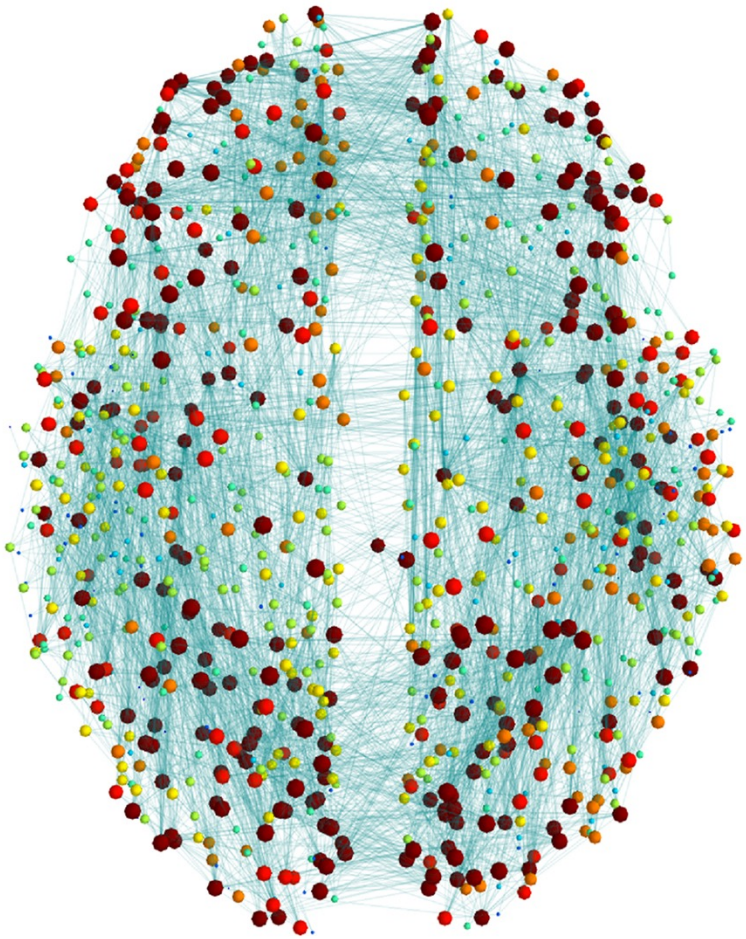
A simplified food web for the Northwest Atlantic. © IMMA

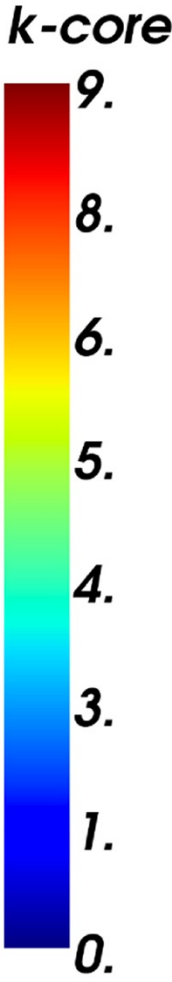
Graphs

Neural connections
in the brain









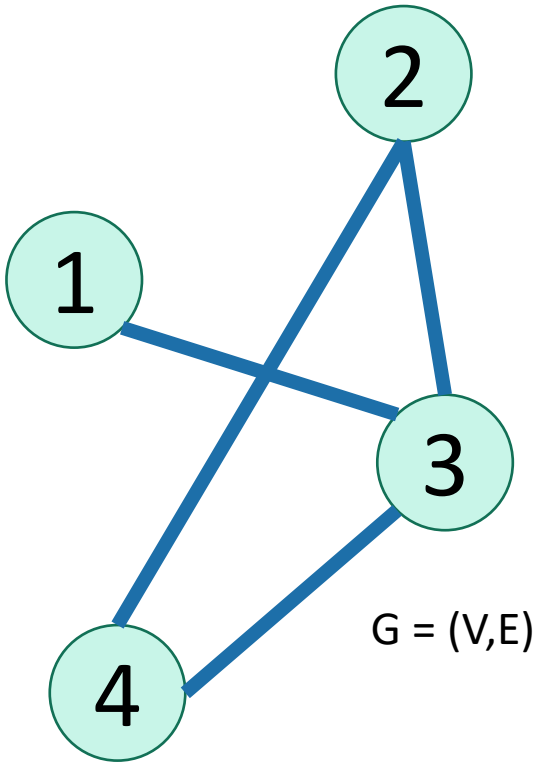
Graphs

- **There are a lot of graphs.**

- We want to answer questions about them.
  - Efficient routing?
  - Community detection/clustering?
    - Computing Bacon numbers
    - Signing up for classes without violating pre-req constraints
    - How to distribute fish in tanks so that none of them will fight.

Undirected Graphs

- Has vertices and edges
  - V is the set of vertices
  - E is the set of edges
  - Formally, a graph is G = (V,E)

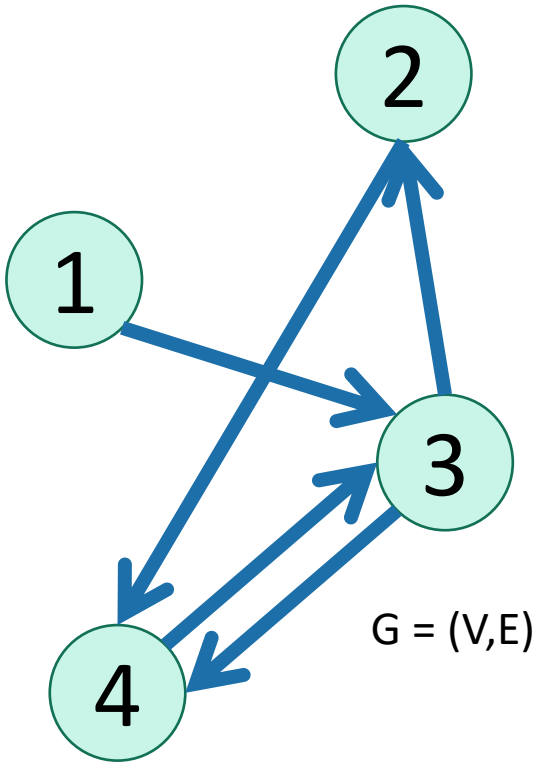


- Example
  - V = {1,2,3,4}
  - E = { {1,3}, {2,4}, {3,4}, {2,3} }

- The **degree** of vertex 4 is 2.
  - There are 2 edges coming out
- Vertex 4's **neighbors** are 2 and 3

Directed Graphs

- Has vertices and edges
  - V is the set of vertices
  - E is the set of **DIRECTED** edges
  - Formally, a graph is G = (V,E)



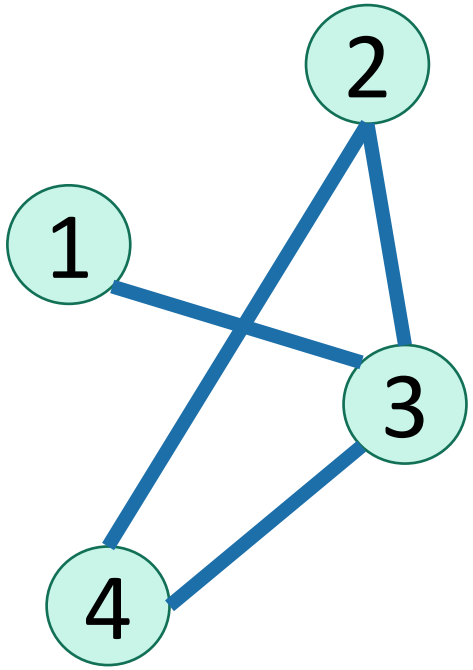
- Example
  - V = {1,2,3,4}
  - E = { (1,3), (2,4), (3,4), (4,3), (3,2) }

- The **in-degree** of vertex 4 is 2.
- The **out-degree** of vertex 4 is 1.
- Vertex 4's **incoming neighbors** are 2,3
- Vertex 4's **outgoing neighbor** is 3.

How do we represent graphs?

- Option 1: adjacency matrix

	1	2	3	4
1	0	0	1	0
2	0	0	1	1
3	1	1	0	1
4	0	1	1	0



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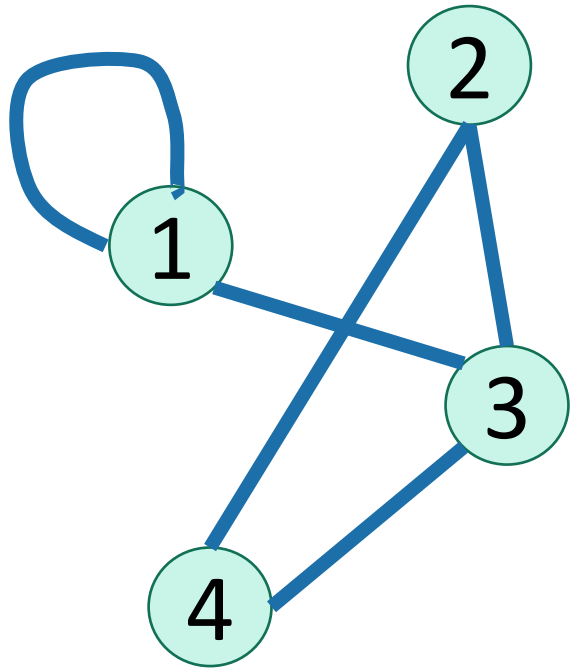

How do we represent graphs?

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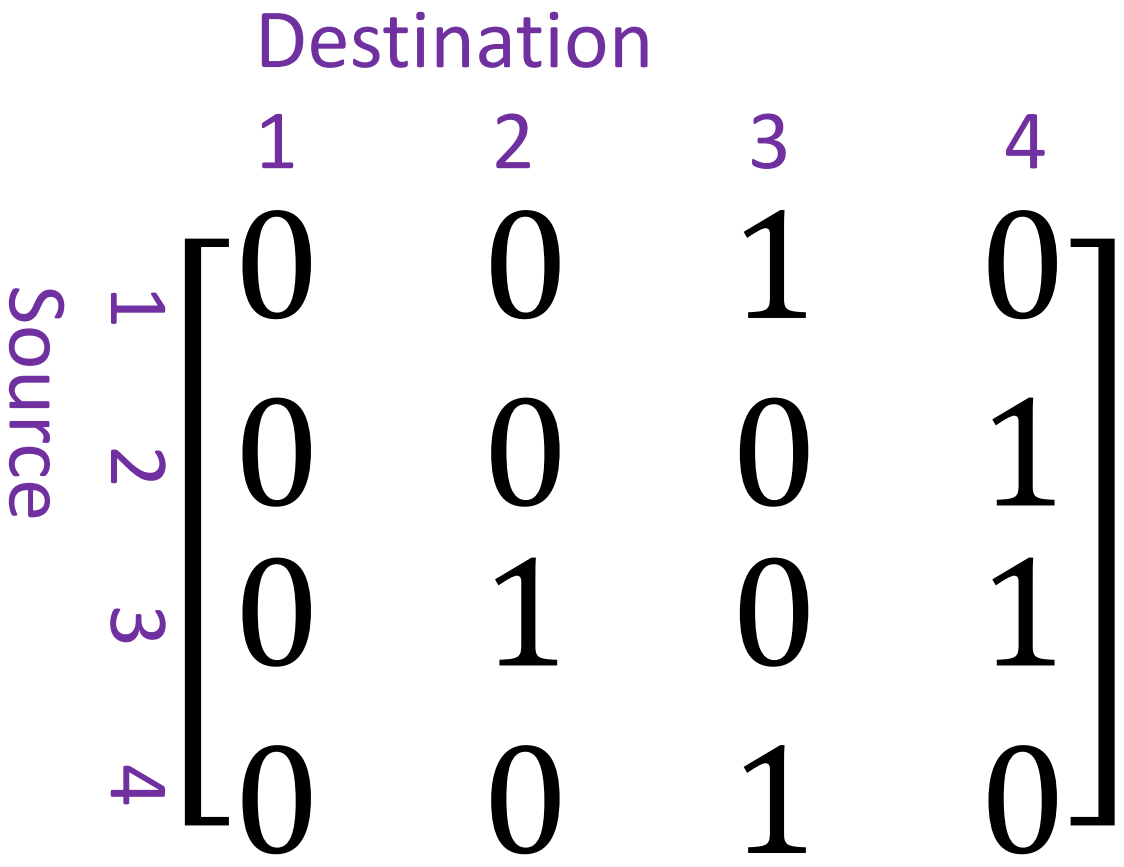
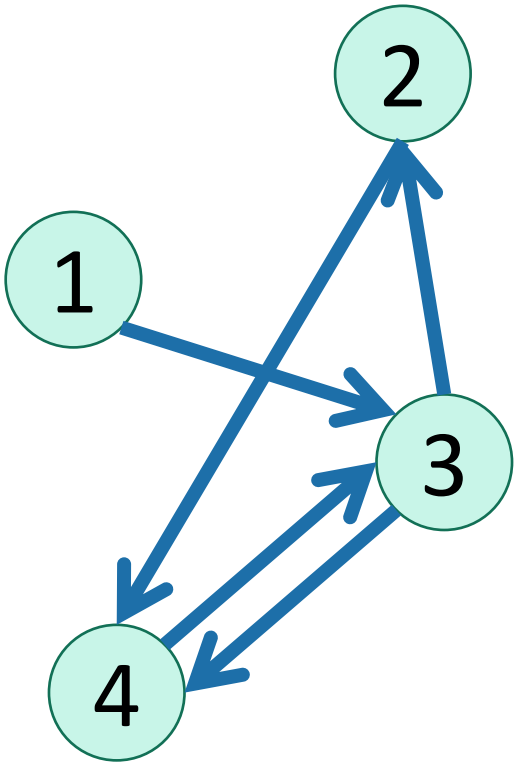
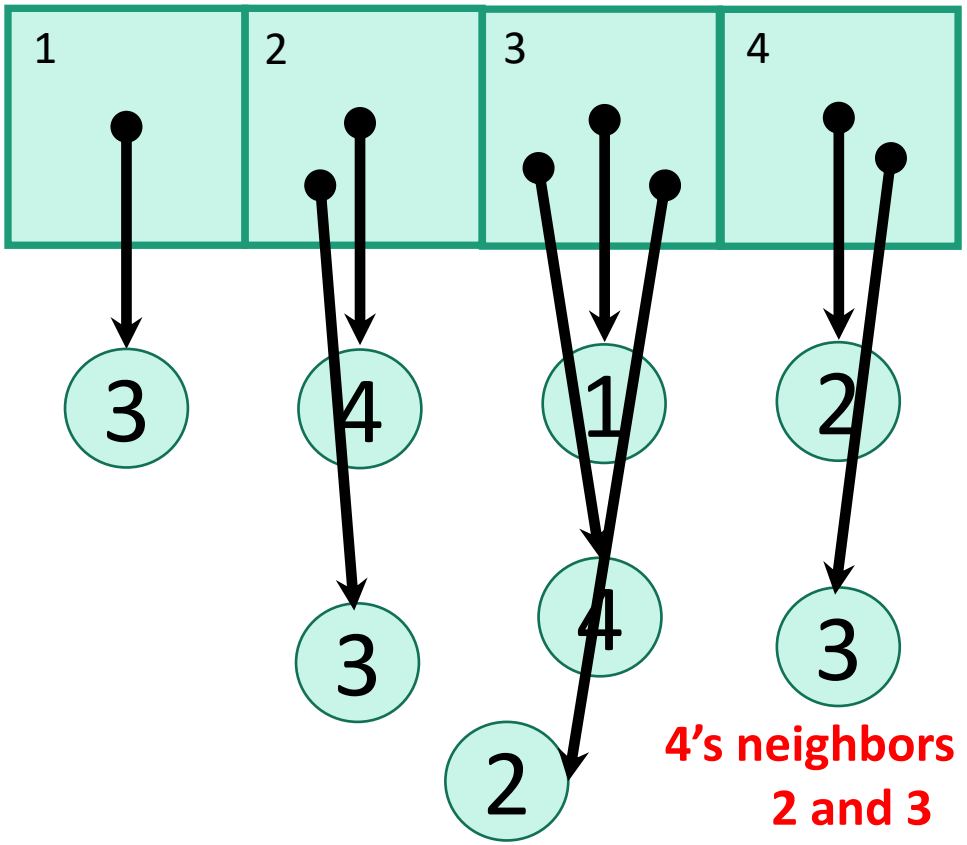


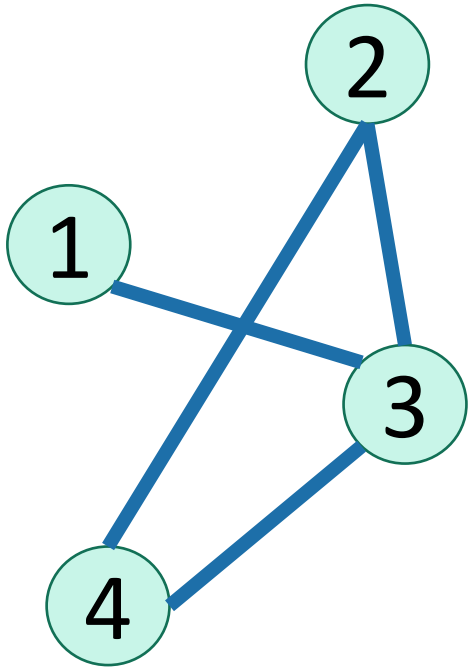


How do we represent graphs?

- Option 2: linked lists.



4's neighbors are 2 and 3



How would you modify this for directed graphs?

In either case

- Vertices can store other information
  - Attributes (name, IP address, …)
  - helper info for algorithms that we will perform on the graph

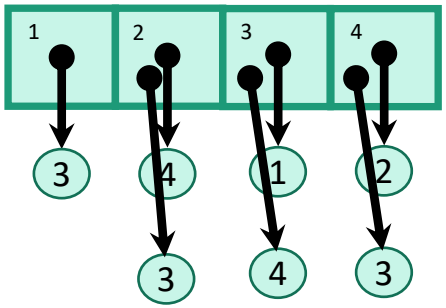
- Want to be able to do the following operations:
  - **Edge Membership**: Is edge e in E?
  - **Neighbor Query**: What are the neighbors of vertex v?

Trade-offs

Say there are n vertices and m edges.

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Generally better for **sparse** graphs

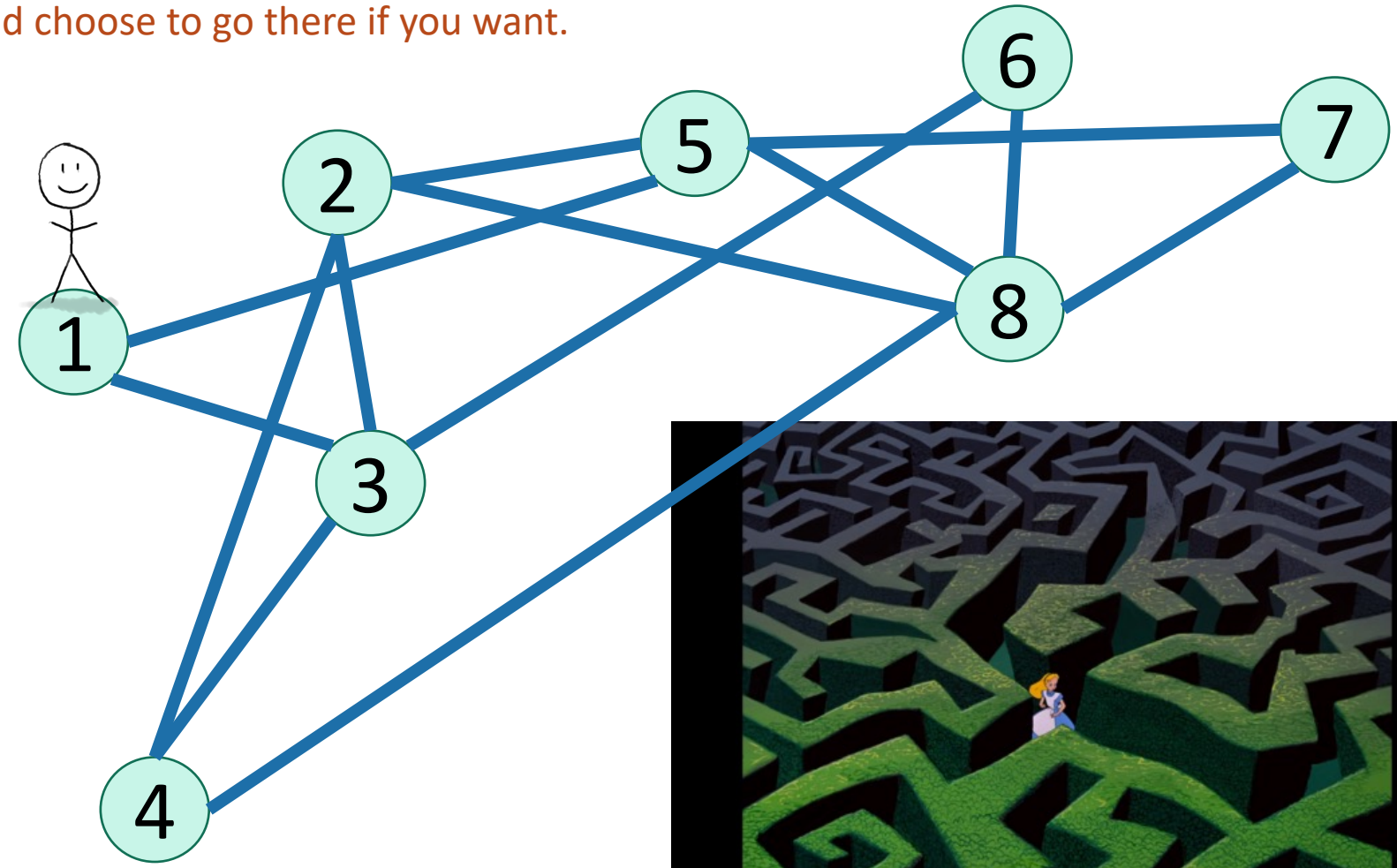


Edge membership Is e = {v,w} in E?	O(1)	O(deg(v)) or O(deg(w))
Neighbor query Give me v's neighbors.	O(n)	O(deg(v))
Space requirements	$O(n^2)$	O(n + m)

Part 1: Depth-first search

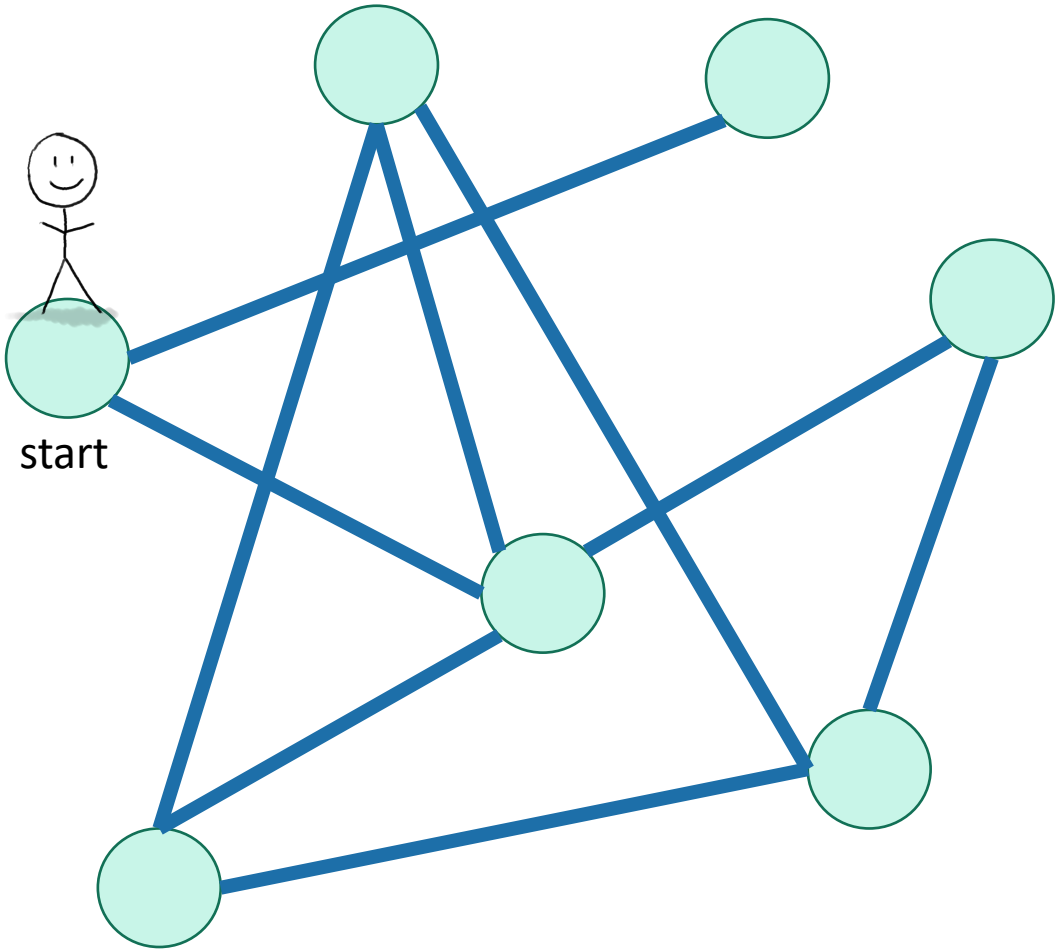
How do we explore a graph?

At each node, you can get a list of neighbors,
and choose to go there if you want.



Depth First Search

Exploring a labyrinth with chalk and a piece of string



Not been there yet

Been there, haven't explored all the paths out.

Been there, have explored all the paths out.

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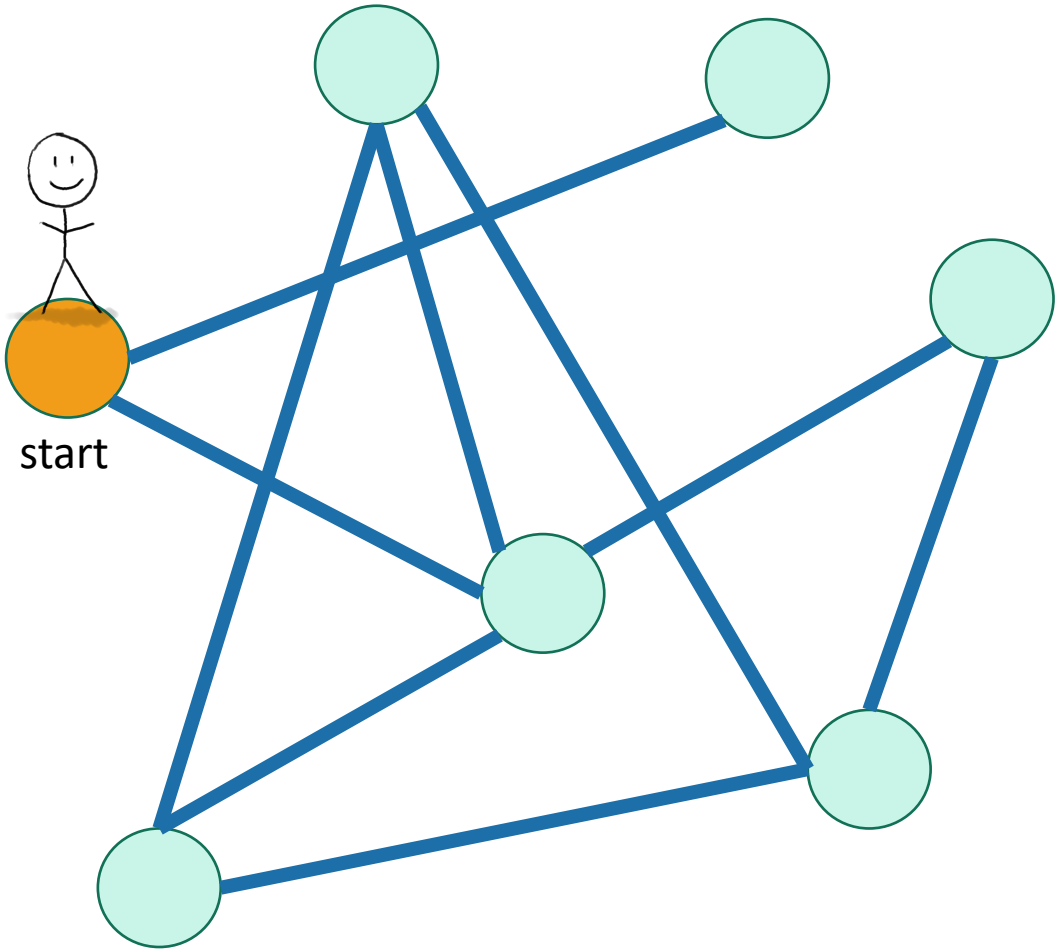

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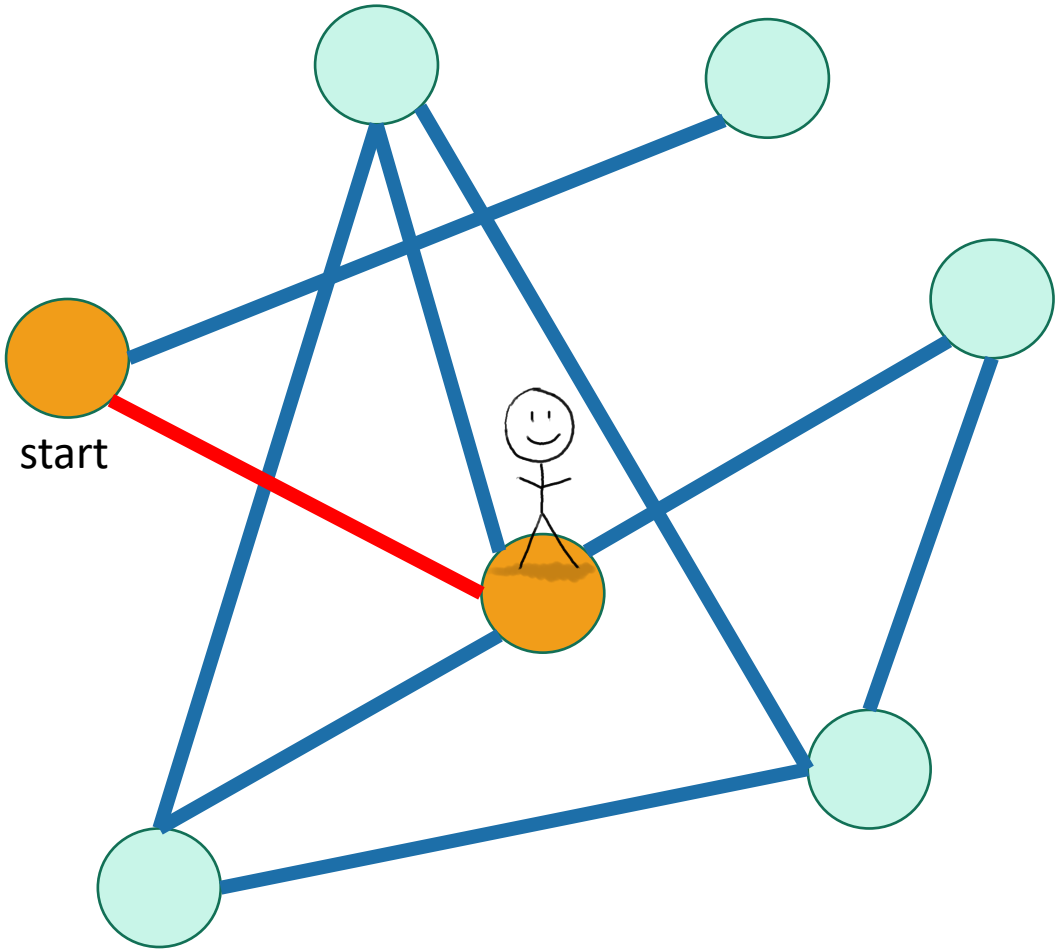

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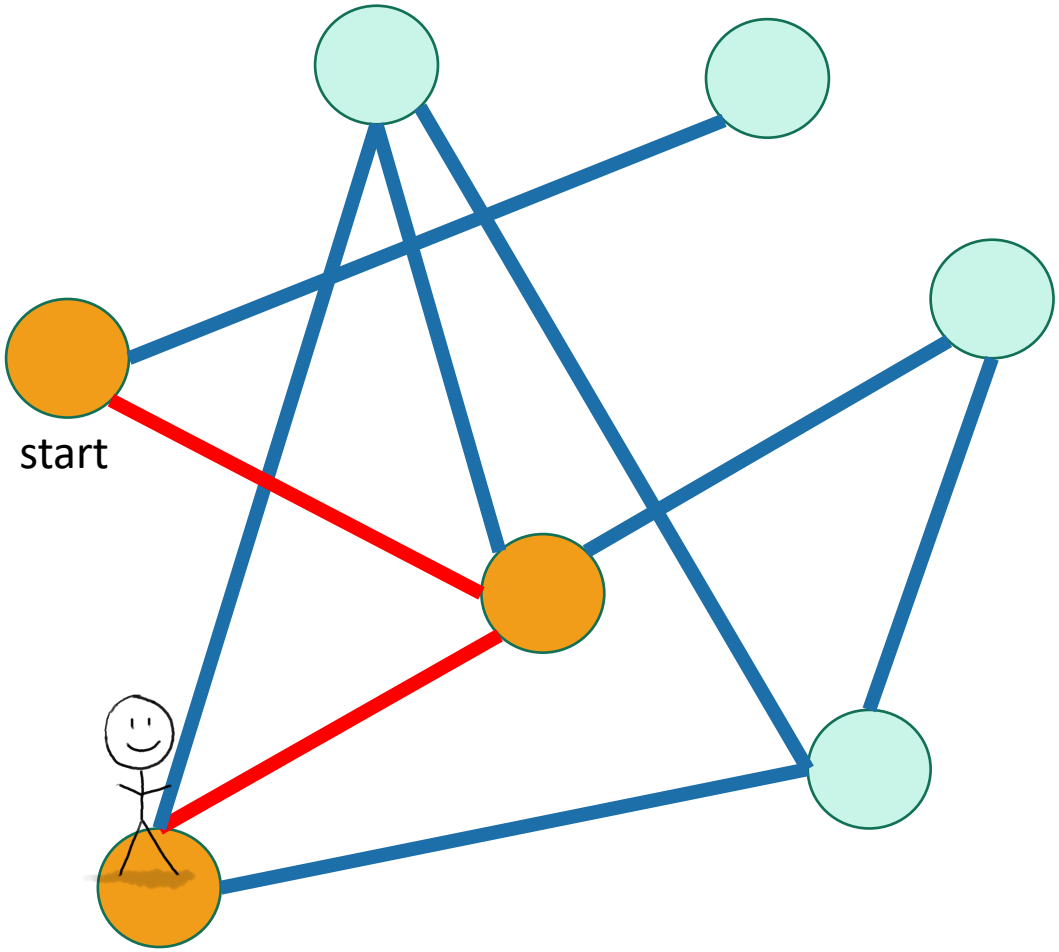

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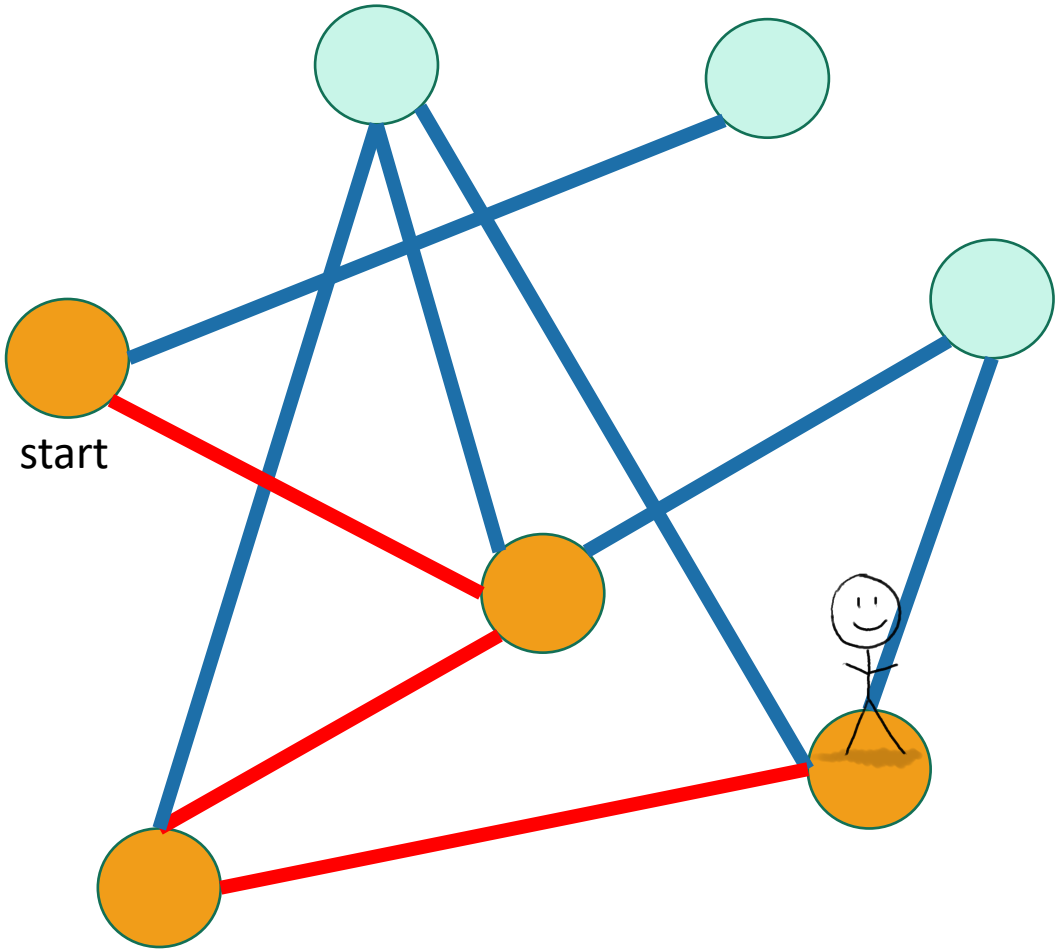

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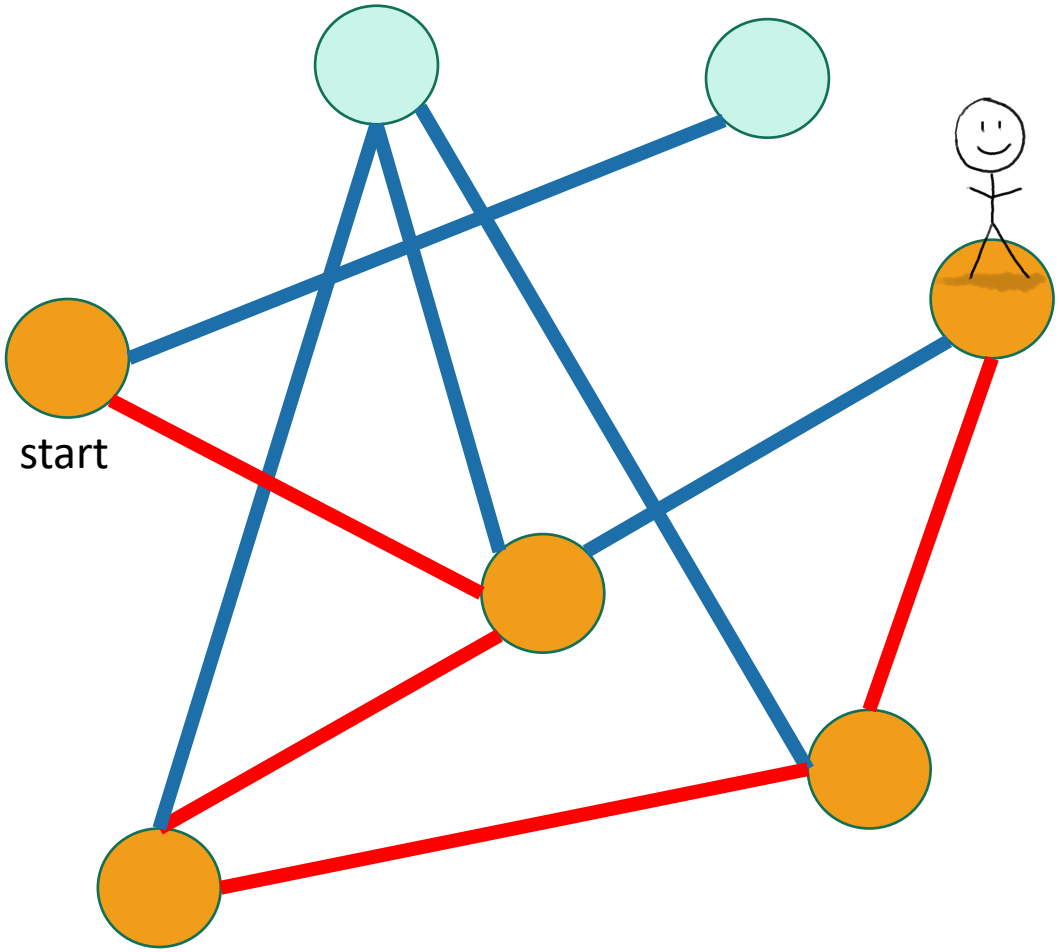

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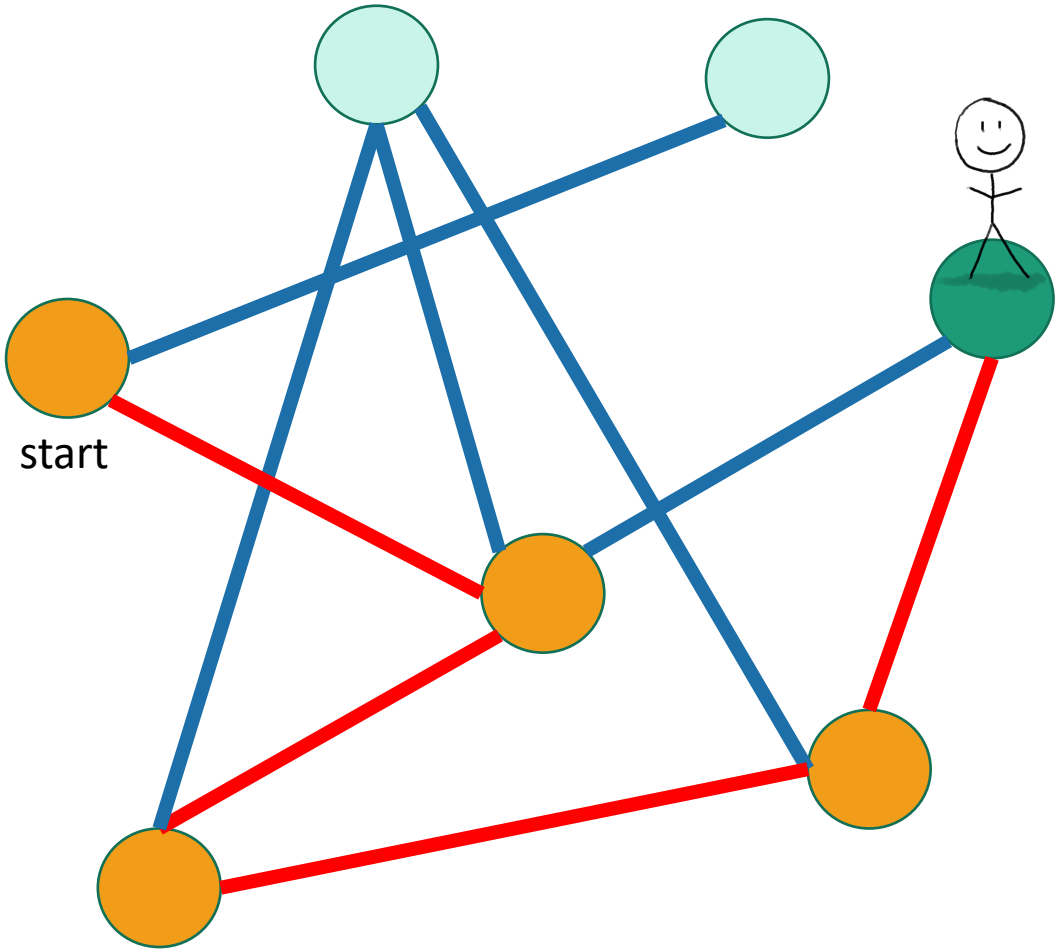

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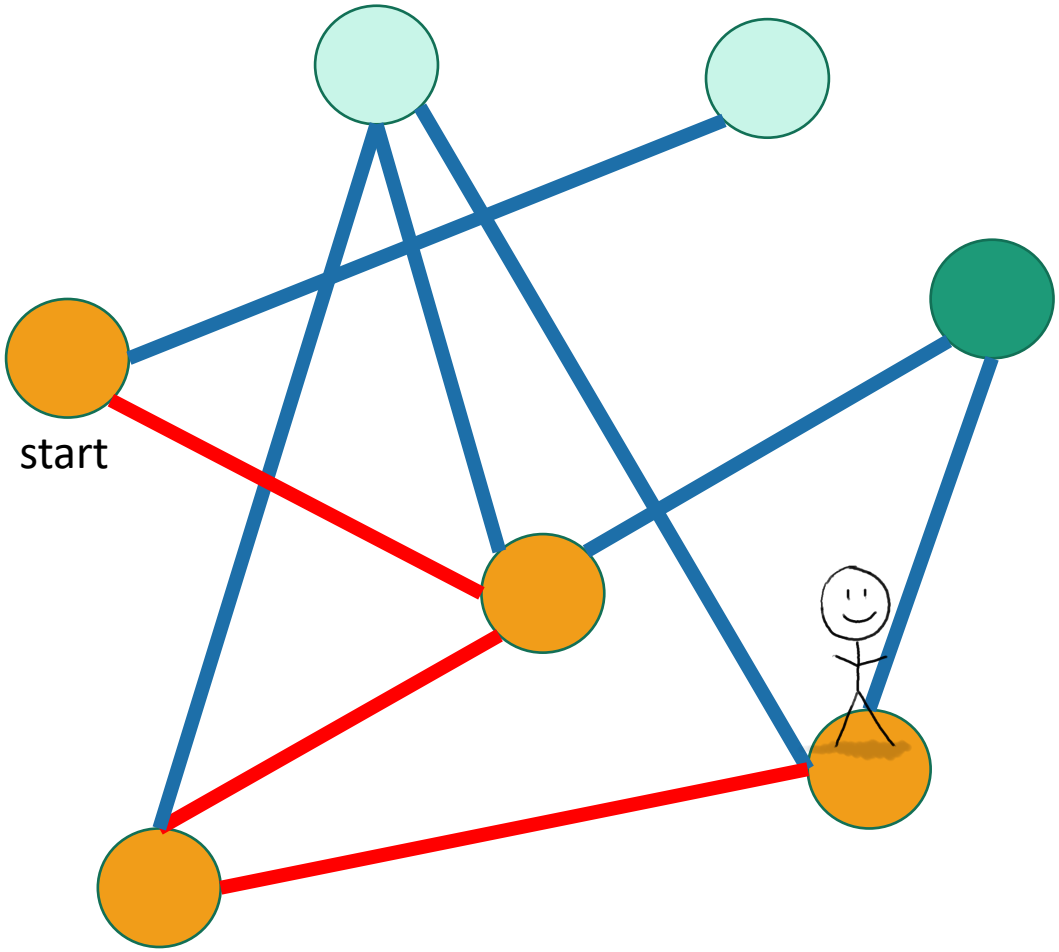

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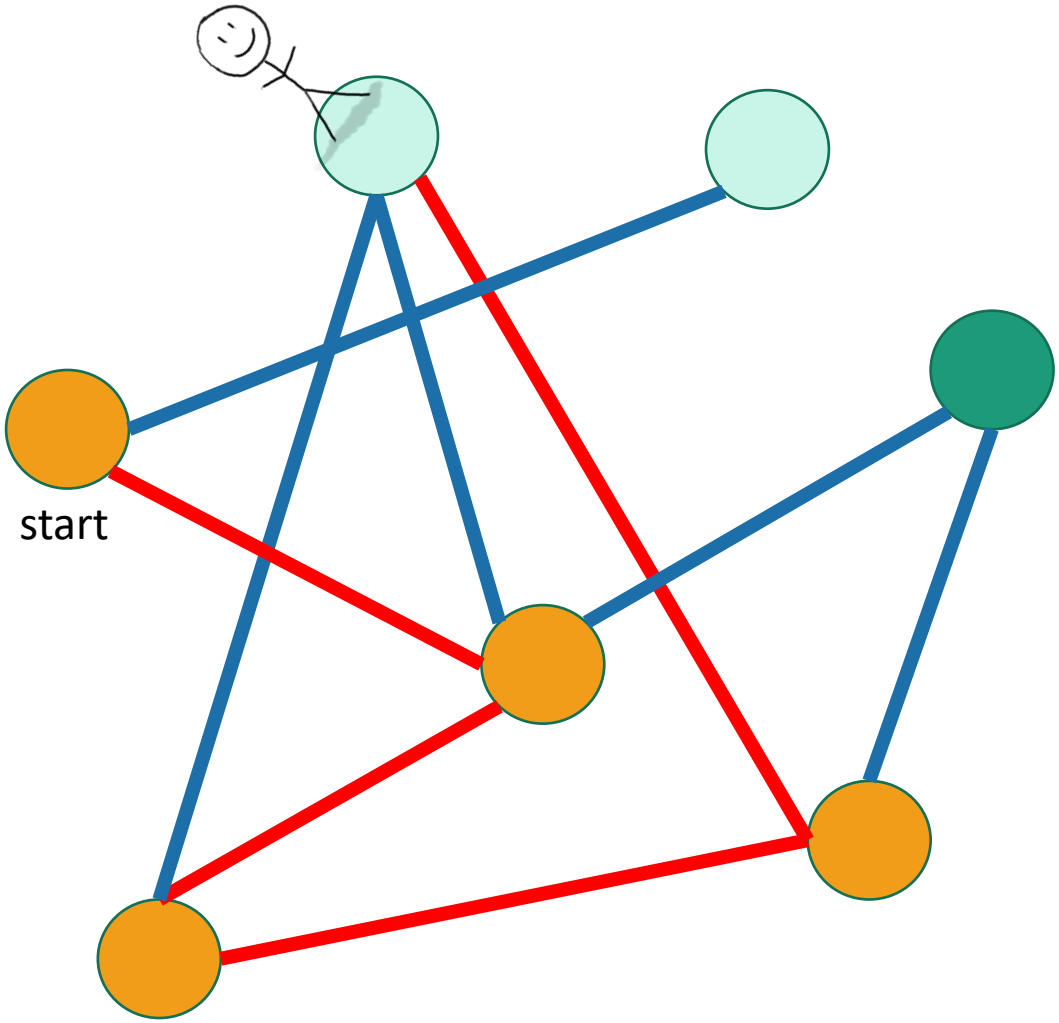

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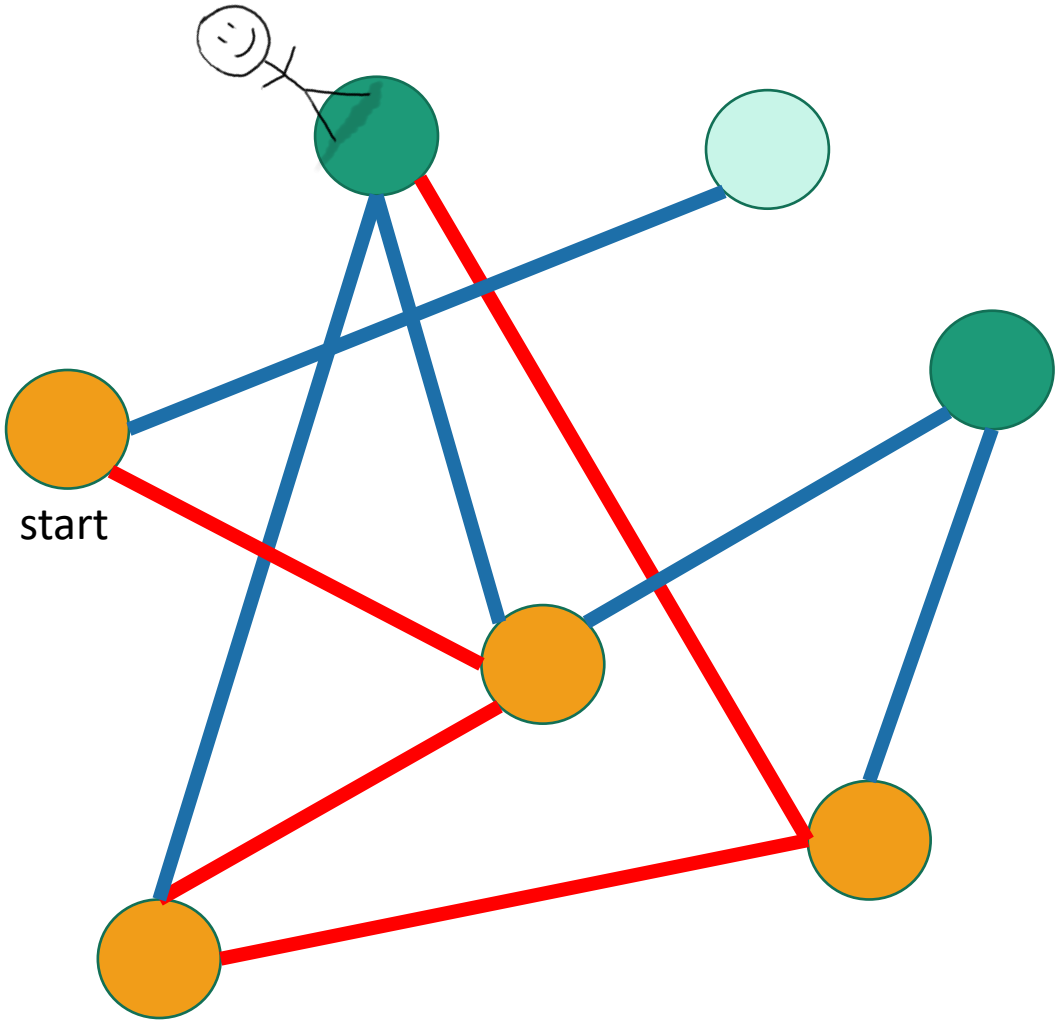

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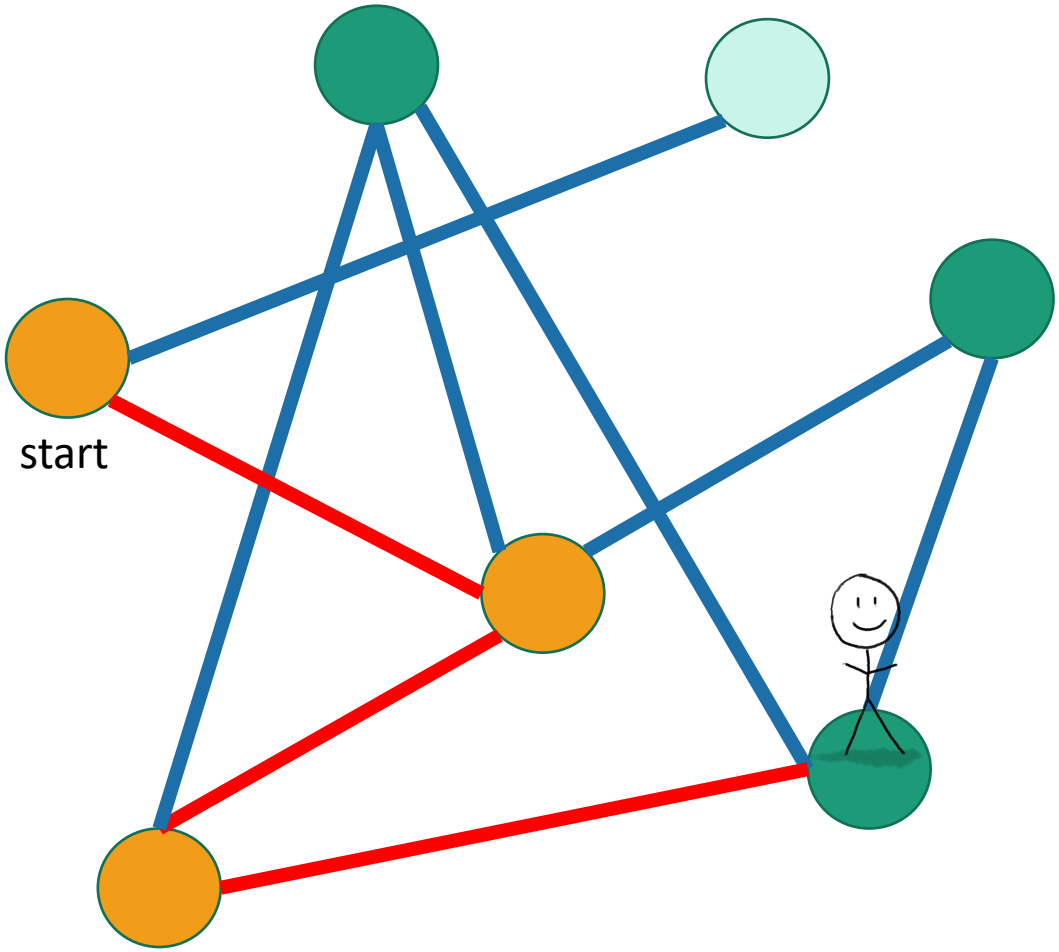

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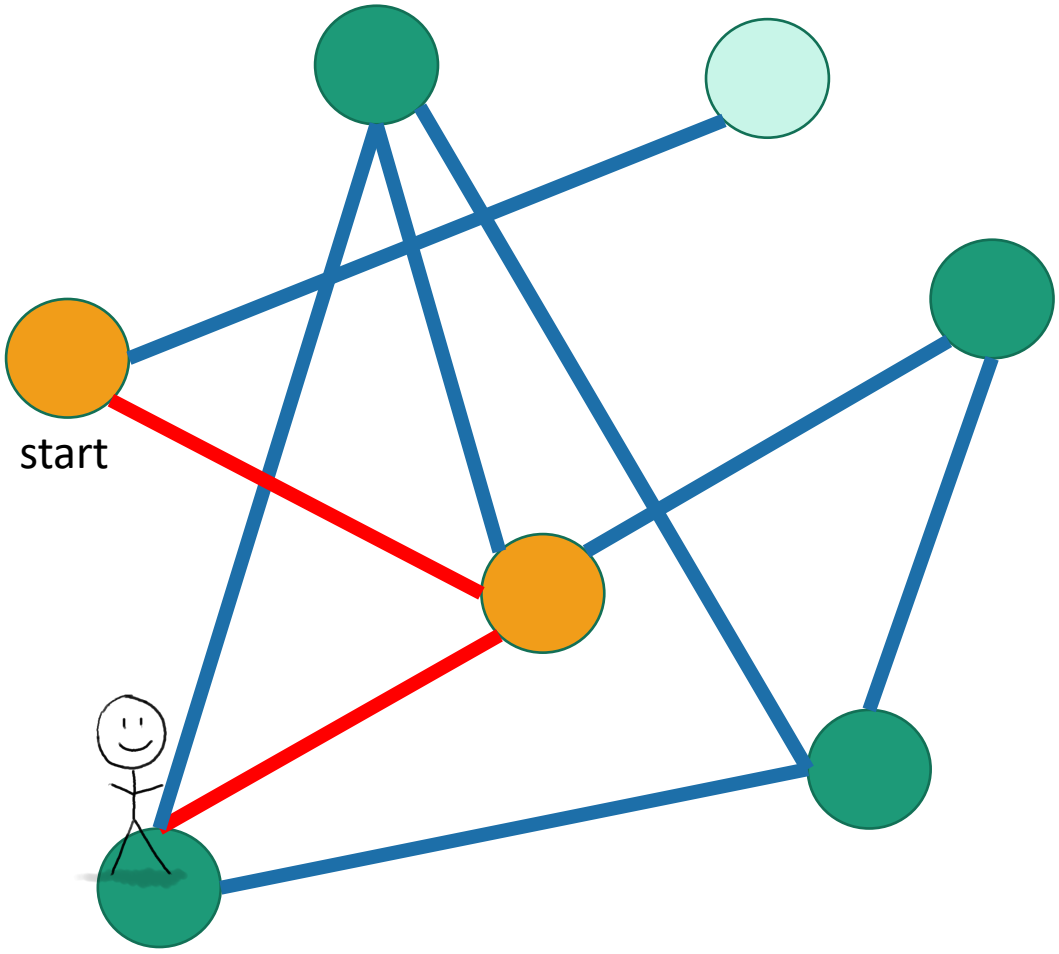

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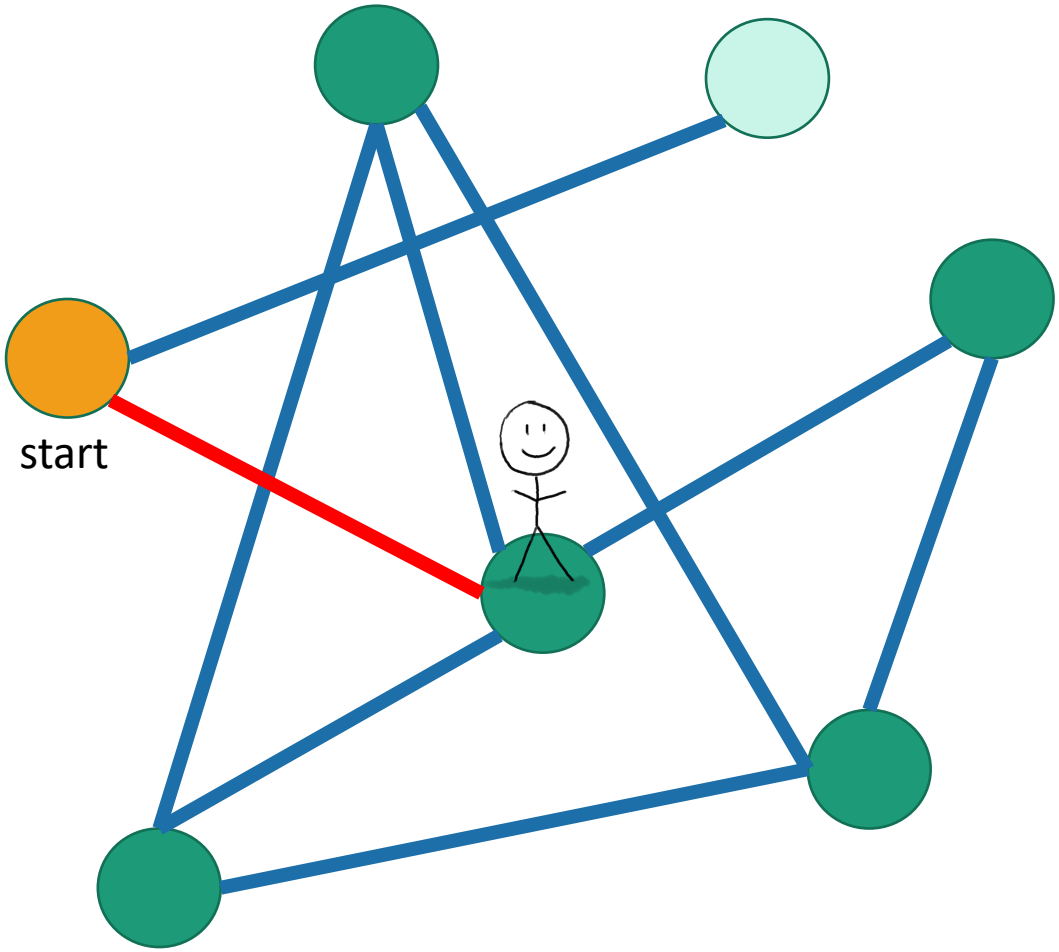

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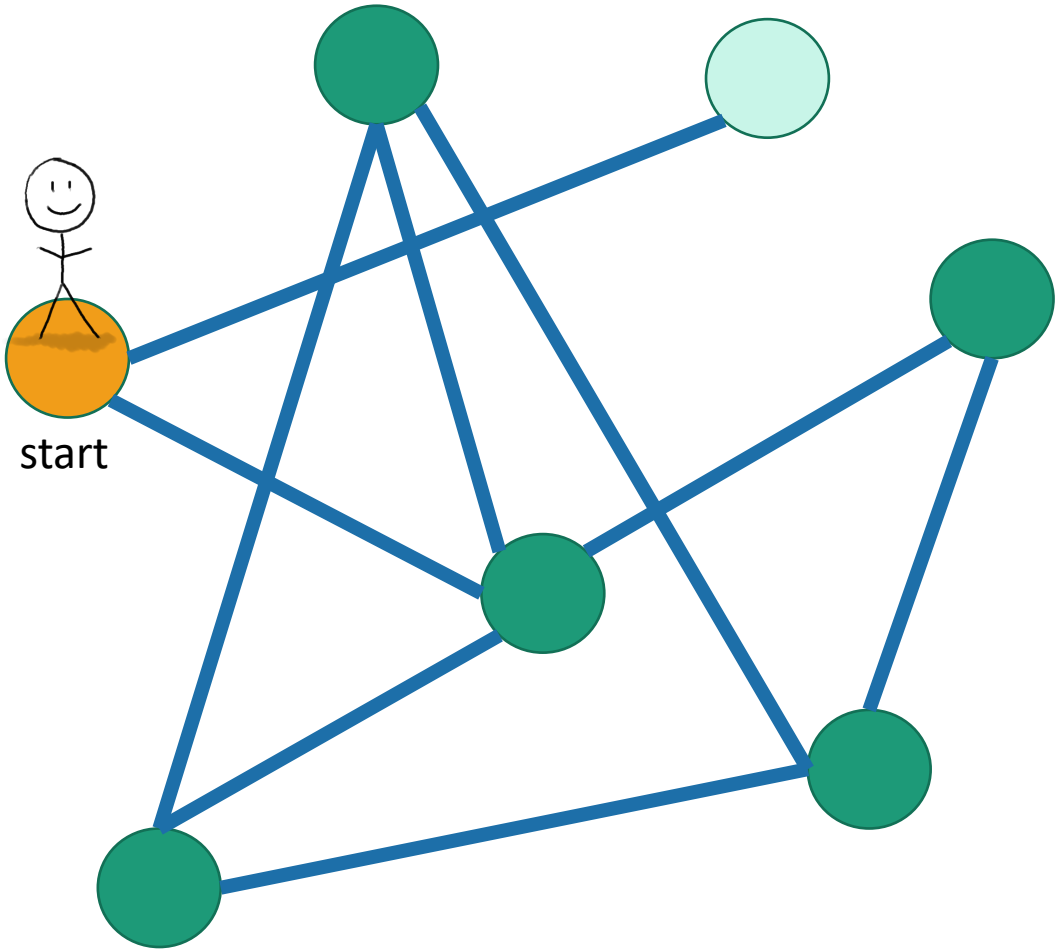

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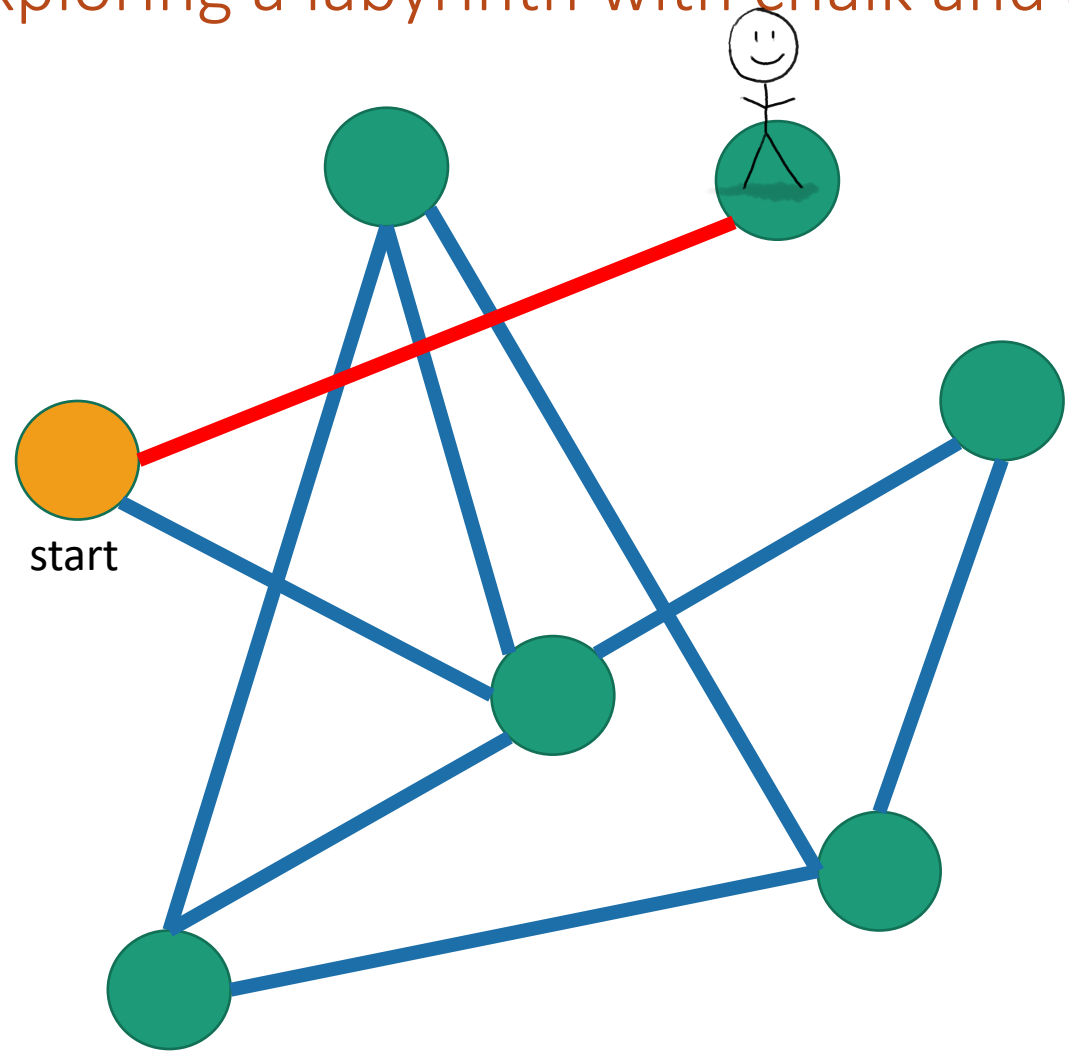
Depth First Search

Exploring a labyrinth with chalk and a piece of string



Not been there yet

Been there, haven't explored all the paths out.

Been there, have explored all the paths out.

Depth First Search

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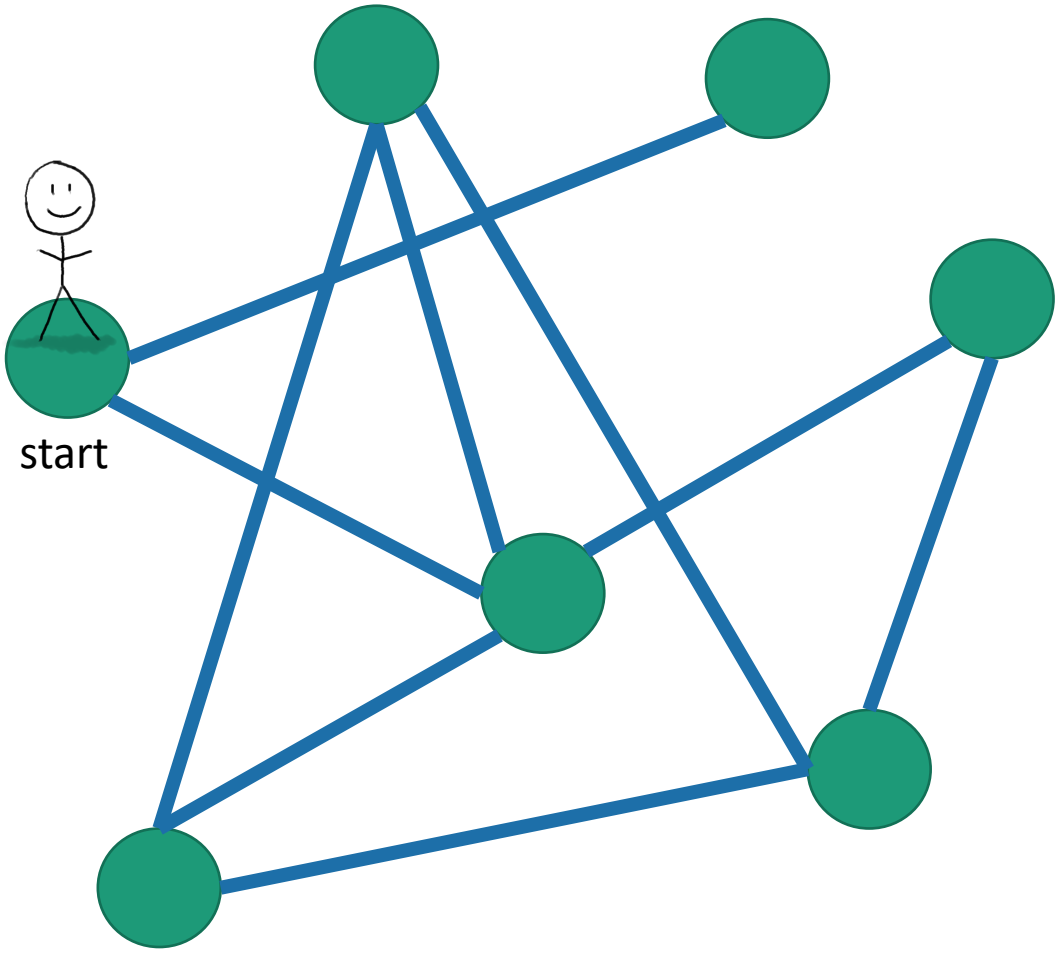


Not been there yet

Been there, haven't explored all the paths out.

Been there, have explored all the paths out.

Labyrinth:

EXPLORED!

Depth First Search

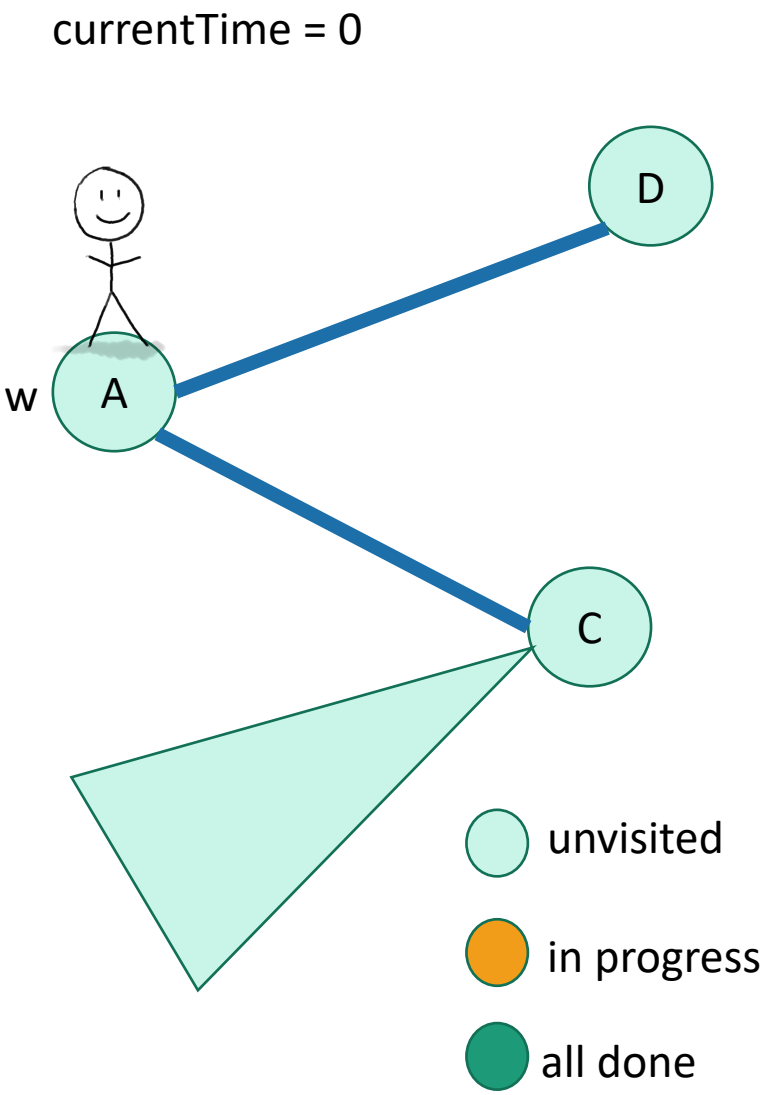
Exploring a labyrinth with pseudocode

- Each vertex keeps track of whether it is:
  - Unvisited
  - In progress
  - All done

- Each vertex will also keep track of:
  - The time we **first enter it**.
  - The time we finish with it and mark it **all done**.

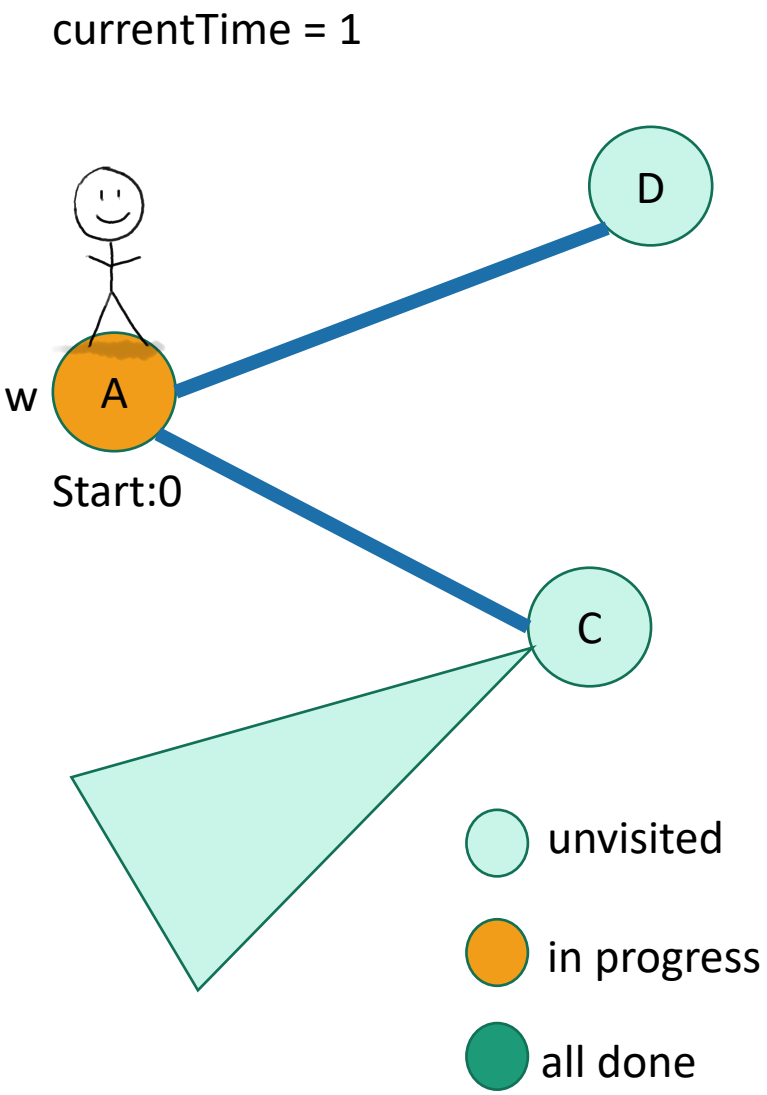
You might have seen other ways to implement DFS than what we are about to go through. This way has more bookkeeping, but more intuition – also, the bookkeeping will be useful later!

Depth First Search



- **DFS**(w, currentTime):
  - w.startTime = currentTime
  - currentTime ++
  - Mark w as **in progress**.
  - **for** v in w.neighbors:
    - **if** v is **unvisited**:
      - currentTime = **DFS**(v, currentTime)
      - currentTime ++
  - w.finishTime = currentTime
  - Mark w as **all done**
  - **return** currentTime

Depth First Search



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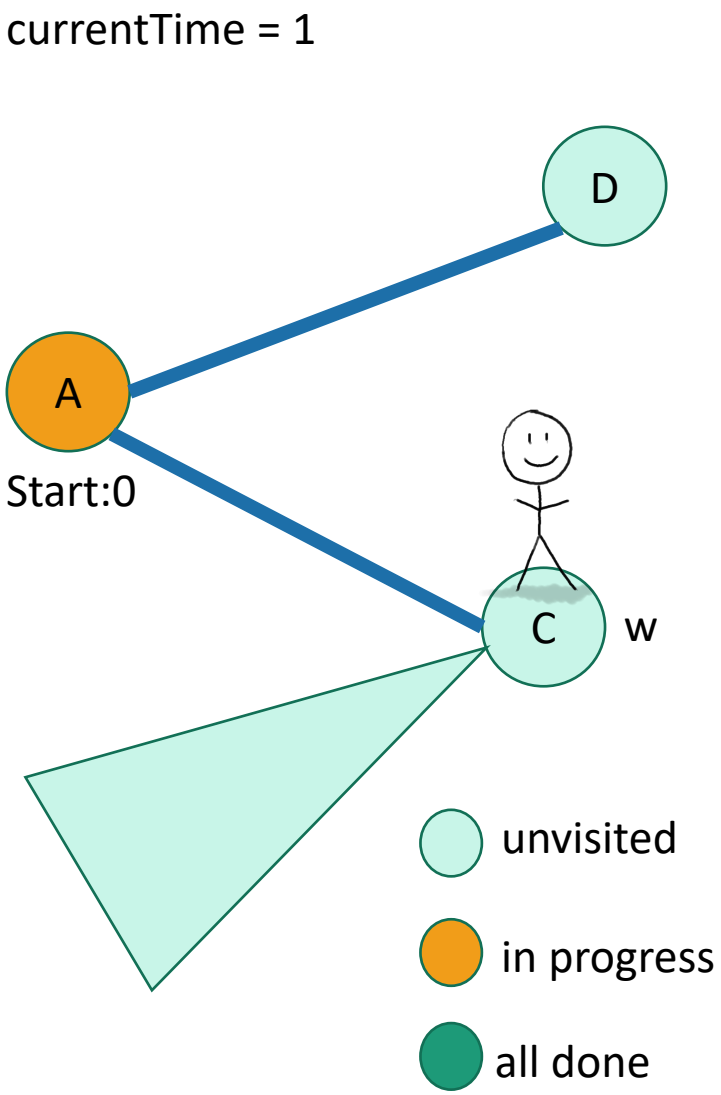
Depth First Search



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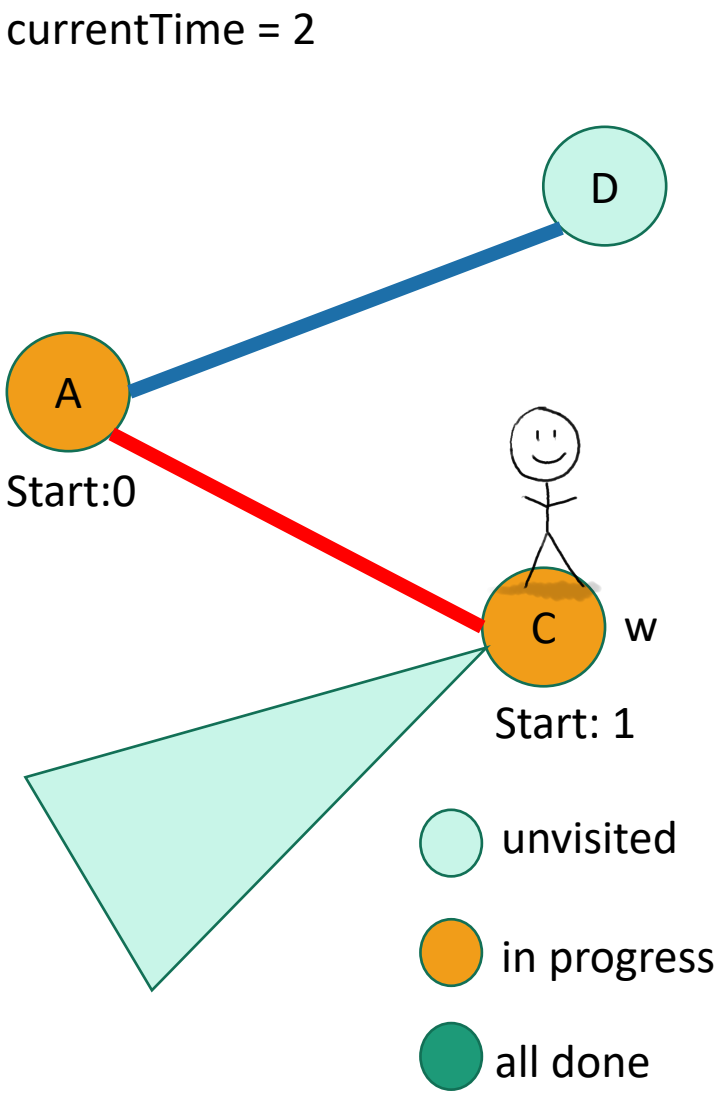
Depth First Search



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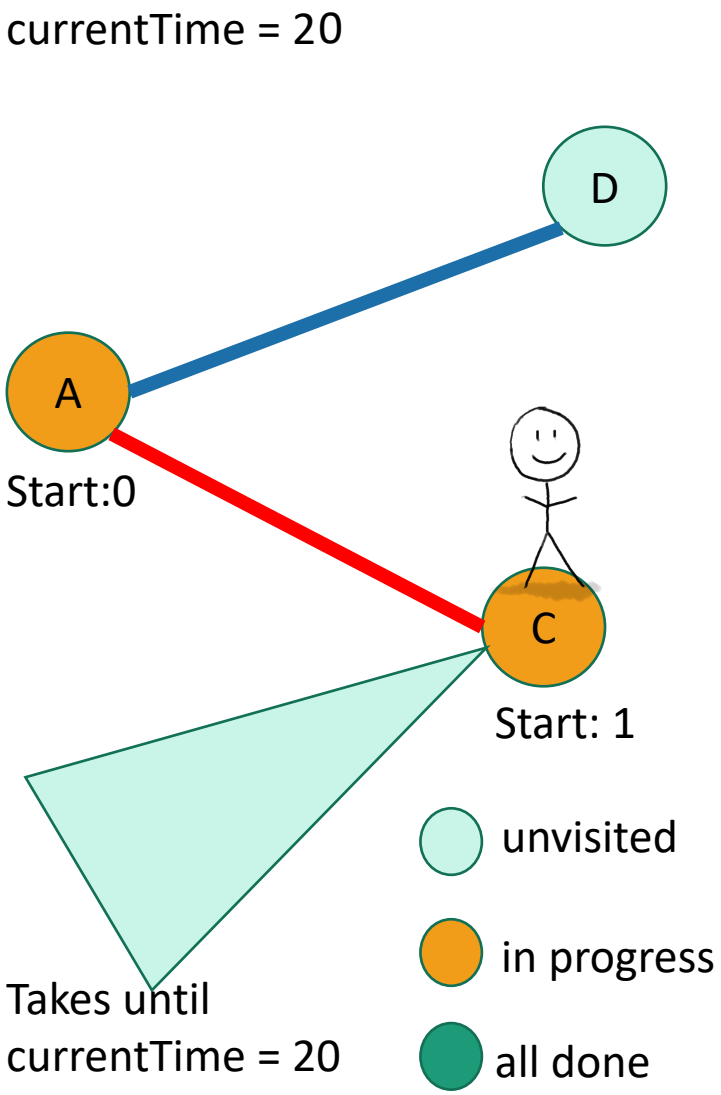
Depth First Search



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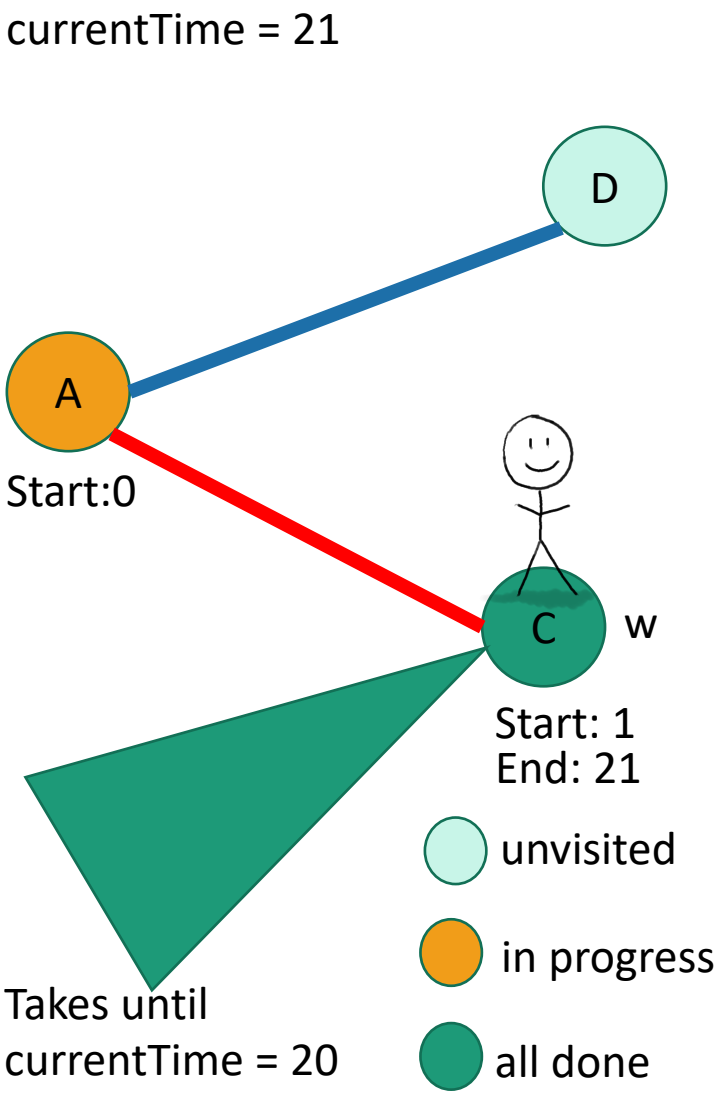
Depth First Search



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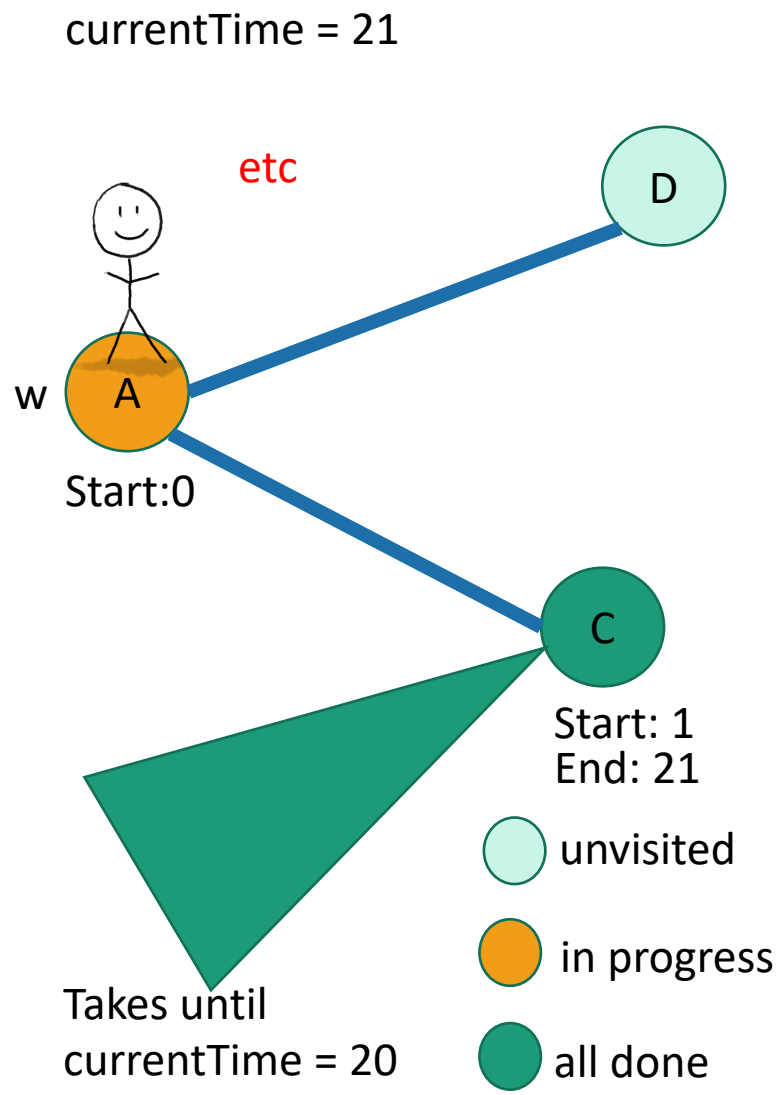
Depth First Search



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  - w.finishTime = currentTime
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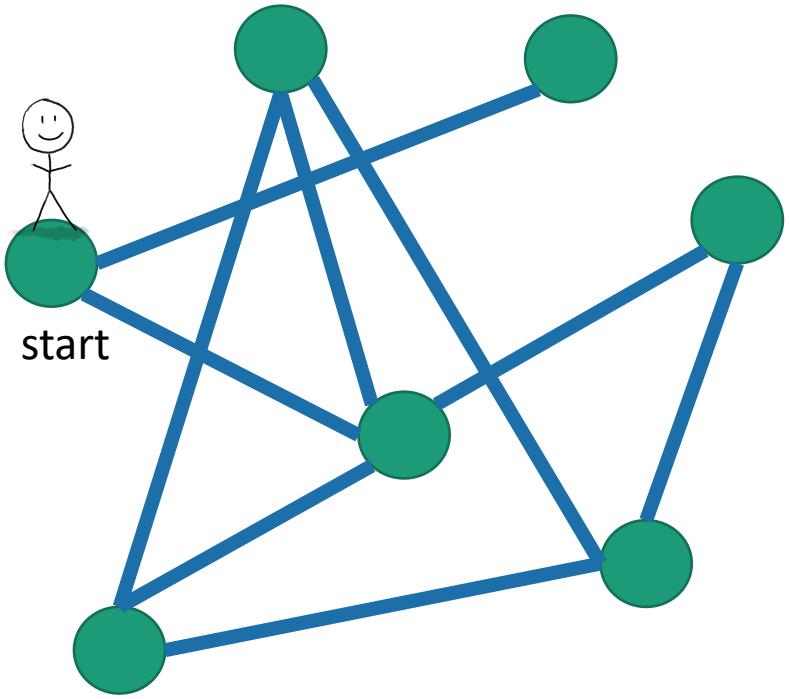
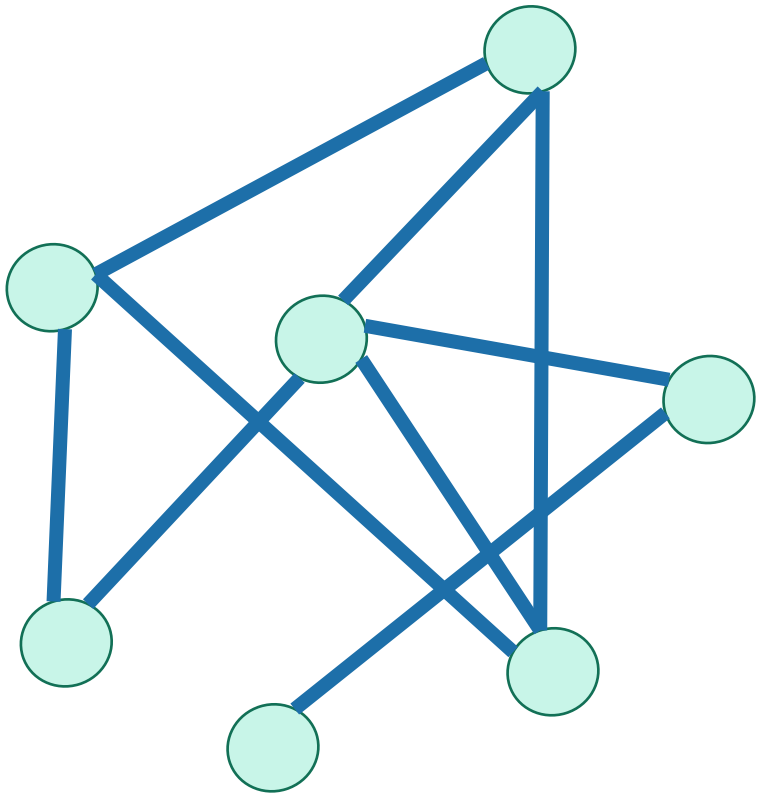
  - **return** currentTime

DFS finds all the nodes reachable from the starting point



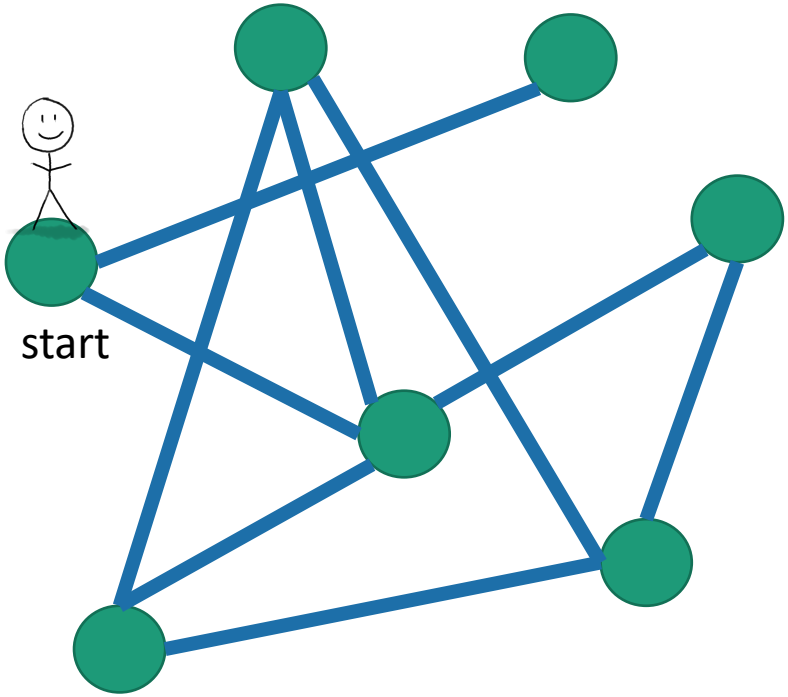
In an undirected graph, this is called a **connected component.**

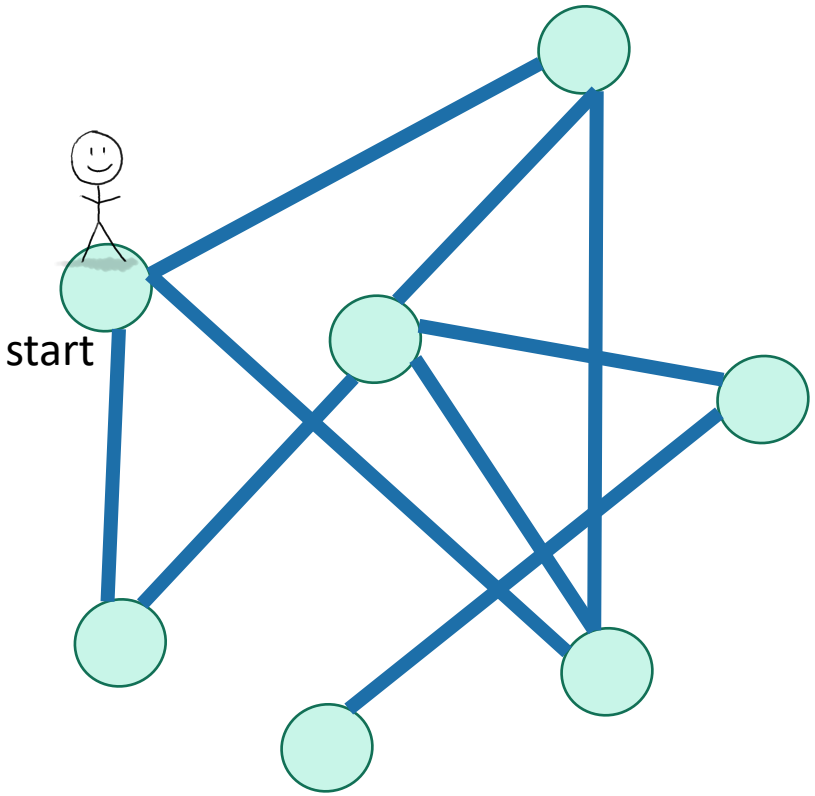


One application: finding connected components.

To explore the whole graph

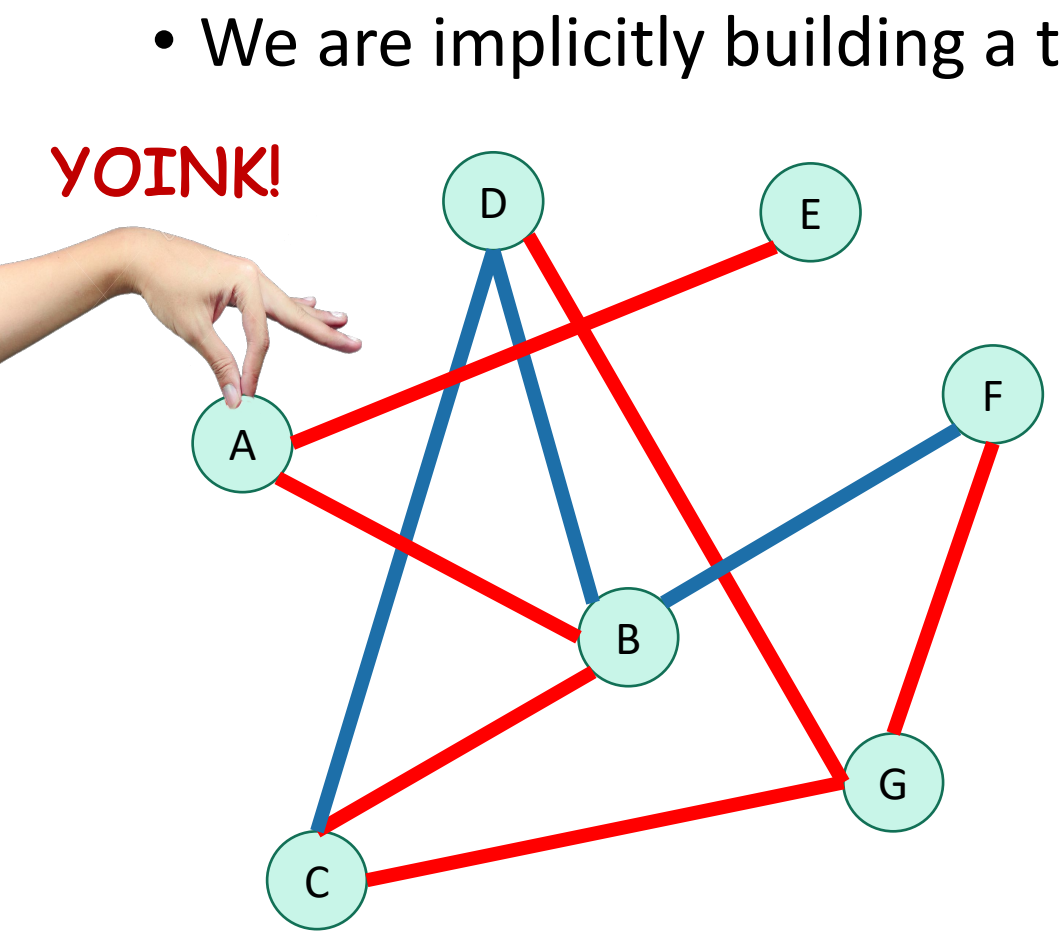
- Do it repeatedly!

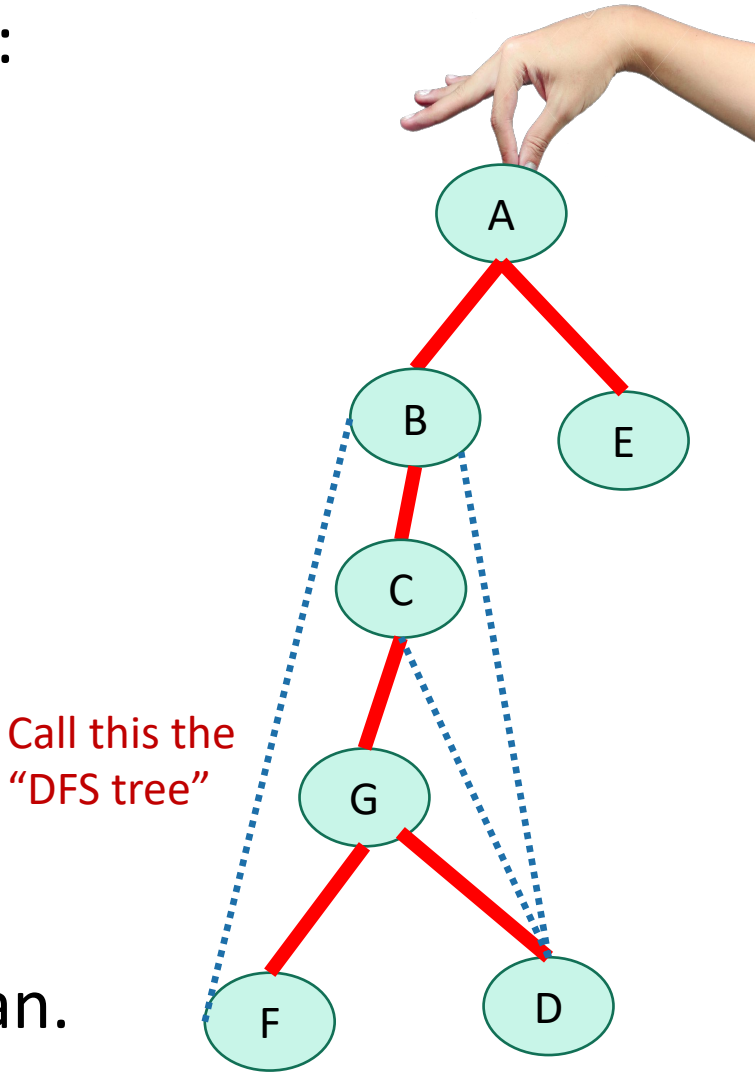




Why is it called depth-first?

- We are implicitly building a tree:





Call this the "DFS tree"

- And first we go as deep as we can.

Running time

To explore **just the connected component** we started in

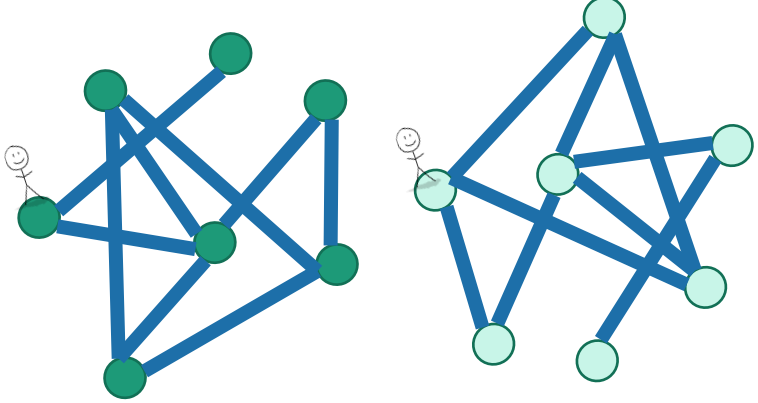
- We look at each edge only once.
- And basically don't do anything else.
- So…

$$O(m)$$

- (Assuming we are using the linked-list representation)

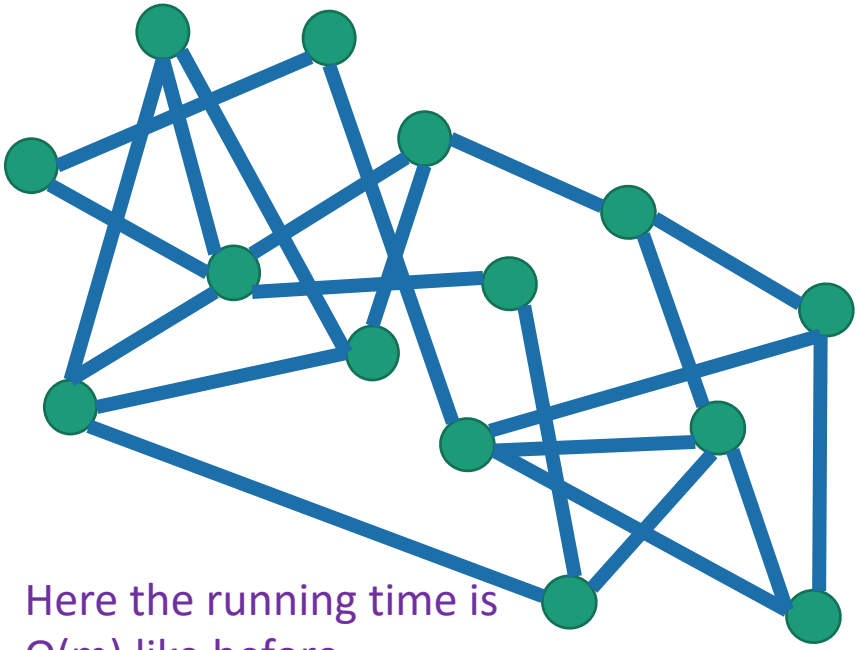
Running time

To explore the whole thing



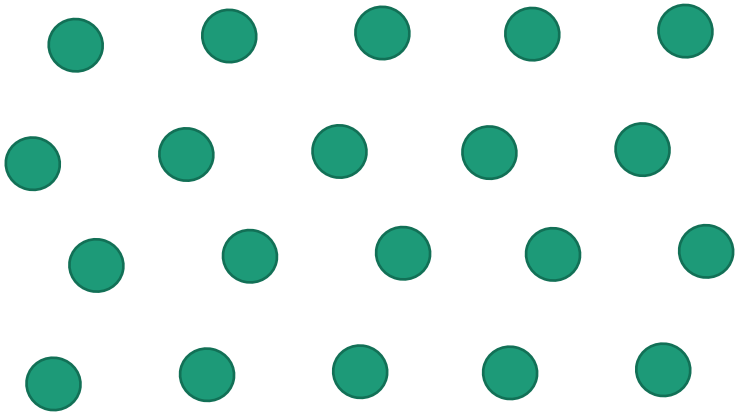
- Explore the connected components one-by-one.
- This takes time

$$O(n + m)$$



Here the running time is O(m) like before

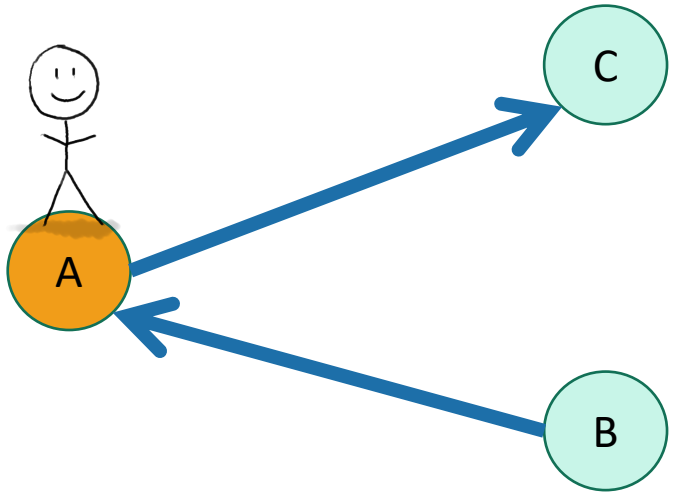
or



Here m=0 but it still takes time O(n) to explore the graph.

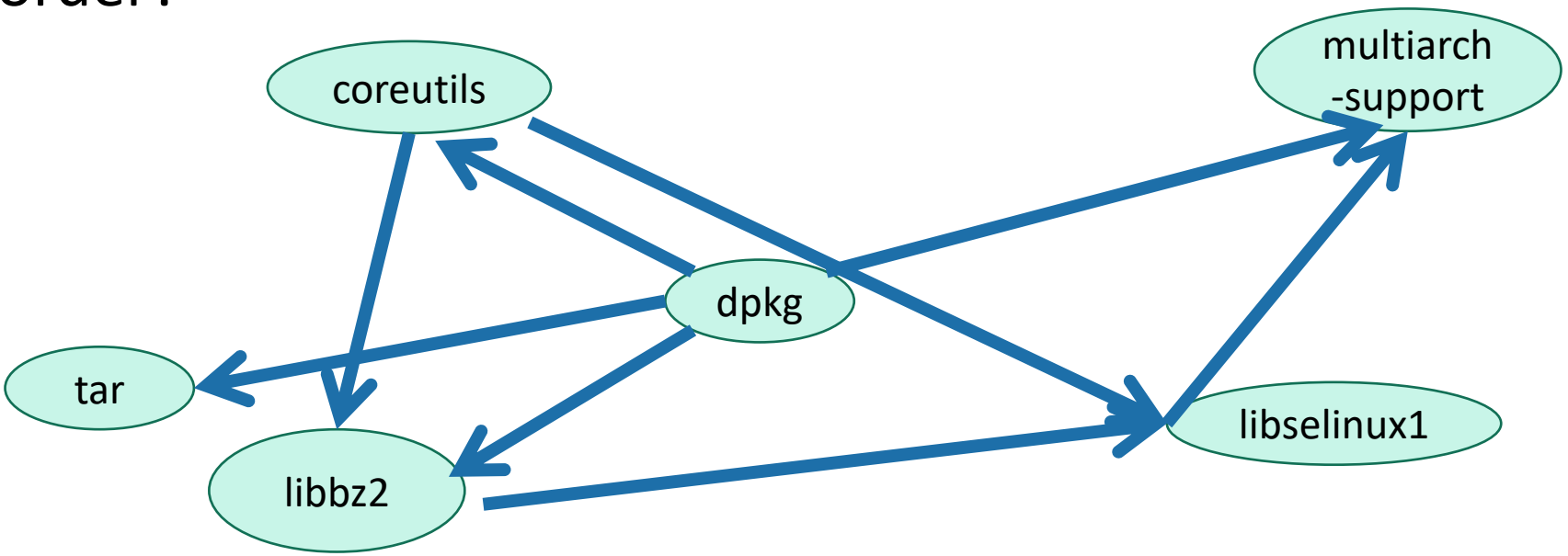
You check:

DFS works fine on directed graphs too!



Only walk to C, not to B.

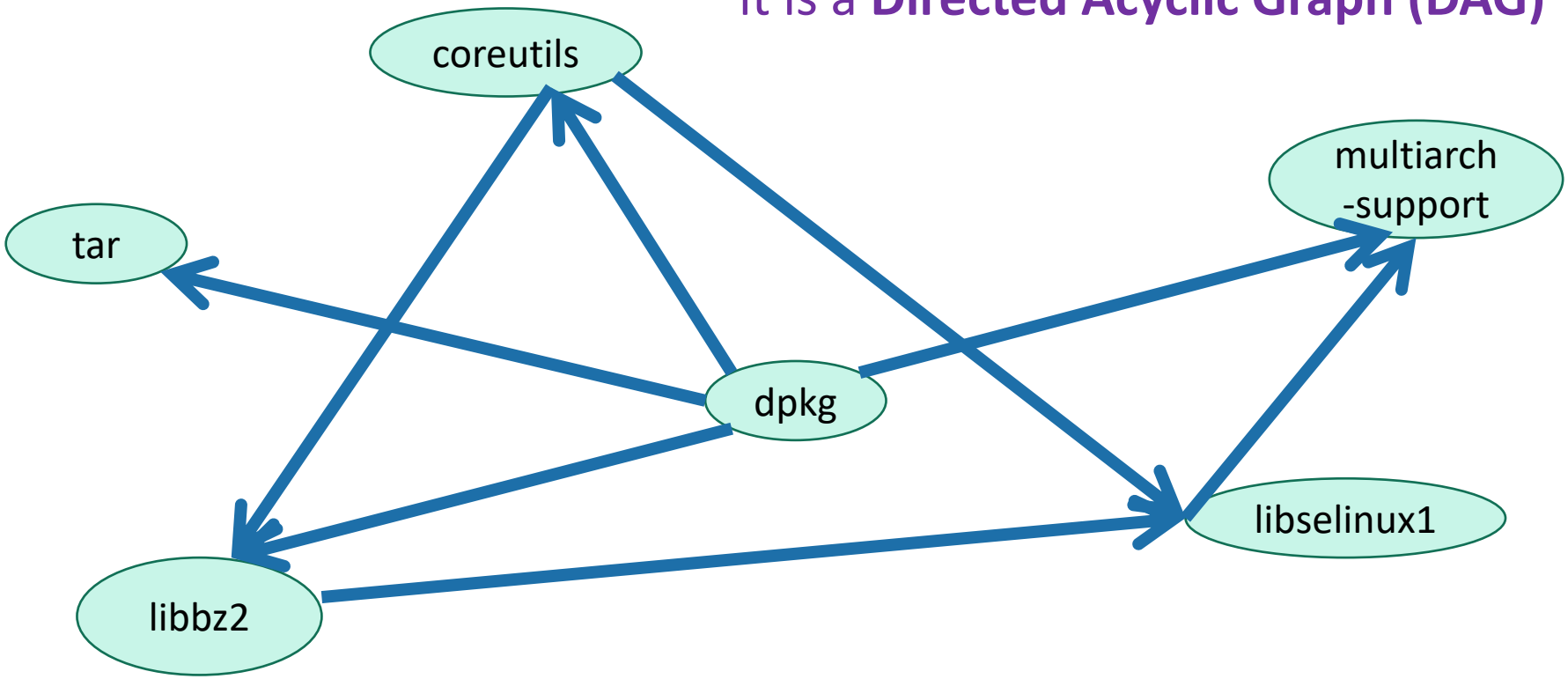
- How can you sign up for classes so that you never violate the pre-req requirements?
- More practically, given a package dependency graph, how do you install packages in the correct order?



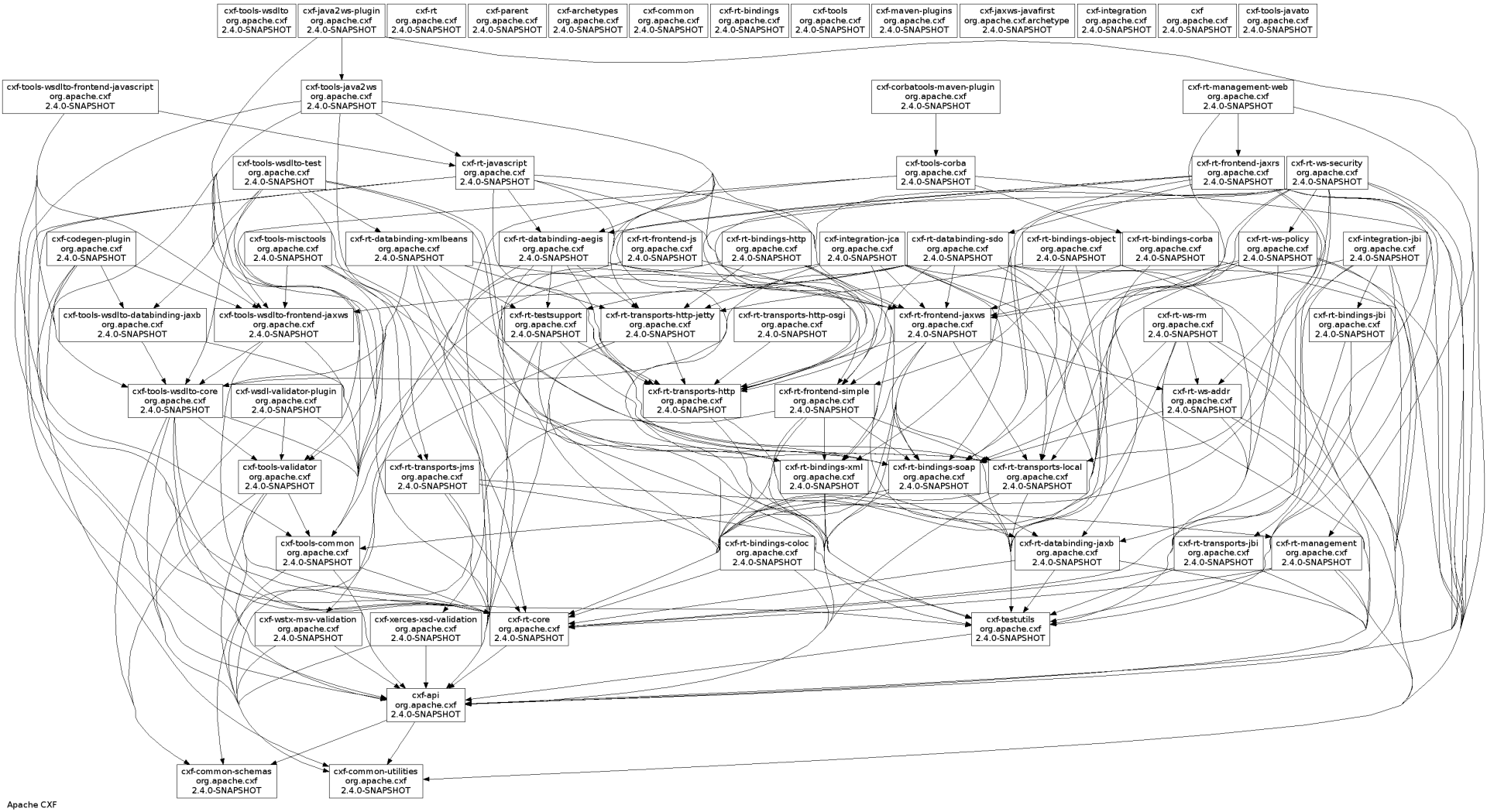
Application: topological sorting

- Question: in what order should I install packages?

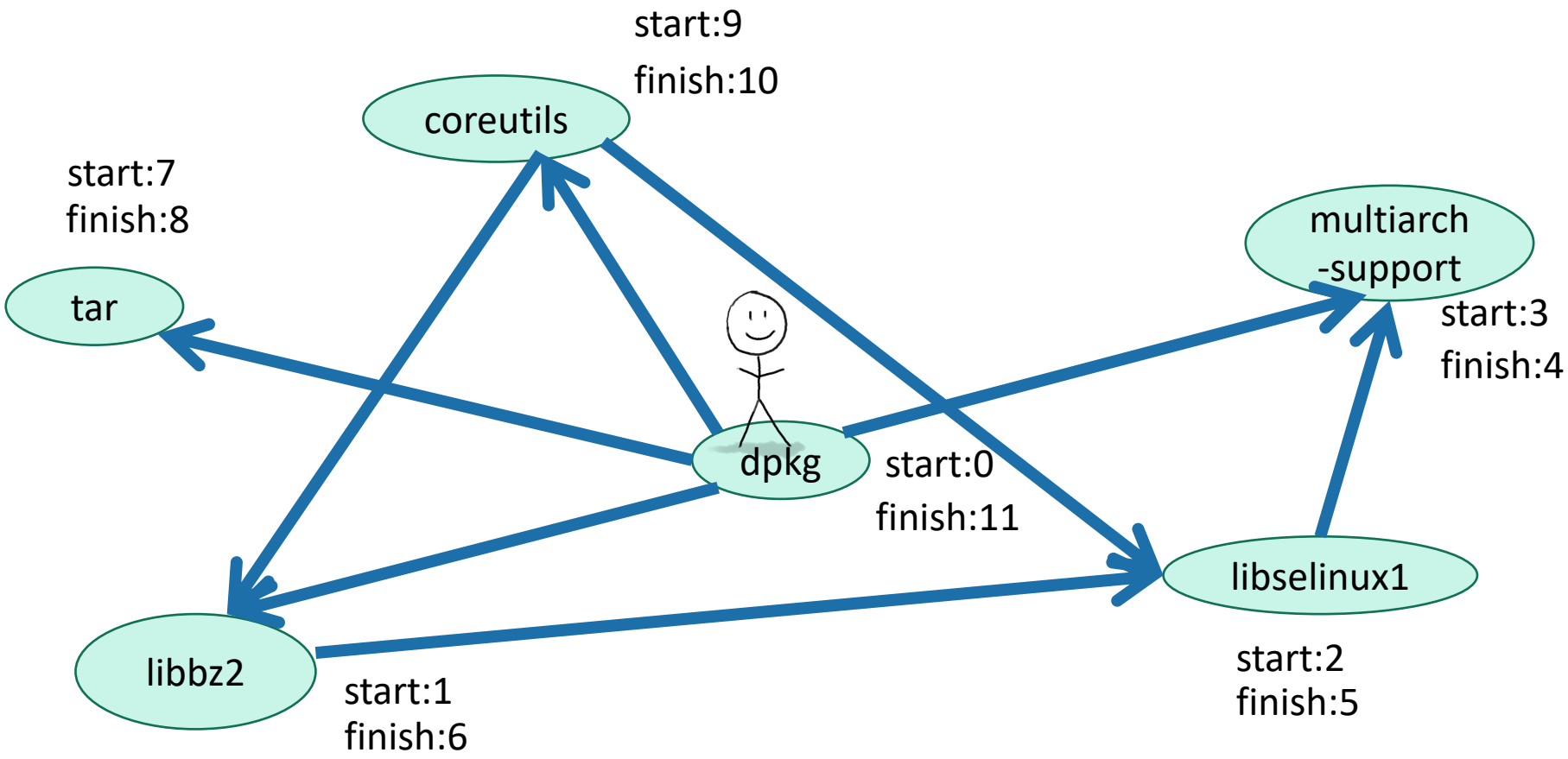
Suppose the dependency graph has no cycles:
it is a **Directed Acyclic Graph (DAG)**



Can’t always eyeball it.



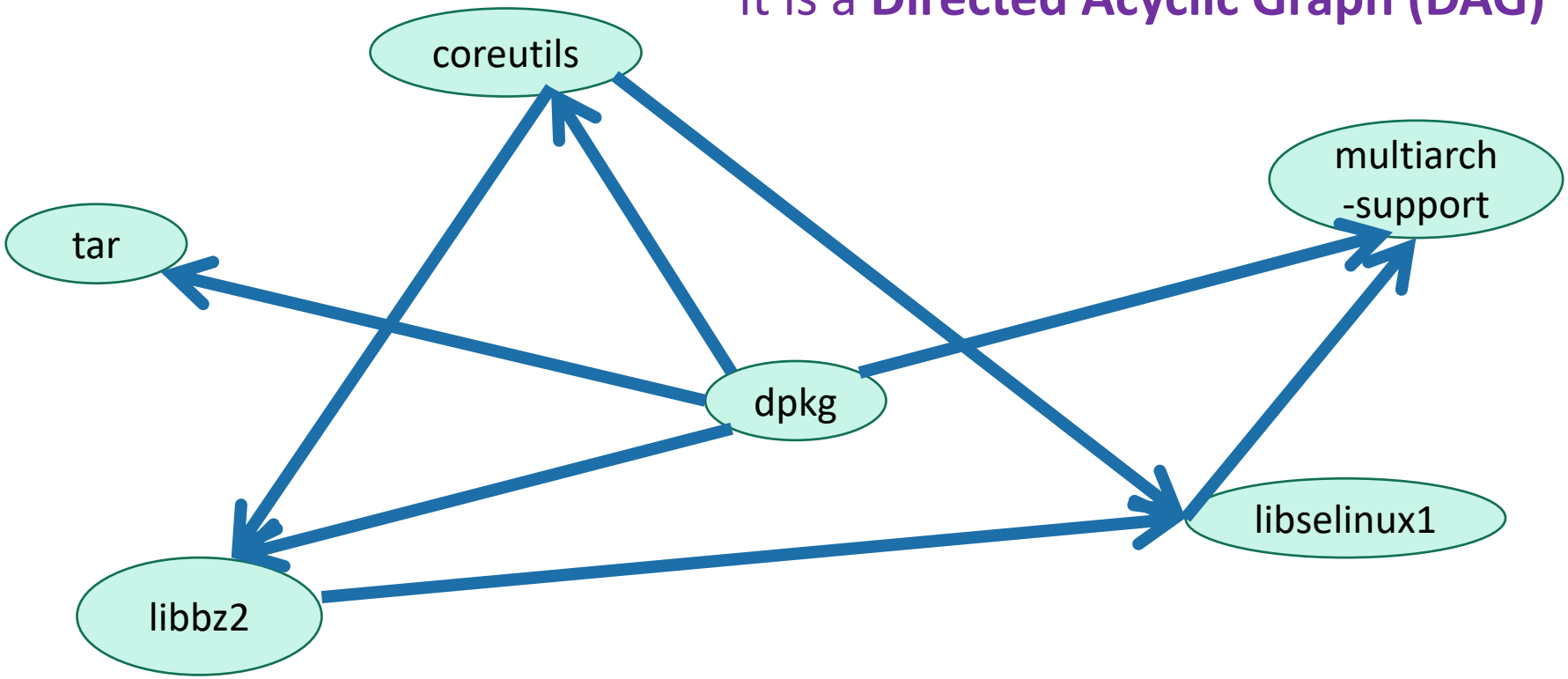
Let’s do DFS


start:9
finish:10
coreutils
start:7
finish:8
tar
multiarch
-support
start:3
finish:4
dpkg
start:0
finish:11
libselinux1
libbz2
start:1
finish:6
start:2
finish:5

Back to this problem

- Question: in what order should I install packages?

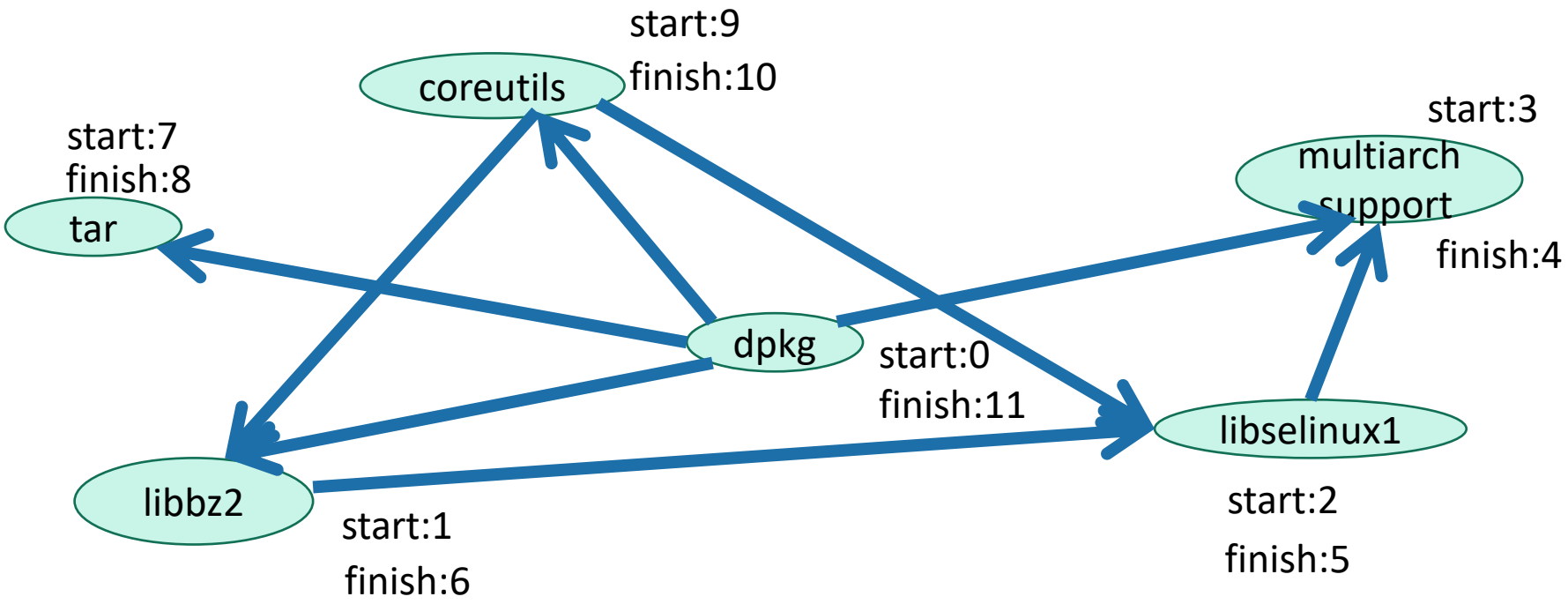
Suppose the dependency graph has no cycles:
it is a **Directed Acyclic Graph (DAG)**



In reverse order of finishing time

- Do DFS
- Maintain a list of packages, in the order you want to install them.
- When you mark a vertex as **all done**, put it at the **beginning** of the list.

- `dpkg`
- `coreutils`
- `tar`
- `libbz2`
- `libselinux1`
- `multiarch_support`

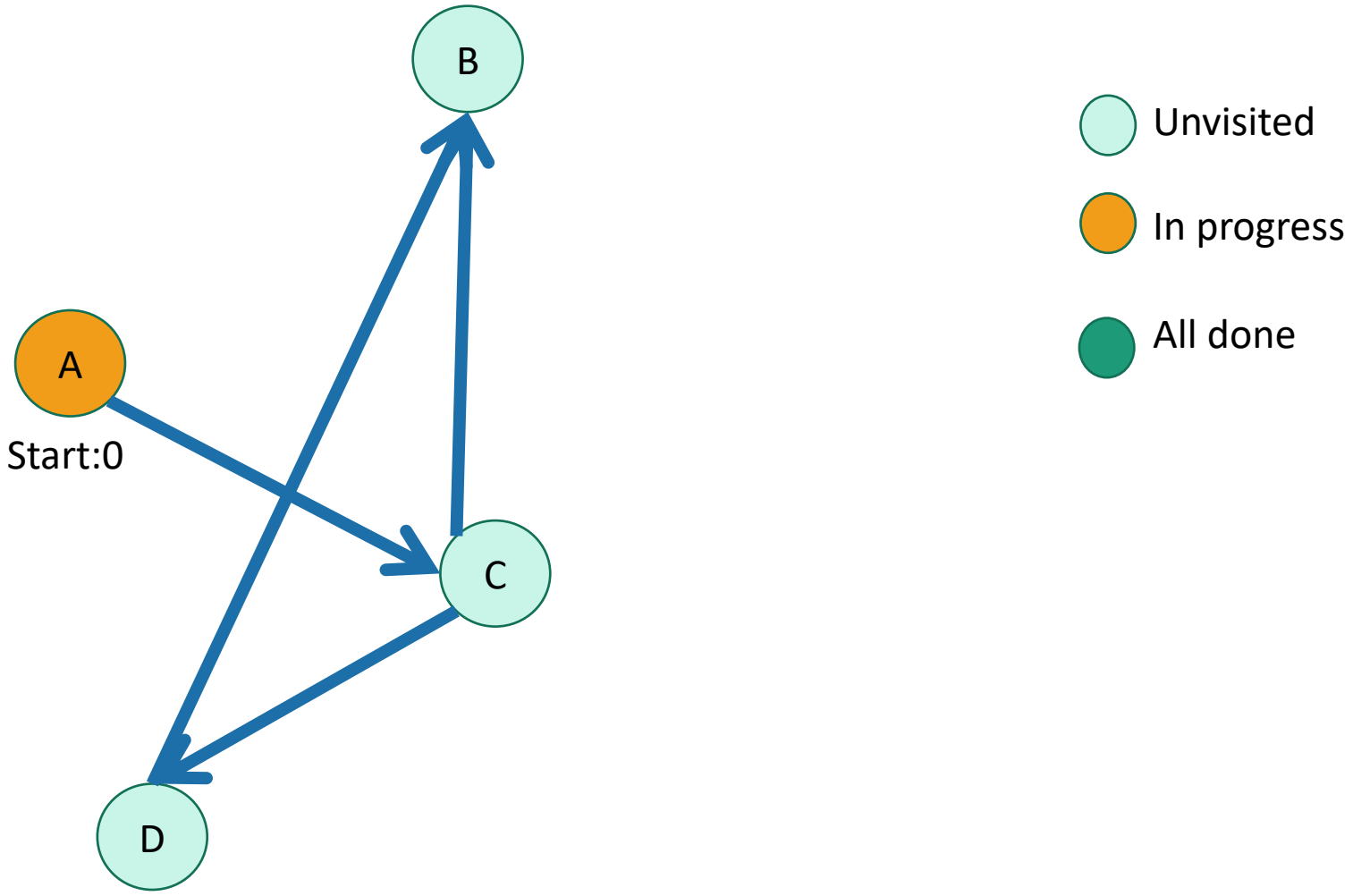


What did we just learn?

- DFS can help you solve the **TOPOLOGICAL SORTING PROBLEM**
  - That's the fancy name for the problem of finding an ordering that respects all the dependencies

- Thinking about the DFS tree is helpful.

Example:



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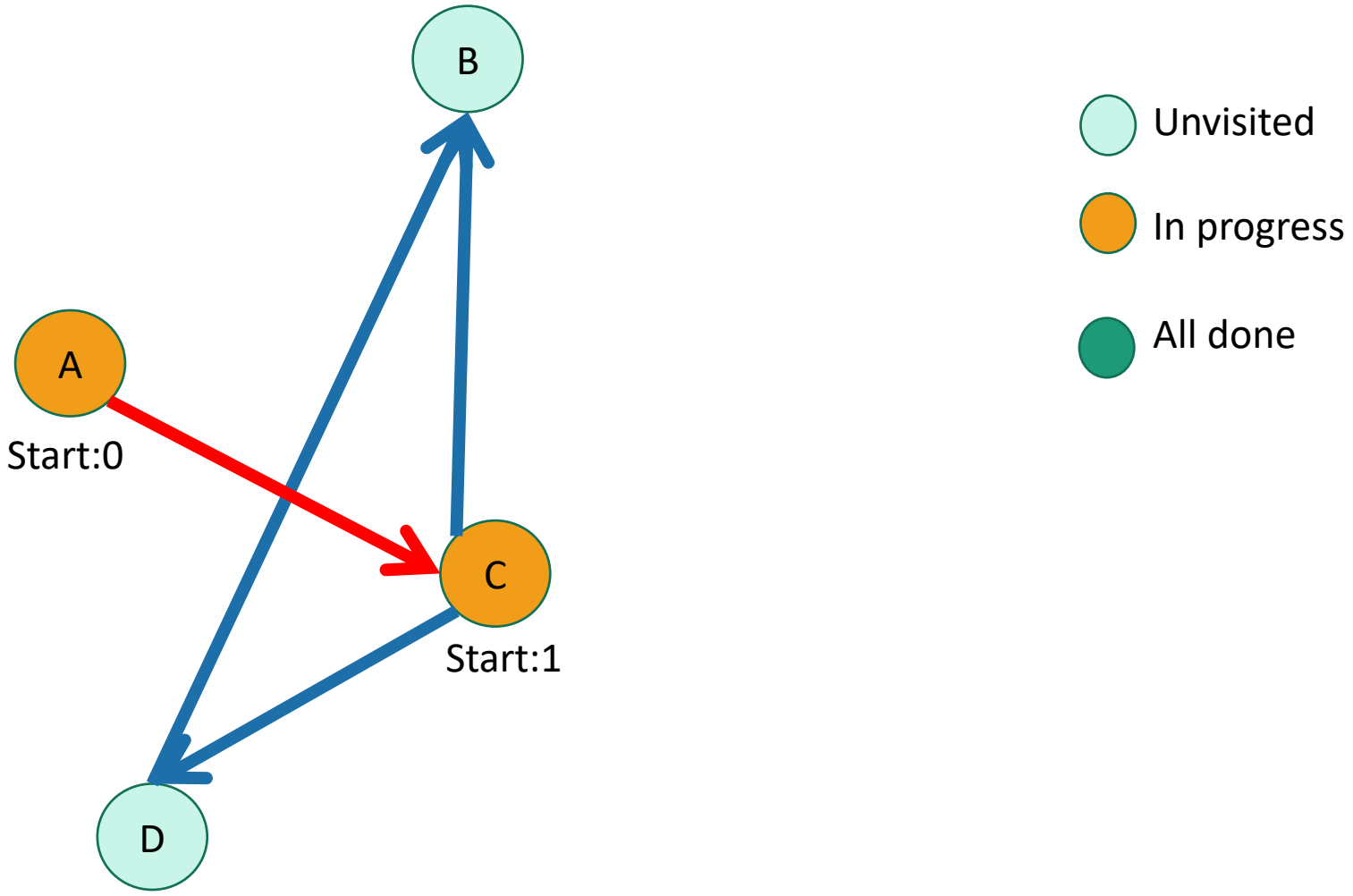
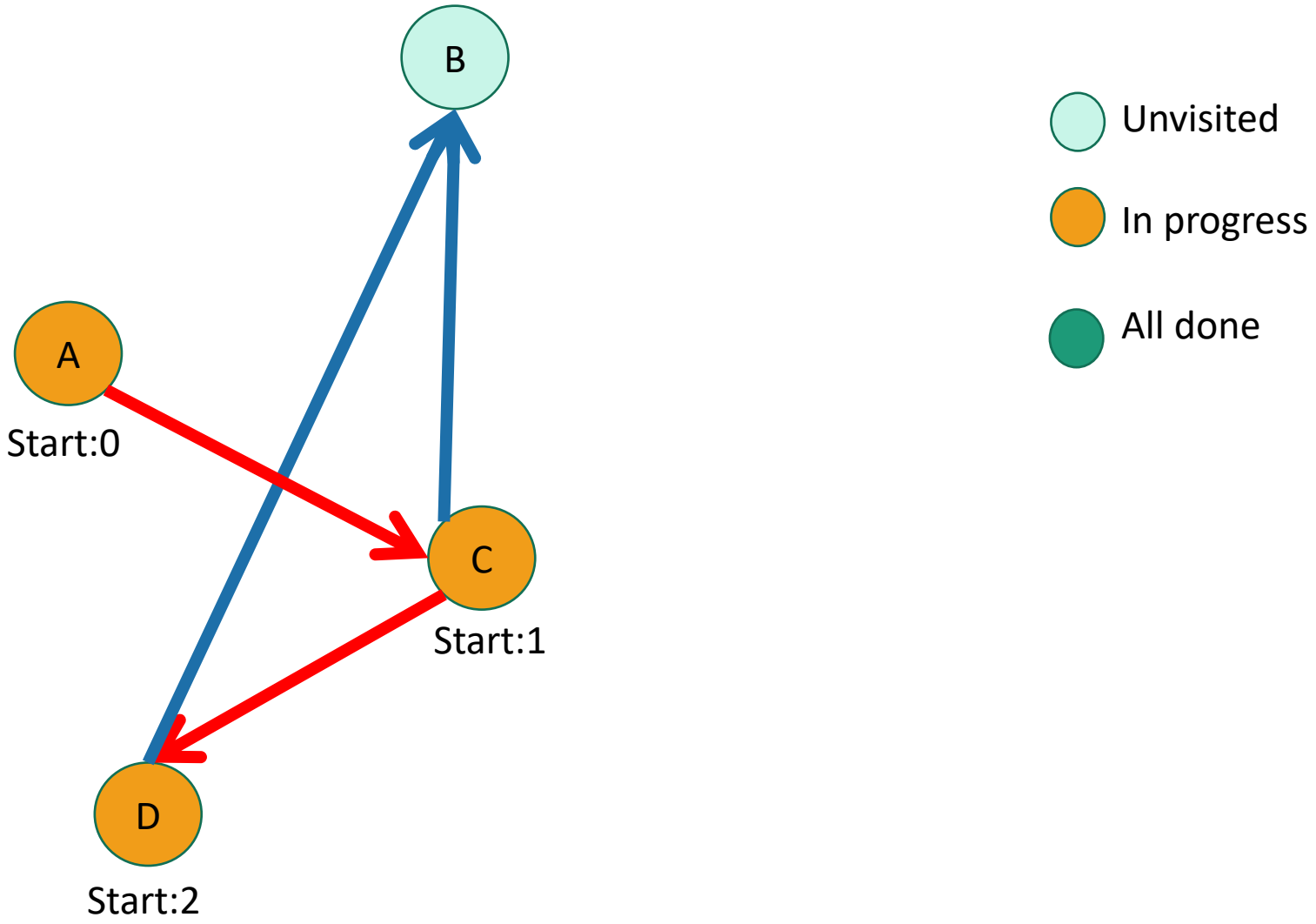
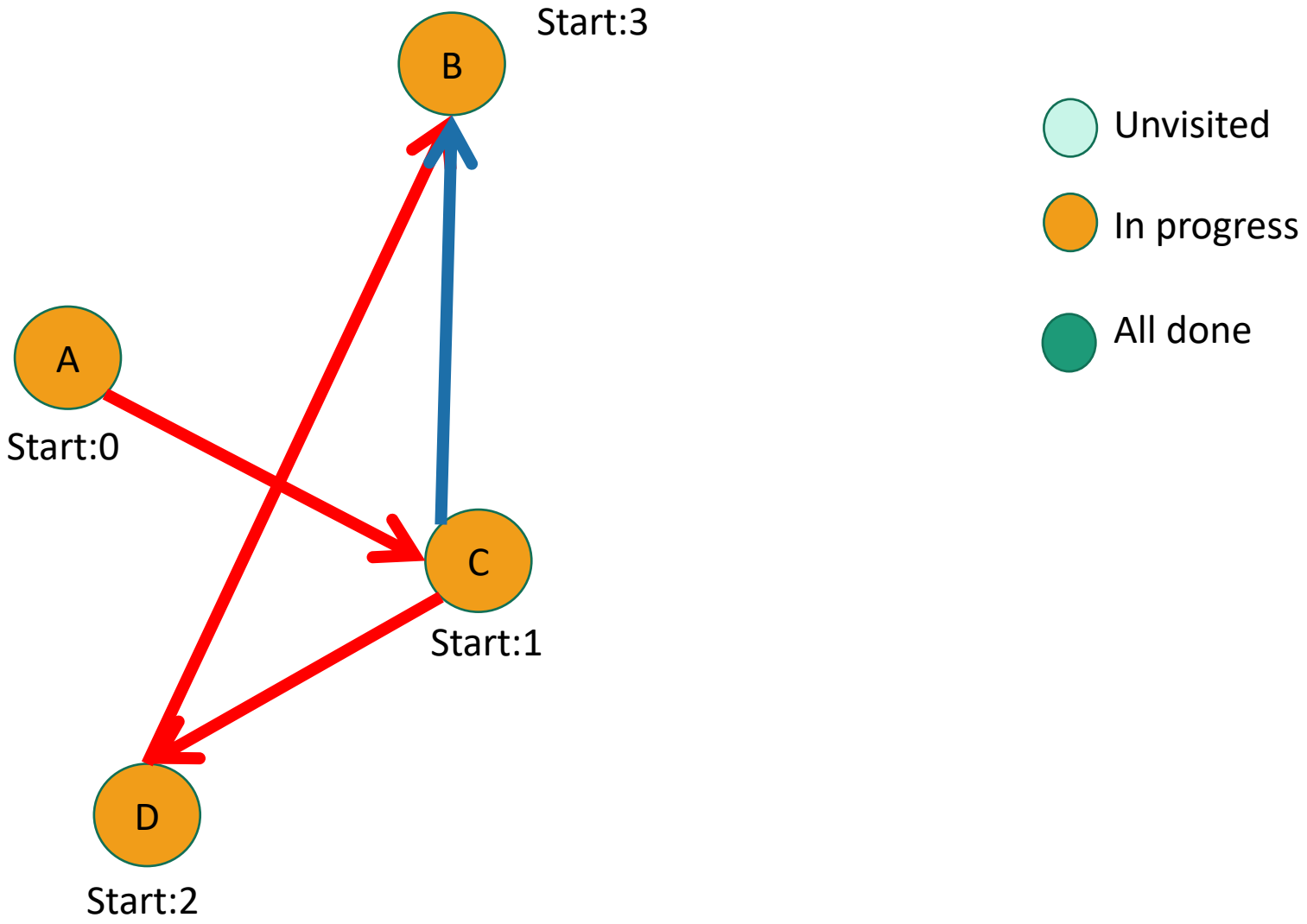


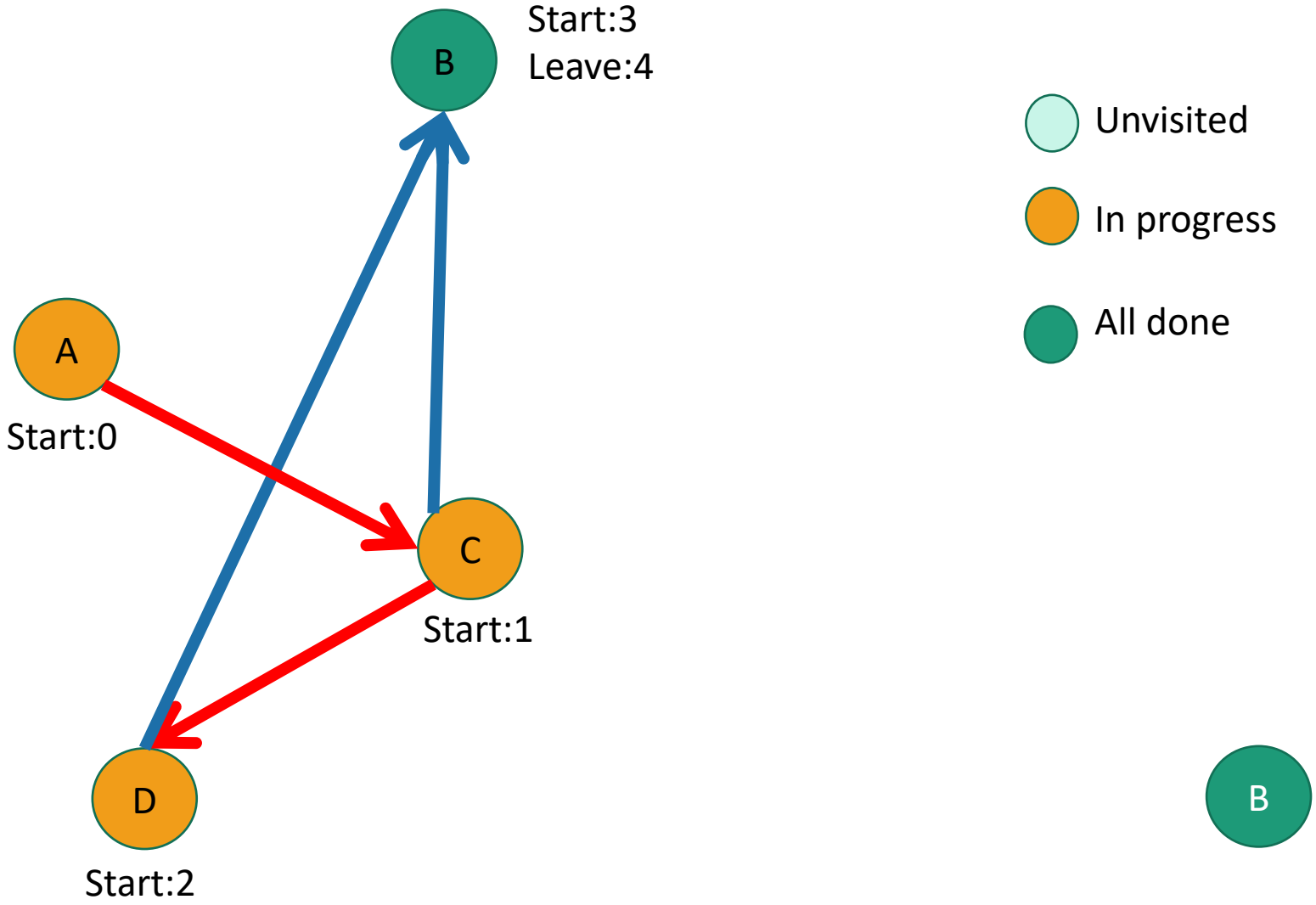
Example



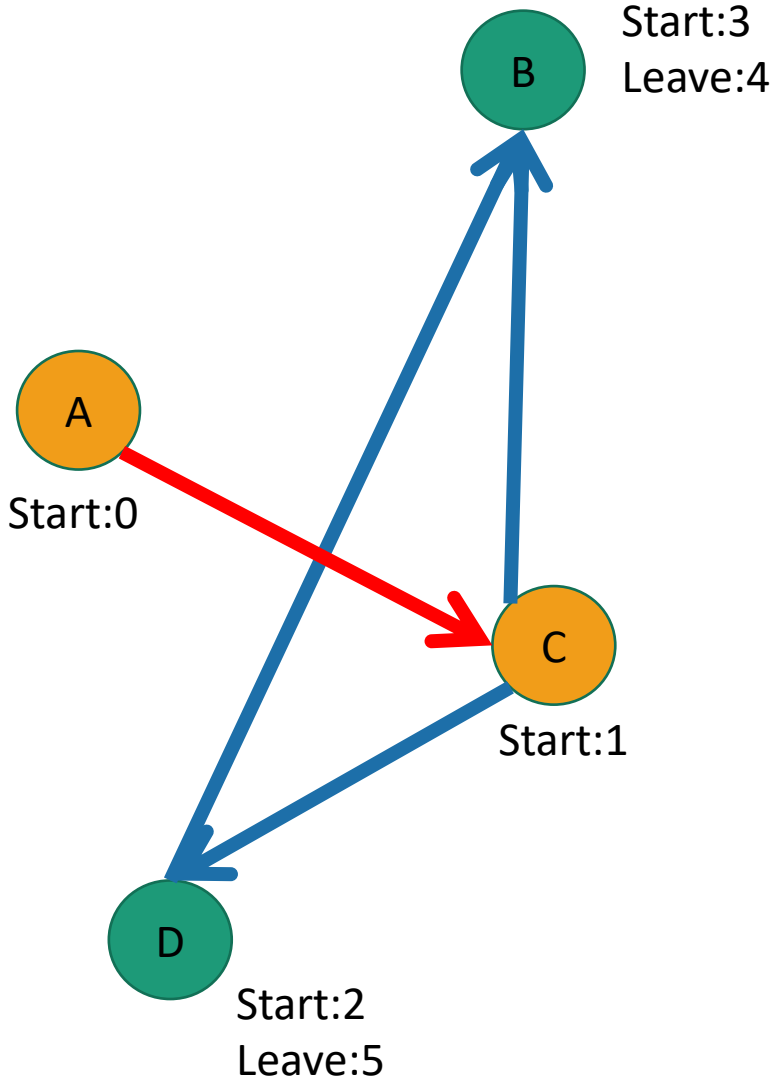
Example


Start:3
B
A
Start:0
C
Start:1
D
Start:2
Unvisited
In progress
All done

Example


B
Start:3
Leave:4
A
Start:0
C
Start:1
D
Start:2
Unvisited
In progress
All done
B

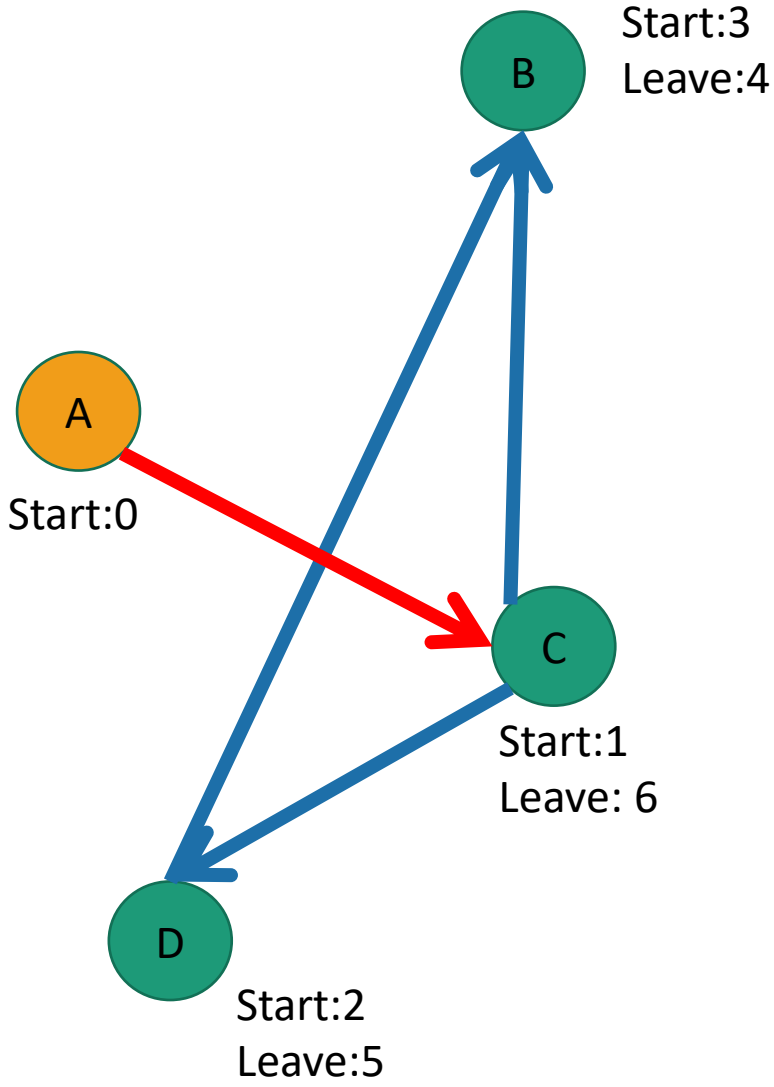
Example


B
Start:3
Leave:4
A
Start:0
C
Start:1
D
Start:2
Leave:5


Unvisited
In progress
All done


D
B

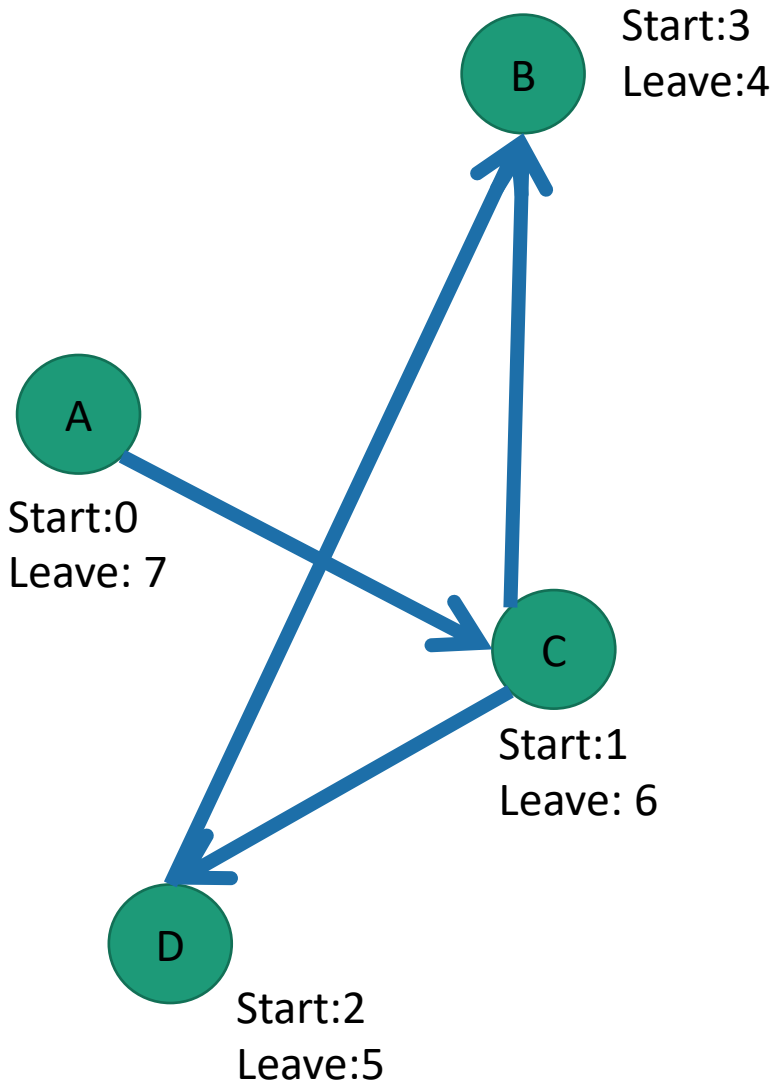
Example


B
Start:3
Leave:4
A
Start:0
C
Start:1
Leave: 6
D
Start:2
Leave:5


Unvisited
In progress
All done


C
D
B

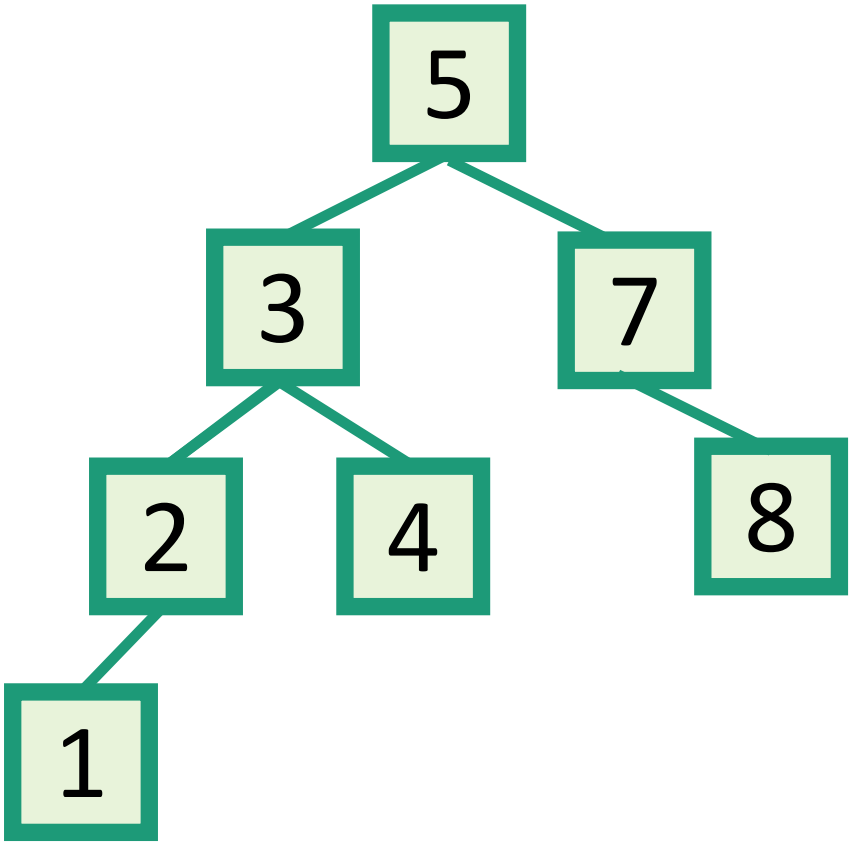
Example



Do them in this order:

Another use of DFS

- In-order enumeration of binary search trees



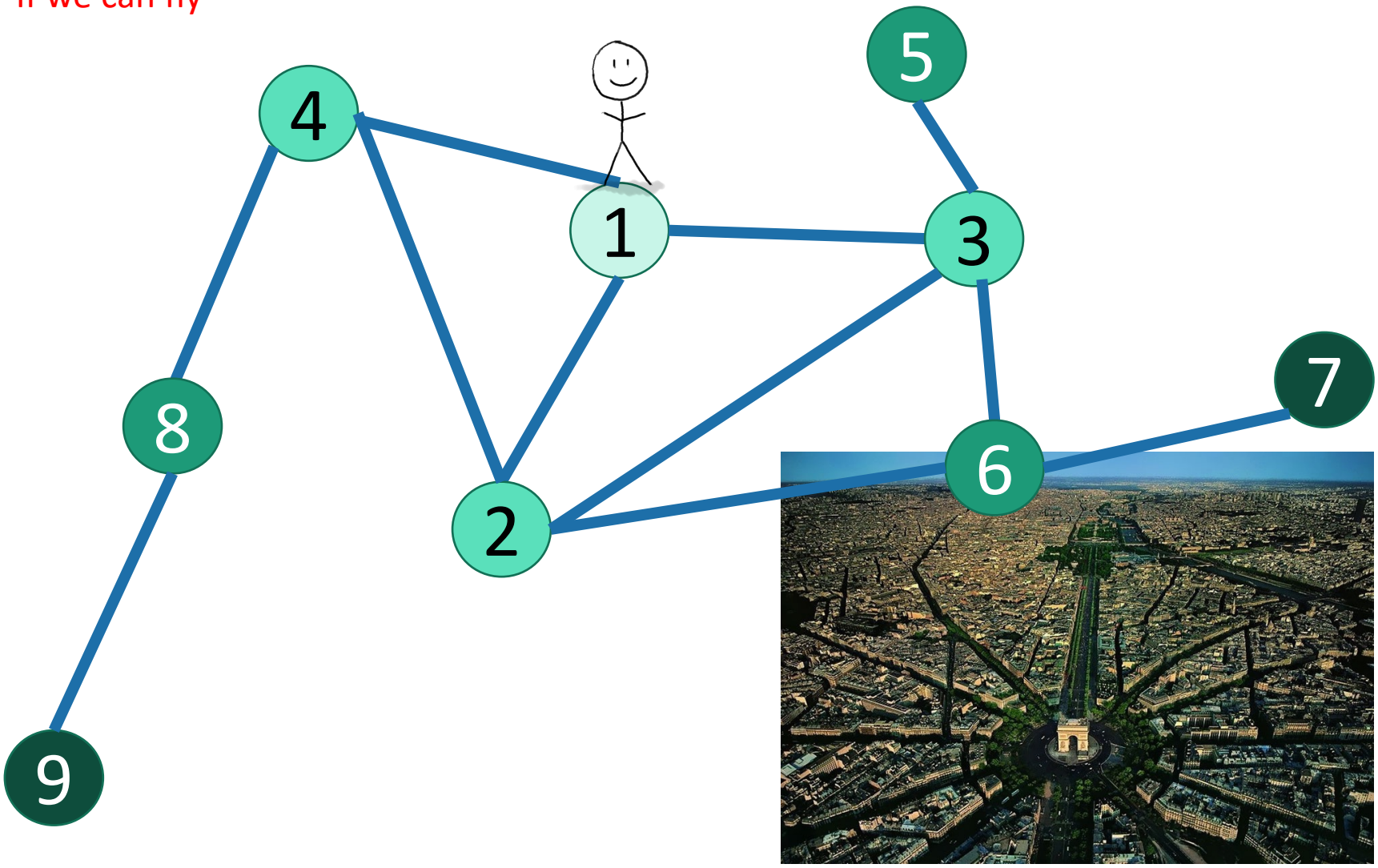
Given a binary search tree, output all the nodes **in order.**

Instead of outputting a node when you are done with it, output it when you are done with the left child and before you begin the right child.

Part 2: breadth-first search

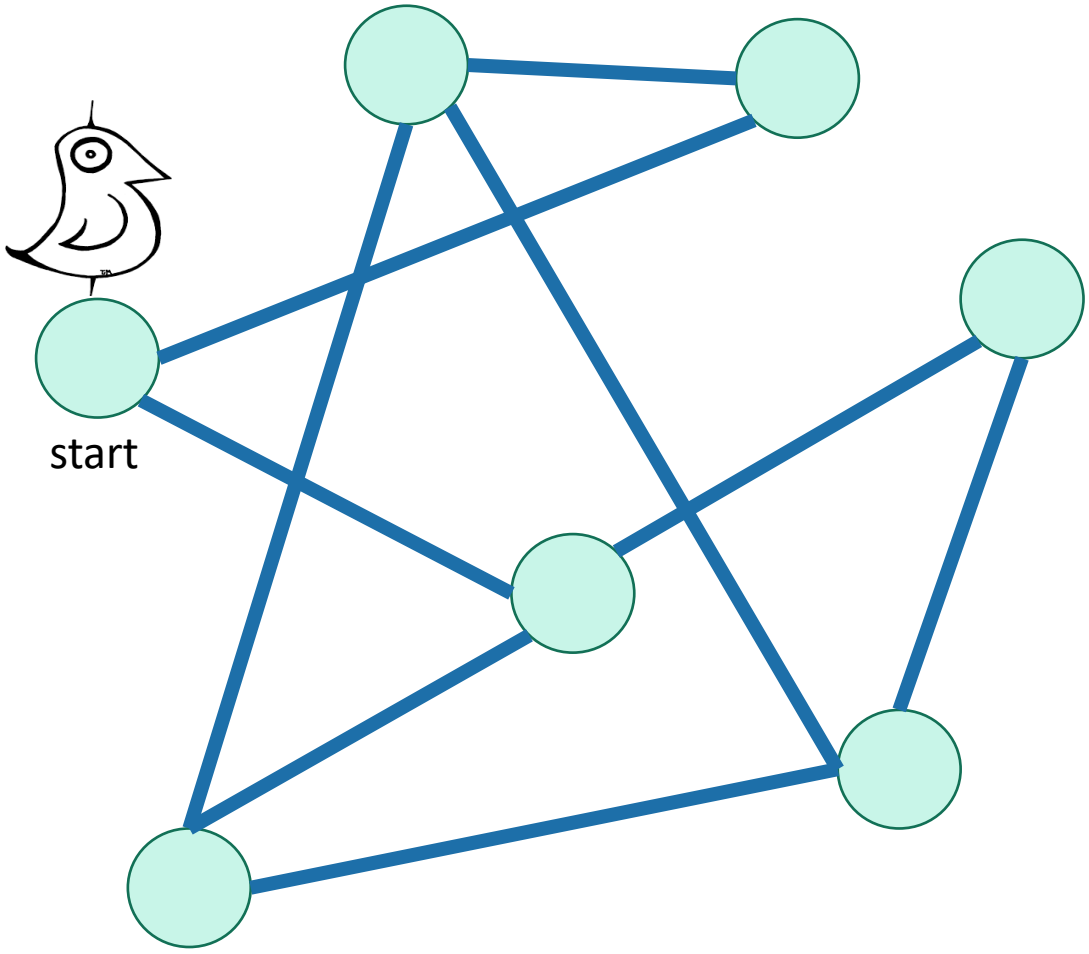
How do we explore a graph?

If we can fly



Breadth-First Search

Exploring the world with a bird's-eye view



Not been there yet

Can reach there in zero steps

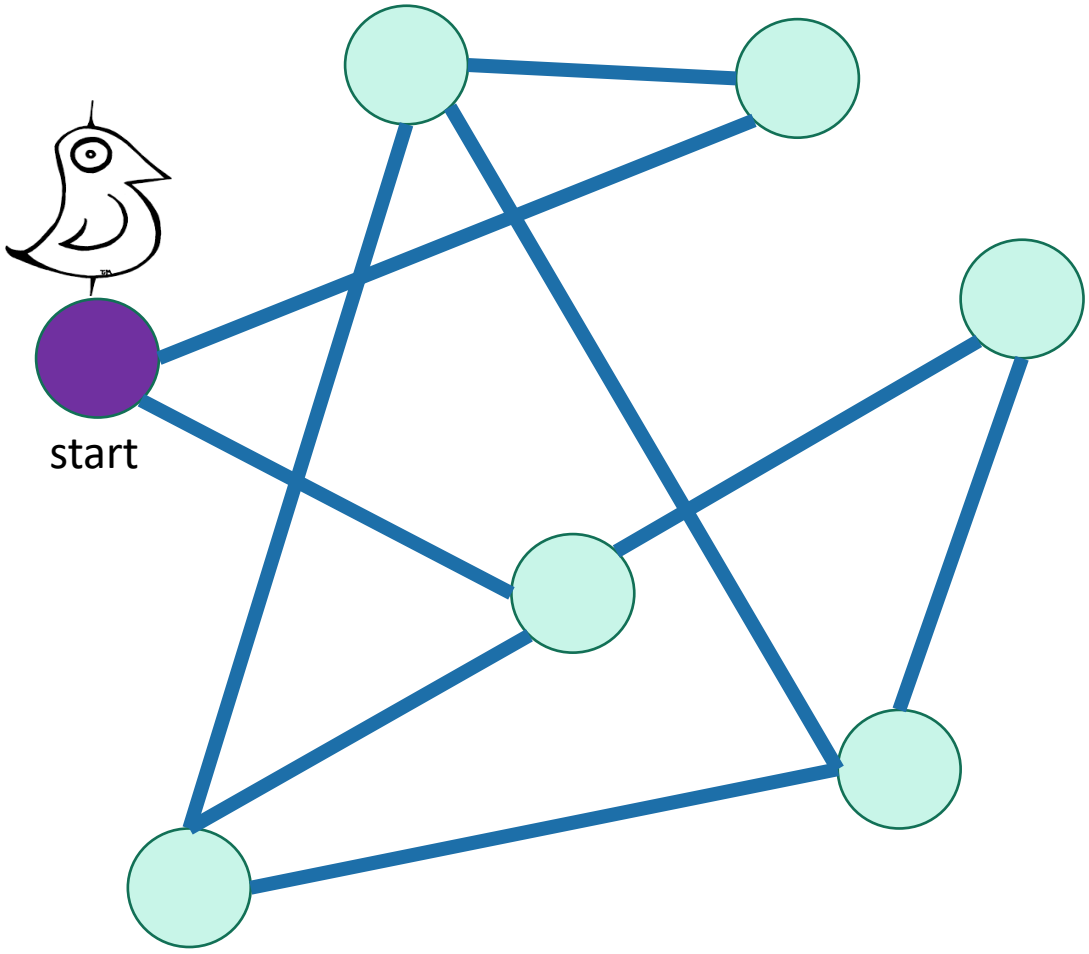
Can reach there in one step

Can reach there in two steps

Can reach there in three steps

Breadth-First Search

Exploring the world with a bird's-eye view



Not been there yet

Can reach there in zero steps

Can reach there in one step

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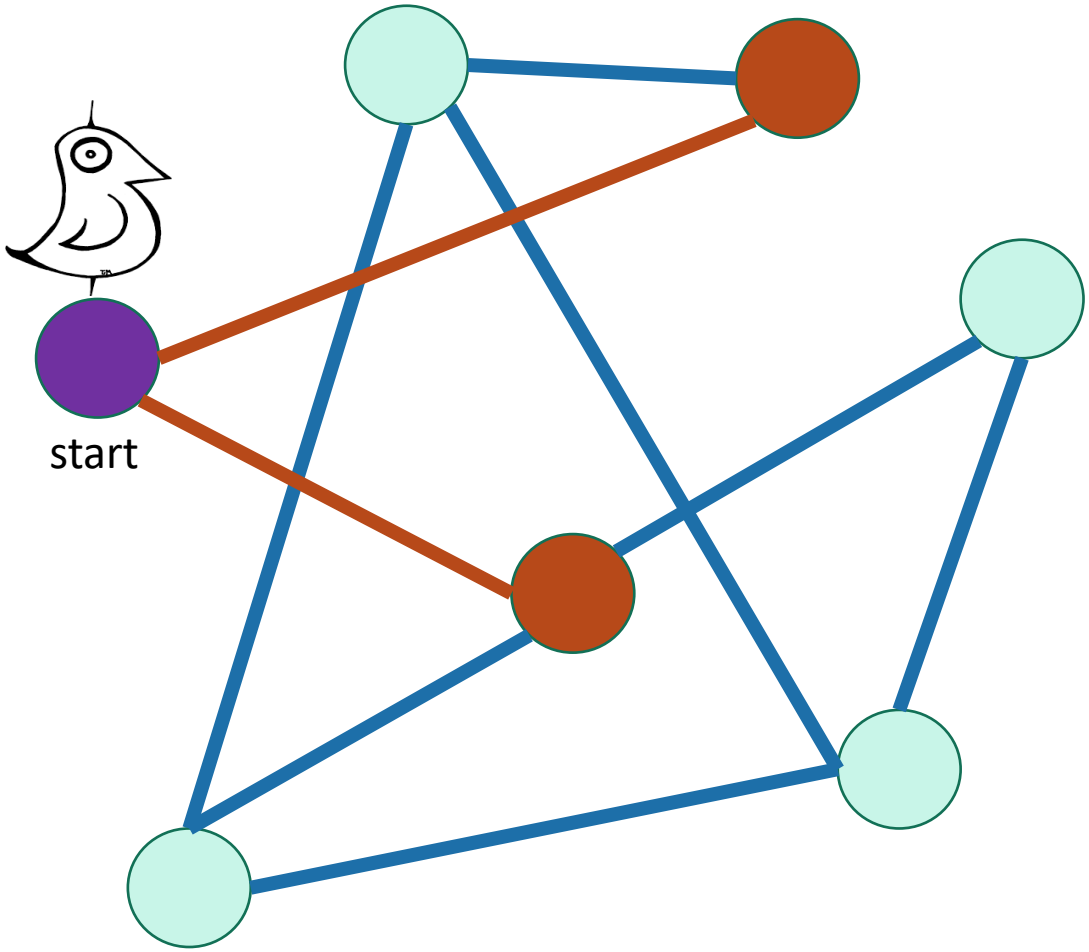
Breadth-First Search

Exploring the world with a bird's-eye view



Not been there yet

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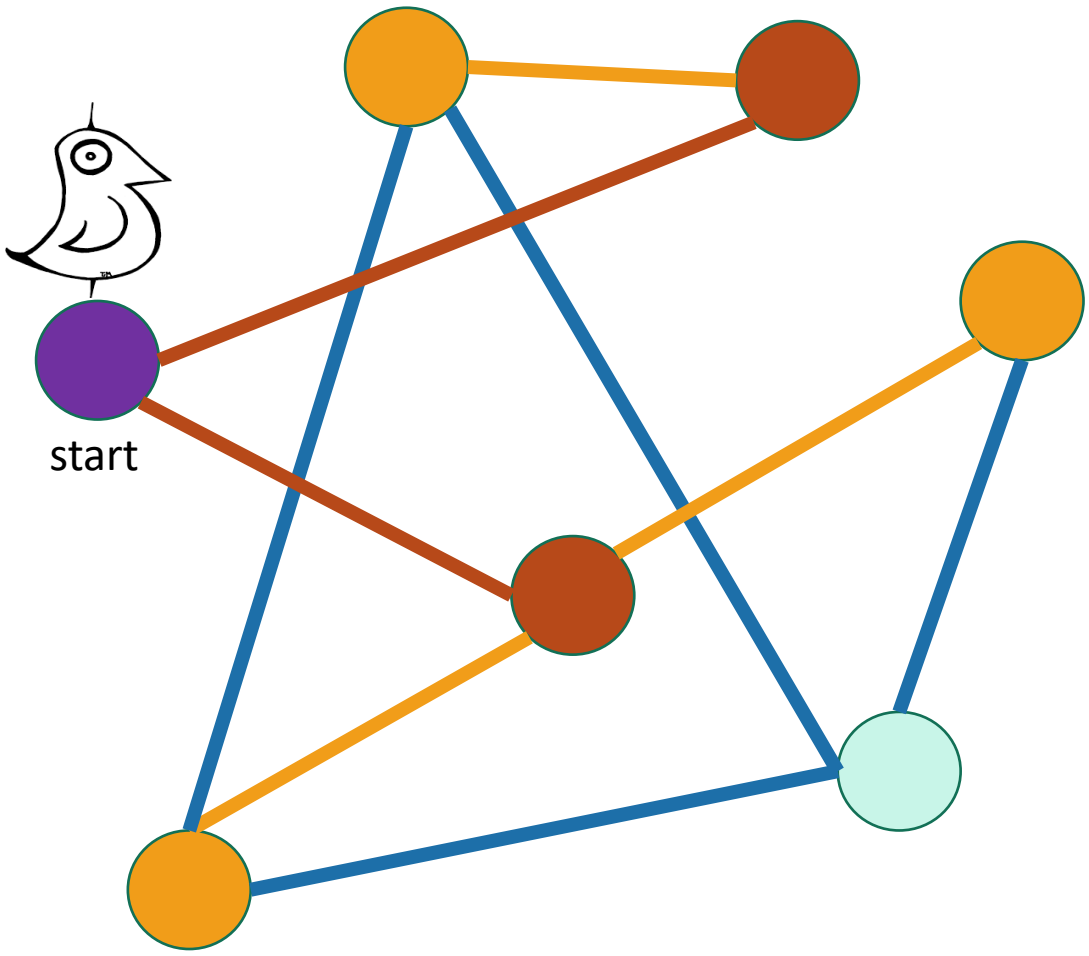
Can reach there in three steps

Breadth-First Search

Exploring the world with a bird's-eye view



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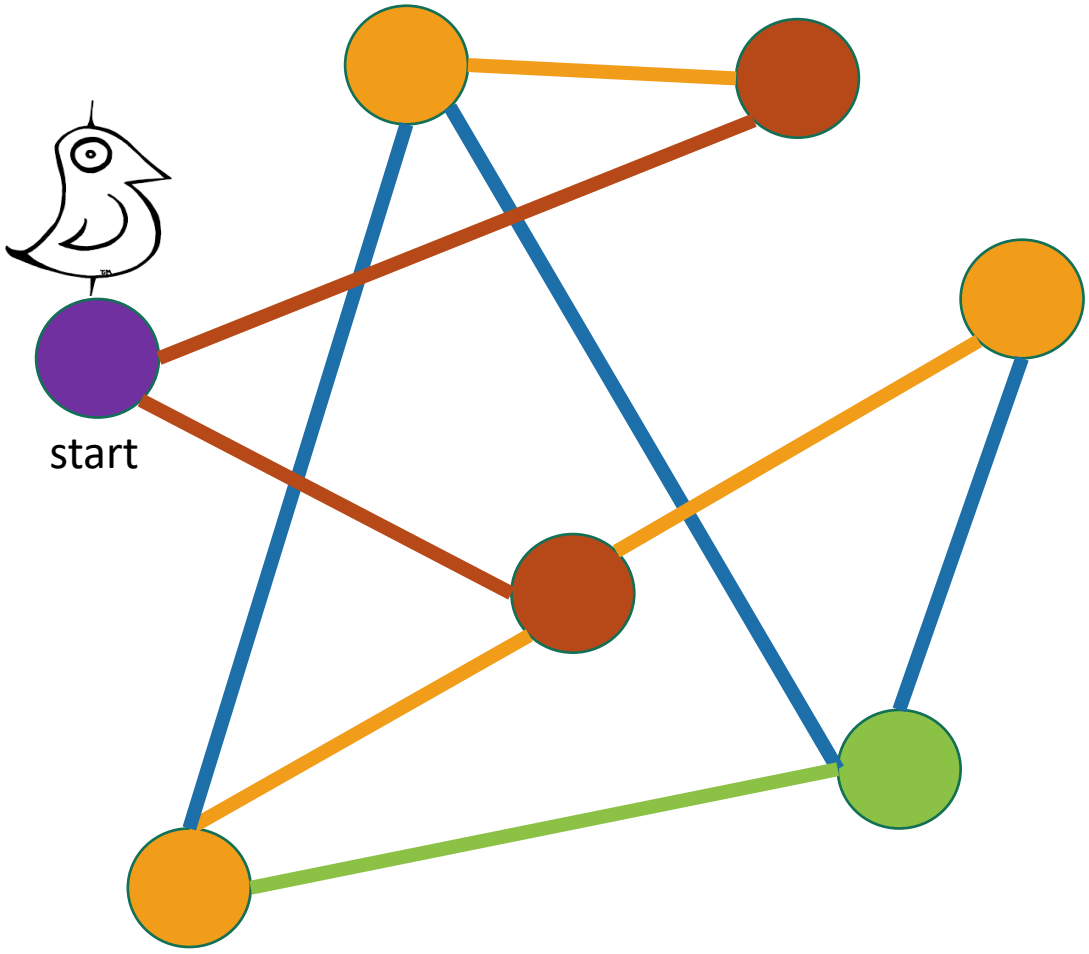

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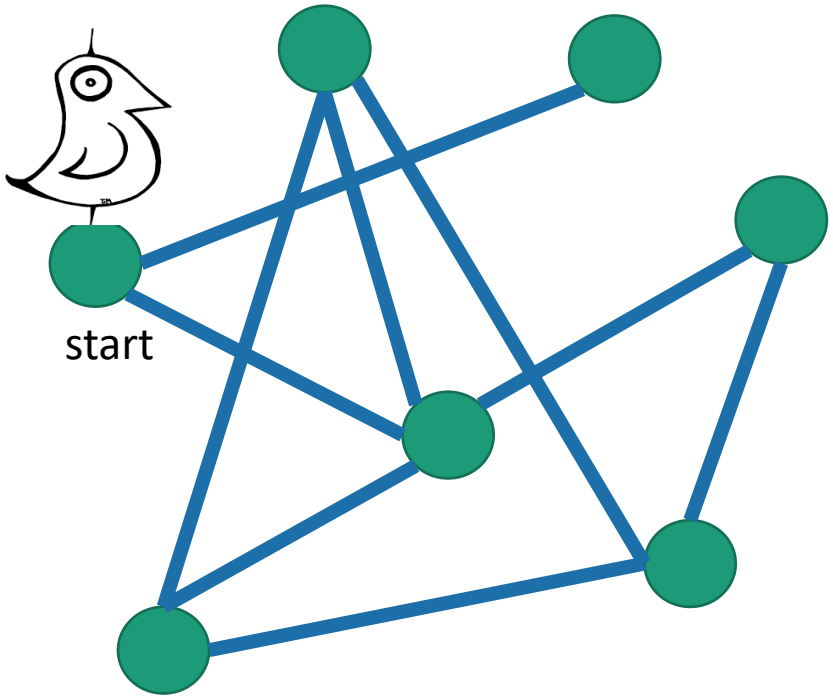
World:

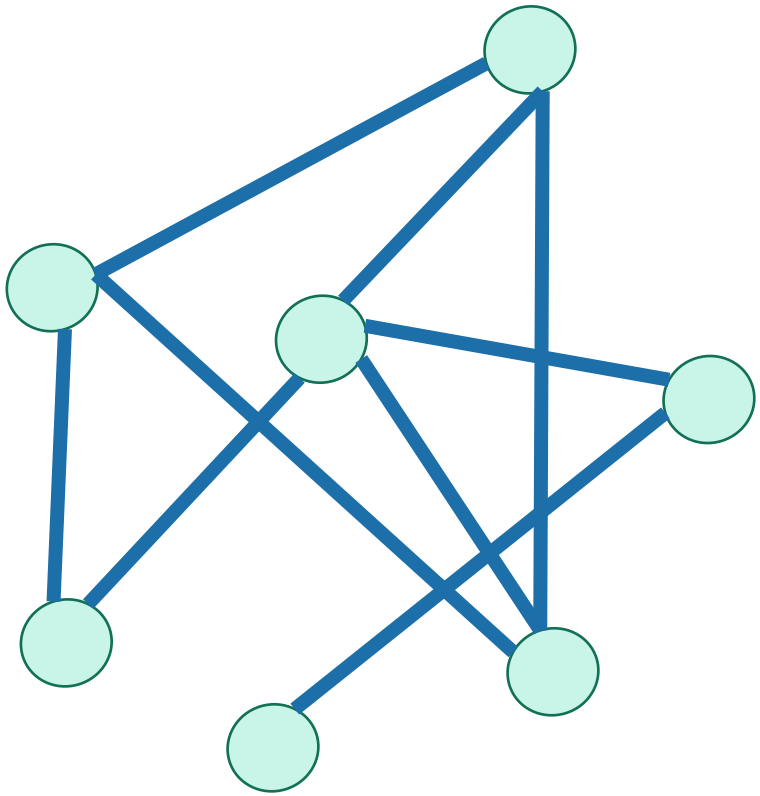
EXPLORED!

```
def DFS(root):

    if root == Null:
        return

    root.visited = True

    for n in root.neighbours:
        if n.visited == False:
            DFS(n)
```

```
def BFS(root):

    visited = [False] * (num_nodes_in_graph)

    # Create a queue for BFS
    queue = []

    queue.append(root)
    visited[root] = True

    while queue:

        r = queue.pop(0)
        print(r)

        for n in r.adjacent:

            if visited[n] == False:

                visited[n] = True
                queue.append(n)

```

BFS also finds all the nodes reachable from the starting point





It is also a good way to find all the **connected components.**

Running time

To explore the whole thing

- Explore the connected components one-by-one.
- Same argument as DFS: running time is

$$O(n + m)$$

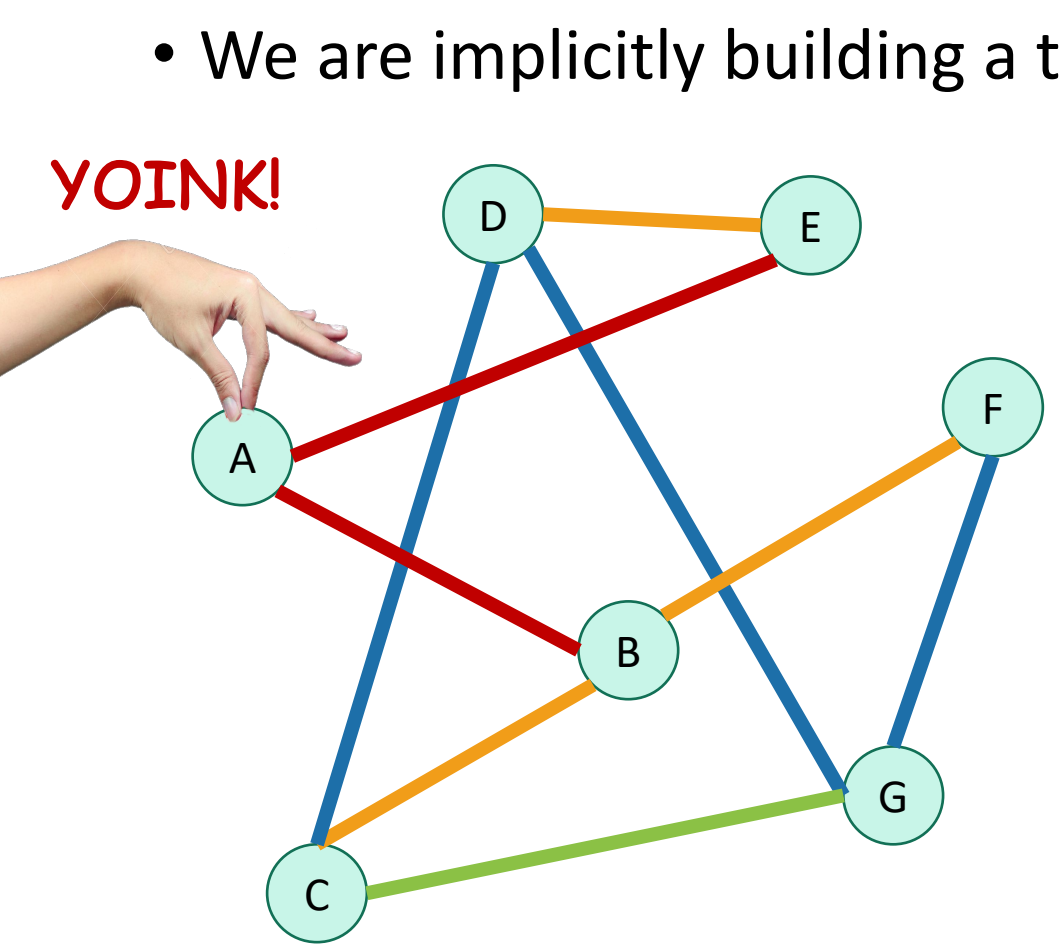
Verify these!

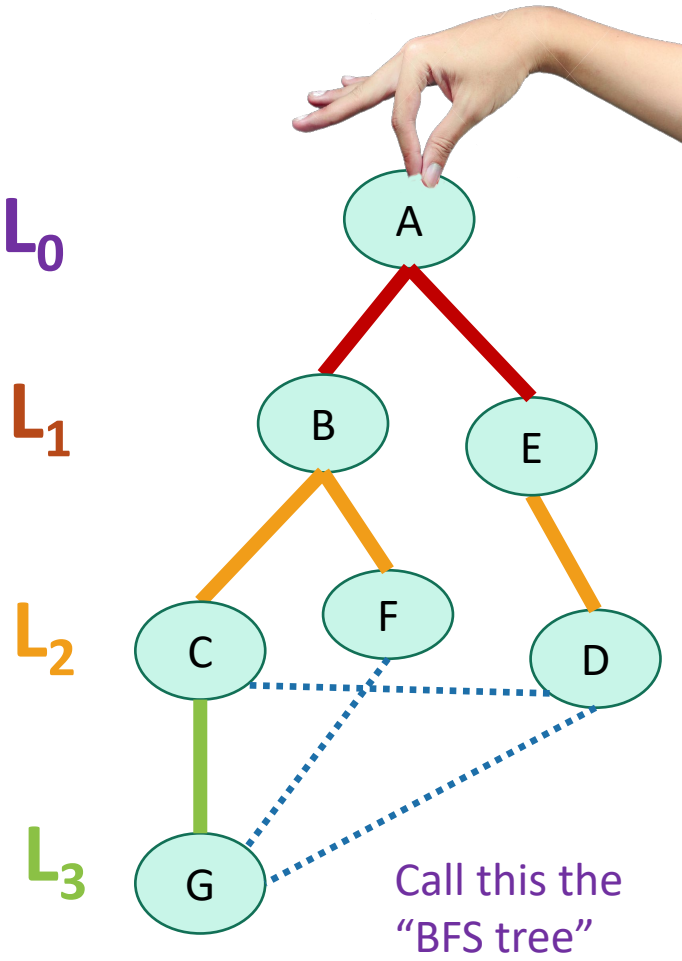
- Like DFS, BFS also works fine on directed graphs.

Siggi the Studious Stork

Why is it called breadth-first?

- We are implicitly building a tree:

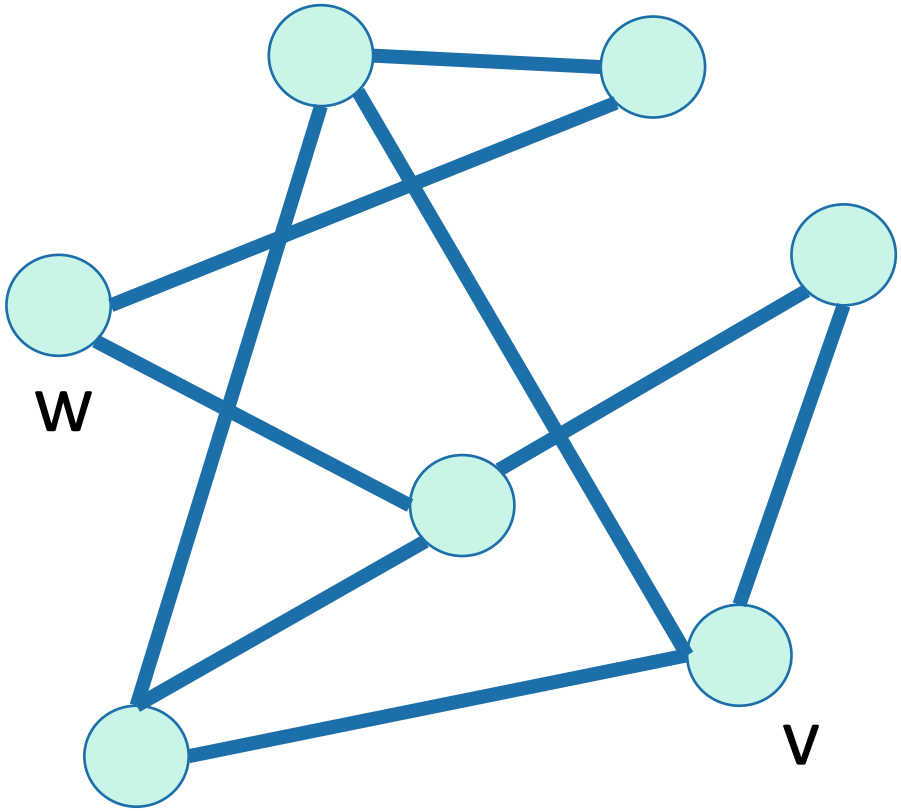




- And first we go as broadly as we can.

Application: shortest path

- How long is the shortest path between w and v?



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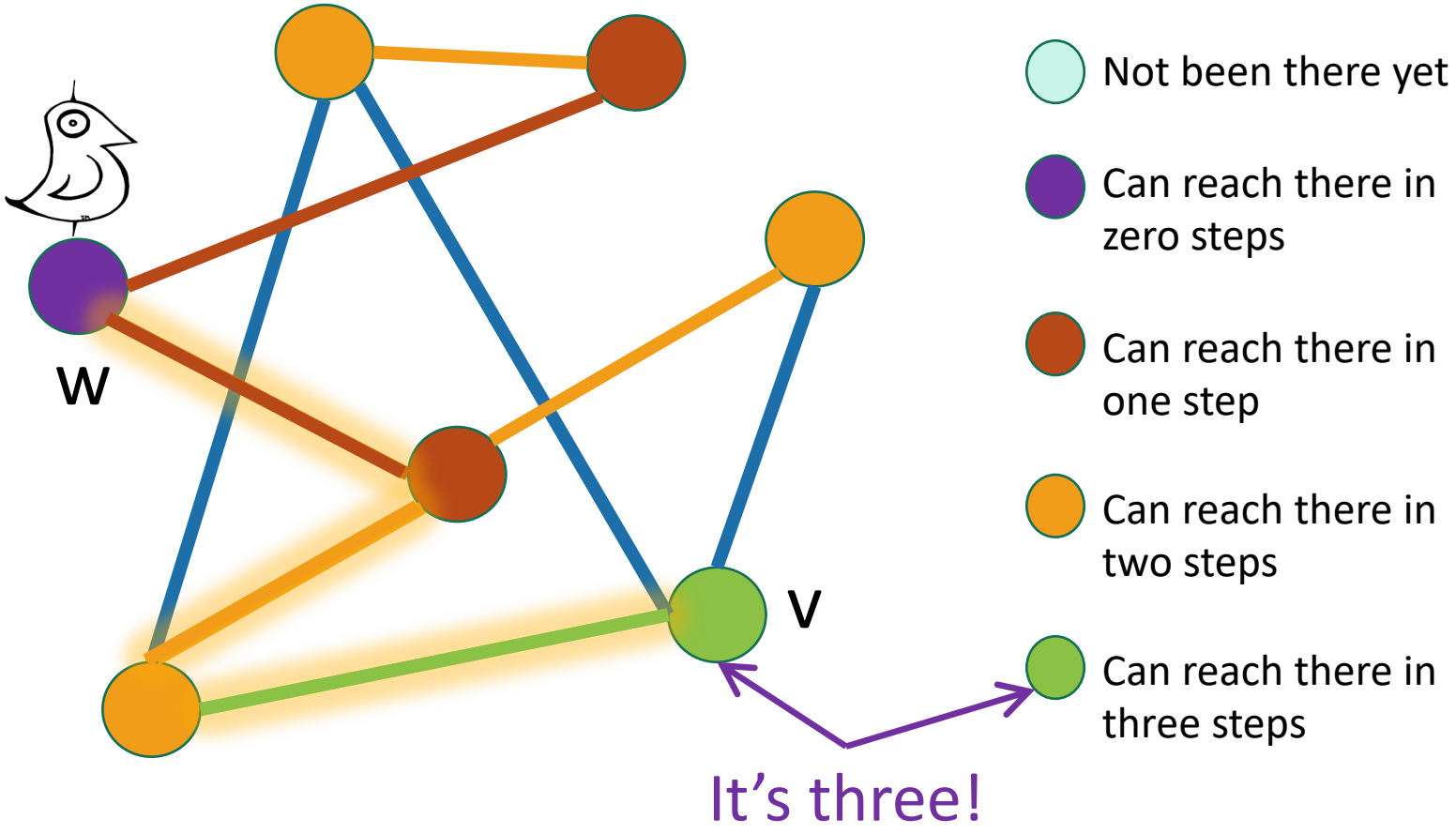
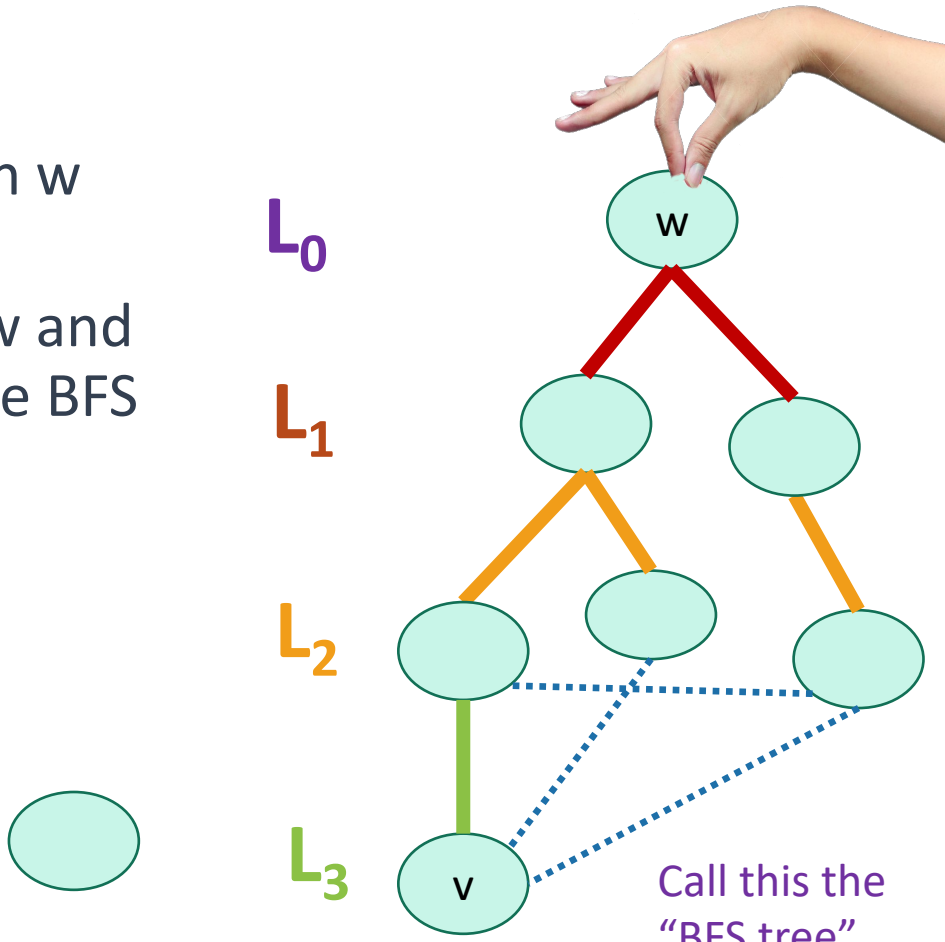


To find the distance between w and all other vertices v

The **distance** between two vertices is the length of the shortest path between them.

- Do a BFS starting at w
- For all v in L_i
  - The shortest path between w and v has length i
  - A shortest path between w and v is given by the path in the BFS tree.
- If we never found v, the distance is infinite.



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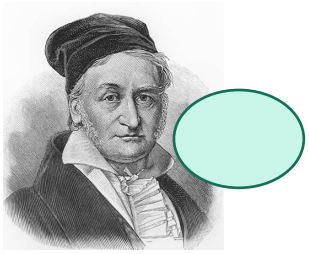
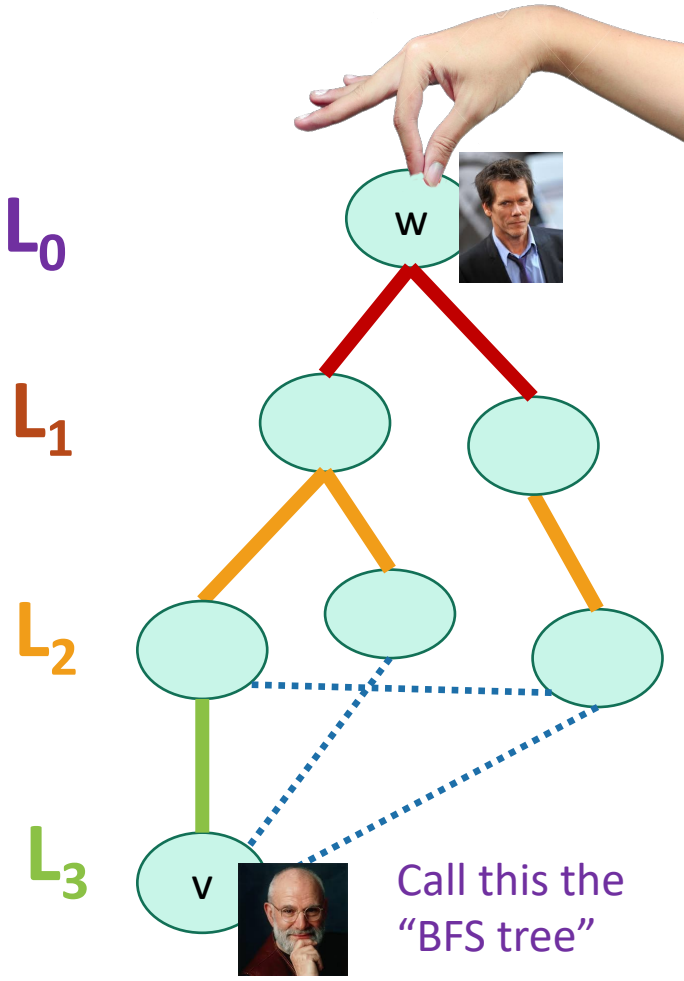


Gauss has no Bacon number



Call this the "BFS tree"

What did we just learn?

- The BFS tree is useful for computing distances between pairs of vertices.
- We can find the shortest path between u and v in time O(m).

The BSF tree is also helpful for:

- Testing if a graph is bipartite or not.

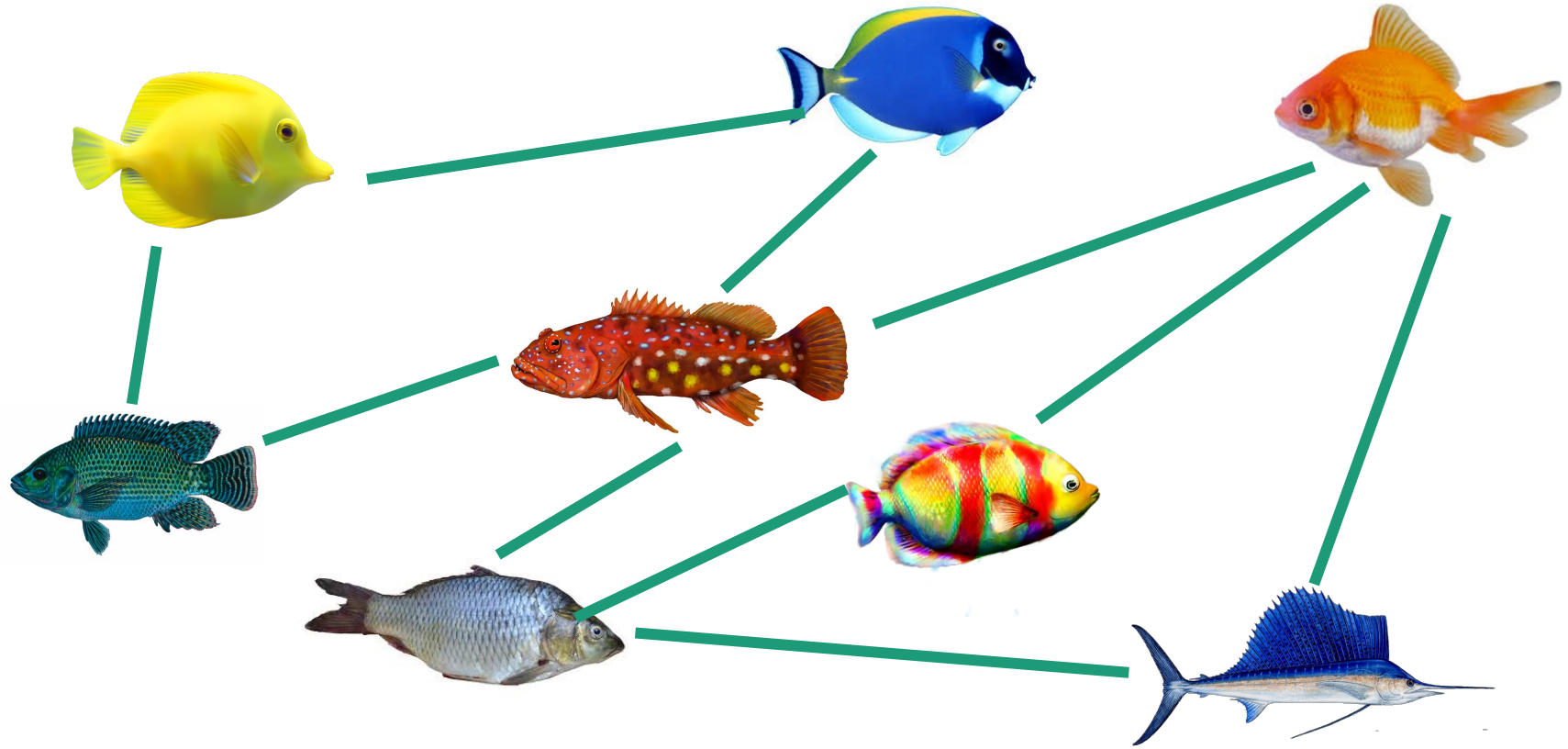
Recap

- Depth-first search
  - Useful for topological sorting
  - Also in-order traversals of BSTs
- Breadth-first search
  - Useful for finding shortest paths
  - Also for testing bipartiteness
- Both DFS, BFS:
  - Useful for exploring graphs, finding connected components, etc

Part 3: Is Bi-partite?

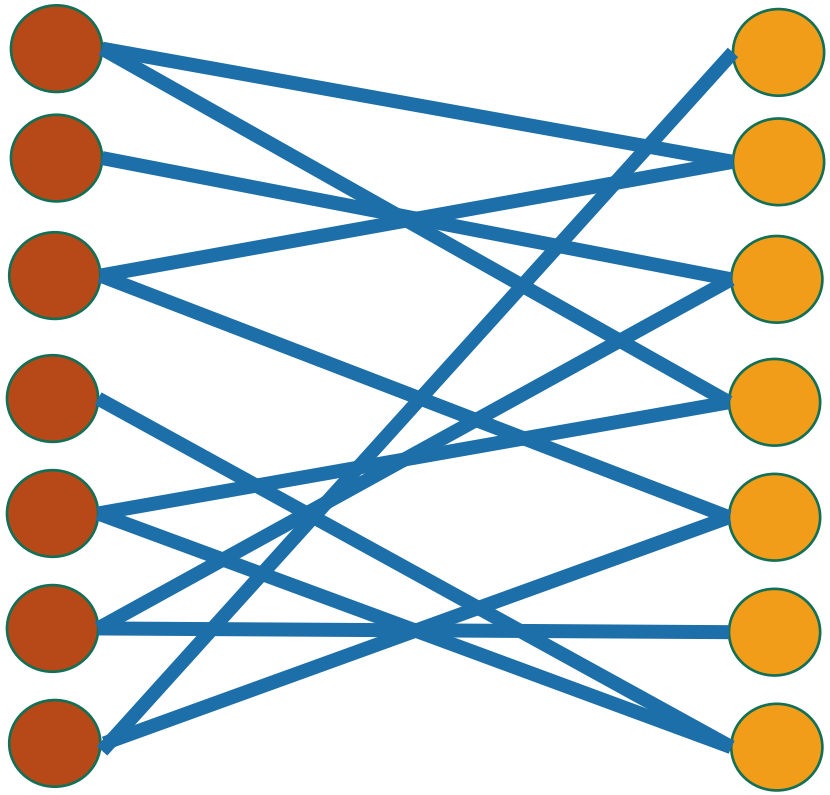
BFS - Application

- Some pairs of species will fight if put in the same tank.
- You only have two tanks.
- Connected fish will fight.



Application: testing if a graph is bipartite

- Bipartite means it looks like this:



Can color the vertices red and orange so that there are no edges between any same-colored vertices

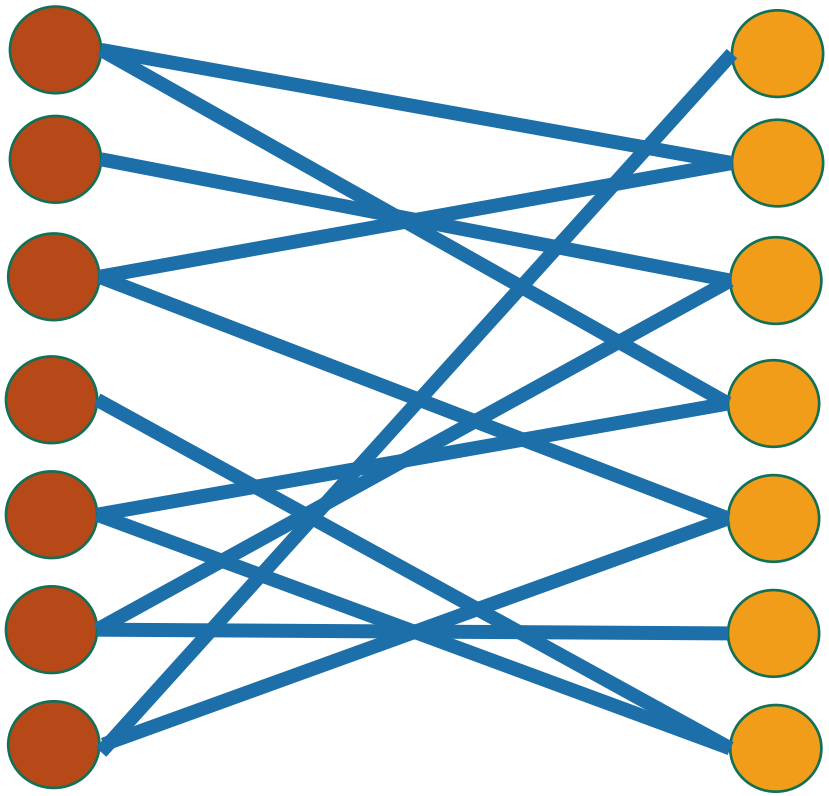
Example:

are in tank A

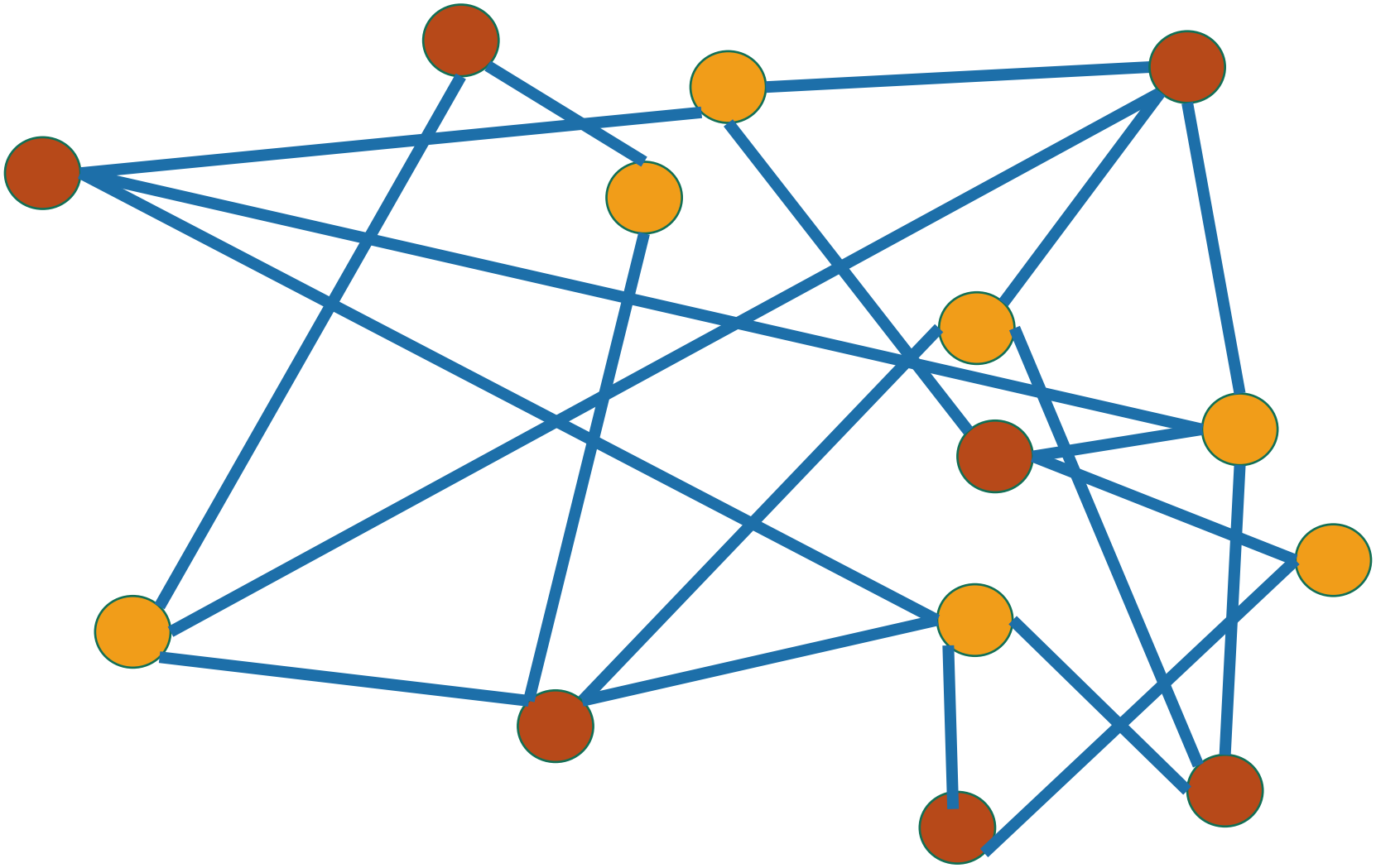
are in tank B

if the fish fight

Is this graph bipartite?



How about this one?



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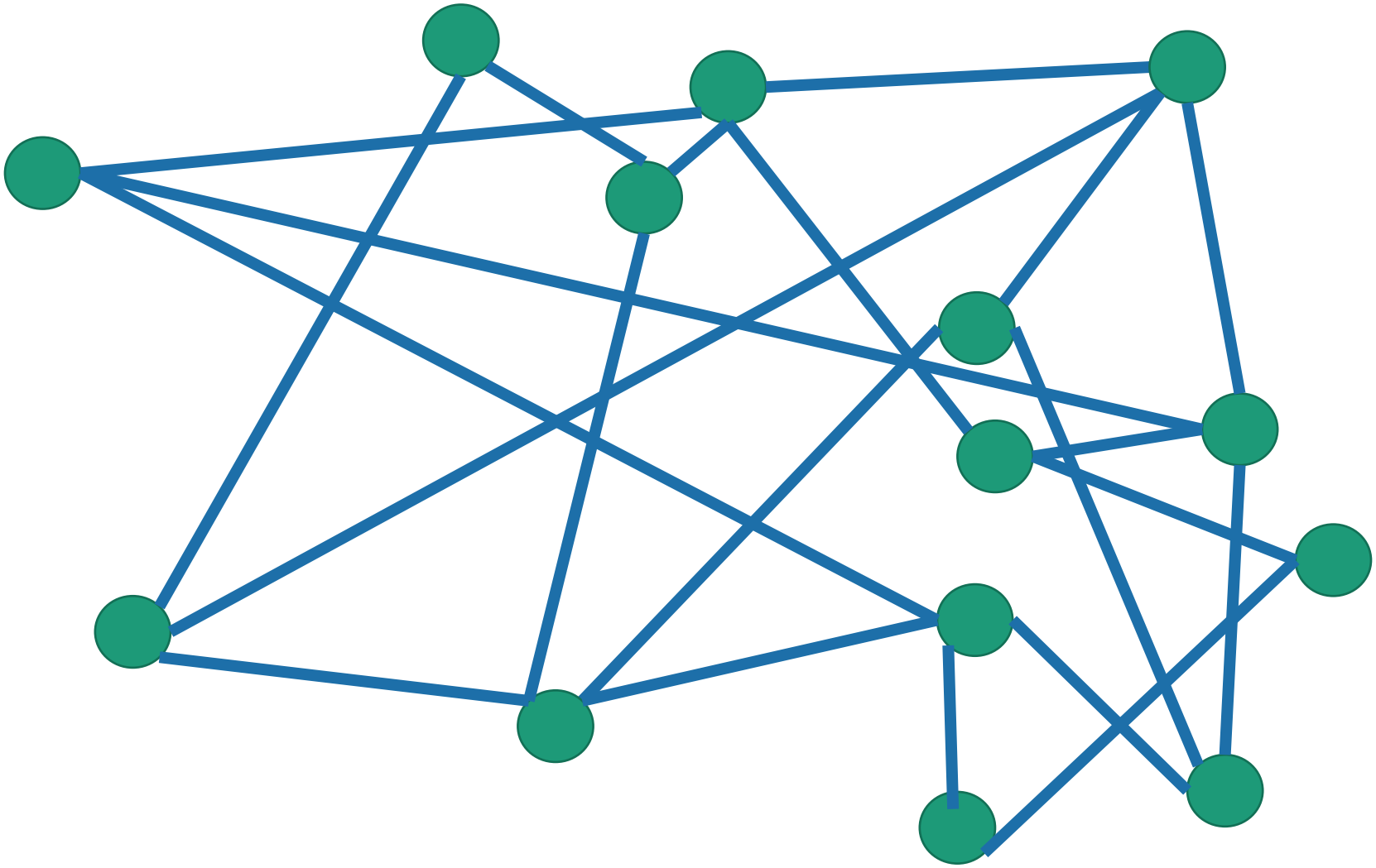
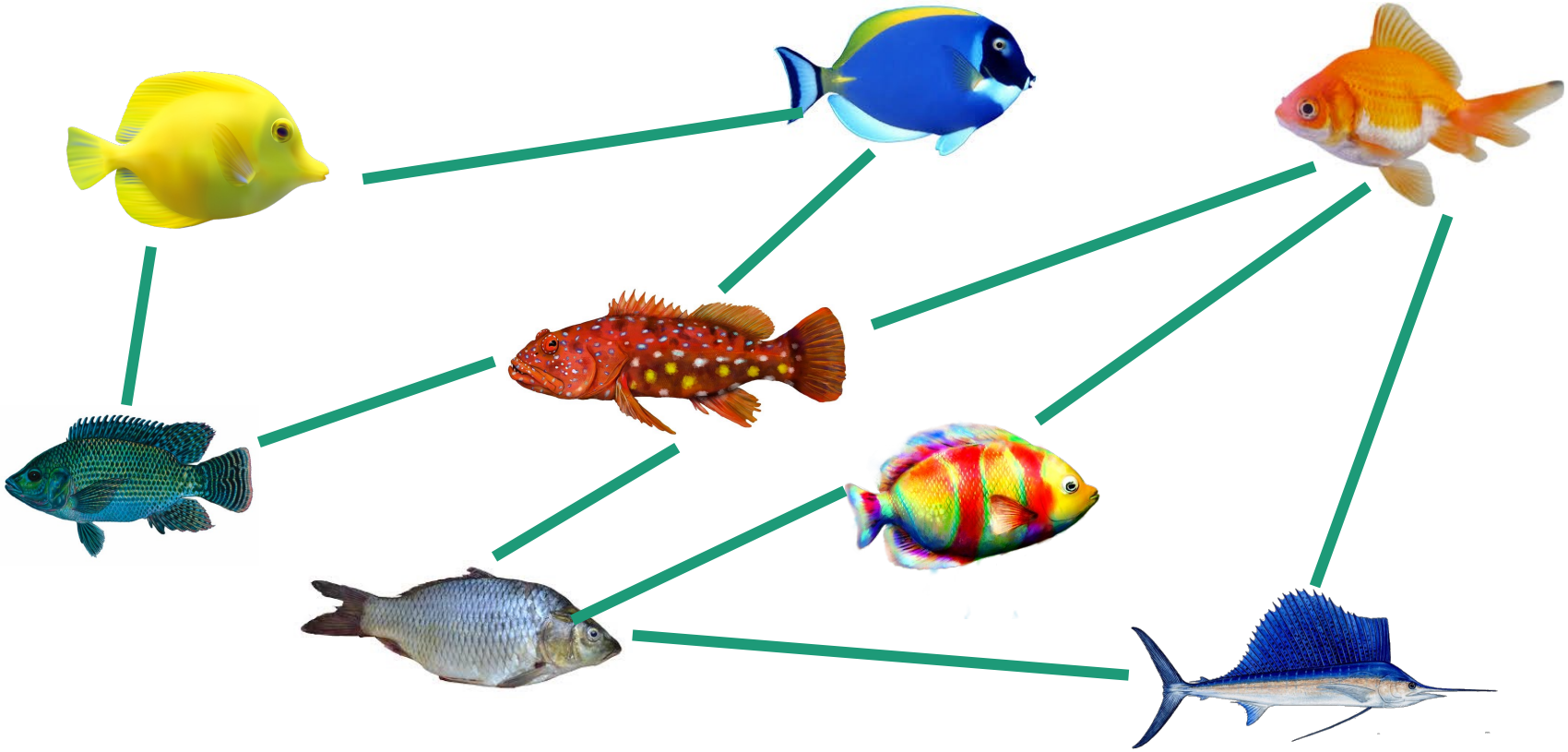
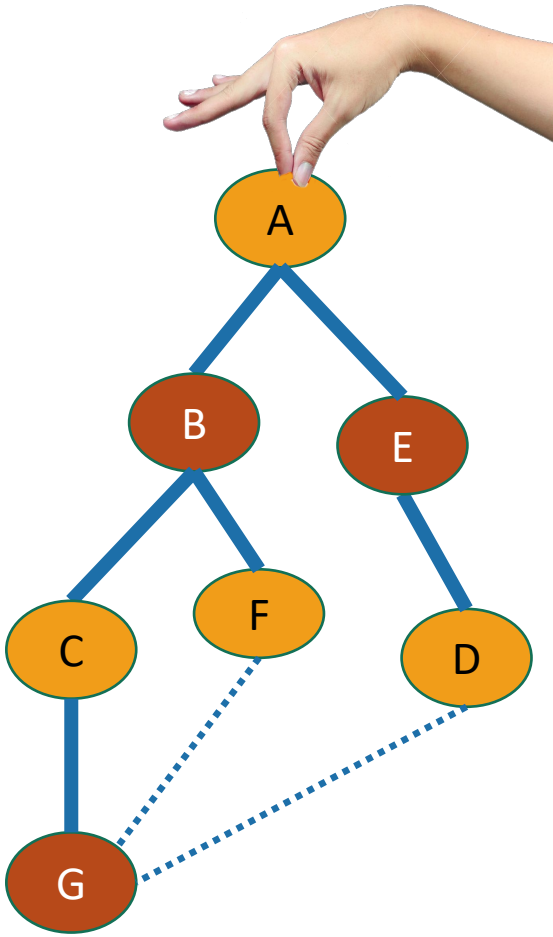


This one?



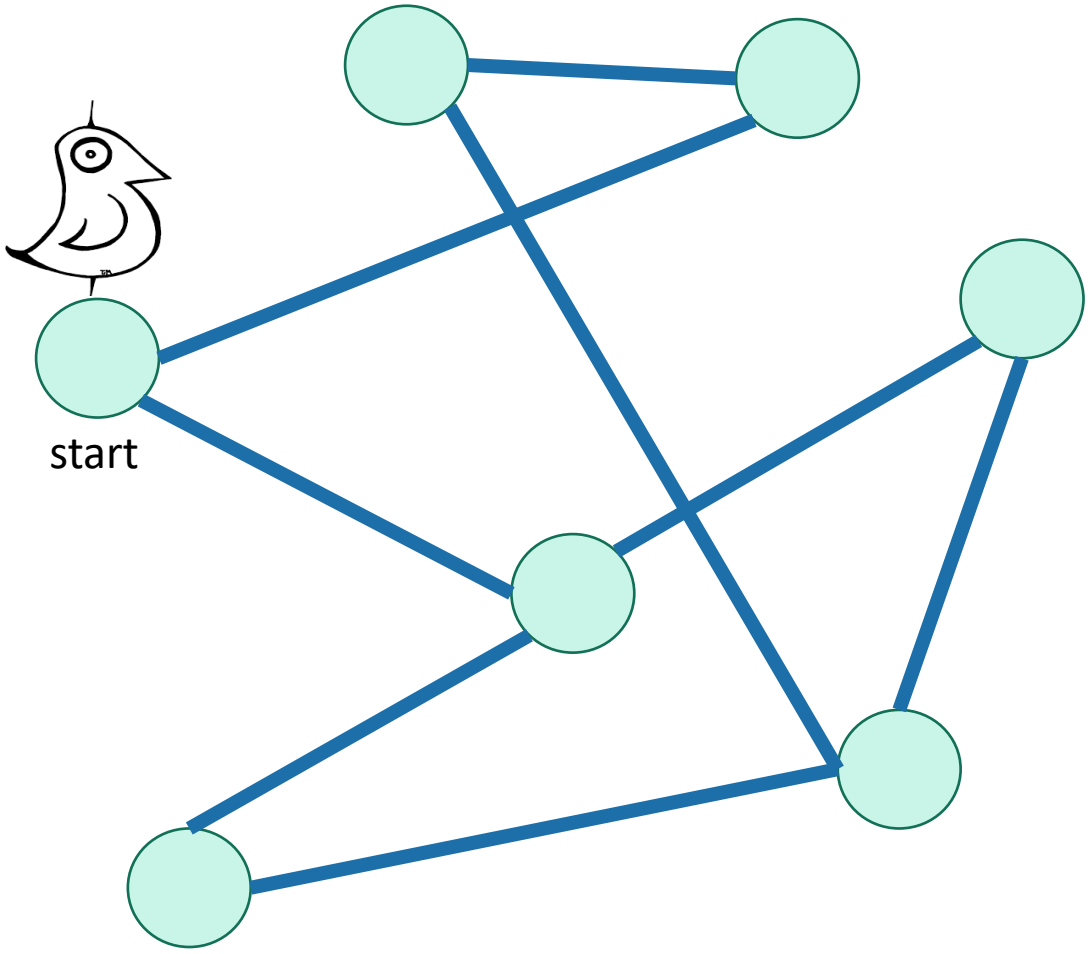
Solution using BFS

- Color the levels of the BFS tree in alternating colors.
- If you never color two connected nodes the same color, then it is bipartite.
- Otherwise, it's not.



Breadth-First Search

For testing bipartite-ness



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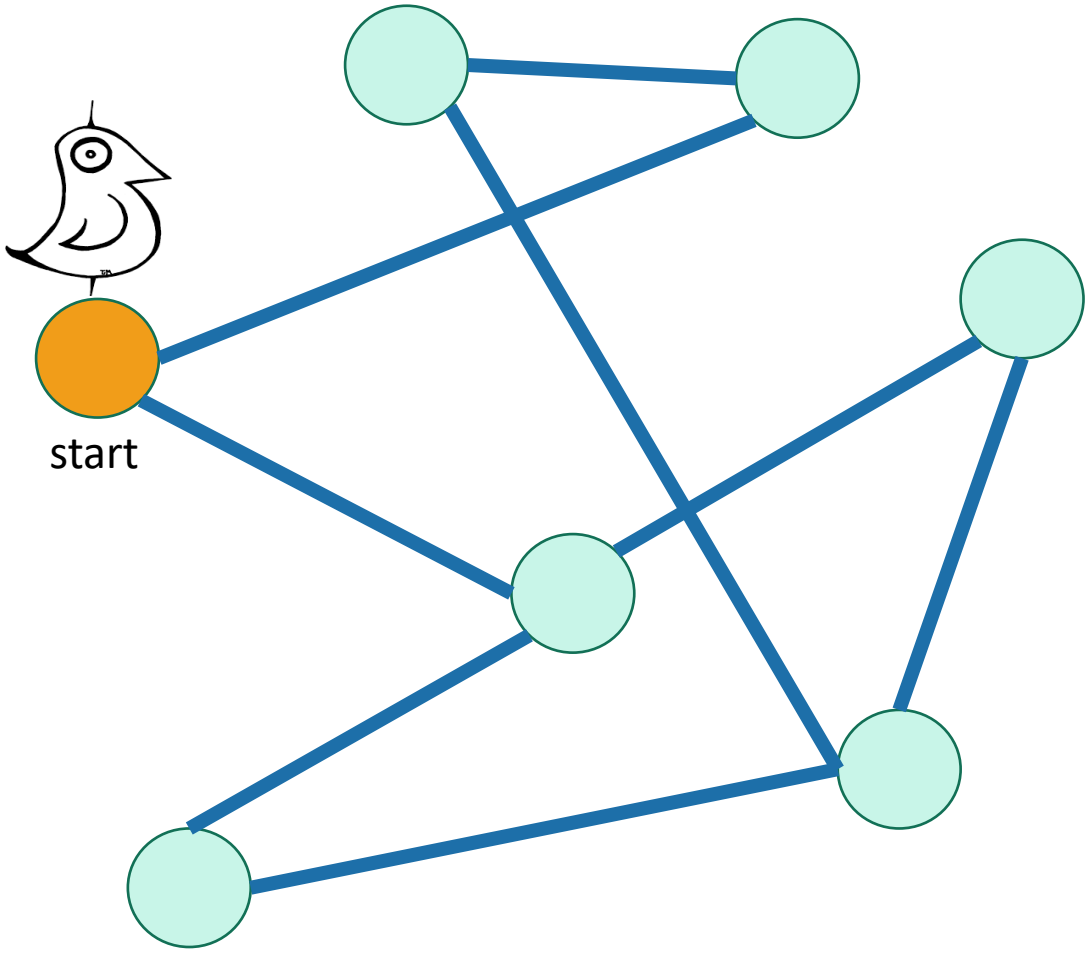

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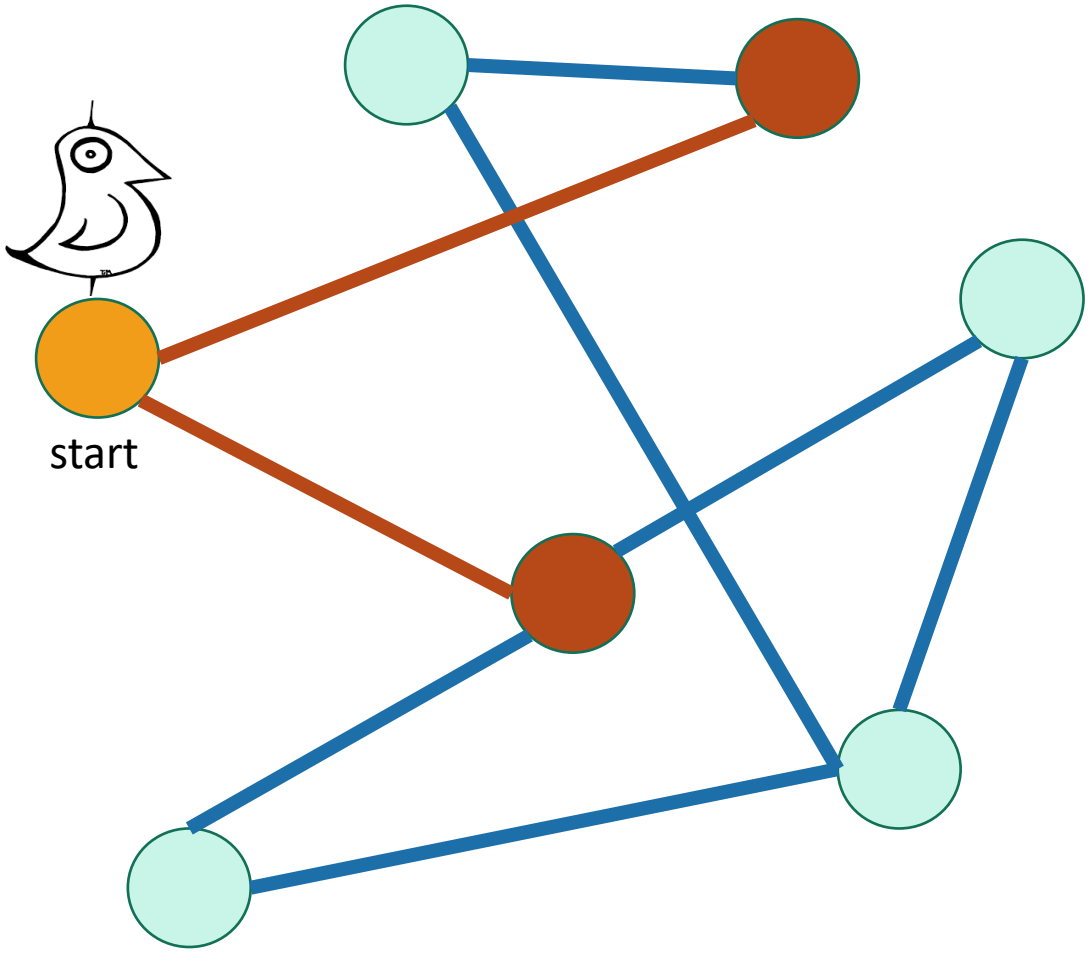

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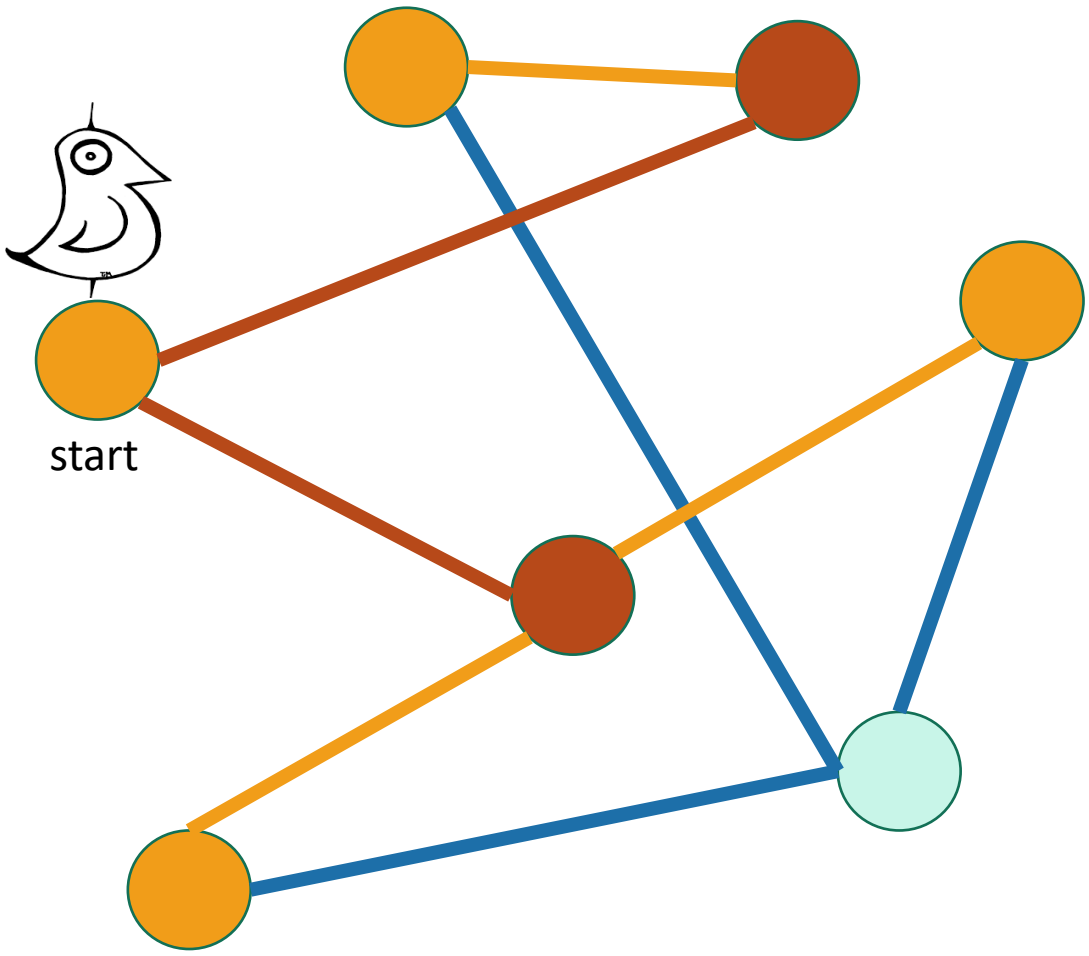

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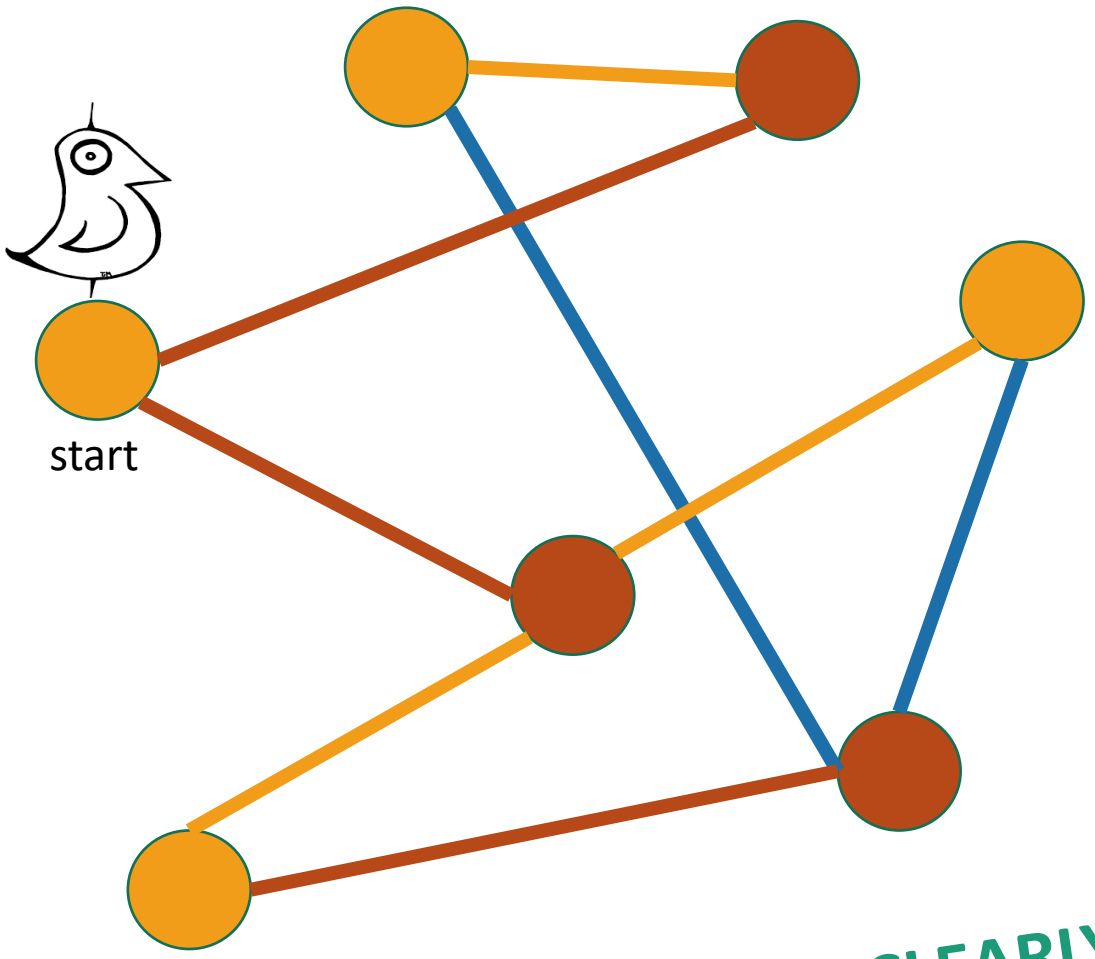

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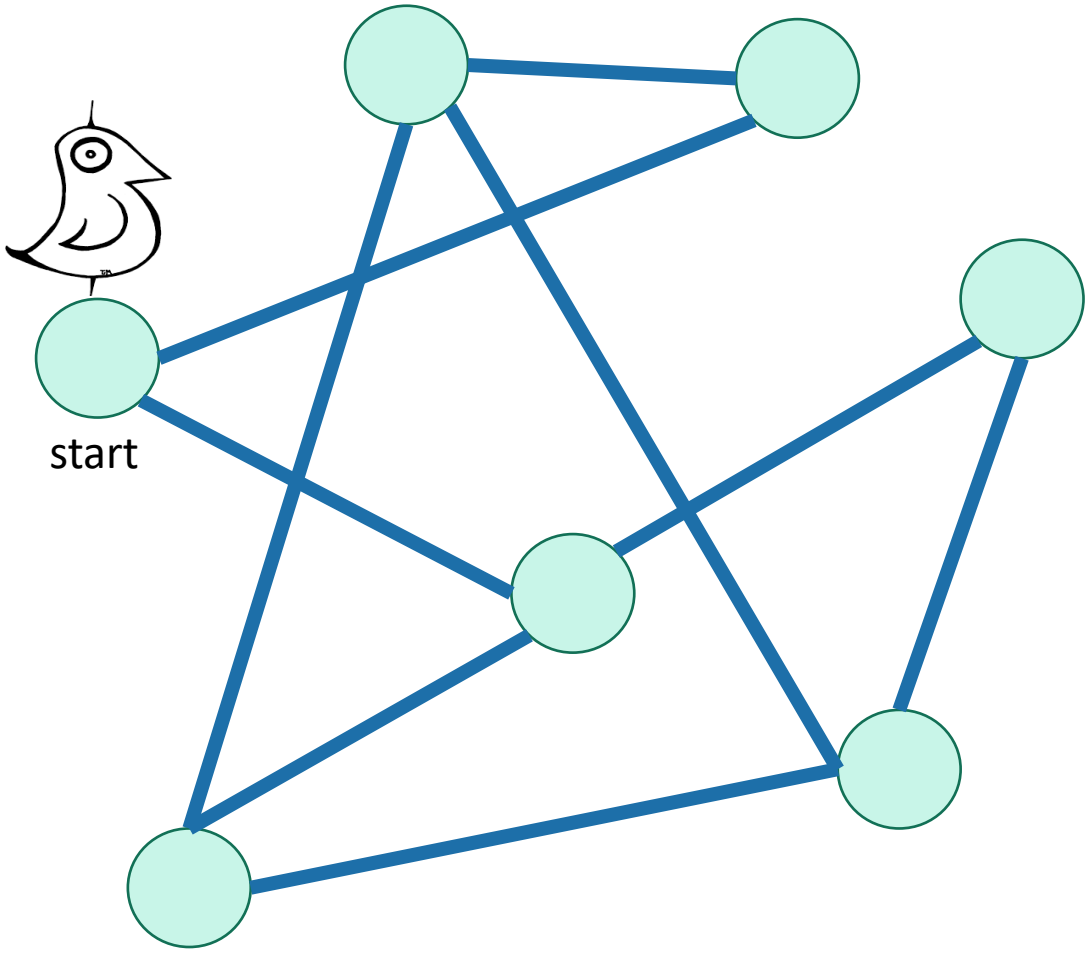
CLEARLY BIPARTITE!

Breadth-First Search

For testing bipartite-ness



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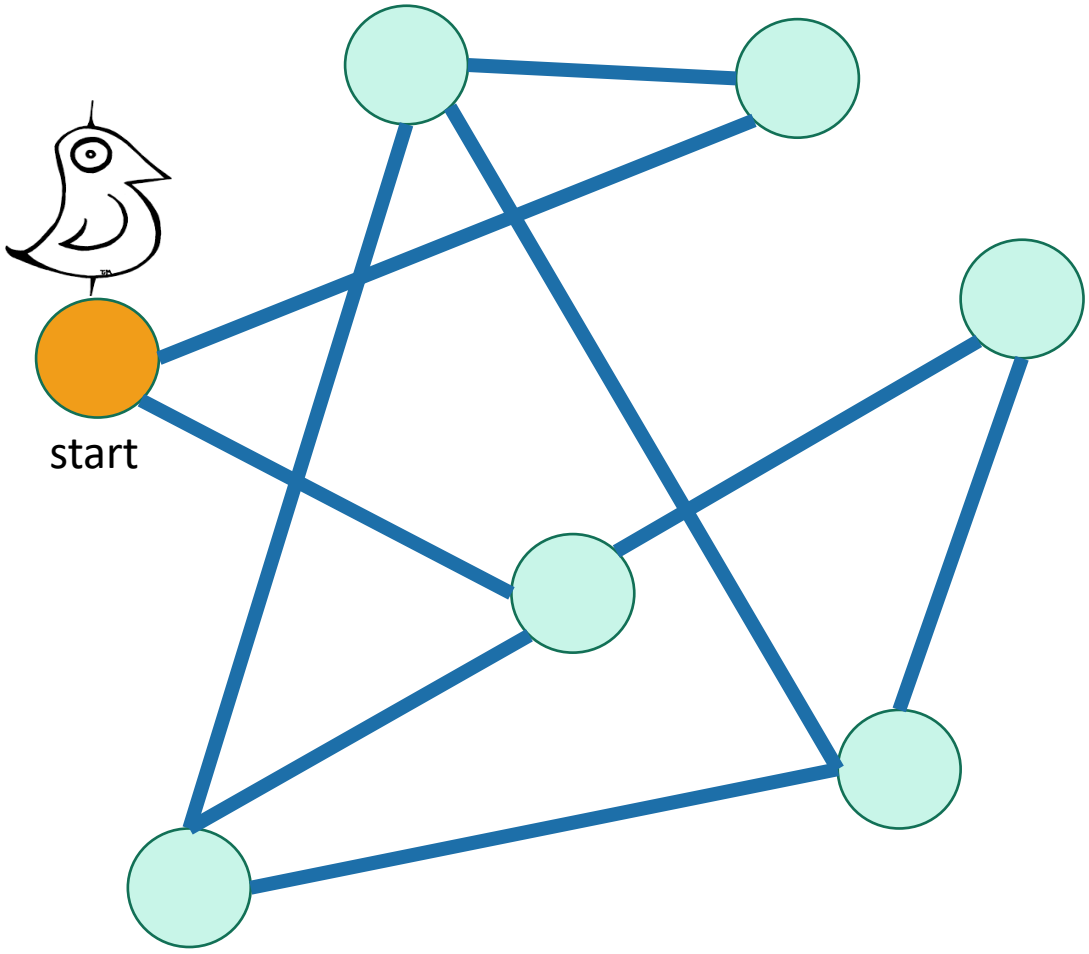

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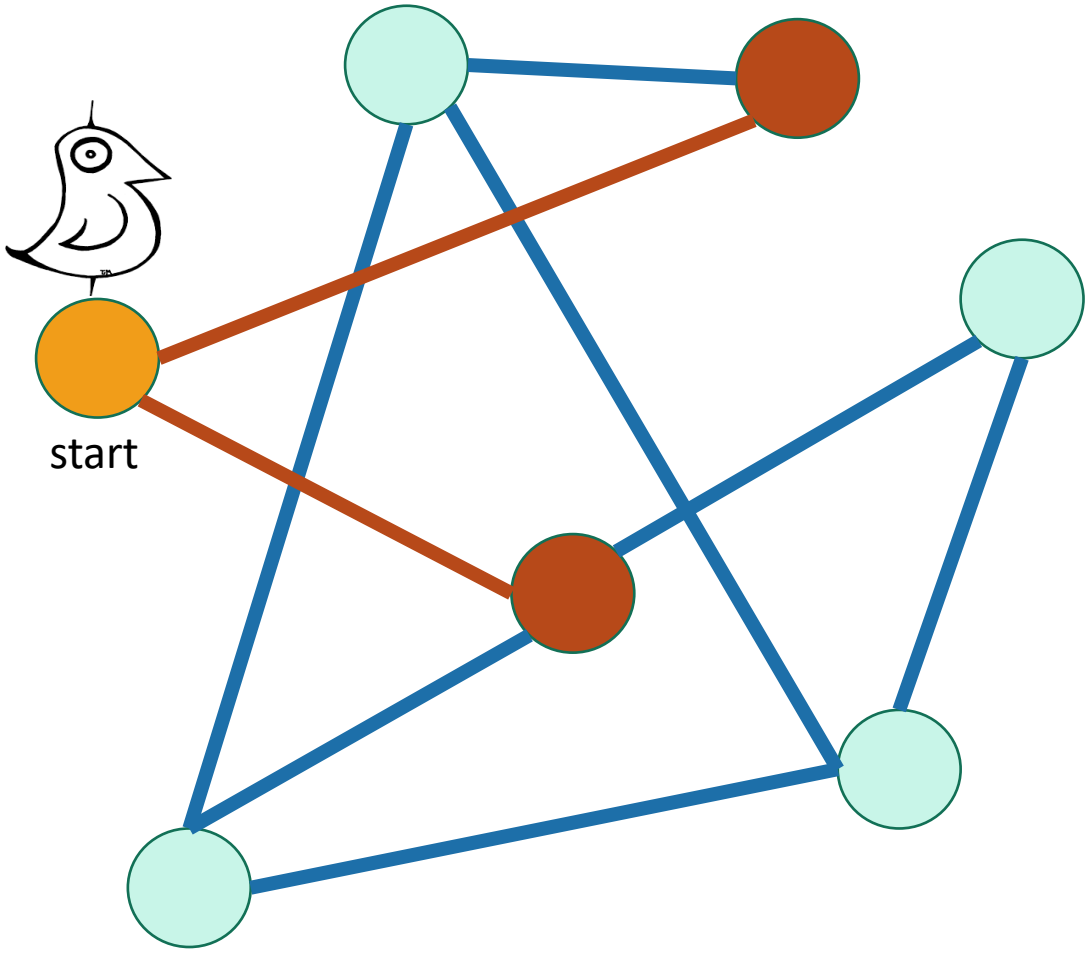

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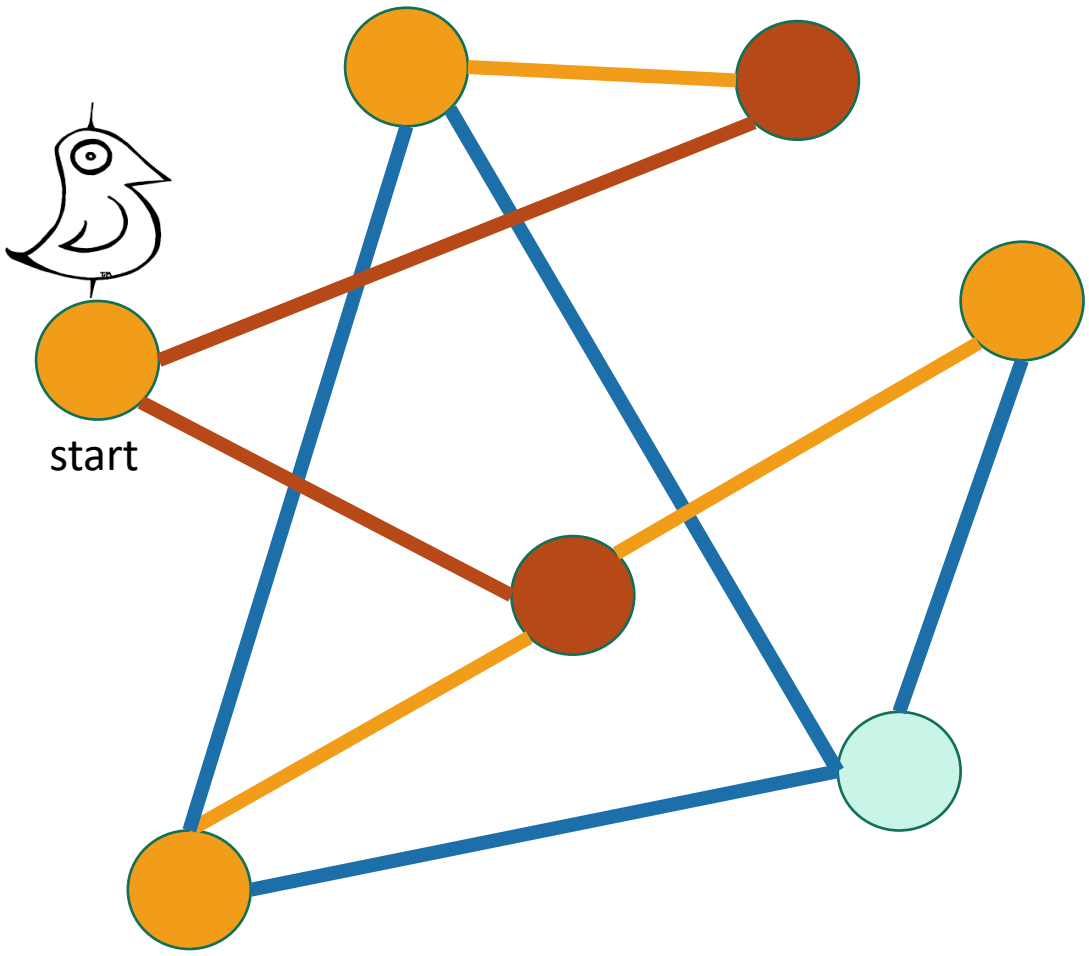

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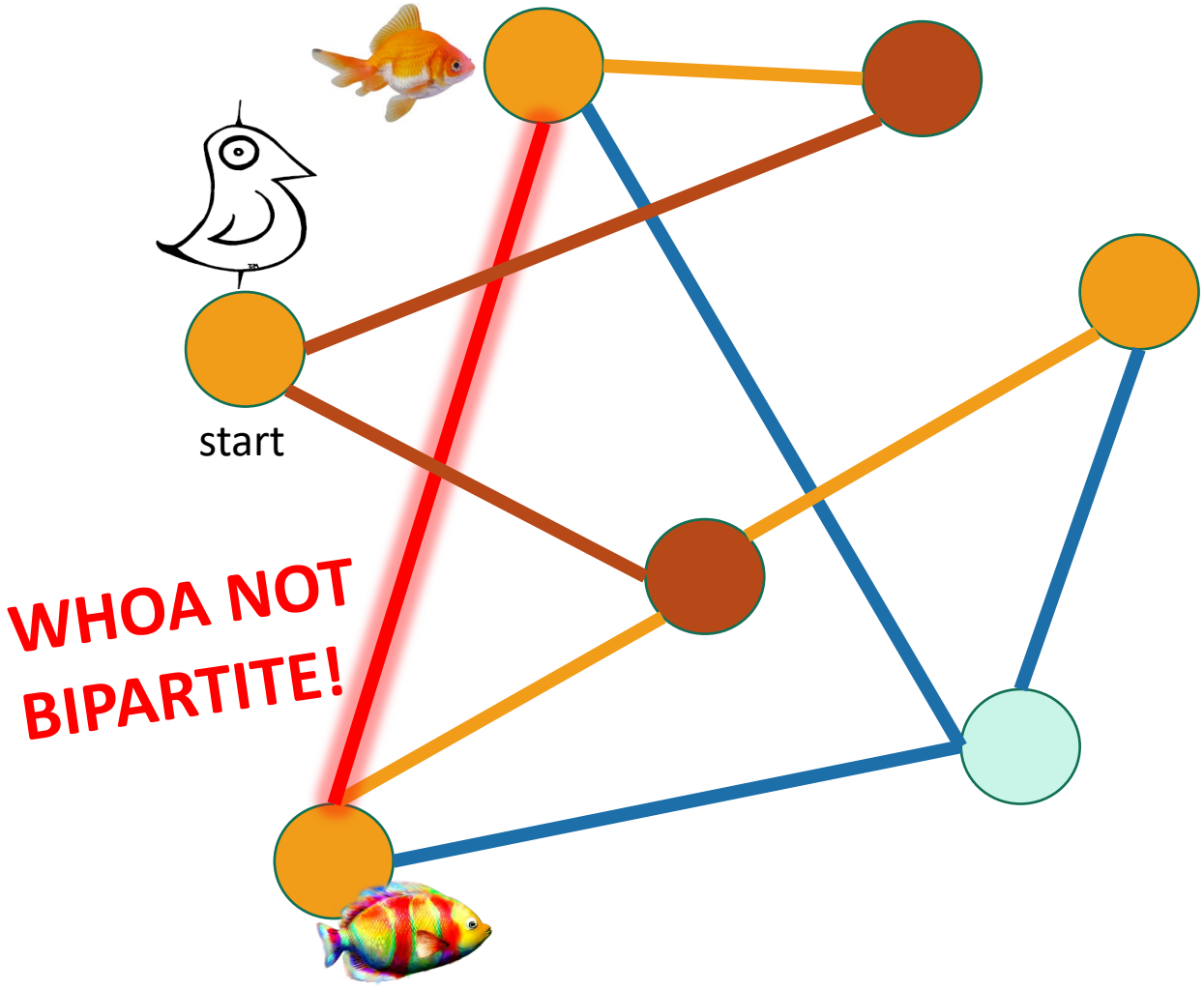

- Not been there yet
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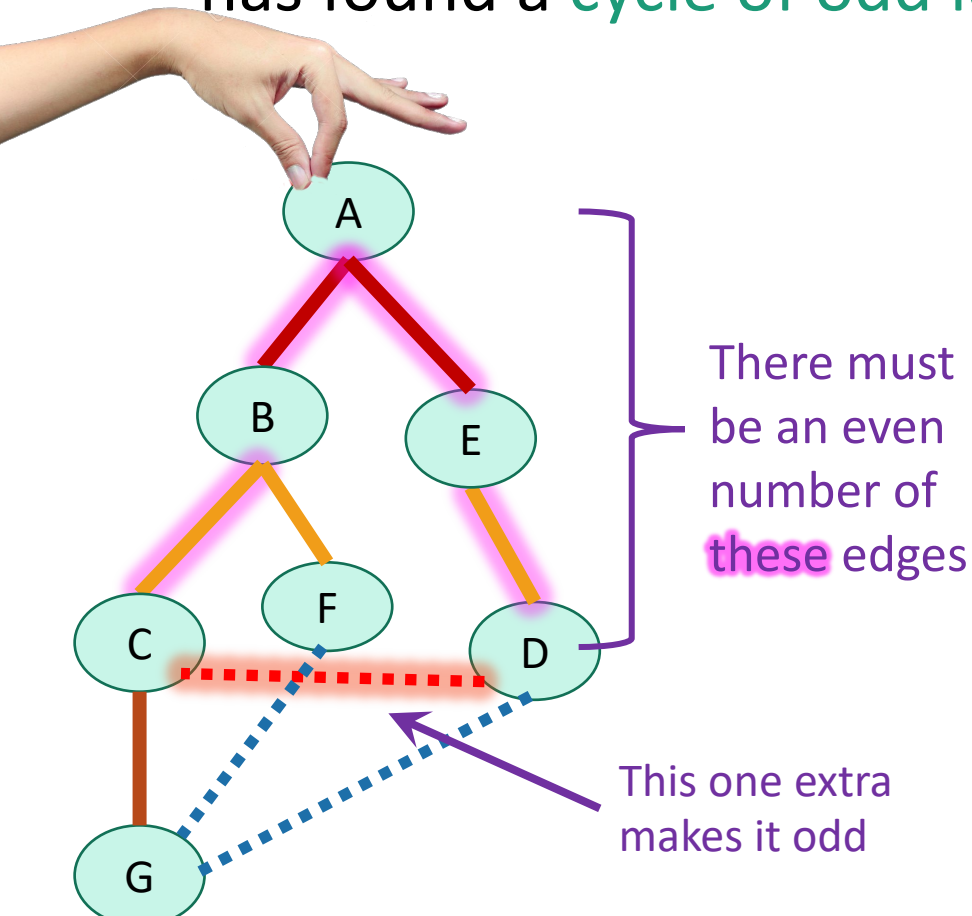
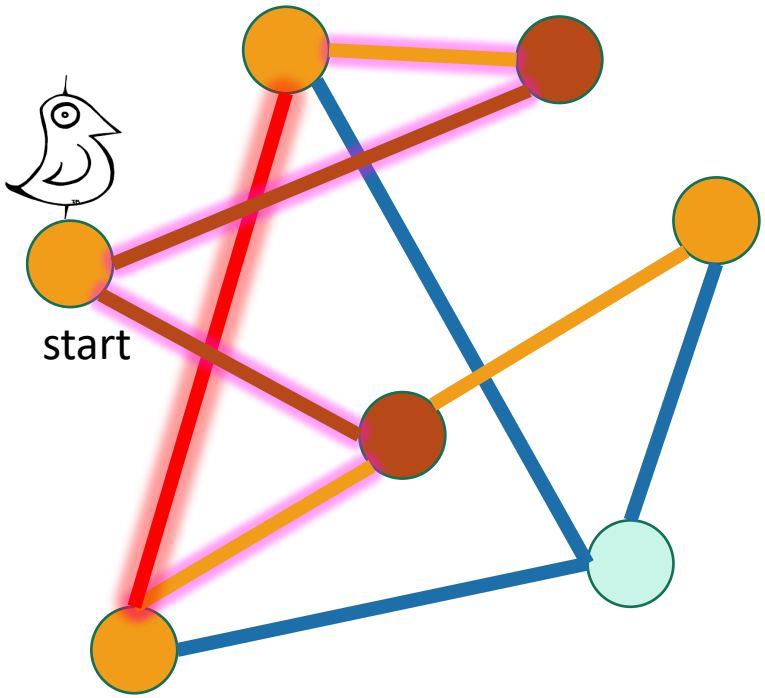
Some proof required

- If BFS colors two neighbors the same color, then it has found a cycle of odd length in the graph.

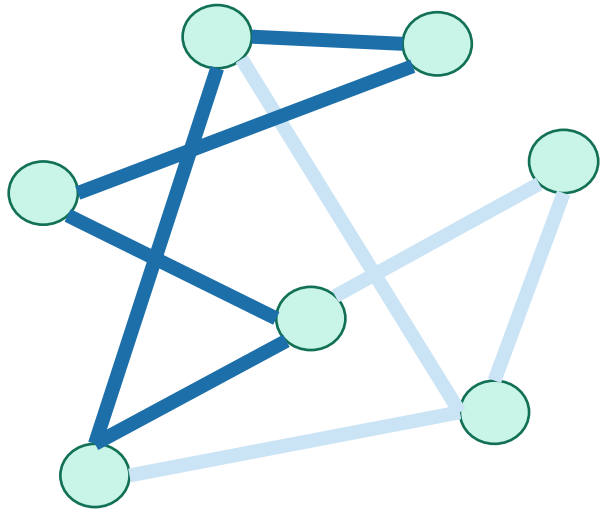




Some proof required

- If BFS colors two neighbors the same color, then it's found an cycle of odd length in the graph.
- So the graph has an odd cycle as a subgraph.
- But you can **never** color an odd cycle with two colors so that no two neighbors have the same color.

- So you can't legitimately color the whole graph either.
- **Thus it's not bipartite.**



What did we just learn?

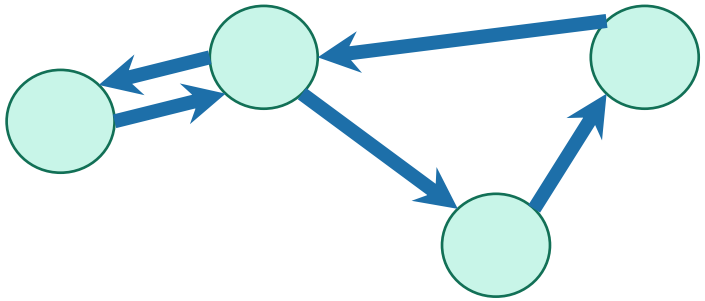
BFS can be used to detect bipartite-ness in time O(n + m).

Part 4: Finding Strongly Connected Components

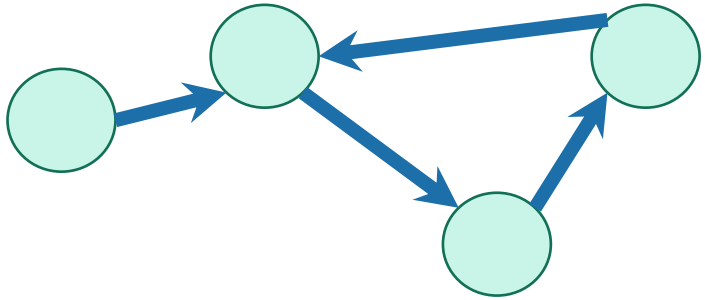
Strongly Connected Components

Strongly connected components

- A directed graph G = (V,E) is **strongly connected** if:
- for all v,w in V:
  - there is a path from v to w and
  - there is a path from w to v.

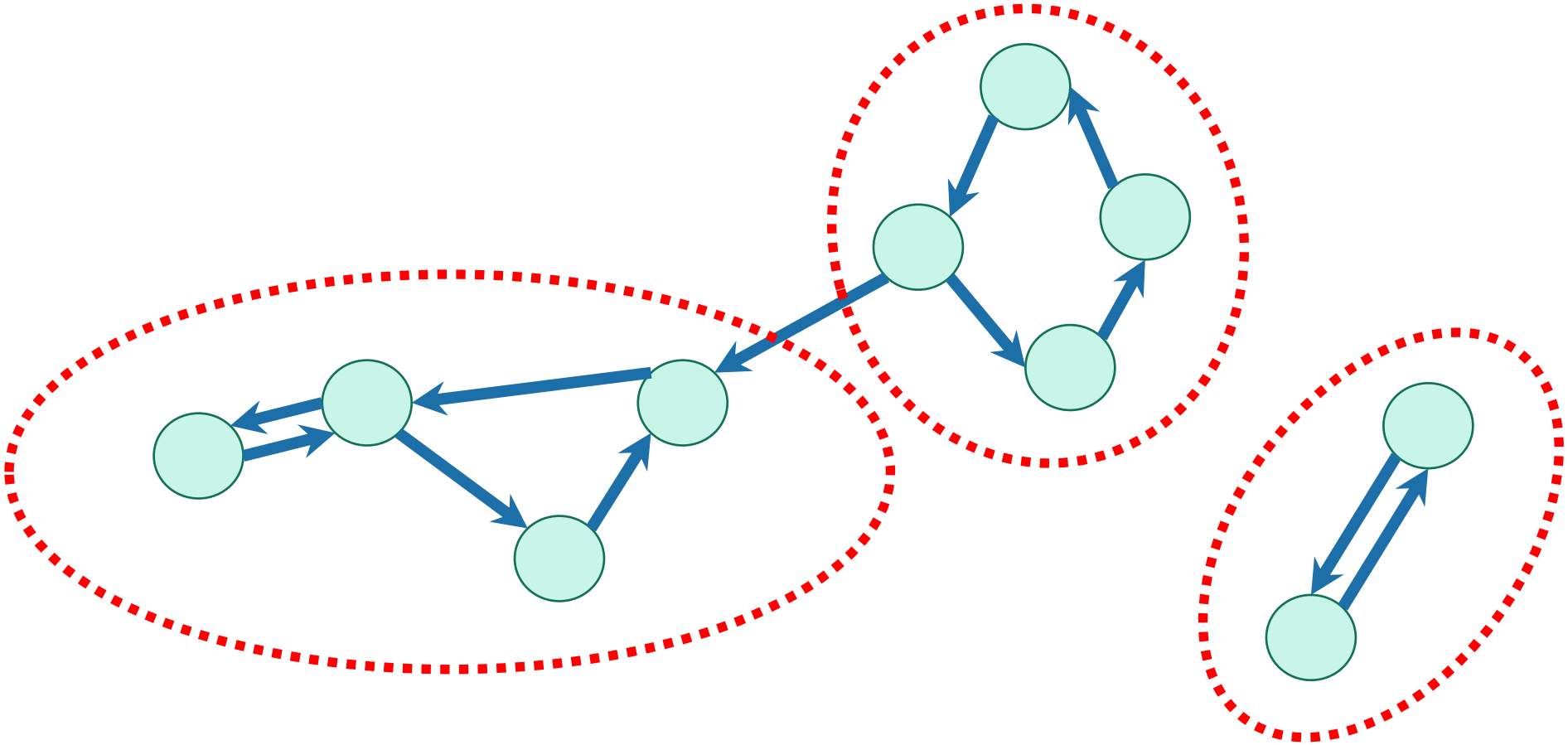


strongly connected



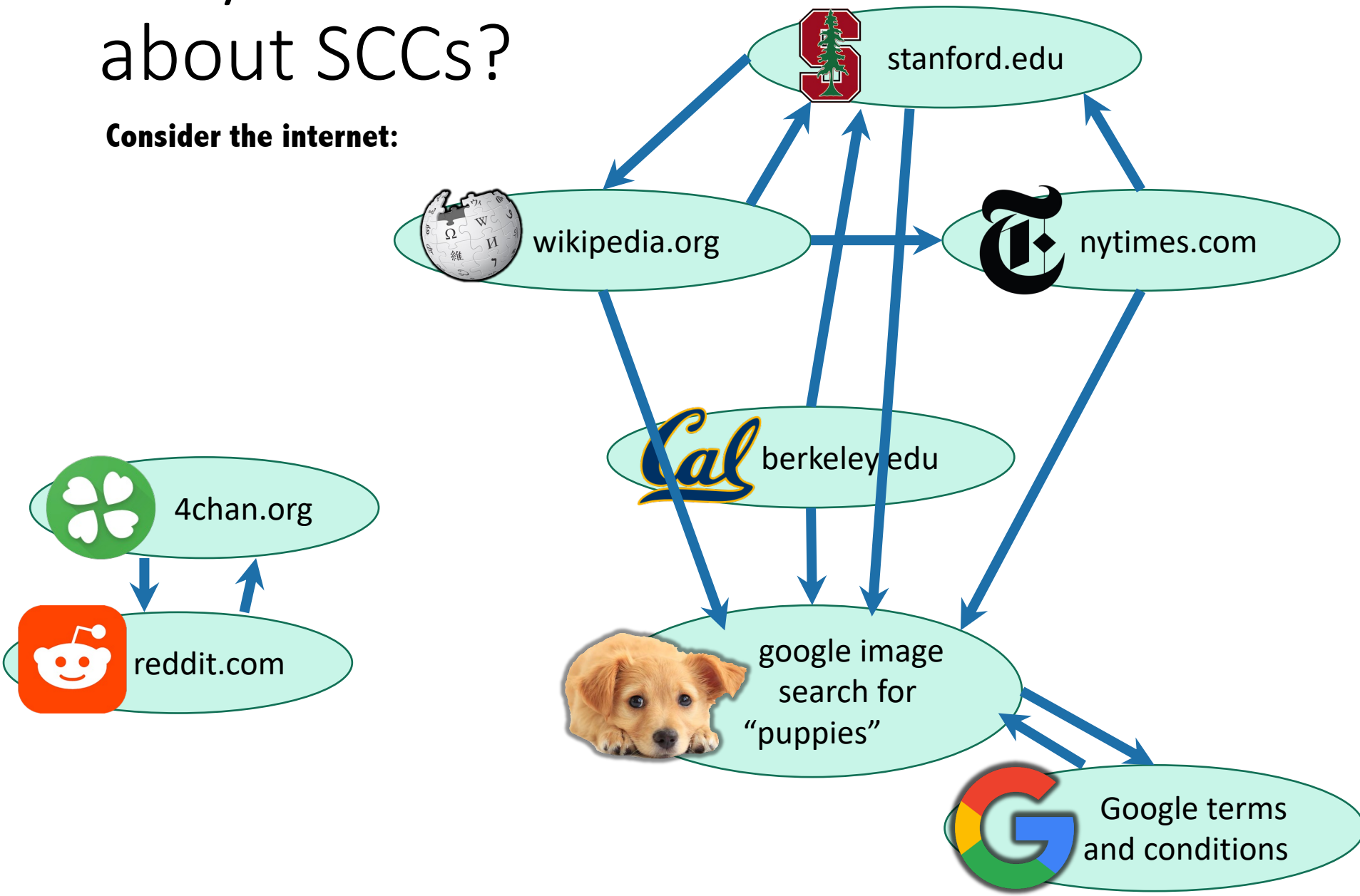
not strongly connected

We can decompose a graph into
strongly connected components (SCCs)



Why do we care about SCCs?

Consider the internet:



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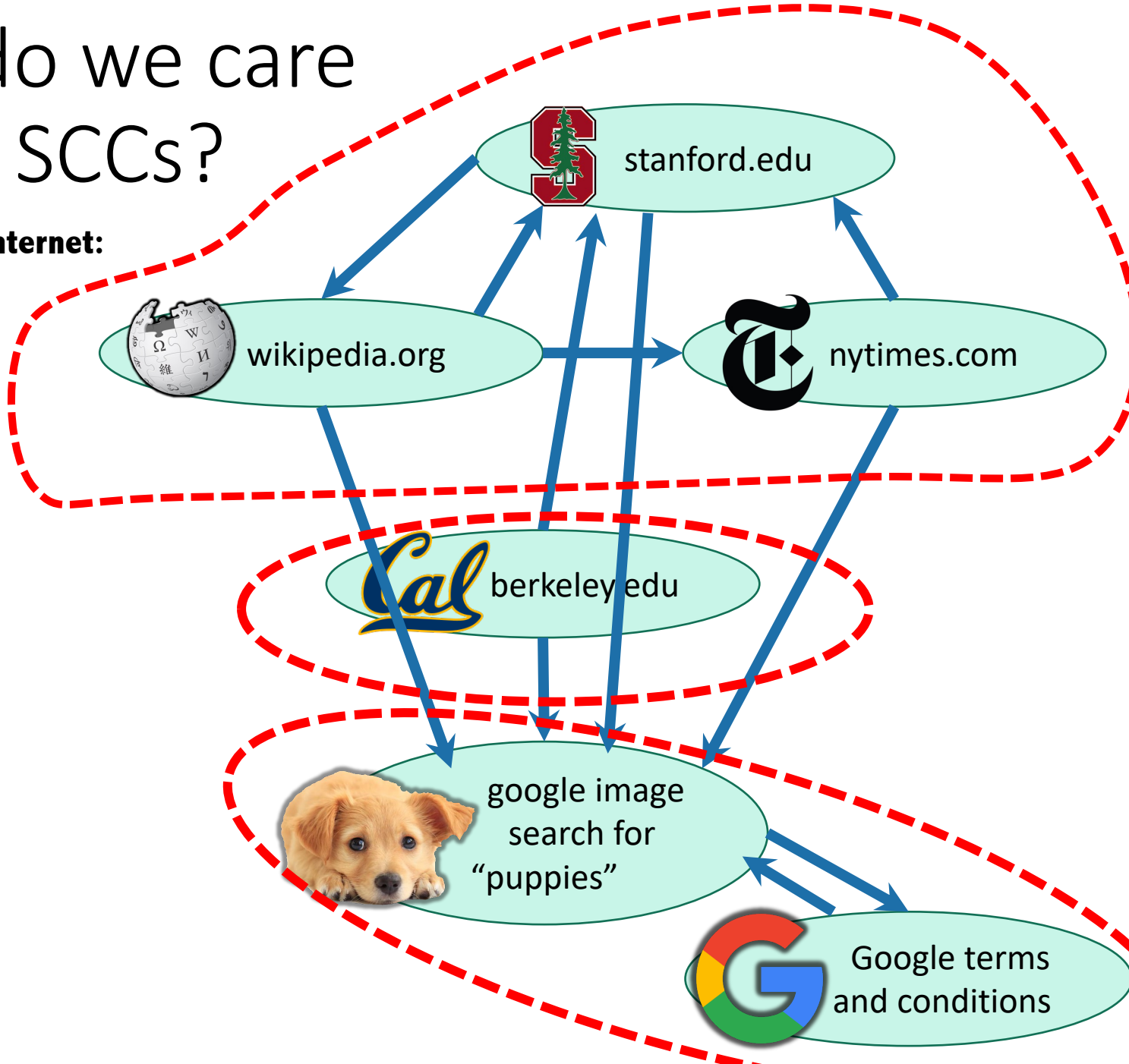


What are the SCCs of the internet?

- In real life, turns out there's one "giant" one.
  - and then a bunch of tendrils.
- More generally:
  - Strongly connected components tell you about **communities.**
- Lots of graph algorithms only make sense on SCCs.
  - (So some times we want to find the SCCs as a first step)

How to find SCCs?

Try 1:

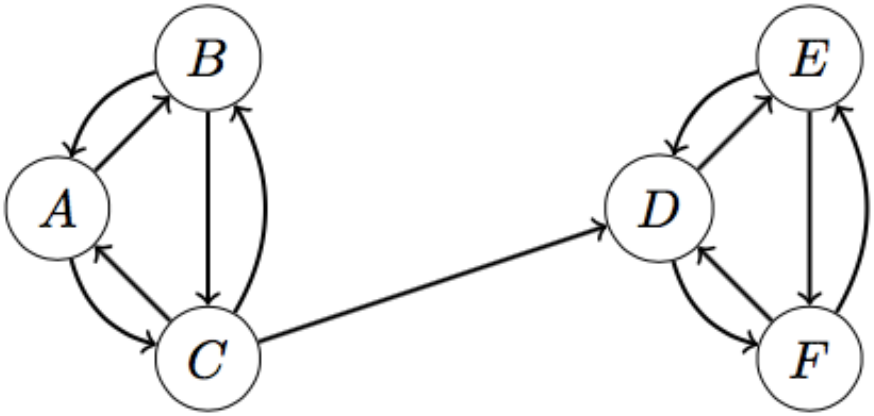
- Consider all possible decompositions and check.

Try 2:

- For each pair (u,v),
  - use DFS to find if there are paths u to v and v to u.
- Aggregate accordingly.
- Running time:

(Definitely ***not*** any better than $O(n^2)$)

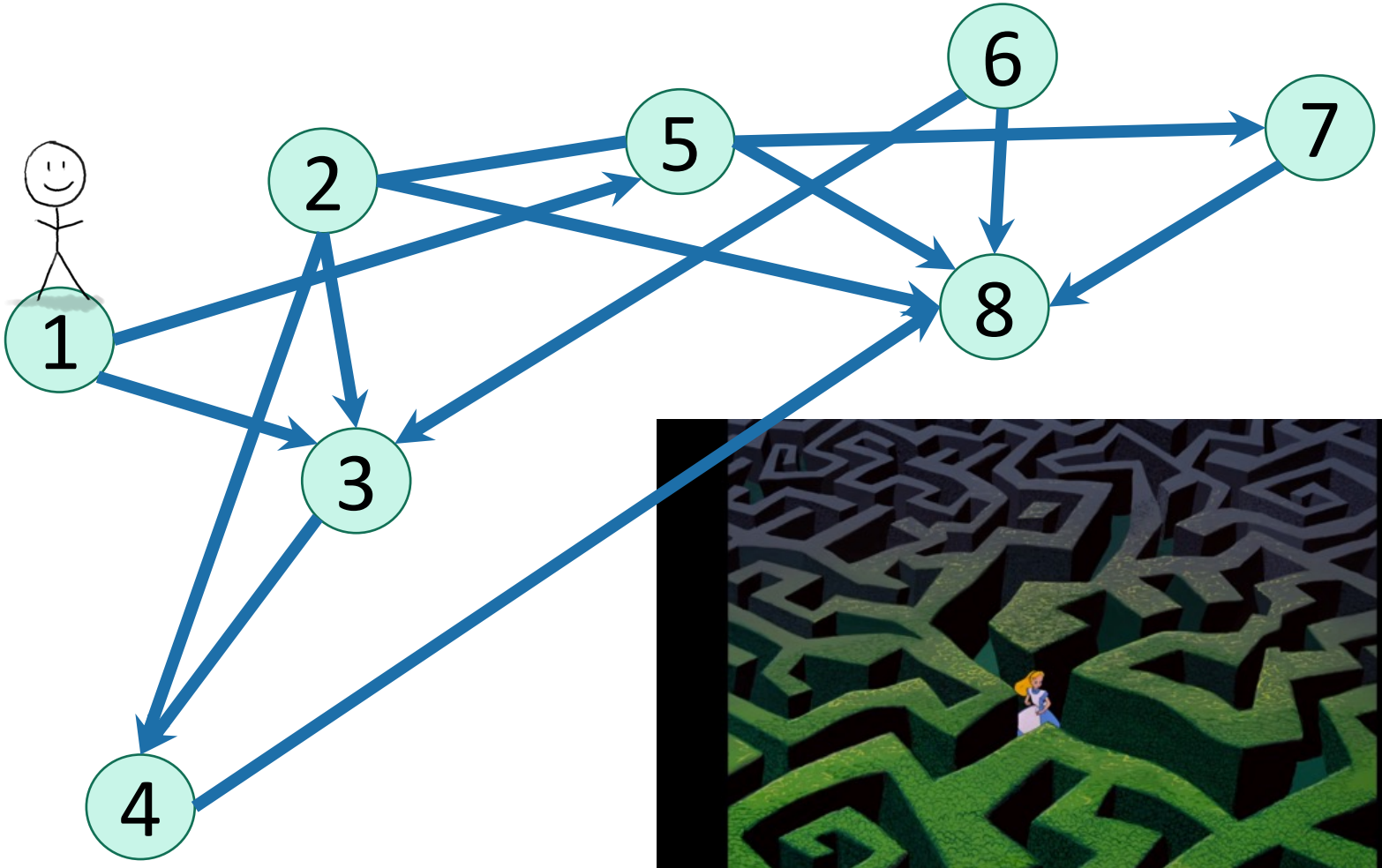
- Run DFS starting at D:



- That will identify SCCs…
- Issues:
  - How do we know where to start DFS?
  - It wouldn't have found the SCCs if we started from A.

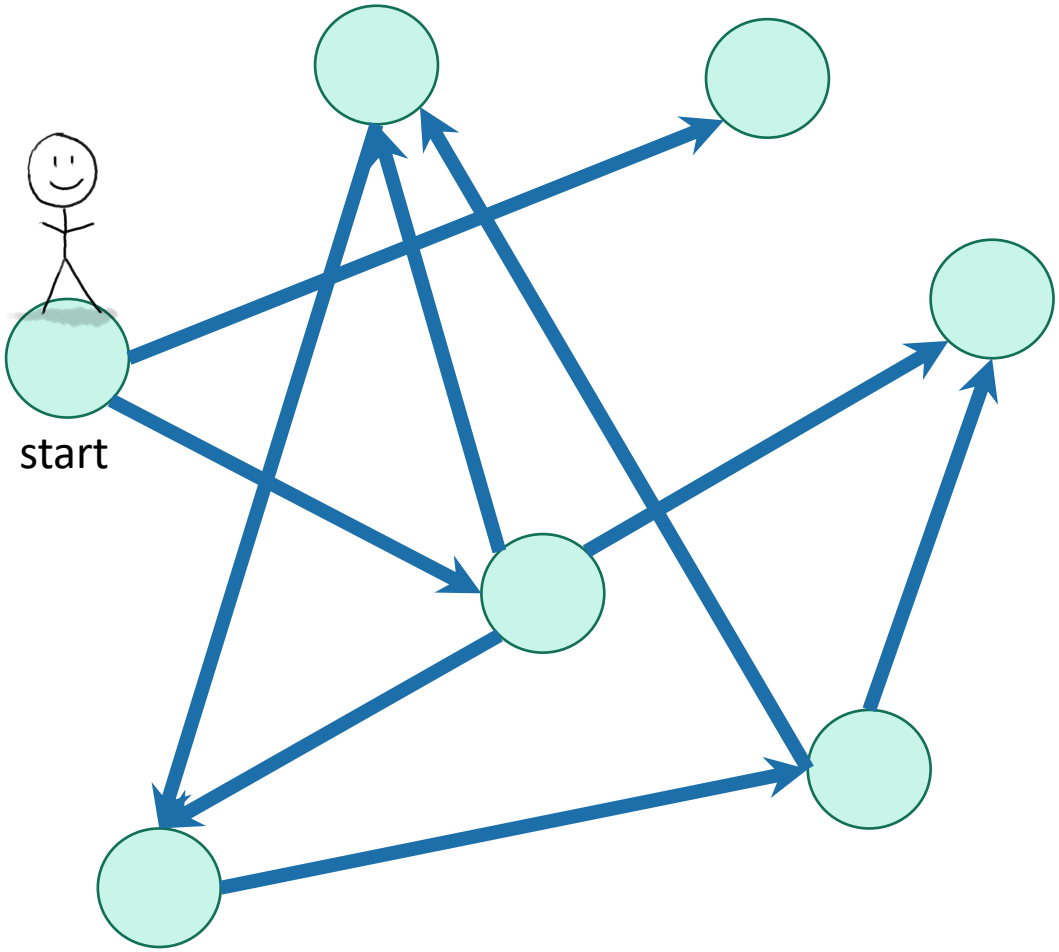
Recall: DFS

It's how you'd explore a labyrinth with chalk and a piece of string.



Depth First Search

Exploring a maze with chalk and a piece of string



Not been there yet

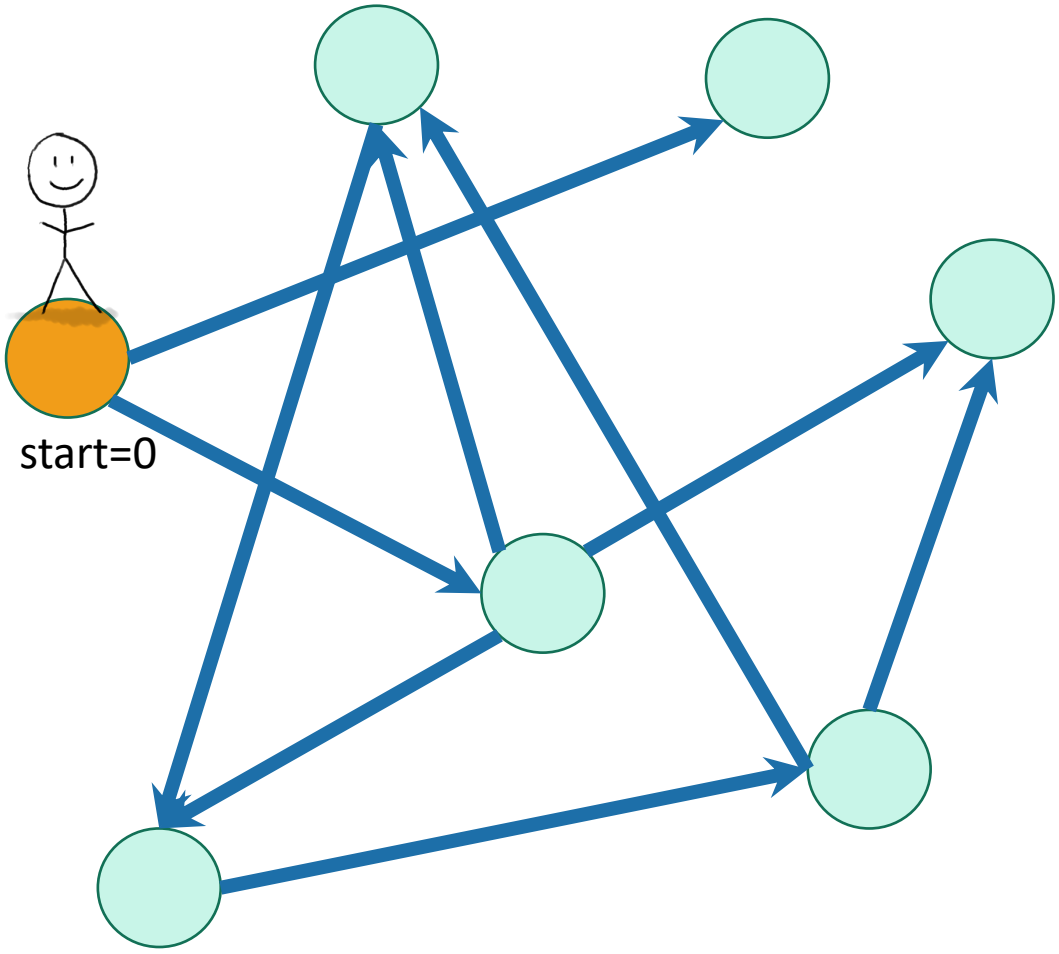
Been there, haven't explored all the paths out.

Been there, have explored all the paths out.

This is the same picture we had Monday, except I've directed all of the edges. Notice that there **ARE** cycles.

Depth First Search

Exploring a labyrinth with chalk and a piece of string



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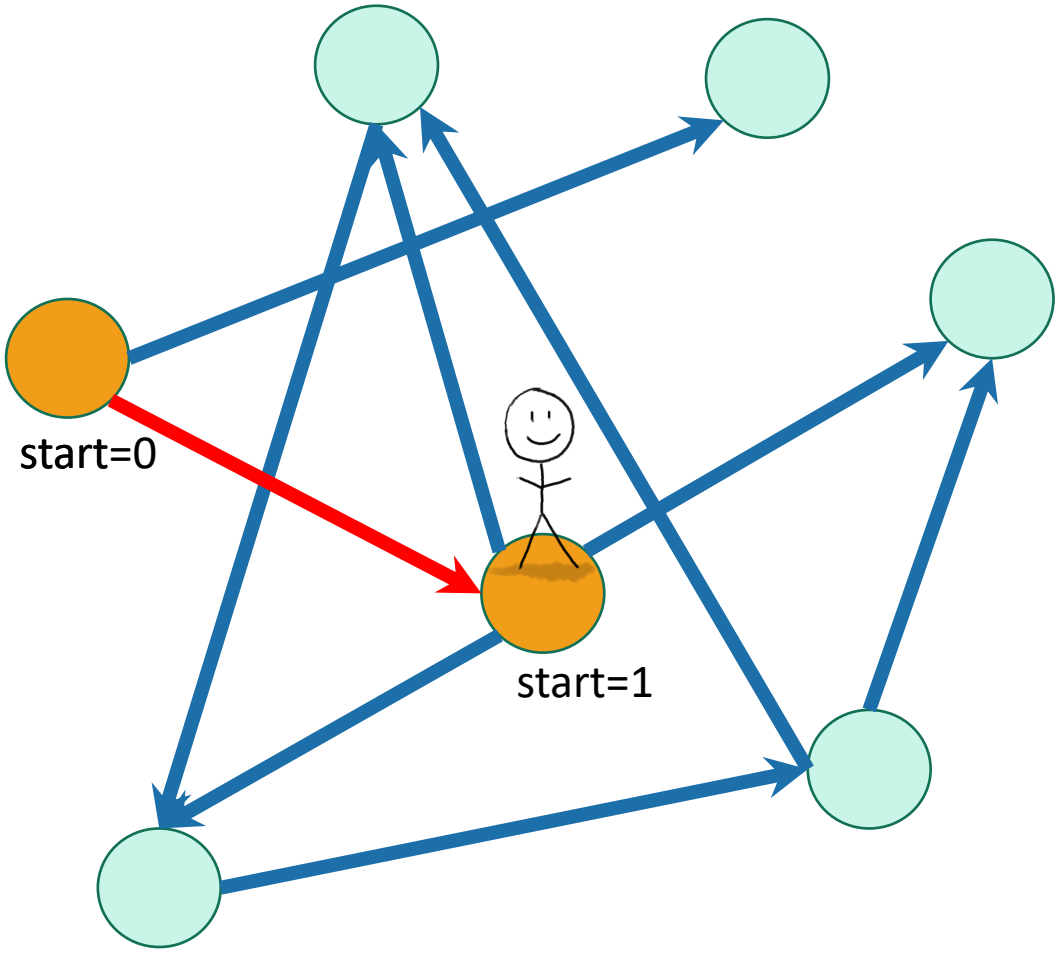

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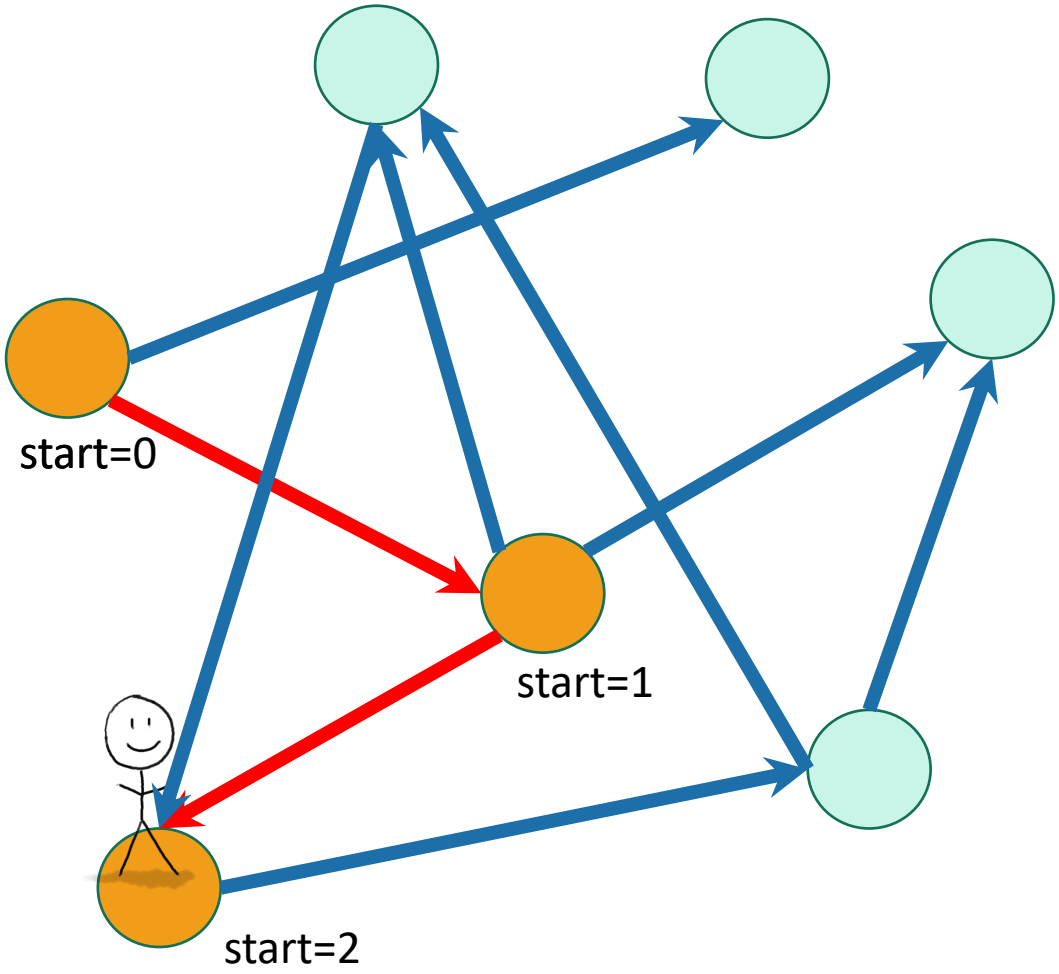

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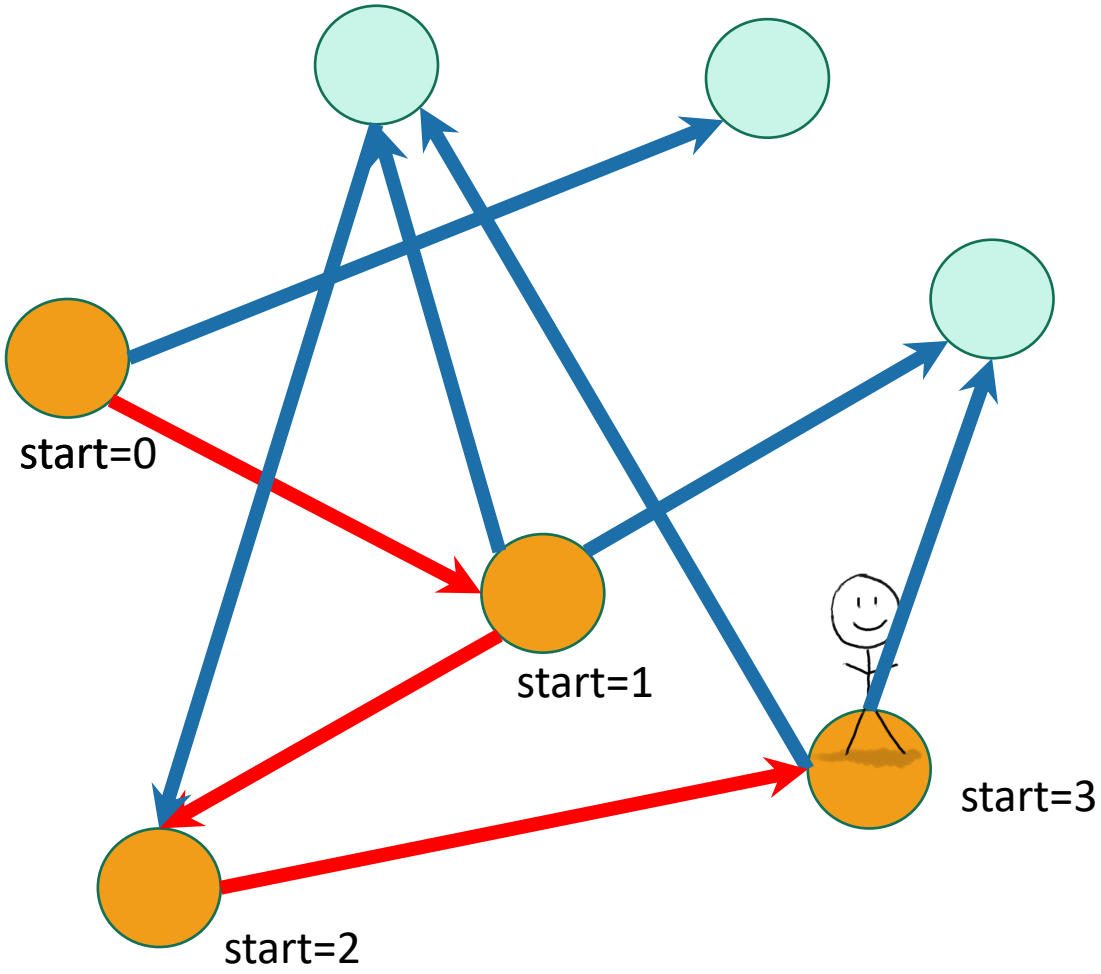

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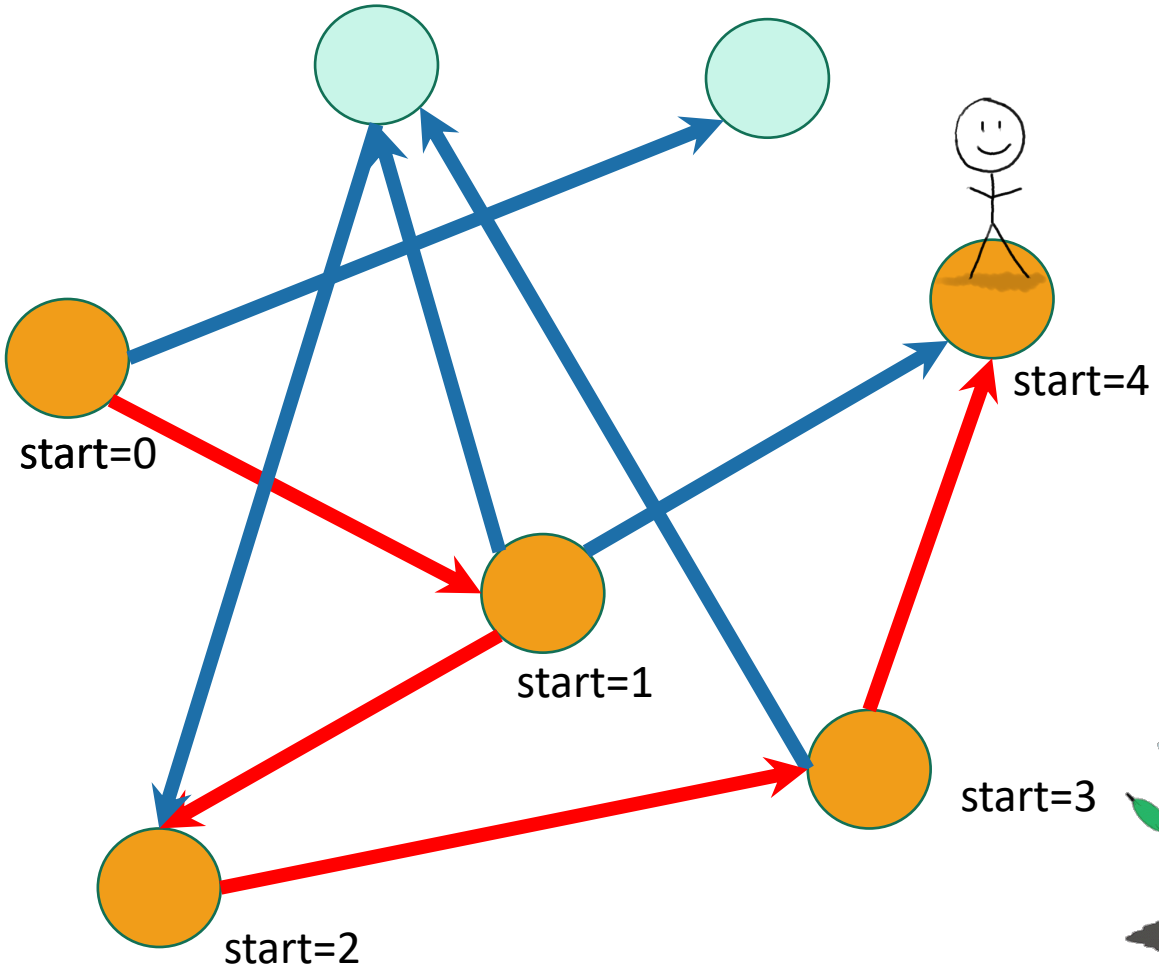

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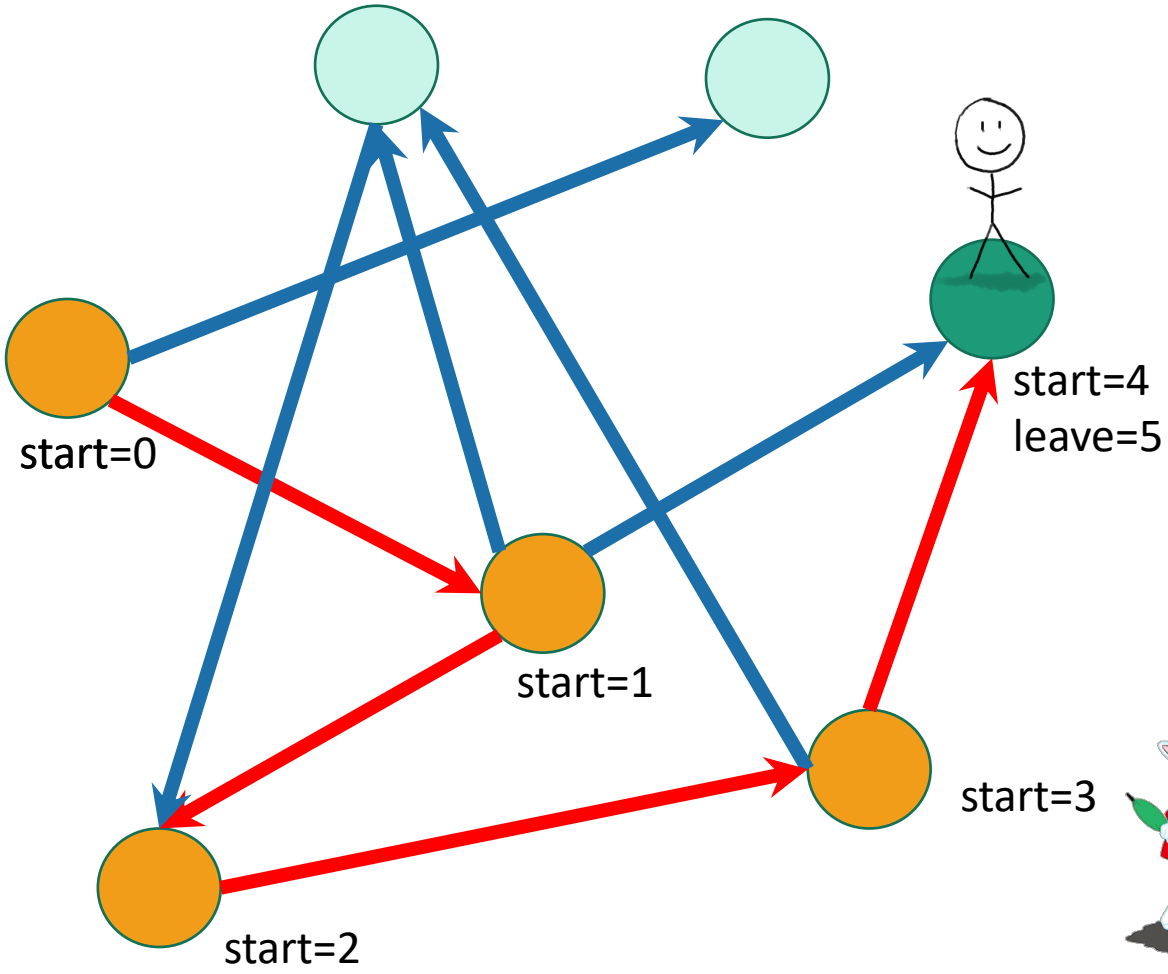

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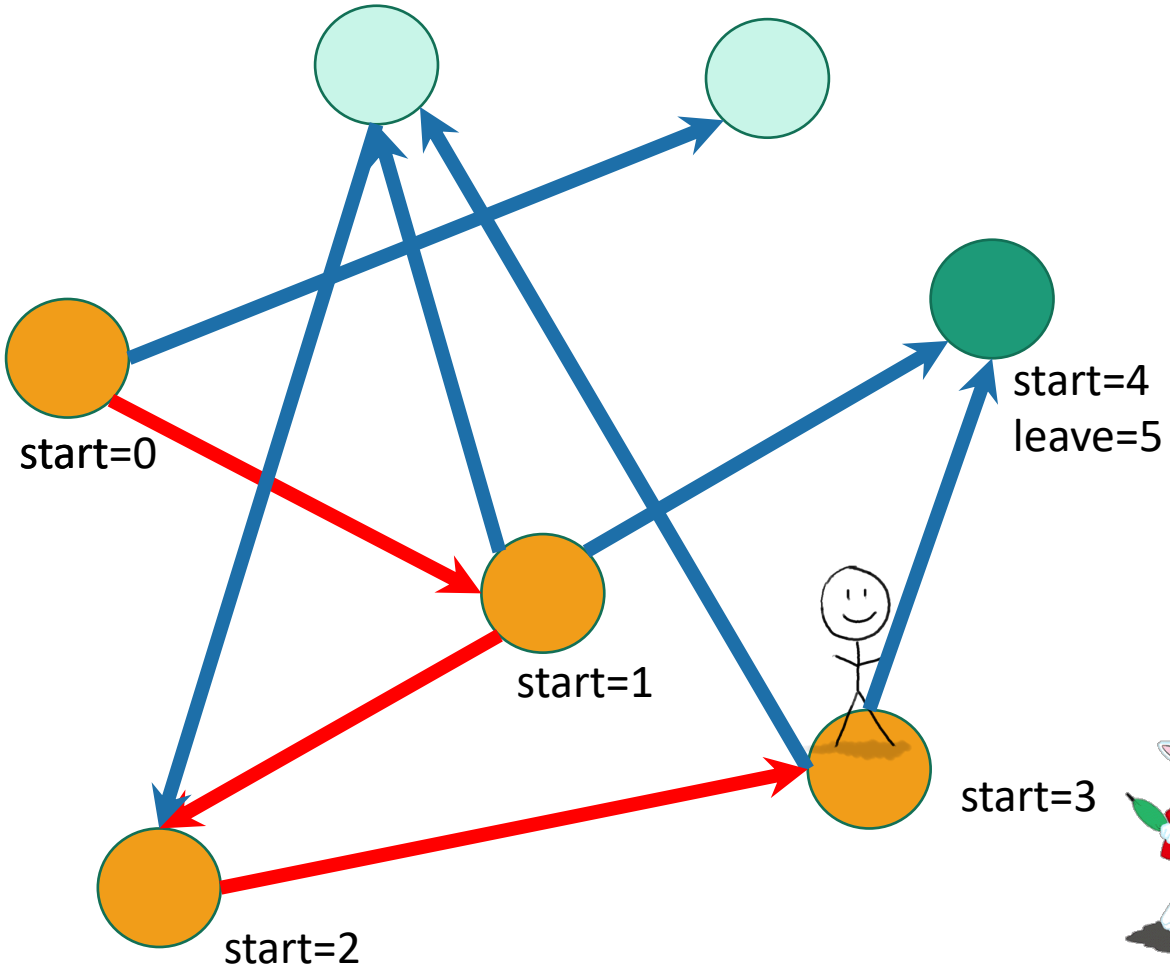

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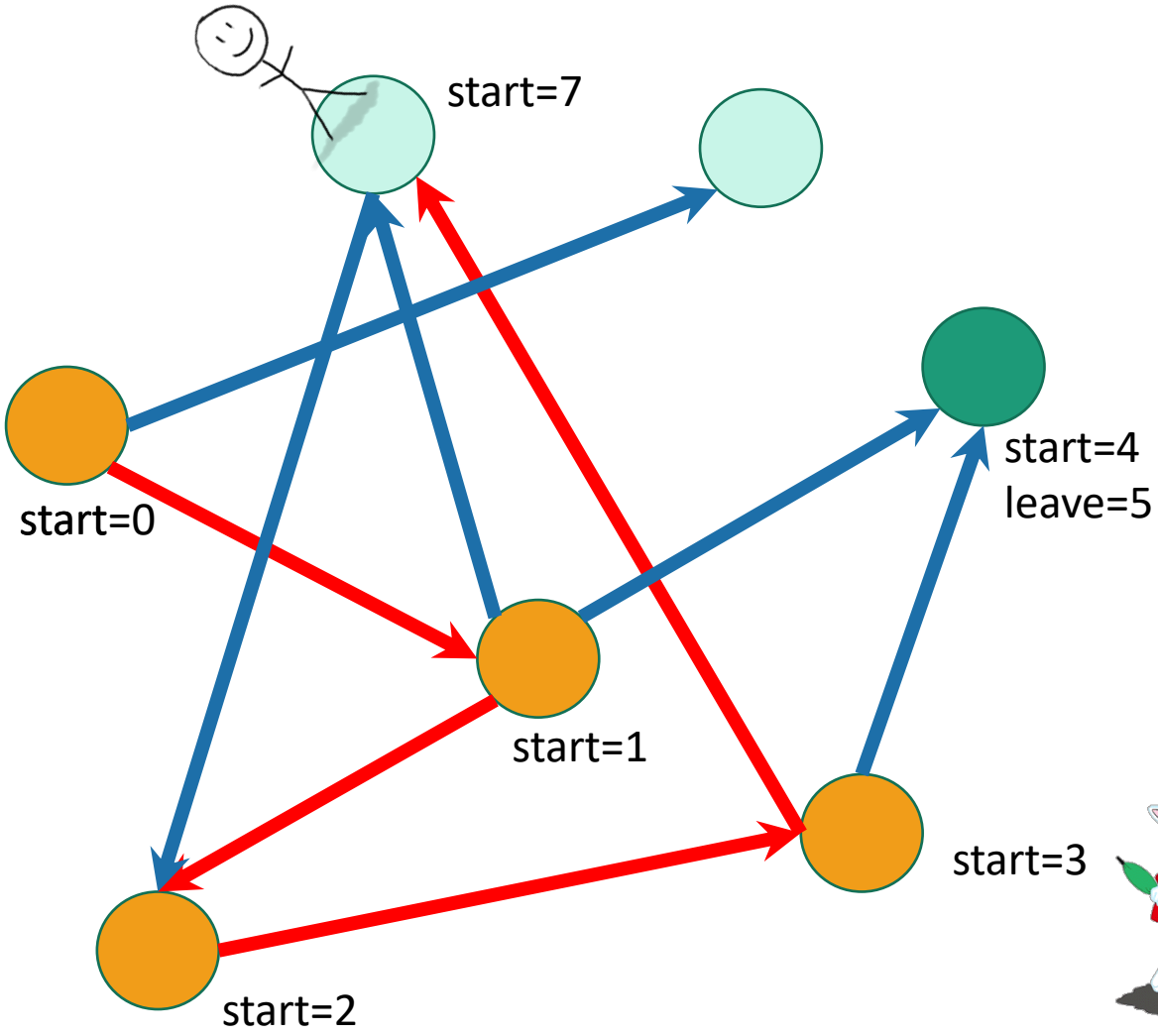

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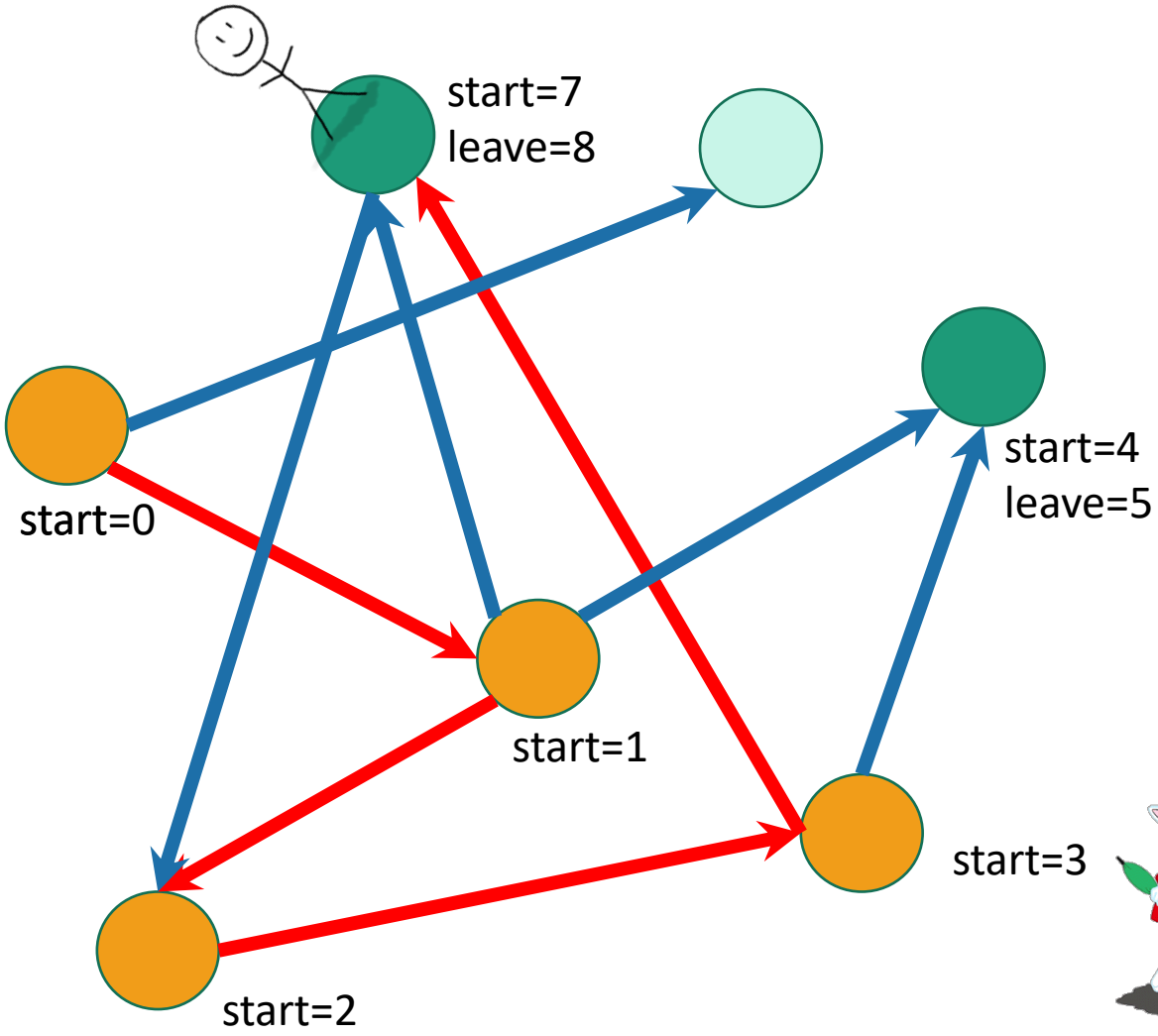

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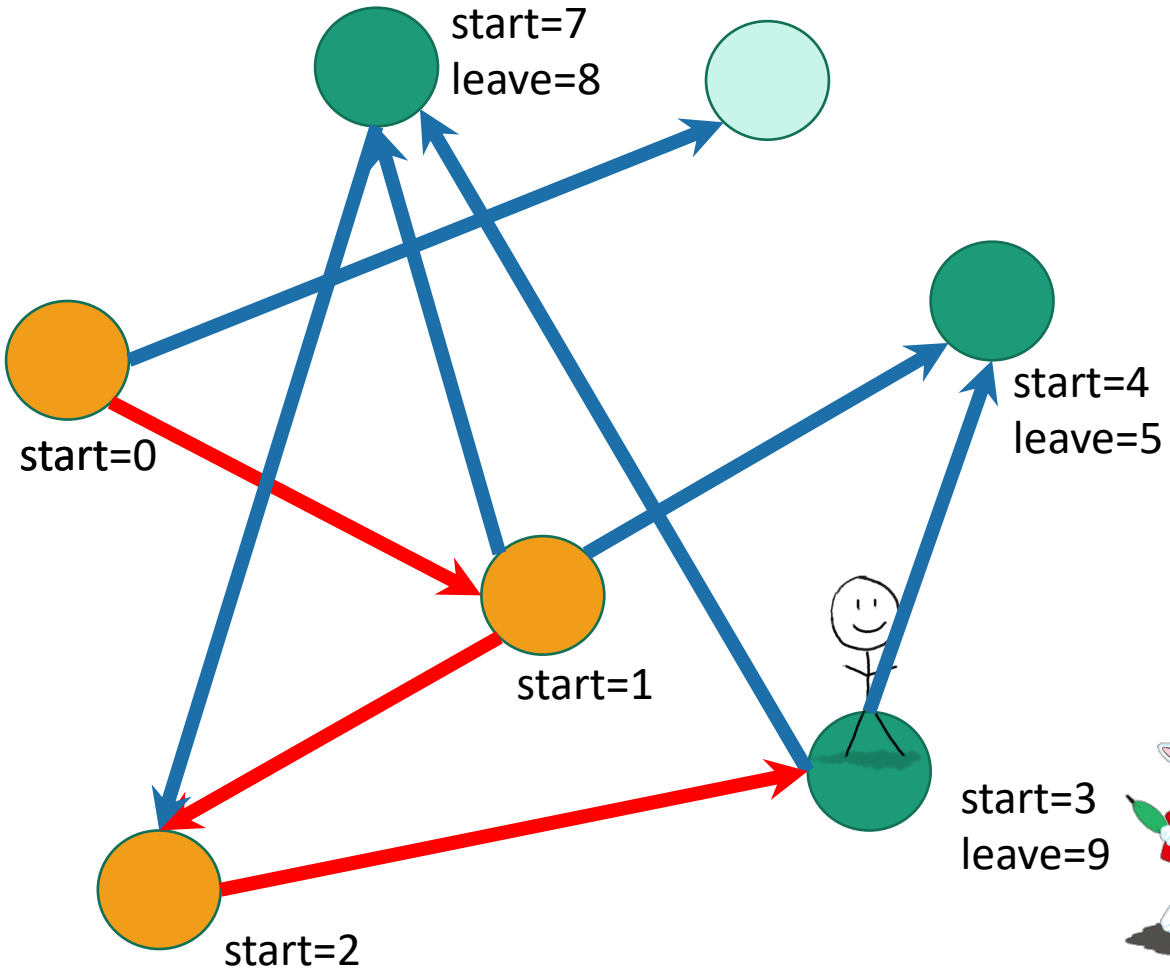

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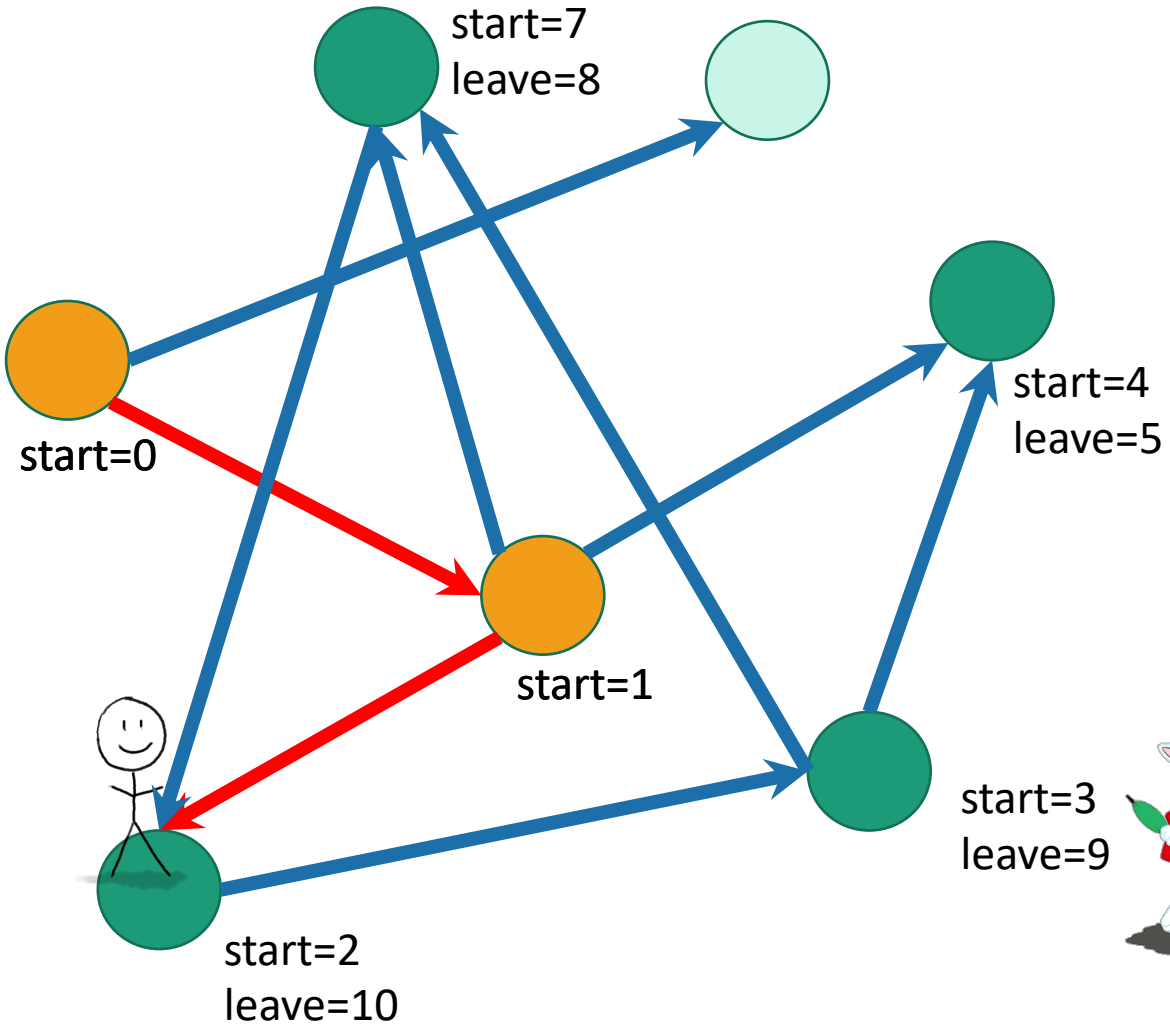

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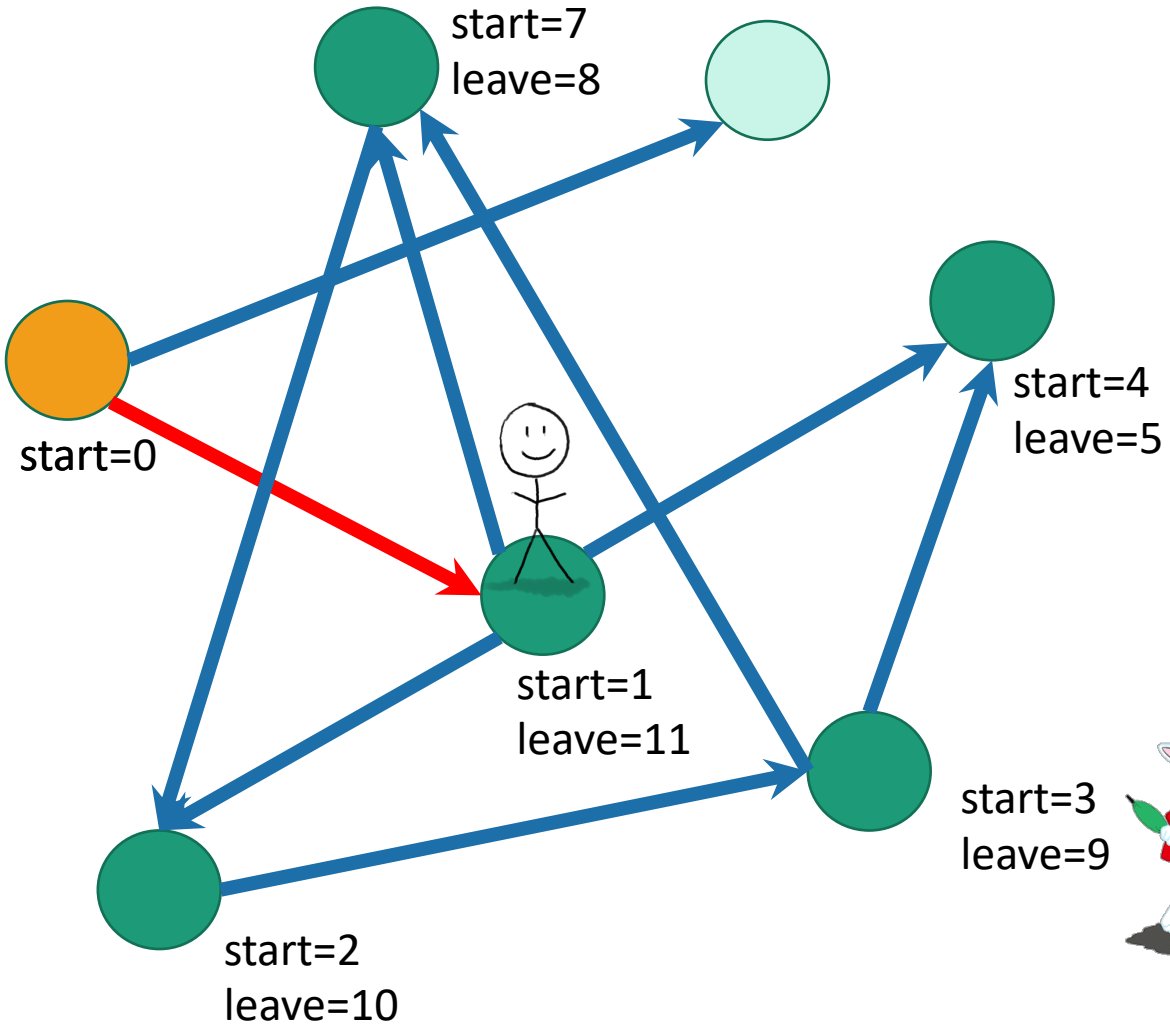

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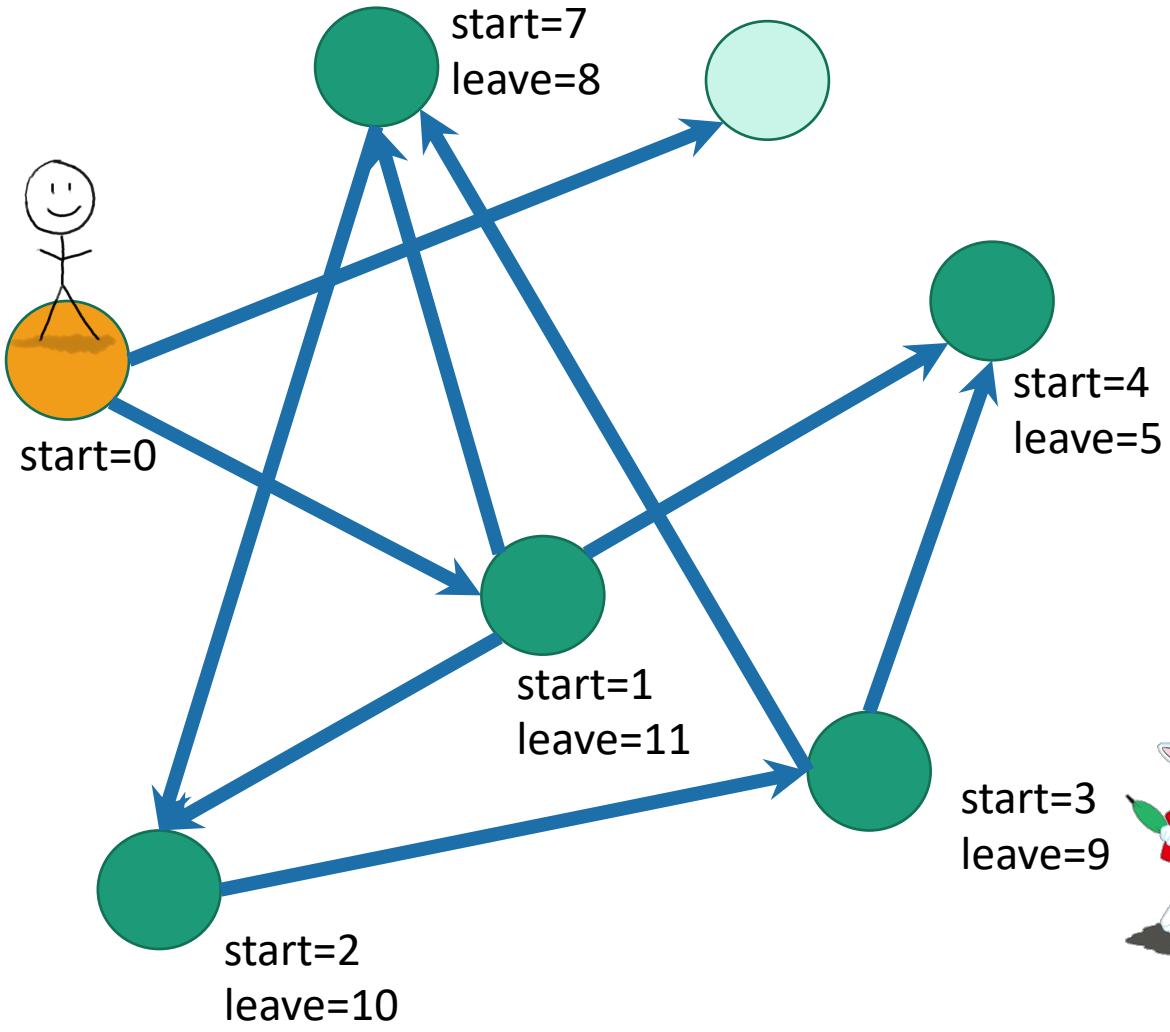

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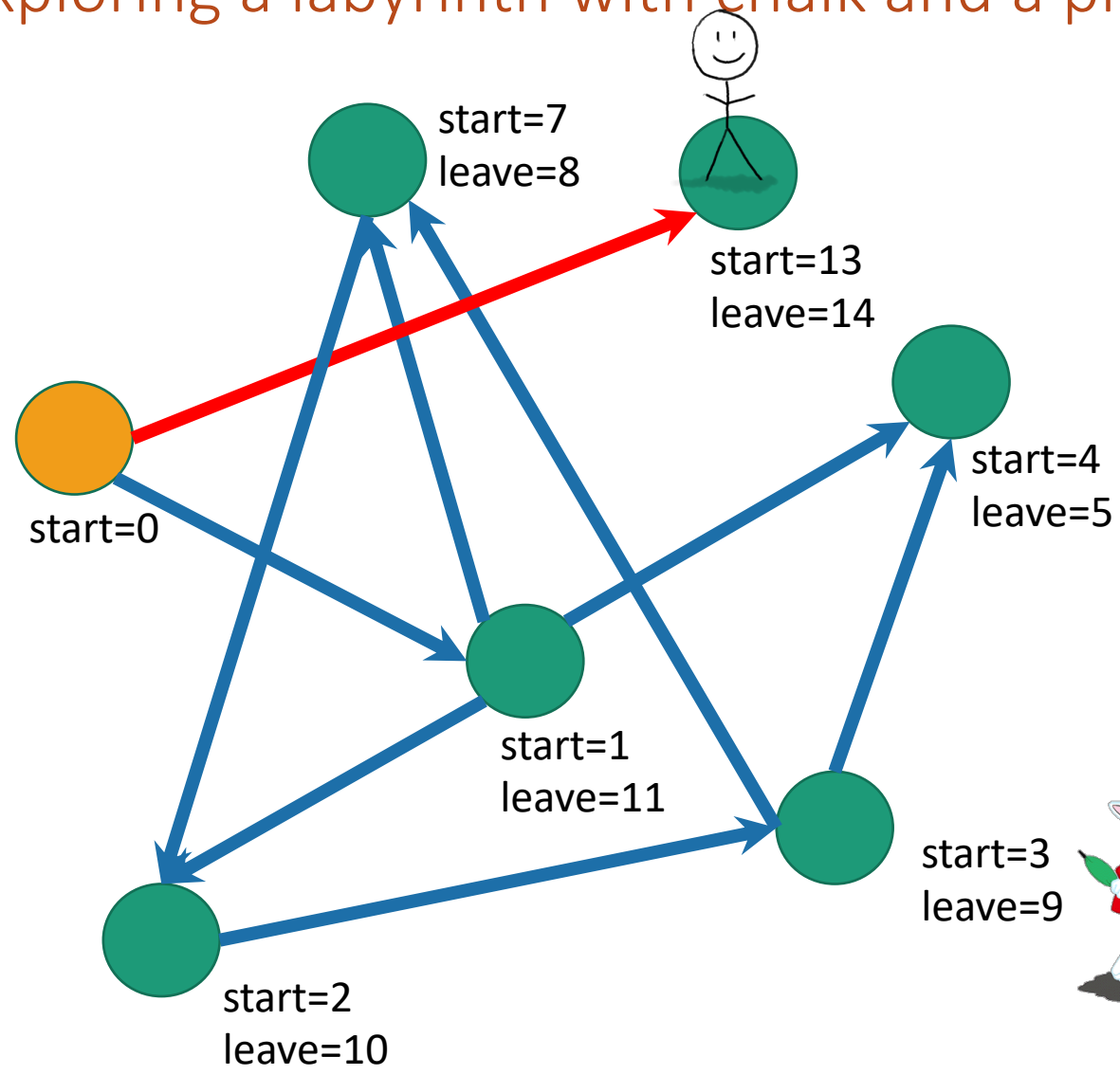

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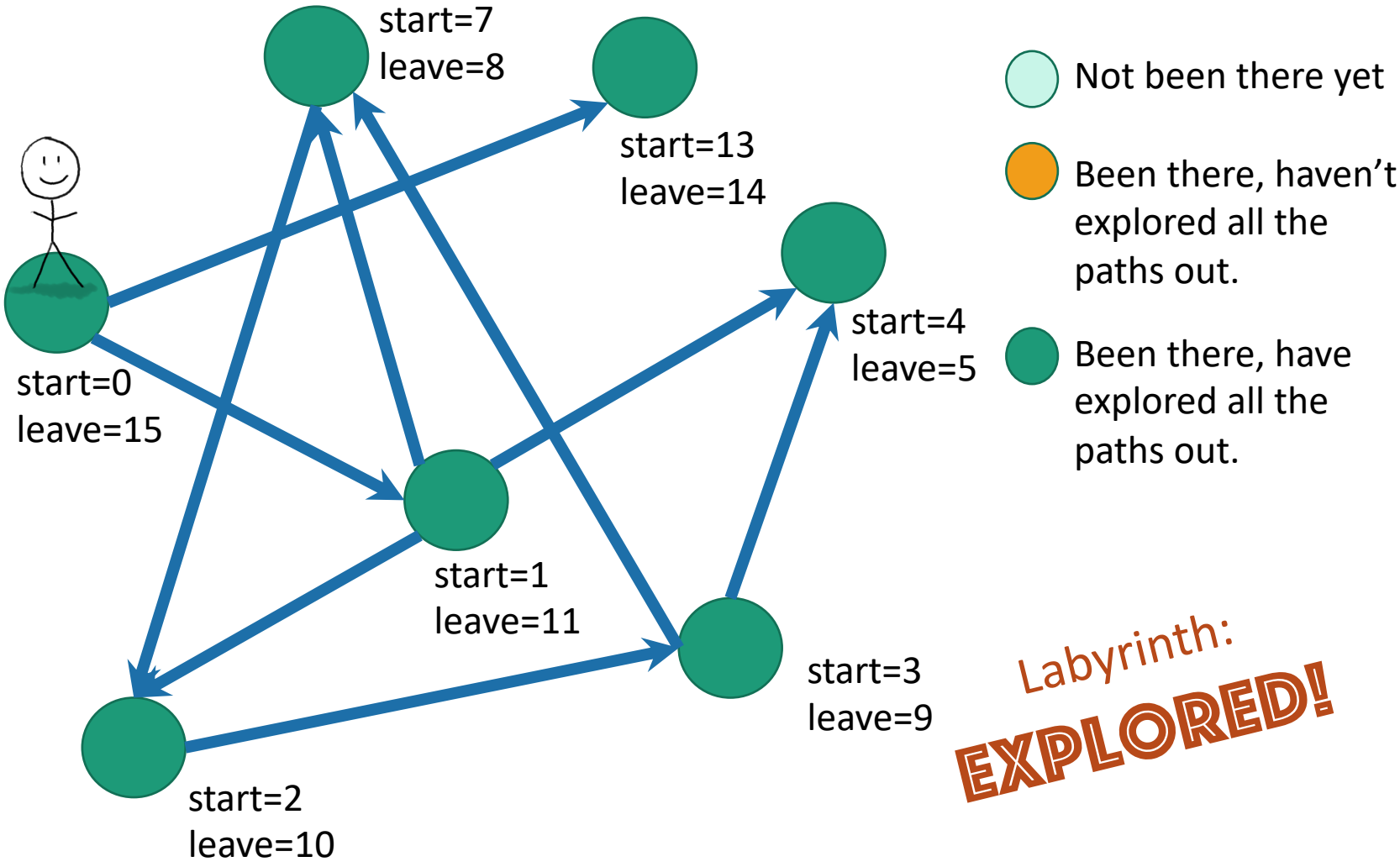
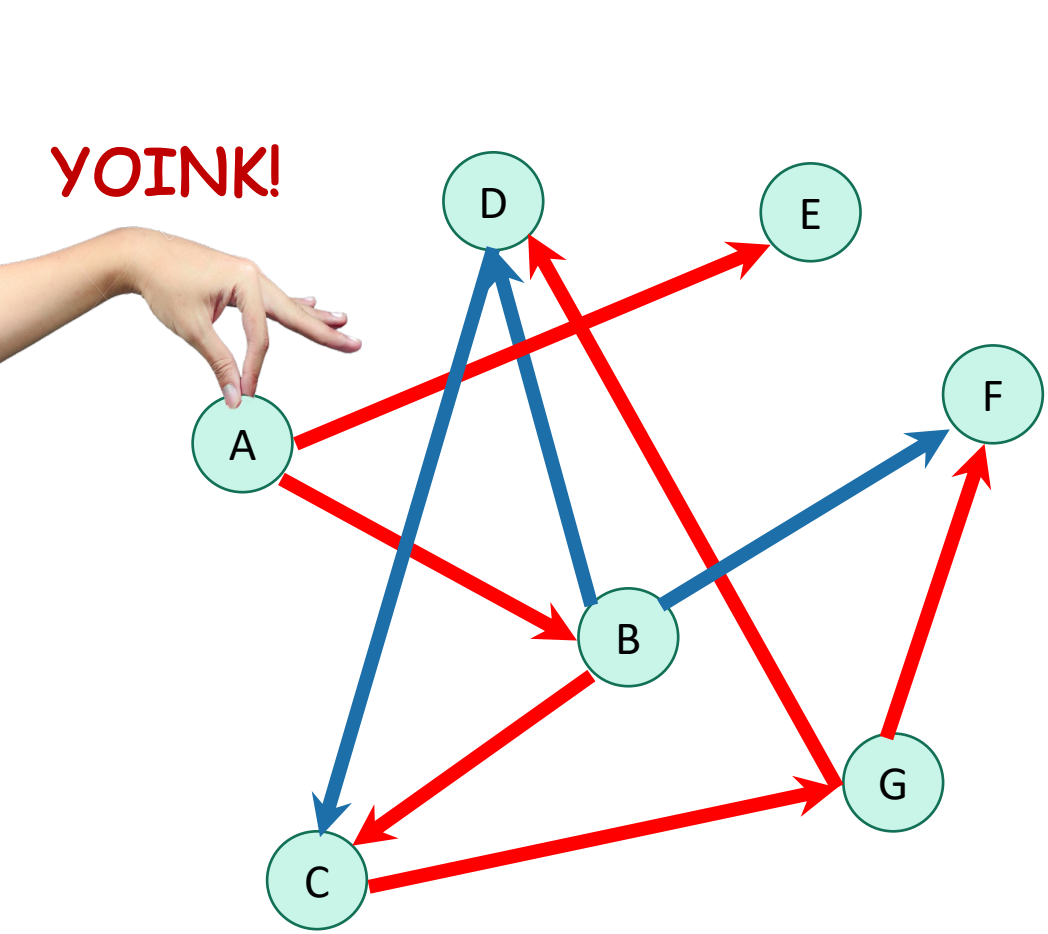
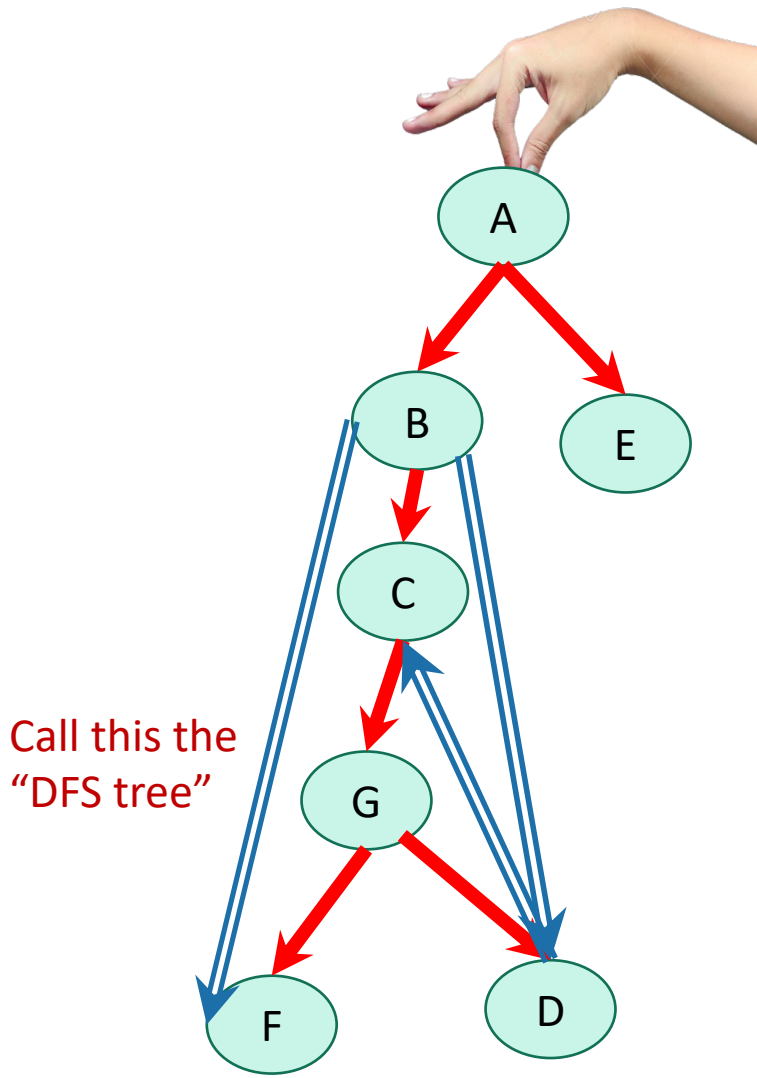


Depth first search

implicitly creates a tree on everything you can reach

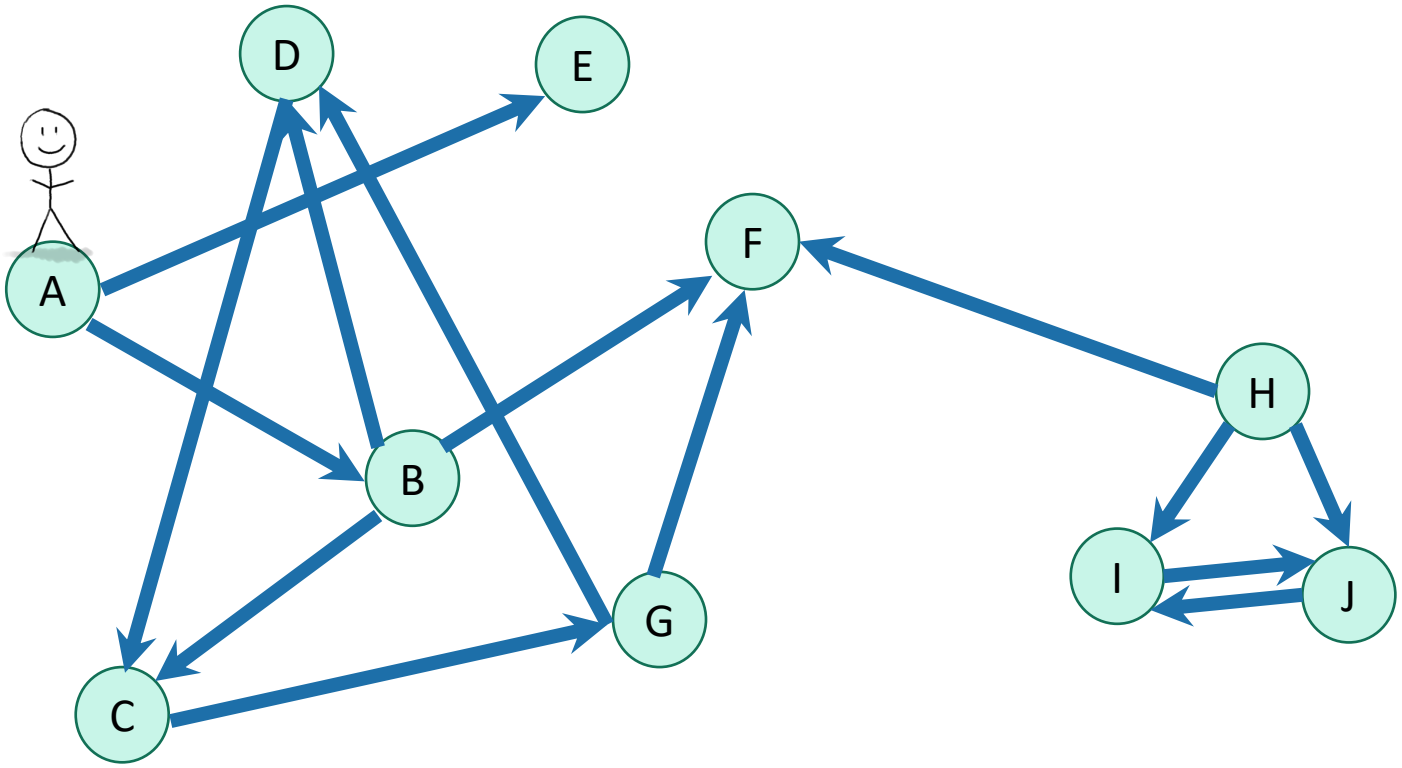




Call this the
"DFS tree"

When you can’t reach everything

- Run DFS repeatedly to get a depth-first forest



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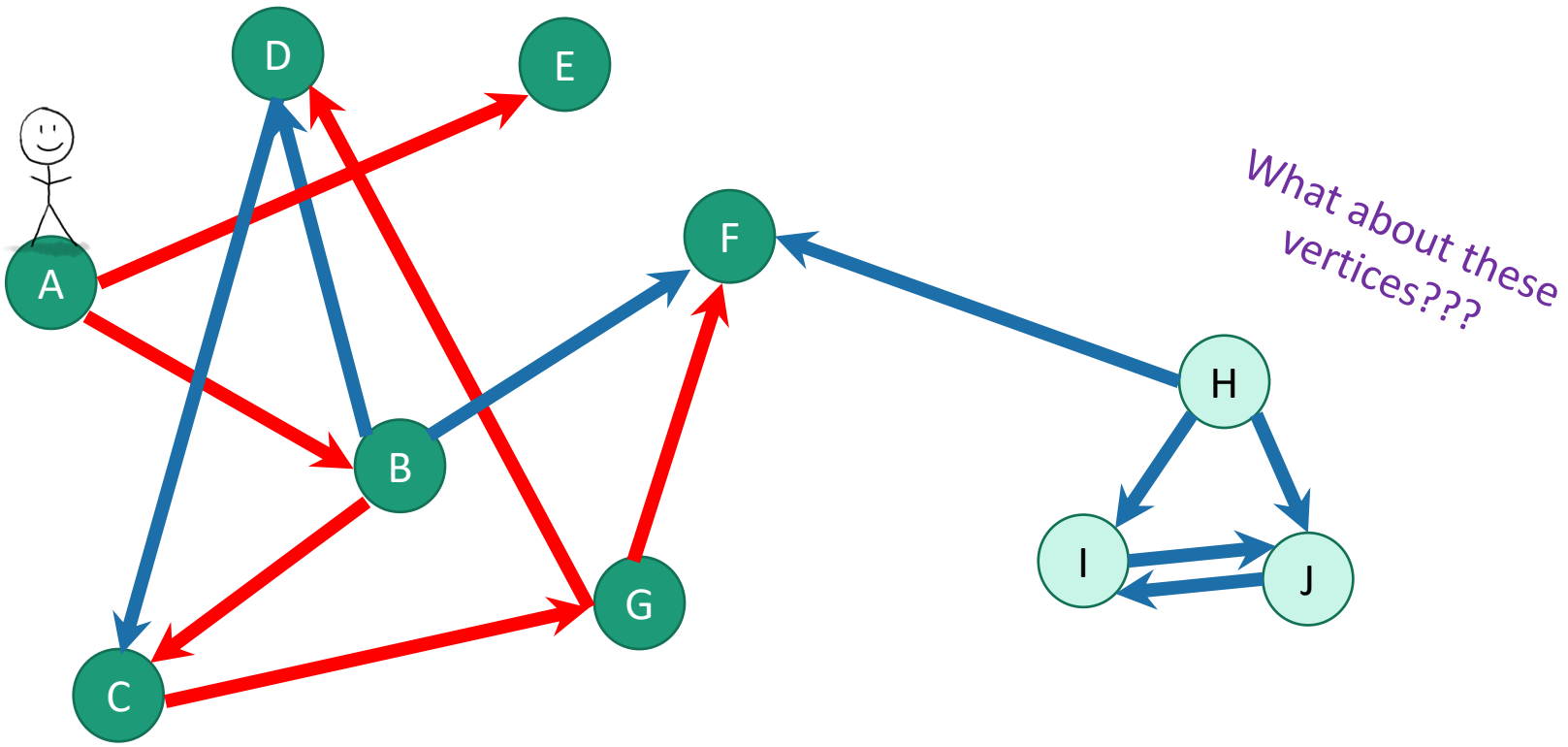

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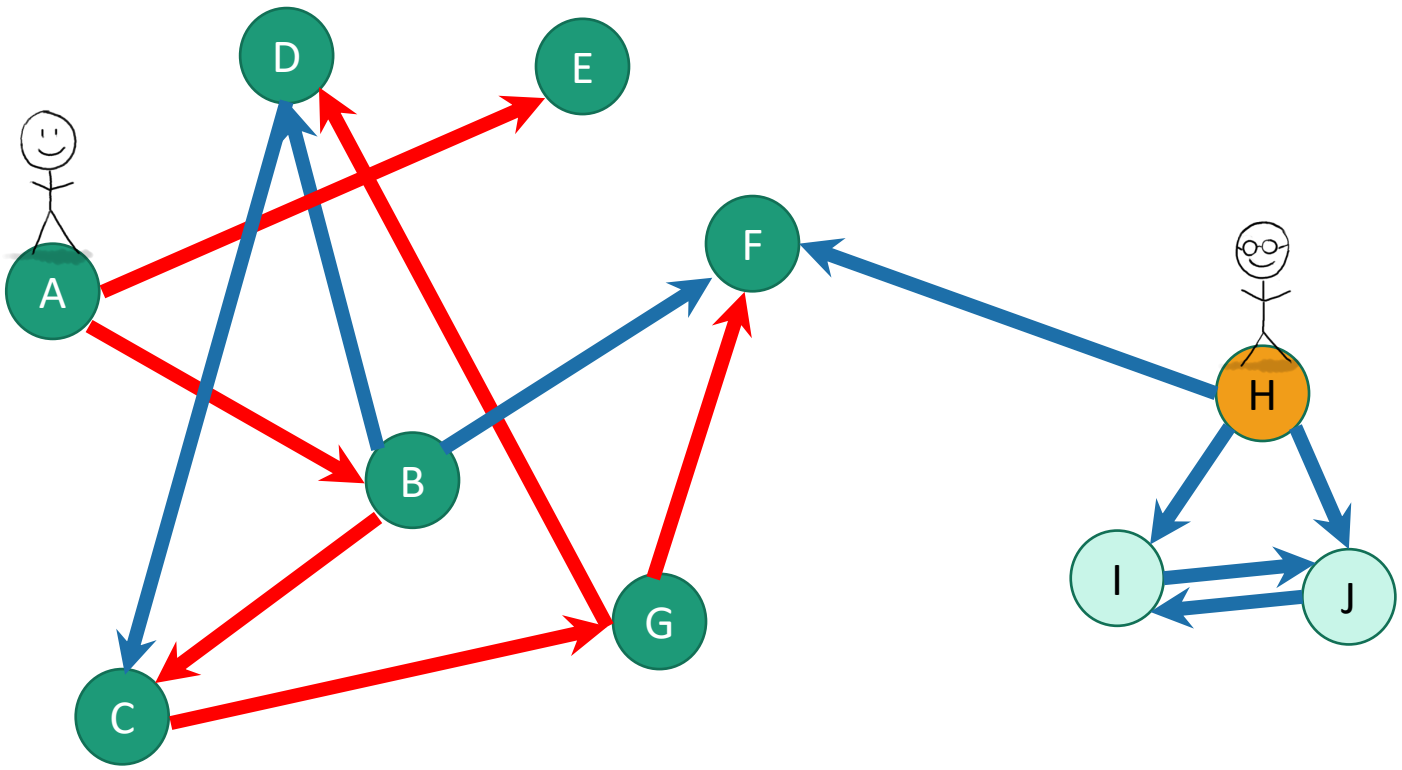

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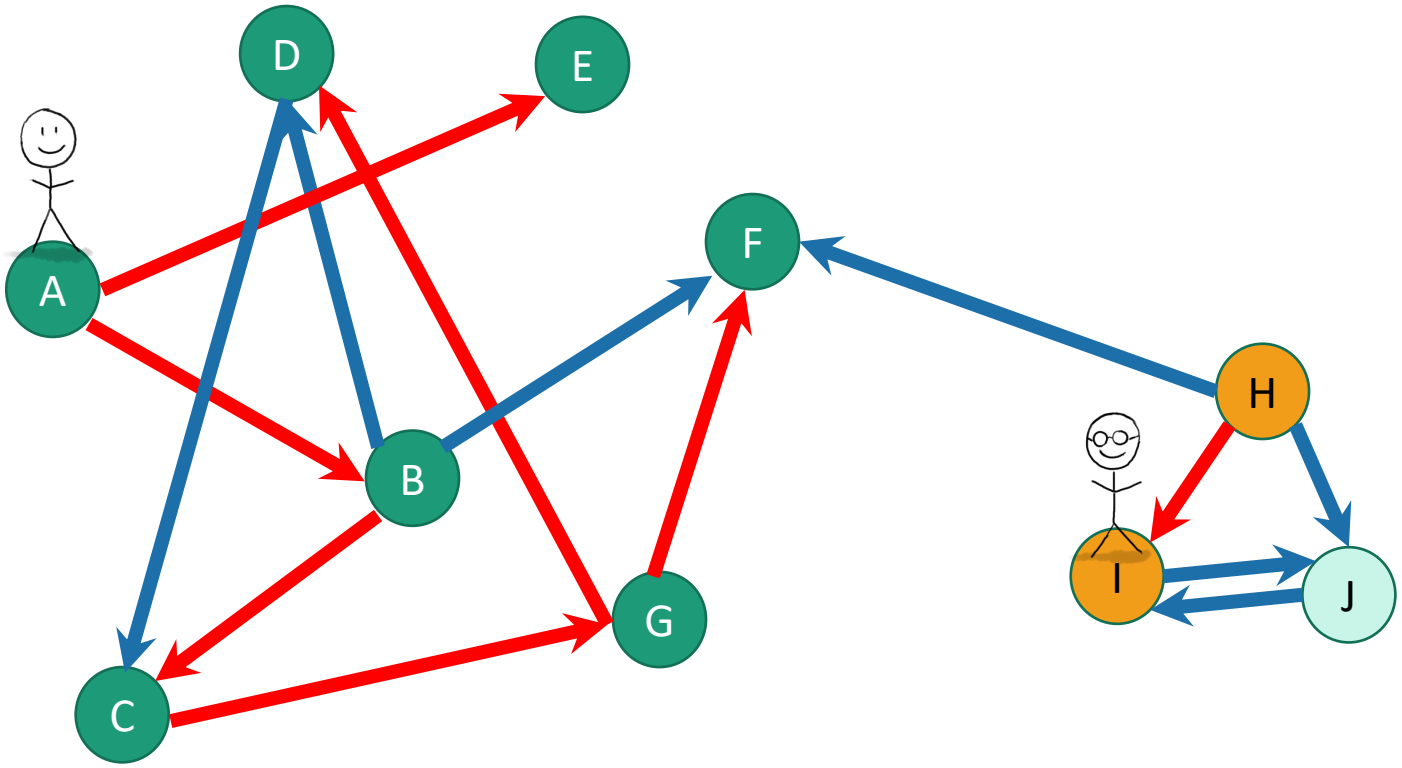

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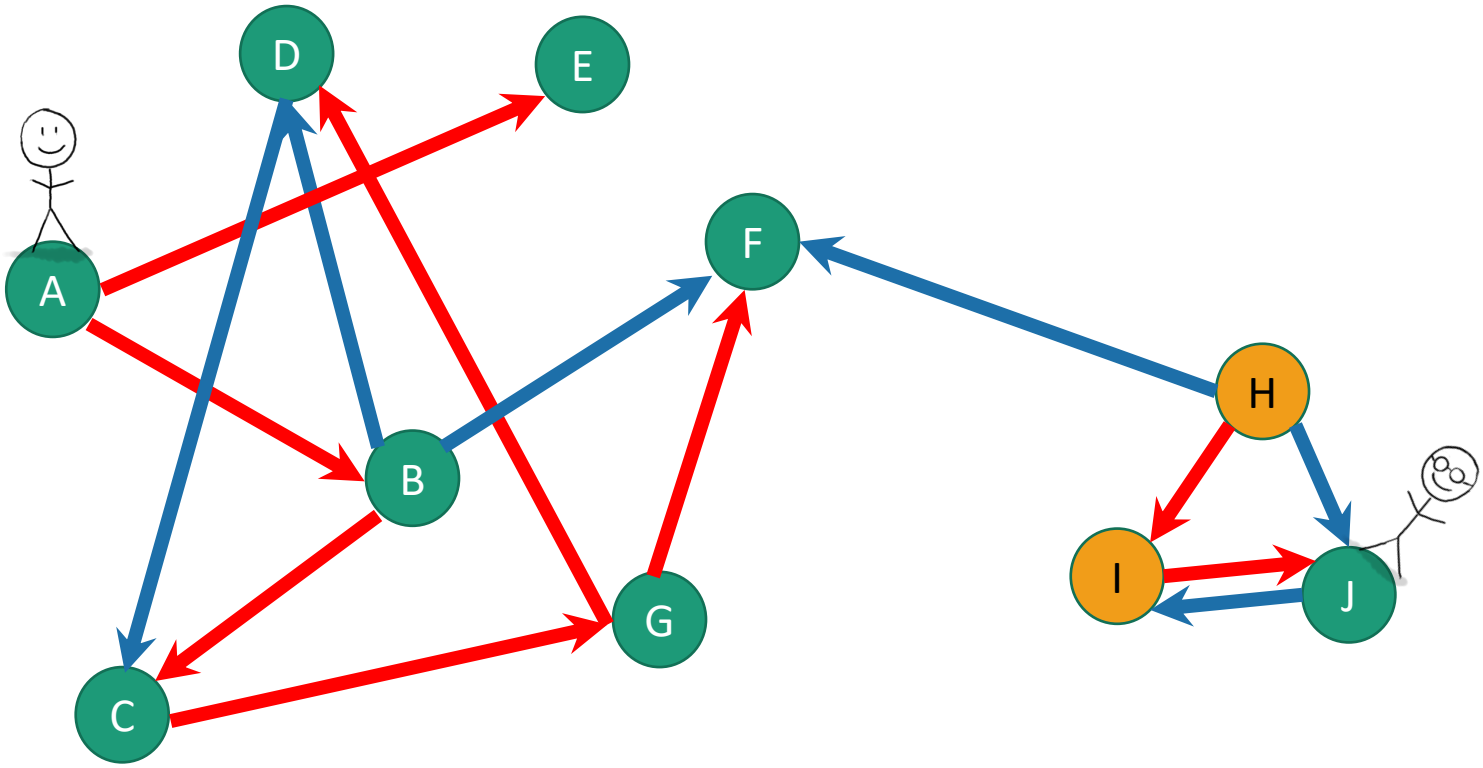

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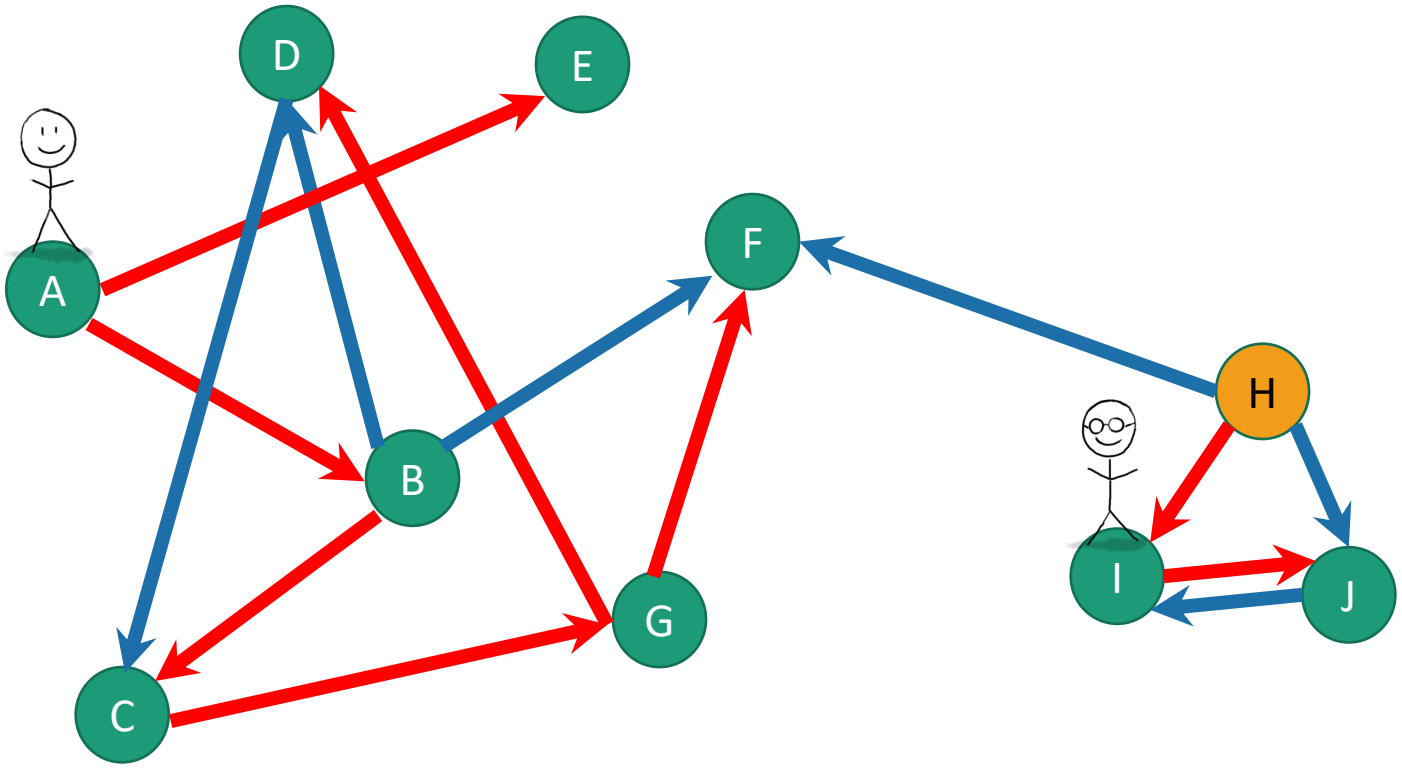

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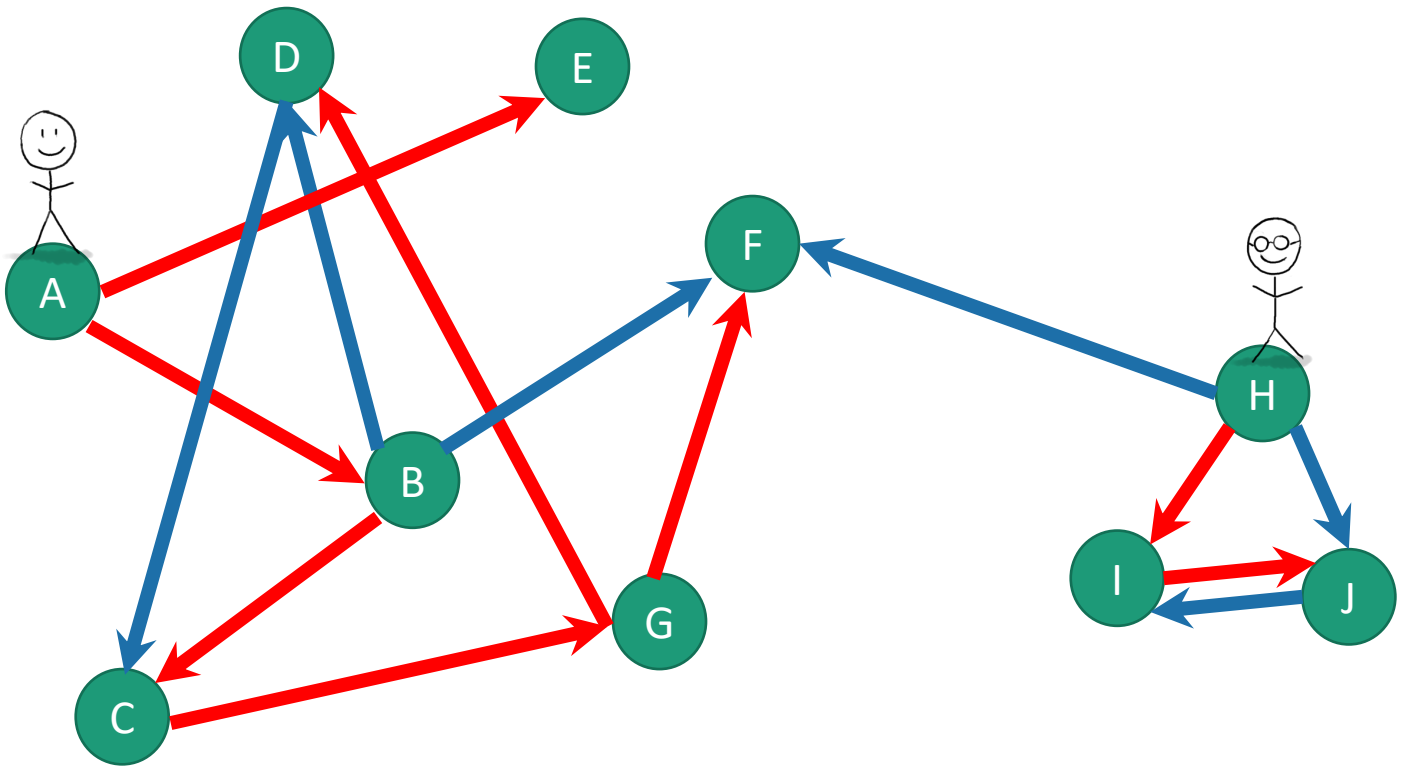

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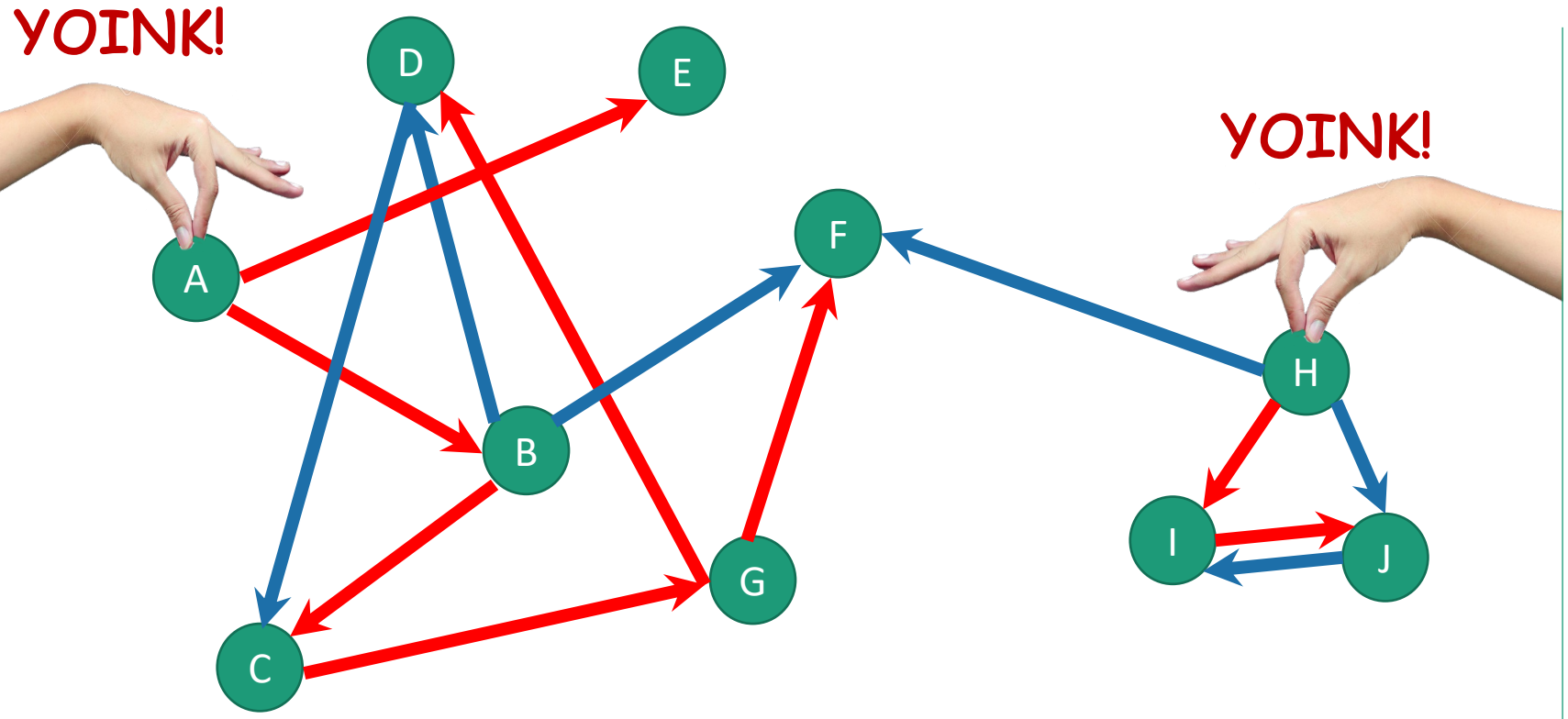

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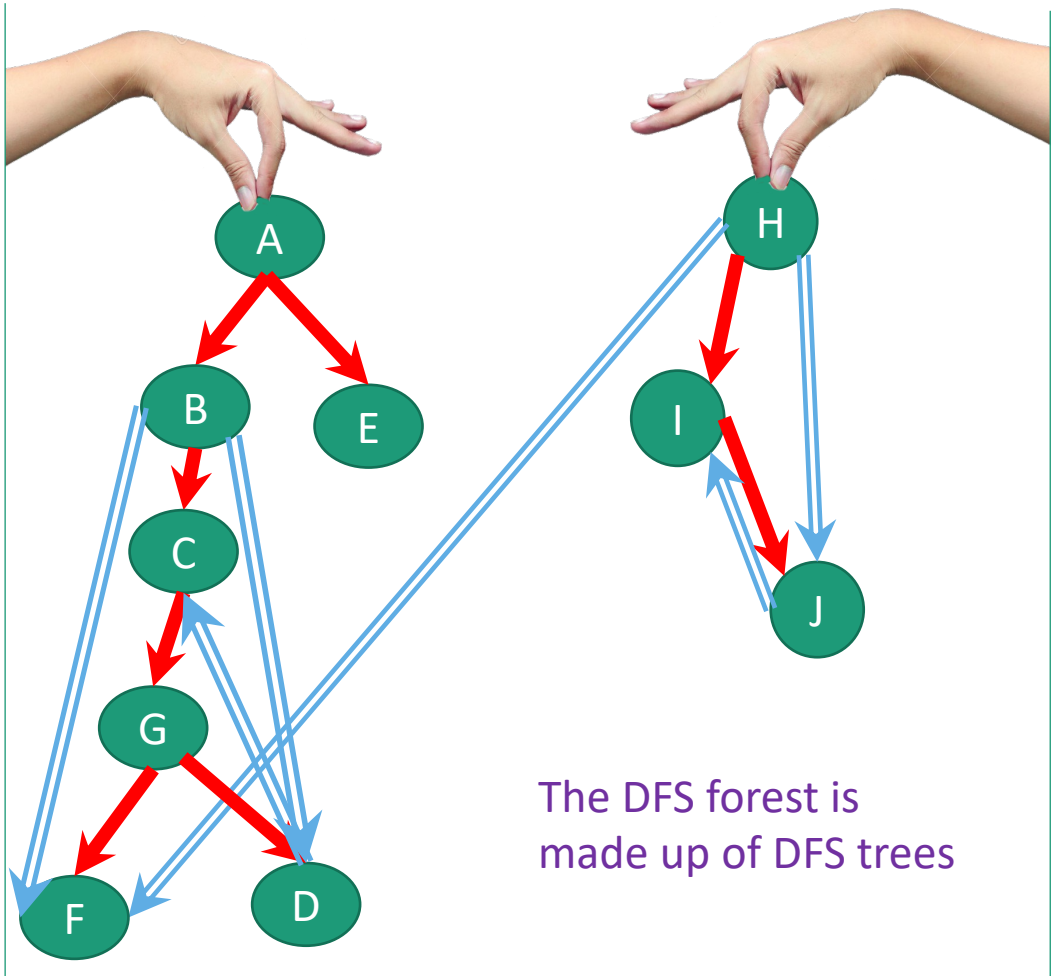
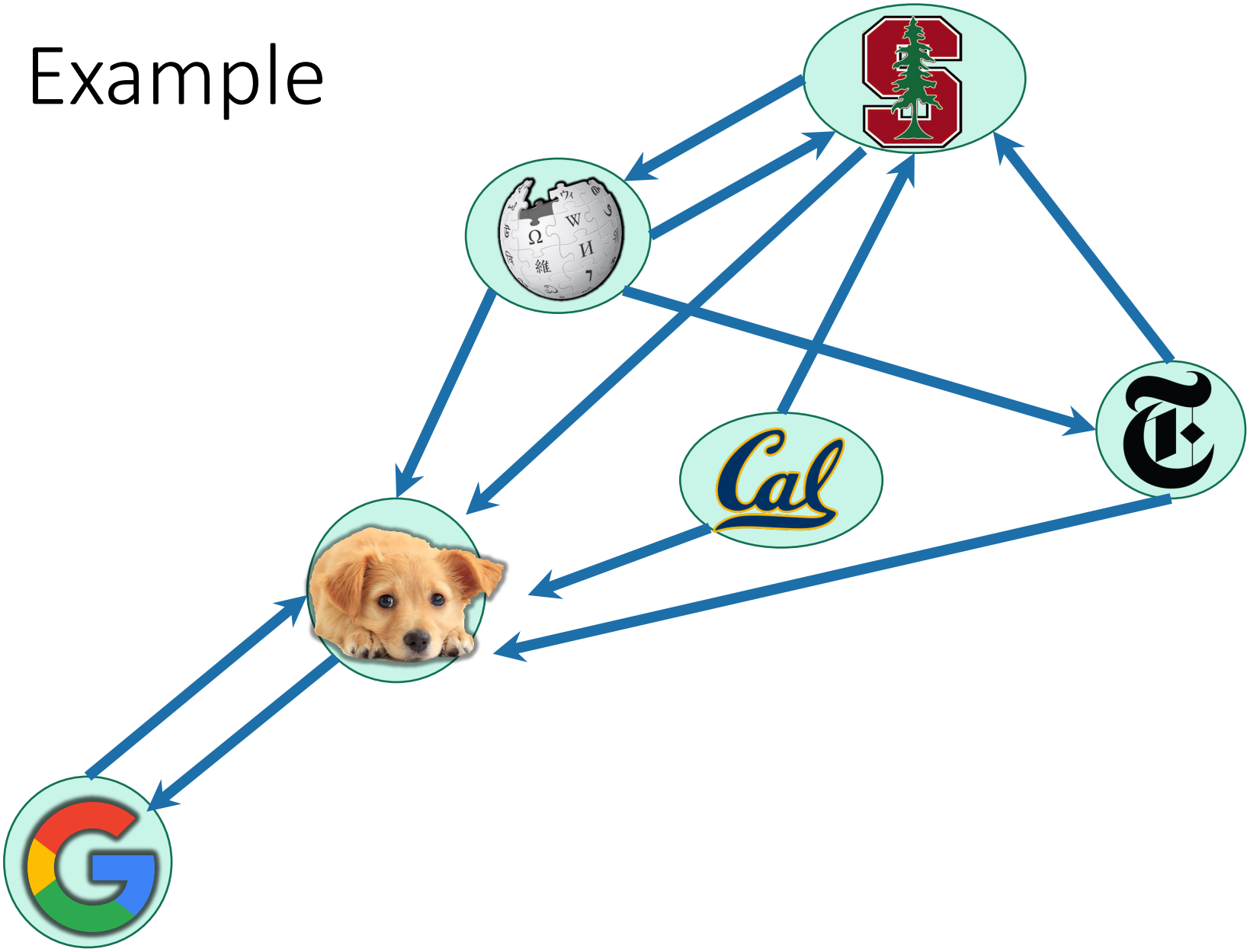


Algorithm

Running time: O(n + m)

- Do DFS to create a **DFS forest**.
  - Choose starting vertices in any order.
  - Keep track of finishing times.
- Reverse all the edges in the graph.
- Do DFS again to create **another DFS forest**.
  - This time, order the nodes in the reverse order of the finishing times that they had from the first DFS run.
- The SCCs are the different trees in the **second DFS forest.**

Example



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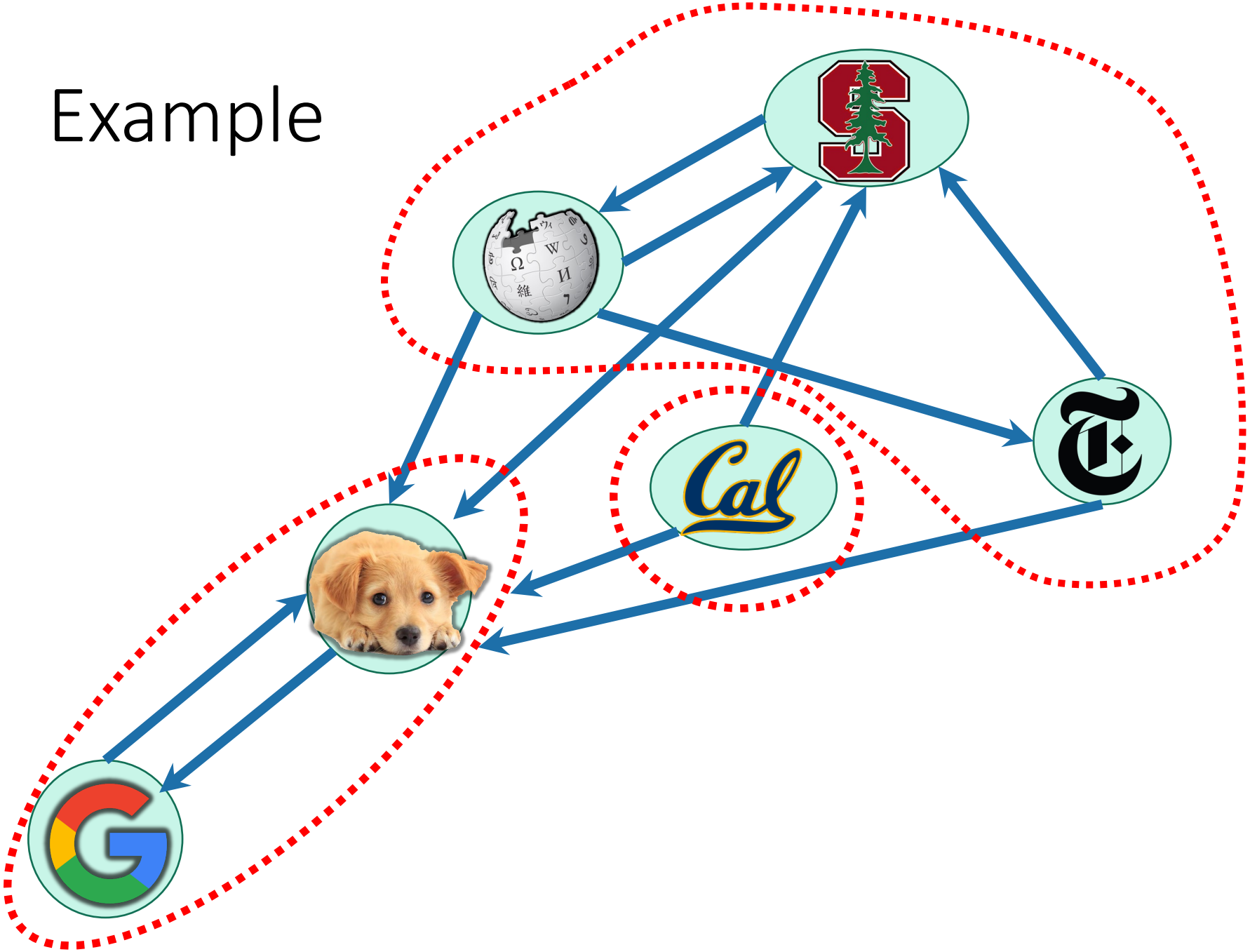

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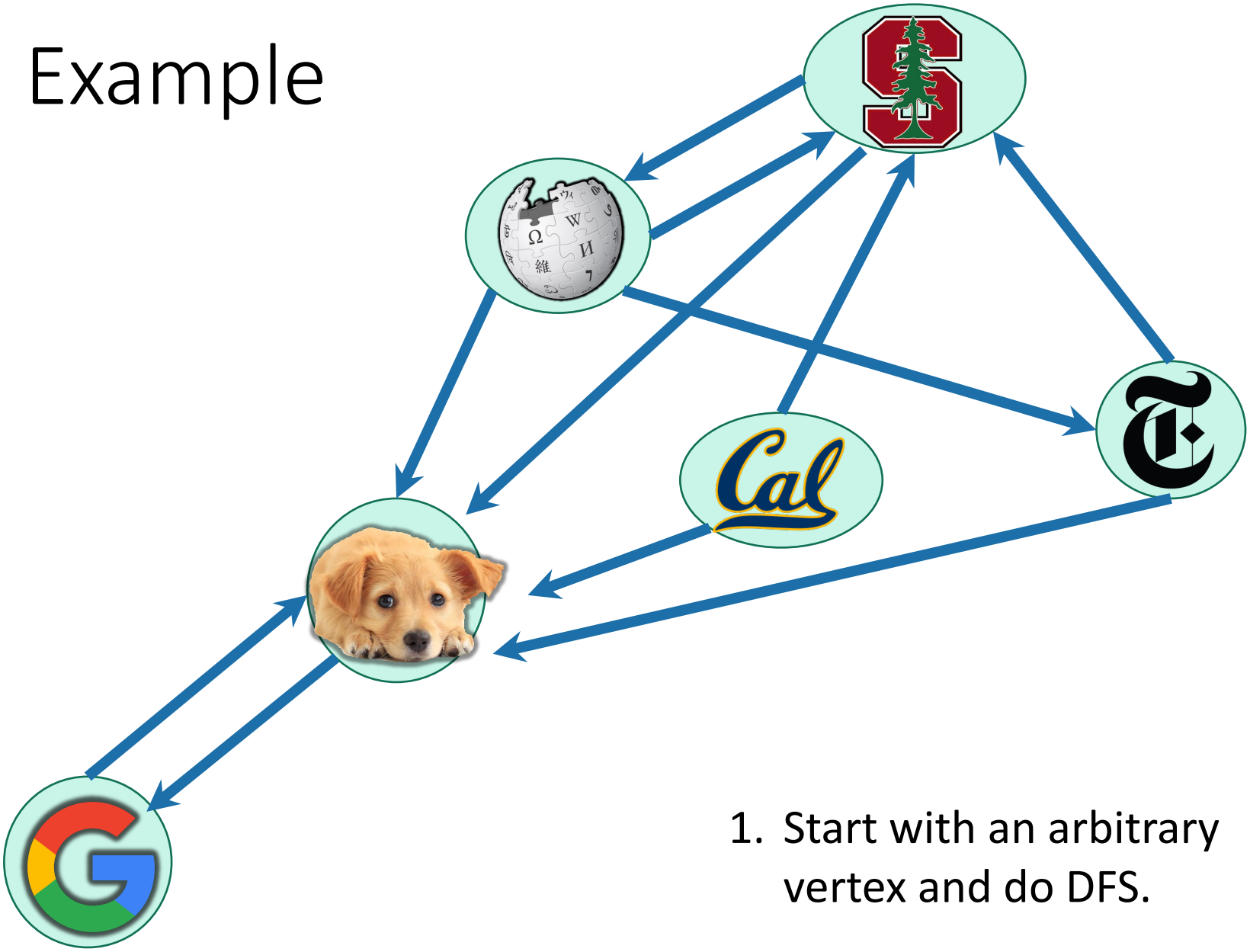

1. Start with an arbitrary vertex and do DFS.

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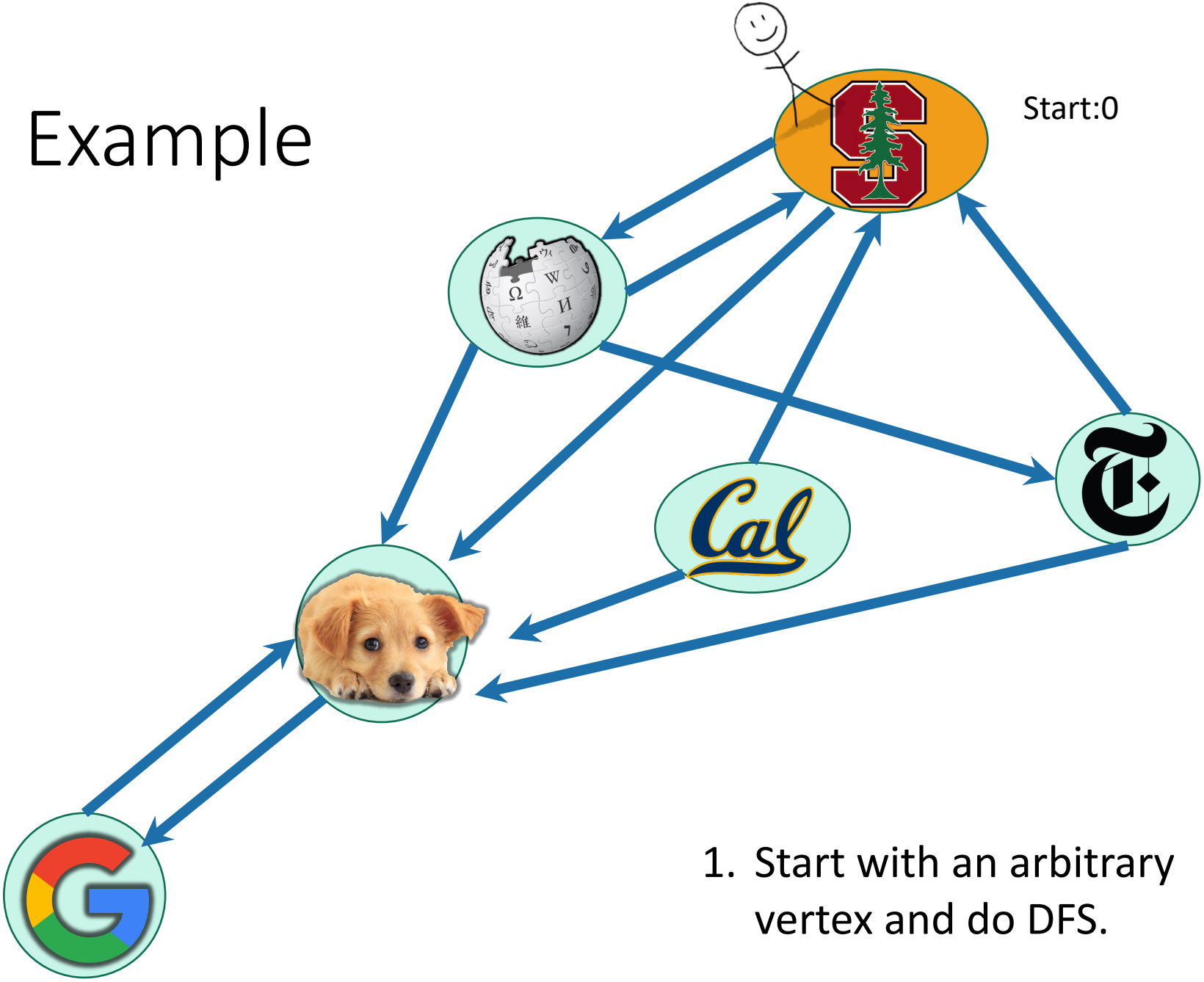

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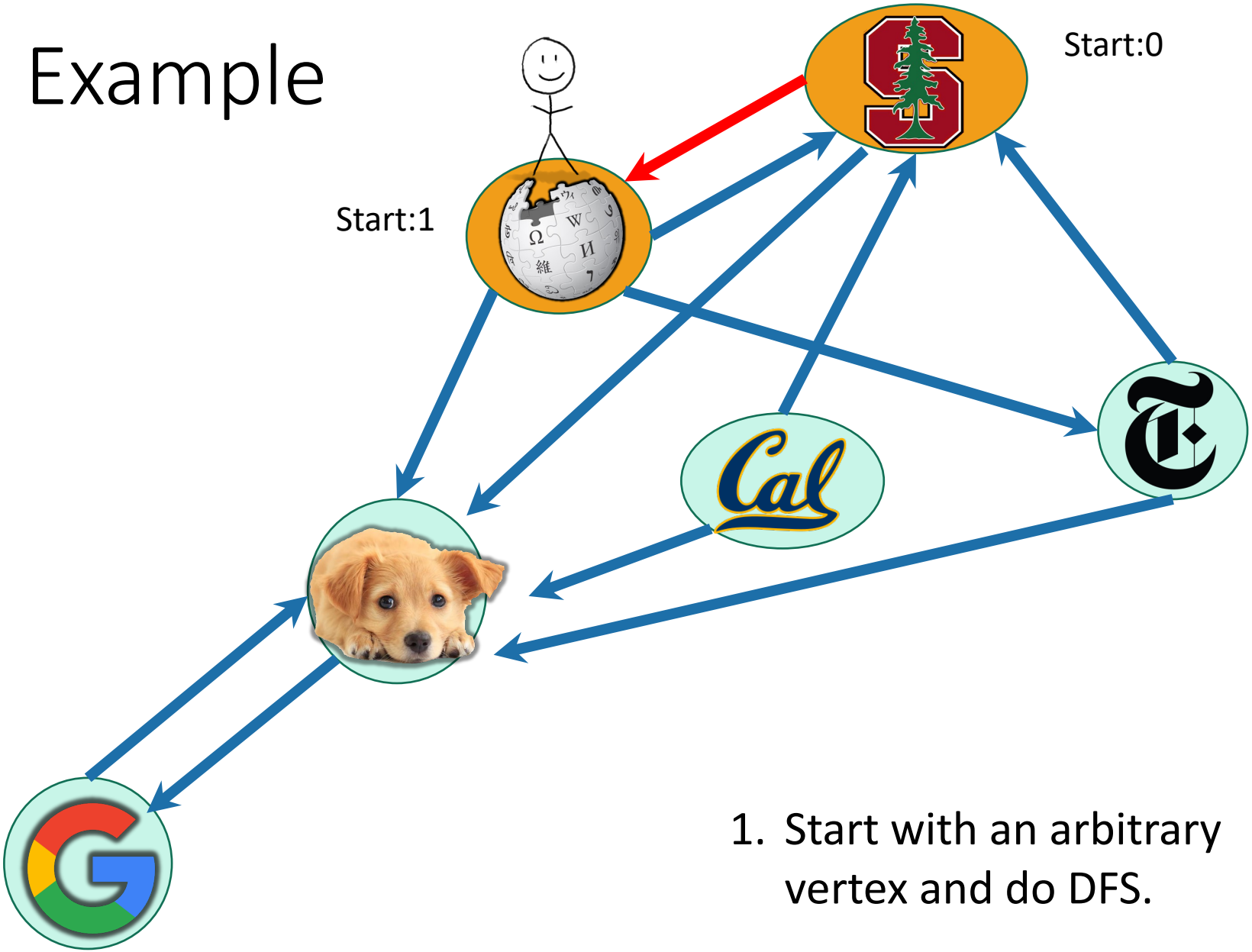

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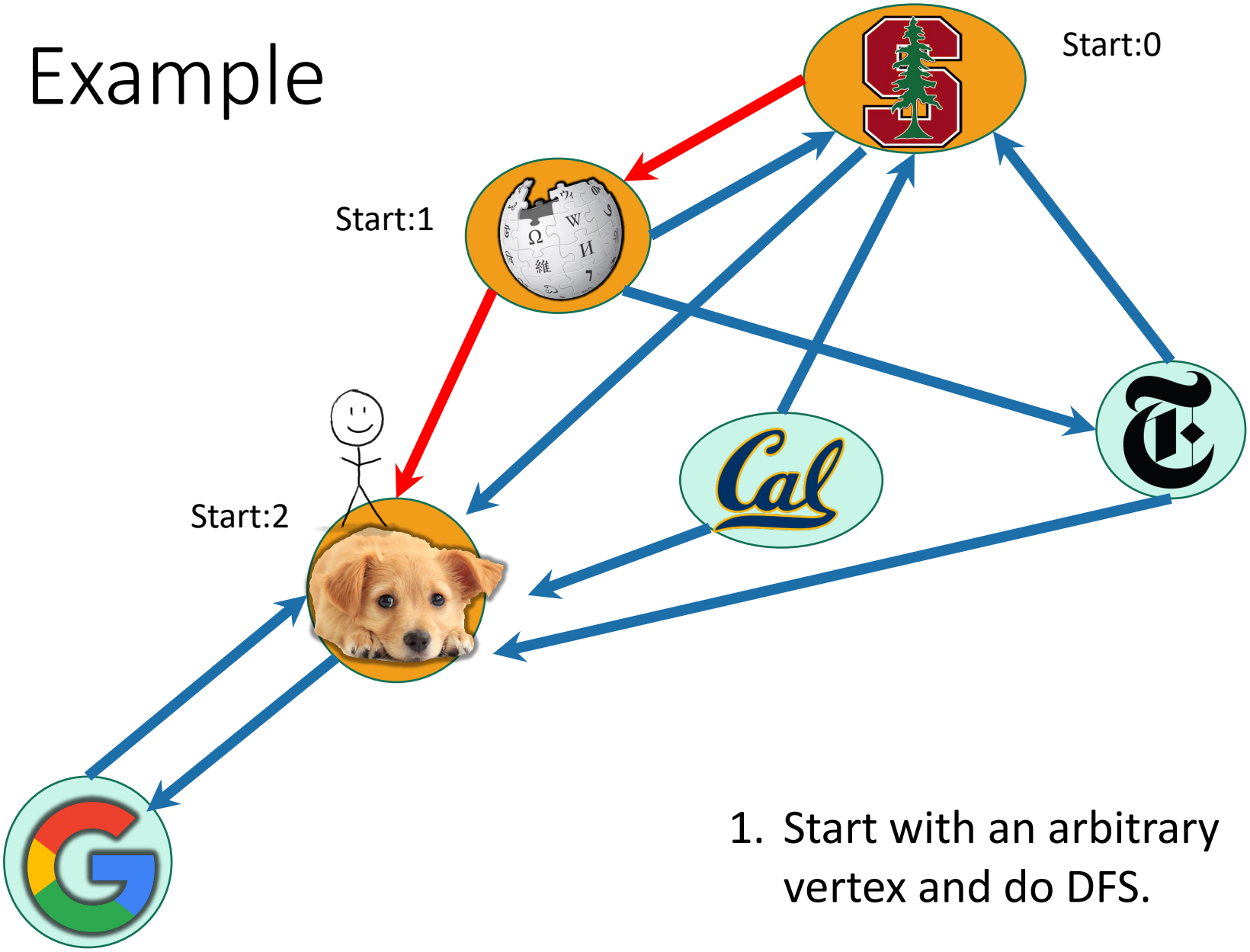

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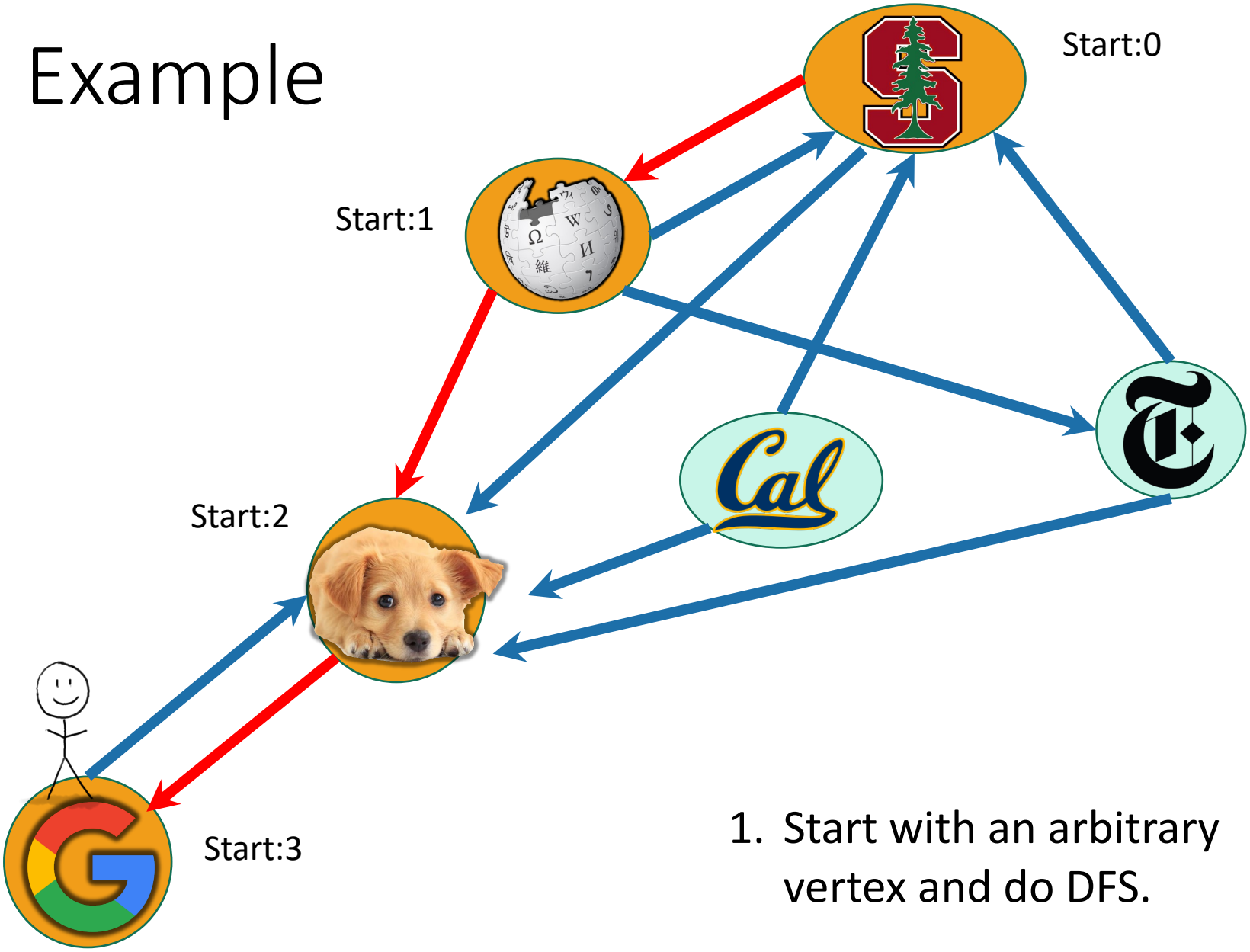

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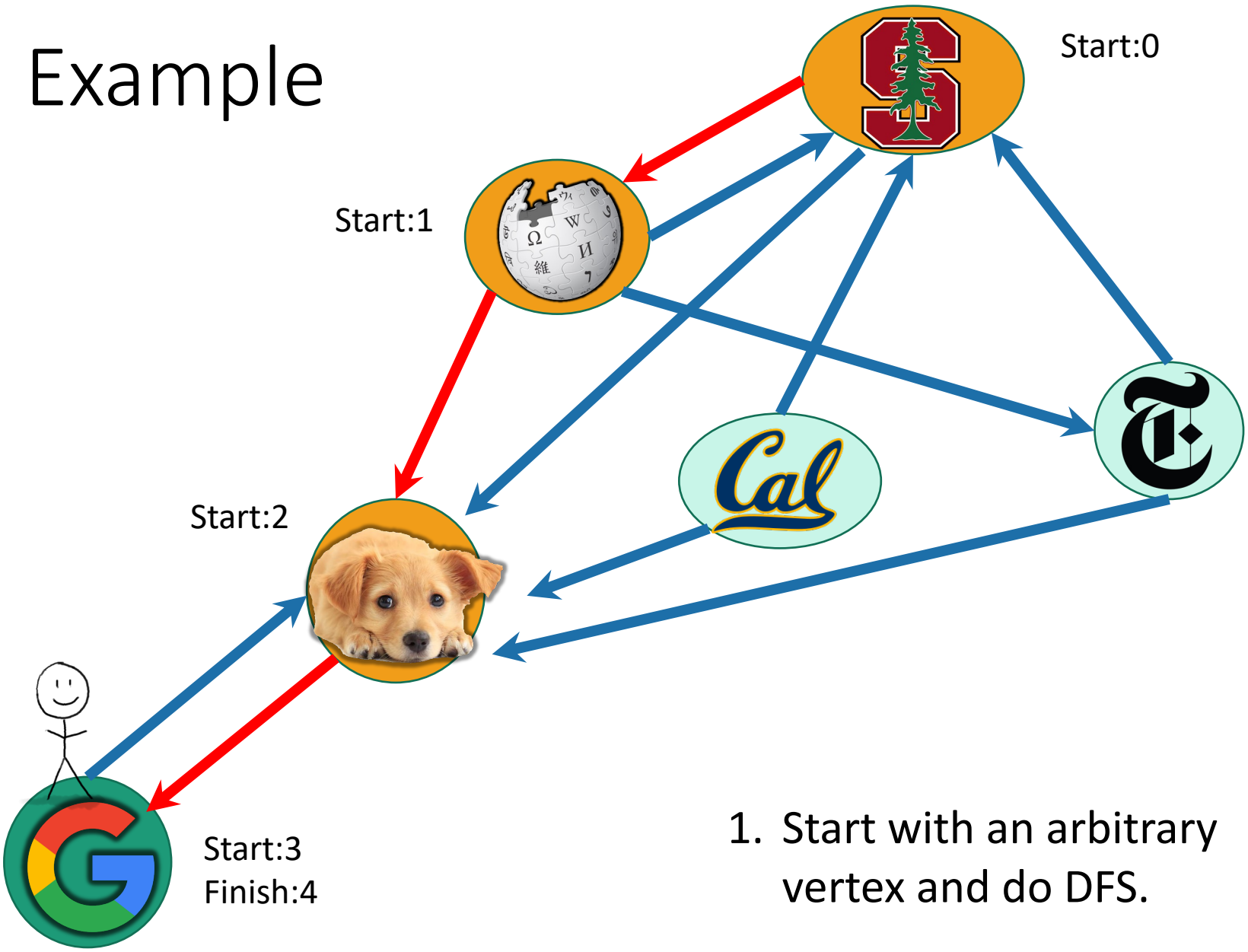

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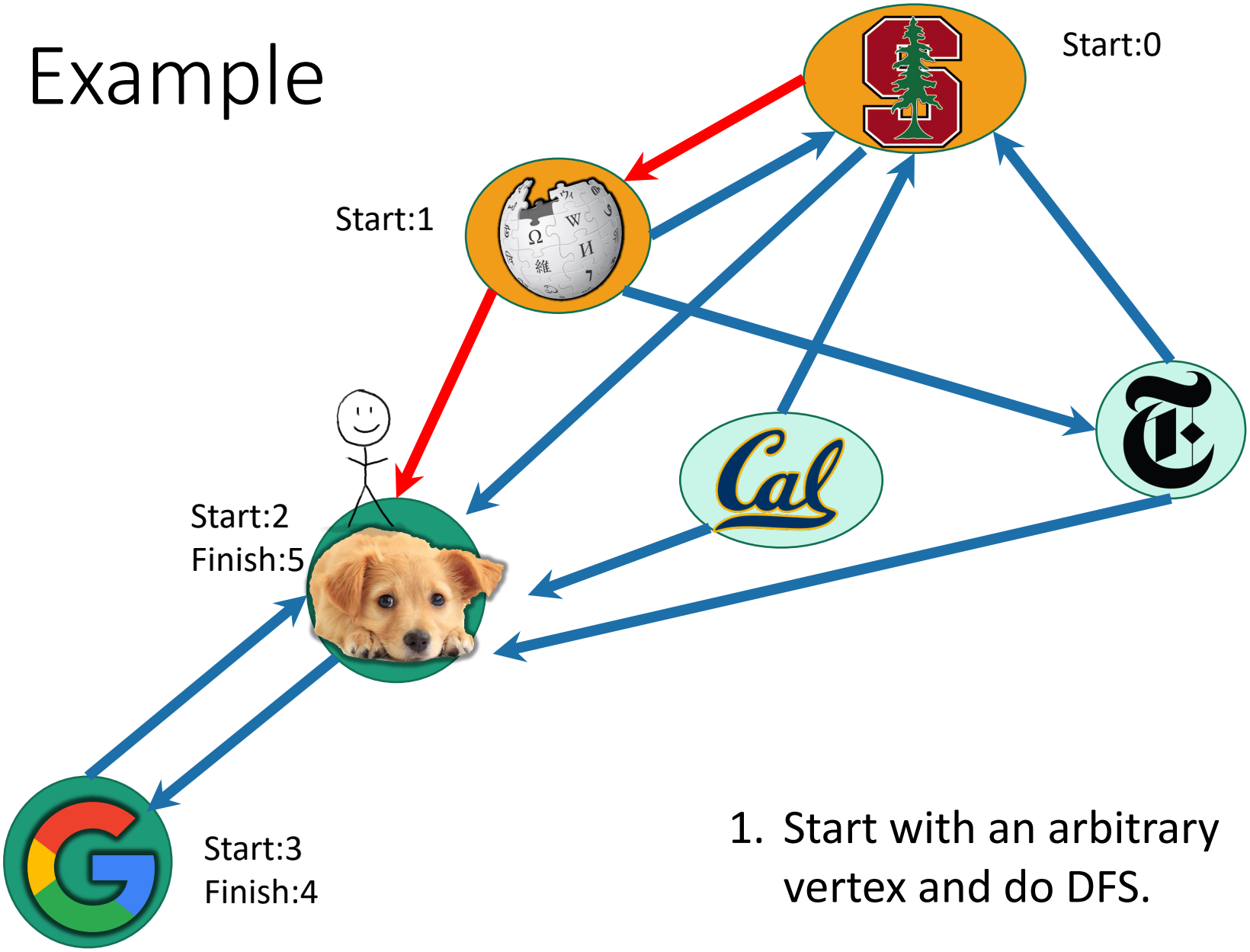

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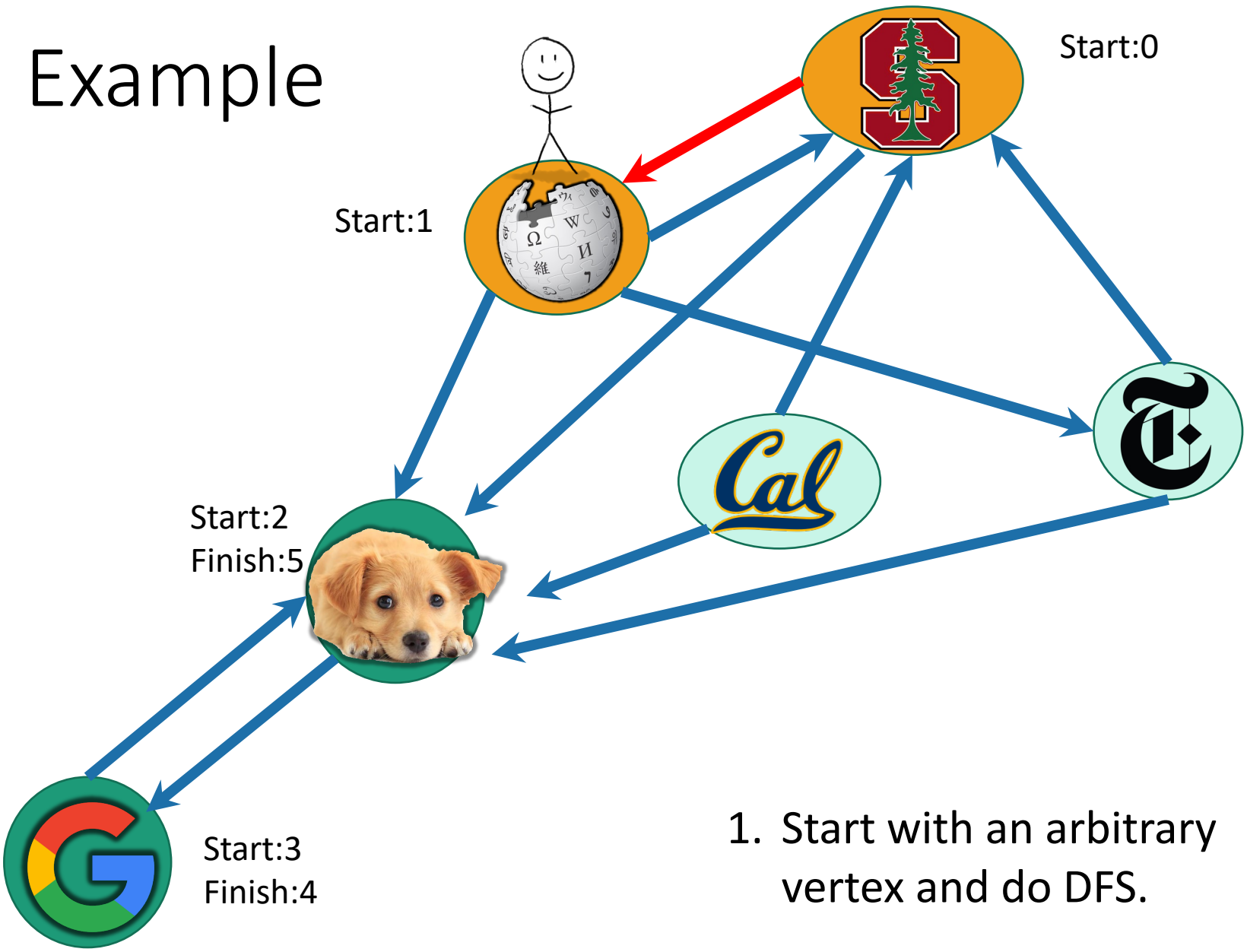

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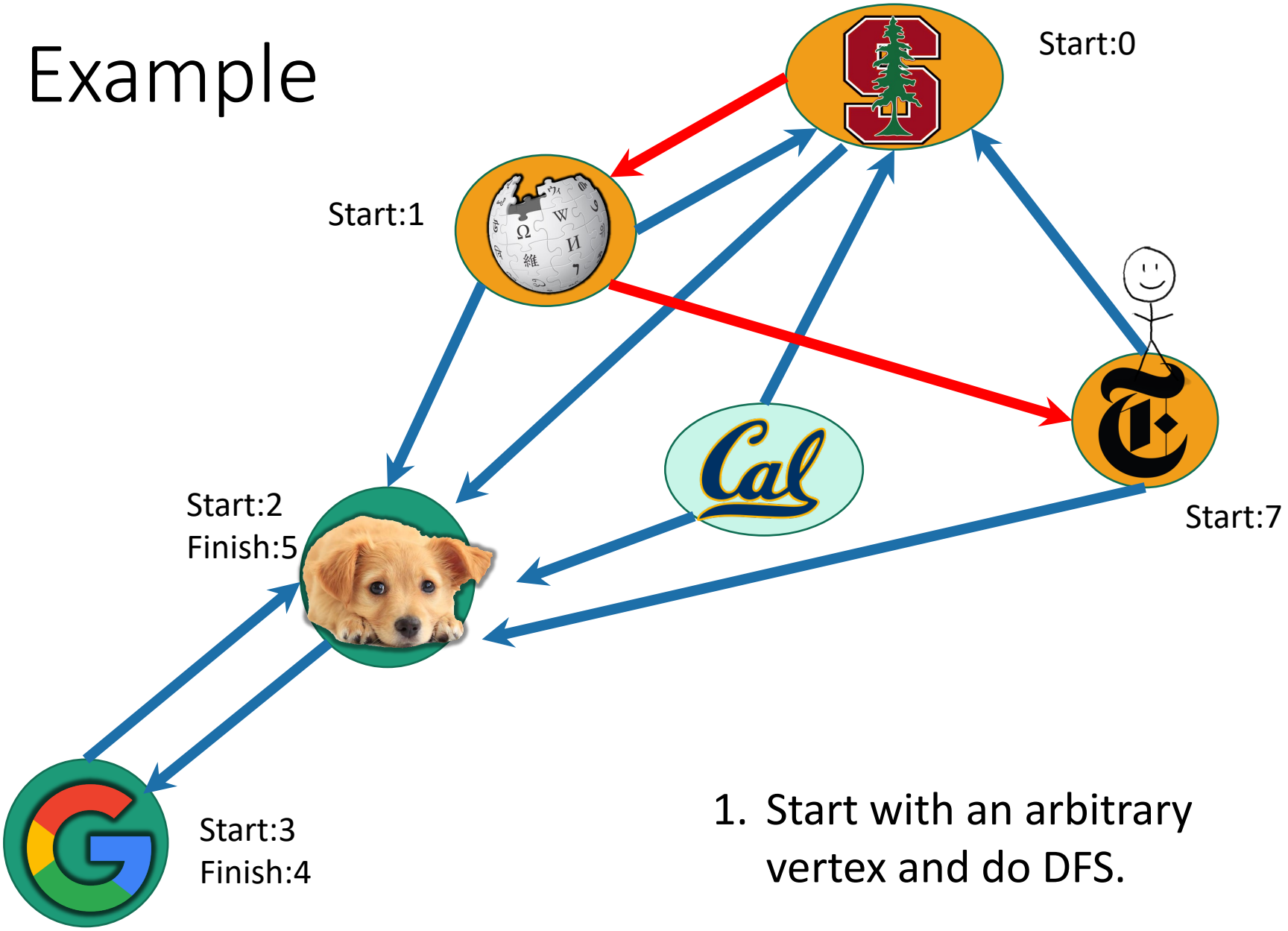

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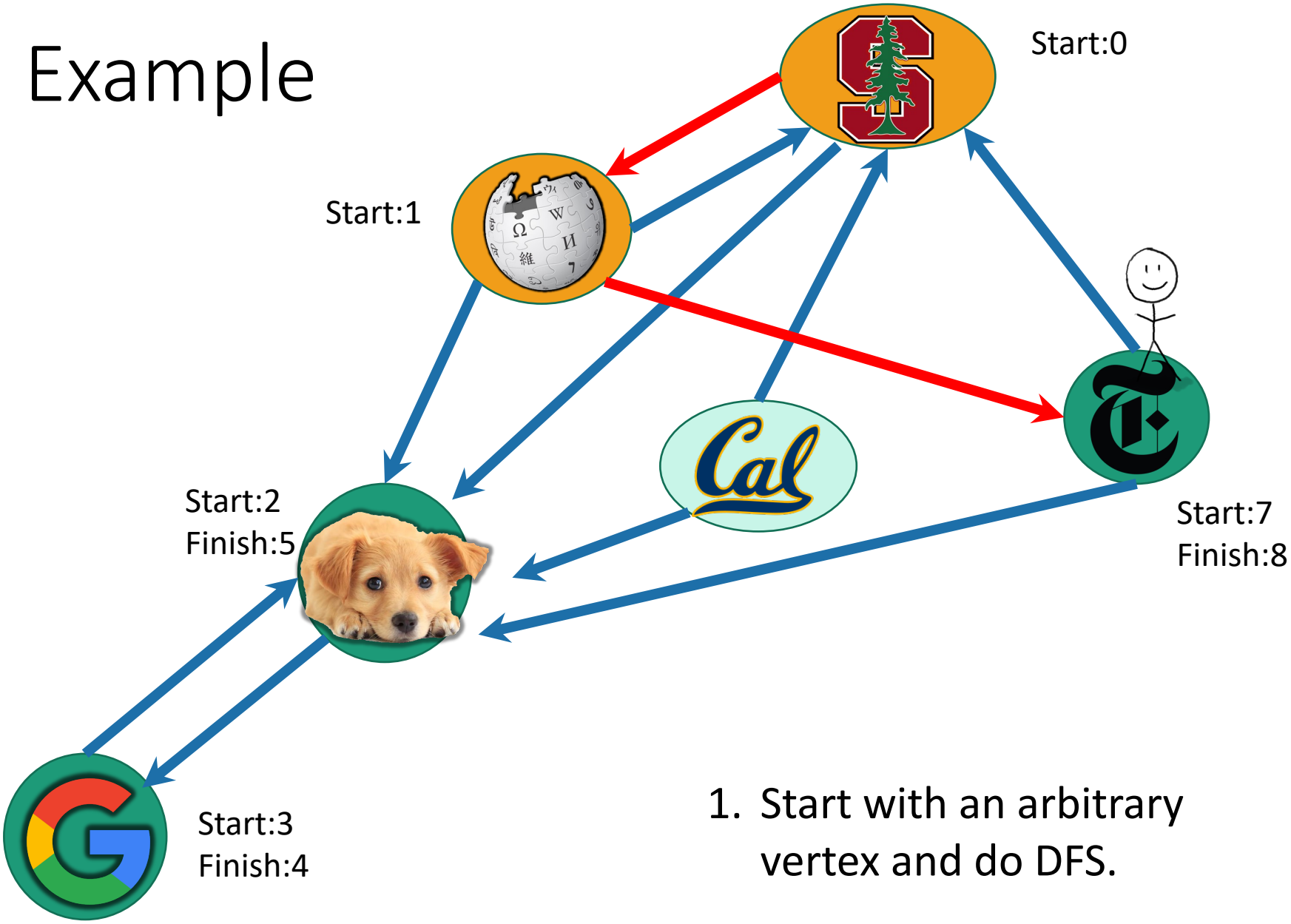

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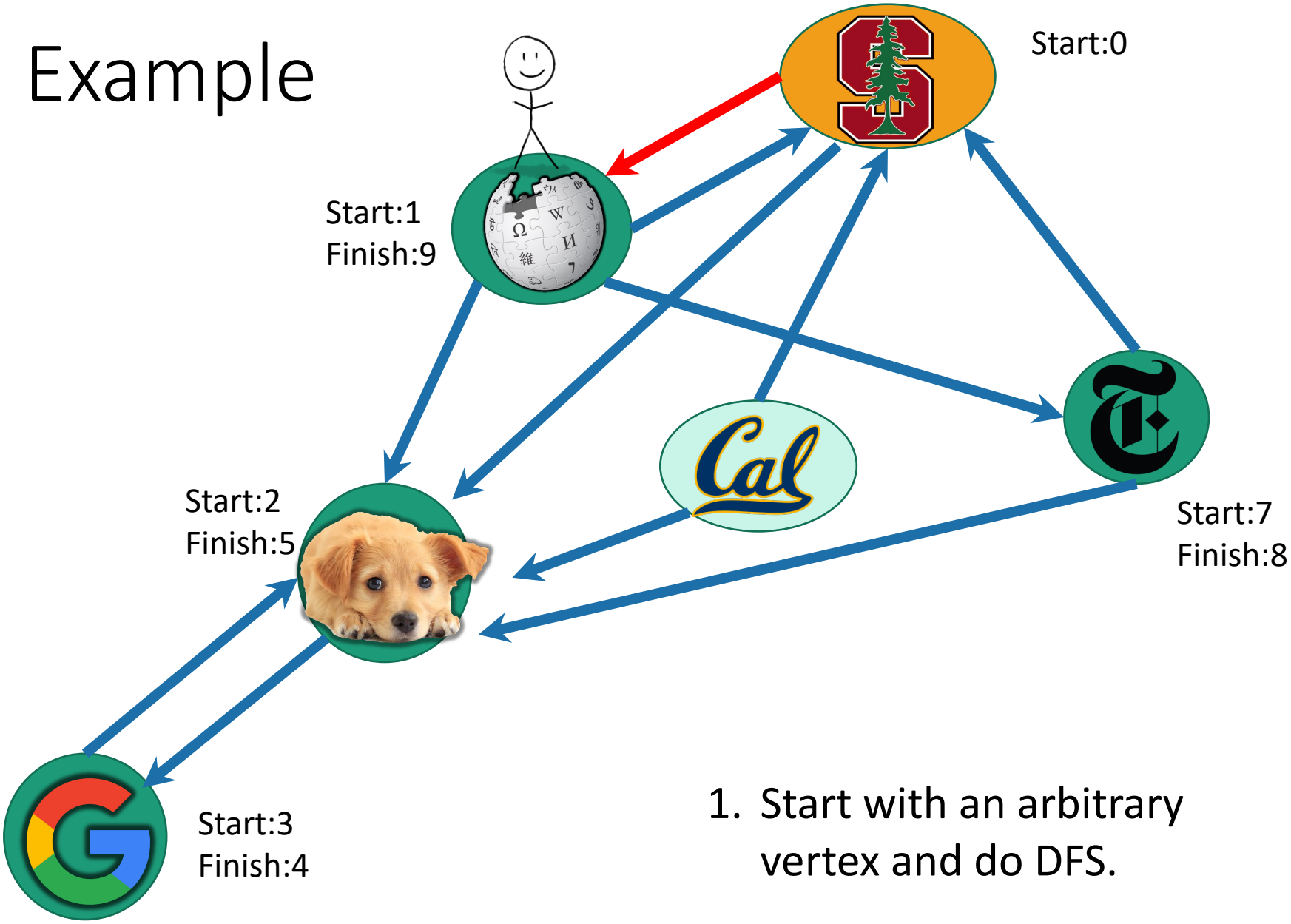

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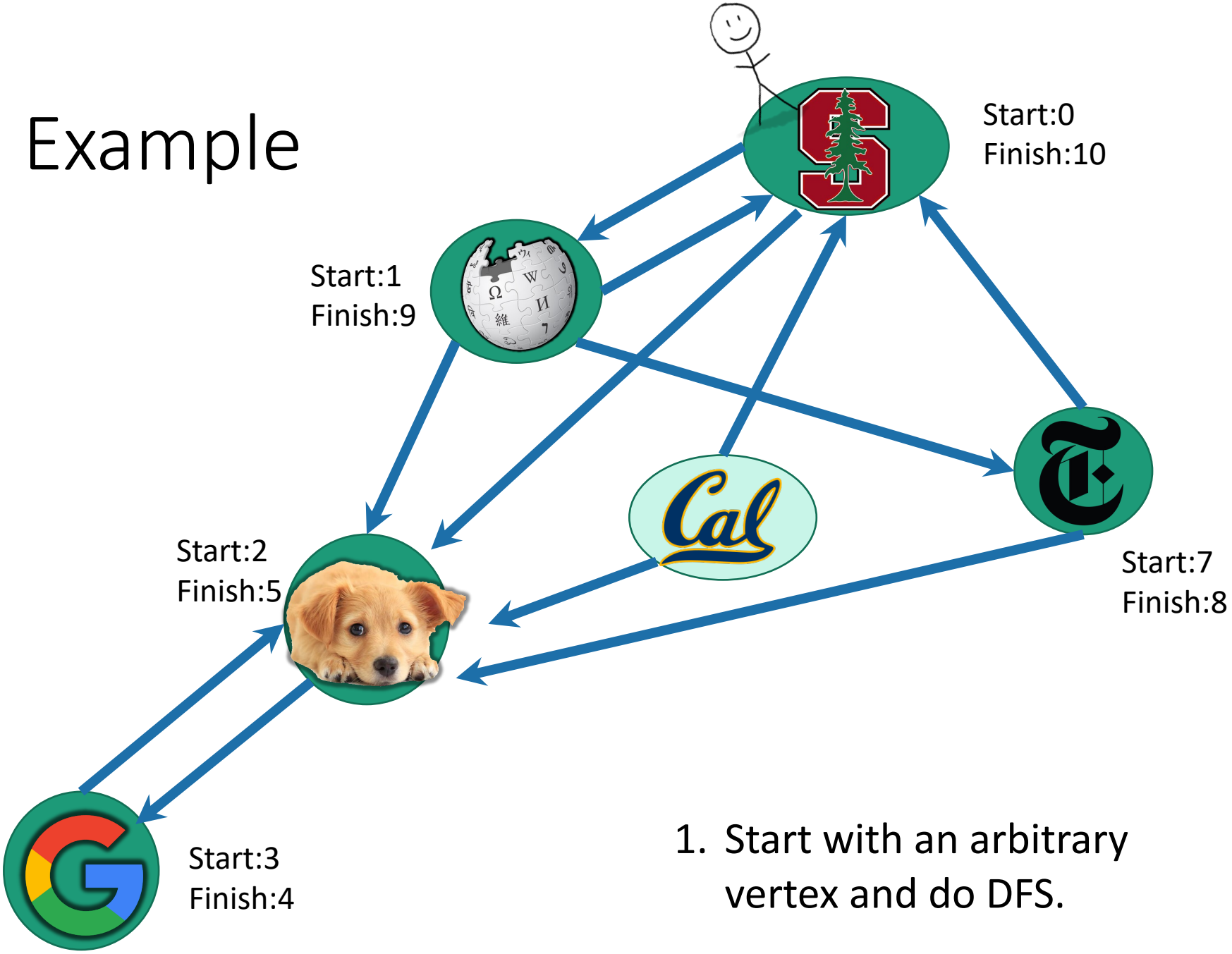

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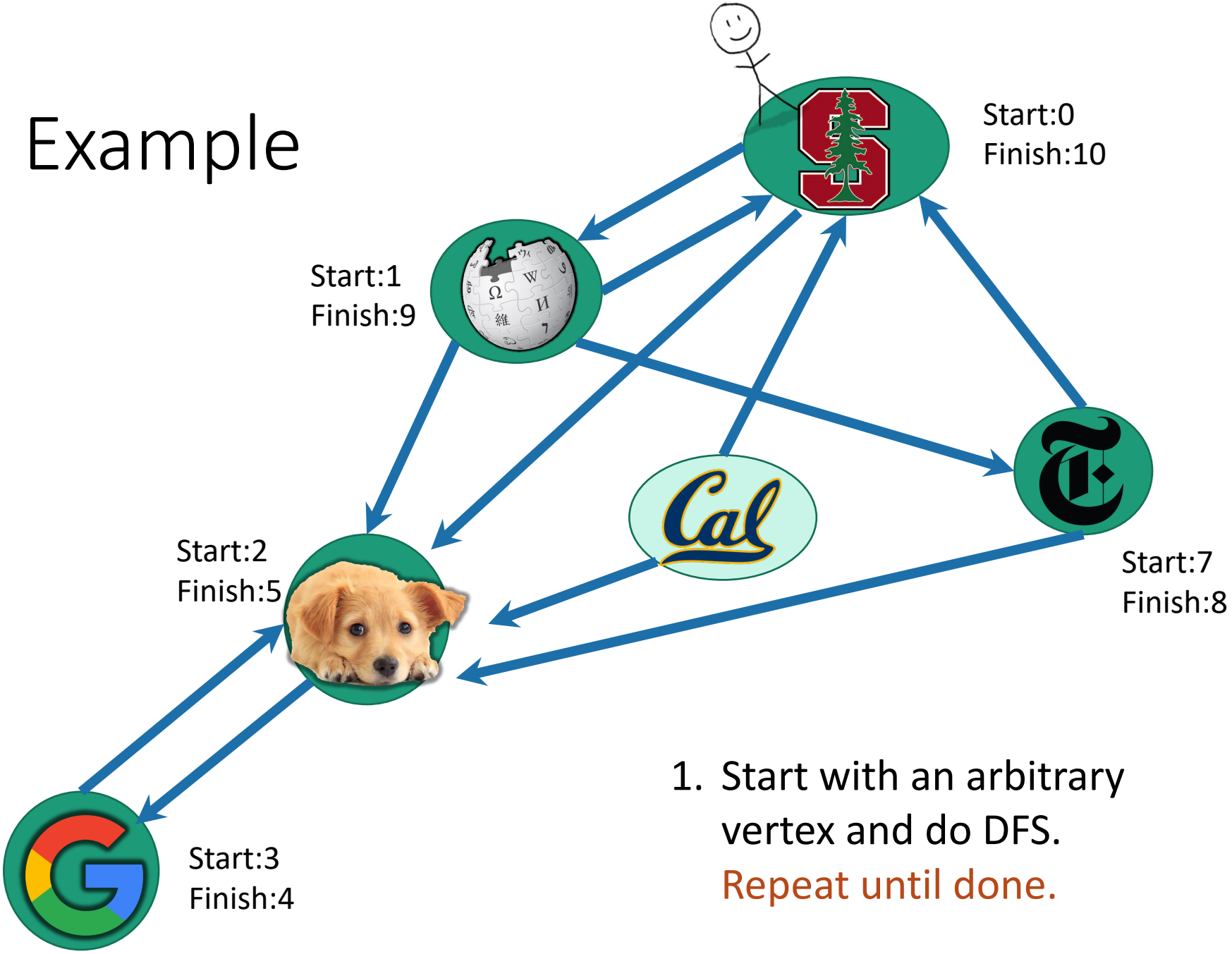

1. Start with an arbitrary vertex and do DFS.
   Repeat until done.

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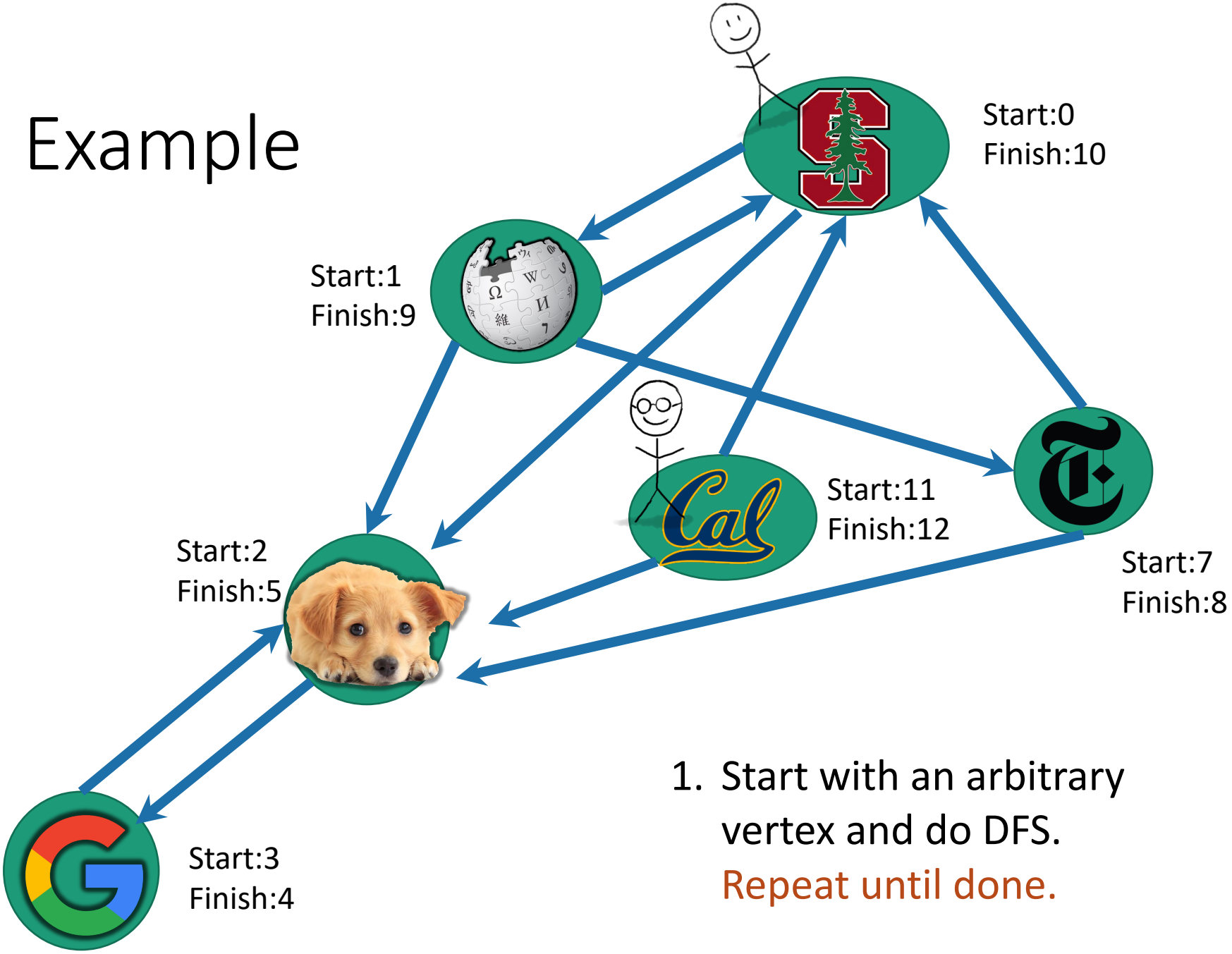

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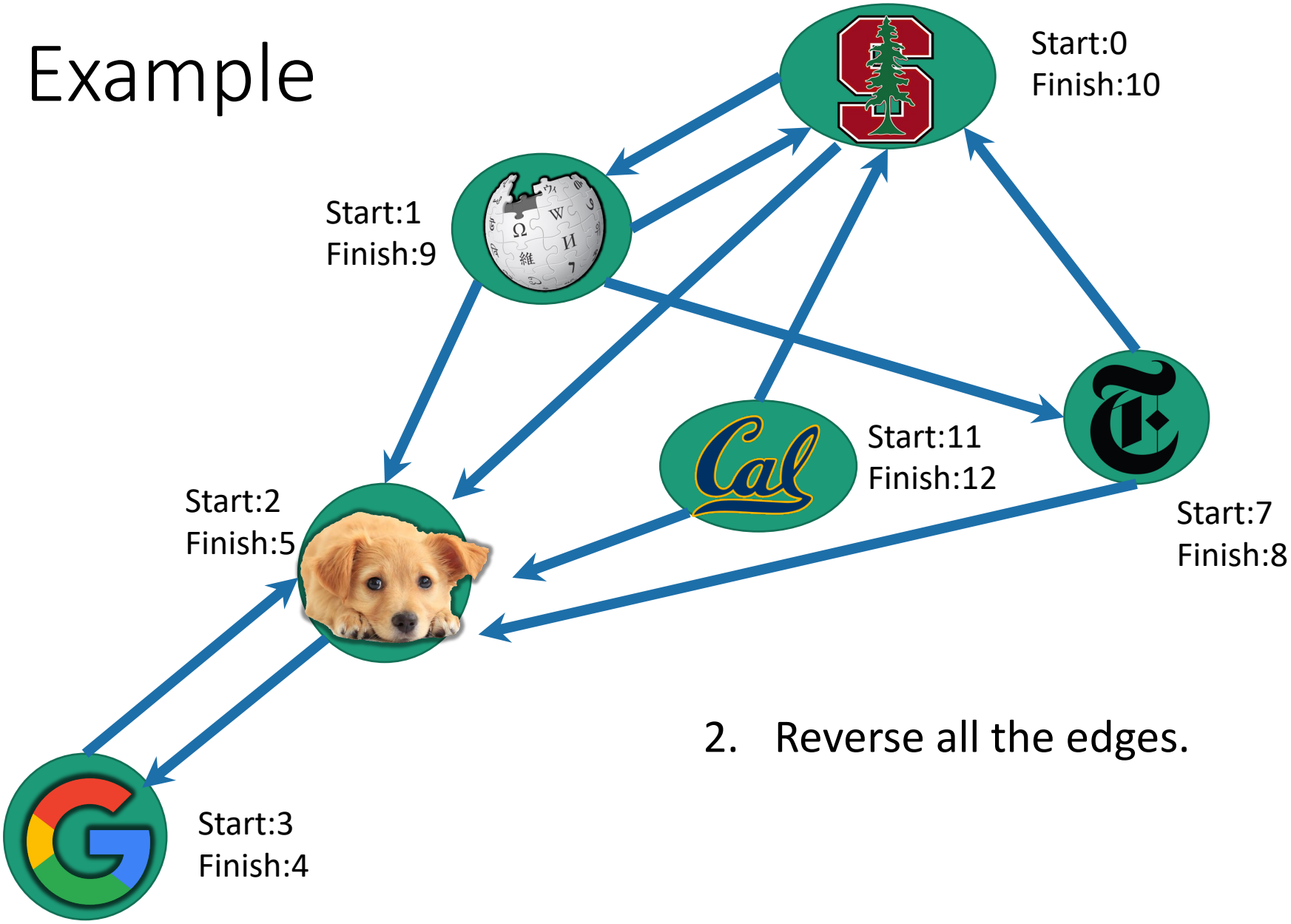
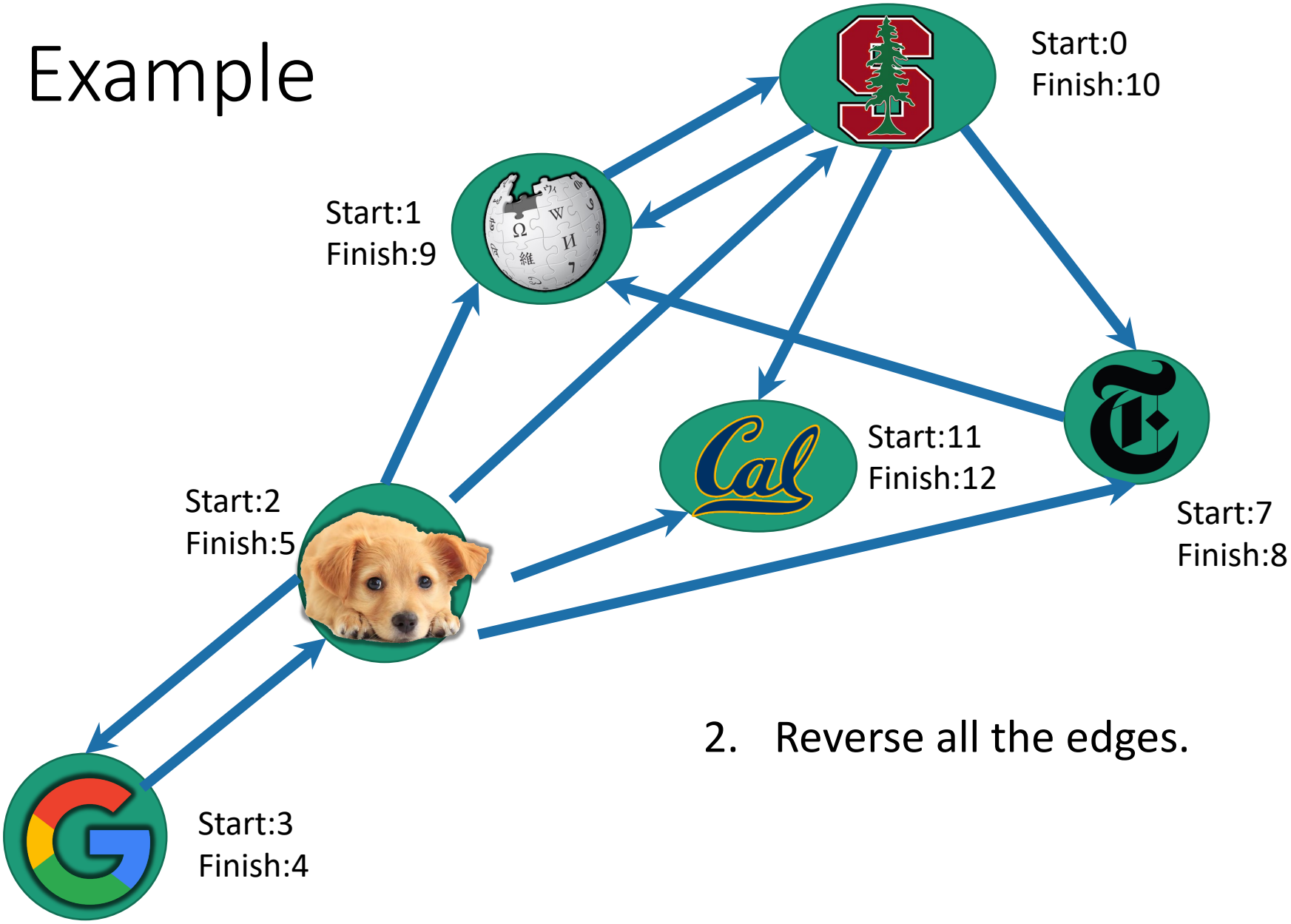


2. Reverse all the edges.

Example



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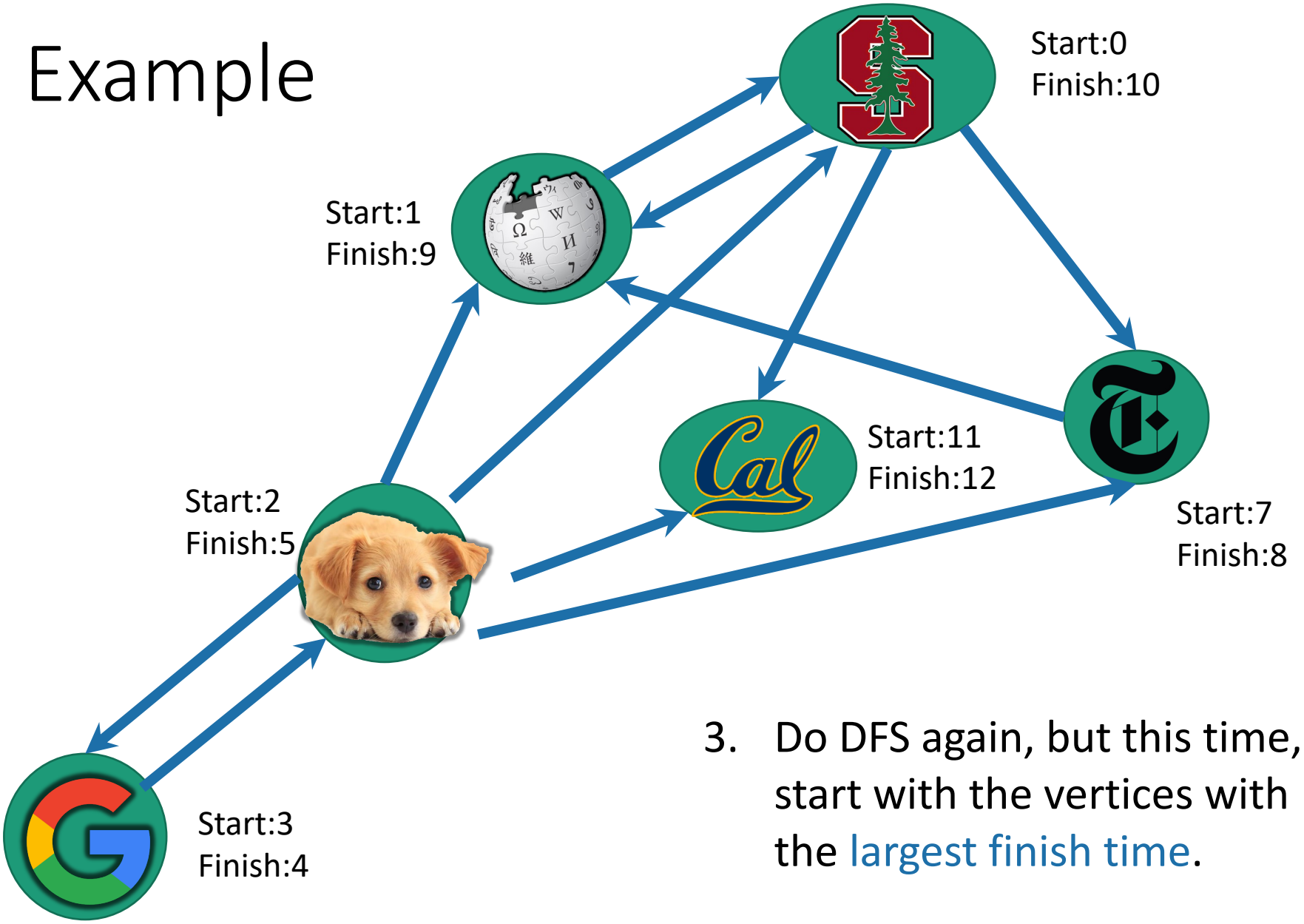
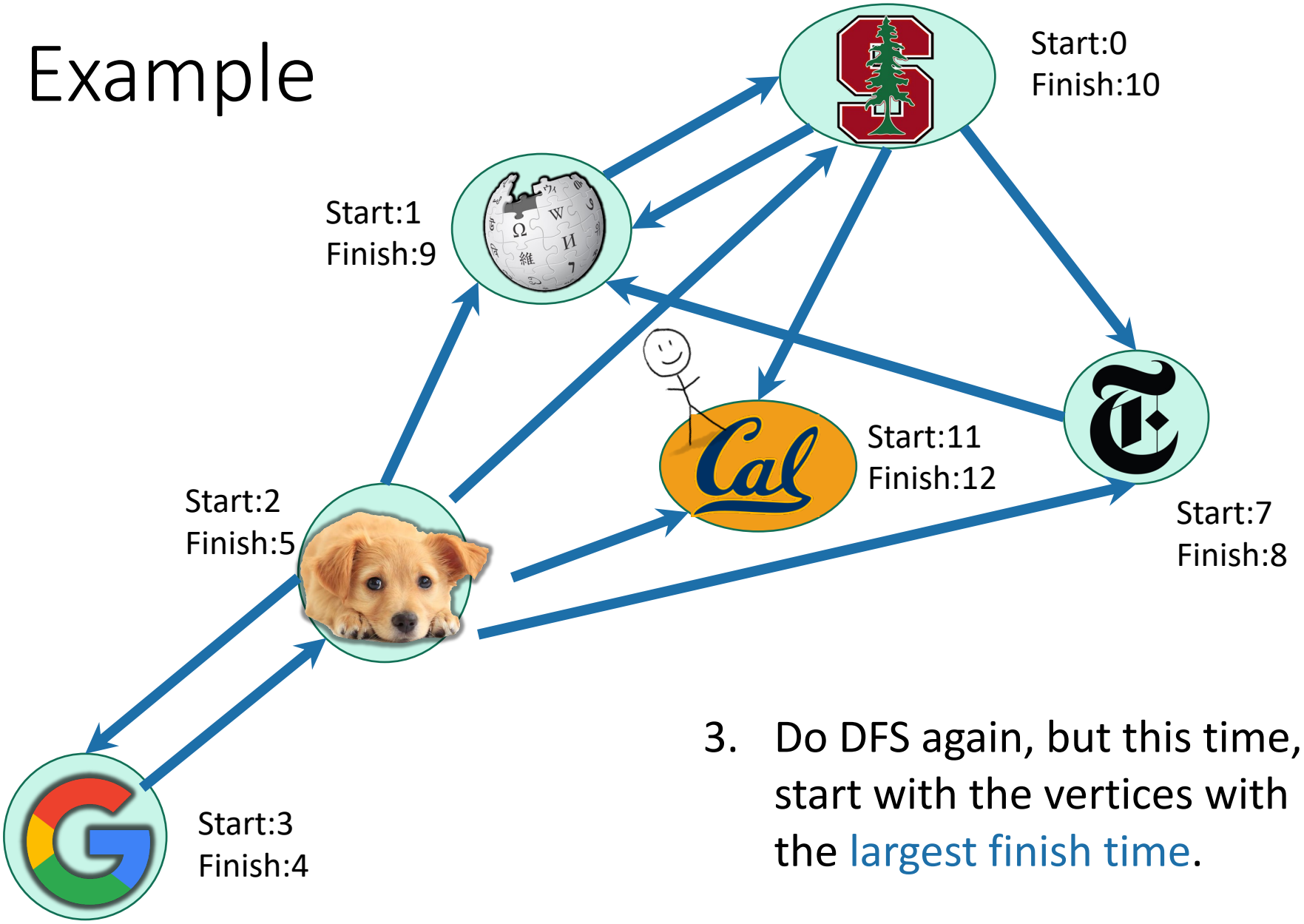


3. Do DFS again, but this time, start with the vertices with the largest finish time.

Example



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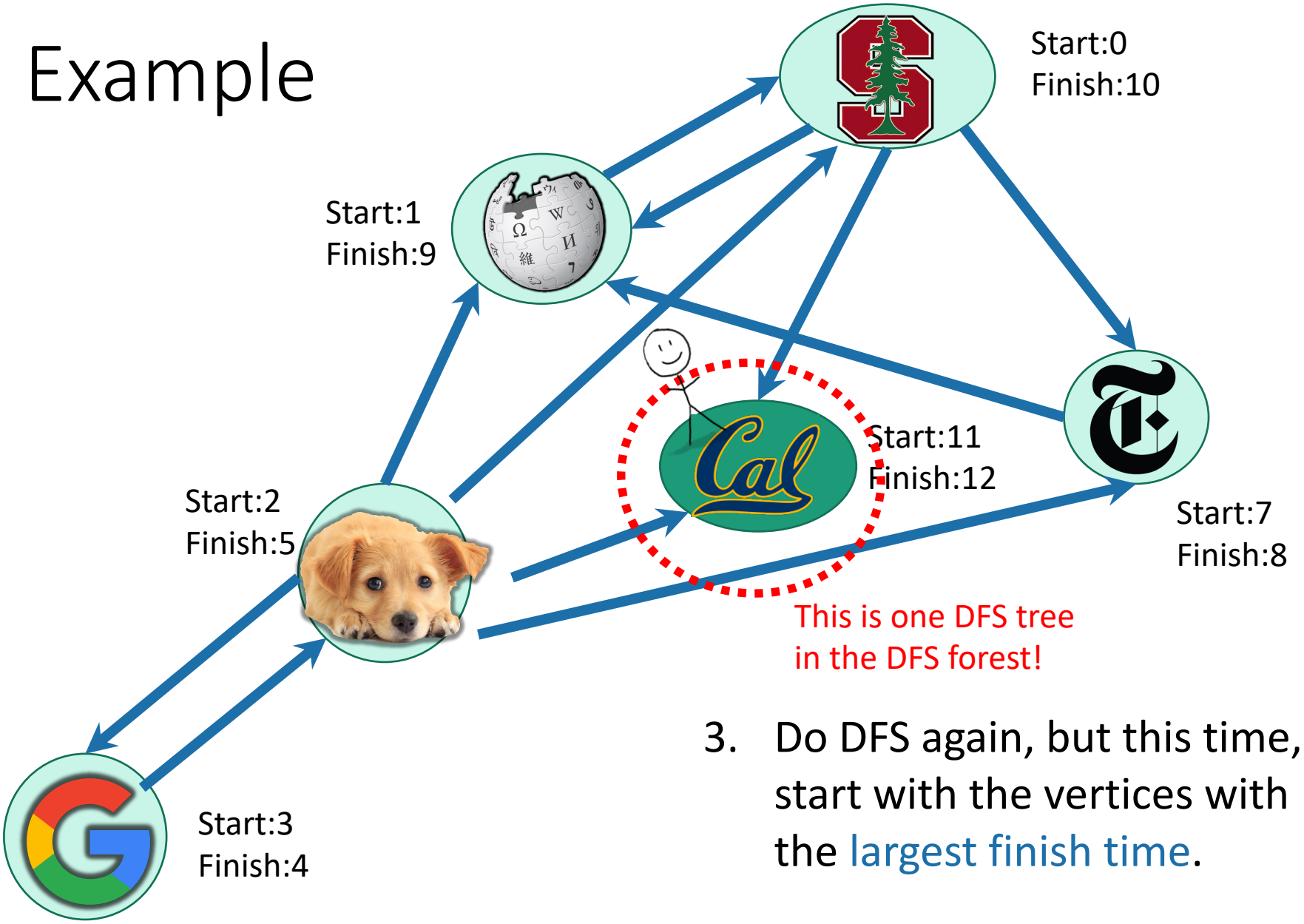

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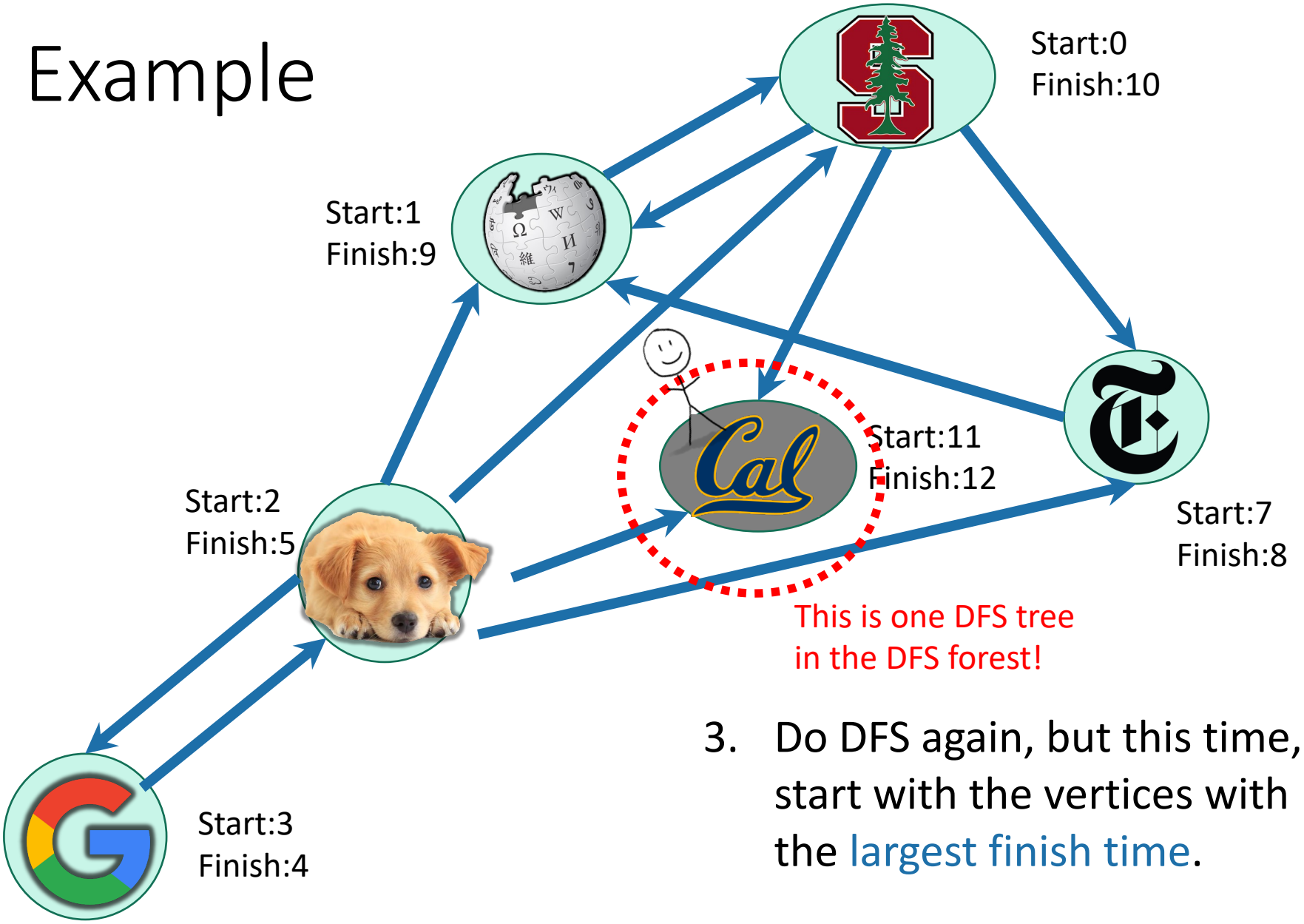

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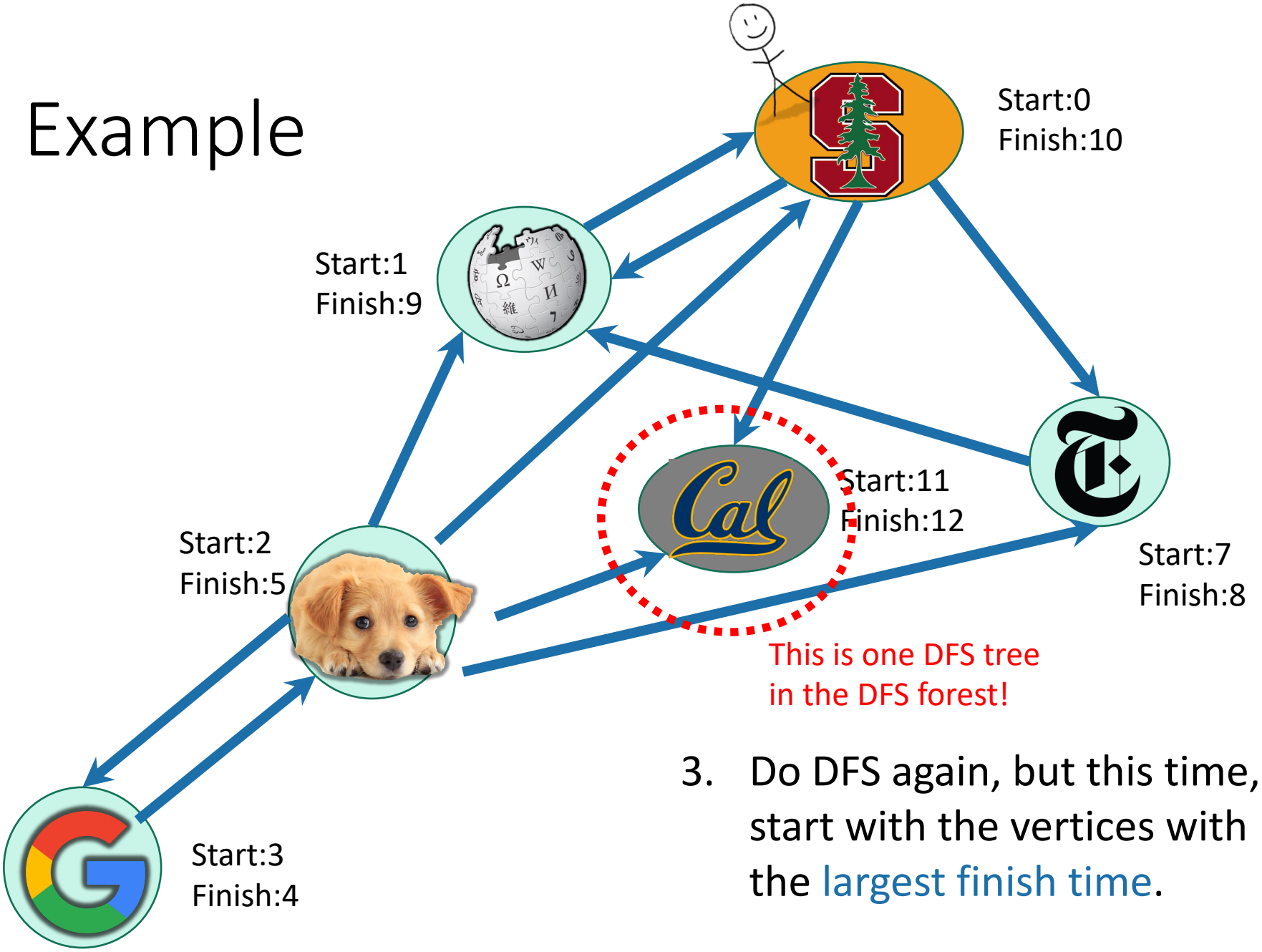

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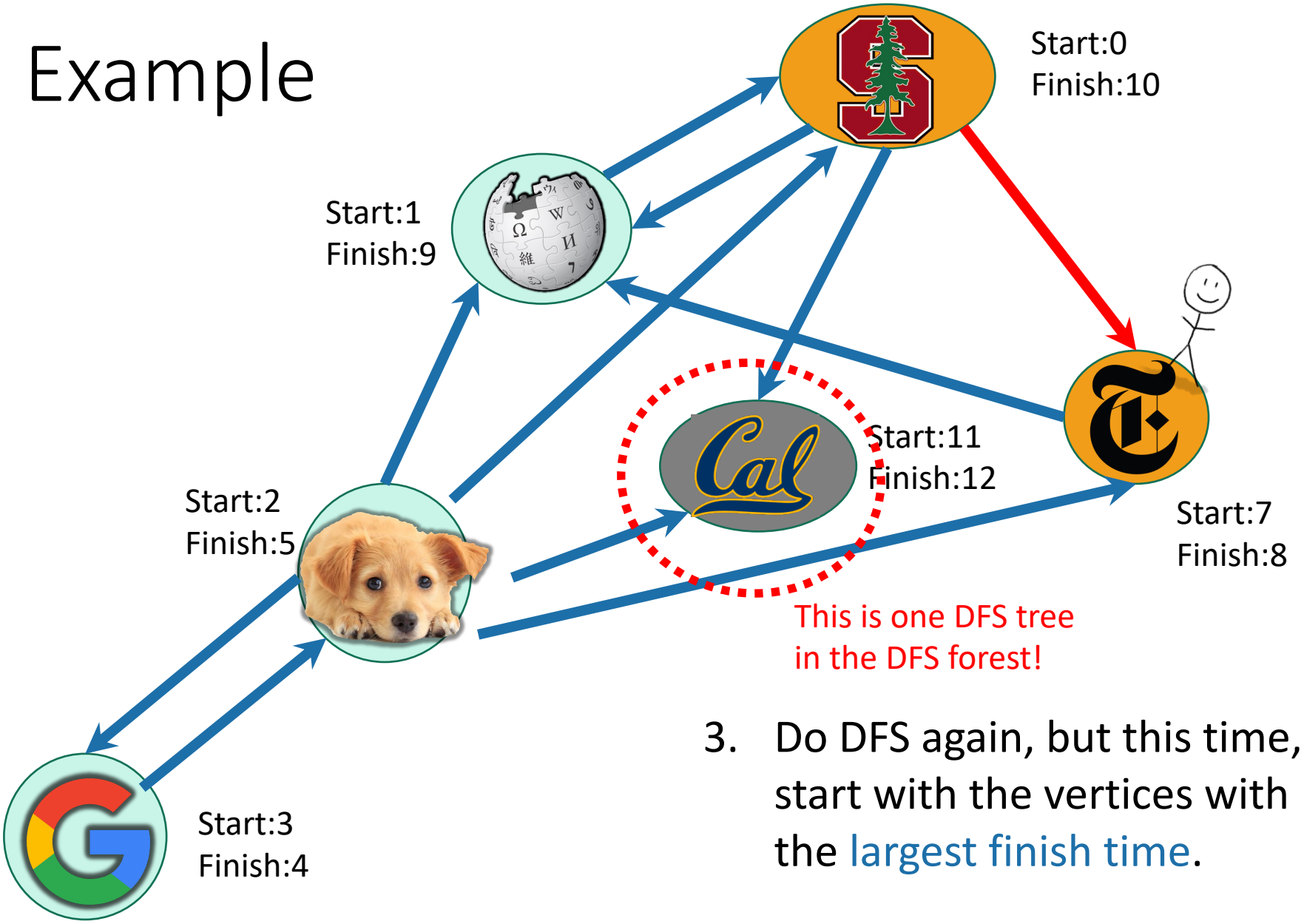

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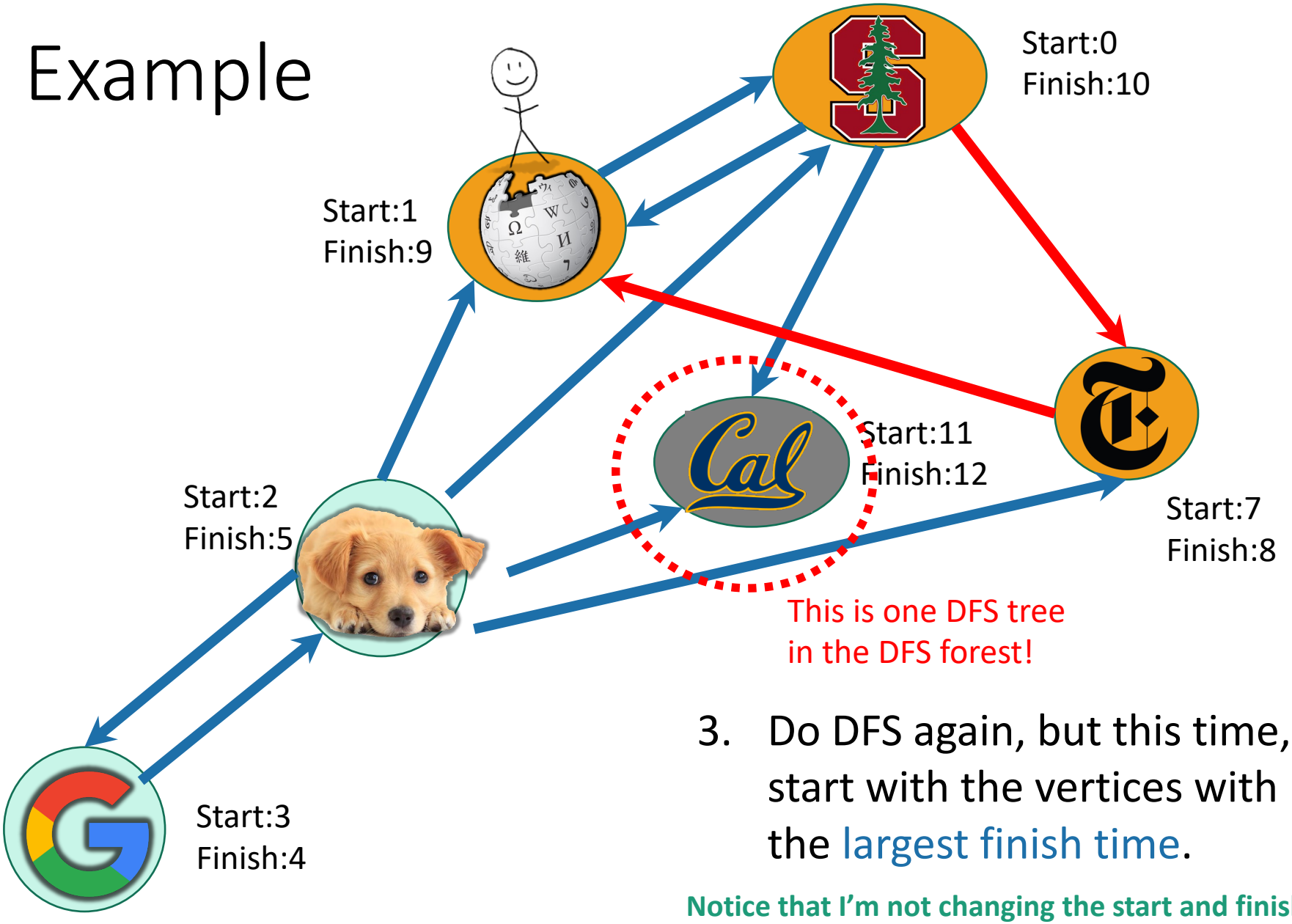

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Notice that I'm not changing the start and finish times – I'm keeping them from the first run.

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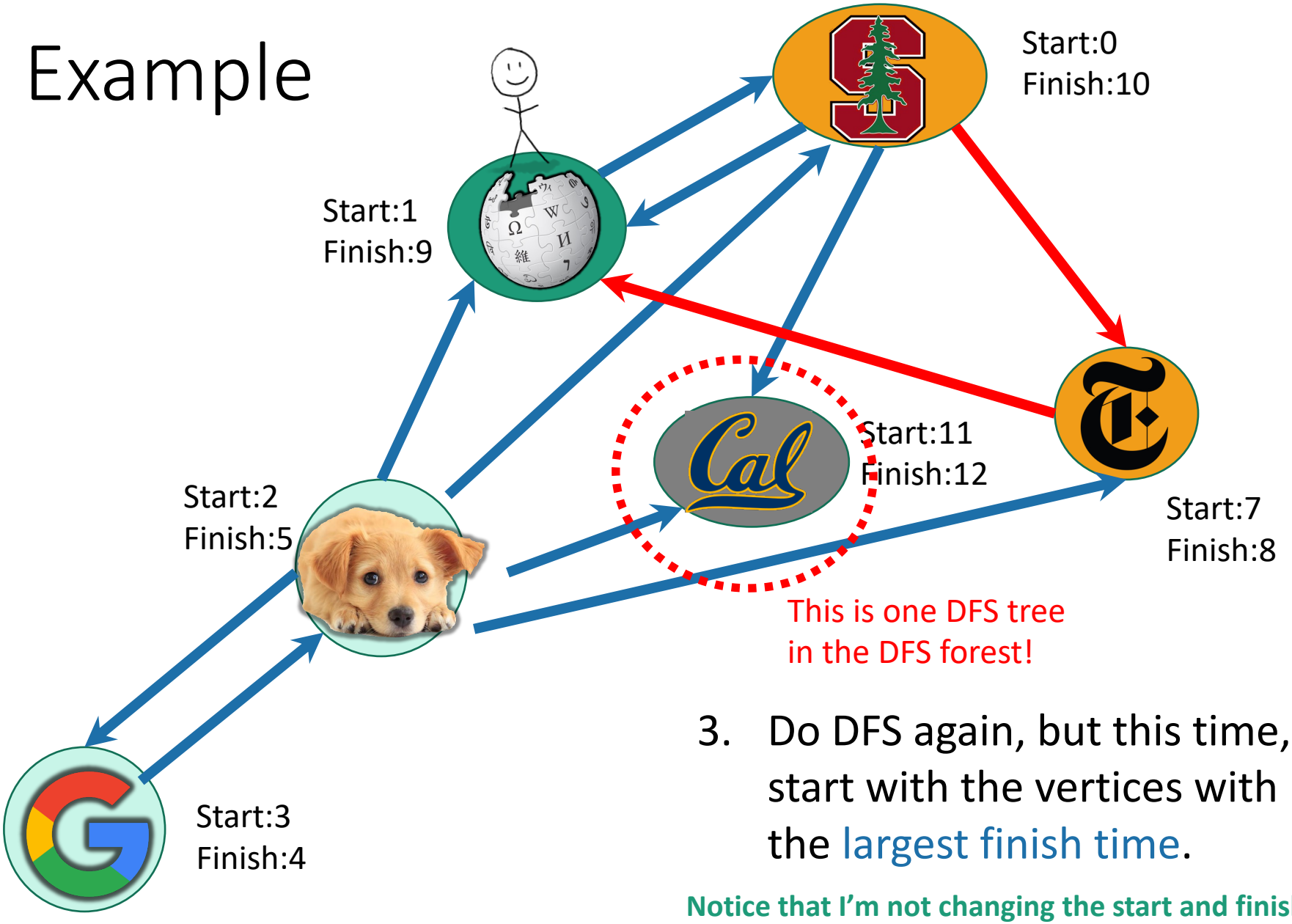

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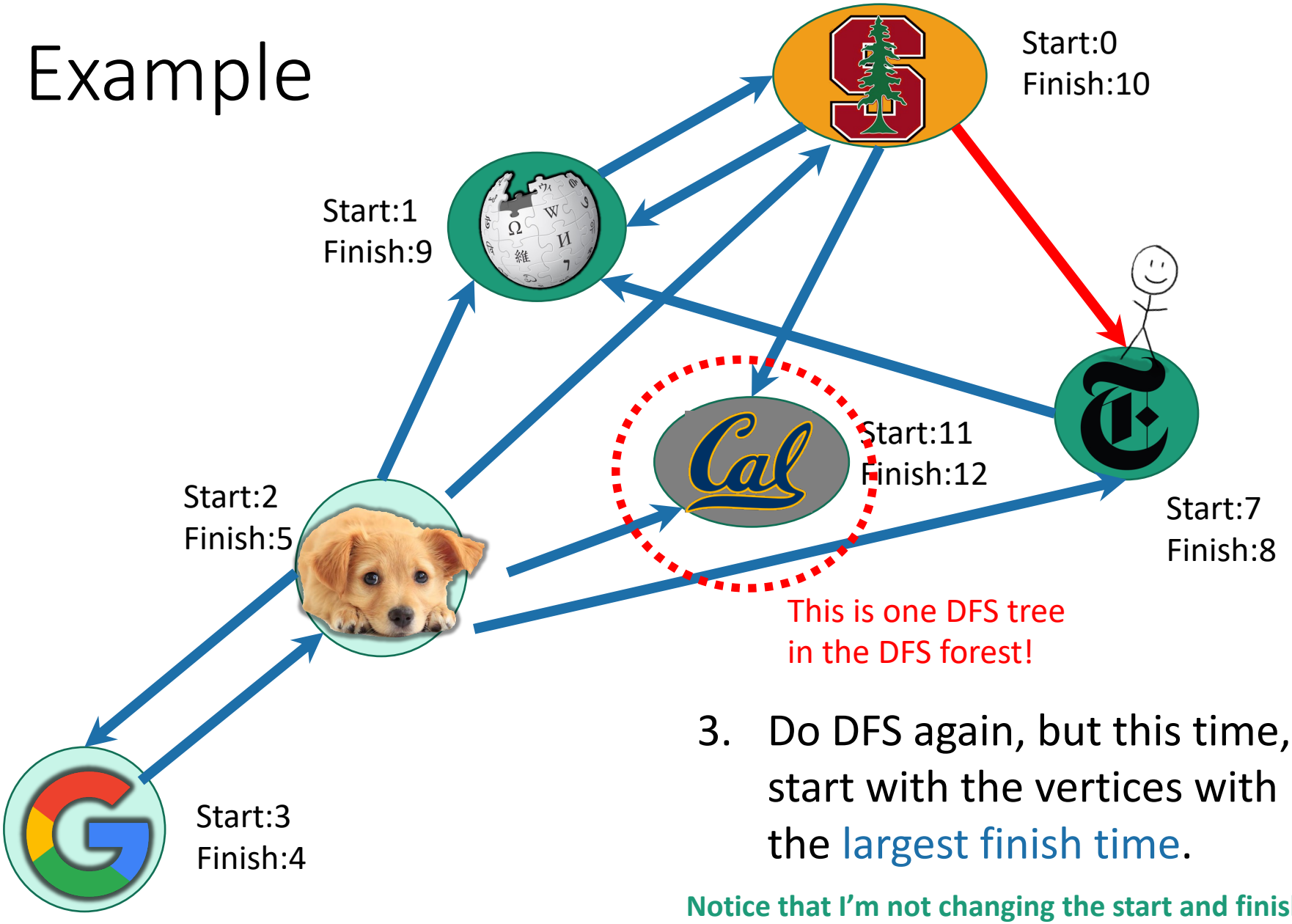

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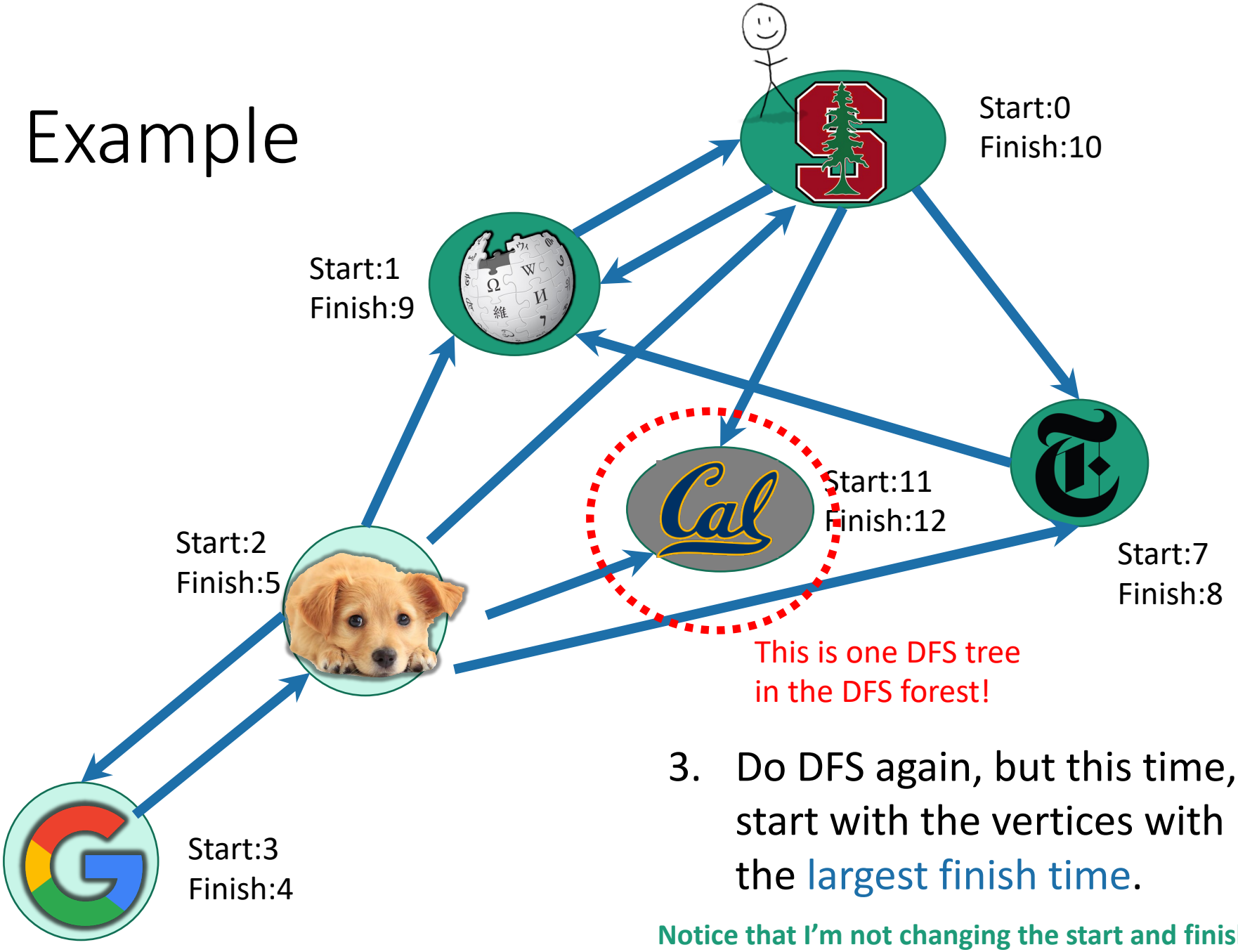

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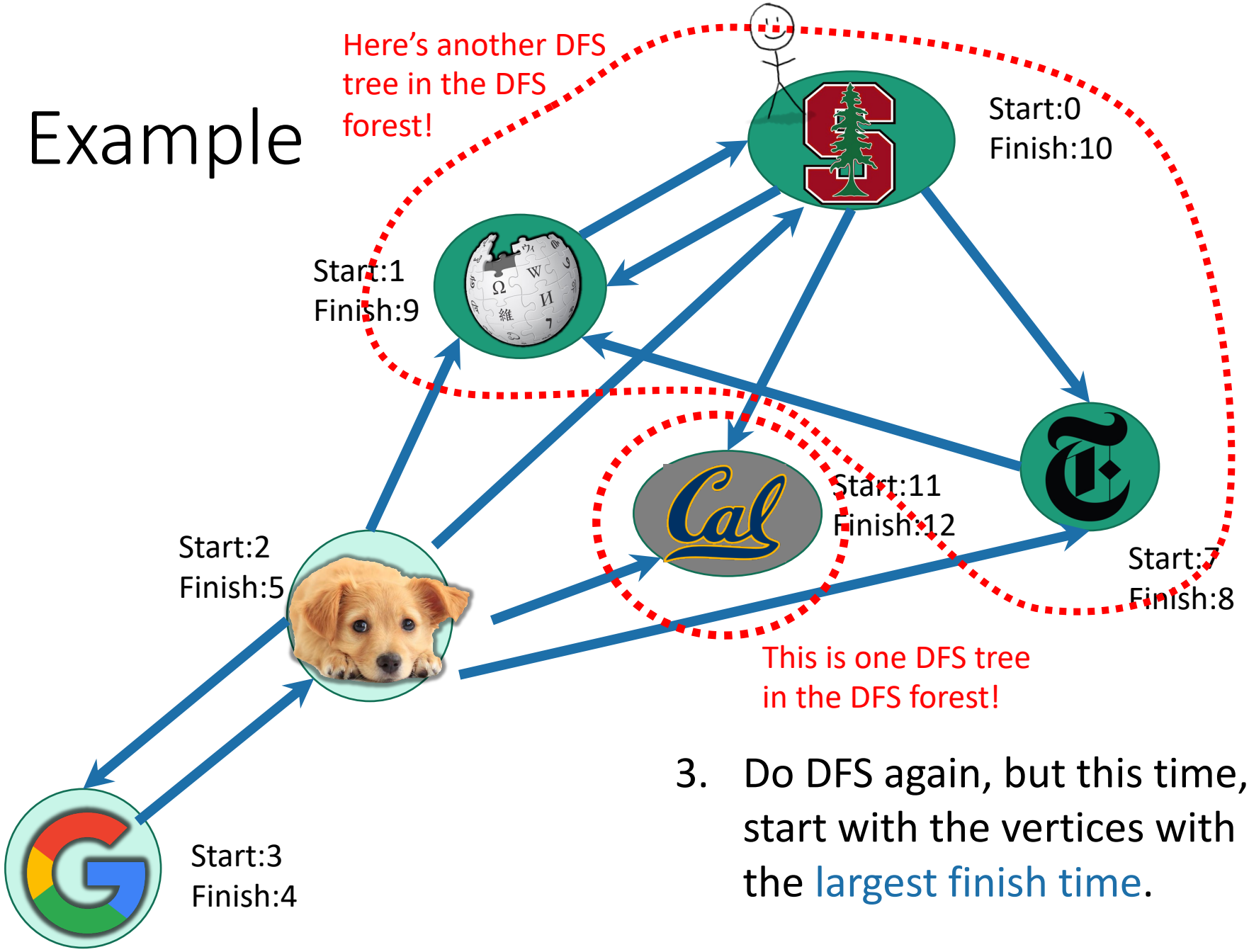

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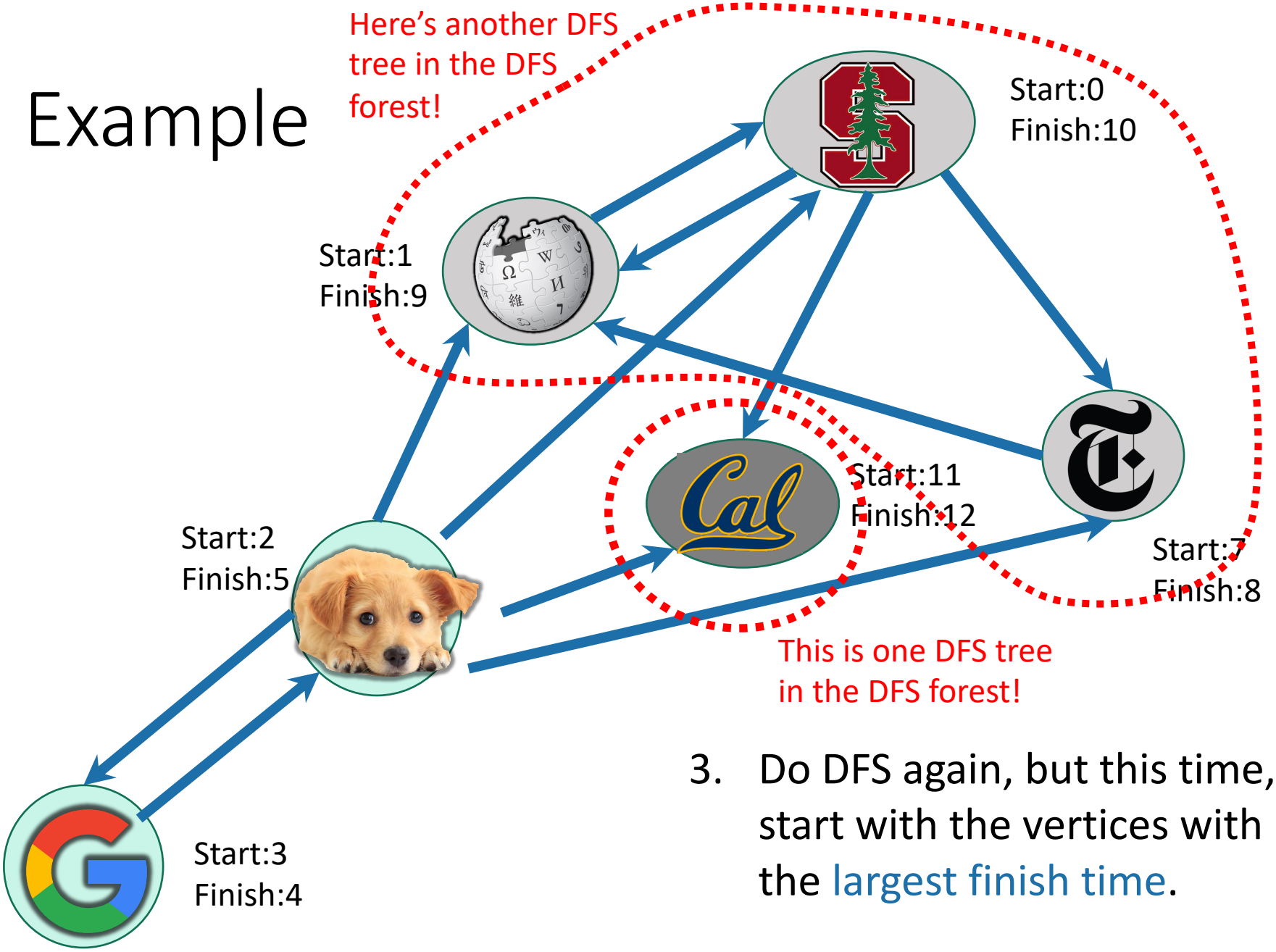

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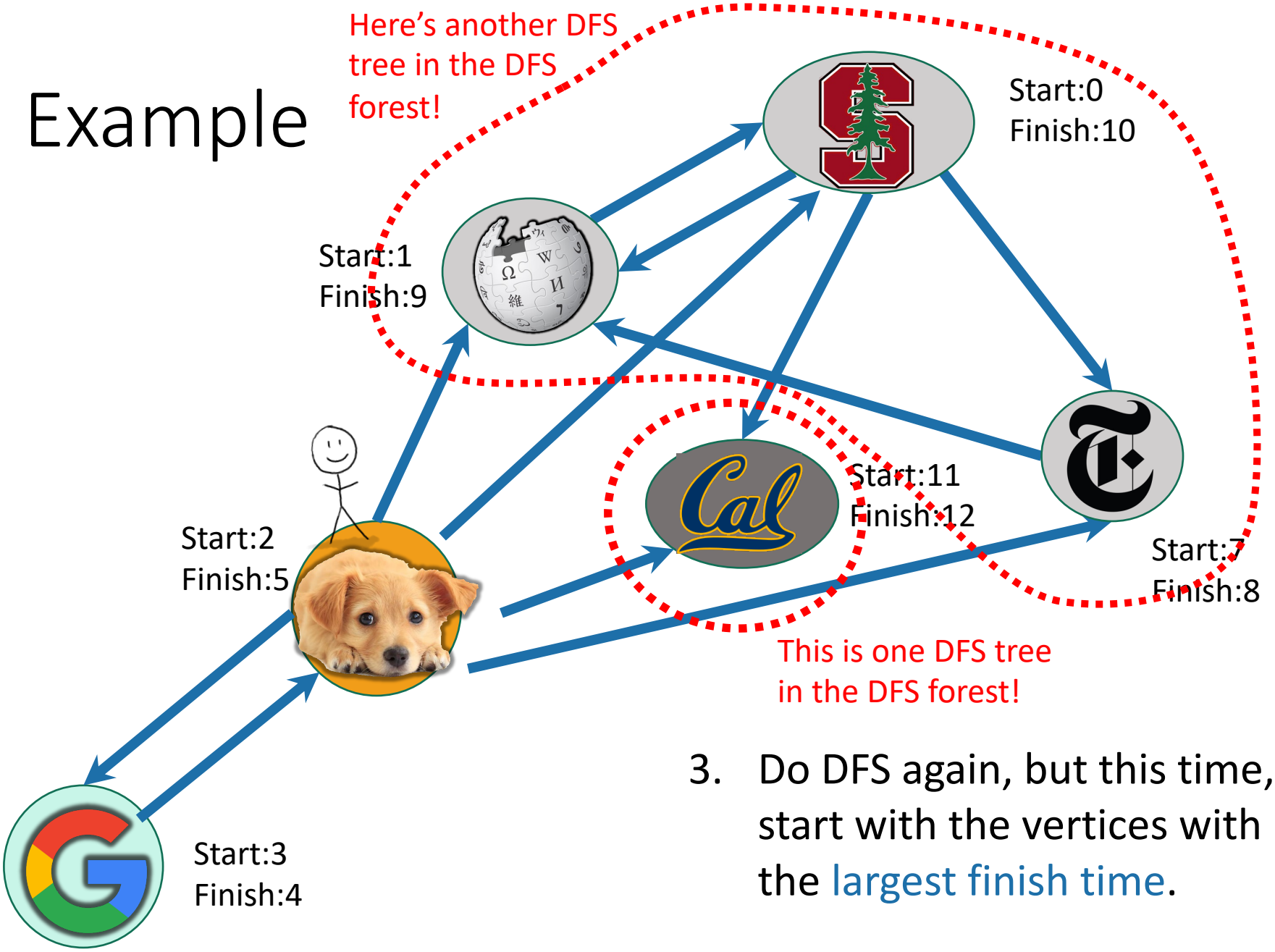

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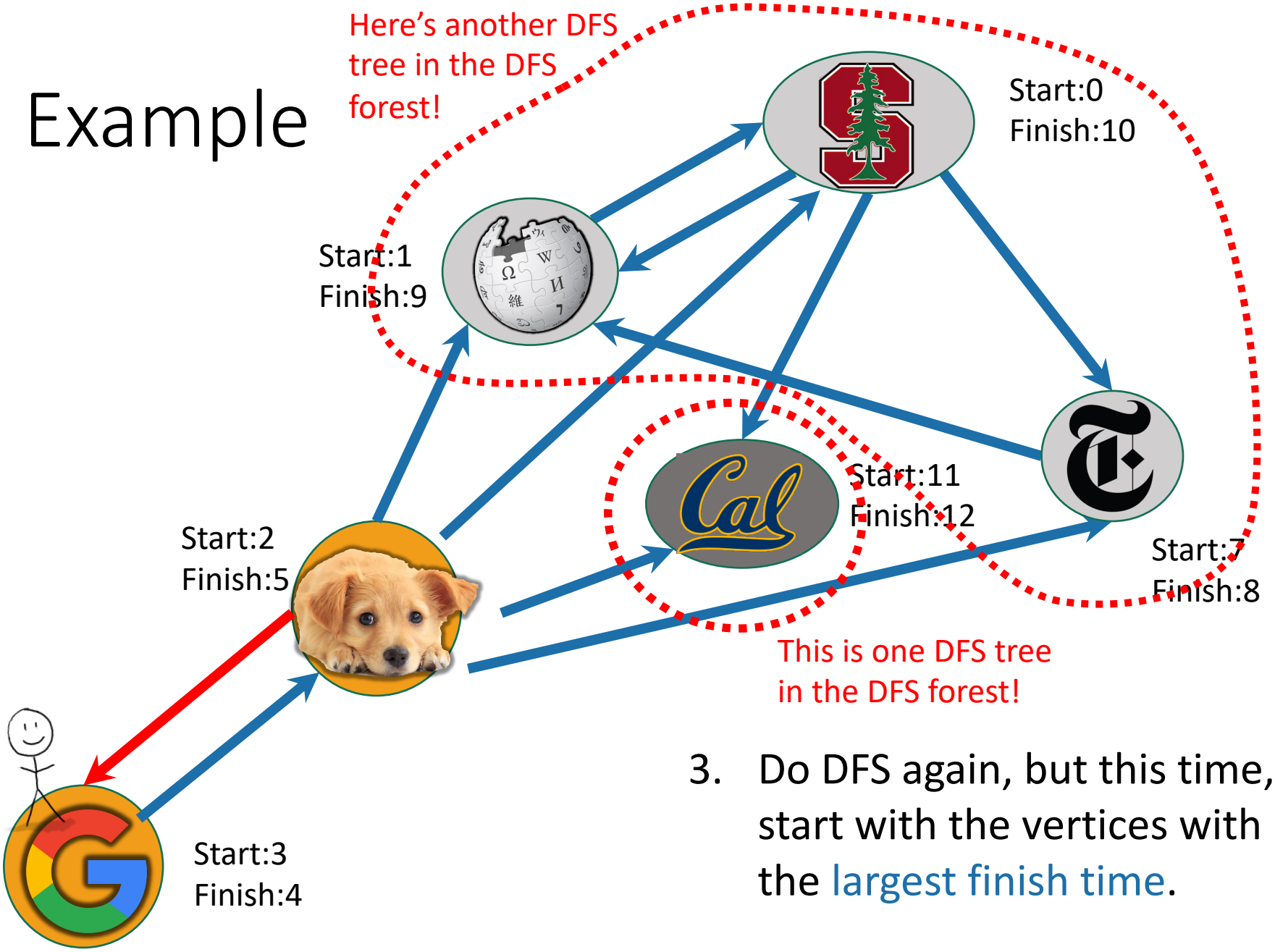

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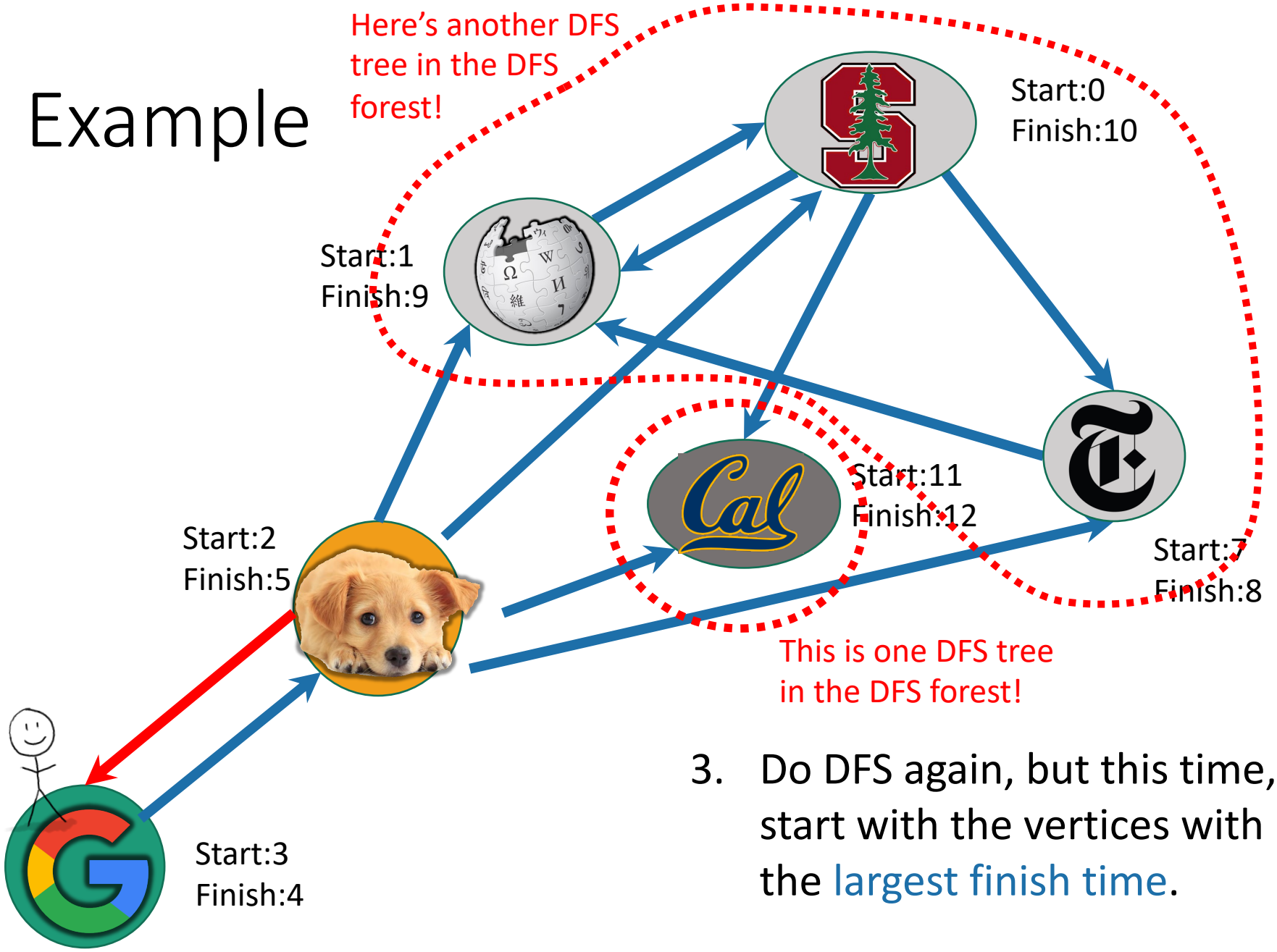

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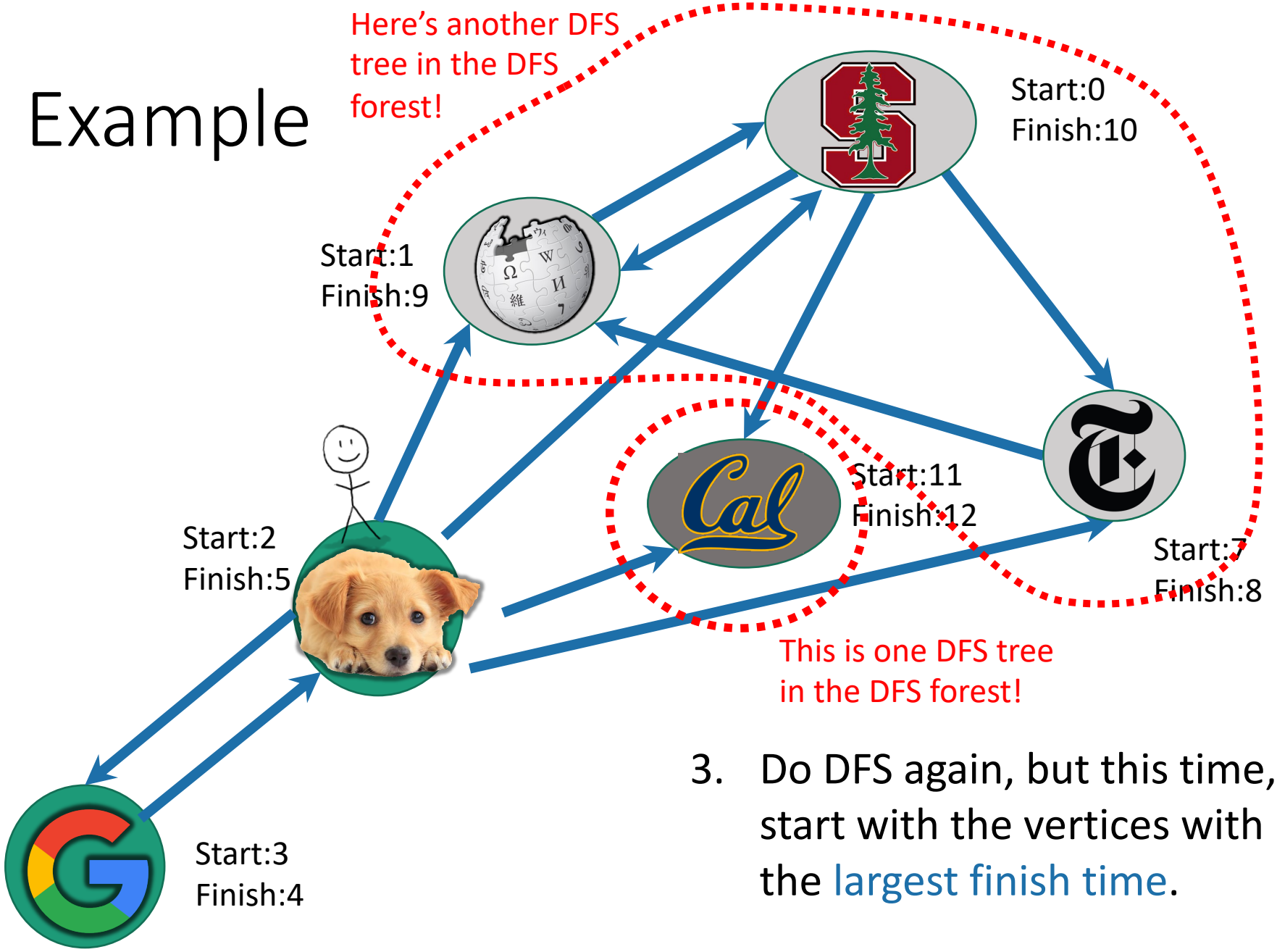

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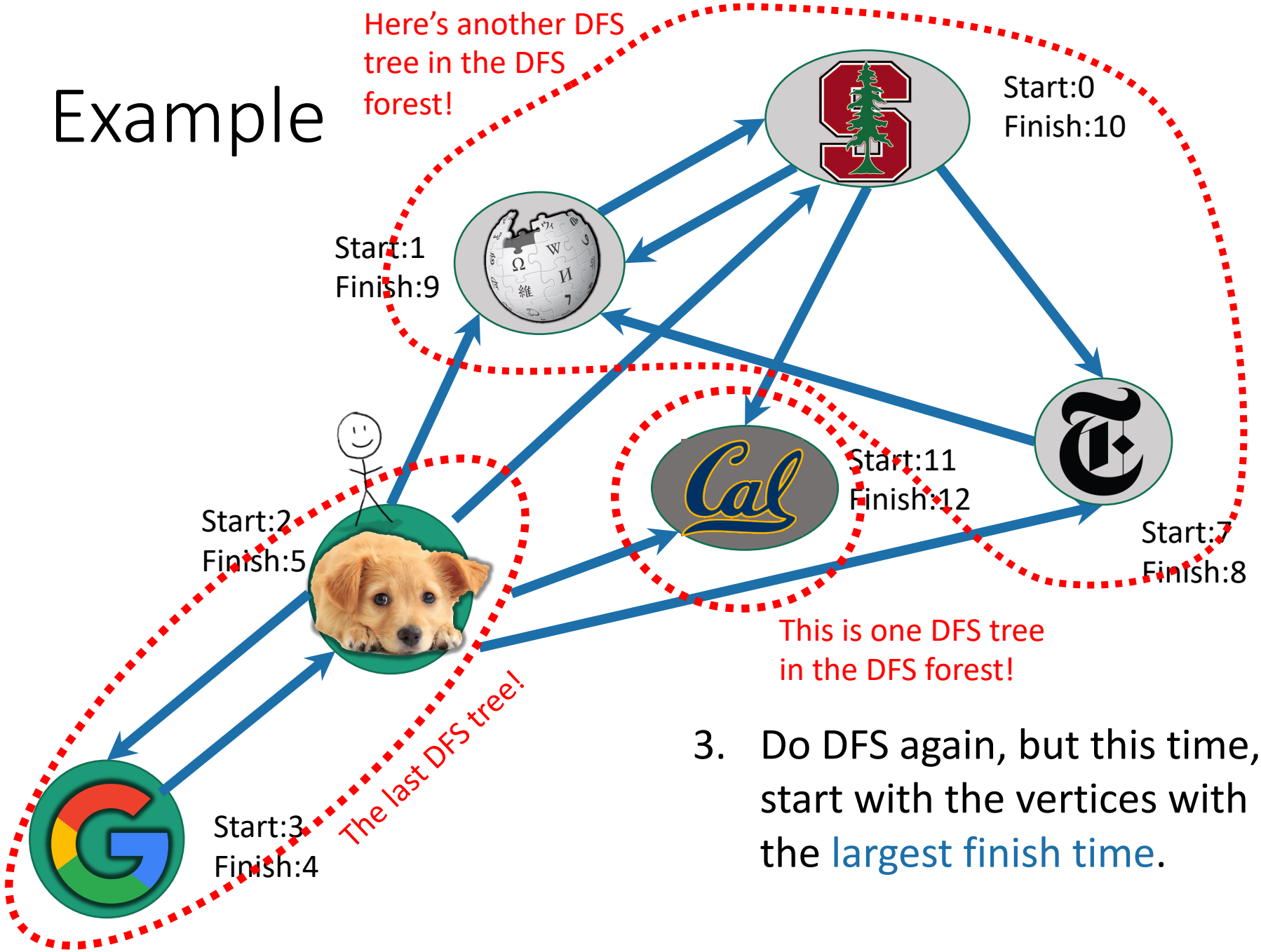

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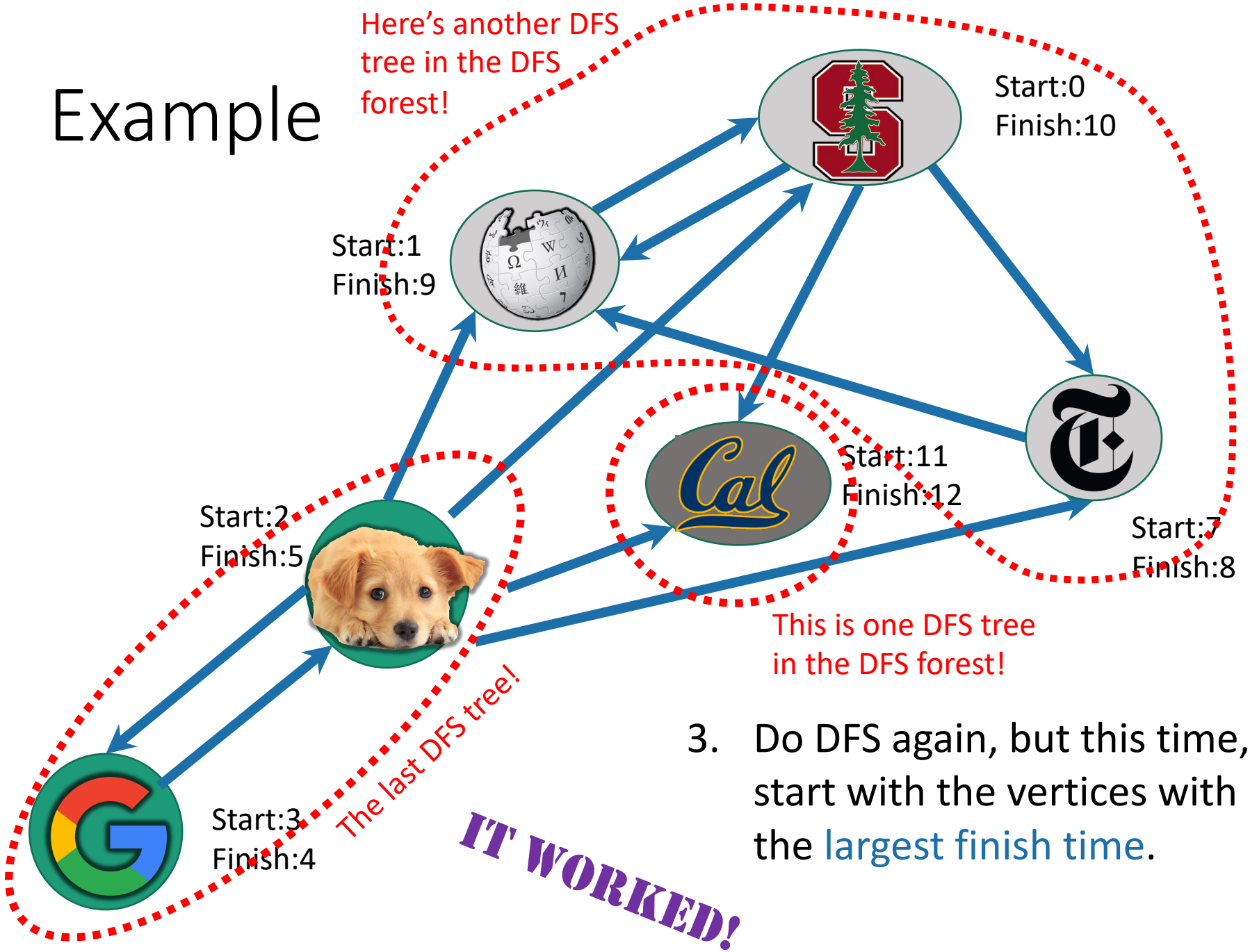

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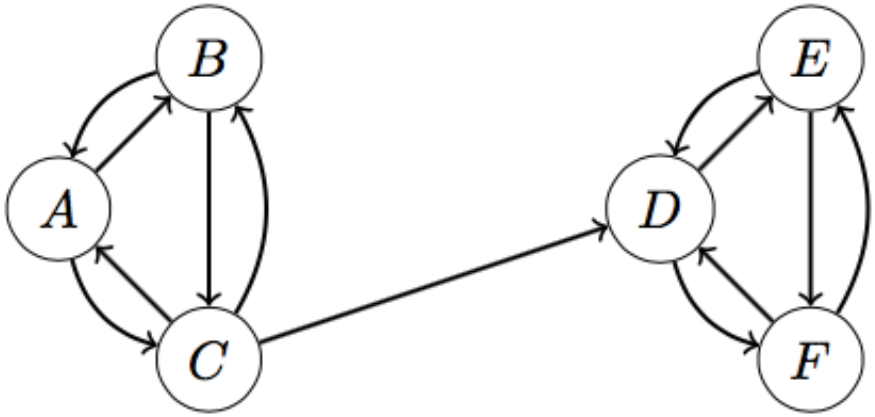
B
A
C
D
E
F

Punchline:
we can find SCCs in time O(n + m)

`Algorithm:`

- Do DFS to create a **DFS forest**.
  - Choose starting vertices in any order.
  - Keep track of finishing times.
- Reverse all the edges in the graph.
- Do DFS again to create **another DFS forest**.
  - This time, order the nodes in the reverse order of the finishing times that they had from the first DFS run.
- The SCCs are the different trees in the **second DFS forest.**

Recap

- Depth First Search reveals a very useful structure!
  - It can be used to do **Topological Sorting** in time O(n+m)
  - It can also find **Strongly Connected Components** in time O(n + m)
  - This was pretty non-trivial.