Module – Dynamic Programming

Modified/Inspired from Stanford's CS161 by Nayyar Zaidi

From Mod3b

- Dynamic programming is an algorithm design paradigm.
- Basic idea:
  - Identify **optimal sub-structure**
    - Optimum to the big problem is built out of optima of small sub-problems
  - Take advantage of **overlapping sub-problems**
    - Only solve each sub-problem once, then use it again and again
  - Keep track of the solutions to sub-problems in a table as you build to the final solution.

Today

- Examples of dynamic programming:
    1. Longest common subsequence
    2. Knapsack problem
        - Two versions!

Longest Common Subsequence

- How similar are these two species?

DNA:
AGCCCTAAGGGCTACCTAGCTT

DNA:
GACAGCCTACAAGCGTTAGCTTG

Longest Common Subsequence

- How similar are these two species?

DNA:
AGCCCTAAGGGCTACCTAGCTT

DNA:
GACAGCCTACAAGCGTTAGCTTG

- Pretty similar, their DNA has a long common subsequence:

AGCCTAAGCTTAGCTT

Longest Common Subsequence

- Subsequence:
  - BDFH is a **subsequence** of ABCDEFGH
- If X and Y are sequences, a **common subsequence** is a sequence which is a subsequence of both.
  - BDFH is a **common subsequence** of ABCDEFGH and of ABDFGHI
- A **longest common subsequence**…
  - …is a common subsequence that is longest.
  - The **longest common subsequence** of ABCDEFGH and ABDFGHI is ABDFGH.

We sometimes want to find these

- Applications in bioinformatics

- The unix command `diff`
- Merging in version control
  - svn, git, etc…

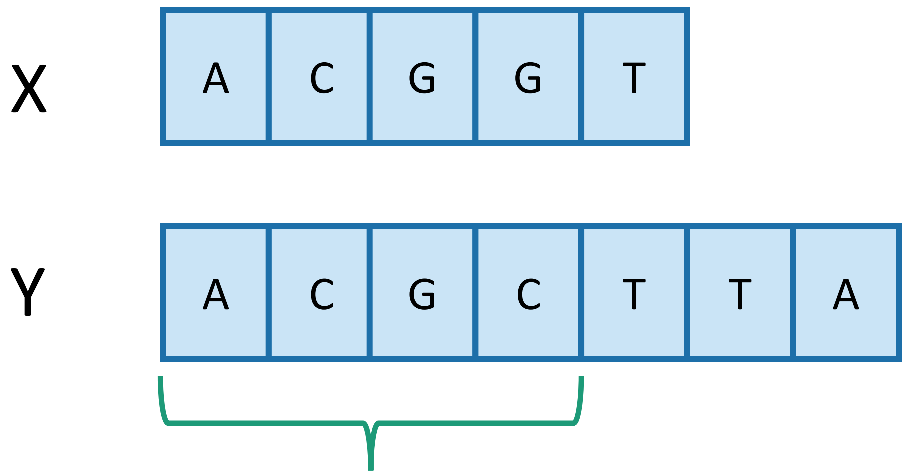
```
DN0a22a660:~ mary$ cat file1
A
B
C
D
E
F
G
H
DN0a22a660:~ mary$ cat file2
A
B
D
F
G
H
I
DN0a22a660:~ mary$ diff file1 file2
3d2
< C
5d3
< E
8a7
> I
DN0a22a660:~ mary$
```

Recipe for applying Dynamic Programming

- **Step 1:** Identify optimal substructure.
- **Step 2:** Find a recursive formulation for the length of the longest common subsequence.
- **Step 3:** Use dynamic programming to find the length of the longest common subsequence.
- **Step 4:** If needed, keep track of some additional info so that the algorithm from Step 3 can find the actual LCS.
- **Step 5:** If needed, code this up like a reasonable person.

Step 1: Optimal substructure

Prefixes:



Notation: denote this prefix **ACGC** by Y_4

- Our sub-problems will be finding LCS's of prefixes to X and Y.
- Let C[i,j] = length_of_LCS(X_i, Y_j)

Optimal substructure ctd.

- Subproblem:
  - finding LCS's of prefixes of X and Y.
- Why is this a good choice?
  - There's some relationship between LCS's of prefixes and LCS's of the whole things.
  - These subproblems overlap a lot.

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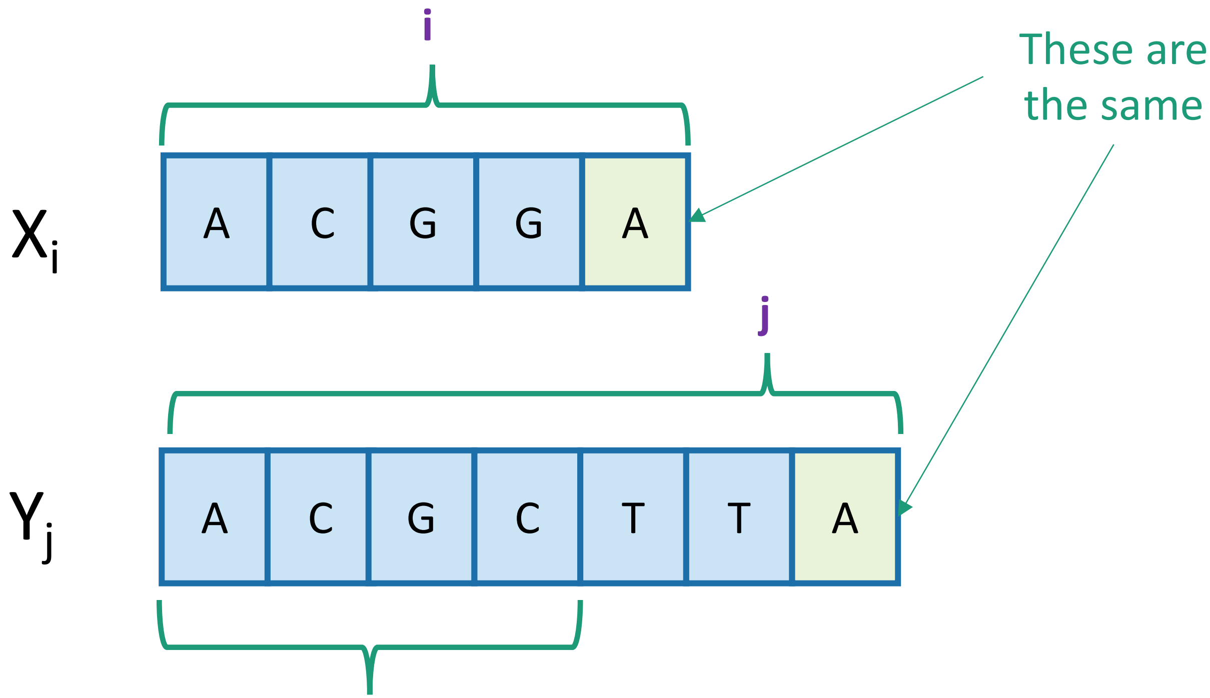
Case 1: X[i] = Y[j]

Two cases

- Our sub-problems will be finding LCS's of prefixes to X and Y.
- Let C[i,j] = length_of_LCS(X_i, Y_j)



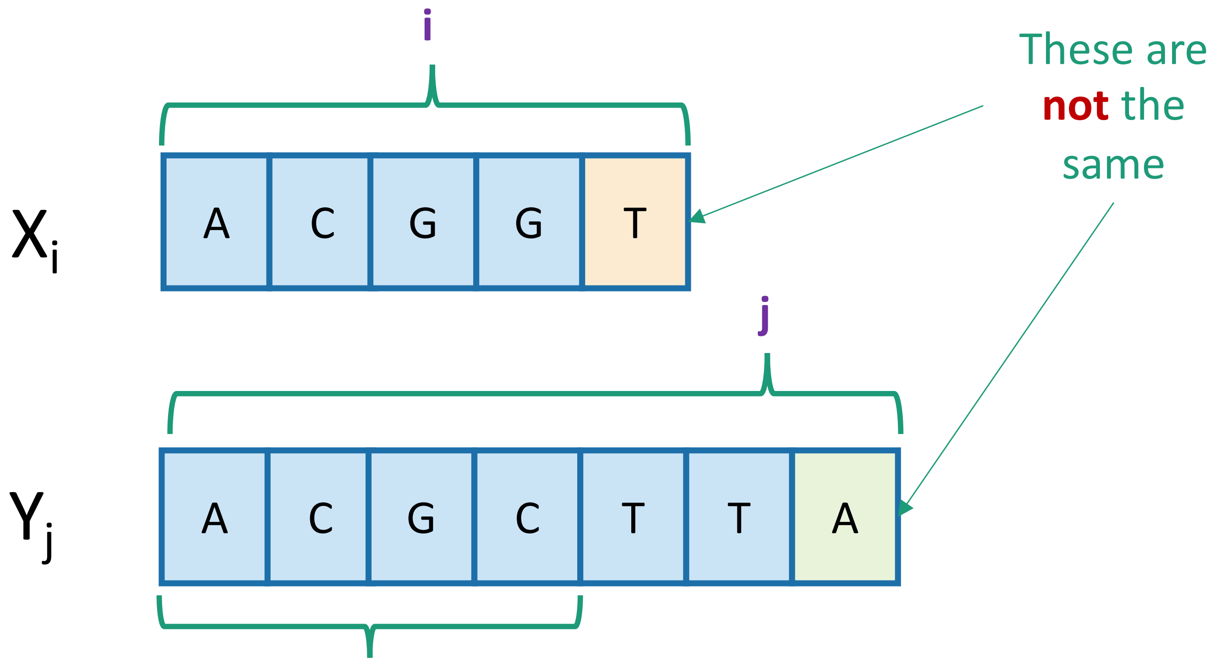
Notation: denote this prefix **ACGC** by Y_4

- Then C[i,j] = 1 + C[i-1,j-1].
  - because LCS(X_i,Y_j) = LCS(X_{i-1},Y_{j-1}) followed by $\boxed{A}$

Case 2: X[i] != Y[j]

Two cases

- Our sub-problems will be finding LCS's of prefixes to X and Y.
- Let C[i,j] = length_of_LCS(X_i, Y_j)



Notation: denote this prefix **ACGC** by Y_4

- Then C[i,j] = max{ C[i-1,j], C[i,j-1] }.
  - either LCS(X_i,Y_j) = LCS(X_{i-1},Y_j) and $\boxed{T}$ is not involved,
  - or LCS(X_i,Y_j) = LCS(X_i,Y_{j-1}) and $\boxed{A}$ is not involved,

Recursive formulation
of the optimal solution

X_0 | |

Y_j | A | C | G | C | T | T | A |

Case 0

- $$C[i,j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ C[i-1, j-1] + 1 & \text{if } X[i] = Y[j] \text{ and } i, j > 0 \\ \max\{ C[i, j-1], C[i-1, j]\} & \text{if } X[i] \neq Y[j] \text{ and } i, j > 0 \end{cases}$$

Case 1

X_i | A | C | G | G | A |

Y_j | A | C | G | C | T | T | A |

Case 2

X_i | A | C | G | G | T |

Y_j | A | C | G | C | T | T | A |

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LCS DP OMG BBQ

- **LCS**(X, Y):
  - C[i,0] = C[0,j] = 0 for all i = 1,…,m, j=1,…n.
  - **For** i = 1,…,m and j = 1,…,n:
    - **If** X[i] = Y[j]:
      - C[i,j] = C[i-1,j-1] + 1
    - **Else:**
      - C[i,j] = max{ C[i,j-1], C[i-1,j] }

Running time: O(nm)

$$C[i,j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ C[i-1, j-1] + 1 & \text{if } X[i] = Y[j] \text{ and } i, j > 0 \\ \max\{ C[i, j-1], C[i-1, j] \} & \text{if } X[i] \neq Y[j] \text{ and } i, j > 0 \end{cases}$$

Example

X: | A | C | G | G | A |

Y: | A | C | T | G |

		A	C	T	G
	0	0	0	0	0
A	0				
C	0				
G	0				
G	0				
A	0				

$$C[i,j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ C[i-1, j-1] + 1 & \text{if } X[i] = Y[j] \text{ and } i,j > 0 \\ \max\{ C[i, j-1], C[i-1, j]\} & \text{if } X[i] \neq Y[j] \text{ and } i,j > 0 \end{cases}$$

Example

X	A	C	G	G	A

Y	A	C	T	G

		Y: A	C	T	G
	0	0	0	0	0
X: A	0	1	1	1	1
C	0	1	2	2	2
G	0	1	2	2	3
G	0	1	2	2	3
A	0	1	2	2	3

So the LCM of X and Y has length 3.

$$C[i,j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ C[i-1, j-1] + 1 & \text{if } X[i] = Y[j] \text{ and } i,j > 0 \\ \max\{ C[i, j-1], C[i-1, j]\} & \text{if } X[i] \neq Y[j] \text{ and } i,j > 0 \end{cases}$$

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Example

X	A	C	G	G	A

Y	A	C	T	G

X \ Y		A	C	T	G
	0	0	0	0	0
A	0				
C	0				
G	0				
G	0				
A	0				

$$C[i,j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ C[i-1, j-1] + 1 & \text{if } X[i] = Y[j] \text{ and } i,j > 0 \\ \max\{\, C[i, j-1], C[i-1, j]\} & \text{if } X[i] \neq Y[j] \text{ and } i,j > 0 \end{cases}$$

Example

X	A	C	G	G	A
Y	A	C	T	G	

Y

X		A	C	T	G
	0	0	0	0	0
A	0	1	1	1	1
C	0	1	2	2	2
G	0	1	2	2	3
G	0	1	2	2	3
A	0	1	2	2	3

$$C[i,j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ C[i-1, j-1] + 1 & \text{if } X[i] = Y[j] \text{ and } i,j > 0 \\ \max\{\, C[i, j-1], C[i-1, j]\} & \text{if } X[i] \neq Y[j] \text{ and } i,j > 0 \end{cases}$$

Example

X	A	C	G	G	A

Y	A	C	T	G

Y

X		A	C	T	G
	0	0	0	0	0
A	0	1	1	1	1
C	0	1	2	2	2
G	0	1	2	2	3
G	0	1	2	2	3
A	0	1	2	2	3

- Once we've filled this in, we can work backwards.

$$C[i,j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ C[i-1, j-1] + 1 & \text{if } X[i] = Y[j] \text{ and } i,j > 0 \\ \max\{ C[i, j-1], C[i-1, j]\} & \text{if } X[i] \neq Y[j] \text{ and } i,j > 0 \end{cases}$$

Example

X	A	C	G	G	A

Y	A	C	T	G

Y

		A	C	T	G
	0	0	0	0	0
A	0	1	1	1	1
C	0	1	2	2	2
X G	0	1	2	2	3
G	0	1	2	2	3
A	0	1	2	2	3

- Once we've filled this in, we can work backwards.

That 3 must have come from the 3 above it.

$$C[i,j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ C[i-1, j-1] + 1 & \text{if } X[i] = Y[j] \text{ and } i,j > 0 \\ \max\{\, C[i, j-1], C[i-1, j]\} & \text{if } X[i] \neq Y[j] \text{ and } i,j > 0 \end{cases}$$

Example

X	A	C	G	G	A

Y	A	C	T	G

Y

X \ Y		A	C	T	G
	0	0	0	0	0
A	0	1	1	1	1
C	0	1	2	2	2
G	0	1	2	2	3
G	0	1	2	2	3
A	0	1	2	2	3

- Once we've filled this in, we can work backwards.
- A diagonal jump means that we found an element of the LCS!

This 3 came from that 2 – we found a match!

$$C[i,j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ C[i-1, j-1] + 1 & \text{if } X[i] = Y[j] \text{ and } i, j > 0 \\ \max\{ C[i, j-1], C[i-1, j]\} & \text{if } X[i] \neq Y[j] \text{ and } i, j > 0 \end{cases}$$

Example

X	A	C	G	G	A

Y	A	C	T	G

Y

X		A	C	T	G
	0	0	0	0	0
A	0	1	1	1	1
C	0	1	2	2	2
G	0	1	2	2	3
G	0	1	2	2	3
A	0	1	2	2	3

- Once we've filled this in, we can work backwards.
- A diagonal jump means that we found an element of the LCS!

That 2 may as well have come from this other 2.

G

$$C[i,j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ C[i-1,j-1]+1 & \text{if } X[i] = Y[j] \text{ and } i,j > 0 \\ \max\{\, C[i,j-1], C[i-1,j]\} & \text{if } X[i] \neq Y[j] \text{ and } i,j > 0 \end{cases}$$

Example

X: | A | C | G | G | A |

Y: | A | C | T | G |

X \ Y		A	C	T	G
	0	0	0	0	0
A	0	1	1	1	1
C	0	1	2	2	2
G	0	1	2	2	3
G	0	1	2	2	3
A	0	1	2	2	3

- Once we've filled this in, we can work backwards.
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G

$$C[i,j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ C[i-1, j-1] + 1 & \text{if } X[i] = Y[j] \text{ and } i, j > 0 \\ \max\{ C[i, j-1], C[i-1, j] \} & \text{if } X[i] \neq Y[j] \text{ and } i, j > 0 \end{cases}$$

Example

X	A	C	G	G	A

Y	A	C	T	G

		A	C	T	G
	0	0	0	0	0
A	0	1	1	1	1
C	0	1	2	2	2
G	0	1	2	2	3
G	0	1	2	2	3
A	0	1	2	2	3

- Once we've filled this in, we can work backwards.
- A diagonal jump means that we found an element of the LCS!

C G

$$C[i,j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ C[i-1, j-1] + 1 & \text{if } X[i] = Y[j] \text{ and } i,j > 0 \\ \max\{ C[i, j-1], C[i-1, j] \} & \text{if } X[i] \neq Y[j] \text{ and } i,j > 0 \end{cases}$$

Example

X: | A | C | G | G | A |

Y: | A | C | T | G |

		Y: A	C	T	G
	0	0	0	0	0
X: A	0	1	1	1	1
C	0	1	2	2	2
G	0	1	2	2	3
G	0	1	2	2	3
A	0	1	2	2	3

- Once we've filled this in, we can work backwards.
- A diagonal jump means that we found an element of the LCS!

| A | C | G |

This is the LCS!

$$C[i,j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ C[i-1,j-1]+1 & \text{if } X[i] = Y[j] \text{ and } i,j > 0 \\ \max\{ C[i,j-1], C[i-1,j]\} & \text{if } X[i] \neq Y[j] \text{ and } i,j > 0 \end{cases}$$

This gives an algorithm to recover the actual LCS not just its length

- See lecture notes for pseudocode
- It runs in time O(n + m)
  - We walk up and left in an n-by-m array
  - We can only do that for n + m steps.
- So actually recovering the LCS from the table is much faster than building the table was.
- We can find LCS(X,Y) in time O(mn).

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This pseudocode actually isn't so bad

- If we are only interested in the length of the LCS:
  - Since we go across the table one-row-at-a-time, we can only keep two rows if we want.
  - If we want to recover the LCS, we need to keep the whole table.

- Can we do better than O(mn) time?
  - A bit better.
    - By a log factor or so.
  - But doing much better (polynomially better) is an open problem!
    - If you can do it let me know :D

What have we learned?

- We can find LCS(X,Y) in time O(nm)
  - if |Y|=n, |X|=m
- We went through the steps of coming up with a dynamic programming algorithm.
  - We kept a 2-dimensional table, breaking down the problem by decrementing the length of X and Y.

Example 2: Knapsack Problem

- We have n items with weights and values:

Item:					
Weight:	6	2	4	3	11
Value:	20	8	14	13	35

- And we have a knapsack:
  - it can only carry so much weight:

Capacity: 10

Capacity: 10

Item:					
Weight:	6	2	4	3	11
Value:	20	8	14	13	35

- Unbounded Knapsack:
  - Suppose I have infinite copies of all of the items.
  - What's the most valuable way to fill the knapsack?

Total weight: 10
Total value: 42

- 0/1 Knapsack:
  - Suppose I have only one copy of each item.
  - What's the most valuable way to fill the knapsack?

Total weight: 9
Total value: 35

Some notation

Item:					
Weight:	w_1	w_2	w_3	…	w_n
Value:	v_1	v_2	v_3		v_n

Capacity: W

Recipe for applying Dynamic Programming

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Optimal substructure

- Sub-problems:
  - Unbounded Knapsack with a smaller knapsack.

First solve the problem for small knapsacks

Then larger knapsacks

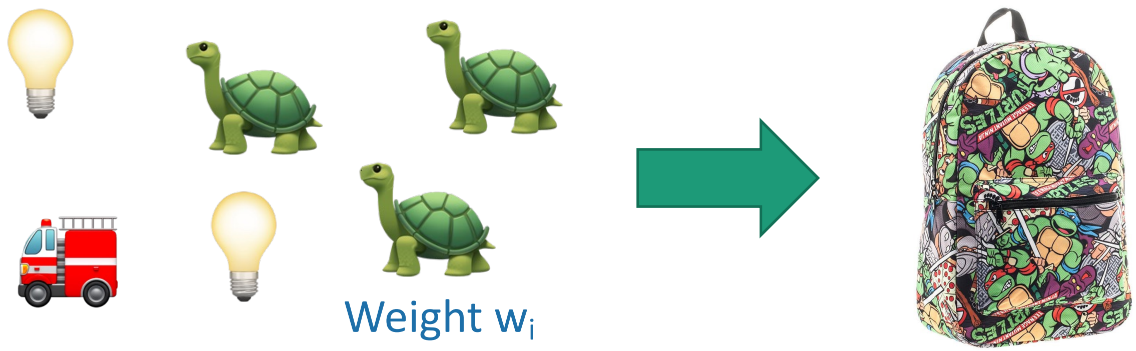
Then larger knapsacks

Optimal substructure

item i

- Suppose this is an optimal solution for capacity x:

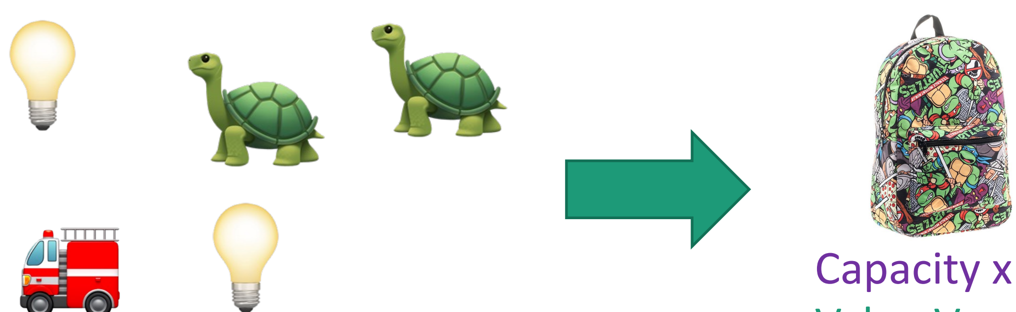
Say that the optimal solution contains at least one copy of item i.



Weight w_i
Value v_i

Capacity x
Value V

- Then this optimal for capacity x - w_i:



Capacity x – w_i
Value V - v_i

If I could do better than the second solution, then adding a turtle to that improvement would improve the first solution.

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Recursive relationship

- Let K[x] be the optimal value for capacity x.

$K[x] = \max_i \{$  $+$  $\}$

The maximum is over all i so that $w_i \leq x$.

Optimal way to fill the smaller knapsack

The value of item i.

$$K[x] = \max_i \{ K[x - w_i] + v_i \}$$

- (And K[x] = 0 if the maximum is empty).
  - That is, there are no i so that $w_i \leq x$

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Let’s write a bottom-up DP algorithm

- UnboundedKnapsack(W, n, weights, values):
  - K[0] = 0
  - **for** x = 1, …, W:
    - K[x] = 0
    - **for** i = 1, …, n:
      - **if** $w_i \leq x$:
        - $K[x] = \max\{ K[x], K[x - w_i] + v_i \}$
  - **return** K[W]

Running time: O(nW)

$K[x] = \max_i \{$  $+$  $\}$

$= \max_i \{ K[x - w_i] + v_i \}$

Why does this work?

Because our recursive relationship makes sense.

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  - **for** x = 1, …, W:
    - K[x] = 0
    - **for** i = 1, …, n:
      - **if** $w_i \leq x$:
        - $K[x] = \max\{ K[x], K[x - w_i] + v_i \}$
  - **return** K[W]

K[x] = max$_i$ {  +  }

= max$_i$ { K[x – w$_i$] + v$_i$ }

Let’s write a bottom-up DP algorithm

- UnboundedKnapsack(W, n, weights, values):
  - K[0] = 0
  - ITEMS[0] = Ø
  - **for** x = 1, …, W:
    - K[x] = 0
    - **for** i = 1, …, n:
      - **if** $w_i \leq x$:
        - $K[x] = \max\{\, K[x], K[x - w_i] + v_i \,\}$
        - If K[x] was updated:
          - ITEMS[x] = ITEMS[$x - w_i$] ∪ { item i }
  - **return** ITEMS[W]

$K[x] = \max_i \{$  +  $\}$

$= \max_i \{ K[x - w_i] + v_i \}$

Example

	0	1	2	3	4
K	0				
ITEMS					

- UnboundedKnapsack(W, n, weights, values):
  - K[0] = 0
  - ITEMS[0] = ∅
  - **for** x = 1, …, W:
    - K[x] = 0
    - **for** i = 1, …, n:
      - **if** $w_i \leq x$:
        - $K[x] = \max\{ K[x], K[x - w_i] + v_i \}$
        - If K[x] was updated:
          - ITEMS[x] = ITEMS[x – w_i] ∪ { item i }
  - **return** ITEMS[W]

Item:			
Weight:	1	2	3
Value:	1	4	6

Capacity: 4

Example

	0	1	2	3	4
K	0	1			
ITEMS					

ITEMS[1] = ITEMS[0] + 

- UnboundedKnapsack(W, n, weights, values):
  - K[0] = 0
  - ITEMS[0] = Ø
  - **for** x = 1, …, W:
    - K[x] = 0
    - **for** i = 1, …, n:
      - **if** $w_i \leq x$:
        - $K[x] = \max\{\, K[x], K[x - w_i] + v_i \,\}$
        - If K[x] was updated:
          - ITEMS[x] = ITEMS[x – w_i] ∪ { item i }
  - **return** ITEMS[W]

Item:			
Weight:	1	2	3
Value:	1	4	6

Capacity: 4

Example

- UnboundedKnapsack(W, n, weights, values):
  - K[0] = 0
  - ITEMS[0] = Ø
  - **for** x = 1, …, W:
    - K[x] = 0
    - **for** i = 1, …, n:
      - **if** $w_i \leq x$:
        - $K[x] = \max\{\, K[x], K[x - w_i] + v_i \,\}$
        - If K[x] was updated:
          - ITEMS[x] = ITEMS[x – w_i] ∪ { item i }
  - **return** ITEMS[W]

	0	1	2	3	4
K	0	1	2		
ITEMS			 		

ITEMS[2] = ITEMS[1] + 

Item:			
Weight:	1	2	3
Value:	1	4	6

Capacity: 4

Example

- UnboundedKnapsack(W, n, weights, values):
  - K[0] = 0
  - ITEMS[0] = Ø
  - **for** x = 1, …, W:
    - K[x] = 0
    - **for** i = 1, …, n:
      - **if** $w_i \leq x$:
        - $K[x] = \max\{\, K[x], K[x \; - w_i] + v_i \,\}$
        - If K[x] was updated:
          - ITEMS[x] = ITEMS[x – w_i] ∪ { item i }
  - **return** ITEMS[W]

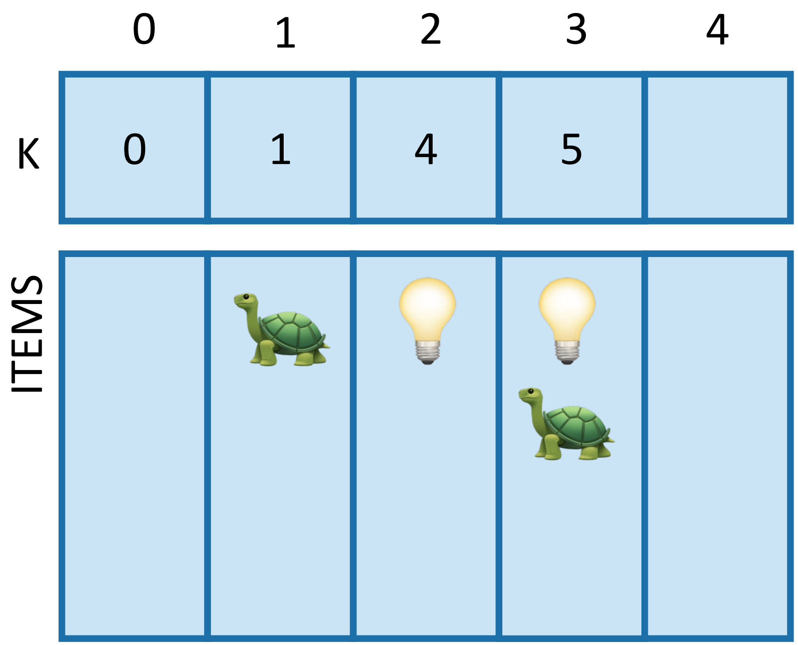
	0	1	2	3	4
K	0	1	4		
ITEMS					

ITEMS[2] = ITEMS[0] + 

Item:			
Weight:	1	2	3
Value:	1	4	6

Capacity: 4

Example



ITEMS[3] = ITEMS[2] + 

- UnboundedKnapsack(W, n, weights, values):
  - K[0] = 0
  - ITEMS[0] = Ø
  - **for** x = 1, …, W:
    - K[x] = 0
    - **for** i = 1, …, n:
      - **if** $w_i \leq x$:
        - $K[x] = \max\{\, K[x], K[x - w_i] + v_i \,\}$
        - If K[x] was updated:
          - ITEMS[x] = ITEMS[x – w_i] ∪ { item i }
  - **return** ITEMS[W]

Item: 

Weight:	1	2	3
Value:	1	4	6

Capacity: 4

Example

	0	1	2	3	4
K	0	1	4	6	
ITEMS					

ITEMS[3] = ITEMS[0] + 

- UnboundedKnapsack(W, n, weights, values):
  - K[0] = 0
  - ITEMS[0] = Ø
  - **for** x = 1, …, W:
    - K[x] = 0
    - **for** i = 1, …, n:
      - **if** $w_i \leq x$:
        - $K[x] = \max\{\, K[x], K[x - w_i] + v_i \,\}$
        - If K[x] was updated:
          - ITEMS[x] = ITEMS[x – w_i] ∪ { item i }
  - **return** ITEMS[W]

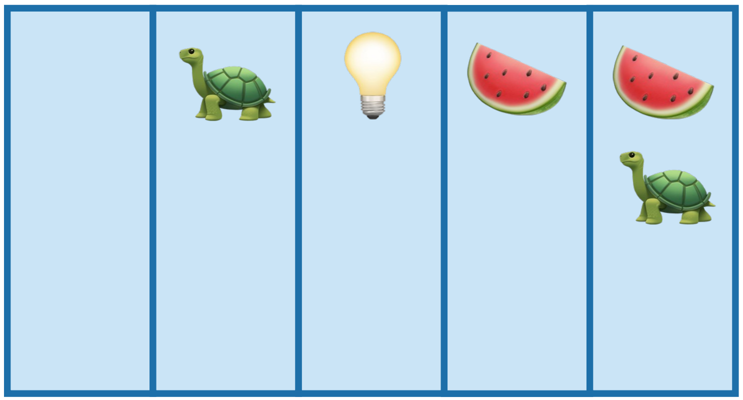
Item:			
Weight:	1	2	3
Value:	1	4	6

Capacity: 4

Example

	0	1	2	3	4
K	0	1	4	6	7

ITEMS



ITEMS[4] = ITEMS[3] + 

- UnboundedKnapsack(W, n, weights, values):
  - K[0] = 0
  - ITEMS[0] = Ø
  - **for** x = 1, …, W:
    - K[x] = 0
    - **for** i = 1, …, n:
      - **if** $w_i \leq x$:
        - $K[x] = \max\{ K[x], K[x - w_i] + v_i \}$
        - If K[x] was updated:
          - ITEMS[x] = ITEMS[x – w_i] ∪ { item i }
  - **return** ITEMS[W]

Item: 

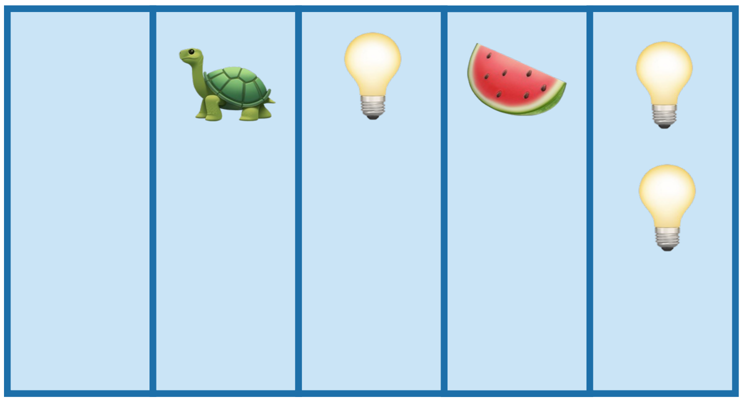
Weight:	1	2	3
Value:	1	4	6

Capacity: 4

Example

	0	1	2	3	4
K	0	1	4	6	8

ITEMS



- UnboundedKnapsack(W, n, weights, values):
    - K[0] = 0
    - ITEMS[0] = Ø
    - **for** x = 1, …, W:
        - K[x] = 0
        - **for** i = 1, …, n:
            - **if** $w_i \leq x$:
                - $K[x] = \max\{ K[x], K[x - w_i] + v_i \}$
                - If K[x] was updated:
                    - ITEMS[x] = ITEMS[x – w_i] ∪ { item i }
    - **return** ITEMS[W]

Item: 

Weight:	1	2	3
Value:	1	4	6

ITEMS[4] = ITEMS[2] + 

Capacity: 4

Recipe for applying Dynamic Programming

- **Step 1:** Identify optimal substructure.
- **Step 2:** Find a recursive formulation for the value of the optimal solution.
- **Step 3:** Use dynamic programming to find the value of the optimal solution.
- **Step 4:** If needed, keep track of some additional info so that the algorithm from Step 3 can find the actual solution.
- **Step 5:** If needed, code this up like a reasonable person.

(Pass)

What have we learned?

- We can solve unbounded knapsack in time O(nW).
    - If there are n items and our knapsack has capacity W.
- We again went through the steps to create DP solution:
    - We kept a one-dimensional table, creating smaller problems by making the knapsack smaller.

Capacity: 10

Item:	🐢	💡	🍉	🌮	🚒
Weight:	6	2	4	3	11
Value:	20	8	14	13	35

- Unbounded Knapsack:
  - Suppose I have infinite copies of all of the items.
  - What's the most valuable way to fill the knapsack?

🌮 🌮 💡 💡

Total weight: 10
Total value: 42

- 0/1 Knapsack:
  - Suppose I have only one copy of each item.
  - What's the most valuable way to fill the knapsack?

💡 🍉 🌮

Total weight: 9
Total value: 35

Recipe for applying Dynamic Programming

- **Step 1:** Identify optimal substructure.
- **Step 2:** Find a recursive formulation for the value of the optimal solution.
- **Step 3:** Use dynamic programming to find the value of the optimal solution.
- **Step 4:** If needed, keep track of some additional info so that the algorithm from Step 3 can find the actual solution.
- **Step 5:** If needed, code this up like a reasonable person.

Optimal substructure: try 1

- Sub-problems:
  - Unbounded Knapsack with a smaller knapsack.

First solve the problem for small knapsacks

Then larger knapsacks

Then larger knapsacks

This won’t quite work…

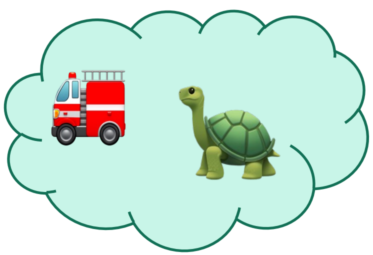
- We are only allowed **one copy of each item**.
- The sub-problem needs to “know” what items we’ve used and what we haven’t.

Optimal substructure: try 2

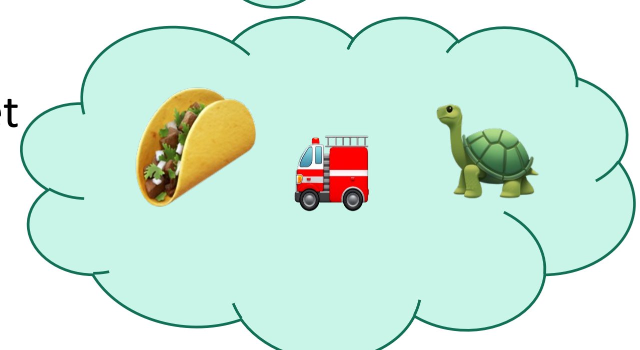
- Sub-problems:
  - 0/1 Knapsack with fewer items.

First solve the problem with few items

Then more items



Then yet more items

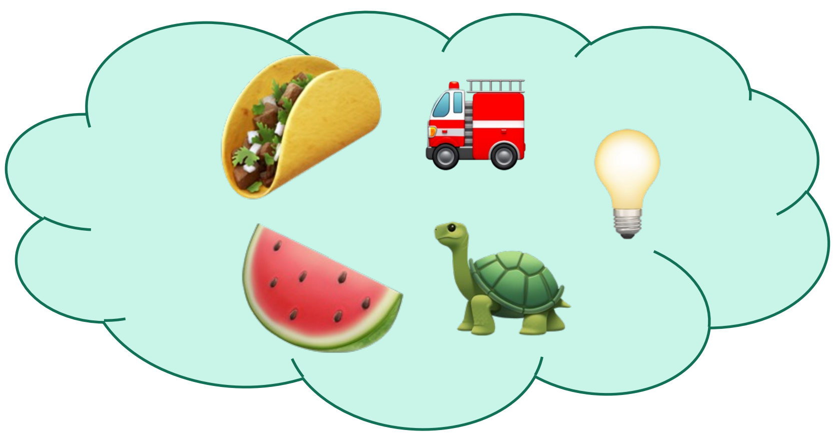


We'll still increase the size of the knapsacks.

(We'll keep a two-dimensional table).

Our sub-problems:

- Indexed by x and j



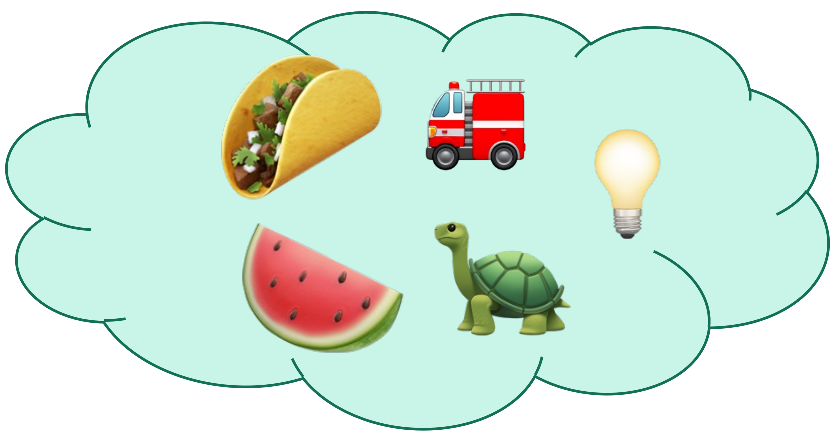
First j items

Capacity x

Two cases

item j

- **Case 1**: Optimal solution for j items does not use item j.
- **Case 2**: Optimal solution for j items does use item j.



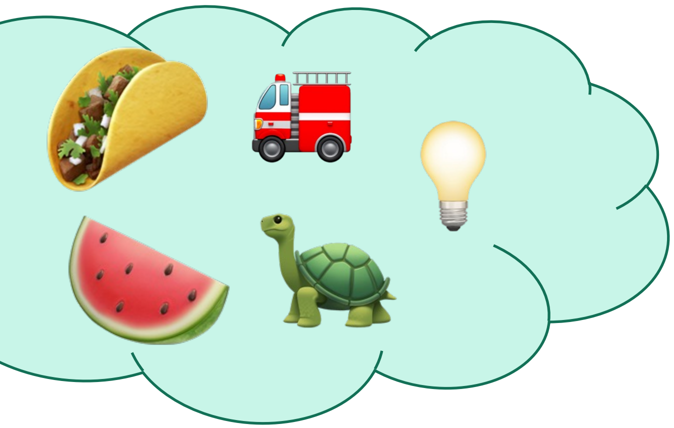
First j items

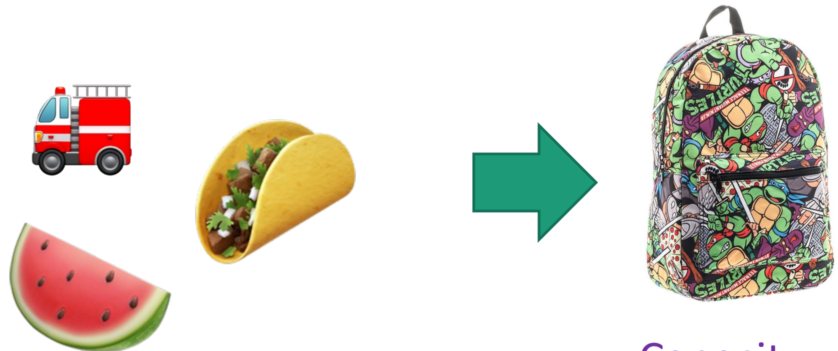
Capacity x

Two cases

item j

- **Case 1**: Optimal solution for j items does not use item j.

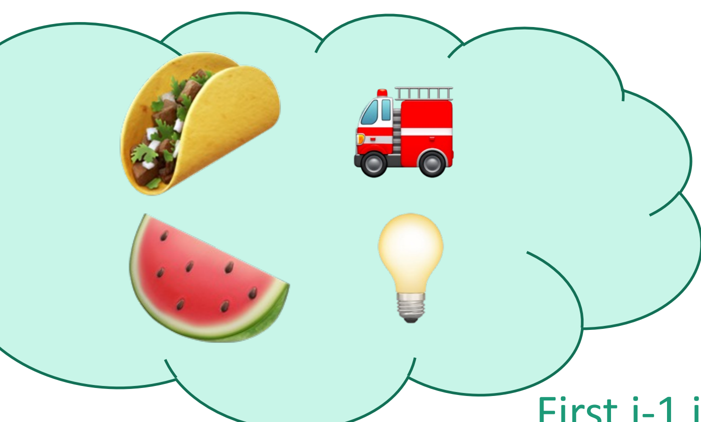




First j items

Capacity x
Value V
Use only the first j items

- Then this is an optimal solution for j-1 items:



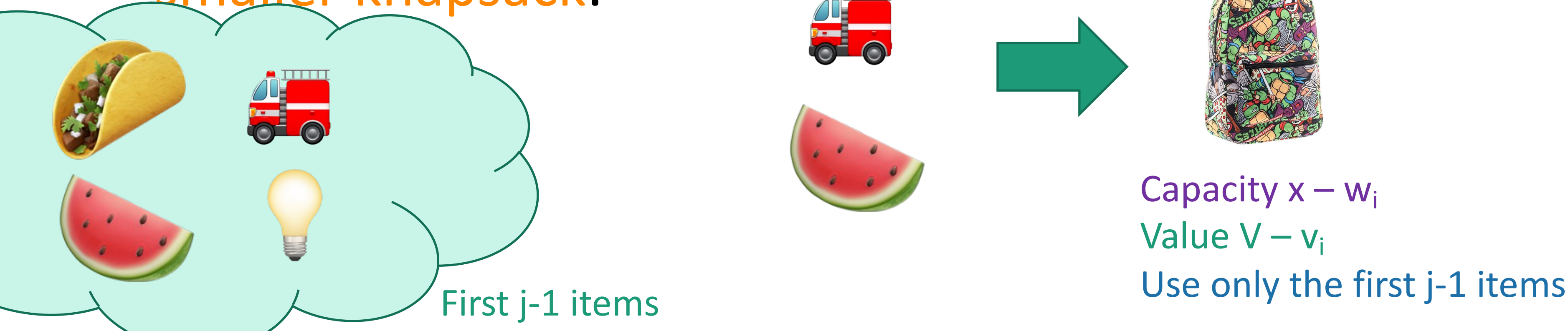
First j-1 items

Capacity x
Value V
Use only the first j-1 items.

Two cases

- **Case 2**: Optimal solution for j items uses item j.

- Then this is an optimal solution for j-1 items and a smaller knapsack:



Recipe for applying Dynamic Programming

- **Step 1:** Identify optimal substructure.
- **Step 2:** Find a recursive formulation for the value of the optimal solution.
- **Step 3:** Use dynamic programming to find the value of the optimal solution.
- **Step 4:** If needed, keep track of some additional info so that the algorithm from Step 3 can find the actual solution.
- **Step 5:** If needed, code this up like a reasonable person.

Recursive relationship

- Let K[x,j] be the optimal value for:
  - capacity x,
  - with j items.

$$K[x,j] = \max\{\ \underbrace{K[x, j-1]}_{\text{Case 1}}\ ,\ \underbrace{K[x - w_{j,}\ j-1] + v_j}_{\text{Case 2}}\ \}$$

- (And K[x,0] = 0 and K[0,j] = 0).

Recipe for applying Dynamic Programming

- **Step 1:** Identify optimal substructure.
- **Step 2:** Find a recursive formulation for the value of the optimal solution.
- **Step 3:** Use dynamic programming to find the value of the optimal solution.
- **Step 4:** If needed, keep track of some additional info so that the algorithm from Step 3 can find the actual solution.
- **Step 5:** If needed, code this up like a reasonable person.

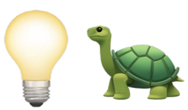
Bottom-up DP algorithm

- Zero-One-Knapsack(W, n, w, v):
  - $K[x,0] = 0$ for all $x = 0,\ldots,W$
  - $K[0,i] = 0$ for all $i = 0,\ldots,n$
  - **for** $x = 1,\ldots,W$:
    - **for** $j = 1,\ldots,n$:
      - $K[x,j] = K[x, j-1]$ Case 1
      - **if** $w_j \leq x$:
        - $K[x,j] = \max\{ K[x,j], K[x - w_j, j-1] + v_j \}$ Case 2
  - **return** $K[W,n]$

Running time $O(nW)$

Example

- Zero-One-Knapsack(W, n, w, v):
  - K[x,0] = 0 for all x = 0,…,W
  - K[0,i] = 0 for all i = 0,…,n
  - **for** x = 1,…,W:
    - **for** j = 1,…,n:
      - K[x,j] = K[x, j-1]
      - **if** $w_j \leq x$:
        - K[x,j] = max{ K[x,j], K[x – w_j, j-1] + v_j }
  - **return** K[W,n]

		x=0	x=1	x=2	x=3
	j=0	0	0	0	0
	j=1	0			
	j=2	0			
	j=3	0			

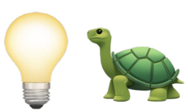
current entry

relevant previous entry

Item:				
Weight:	1	2	3	
Value:	1	4	6	Capacity: 3

Example

- Zero-One-Knapsack(W, n, w, v):
  - K[x,0] = 0 for all x = 0,…,W
  - K[0,i] = 0 for all i = 0,…,n
  - **for** x = 1,…,W:
    - **for** j = 1,…,n:
      - K[x,j] = K[x, j-1]
      - **if** $w_j \leq x$:
        - K[x,j] = max{ K[x,j], $K[x - w_j, j\text{-}1] + v_j$ }
  - **return** K[W,n]

	x=0	x=1	x=2	x=3
j=0	0	0	0	0
 j=1	0	0		
 j=2	0			
 j=3	0			

current entry

relevant previous entry

Item:				
Weight:	1	2	3	
Value:	1	4	6	Capacity: 3

Example

- Zero-One-Knapsack(W, n, w, v):
  - K[x,0] = 0 for all x = 0,…,W
  - K[0,i] = 0 for all i = 0,…,n
  - **for** x = 1,…,W:
    - **for** j = 1,…,n:
      - K[x,j] = K[x, j-1]
      - **if** $w_j \leq x$:
        - K[x,j] = max{ K[x,j], K[x – w_j, j-1] + v_j }
  - **return** K[W,n]

	x=0	x=1	x=2	x=3
j=0	0	0	0	0
j=1	0	1		
j=2	0			
j=3	0			

current entry

relevant previous entry

Item:			
Weight:	1	2	3
Value:	1	4	6

Capacity: 3

Example

- Zero-One-Knapsack(W, n, w, v):
  - K[x,0] = 0 for all x = 0,…,W
  - K[0,i] = 0 for all i = 0,…,n
  - **for** x = 1,…,W:
    - **for** j = 1,…,n:
      - K[x,j] = K[x, j-1]
      - **if** $w_j \leq x$:
        - K[x,j] = max{ K[x,j], $K[x - w_j, j\text{-}1] + v_j$ }
  - **return** K[W,n]

	x=0	x=1	x=2	x=3
j=0	0	0	0	0
j=1	0	1		
j=2	0	1		
j=3	0			

current entry

relevant previous entry

Item:			
Weight:	1	2	3
Value:	1	4	6

Capacity: 3

Example

- Zero-One-Knapsack(W, n, w, v):
  - K[x,0] = 0 for all x = 0,…,W
  - K[0,i] = 0 for all i = 0,…,n
  - **for** x = 1,…,W:
    - **for** j = 1,…,n:
      - K[x,j] = K[x, j-1]
      - **if** $w_j \leq x$:
        - K[x,j] = max{ K[x,j], $K[x - w_j, j-1] + v_j$ }
  - **return** K[W,n]

	x=0	x=1	x=2	x=3
j=0	0	0	0	0
j=1	0	1		
j=2	0	1		
j=3	0	1		

current entry

relevant previous entry

Item:			
Weight:	1	2	3
Value:	1	4	6

Capacity: 3

Example

- Zero-One-Knapsack(W, n, w, v):
  - K[x,0] = 0 for all x = 0,…,W
  - K[0,i] = 0 for all i = 0,…,n
  - **for** x = 1,…,W:
    - **for** j = 1,…,n:
      - K[x,j] = K[x, j-1]
      - **if** $w_j \leq x$:
        - K[x,j] = max{ K[x,j], $K[x - w_j, j-1] + v_j$ }
  - **return** K[W,n]

	x=0	x=1	x=2	x=3
j=0	0	0	0	0
j=1	0	1	0	
j=2	0	1		
j=3	0	1		

current entry

relevant previous entry

Item:			
Weight:	1	2	3
Value:	1	4	6

Capacity: 3

Example

- Zero-One-Knapsack(W, n, w, v):
  - K[x,0] = 0 for all x = 0,…,W
  - K[0,i] = 0 for all i = 0,…,n
  - **for** x = 1,…,W:
    - **for** j = 1,…,n:
      - K[x,j] = K[x, j-1]
      - **if** $w_j \leq x$:
        - K[x,j] = max{ K[x,j], $K[x - w_j, j-1] + v_j$ }
  - **return** K[W,n]

	x=0	x=1	x=2	x=3
j=0	0	0	0	0
j=1	0	1	1	
j=2	0	1		
j=3	0	1		

current entry

relevant previous entry

Item:			
Weight:	1	2	3
Value:	1	4	6

Capacity: 3

Example

- Zero-One-Knapsack(W, n, w, v):
  - K[x,0] = 0 for all x = 0,…,W
  - K[0,i] = 0 for all i = 0,…,n
  - **for** x = 1,…,W:
    - **for** j = 1,…,n:
      - K[x,j] = K[x, j-1]
      - **if** $w_j \leq x$:
        - K[x,j] = max{ K[x,j], $K[x - w_j, j\text{-}1] + v_j$ }
  - **return** K[W,n]

	x=0	x=1	x=2	x=3
j=0	0	0	0	0
j=1	0	1	1	
j=2	0	1	1	
j=3	0	1		

current entry

relevant previous entry

Item: 

Weight: 1 2 3

Value: 1 4 6

Capacity: 3

Example

- Zero-One-Knapsack(W, n, w, v):
  - K[x,0] = 0 for all x = 0,…,W
  - K[0,i] = 0 for all i = 0,…,n
  - **for** x = 1,…,W:
    - **for** j = 1,…,n:
      - K[x,j] = K[x, j-1]
      - **if** $w_j \leq x$:
        - K[x,j] = max{ K[x,j], K[$x - w_j$, j-1] + v_j }
  - **return** K[W,n]

		x=0	x=1	x=2	x=3
	j=0	0	0	0	0
🐢	j=1	0	1 🐢	1 🐢	
💡 🐢	j=2	0	1 🐢	4 💡	
🍉 💡 🐢	j=3	0	1 🐢		

current entry

relevant previous entry

Item:	🐢	💡	🍉
Weight:	1	2	3
Value:	1	4	6

Capacity: 3

Example

- Zero-One-Knapsack(W, n, w, v):
  - K[x,0] = 0 for all x = 0,…,W
  - K[0,i] = 0 for all i = 0,…,n
  - **for** x = 1,…,W:
    - **for** j = 1,…,n:
      - K[x,j] = K[x, j-1]
      - **if** $w_j \leq x$:
        - K[x,j] = max{ K[x,j], $K[x - w_j, j-1] + v_j$ }
  - **return** K[W,n]

	x=0	x=1	x=2	x=3
j=0	0	0	0	0
j=1	0	1	1	
j=2	0	1	4	
j=3	0	1	4	

current entry

relevant previous entry

Item:	🐢	💡	🍉
Weight:	1	2	3
Value:	1	4	6

Capacity: 3

Example

- Zero-One-Knapsack(W, n, w, v):
  - K[x,0] = 0 for all x = 0,…,W
  - K[0,i] = 0 for all i = 0,…,n
  - **for** x = 1,…,W:
    - **for** j = 1,…,n:
      - K[x,j] = K[x, j-1]
      - **if** $w_j \leq x$:
        - K[x,j] = max{ K[x,j], $K[x - w_j, j-1] + v_j$ }
  - **return** K[W,n]

	x=0	x=1	x=2	x=3
j=0	0	0	0	0
j=1	0	1	1	0
j=2	0	1	4	
j=3	0	1	4	

current entry

relevant previous entry

Item:			
Weight:	1	2	3
Value:	1	4	6

Capacity: 3

Example

- Zero-One-Knapsack(W, n, w, v):
  - K[x,0] = 0 for all x = 0,…,W
  - K[0,i] = 0 for all i = 0,…,n
  - **for** x = 1,…,W:
    - **for** j = 1,…,n:
      - K[x,j] = K[x, j-1]
      - **if** $w_j \leq x$:
        - K[x,j] = max{ K[x,j], $K[x - w_j, j-1] + v_j$ }
  - **return** K[W,n]

		x=0	x=1	x=2	x=3
	j=0	0	0	0	0
🐢	j=1	0	1 🐢	1 🐢	1 🐢
💡🐢	j=2	0	1 🐢	4 💡	
🍉💡🐢	j=3	0	1 🐢	4 💡	

current entry

relevant previous entry

Item:	🐢	💡	🍉
Weight:	1	2	3
Value:	1	4	6

Capacity: 3

Example

- Zero-One-Knapsack(W, n, w, v):
  - K[x,0] = 0 for all x = 0,…,W
  - K[0,i] = 0 for all i = 0,…,n
  - **for** x = 1,…,W:
    - **for** j = 1,…,n:
      - K[x,j] = K[x, j-1]
      - **if** $w_j \leq x$:
        - K[x,j] = max{ K[x,j], K[x – w_j, j-1] + v_j }
  - **return** K[W,n]

		x=0	x=1	x=2	x=3
	j=0	0	0	0	0
🐢	j=1	0	1 🐢	1 🐢	1 🐢
💡 🐢	j=2	0	1 🐢	4 💡	1 🐢
🍉 💡 🐢	j=3	0	1 🐢	4 💡	

current entry

relevant previous entry

Item:	🐢	💡	🍉
Weight:	1	2	3
Value:	1	4	6

Capacity: 3

Example

- Zero-One-Knapsack(W, n, w, v):
  - K[x,0] = 0 for all x = 0,…,W
  - K[0,i] = 0 for all i = 0,…,n
  - **for** x = 1,…,W:
    - **for** j = 1,…,n:
      - K[x,j] = K[x, j-1]
      - **if** $w_j \leq x$:
        - K[x,j] = max{ K[x,j], K[x – w_j, j-1] + v_j }
  - **return** K[W,n]

	x=0	x=1	x=2	x=3
j=0	0	0	0	0
j=1	0	1	1	1
j=2	0	1	4	5
j=3	0	1	4	

current entry

relevant previous entry

Item:

Weight: 1 2 3

Value: 1 4 6

Capacity: 3

Example

- Zero-One-Knapsack(W, n, w, v):
  - K[x,0] = 0 for all x = 0,…,W
  - K[0,i] = 0 for all i = 0,…,n
  - **for** x = 1,…,W:
    - **for** j = 1,…,n:
      - K[x,j] = K[x, j-1]
      - **if** $w_j \leq x$:
        - K[x,j] = max{ K[x,j], $K[x - w_j, j-1] + v_j$ }
  - **return** K[W,n]

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current entry

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j=3	0	1	4	6

So the optimal solution is to put one watermelon in your knapsack!

current entry

relevant previous entry

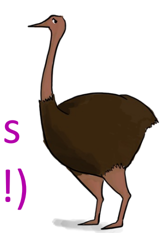
Item:			
Weight:	1	2	3
Value:	1	4	6

Capacity: 3

Recipe for applying Dynamic Programming

- **Step 1:** Identify optimal substructure.
- **Step 2:** Find a recursive formulation for the value of the optimal solution.
- **Step 3:** Use dynamic programming to find the value of the optimal solution.
- **Step 4:** If needed, keep track of some additional info so that the algorithm from Step 3 can find the actual solution.
- **Step 5:** If needed, code this up like a reasonable person.

You do this one!
(We did it on the slide in the previous example, just not in the pseudocode!)



What have we learned?

- We can solve 0/1 knapsack in time O(nW).
  - If there are n items and our knapsack has capacity W.
- We again went through the steps to create DP solution:
  - We kept a two-dimensional table, creating smaller problems by restricting the set of allowable items.

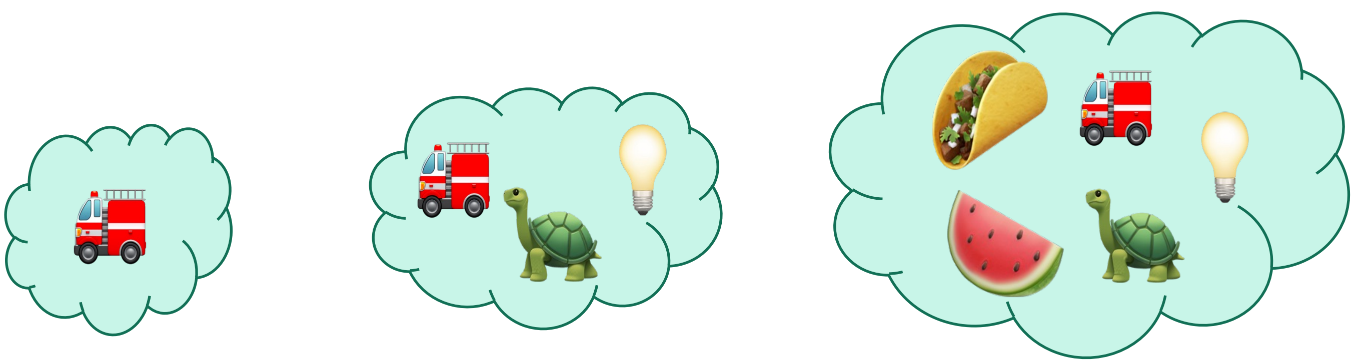
Question

- How did we know which substructure to use in which variant of knapsack?

Answer in retrospect:

This one made sense for unbounded knapsack because it doesn't have any memory of what items have been used.

vs.



In 0/1 knapsack, we can only use each item once, so it makes sense to leave out one item at a time.

Operational Answer: try some stuff, see what works!

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