2 Order of Magnitude Estimates

This final problem set involves a number of challenges in order-of-magnitude thinking. As a reminder, when doing street fighting estimates, the goal is to do simple arithmetic of the kind that all numbers take the values 1, few (f) or 10. few × few = 10, etc. Please do not provide estimates with multiple "significant" digits that are meaningless. Be thoughtful about what you know and what you don't know. You may use the Bionumbers website (http://bionumbers.hms.harvard.edu/) to find key numbers (examples are masses of amino acids (BNID 104877) and nucleotides (BNID 103828), the speed of the ribosome (BNID 100059), etc.), but please provide a citation to the Bionumber of interest as shown above if you do decide to use those numbers. If you have some other source for your numbers, please cite them. However, for many of these problems the essence of things is to do simple estimates, not to look quantities up.

Choose any 15 of the problems below and provide estimates in the style we have been doing all term or the style that can be found in the vignettes of Cell Biology by the Numbers. Remember, we are not fixated on your answers as much as on the clarity of your assumptions, your skill with 1, f and 10 arithmetic, your common sense and intuition. If after performing your estimate you want to comment on data and how your results jibe with that data, brief discussions are welcome. Try to spend less than 5 minutes on each of the questions — this is meant to be a fun, rapid drill in exercising your estimation skills.

Before we go through the questions:

Knowing that k_B is approximately $1.38 \times 10^{-23} J/K$, and in these processes temperature T is often considered to be 300K.

For $T \approx 300K$, we have $k_BT \approx 4 \times 10^{-21} J$

Also, $1k_BT$ is usually estimated as $4pN \cdot nm$, as we derived and emphasized in the class.

Question 4 pH and ion numbers.

At the typical pH of a bacterial cell, how many hydrogen ions does that correspond to?

Answer:

According to BNID 107037, the internal pH of E.coli is 7.85 ± 0.05.

Since pH is the minus value of concentration of hydrogen ion: $[H^+] = 10^{-pH}$.

Thus, the concentration of proton is approximately:

$$[H^+] = 10^{-7.85} \approx 1.4 \times 10^{-8} mol/L$$

As we estimated in the class, an E.coli can be simplified as a cubic with $1\mu m$ side length. The volume can be estimated as $1\mu m^3$.

$1\mu m^3 = 10^{-18} m^3 = 10^{-15} L.$

The molar number of proton is:

$$N_{moler} = 1.4 \times 10^{-8} mol/L \times 10^{-15} L = 1.4 \times 10^{-23} mol$$

The actual number of hydrogen ion in an E.coli at pH=7.85 would be:

$$N = 1.4 \times 10^{-23} \times 6.02 \times 10^{23} \approx 8 \text{ ions/cell}$$

If we simply estimate the **pH value to be 7**, the answer would be approximately **60** ions per cell.

Question 6 Lipids per cell

How many lipids are there in a bacterial cell?

Answer:

As we estimated in the class, an E.coli can be simplified as a cubic with $1\mu m$ side length. Thus the surface of an E.coli is approximately $6\mu m^2$.

According to BNID 106993, surface area per lipid molecule is $0.5nm^2$.

Note that most of the lipids locate in the cell membrane in the form of **bilayer** lipids.

Thus, we can calculate calculate the number of lipids in a single layer:

Surface area: $6\mu m^2 = 6 \times 10^6 nm^2$.

$$\text{number of lipids in single layer per cell} = \frac{6 \times 10^6 \text{nm}^2}{0.5 \text{nm}^2/\text{lipid}} = 1.2 \times 10^7 \text{ lipids}$$

Consider the **bilayer** lipids structure of plasma membrane:

$$\text{number of lipids in bilayer per cell} = \ 1.2 \times 10^7 \times 2 \text{ lipids} = 2.4 \times 10^7 \text{lipids}$$

Consider the lipids do not only exist in the plasma membrane but occupies most of the lipids usage:

A reasonable estimation is: an E.coli has $\text{few} \times 10^7$ lipids.

Question 8 Water in the Central Valley?

How much water is used to irrigate the Central Valley of California every year?

Answer:

Consider the Mediterranean climate of California, with the average rainfall of about 500 nm, we can just assume central valley grows commercial crops in the whole year. From my experience while driving along Highway 5 in January, most of the fields along with Highway 5 is utilized in January. Which means the crop season could just be whole year in our estimation.

From empirical experience of planting in flowerpot, a cup of water (approximately $250mL = 0.25L$) is needed for irritation per day.

From homework 1, the area of central valley is approximately $300 \times 100km^2$ which is $3 \times 10^{10}\ m^2$.

We can first calculate the water needed per day:

$$\text{water needed per day in a unit area} = \frac{0.25\text{L/day}}{20 \times 20\text{cm}^2} = \frac{0.25\text{L/day}}{0.04m^2} = 6.25 \text{ L/m}^2\text{/day}$$

Then the annual water usage can be estimated as:

$$3 \times 10^{10}\text{m}^2 \times 6.25\text{L/day} \times 365\text{day/year} \approx 6.84 \times 10^{13}\text{L/year}$$

Consider there will be rainfall and there would be different water usage for various crops, $\text{few} \times 10^{13}\text{L}$ would be a reasonable estimation.

Question 9 Mass of mRNA and proteins.

What is the mass of an mRNA and a protein?

Answer:

Only consider the general case of mRNA and protein.

Let's assume a typical protein is around 300 amino acid.

According to BNID 103248, the range of amino acid is from 57-186 Dalton.

As a rule of thumb, the average amino acid mass would be **100 Dalton** (BNID 101830).

The mass of a protein is:

$$m_{protein} = \frac{100Da/aa \times 300aa}{6.02 \times 10^{23}/mol} = \frac{3 \times 10^4 g/mol}{6.02 \times 10^{23}/mol} = 5 \times 10^{-20} g$$

Similarly, an mRNA is composed of nucleotides.

According to BNID 104886, average molecular mass of RNA nucleotides in E.coli is 324.3 Dalton.

Let's take $m_{nt} = 300Da$ for simplification.

Knowing that 3 nucleotide coding for a protein, $N_{nt} \approx 3 \times N_{aa}$ and $m_{nt} \approx 3 \times m_{aa}$.

The corresponding mass of mRNA could be:

$$\begin{aligned} \text{mass of mRNA } &= N_{nt} \times m_{nt} = 3 \times N_{aa} \times 3 \times m_{aa} \approx 10 \times N_{aa} m_{aa} \\ &= 10 \times m_{protein} = 5 \times 10^{-19} g \end{aligned}$$

Question 10 Water uptake in dividing bacteria

How many water molecules are taken up each second during the growth of a rapidly growing bacterium?

Answer:

Take E.coli as a bacterial model. In our estimations, E.coli is often considered as a cubic with $1\mu m$ side length. The volume is $1\text{fL} = 1 \times 10^{-15} L$.

Let's consider the cell growth rate to derive the growth of a rapidly growing bacterium.

According to BNID 111284, the experimentally observed specific growth rate of E.coli is **1.14 per hour**.

If we consider the division means a doubled volume, and most of the volume increased is considered water.

Knowing that:

Within 1 hour, E.coli will grow 1.14 volume of E.coli.

Volume increased in 1 hour: $1.14 \times 1\text{fL}/\text{hour} = 1.14 \times 10^{-15} L/hour$

1 hour is 3600 second.

Volume of water taken in each second:

$$\frac{1.14 \times 10^{-15}\text{L}/\text{hour}}{3600\text{sec}/\text{hour}} \approx 3.2 \times 10^{-19}\text{L}/\text{sec}$$

The density of water is 1g/L.

Mass of water taken per second during rapid growth: $3.2 \times 10^{-19}\mathrm{g/sec}$

The molecular mass of water (H_2O) is 18g/mol.

Mole of water taken per second during rapid growth:

$$\frac{3.2 \times 10^{-19}\mathrm{g/sec}}{18\mathrm{g/mol}} \approx 1.78 \times 10^{-20} mol$$

Convert mol into exact value:

$$1.78 \times 10^{-20} mol \times 6.02 \times 10^{23}\mathrm{molecules}/mol \approx 1 \times 10^{4}\mathrm{molecules}$$

Consider the growth rate and the conditions in E.coli growth, **few hundreds** or **few thousands** water molecules taken per second during rapid growth.

Question 12 Length of DNA in your cells

What is the length of DNA in one of your cells?

Answer:

Let's just consider a cell with normal autosomal chromosomes in my body. (If we consider cells undergoing cell division or germ cells, the condition would be different.)

According to BNID 103777, the "Rule of thumb" for length of DNA per base pair is $0.333nm/bp$.

According to BNID 110200, "the size of the total human genome is estimated to be about 3.1Gb".

Gb stands for Giga base pairs. One gigabase is equal to **1 billion bases**.

Length of DNA in my cell in bps is: $3.1Gb = 3.1\mathrm{billion\ bases} = 3.1 \times 10^{9}\mathrm{bps}$.

Length of DNA in a typical cell of mine in meter would be:

$$3.1 \times 10^{9}\mathrm{bps} \times 0.333\mathrm{nm/bp} \approx 1 \times 10^{9} nm = 1\mathrm{m}$$

For a sanity check, according to BNID 102980, the length of longest chromosome in human homo sapiens is $10cm = 0.1m$. Since we have 46 chromosomes in somatic cells, **few meters** would be a reasonable estimation.

Question 18 Size of sequence space

How many times the volume of our universe would it take to make a single copy of every 300 aa protein?

Answer:

Consider mainly 20 types of animo acids that can compose the protein. For each position among 300 amino acids, there are 20 choice for each animo acid position. Assume that the probability of occurrence is equal and the position correlation is not considered, we can have:

$$Seq = 20^{300} \approx 2.04 \times 10^{390}$$

By consulting Jena Library in BioNumber (BNID 103241), we can estimate the volume for each amino acid to be $100Å^{3} = 0.1nm^{3}$. Thus, the volume of such a protein could be approximately

$$300 \times 0.1nm^3 = 30nm^3 = 3 \times 10^{-26}m^3$$

As a sanity check, typical protein with around 300 aa is Green Fluorescent Protein (GFP) with 238 amino acids. According to BNID 108461 and BNID 110498, if we consider the volume of GFP can be considered as a sphere, by utilizing $V = \frac{4}{3}\pi r^3$,the volume is $\text{few} \times 10^{-26}m^3$.

By checking the google website and NASA website, the volume of observable universe is $4 \times 10^{80}m^3$.

Consider all the possibilities we could have, in total the volume takes place:

$$3 \times 10^{-26}m^3 \times 2.04 \times 10^{390} = 6.12 \times 10^{364}m^3$$

Thus it occupies:

$$\text{\# of observable universe} = \frac{6.12 \times 10^{364}}{4 \times 10^{80}} = 1.53 \times 10^{284}$$

It needs approximately $\text{few} \times 10^{284}$ observable universe to contain all the permutations of the proteins.

Question 20 Fraction of body mass in DNA

What fraction of your body mass is DNA?

Answer:

According to BNID 110200, "the size of the total human genome is estimated to be about 3.1Gb".

Gb stands for Giga base pairs. One gigabase is equal to **1 billion bases**.

Length of DNA in my cell in bps is: $3.1Gb = 3.1\text{billion bases} = 3.1 \times 10^9\text{bps}$.

By checking the chemical formula of A/T/C/G, the average molecular weight of base pairs is $(135 + 126 + 111 + 151)/4 \approx 131g/mol$.

Thus, we have the DNA weight per cell is:

$$3.1 \times 10^9\text{bps/cell} \times 131\text{g /mol} \times \frac{1mol}{6.02 \times 10^{23}\text{bps}} \approx 6.75 \times 10^{-13}\text{g/cell} = 6.75 \times 10^{-16}\text{kg/cell}$$

According to BNID 113040, the number of cells in human body is approximately 3.7×10^{13}.

Thus the total mass of DNA is:

$$m_{DNA} = 3.7 \times 10^{13} \times 6.75 \times 10^{-16}\text{kg} \approx 0.025\text{kg}$$

A typical average body weight of human is $70kg$ (so do I).

Fraction of DNA mass in human body can be estimated as:

$$\frac{0.025\text{kg}}{70\text{kg}} \times 100\% \approx 0.036\%$$

The mass of human DNA in body mass is ~0.036%.

Question 22 Number of humans born per year

Make the spurious assumption of a steady state human population size and use Little’s theorem to work out the number of humans born every year.

Answer:

By consulting the web source, Little's theorem, commonly applied in queuing theory, states that the long-term average number L of customers in a stationary system is equal to the long-term average effective arrival rate λ multiplied by the average time W a customer spends in the system.

$L = \lambda W$

where L represents the total human population at any given time, W is the average lifespan of human, and λ is the rate of new individuals (births) entering the population.

In the steady-state assumption, by checking the website, there is approximately 8.1 billion people in the world. $L = 8.1 \times 10^9$.

According to the World Bank website, the average human lifespan worldwide is 71.33 years. $W = 71.33$ years.

$$\lambda = \frac{L}{W} = \frac{8.1 \times 10^9 \text{people}}{71.33 \text{years}} = 1.14 \times 10^8 \text{ births per year}$$

110 million births per year worldwide would be the result in this context.

Question 29 Intuition for concentration

If there is one copy of a molecule of interest per bacterial cell, what is the concentration in M units?

Answer:

Take the E.coli as a bacterial model.

As we estimated in lectures and previous answers, the volume of an E.coli is $1\mu m^3 = 1 \times 10^{-15} L$.

If there is one molecules in each E.coli, it corresponds to $1 \text{ molecule}/10^{-15} L$

The concentration is: $10^{15} \text{molecules/L}$.

Convert the concentration into mol:

$$\text{concentration in mol} = \frac{10^{15} \text{molecules/L}}{6.02 \times 10^{23} \text{molecules/mol}} \approx 1.66 \times 10^{-9} \text{mol/L} = 1.66 \times 10^{-9} \text{M} = 1.66 \text{nM}$$

This is the molar concentration of this molecule in bacteria, which is about 1.66 nanomoles per liter (nM).

Question 30 Volume of a human body

In m^3 units, what is the volume of a human body?

Answer:

To simplify and model my body, I'd like to consider my body as a sphere and a cube. The sphere is my head and the cube is my body (from leg to shoulder).

The radius of the sphere can be estimated as $r = 7cm = 0.07m$.

The thickness of my body (measured by roommate) is $a = 15cm = 0.15m$. For the width, we apply the "the shoulder width minus the arms", and $b = 30cm = 0.3m$. For the height, from my feet to shoulder is approximately $c = 150cm = 1.5m$

The estimated volume of the sphere (head) is: $V_{sphere} = \frac{4}{3}\pi r^3 \approx 1.4 \times 10^{-3} m^3$.

The estimated volume of the cube (under the shoulder) is:
$V_{cube} = 0.15 \times 0.3 \times 1.5m^3 = 0.0675m^3 = 6.75 \times 10^{-2}m^3$.

Thus, the total volume estimated can be:

$$V_{total} = V_{sphere} + V_{cube} = (6.75 + 0.14) \times 10^{-3}m^3 \approx 6.9 \times 10^{-3}m^3$$

Question 31 Number of red blood cells in a human

How many red blood cells in a human body?

Answer:

According to BNID 101707, volume of blood in average-sized (70 kg) person is $5.5L$.

According to BNID 102745, concentration of red blood cells in human blood is $5.6 \times 10^6 \text{cells/mm}^3$.

Convert the unit for concentration: since $1mm^3 = 10^{-6}L$, concentration: $5.6 \times 10^{12}\text{cells/L}$.

Total number of red blood cell in a human body:

$$N_{\text{RBC}} = 5.6 \times 10^{12}\text{cells/L} \times 5.5L = 3.08 \times 10^{13} \text{ cells}$$

Question 34 Translation time

How long does it take for a ribosome to translate all of the proteins that are needed to make another ribosome?

Answer:

Consider the E.coli case.

According to BNID 101175, total number of amino acids in the ribosome of E.coli is 7459 amino acids.

According to BNID 111689, the average translation rate of E.coli is ~8 amino acids/sec.

Now we can calculate the time needed:

$$\frac{7459 \text{ amino acids}}{8 \text{ amino acids/sec}} \approx 930 \text{ seconds}$$

It takes approximately 930 seconds (15.5 mins) for a ribosome to translate all of the proteins required for another ribosome.

Note that translation rate differs at different condition. Few tens minutes or 10 mins is logically acceptable.

Question 36 Areas and volumes of your cells

Crudely estimate the total (membrane) surface area, and total (cellular) volume, of all bacterial cells in your body (assuming they are rods like E coli).

Do the same for your eukaryotic cells; how different are these bacterial & human values?

Answer:

According to BNID 113000, the number of bacteria across the whole body of the "reference man" is 3.8×10^{13} bacteria/human.

Assume all of these bacteria are like E.coli, with $6\mu m^2$ surface area and $1\mu m^3$ volume (as we always did in our lecture).

$$A_{bacterial} = 3.8 \times 10^{13} \times 6\mu m^2 \approx 2.3 \times 10^{14} \mu m^2$$
$$V_{bacterial} = 3.8 \times 10^{13} \times 1\mu m^3 = 3.8 \times 10^{13} \mu m^3$$

For eukaryotic cells in my body, let's just use **typical mammalian tissue culture cell or HeLa cells** as our reference.

According to BNID 105906, the volume of a typical cell is $4000\mu m^3$ (in mammalian tissue culture cell). (Sanity check: Volume of HeLa cell is $3700 \pm 1500\mu m^3$). According to BNID 103718, the surface area of HeLa Cell is $1600 \pm 500\mu m^2$.

According to BNID 109716, total cell number in body for human is $(3.7 \pm 0.8) \times 10^{13}$ cells.

Sanity check: BNID 113005:

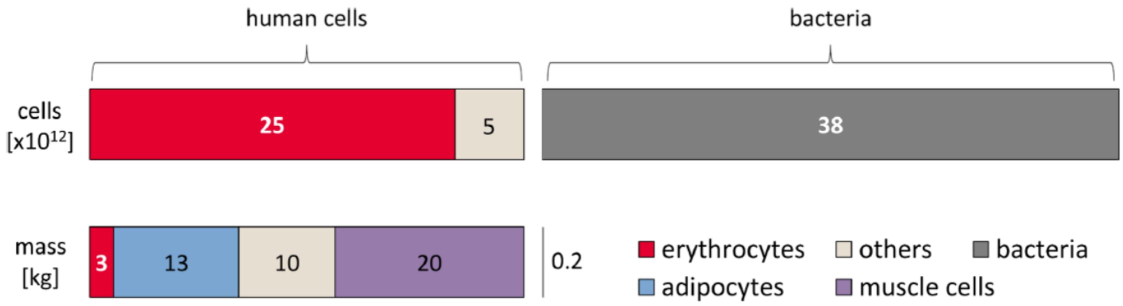

Fig 3. Distribution of cell number and mass for different cell types in the human body (for a 70 kg adult man). The upper bar displays the number of cells, while the lower bar displays the contribution from each of the main cell types comprising the overall cellular body mass (not including extracellular mass that adds another ≈24 kg). For comparison, the contribution of bacteria is shown on the right, amounting to only 0.2 kg, which is about 0.3% of the body weight.

doi:10.1371/journal.pbio.1002533.g003

We can calculate:

$$A_{eukaryotic} = 3.7 \times 10^{13} \times 1600\mu m^2 \approx 5.9 \times 10^{16} \mu m^2$$
$$V_{eukaryotic} = 3.7 \times 10^{13} \times 4000\mu m^3 = 1.48 \times 10^{17} \mu m^3$$

Conclusion:

Although bacteria and human cells are roughly equal in number, the total volume and total surface area of human cells far exceed that of bacterial cells.

Consider the bacterial cell has a higher ratio of Surface Area to Volume than eukaryotic cells in human body; This allows them to be widely distributed in the human body and play important roles in the human microecology.

Question 38 Food and mountain climbing

What is the maximum height of a mountain you can climb after eating just a bowl of ramen?

Answer:

Let's clarify this question: if we only consume the calories by a bowl of ramen, what is the height of mountain that the energy derived allows me to climb?

Combining the search result from Google and menus supplied by Yelp, a typical bowl of Ramen contains 450-600 kcal.

Knowing that 1 calories is approximately 4.2 Joules: $1 \text{ calories} \approx 4.2J$.

Let's take 500 kcal for estimation in this question.

Energy derived: $500\text{kcal} = 2100kJ = 2.1 \times 10^6 J$

My weight is $m = 70kg$, the acceleration of gravity is $g = 9.8m/s^2$. Assume the maximum height is h(m).

Consider all of the energy are transferred into the gravitational potential energy:

$$mgh = 2.1 \times 10^6 J$$
$$70kg \times 9.8m/s^2 \times h = 2.1 \times 10^6 J$$
$$h = \frac{2.1 \times 10^6 J}{686kg \cdot m/s^2} \approx 3061m$$

Incredible. However, in the real case, not all energy can be transferred into the gravitational potential energy. Kinetic energy also account for a certain portion of energy consumption.

By consulting Google and discussing in last lecture, only approximately 25% of energy taken are transferred into mechanical energy, base on this, a more reasonable estimation if all of the energy are transferred into gravitational potential energy would be:

$$h = 3061m \times 25\% = 765.25m$$

Although it is still higher than my gut feeling, but it is more reasonable.