Kinematics

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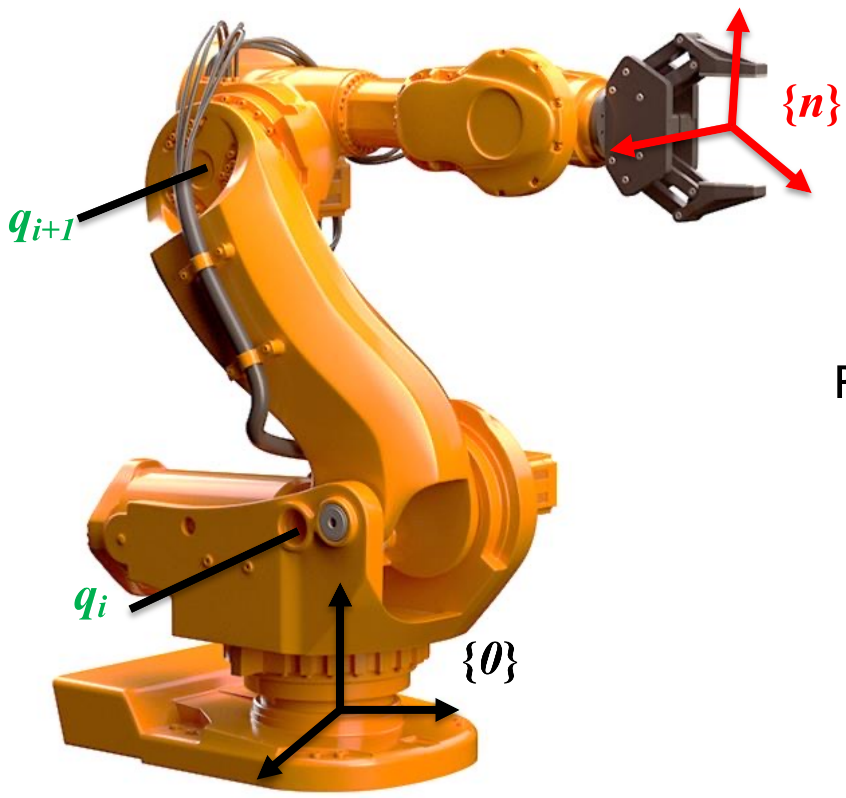
Kinematics of Robot Manipulators

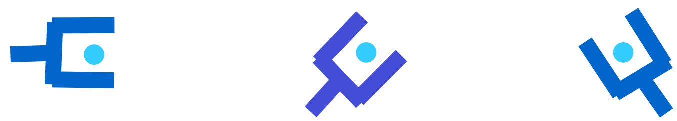


Relation between joints (q_i) and the pose (position/orientation) of some point (e.g.: frame $\{n\}$)

Workspace

- Primary Workspace (*reachable*): WS_1

  Positions that can be reached with <u>at least one orientation</u>

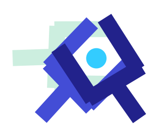


Each point can be reached (orientation "does not matter")

  - Out of WS_1 there is no solution to the problem
  - For all $\mathbf{p}$ WS_1 (using a proper orientation), there is at least one solution

- Secondary Workspace (*dexterous*): WS_2

  Positions can be reached with <u>any orientation</u>



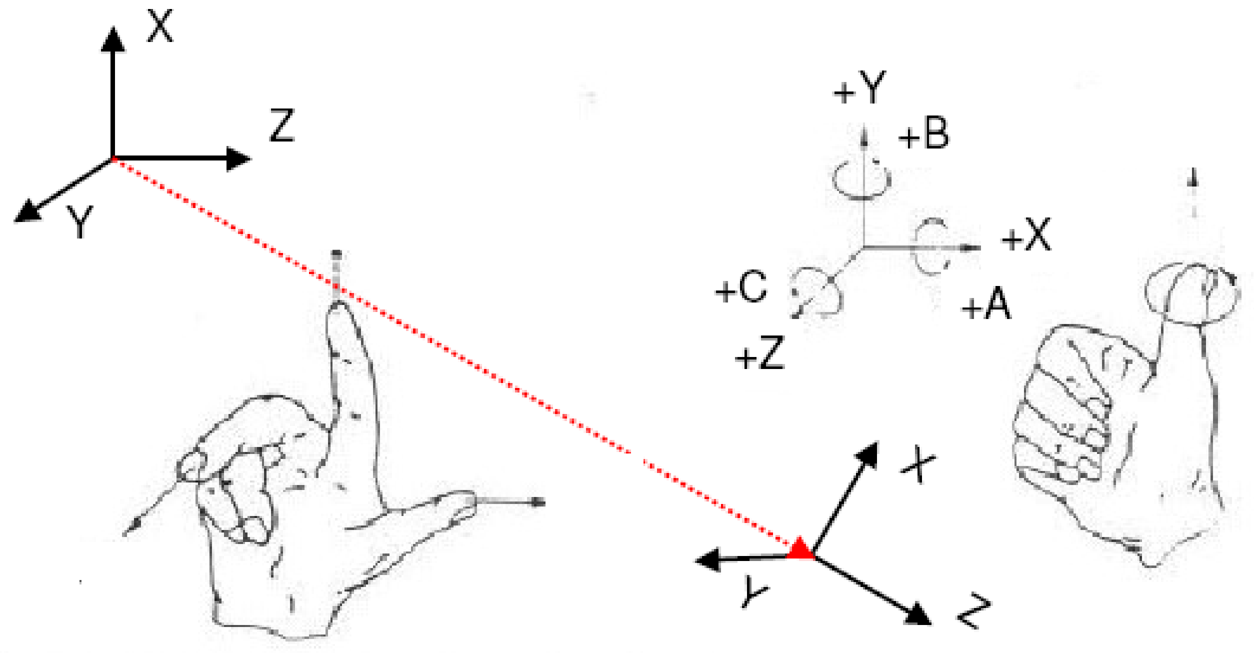
Reach every poing with all possible orientations

  - For all $\mathbf{p}$ WS_2 there is (at least) one solution for every orientation

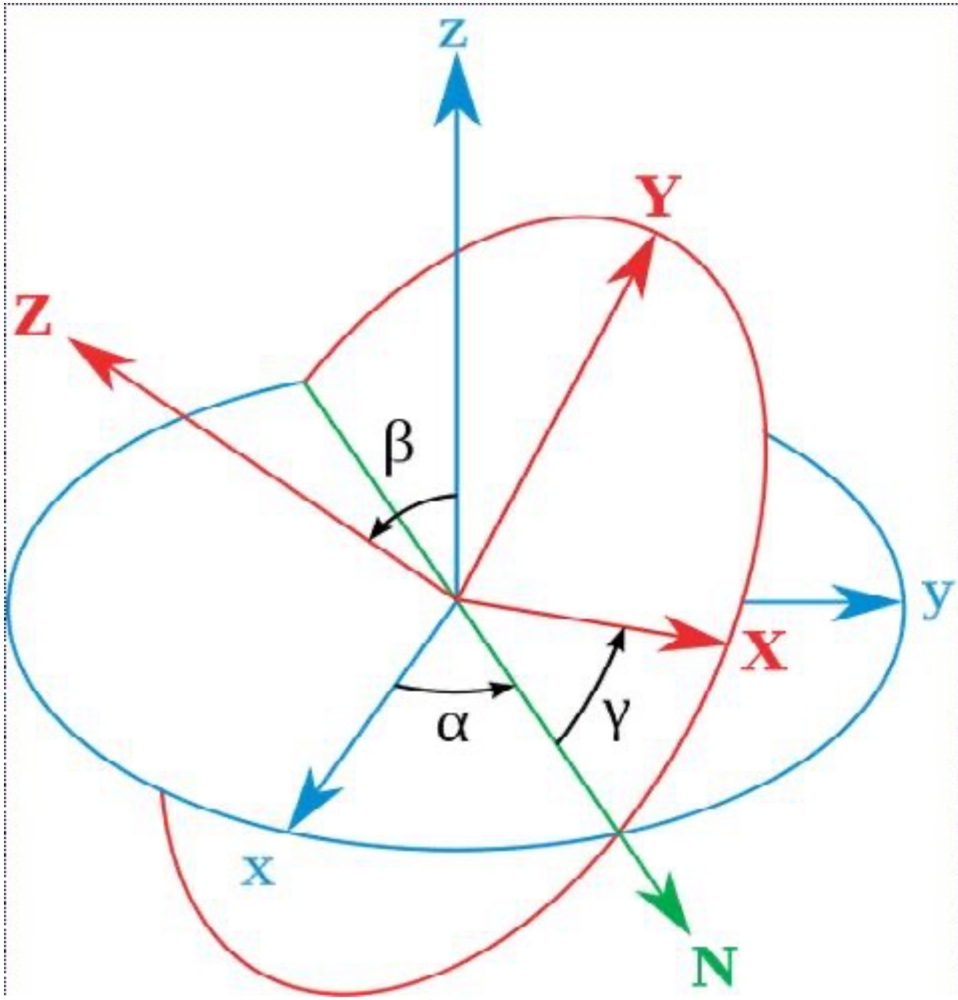
- Relation between WS_1 y WS_2:

$$WS_2 \subseteq WS_1$$

Degrees of Freedom N – number of independent motion parameters of a body in space



Rotation matrix



Pose of an object in space

- q = (position, orientation) = $(x, y, z, \textbf{???})$

- Parametrization of orientations by matrix:

  $q = (r_{11}, r_{12}, \ldots, r_{33}, r_{33})$ where $r_{11}, r_{12}, \ldots, r_{33}$ are the elements of rotation matrix

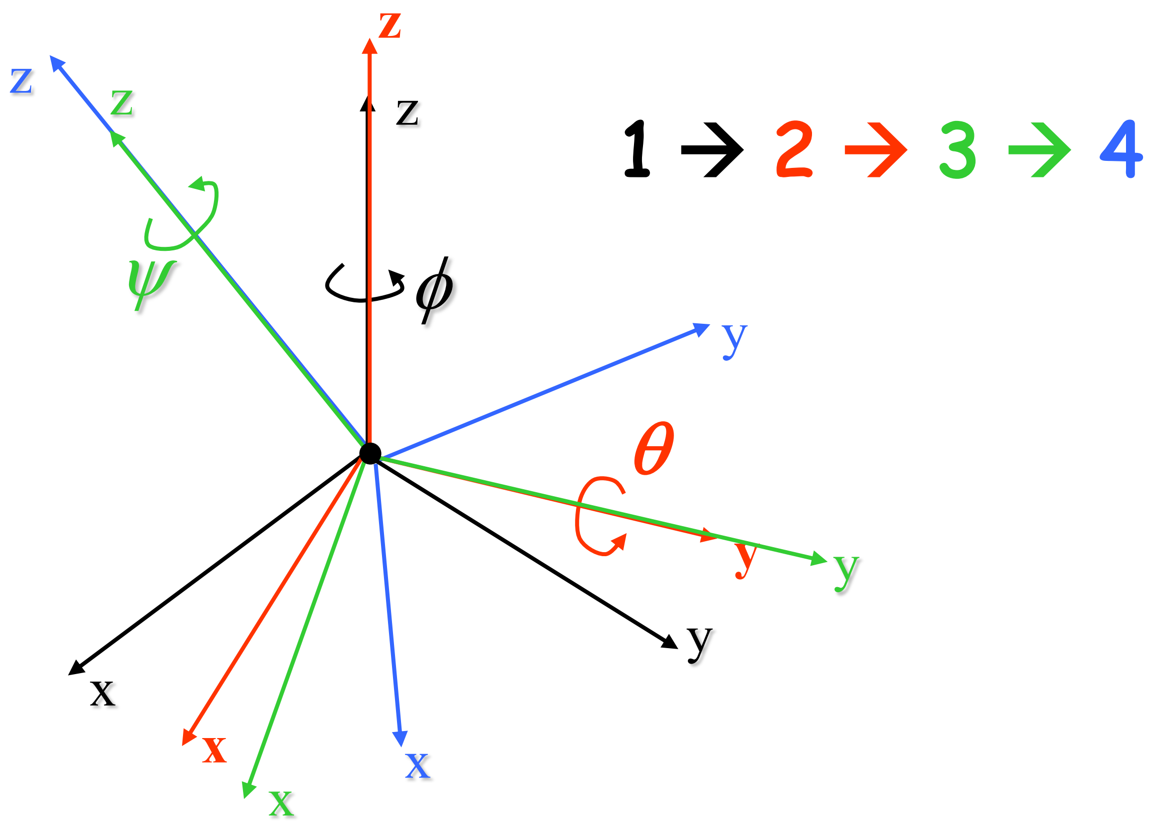
$$R = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix}$$

  with

  - $r_{1i}{}^2 + r_{2i}{}^2 + r_{3i}{}^2 = 1$ for all i ,
  - $r_{1i}\, r_{1j} + r_{2i}\, r_{2j} + r_{3i}\, r_{3j} = 0$ for all $i \neq j$,
  - $\det(R) = +1$

Example: rigid robot in 3-D workspace

- Parametrization of orientations by Euler angles: (ϕ, θ, ψ)



Example of a singularity in Euler representation

$$
\begin{aligned}
{}^A_B R_{XYZ}(\gamma, \beta, \alpha) &= R_Z(\alpha) R_Y(\beta) R_X(\gamma) \\
&= \begin{bmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\gamma & -s\gamma \\ 0 & s\gamma & c\gamma \end{bmatrix},
\end{aligned}
$$

$$
{}^A_B R_{XYZ}(\gamma, \beta, \alpha) = \begin{bmatrix} c\alpha c\beta & c\alpha s\beta s\gamma - s\alpha c\gamma & c\alpha s\beta c\gamma + s\alpha s\gamma \\ s\alpha c\beta & s\alpha s\beta s\gamma + c\alpha c\gamma & s\alpha s\beta c\gamma - c\alpha s\gamma \\ -s\beta & c\beta s\gamma & c\beta c\gamma \end{bmatrix}
$$

$$\beta = \mathrm{Atan2}(-r_{31}, \sqrt{r_{11}^2 + r_{21}^2}),$$

$$\alpha = \mathrm{Atan2}(r_{21}/c\beta, r_{11}/c\beta),$$

$$\gamma = \mathrm{Atan2}(r_{32}/c\beta, r_{33}/c\beta),$$

$$\beta = 90.0^\circ,$$

$$\alpha = 0.0.$$

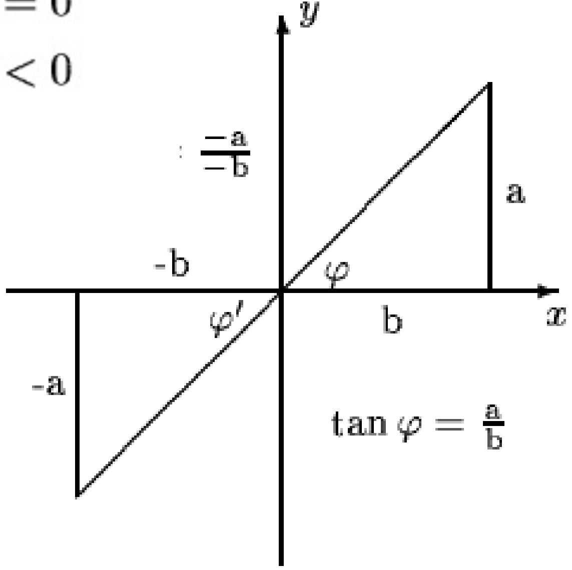
$$\gamma = \mathrm{Atan2}(r_{12}, r_{22}),$$

Euler representations

Proper Euler angles
$X_1 Z_2 X_3 = \begin{bmatrix} c_2 & -c_3 s_2 & s_2 s_3 \\ c_1 s_2 & c_1 c_2 c_3 - s_1 s_3 & -c_3 s_1 - c_1 c_2 s_3 \\ s_1 s_2 & c_1 s_3 + c_2 c_3 s_1 & c_1 c_3 - c_2 s_1 s_3 \end{bmatrix}$
$X_1 Y_2 X_3 = \begin{bmatrix} c_2 & s_2 s_3 & c_3 s_2 \\ s_1 s_2 & c_1 c_3 - c_2 s_1 s_3 & -c_1 s_3 - c_2 c_3 s_1 \\ -c_1 s_2 & c_3 s_1 + c_1 c_2 s_3 & c_1 c_2 c_3 - s_1 s_3 \end{bmatrix}$
$Y_1 X_2 Y_3 = \begin{bmatrix} c_1 c_3 - c_2 s_1 s_3 & s_1 s_2 & c_1 s_3 + c_2 c_3 s_1 \\ s_2 s_3 & c_2 & -c_3 s_2 \\ -c_3 s_1 - c_1 c_2 s_3 & c_1 s_2 & c_1 c_2 c_3 - s_1 s_3 \end{bmatrix}$
$Y_1 Z_2 Y_3 = \begin{bmatrix} c_1 c_2 c_3 - s_1 s_3 & -c_1 s_2 & c_3 s_1 + c_1 c_2 s_3 \\ c_3 s_2 & c_2 & s_2 s_3 \\ -c_1 s_3 - c_2 c_3 s_1 & s_1 s_2 & c_1 c_3 - c_2 s_1 s_3 \end{bmatrix}$
$Z_1 Y_2 Z_3 = \begin{bmatrix} c_1 c_2 c_3 - s_1 s_3 & -c_3 s_1 - c_1 c_2 s_3 & c_1 s_2 \\ c_1 s_3 + c_2 c_3 s_1 & c_1 c_3 - c_2 s_1 s_3 & s_1 s_2 \\ -c_3 s_2 & s_2 s_3 & c_2 \end{bmatrix}$
$Z_1 X_2 Z_3 = \begin{bmatrix} c_1 c_3 - c_2 s_1 s_3 & -c_1 s_3 - c_2 c_3 s_1 & s_1 s_2 \\ c_3 s_1 + c_1 c_2 s_3 & c_1 c_2 c_3 - s_1 s_3 & -c_1 s_2 \\ s_2 s_3 & c_3 s_2 & c_2 \end{bmatrix}$

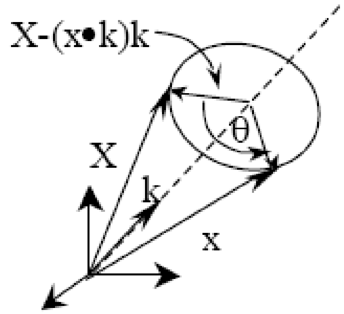
ATAN2 Function

$$
\mathrm{ATAN2}(a,b) = \begin{cases} arctan(\frac{a}{b}) & \text{falls } b > 0 \\ \frac{\pi}{2} & \text{falls } b = 0, a > 0 \\ \text{undefiniert} & \text{falls } b = 0, a = 0 \\ -\frac{\pi}{2} & \text{falls } b = 0, a < 0 \\ arctan(\frac{a}{b}) + \pi & \text{falls } b < 0 \end{cases}
$$



Axis-angle representation of Rotation

Let's start from a geometric view point. Imagine a coordinate with a vector $\vec{X}$ where $\vec{k}$ is the unit vector representing the axis of rotation. Let the vector $\vec{x}$ be the result of rotating $\vec{X}$ by an angle θ about $\vec{k}$. You can imagine a circle created by $\vec{X}$ and $\vec{x}$ with the axis of rotation going through its center (see Figure 1).



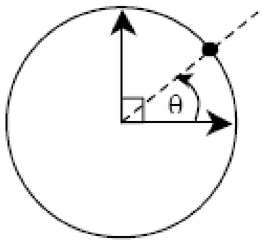


Figure 1: Axis and angle of rotation

Hence $\vec{x} = (\vec{X} \bullet \vec{k})\vec{k} + (\vec{X} - (\vec{X} \bullet \vec{k})\vec{k})\cos\theta + (\vec{k} \times \vec{X})\sin\theta$ (1)

(Also a good exercise to prove that $\vec{k} \times \vec{X}$ is perpendicular to $\vec{X} - (\vec{X} \bullet \vec{k})\vec{k}$)

Let define a skew symmetric matrix K such that $K = \mathrm{J}(\vec{k})$. This means

$$K = \begin{bmatrix} 0 & -k_3 & k_2 \\ k_3 & 0 & -k_1 \\ -k_2 & k_1 & 0 \end{bmatrix}$$ and we know that $K\vec{v} = \vec{k} \times \vec{v}$

Now we can write $\vec{x}$ as

$$
\begin{aligned}
\vec{x} &= \vec{X} - \vec{X} + (\vec{X} \bullet \vec{k})\vec{k} + (\vec{X} - (\vec{X} \bullet \vec{k})\vec{k})\cos\theta + (K\vec{X})\sin\theta \\
&= \vec{X} - (\vec{X} - (\vec{X} \bullet \vec{k})\vec{k}) + (\vec{X} - (\vec{X} \bullet \vec{k})\vec{k})\cos\theta + (K\vec{X})\sin\theta \\
&= \vec{X} - (1 - \cos\theta)(\vec{X} - (\vec{X} \bullet \vec{k})\vec{k}) + (K\vec{X})\sin\theta
\end{aligned} \tag{2}
$$

There exists an identity that $a \times (a \times b) = (a \bullet a)b - (a \bullet b)a$. You can also try to prove this for exercise as well. Now we can rewrite $(\vec{X} - (\vec{X} \bullet \vec{k})\vec{k})$ using this identity as

$$
(\vec{X} - (\vec{X} \bullet \vec{k})\vec{k}) = (\vec{k} \bullet \vec{k})\vec{X} - (\vec{X} \bullet \vec{k})\vec{k} = \vec{k} \times (\vec{X} \times \vec{k}) = -\vec{k} \times (\vec{k} \times \vec{X}) \tag{3}
$$

Note here that $(\vec{k} \bullet \vec{k})$ is just 1, so this doesn't change anything. Then rewrite the result using the property of the skew symmetric matrix K, we get

$$-\vec{k}\times(\vec{k}\times\vec{X}) = -\vec{k}\times K\vec{X} = -(K(K\vec{X})) = -K^2\vec{X} \qquad (4)$$

Substitute (4) in (2), we get

$$\begin{aligned}\vec{x} &= \vec{X} - (1-\cos\theta)(K^2\vec{X}) + (K\vec{X})\sin\theta \\ &= (I + (1-\cos\theta)K^2 + \sin\theta K)\vec{X}\end{aligned} \qquad (5)$$

Since $\vec{x} = R\vec{X}$, therefore, the rotation matrix is described by

$$\boxed{R = (I + (1-\cos\theta)K^2 + \sin\theta K)} \qquad (6)$$

Rodrigues formula.

Now we can use this formula to find back $\vec{k}$ and θ. Knowing that $R^T(\vec{k},\theta) = R(\vec{k},-\theta)$ applying Rodrigues formula for both sides, we will get

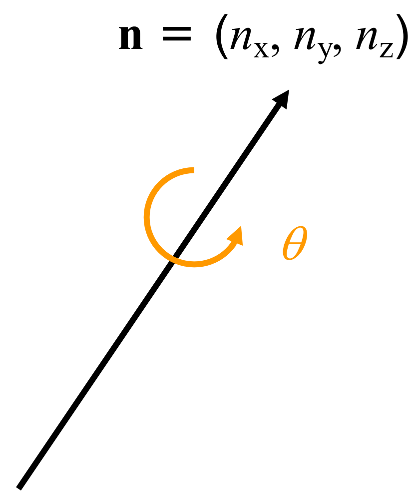
$$\begin{aligned}R - R^T &= 2\sin\theta K \\ K &= \frac{R-R^T}{2\sin\theta}\end{aligned} \qquad (7)$$

Hence, $\vec{k} = \dfrac{1}{2\sin\theta} vect(K)$ and θ can be determined by solving $2\sin\theta = \left\| vect(R - R^T) \right\|$

Note: Problems arise when θ is small since the axis of rotation is ill-defined and that $(\vec{k},\theta)$ and $(-\vec{k},-\theta)$ result in the same orientation.

Example: rigid obot in 3-D workspace

- Parametrization of orientations by **unit quaternion**: $u = (u_1, u_2, u_3, u_4)$ with $u_1^2 + u_2^2 + u_3^2 + u_4^2 = 1$.
  - Note $(u_1, u_2, u_3, u_4) = (\cos\theta/2, n_x \sin\theta/2, n_y \sin\theta/2, n_z \sin\theta/2)$ with $n_x^2 + n_y^2 + n_z^2 = 1$.
  - Compare with representation of orientation in 2-D:
    $(u_1, u_2) = (\cos\theta, \sin\theta)$



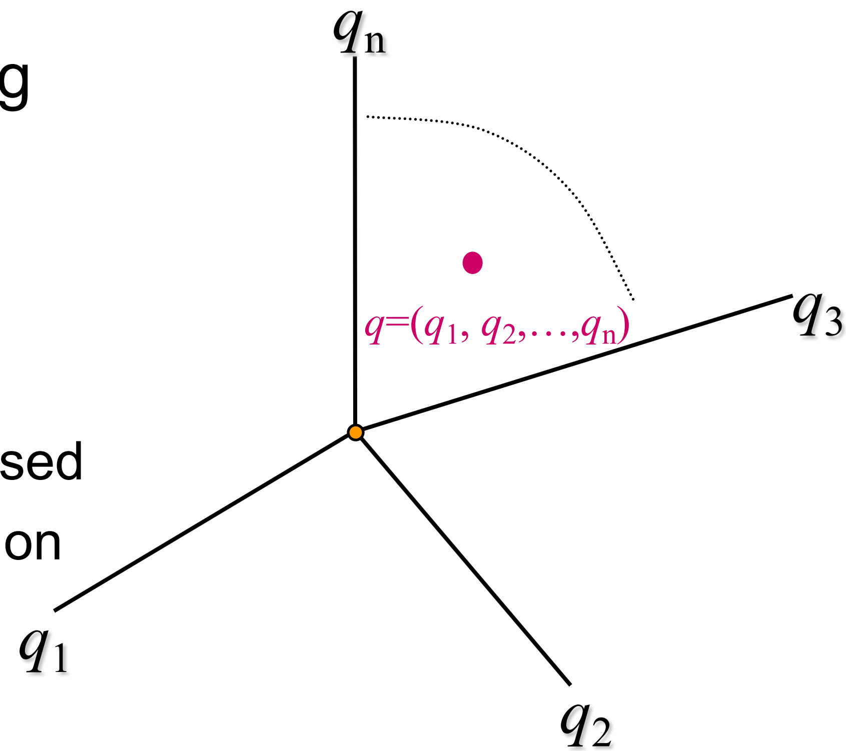
Example: rigid robot in 3-D workspace

- Advantage of unit quaternion representation
  - Compact
  - No singularity
  - Naturally reflect the topology of the space of orientations

- Number of dofs = 6
- Topology: $R^3 \times SO(3)$

Configuration space

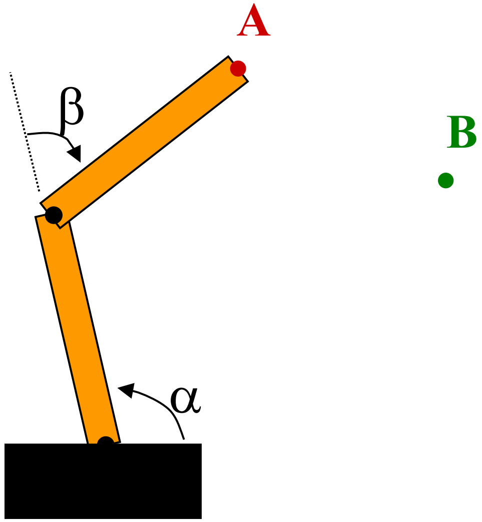
- The **configuration** of a moving object is a specification of the position of **every** point on the object.
  - Usually a configuration is expressed as a vector of position & orientation parameters: $q = (q_1, q_2,\ldots,q_n)$



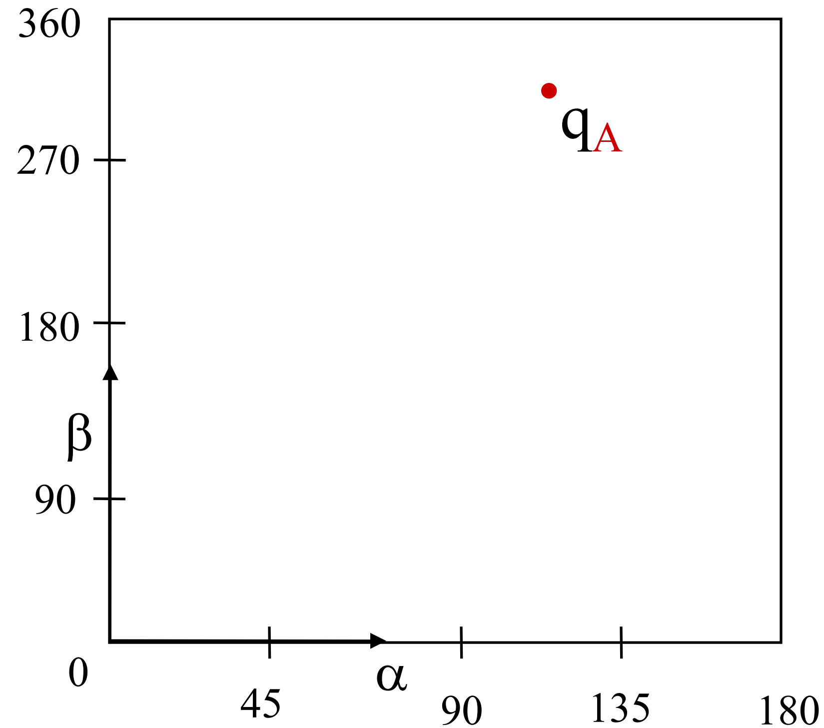
- The **configuration space** C is the set of all possible configurations.
  - A configuration is a point in C.

Configuration Space

Where can we put • q_B ?



An obstacle in the robot's workspace

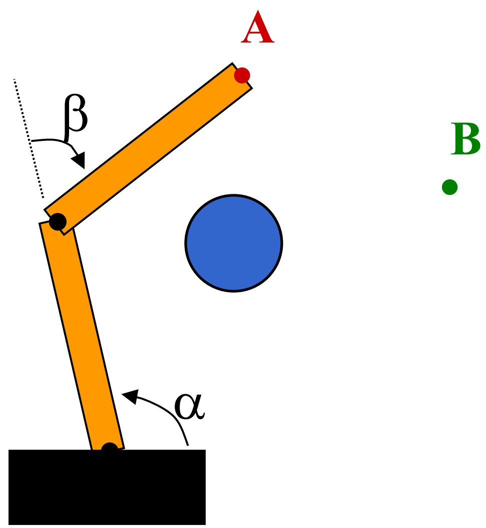


Torus

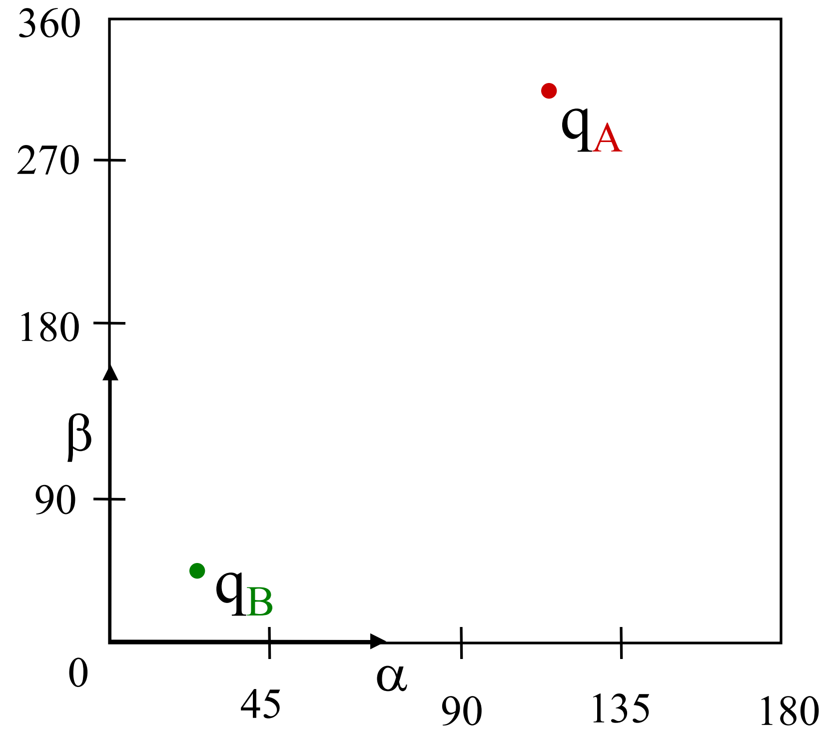
(wraps horizontally and vertically)

Configuration Space "Quiz"

Where do we put ?



An obstacle in the robot's workspace

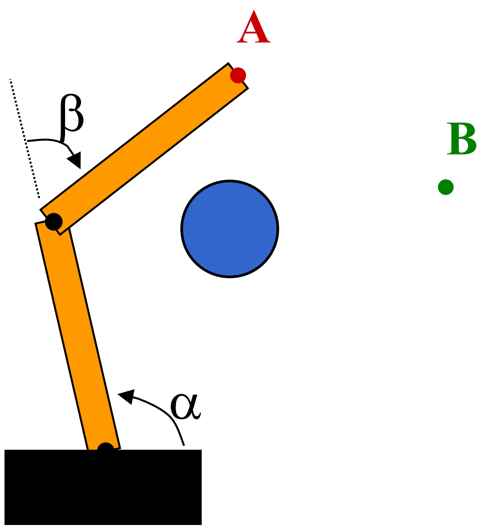


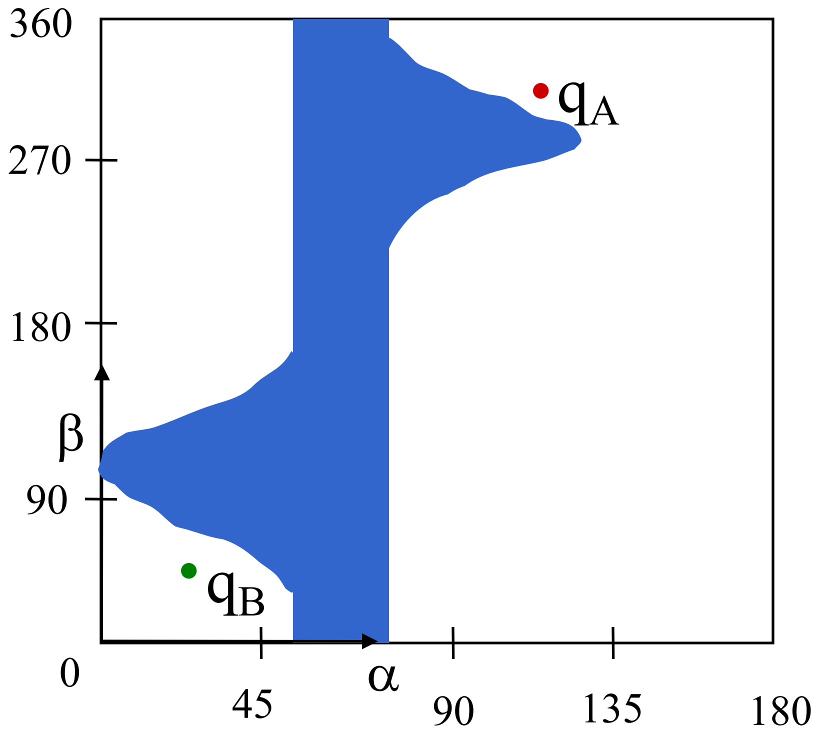
Torus

(wraps horizontally and vertically)

Configuration Space

How do we get from A to B ?





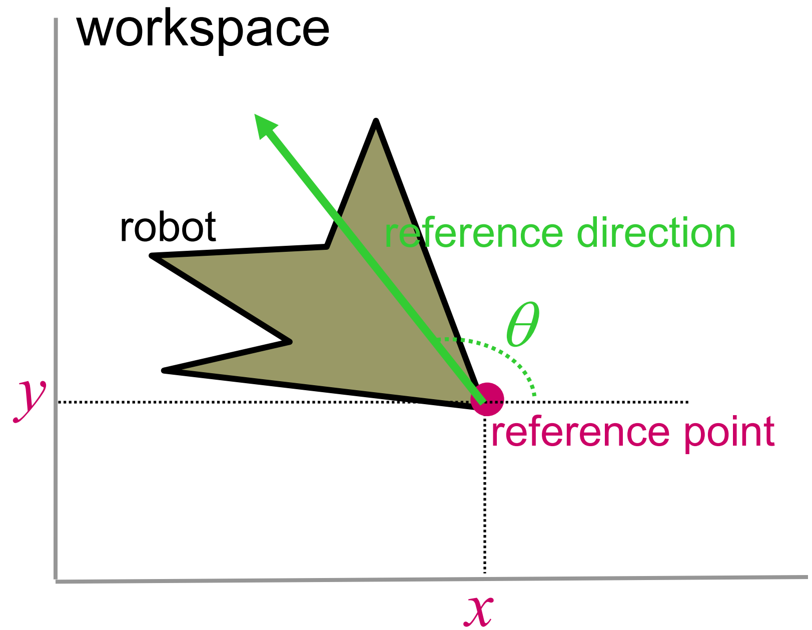
An obstacle in the robot's workspace

The C-space representation of this obstacle…

Dimension of configuration space

- The **dimension of a configuration space** is the **minimum** number of parameters needed to specify the configuration of the object completely.
- It is also called the **number of degrees of freedom** (dofs) of a moving object.

Example: rigid robot in 2-D workspace



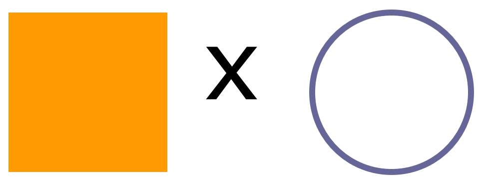
- 3-parameter specification: $q = (x,\ y,\ \theta)$ with $\theta \in [0,\ 2\pi)$.
  - 3-D configuration space

Example: rigid robot in 2-D workspace

- 4-parameter specification: $q = (x, y, u, v)$ with $u^2+v^2 = 1$. Note $u = \cos\theta$ and $v = \sin\theta$.

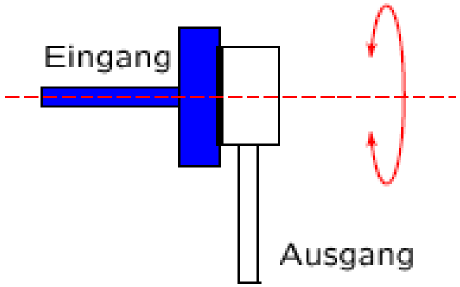
- dim of configuration space = **???** **3**
  - Does the dimension of the configuration space (number of dofs) depend on the parametrization?

- Topology: a 3-D cylinder $C = \mathrm{R}^2 \times \mathrm{S}^1$

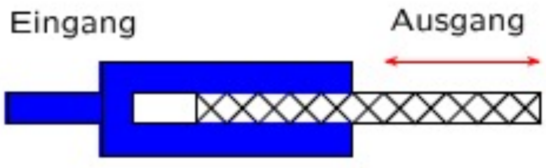


  - Does the topology depend on the parametrization?

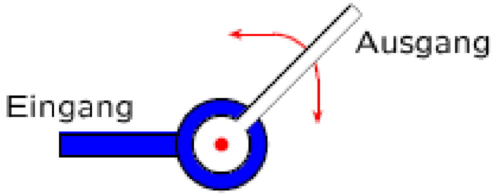
Motion of the robot is generated by its joints



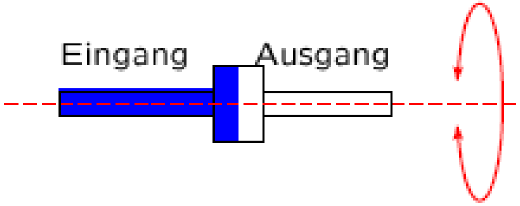
revolving joint



linear joint



rotational joint

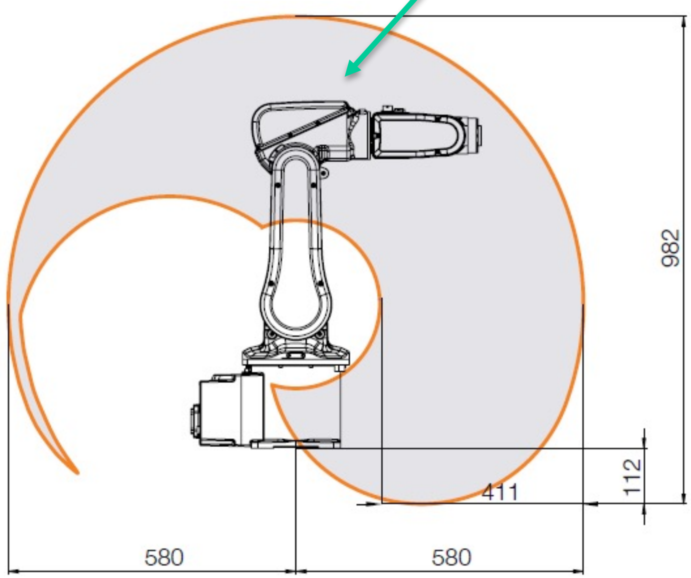


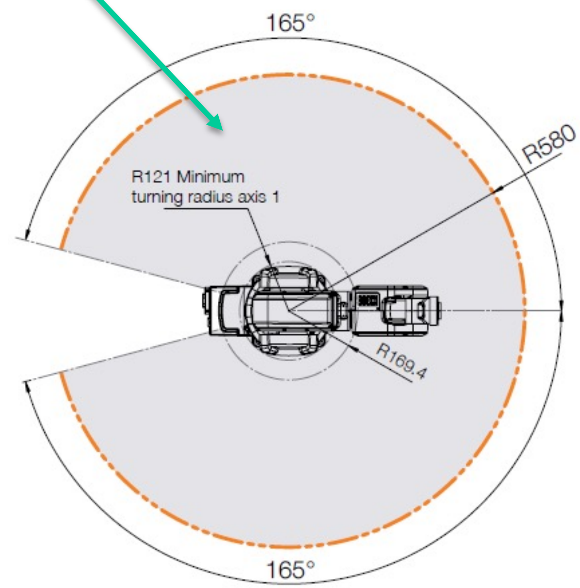
twisting joint

Workspace

- Example: ABB's IRB 120 robot

WS_1 (=WS_2 for spherical wrist)



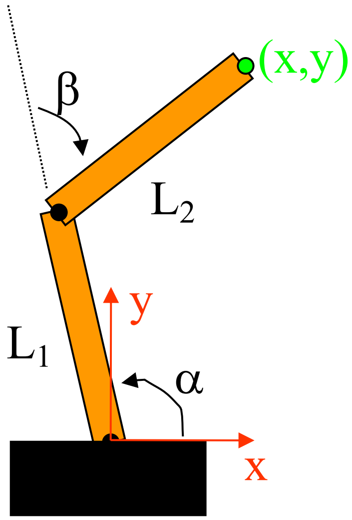


It is used to evaluate the robot for a specific application

[ABB IRB 120 Industrial Robot – Datasheet]

Robot Manipulators

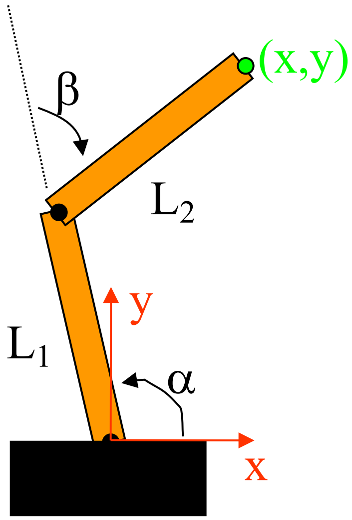
What are this arm’s forward kinematics?



(How does its position depend on its joint angles?)

Robot Manipulators

What are this arm's forward kinematics?



Find (x,y) in terms of α and β ...

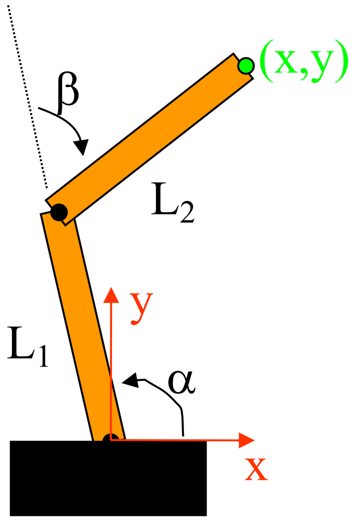
Keeping it "simple"

$c_\alpha = \cos(\alpha)$, $s_\alpha = \sin(\alpha)$

$c_\beta = \cos(\beta)$, $s_\beta = \sin(\beta)$

$c_+ = \cos(\alpha+\beta)$, $s_+ = \sin(\alpha+\beta)$

Manipulator kinematics



$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} L_1 c_\alpha \\ L_1 s_\alpha \end{pmatrix} + \begin{pmatrix} L_2 c_+ \\ L_2 s_+ \end{pmatrix}$$ Position

Keeping it "simple"

$c_\alpha = \cos(\alpha)$, $s_\alpha = \sin(\alpha)$

$c_\beta = \cos(\beta)$, $s_\beta = \sin(\beta)$

$c_+ = \cos(\alpha+\beta)$, $s_+ = \sin(\alpha+\beta)$

In general, a point in n-D space transforms by

P' = rotate(point) + translate(point)

In 2-D space, this can be written as a matrix equation:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} Cos(\theta) & -Sin(\theta) \\ Sin(\theta) & Cos(\theta) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} tx \\ ty \end{pmatrix}$$

In 3-D space (or n-D), this can generalized as a matrix equation:

$p' = R\,p + T$ or $p = R^t\,(p' - T)$

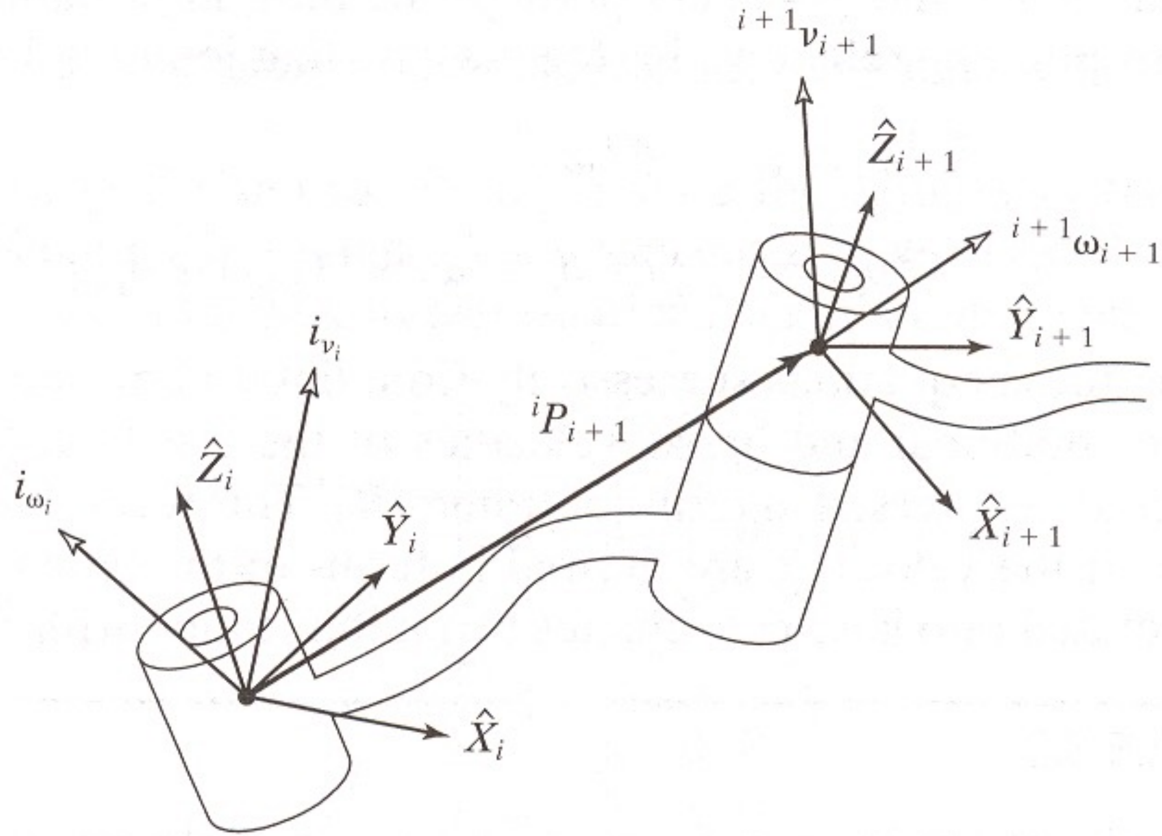
Now, using the idea of homogeneous transforms,
we can write:

$$p' = \begin{pmatrix} & R & & T \\ 0 & 0 & 0 & 1 \end{pmatrix} p$$

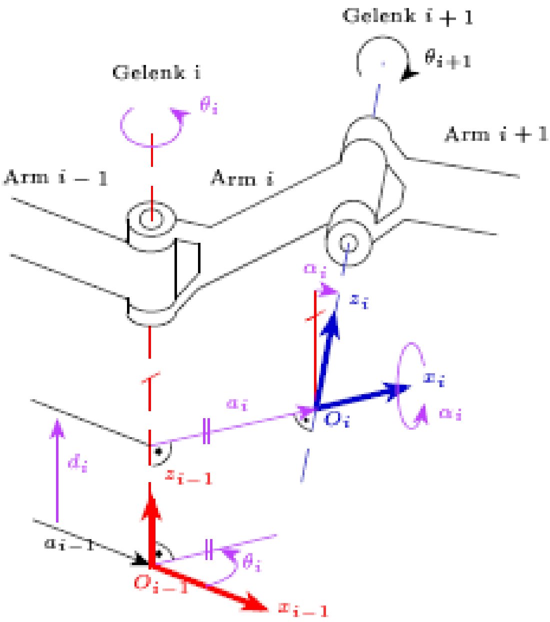
The group of rigid body rotations SO(3) $\times$ $\Re$(3) is
denoted SE(3) (for special Euclidean group)

What does the inverse transformation look like?

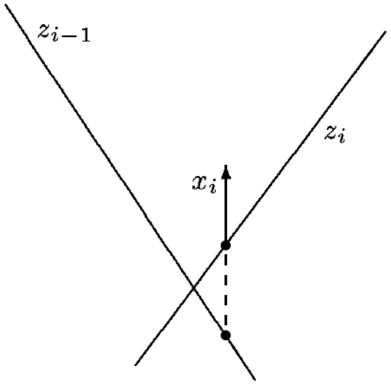
Coordinate frames



z-axis along the axis of motion
x-axis perpendicular on the two consecutive motion axes
y in a direction defined by a right hand system



DH Rules



1. Translation along z_i
2. Rotation around z_i
3. Translation along to the origin of the next frame
4. Rotation between the coordinate frames

$\boldsymbol{T}(0,0,d_i)$
$\boldsymbol{R}(z,\theta_i)$
$\boldsymbol{T}(a_i,0,0)$
$\boldsymbol{R}(x,\alpha_i)$

$$^{i-1}\boldsymbol{A}_i = \boldsymbol{T}(0,0,d_i)\cdot\boldsymbol{R}(z,\theta_i)\cdot\boldsymbol{T}(a_i,0,0)\cdot\boldsymbol{R}(x,\alpha_i)$$

$$^{i-1}\boldsymbol{A}_i = \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&d_i\\0&0&0&1\end{pmatrix}\cdot\begin{pmatrix}C\theta_i&-S\theta_i&0&0\\S\theta_i&C\theta_i&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\cdot\begin{pmatrix}1&0&0&a_i\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\cdot\begin{pmatrix}1&0&0&0\\0&C\alpha_i&-S\alpha_i&0\\0&S\alpha_i&C\alpha_i&0\\0&0&0&1\end{pmatrix}$$

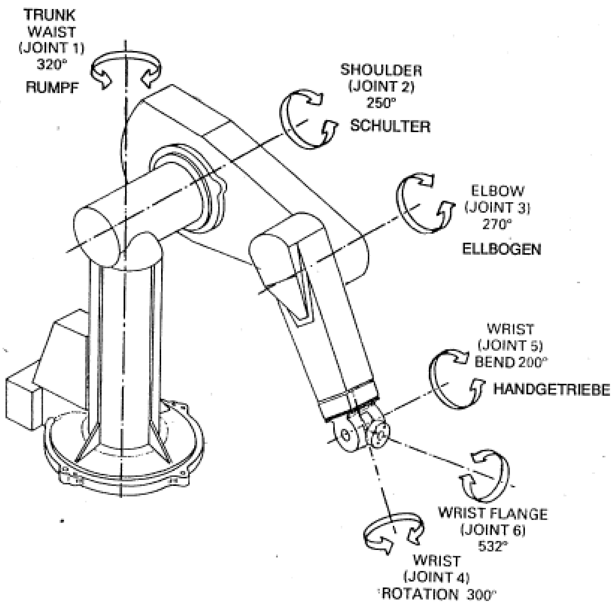
.. results in

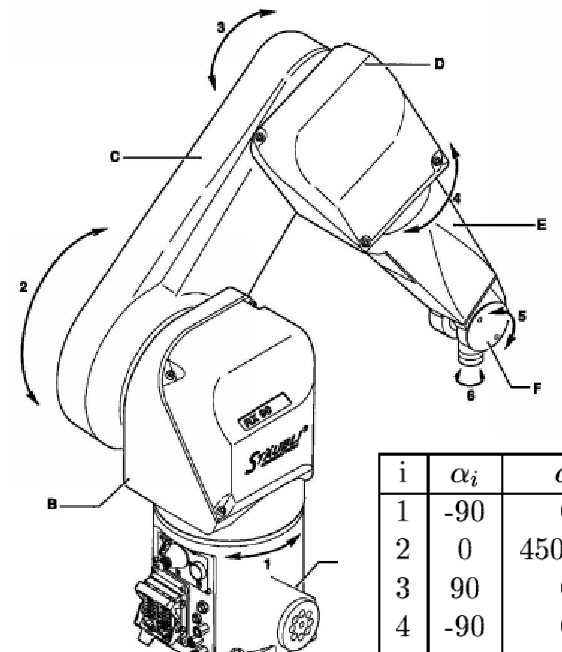
$$
{}^{i-1}\boldsymbol{A}_i = \begin{pmatrix} C\theta_i & -C\alpha_i \cdot S\theta_i & S\alpha_i \cdot S\theta_i & a_i \cdot C\theta_i \\ S\theta_i & C\alpha_i \cdot C\theta_i & -S\alpha_i \cdot C\theta_i & a_i \cdot S\theta_i \\ 0 & S\alpha_i & C\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

… the inverse direction can be calculated to

$$
{}^{i-1}\boldsymbol{A}_i^{-1} = {}^{i}\boldsymbol{A}_{i-1} = \begin{pmatrix} C\theta_i & S\theta_i & 0 & -a_i \\ -C\alpha_i \cdot S\theta_i & C\alpha_i \cdot C\theta_i & -S\alpha_i & -d_i \cdot S\alpha_i \\ S\alpha_i \cdot S\theta_i & -S\alpha_i \cdot C\theta_i & C\alpha_i & -d_i \cdot C\alpha_i \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

RX-90





i	α_i	a_i	d_i	θ_i-Bereich
1	-90	0	0	$-160° \cdots +160°$
2	0	450mm	0	$-227.5° \cdots +47.5°$
3	90	0	0	$-52.5° \cdots +232.5°$
4	-90	0	$450mm$	$-270° \cdots +270°$
5	90	0	0	$-105° \cdots +120°$
6	0	0	$85mm$	$-270° \cdots +270°$

$$
{}^0\boldsymbol{A}_6 = {}^0\boldsymbol{A}_1 \cdot {}^1\boldsymbol{A}_2 \cdot {}^2\boldsymbol{A}_3 \cdot {}^3\boldsymbol{A}_4 \cdot {}^4\boldsymbol{A}_5 \cdot {}^5\boldsymbol{A}_6
$$

$$
{}^0\boldsymbol{A}_3 = {}^0\boldsymbol{A}_1 \cdot {}^1\boldsymbol{A}_2 \cdot {}^2\boldsymbol{A}_3 = \begin{pmatrix} \cos\theta_1 \cdot \cos(\theta_2+\theta_3) & -\sin\theta_1 & \cos\theta_1 \cdot \cos(\theta_2+\theta_3) & a_2 \cdot \cos\theta_1 \cdot \cos\theta_2 \\ \sin\theta_1 \cdot \cos(\theta_2+\theta_3) & \cos\theta_1 & \sin\theta_1 \cdot \sin(\theta_2+\theta_3) & a_2 \cdot \sin\theta_1 \cdot \cos\theta_2 \\ -\sin(\theta_2+\theta_3) & 0 & \cos(\theta_2+\theta_3) & -a_2\sin\theta_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

Forward Kinematics

- Given joint variables $q = (q_1, q_2, \cdots q_n)$
- End-effector position & orientation $Y = (x, y, z, \phi, \theta, \psi)$

Homogeneous matrix T_0^n

- specifies the location of the ith coordinate frame w.r.t. the base coordinate system
- chain product of successive coordinate transformation matrices of T_{i-1}^i

$$T_0^n = T_0^1 T_1^2 \ldots T_{n-1}^n$$

$$= \begin{bmatrix} \boxed{R_0^n} & \boxed{P_0^n} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} n & s & a & P_0^n \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Orientation matrix

Position vector

Yaw-Pitch-Roll representation for orientation

$$T_0^n = \begin{bmatrix} C\phi C\vartheta & C\phi S\theta S\psi - S\phi C\psi & C\phi S\theta C\psi + S\phi S\psi & p_x \\ S\phi C\theta & S\phi S\theta S\psi + C\phi C\psi & S\phi S\theta C\psi - C\phi S\psi & p_y \\ -S\theta & C\theta S\psi & C\theta C\psi & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

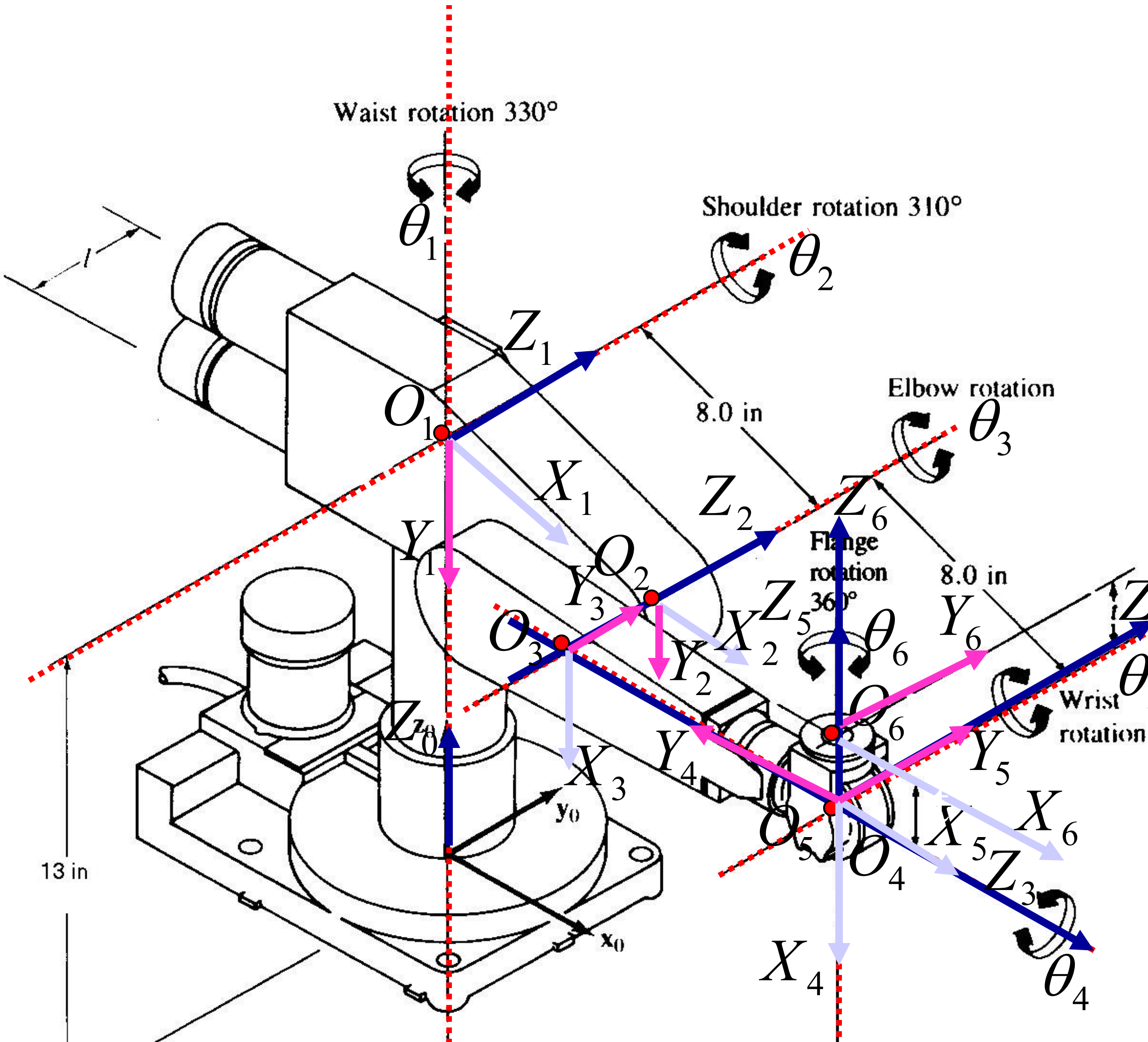
$$T_0^n = \begin{bmatrix} n_x & s_x & a_x & p_x \\ n_y & s_y & a_y & p_y \\ n_z & s_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\theta = \sin^{-1}(-n_z)$$

$$\psi = \cos^{-1}\left(\frac{a_z}{\cos\theta}\right)$$

$$\phi = \cos^{-1}\left(\frac{n_x}{\cos\theta}\right)$$

Example: PUMA 260



PUMA 260

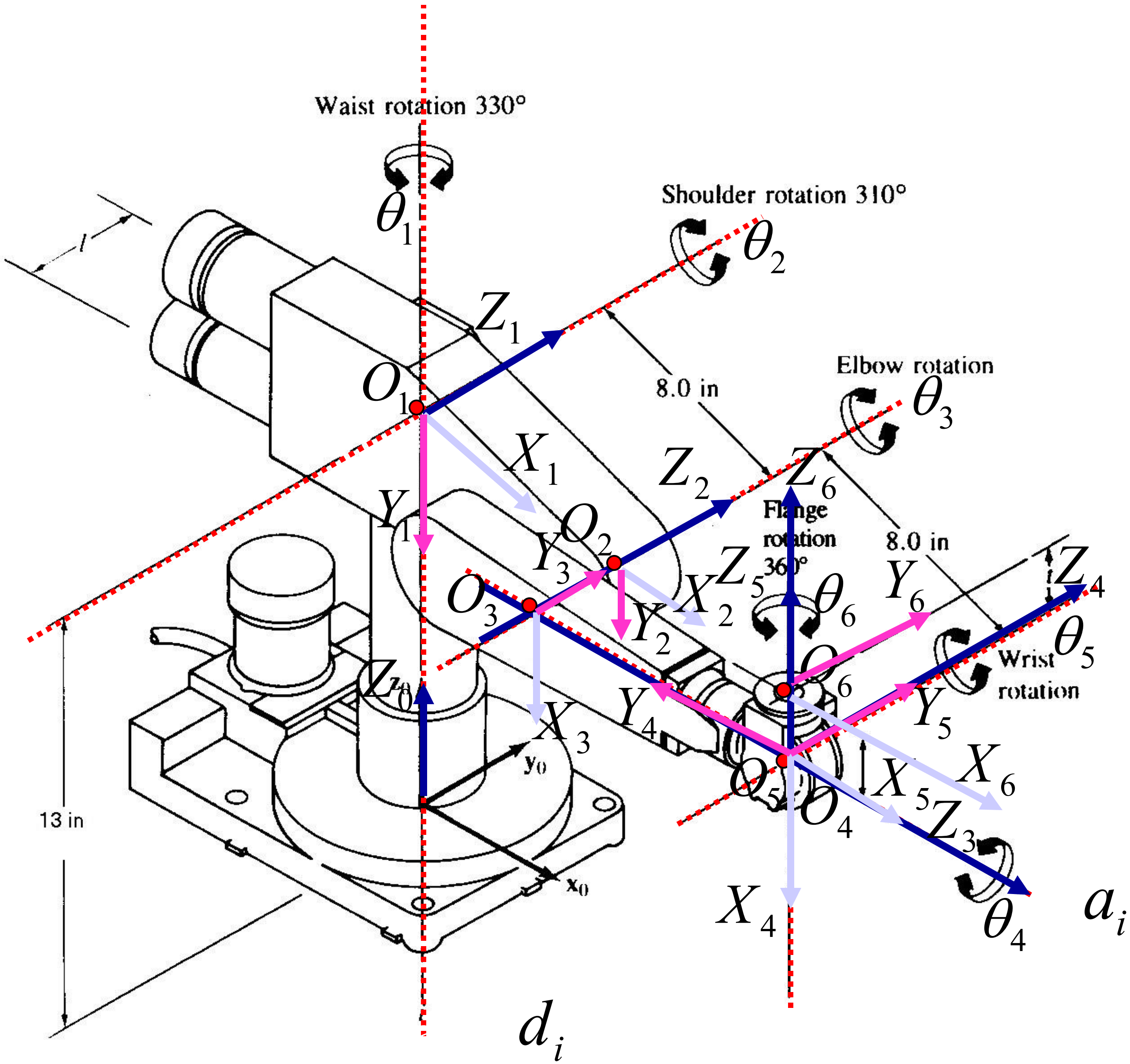
1. Number the joints
2. Establish base frame
3. Establish joint axis Z_i
4. Locate origin, (intersect. of Z_i & Z_{i-1}) OR (intersect of common normal & Z_i)

$$X_i = \pm(Z_{i-1} \times Z_i)/\|Z_{i-1} \times Z_i\|$$

$$Y_i = \pm(Z_i \times X_i)/\|Z_i \times X_i\|$$

5. Establish X_i,Y_i

Link Parameters



J	θ_i	α_i	a_i	d_i
1	θ_1	-90	0	13
2	θ_2	0	8	0
3	θ_3	90	0	
4	θ_4	-90	0	8
5	θ_5	90	0	0
6	θ_6	0	0	t

θ_i

α_i