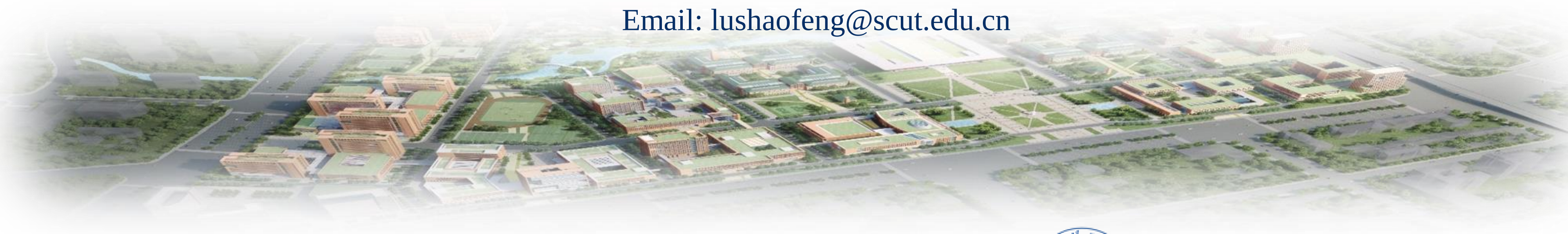
Lecture 2: Motion of Particles: Normal-Tangential and Polar Coordinates

Vm240: Introduction to Dynamics and Vibration

Shaofeng Lu
Shien-Ming Wu School of Intelligent Engineering
South China University of Technology
Email: lushaofeng@scut.edu.cn



吴贤铭智能工程学院
SHIEN-MING WU SCHOOL OF INTELLIGENT ENGINEERING

Contents

- Describing motion of particles: motion along a curved path
- Main concept: using normal-tangential and polar coordinates
  - Review of some aspects of vectors
  - Circular motion
    - Cartesian Coordinates
    - Normal-Tangential Coordinates
  - Motion along an arbitrary planar path: normal/tangential coordinates
  - Motion along an arbitrary planar path: polar coordinates.

Vector Operations: A Quick Review

Quick review of some vector operations

- Dot Product
  - Definition: $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta$
  - Cartesian component form:
    - $\mathbf{a} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$
    - $\mathbf{b} = b_x\mathbf{i} + b_y\mathbf{j} + b_z\mathbf{k}$
    - $\Rightarrow \mathbf{a} \cdot \mathbf{b} = a_x b_x + a_y b_y + a_z b_z$
  - Useful results:
    - Magnitude: $|\mathbf{a}| = \sqrt{a \cdot a} = \sqrt{a_x^2 + a_y^2 + a_z^2}$
    - Unit vector $\mathbf{n}$ parallel to a vector $\mathbf{a}$: $\mathbf{n} = \frac{\mathbf{a}}{|\mathbf{a}|}$
    - Dot products of basis vectors: $\mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k} = \mathbf{1}$ $\quad \mathbf{i} \cdot \mathbf{j} = \mathbf{i} \cdot \mathbf{k} = \mathbf{j} \cdot \mathbf{k} = \mathbf{0}$

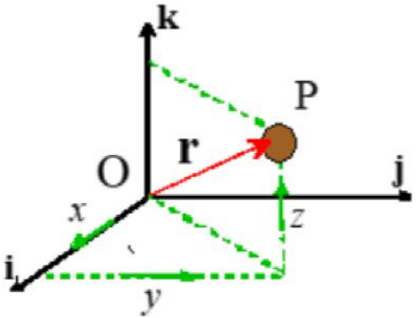
Quick review of some vector operations

- Cross Product
  - If $\mathbf{c} = \mathbf{a} \times \mathbf{b}$, then
    - $|\mathbf{c}| = |\mathbf{a}| \times |\mathbf{b}| \sin \boldsymbol{\theta}$
    - c is a perpendicular to a and b with right hand screw convention
    - Cross products of basis vectors:
      - $\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = \mathbf{0}$
      - $\mathbf{i} \times \mathbf{j} = -\mathbf{j} \times \mathbf{i} = \mathbf{k}$
      - $\mathbf{k} \times \mathbf{i} = -\mathbf{i} \times \mathbf{k} = \mathbf{j}$
      - $\mathbf{j} \times \mathbf{k} = -\mathbf{k} \times \mathbf{j} = \mathbf{i}$
  - Cartesian components:
    - $\mathbf{a} \times \mathbf{b} = \left(a_y b_z - a_z b_\mathbf{y}\right)\mathbf{i} + (a_z b_x - a_x b_z)\mathbf{j} + \left(a_x b_y - a_y b_x\right)\mathbf{k}$

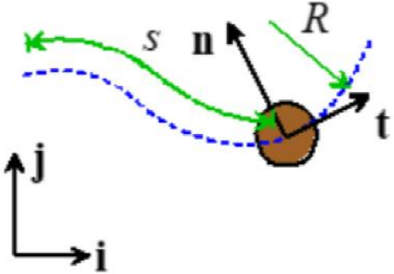
Basis Vectors and Transformation

Basis Vectors – Background

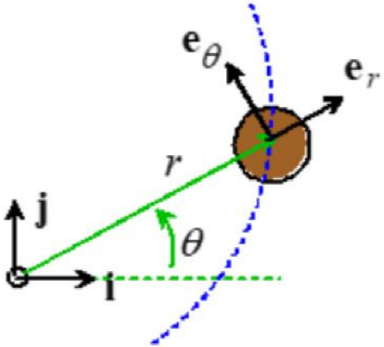
- We use many different coordinate systems in dynamics:



Cartesian



Normal-Tangential



Cylindrical-polar

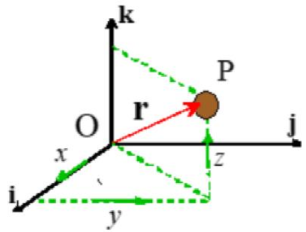
tangential 畅通词汇

英 [tæn'dʒenʃl] 美 [tæn'dʒenʃl]

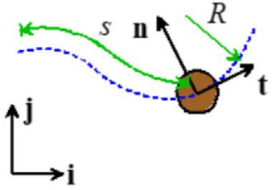
adj. 离题的；肤浅的；切线的；相切的

Basis Vectors – Background

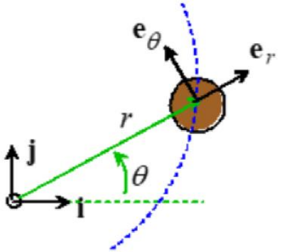
- We need to understand these concepts:
  - Basis vectors
  - Components of a vector in a basis
  - How to transform components from one basis to another?



Cartesian



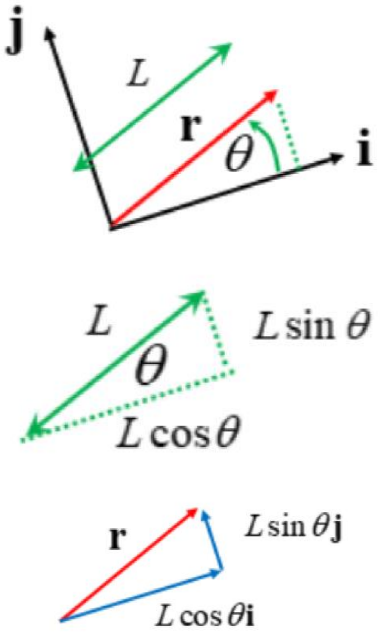
Normal-Tangential



Cylindrical-polar

Basis Vectors

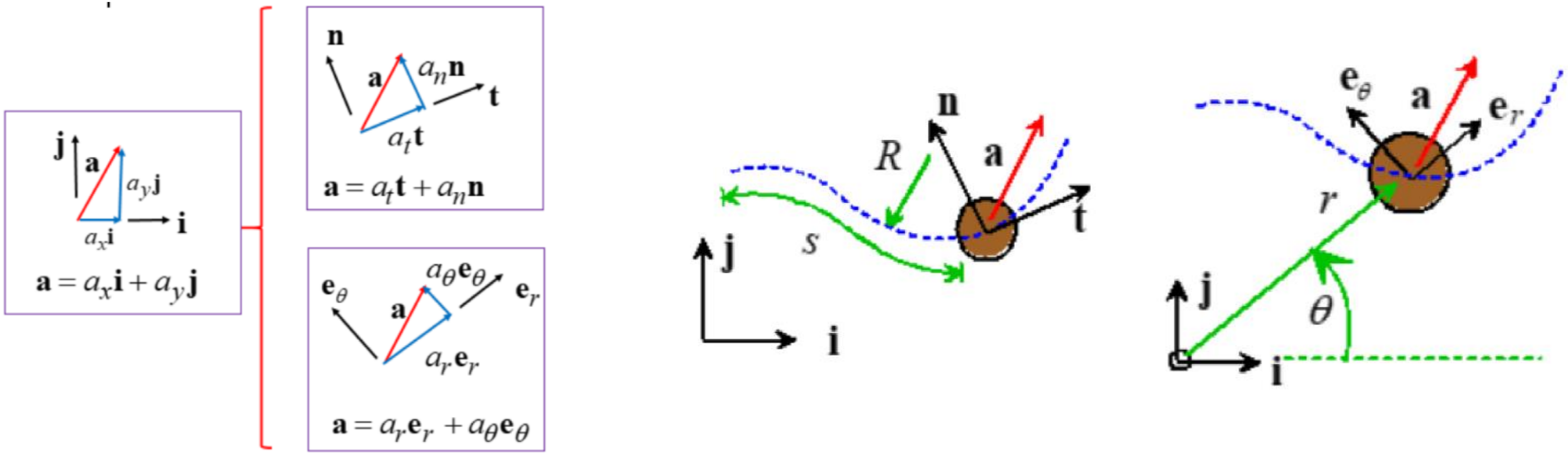
- Definition of a vector basis:
  - Any 3 or (2 in 2D) linearly independent vectors
  - Usually
    - Basis vectors have unit length
    - Basis vectors are mutually perpendicular
    - Example: $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ basis vectors for Cartesian components
  - Vector components (in a basis)
  - Any vector can be created by adding multiples of the basis vectors
  - Example: position vector $r = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$
  - We often do this by projection:



$$\mathbf{r} = L\cos\theta\mathbf{i} + L\sin\theta\mathbf{j}$$

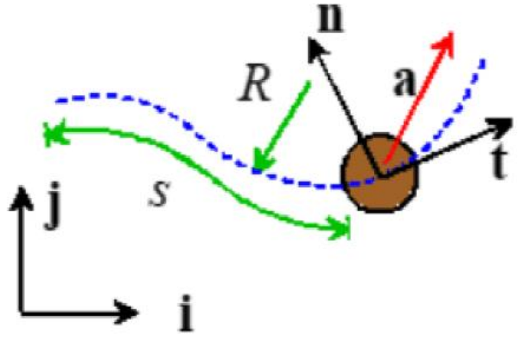
Basis Transformations

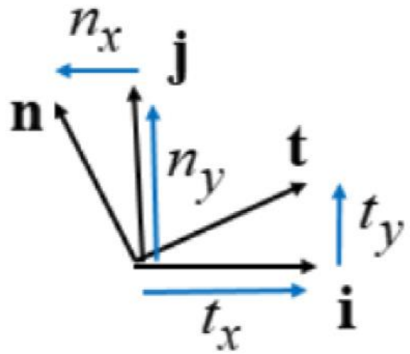
- Using more than one basis:
  - We can express the same vector as components in more than one basis
  - Example:



Basis transformations

- Converting from one basis to another:
  - Use any trick you can find
  - You can often do the projection directly (use trigonometric)
  - For a formal approach, use this:
  - To convert a from $\{\mathbf{i}, \mathbf{j}\}$ to $\{\mathbf{n}, \mathbf{t}\}$
    - **Step 1:** Write $\{\mathbf{n}, \mathbf{t}\}$ in $\{\mathbf{i}, \mathbf{j}\}$ components
    - $\mathbf{n} = n_x\mathbf{i} + n_y\mathbf{j}$ and $\mathbf{t} = t_x\mathbf{i} + t_y\mathbf{j}$
    - **Step 2:** $a_n = \mathbf{n} \cdot \mathbf{a} = \mathrm{n_x a_x} + \mathrm{n_y a_y}$
    - $a_t = \mathbf{t} \cdot \mathbf{a} = t_x a_x + t_y a_y$





Basis Transformations- Proof

- Proof:
  - $\mathbf{a} = a_n\mathbf{n} + a_t\mathbf{t} = a_x\mathbf{i} + a_y\mathbf{j}$
  - $\Rightarrow \mathbf{n}\cdot\mathbf{a} = a_n\mathbf{n}\cdot\mathbf{n} + a_t\mathbf{n}\cdot\mathbf{t} = \left(n_x\mathbf{i} + n_y\mathbf{j}\right)\cdot\left(a_x\mathbf{i} + a_y\mathbf{j}\right)$
  - $\Rightarrow a_n = n_x a_x + n_y a_y$
  - $\mathbf{t}\cdot\mathbf{a} = a_n\mathbf{t}\cdot\mathbf{n} + a_t\mathbf{t}\cdot\mathbf{t} = \left(t_x\mathbf{i} + t_y\mathbf{j}\right)\cdot\left(a_x\mathbf{i} + a_y\mathbf{j}\right)$
  - $\Rightarrow a_t = t_x a_x + t_y a_y$

Vector Operations in Other Bases

- Use all the usual formulas for magnitude, dot and cross products.

$$\mathbf{a} = a_t\mathbf{t} + a_n\mathbf{n} + a_z\mathbf{k} \qquad \mathbf{b} = b_t\mathbf{t} + b_n\mathbf{n} + b_z\mathbf{k}$$

$$\Rightarrow \mathbf{a}\cdot\mathbf{b} = a_t b_t + a_n b_n + a_z b_z$$

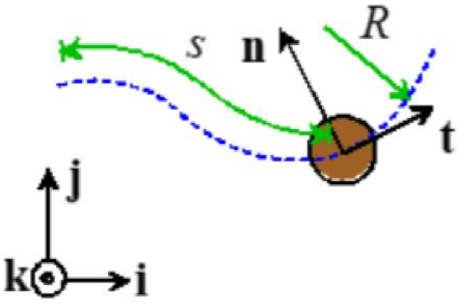
$$|\mathbf{a}| = \sqrt{\mathbf{a}\cdot\mathbf{a}} = \sqrt{a_t^2 + a_n^2 + a_z^2}$$

$$\mathbf{a}\times\mathbf{b} = \pm\left[(a_n b_z - a_z b_n)\mathbf{t} + (a_z b_t - a_t b_z)\mathbf{n} + (a_t b_n - a_n b_t)\mathbf{k}\right]$$

$$\mathbf{t}\times\mathbf{t} = \mathbf{n}\times\mathbf{n} = \mathbf{k}\times\mathbf{k} = 0$$

$$\mathbf{t}\times\mathbf{n} = -\mathbf{n}\times\mathbf{t} = \pm\mathbf{k} \quad \mathbf{k}\times\mathbf{t} = -\mathbf{t}\times\mathbf{k} = \pm\mathbf{n} \quad \mathbf{n}\times\mathbf{k} = -\mathbf{k}\times\mathbf{n} = \pm\mathbf{t}$$

Use + if **n** points to left of **t**, use – if **n** points to right of **t**



Normal-Tangential

Vector Operations in Cylindrical-Polar Bases

$$\mathbf{a} = a_r \mathbf{e}_r + a_\theta \mathbf{e}_\theta + a_z \mathbf{k} \qquad \mathbf{b} = b_r \mathbf{e}_r + b_\theta \mathbf{e}_\theta + b_z \mathbf{k}$$

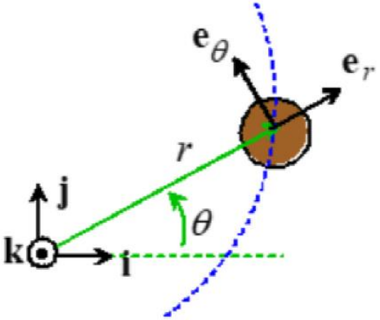
$$\Rightarrow \mathbf{a} \cdot \mathbf{b} = a_r b_r + a_\theta b_\theta + a_z b_z$$

$$|\mathbf{a}| = \sqrt{\mathbf{a} \cdot \mathbf{a}} = \sqrt{a_r^2 + a_\theta^2 + a_z^2}$$

$$\mathbf{a} \times \mathbf{b} = (a_\theta b_z - a_z b_\theta)\mathbf{e}_r + (a_z b_r - a_r b_z)\mathbf{e}_\theta + (a_r b_\theta - a_\theta b_r)\mathbf{k}$$

$$\mathbf{e}_r \times \mathbf{e}_r = \mathbf{e}_\theta \times \mathbf{e}_\theta = \mathbf{k} \times \mathbf{k} = 0$$

$$\mathbf{e}_r \times \mathbf{e}_\theta = -\mathbf{e}_\theta \times \mathbf{e}_r = \mathbf{k} \quad \mathbf{k} \times \mathbf{e}_r = -\mathbf{e}_r \times \mathbf{k} = \mathbf{e}_\theta \quad \mathbf{e}_\theta \times \mathbf{k} = -\mathbf{k} \times \mathbf{e}_\theta = \mathbf{e}_r$$

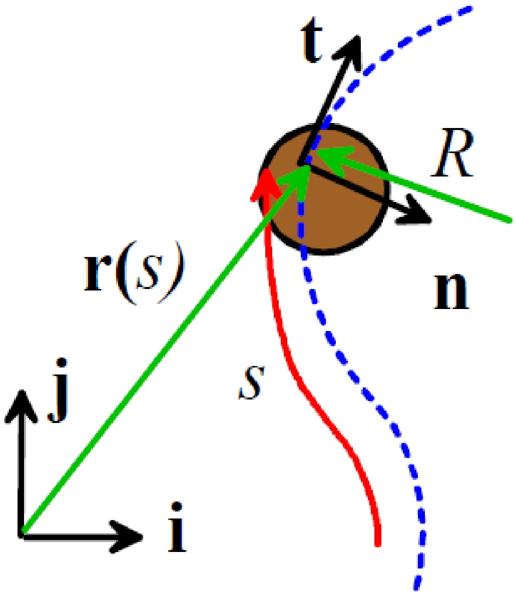


Cylindrical-polar

Normal-tangential Coordinates for Particles

Definition of Normal-Tangential coordinates

- Position vector of a point on the path in terms of the distance s travelled along the path
  - $\mathbf{r}(s) = x(s)\mathbf{i} + y(s)\mathbf{j}$
- Introducing two unit vectors **n** and **t**, with **n** pointing normal to the path.
- Introduce the radius of curvature of the path R (given in most cases).



Deriving the Velocity and Acceleration in n-t Coordinates

- Use the following formulas to calculate speed, velocity and acceleration:
  - $\mathbf{v} = V\mathbf{t}$
  - $\mathbf{a} = \frac{dV}{dt}\mathbf{t} + \frac{V^2}{R}\mathbf{n} = \underbrace{a_t}_{\text{Tangential acceleration}}\mathbf{t} + \underbrace{a_n}_{\text{Normal acceleration}}\mathbf{n}$
- Alternatively, we use formula $V = \frac{ds}{dt}$ to write velocity and acceleration in terms of distance:
  - $\mathbf{v} = \frac{ds}{dt}\mathbf{t}$
  - $\boldsymbol{a} = \frac{dV}{dt}\mathbf{t} + \frac{1}{\mathrm{R}}\left(\frac{ds}{dt}\right)^2\mathbf{n}$

Deriving the n-t Coordinate Formulas

- Formula for the path:

Distance travelled $s(t)$, $\quad \mathbf{r} = x(s)\mathbf{i} + y(s)\mathbf{j}$

Speed $\quad V = \dfrac{ds}{dt}$

Definitions of $\{\mathbf{n}, \mathbf{t}\}$ $\quad \mathbf{t} = \dfrac{d\mathbf{r}}{ds} \qquad \mathbf{n} = R\dfrac{d\mathbf{t}}{ds} = R\dfrac{d^2\mathbf{r}}{ds^2}$

Velocity $\quad \mathbf{v} = \dfrac{d\mathbf{r}}{dt} = \dfrac{d\mathbf{r}}{ds}\dfrac{ds}{dt} \Rightarrow \mathbf{v} = V\mathbf{t}$

Acceleration $\quad \mathbf{a} = \dfrac{d\mathbf{v}}{dt} = \dfrac{d(V\mathbf{t})}{dt} = \dfrac{dV}{dt}\mathbf{t} + V\dfrac{d\mathbf{t}}{dt} = \dfrac{dV}{dt}\mathbf{t} + V\dfrac{d\mathbf{t}}{ds}\dfrac{ds}{dt}$

$$\Rightarrow \mathbf{a} = \frac{dV}{dt}\mathbf{t} + \frac{V^2}{R}\mathbf{n}$$

Radius of curvature

$$\mathbf{n} \cdot \mathbf{n} = 1 \Rightarrow R^2 \frac{d^2\mathbf{r}}{ds^2} \cdot \frac{d^2\mathbf{r}}{ds^2} = 1$$

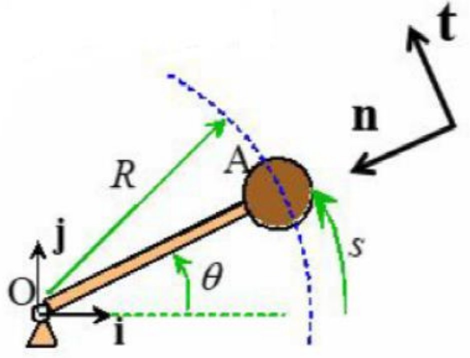
Note: $\dfrac{d^2\mathbf{r}}{ds^2} = \dfrac{d^2x}{ds^2}\mathbf{i} + \dfrac{d^2y}{ds^2}\mathbf{j}$

$$\boxed{\Rightarrow \quad \frac{1}{R^2} = \left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2}$$

Example: Circular Motion at Constant Speed

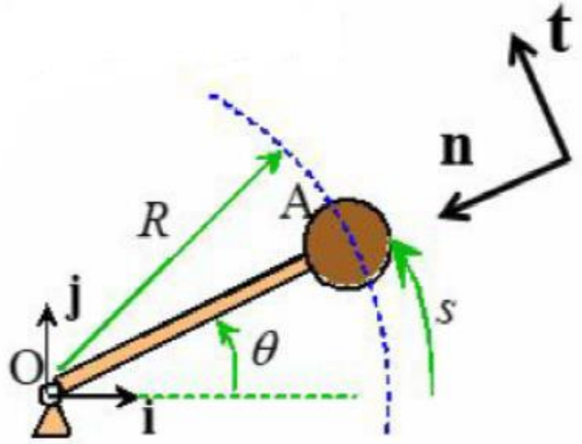
The bar OA rotates with constant angular speed ω
Find the speed, velocity and acceleration vectors of A
(give vectors in both $\{\mathbf{i},\mathbf{j}\}$ and $\{\mathbf{n},\mathbf{t}\}$ bases)

$\underset{\sim}{t}$: Tangent to path
$\underset{\sim}{n}$: Normal to path, towards center of circle
$\{\underset{\sim}{n}, \underset{\sim}{t}\}$ both unit vectors



Example: Circular Motion at Constant Speed

The bar OA rotates with constant angular speed ω
Find the speed, velocity and acceleration vectors of A
(give vectors in both $\{\mathbf{i},\mathbf{j}\}$ and $\{\mathbf{n},\mathbf{t}\}$ bases)



$\underline{t}$: Tangent to path
$\underline{n}$: Normal to path, towards center of circle
$\{\underline{n},\underline{t}\}$ both unit vectors

Preliminaries $\quad \theta = \omega t \quad \omega = \text{constant}$

Geometry $\quad s = R\theta \Rightarrow s = R\omega t$

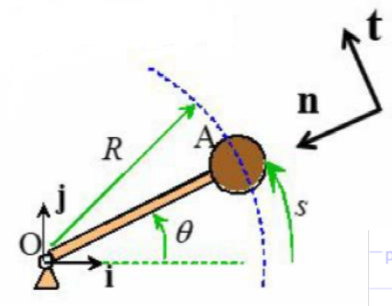
Speed $\quad V = \frac{ds}{dt} = R\frac{d\theta}{dt} \Rightarrow V = R\omega$

Position $\quad \underline{r} = R\cos\omega t\,\underline{i} + R\sin\omega t\,\underline{j}$

Example: Circular Motion at Constant Speed

The bar OA rotates with constant angular speed ω
Find the speed, velocity and acceleration vectors of A
(give vectors in both $\{\mathbf{i},\mathbf{j}\}$ and $\{\mathbf{n},\mathbf{t}\}$ bases)

$\underline{t}$: Tangent to path
$\underline{n}$: Normal to path, towards center of circle
$\{\underline{n},\underline{t}\}$ both unit vectors



Preliminaries $\theta = \omega t$ $\omega = \text{constant}$

Geometry $s = R\theta \Rightarrow s = R\omega t$

Speed $V = \frac{ds}{dt} = R\frac{d\theta}{dt} \Rightarrow V = R\omega$

Position $\underline{r} = R\cos\omega t\, \underline{i} + R\sin\omega t\, \underline{j}$

page 11

Velocity / Accel in Cartesian coords

Given : $\underline{r} = R\cos\omega t\,\underline{i} + R\sin\omega t\,\underline{j}$

Velocity $\underline{v} = \frac{d\underline{r}}{dt} = -R\omega\sin\omega t\,\underline{i} + R\omega\cos\omega t\,\underline{j}$

$\Rightarrow \underline{v} = R\omega(-\sin\omega t\,\underline{i} + \cos\omega t\,\underline{j})$

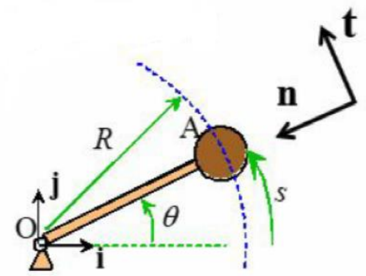
Recall $V = R\omega$

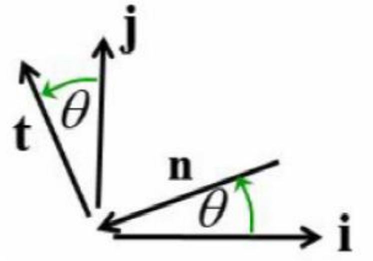
$\Rightarrow \underline{v} = \underbrace{V}_{\text{magnitude}}\underbrace{(-\sin\omega t\,\underline{i} + \cos\omega t\,\underline{j})}_{\text{Direction}}$

Example: Circular Motion at Constant Speed

The bar OA rotates with constant angular speed ω
Find the speed, velocity and acceleration vectors of A
(give vectors in both $\{\mathbf{i},\mathbf{j}\}$ and $\{\mathbf{n},\mathbf{t}\}$ bases)

$\underset{\sim}{t}$: Tangent to path
$\underset{\sim}{n}$: Normal to path, towards center of circle
$\{\underset{\sim}{n},\underset{\sim}{t}\}$ both unit vectors





Velocity/Accel in $\{\mathbf{n},\mathbf{t}\}$ coords.

Substitute in $\{\underset{\sim}{i},\underset{\sim}{j}\}$ component formulas

$\boxed{\underset{\sim}{v} = R\omega\,\underset{\sim}{t}}$ $\boxed{\underset{\sim}{v} = V\underset{\sim}{t}}$

$\boxed{\underset{\sim}{a} = R\omega^2\underset{\sim}{n}}$ $\boxed{\underset{\sim}{a} = V\omega\underset{\sim}{n}}$ $\boxed{\underset{\sim}{a} = (V^2/R)\underset{\sim}{n}}$

NB: V is constant in these formulas

Express $\{\mathbf{n},\mathbf{t}\}$ in $\{\mathbf{i},\mathbf{j}\}$ basis.
Note: $\{\mathbf{n},\mathbf{t}\}$ both have unit length.

$$\underset{\sim}{t} = -\sin\theta\,\underset{\sim}{i} + \cos\theta\,\underset{\sim}{j}$$

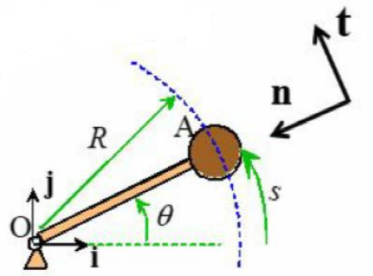
$$\underset{\sim}{n} = -\cos\theta\,\underset{\sim}{i} - \sin\theta\,\underset{\sim}{j}$$

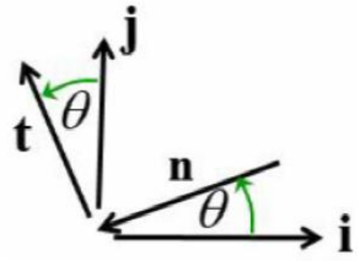
$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = R\omega(-\frac{d\theta}{dt}\cos\theta\mathbf{i} - \frac{d\theta}{dt}\sin\theta\mathbf{j}) = -R\omega^2(\cos\theta\mathbf{i} + \sin\theta\mathbf{j})$$

Example: Circular Motion at Constant Speed

The bar OA rotates with constant angular speed ω
Find the speed, velocity and acceleration vectors of A
(give vectors in both $\{\mathbf{i},\mathbf{j}\}$ and $\{\mathbf{n},\mathbf{t}\}$ bases)

$\underline{t}$: Tangent to path
$\underline{n}$: Normal to path, towards center of circle
$\{\underline{n},\underline{t}\}$ both unit vectors





Velocity/Accel in $\{\mathbf{n},\mathbf{t}\}$ coords.

Interpreting the velocity/accel formulas:

$\underline{v} = v\,\underline{t}$

Magnitude of velocity = speed
Direction is tangent to path

$\underline{a} = \frac{v^2}{R}\,\underline{n}$

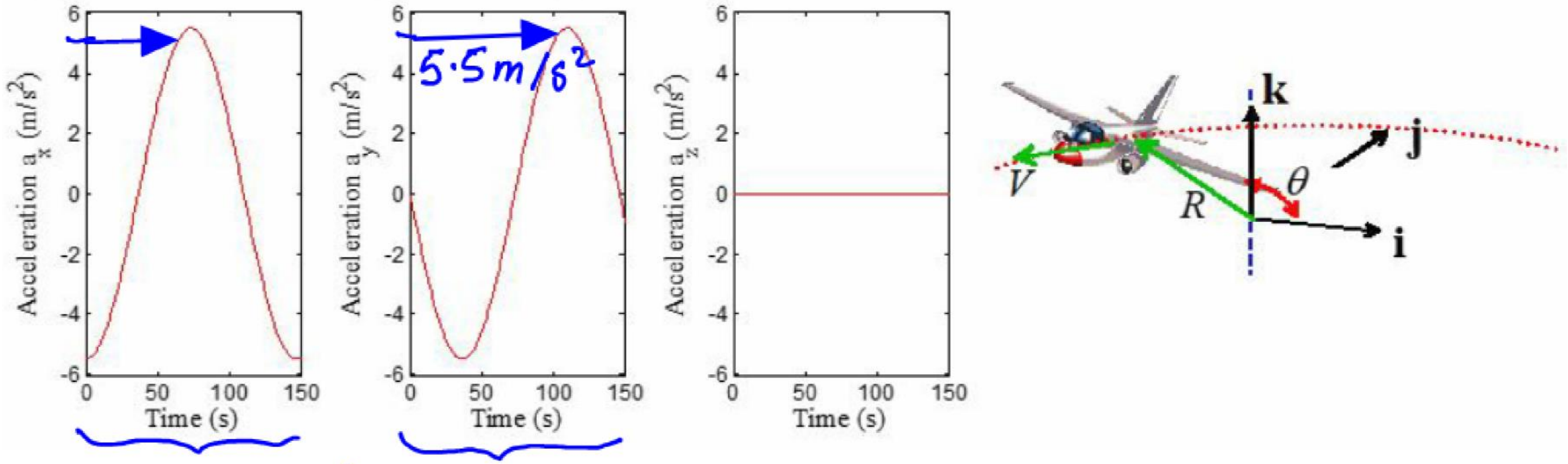
Magnitude of acceleration is v^2/R
Direction is normal to path, towards center of circle

Express $\{\mathbf{n},\mathbf{t}\}$ in $\{\mathbf{i},\mathbf{j}\}$ basis.
Note: $\{\mathbf{n},\mathbf{t}\}$ both have unit length.

Note: $\mathbf{a} \neq \mathbf{0}$ since direction of $\mathbf{v}$ changes

Example: Interpreting Acceleration Data from an Inertial Platform

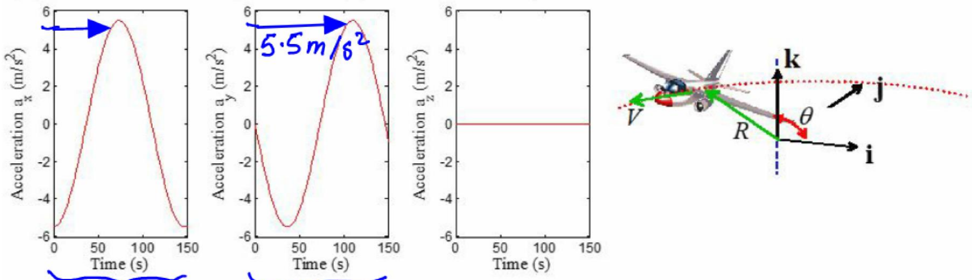
An inertial platform (with fixed orientation) records the accelerations shown. Determine:
(a) The radius of the path; and (b) The aircrafts speed



1 cycle in 150 s

Example: Interpreting Acceleration Data from an Inertial Platform

An inertial platform (with fixed orientation) records the accelerations shown. Determine:
(a) The radius of the path; and (b) The aircrafts speed



1 cycle in 150 s

Measurement: $\underline{a} = -5.5 \cos \frac{2\pi}{150} t \, \underline{i} - 5.5 \sin \frac{2\pi t}{150} \underline{j}$

Formula $\underline{a} = -R\omega^2 \cos \omega t \, \underline{i} - R\omega^2 \sin \omega t \, \underline{j}$

$$\mathbf{a} = -R\omega^2(\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}) = R\omega^2\mathbf{n}$$

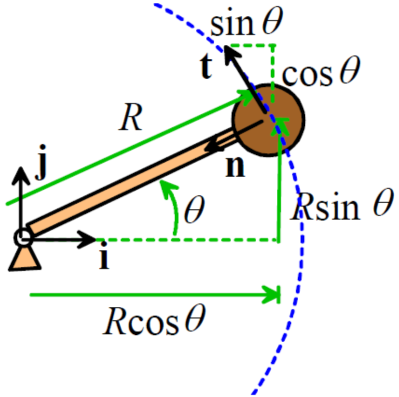
Compare: $R\omega^2 = 5.5 \quad \omega = 2\pi/150 \Rightarrow R = 5.5\left(\frac{150}{2\pi}\right)^2$

$\Rightarrow \boxed{R = 3.1 \text{ km}} \qquad \boxed{V = R\omega = 129 \text{ m/s}}$

Example Circular Motion at Arbitrary Speed

The angle $\theta(t)$ is an arbitrary function of time
Find the speed, velocity and acceleration vectors of A
(give vectors in both $\{\mathbf{i},\mathbf{j}\}$ and $\{\mathbf{n},\mathbf{t}\}$ bases)

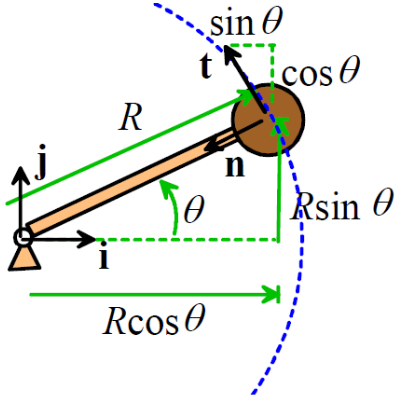
$$\mathbf{t} = -\sin\theta\mathbf{i} + \cos\theta\mathbf{j} \qquad \mathbf{n} = -\cos\theta\mathbf{i} - \sin\theta\mathbf{j}$$



Example Circular Motion at Arbitrary Speed

The angle $\theta(t)$ is an arbitrary function of time
Find the speed, velocity and acceleration vectors of A
(give vectors in both $\{\mathbf{i},\mathbf{j}\}$ and $\{\mathbf{n},\mathbf{t}\}$ bases)

$$\mathbf{t} = -\sin\theta\mathbf{i} + \cos\theta\mathbf{j} \qquad \mathbf{n} = -\cos\theta\mathbf{i} - \sin\theta\mathbf{j}$$



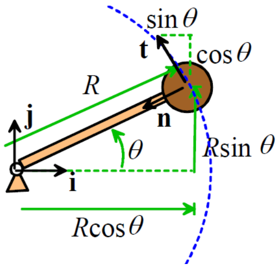
We can write down some useful scalar relations:

- Angular rate: $\omega = \dfrac{d\theta}{dt}$
- Angular acceleration $\alpha = \dfrac{d\omega}{dt} = \dfrac{d^2\theta}{dt^2}$
- Speed $V = R\dfrac{d\theta}{dt} = R\omega$
- Rate of change of speed $\dfrac{dV}{dt} = R\dfrac{d^2\theta}{dt^2} = R\dfrac{d\omega}{dt} = R\alpha$

Example Circular Motion at Arbitrary Speed

The angle $\theta(t)$ is an arbitrary function of time
Find the speed, velocity and acceleration vectors of A
(give vectors in both $\{\mathbf{i},\mathbf{j}\}$ and $\{\mathbf{n},\mathbf{t}\}$ bases)

$$\mathbf{t} = -\sin\theta\mathbf{i} + \cos\theta\mathbf{j} \qquad \mathbf{n} = -\cos\theta\mathbf{i} - \sin\theta\mathbf{j}$$



We can write down some useful scalar relations:

- Angular rate: $\omega = \dfrac{d\theta}{dt}$
- Angular acceleration $\alpha = \dfrac{d\omega}{dt} = \dfrac{d^2\theta}{dt^2}$
- Speed $V = R\dfrac{d\theta}{dt} = R\omega$
- Rate of change of speed $\dfrac{dV}{dt} = R\dfrac{d^2\theta}{dt^2} = R\dfrac{d\omega}{dt} = R\alpha$

We can now calculate vector velocities and accelerations

$$\mathbf{r} = R\cos\theta\mathbf{i} + R\sin\theta\mathbf{j}$$

The velocity can be calculated by differentiating the position vector.

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = -R\frac{d\theta}{dt}\sin\theta\mathbf{i} + R\frac{d\theta}{dt}\cos\theta\mathbf{j} = R\omega(-\sin\theta\mathbf{i} + \cos\theta\mathbf{j})$$

The acceleration vector follows as

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = R\frac{d\omega}{dt}(-\sin\theta\mathbf{i} + \cos\theta\mathbf{j}) + R\omega(-\frac{d\theta}{dt}\cos\theta\mathbf{i} - \frac{d\theta}{dt}\sin\theta\mathbf{j})$$

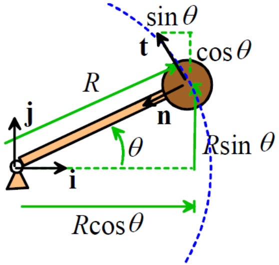
$$= R\alpha(-\sin\theta\mathbf{i} + \cos\theta\mathbf{j}) - R\omega^2(\cos\theta\mathbf{i} + \sin\theta\mathbf{j})$$

$$\mathbf{v} = R\omega\mathbf{t} = V\mathbf{t} \qquad \mathbf{a} = R\alpha\mathbf{t} + R\omega^2\mathbf{n} = \frac{dV}{dt}\mathbf{t} + \frac{V^2}{R}\mathbf{n}$$

Example Circular Motion at Arbitrary Speed

The angle $\theta(t)$ is an arbitrary function of time
Find the speed, velocity and acceleration vectors of A
(give vectors in both $\{\mathbf{i},\mathbf{j}\}$ and $\{\mathbf{n},\mathbf{t}\}$ bases)

$$\mathbf{t} = -\sin\theta\mathbf{i} + \cos\theta\mathbf{j} \qquad \mathbf{n} = -\cos\theta\mathbf{i} - \sin\theta\mathbf{j}$$



We can write down some useful scalar relations:

- Angular rate: $\omega = \dfrac{d\theta}{dt}$
- Angular acceleration $\alpha = \dfrac{d\omega}{dt} = \dfrac{d^2\theta}{dt^2}$
- Speed $V = R\dfrac{d\theta}{dt} = R\omega$
- Rate of change of speed $\dfrac{dV}{dt} = R\dfrac{d^2\theta}{dt^2} = R\dfrac{d\omega}{dt} = R\alpha$

We can now calculate vector velocities and accelerations

Position $\mathbf{r} = R\cos\theta\mathbf{i} + R\sin\theta\mathbf{j}$

The velocity can be calculated by differentiating the position vector.

Velocity $$\mathbf{v} = \frac{d\mathbf{r}}{dt} = -R\frac{d\theta}{dt}\sin\theta\mathbf{i} + R\frac{d\theta}{dt}\cos\theta\mathbf{j} = R\omega(-\sin\theta\mathbf{i} + \cos\theta\mathbf{j})$$

The acceleration vector follows as

Acceleration $$\mathbf{a} = \frac{d\mathbf{v}}{dt} = R\frac{d\omega}{dt}(-\sin\theta\mathbf{i} + \cos\theta\mathbf{j}) + R\omega(-\frac{d\theta}{dt}\cos\theta\mathbf{i} - \frac{d\theta}{dt}\sin\theta\mathbf{j})$$

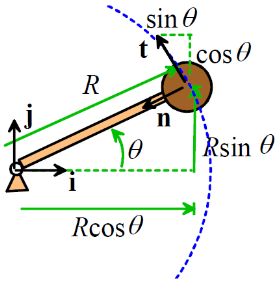
$$= R\alpha(-\sin\theta\mathbf{i} + \cos\theta\mathbf{j}) - R\omega^2(\cos\theta\mathbf{i} + \sin\theta\mathbf{j})$$

Velocity and Acceleration in "n-t" basis

$$\mathbf{v} = R\omega\mathbf{t} = V\mathbf{t} \qquad \mathbf{a} = R\alpha\mathbf{t} + R\omega^2\mathbf{n} = \frac{dV}{dt}\mathbf{t} + \frac{V^2}{R}\mathbf{n}$$

Example Circular Motion at Arbitrary Speed

The angle $\theta(t)$ is an arbitrary function of time
Find the speed, velocity and acceleration vectors of A
(give vectors in both $\{\mathbf{i},\mathbf{j}\}$ and $\{\mathbf{n},\mathbf{t}\}$ bases)



$$\mathbf{t} = -\sin\theta\mathbf{i} + \cos\theta\mathbf{j} \qquad \mathbf{n} = -\cos\theta\mathbf{i} - \sin\theta\mathbf{j}$$

$$\mathbf{v} = R\omega\mathbf{t} = V\mathbf{t} \qquad \mathbf{a} = R\alpha\mathbf{t} + R\omega^2\mathbf{n} = \frac{dV}{dt}\mathbf{t} + \frac{V^2}{R}\mathbf{n}$$

Define

$a_t = \frac{dV}{dt}$ "tangential accel"

$a_n = \frac{V^2}{R}$ "normal accel"

Then $\underline{a} = \underbrace{a_t \underline{t}}_{\text{Component parallel to motion}} + \underbrace{a_n \underline{n}}_{\text{Component perpendicular to motion}}$

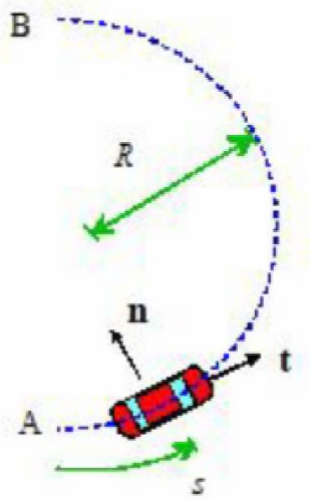
a_t: Represent change in speed
a_n: Represent change in direction

SHIEN-MING WU SCHOOL OF INTELLIGENT ENGINEERING
吴贤铭智能工程学院

Example: Vehicle Accelerating Around a Curve

The vehicle starts at rest at A and travels with constant tangential acceleration a_t

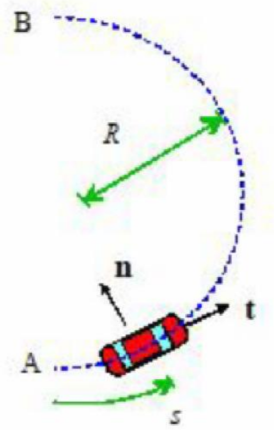
Find a formula for the magnitude of the acceleration at B, in terms of a_t



Example: Vehicle Accelerating Around a Curve

The vehicle starts at rest at A and travels with constant tangential acceleration a_t

Find a formula for the magnitude of the acceleration at B, in terms of a_t



Formula : $\underset{\sim}{a} = a_t \underset{\sim}{t} + \frac{V^2}{R} \underset{\sim}{n} \Rightarrow |\underset{\sim}{a}| = \sqrt{a_t^2 + \left(\frac{V^2}{R}\right)^2}$

Need to find V.

Recall $a_t = V \frac{dV}{ds}$ At B $s = \pi R$

Separate variables

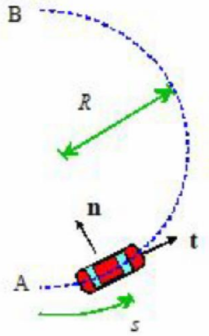
$$\int_0^{V_B} V \, dV = \int_0^{\pi R} a_t \, ds \Rightarrow \frac{1}{2} V_B^2 = \pi R a_t$$

page 20

Example: Vehicle Accelerating Around a Curve

The vehicle starts at rest at A and travels with constant tangential acceleration a_t

Find a formula for the magnitude of the acceleration at B, in terms of a_t



Formula : $\underline{a} = a_t \underline{t} + \frac{V^2}{R}\underline{n} \Rightarrow |\underline{a}| = \sqrt{a_t^2 + \left(\frac{V^2}{R}\right)^2}$

Need to find V.

Recall $a_t = V\frac{dV}{ds}$ At B $s = \pi R$

Separate variables

$$\int_0^{V_B} V\,dV = \int_0^{\pi R} a_t\,ds \Rightarrow \frac{1}{2}V_B^2 = \pi R a_t$$

page 20

Hence $V_B = \sqrt{2\pi R a_t}$

Substitute into formula for $|\underline{a}|$

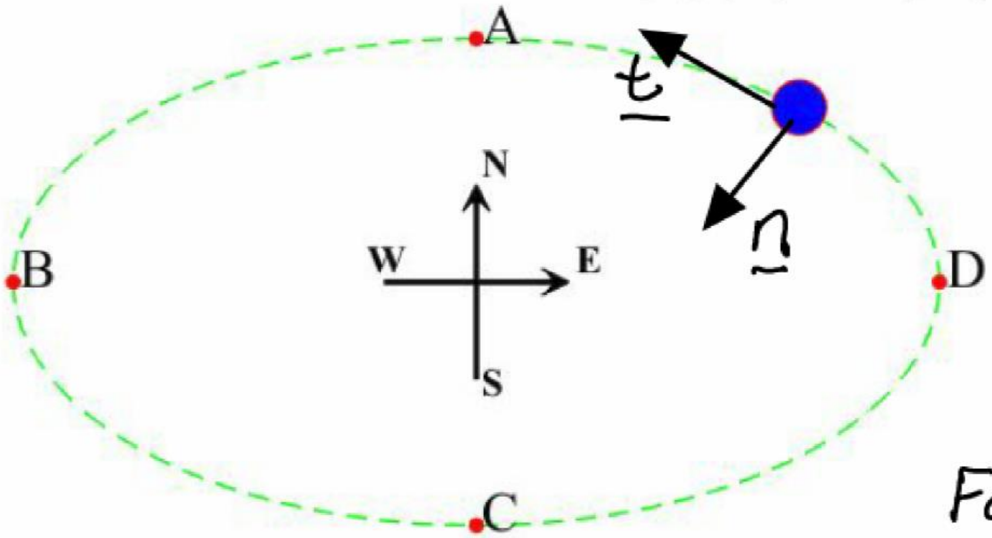
$$|\underline{a}| = \sqrt{a_t^2 + \left(\frac{V_B^2}{R}\right)^2} = \sqrt{a_t^2 + \frac{(2\pi R a_t)^2}{R^2}}$$

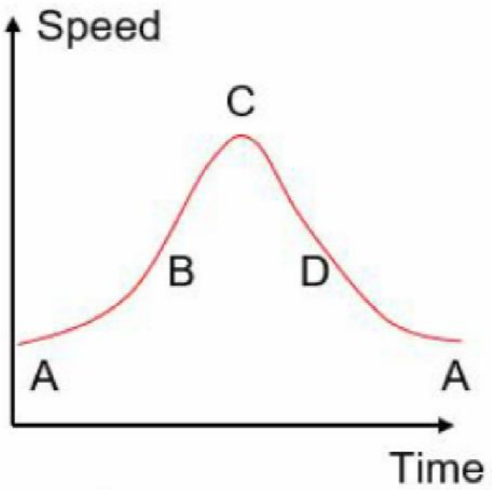
$$\Rightarrow \boxed{|\underline{a}| = a_t\sqrt{1 + 4\pi^2}}$$

Example: Motion Along an Arbitrary Planar Path using n-t Coordinates

Concept question

Give the direction of the acceleration at A,B,C,D (on compass)



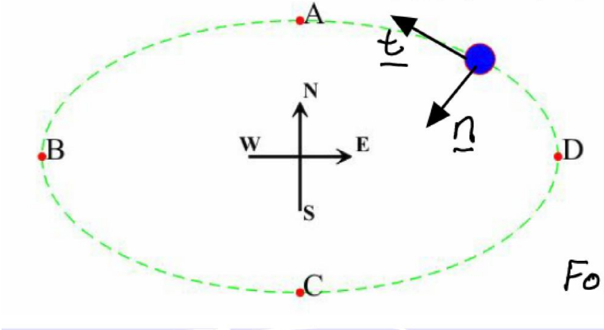


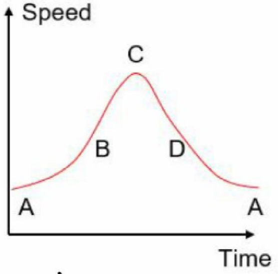
Formula $\underline{a} = \frac{dV}{dt}\underline{t} + \frac{V^2}{R}\underline{n}$

Example: Motion Along an Arbitrary Planar Path using n-t Coordinates

Concept question

Give the direction of the acceleration at A,B,C,D (on compass)



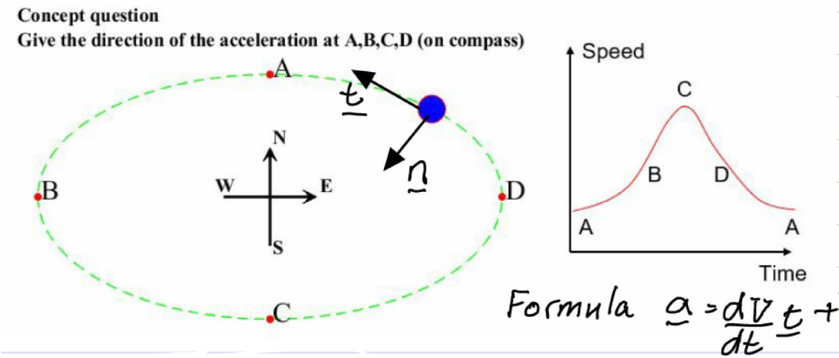


Formula $\underline{a} = \frac{dV}{dt}\underline{t} + \frac{V^2}{R}\underline{n}$

At A: V is min $\Rightarrow dV/dt = 0$

$\Rightarrow$ $\underline{a}$ is South

Example: Motion Along an Arbitrary Planar Path using n-t Coordinates



Formula $\underline{a} = \frac{dV}{dt}\underline{t} + \frac{V^2}{R}\underline{n}$

At A: V is min $\Rightarrow dV/dt = 0$

$\Rightarrow \underline{a}$ is South

At B $dV/dt > 0$ a_t a_n $\Rightarrow \underline{a}$ is SE

At C $dV/dt = 0$ $\Rightarrow \underline{a}$ is North

At D $dV/dt < 0$ a_t a_n $\Rightarrow \underline{a}$ is SW

page 23

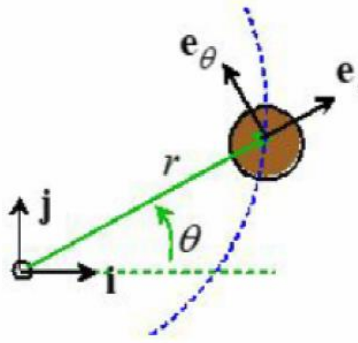
Analyzing Motion using Polar and Cylindrical Coordinates

Analyzing motion using polar coords

Problem: Given $r(t)$, $\theta(t)$

Find $\underline{r}$, $\underline{v}$, $\underline{a}$

in $\{\underline{e_r}, \underline{e_\theta}, \underline{k}\}$



$$r = \sqrt{x^2 + y^2}$$

$$\theta = \tan^{-1}(y/x)$$

$$x = r\cos\theta$$

$$y = r\sin\theta$$

Position $\underline{r} = r(t)\,\underline{e_r}$

Velocity $\underline{v} = \frac{dr}{dt}\underline{e_r} + r\frac{d\theta}{dt}\underline{e_\theta}$

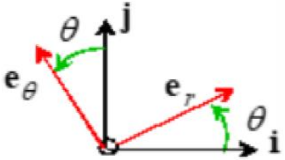
Acceleration

$$\underline{a} = \left\{\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right\}\underline{e_r} + \left\{r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right\}\underline{e_\theta}$$

Deriving the Polar Coordinate Formulas

Time derivatives of $\{\mathbf{e}_r, \mathbf{e}_\theta\}$

$$\mathbf{e}_r = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}$$
$$\mathbf{e}_\theta = -\sin\theta\mathbf{i} + \cos\theta\mathbf{j}$$



$$\frac{d\mathbf{e}_r}{dt} = -\frac{d\theta}{dt}\sin\theta\mathbf{i} + \frac{d\theta}{dt}\cos\theta\mathbf{j} \quad = \frac{d\theta}{dt}\mathbf{e}_\theta$$
$$\frac{d\mathbf{e}_\theta}{dt} = -\frac{d\theta}{dt}\cos\theta\mathbf{i} - \frac{d\theta}{dt}\sin\theta\mathbf{j} \quad = -\frac{d\theta}{dt}\mathbf{e}_r$$

Position $\quad \mathbf{r} = r(t)\mathbf{e}_r(t)$

Velocity $\quad \mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{dr}{dt}\mathbf{e}_r + r\frac{d\mathbf{e}_r}{dt} = \frac{dr}{dt}\mathbf{e}_r + r\frac{d\theta}{dt}\mathbf{e}_\theta$

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2r}{dt^2}\mathbf{e}_r + \frac{dr}{dt}\frac{d\mathbf{e}_r}{dt} + \frac{dr}{dt}\frac{d\theta}{dt}\mathbf{e}_\theta + r\frac{d^2\theta}{dt^2}\mathbf{e}_\theta + r\frac{d\theta}{dt}\frac{d\mathbf{e}_\theta}{dt}$$
$$= \frac{d^2r}{dt^2}\mathbf{e}_r + \frac{dr}{dt}\frac{d\theta}{dt}\mathbf{e}_\theta + \frac{dr}{dt}\frac{d\theta}{dt}\mathbf{e}_\theta + r\frac{d^2\theta}{dt^2}\mathbf{e}_\theta - r\left(\frac{d\theta}{dt}\right)^2\mathbf{e}_r$$
$$= \left\{\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right\}\mathbf{e}_r + \left\{r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right\}\mathbf{e}_\theta$$

Example:

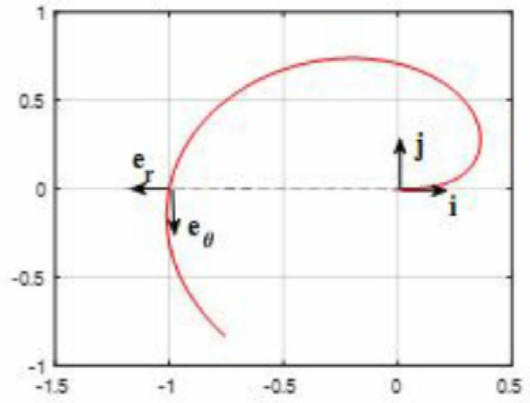
2.16 Example: The particle has polar coordinates

$$\theta = t^2 \qquad r = t / \sqrt{\pi}$$

At the instant when $\theta = \pi$ calculate

(i) the position, velocity and acceleration vectors in the polar basis $\mathbf{e}_r, \mathbf{e}_\theta$

(ii) The normal and tangential components of acceleration



Example:

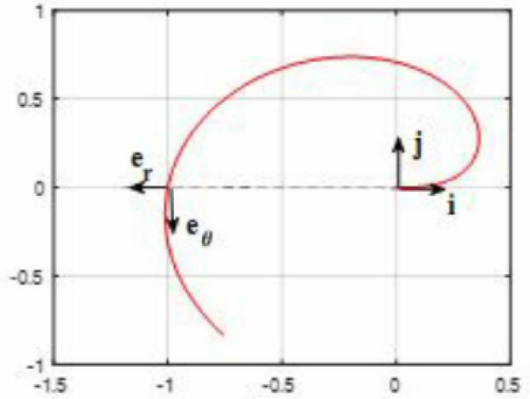
2.16 Example: The particle has polar coordinates

$$\theta = t^2 \qquad r = t/\sqrt{\pi}$$

At the instant when $\theta = \pi$ calculate

(i) the position, velocity and acceleration vectors in the polar basis $\mathbf{e}_r, \mathbf{e}_\theta$

(ii) The normal and tangential components of acceleration



Prelimmaries: Note $t = \sqrt{\theta} \Rightarrow$ when $\theta = \pi$ $t = \sqrt{\pi}$

$dr/dt = 1/\sqrt{\pi}$ $\quad d^2r/dt^2 = 0$ $\quad d\theta/dt = 2t$ $\quad d^2\theta/dt^2 = 2$

Example:

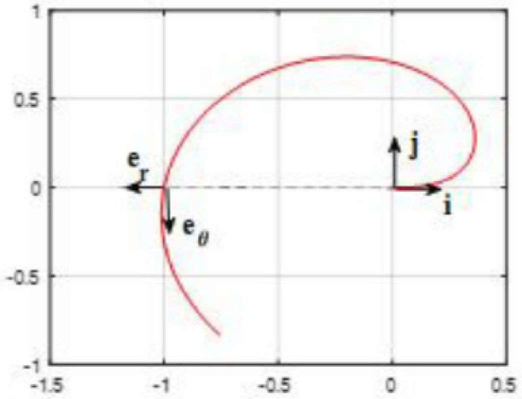
2.16 Example: The particle has polar coordinates

$$\theta = t^2 \qquad r = t/\sqrt{\pi}$$

At the instant when $\theta = \pi$ calculate

(i) the position, velocity and acceleration vectors in the polar basis $\mathbf{e}_r, \mathbf{e}_\theta$

(ii) The normal and tangential components of acceleration



Preliminaries: Note $t = \sqrt{\theta} \Rightarrow$ when $\theta = \pi$ $t = \sqrt{\pi}$

$dr/dt = 1/\sqrt{\pi}$ $\quad d^2r/dt^2 = 0$ $\quad d\theta/dt = 2t$ $\quad d^2\theta/dt^2 = 2$

Position $\quad \underline{r} = r\underline{e_r} = \underline{e_r}$

Velocity $\quad \underline{V} = dr/dt\,\underline{e_r} + r\,d\theta/dt\,\underline{e_\theta}$

$\Rightarrow \underline{V} = \frac{1}{\sqrt{\pi}}\underline{e_r} + 2\sqrt{\pi}\,\underline{e_\theta}$

Example:

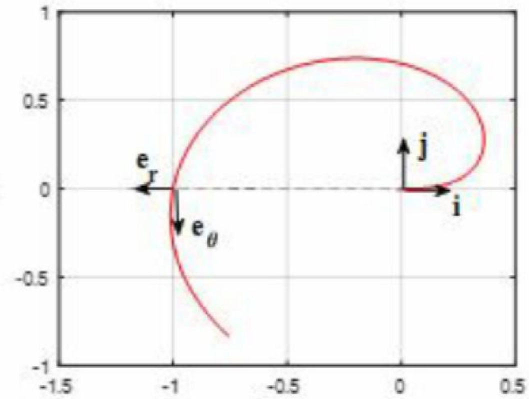
2.16 Example: The particle has polar coordinates

$$\theta = t^2 \qquad r = t/\sqrt{\pi}$$

At the instant when $\theta = \pi$ calculate

(i) the position, velocity and acceleration vectors in the polar basis $\mathbf{e}_r, \mathbf{e}_\theta$

(ii) The normal and tangential components of acceleration



Preliminaries: Note $t = \sqrt{\theta}$ $\Rightarrow$ when $\theta = \pi$ $t = \sqrt{\pi}$

$dr/dt = 1/\sqrt{\pi}$ $\quad d^2r/dt^2 = 0$ $\quad d\theta/dt = 2t$ $\quad d^2\theta/dt^2 = 2$

Position $\quad \underline{r} = r\underline{e_r} = \underline{e_r}$

Velocity $\quad \underline{v} = dr/dt\, \underline{e_r} + r\, d\theta/dt\, \underline{e_\theta}$

$\Rightarrow \quad \underline{v} = \frac{1}{\sqrt{\pi}} \underline{e_r} + 2\sqrt{\pi}\, \underline{e_\theta}$

page 32

Acceleration

$$\underline{a} = \left\{ \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 \right\} \underline{e_r} + \left\{ r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt} \right\} \underline{e_\theta}$$

$$\underline{a} = \left(0 - 1 \times (2\sqrt{\pi})^2\right) \underline{e_r} + \left(1 \times 2 + 2\frac{1}{\sqrt{\pi}} 2\sqrt{\pi}\right) \underline{e_\theta}$$

$$\boxed{\underline{a} = 4\pi\, \underline{e_r} + 6\, \underline{e_\theta}}$$

Example:

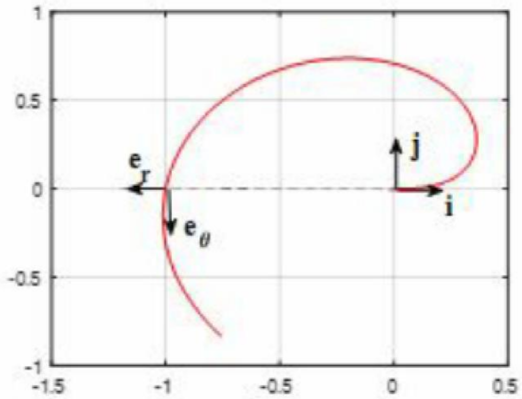
2.16 Example: The particle has polar coordinates

$$\theta = t^2 \qquad r = t / \sqrt{\pi}$$

At the instant when $\theta = \pi$ calculate

(i) the position, velocity and acceleration vectors in the polar basis $\mathbf{e}_r, \mathbf{e}_\theta$

(ii) The normal and tangential components of acceleration



$\underline{n}, \underline{t}$ components

(1) Find $\{\underline{n}, \underline{t}\}$ in $\{\underline{e_r}, \underline{e_\theta}\}$

(2) Use $a_t = \underline{t} \cdot \underline{a} \qquad a_n = \underline{n} \cdot \underline{a}$

Example:

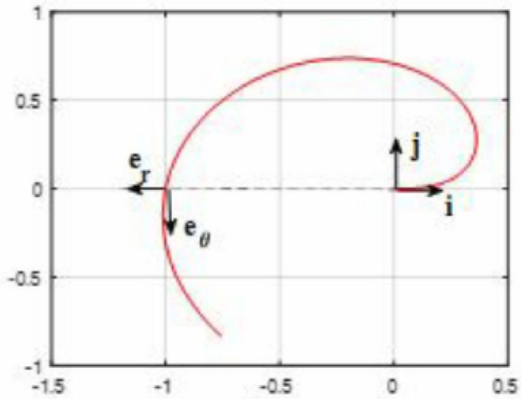
2.16 Example: The particle has polar coordinates

$$\theta = t^2 \qquad r = t/\sqrt{\pi}$$

At the instant when $\theta = \pi$ calculate

(i) the position, velocity and acceleration vectors in the polar basis $\mathbf{e}_r, \mathbf{e}_\theta$

(ii) The normal and tangential components of acceleration



$\underline{n}, \underline{t}$ components

(1) Find $\{\underline{n}, \underline{t}\}$ in $\{\underline{e}_r, \underline{e}_\theta\}$

(2) Use $a_t = \underline{t} \cdot \underline{a} \qquad a_n = \underline{n} \cdot \underline{a}$

Recall $\underline{v} = V\underline{t} \Rightarrow \underline{t} = \underline{v}/V$

and $V = \text{speed} = |\underline{v}|$

$$\Rightarrow \underline{t} = \frac{(1/\sqrt{\pi})\underline{e}_r + 2\sqrt{\pi}\,\underline{e}_\theta}{\sqrt{1/\pi + 4\pi}}$$

$$= (\underline{e}_r + 2\pi\,\underline{e}_\theta)/\sqrt{1+4\pi^2}$$

Example:

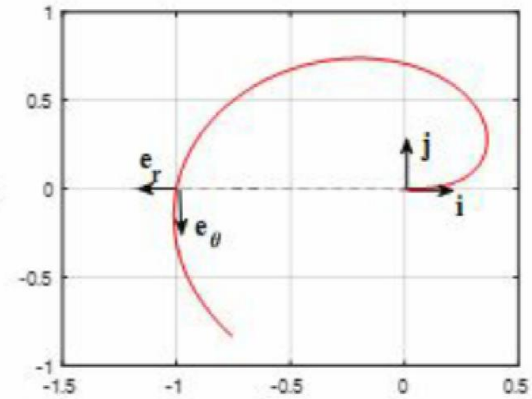
2.16 Example: The particle has polar coordinates

$$\theta = t^2 \qquad r = t/\sqrt{\pi}$$

At the instant when $\theta = \pi$ calculate

(i) the position, velocity and acceleration vectors in the polar basis $\mathbf{e}_r, \mathbf{e}_\theta$

(ii) The normal and tangential components of acceleration



Recall $\underline{v} = V\underline{t} \Rightarrow \underline{t} = \underline{v}/V$

and V = speed = $|\underline{v}|$

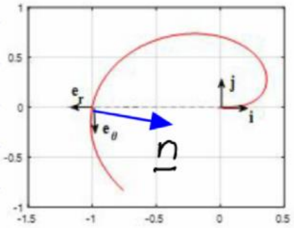
$$\Rightarrow \underline{t} = \frac{(1/\sqrt{\pi})\underline{e_r} + 2\sqrt{\pi}\,\underline{e_\theta}}{\sqrt{1/\pi + 4\pi}}$$

$$= (\underline{e_r} + 2\pi\,\underline{e_\theta})/\sqrt{1+4\pi^2}$$

$\underline{n}$ is perpendicular to $\underline{t}$ and $\underline{k}$

$\Rightarrow \underline{n} = \pm\, \underline{k} \times \underline{t}$

Choose sign so $\underline{n}$ has negative $\underline{e_r}$ component



page 33

Example:

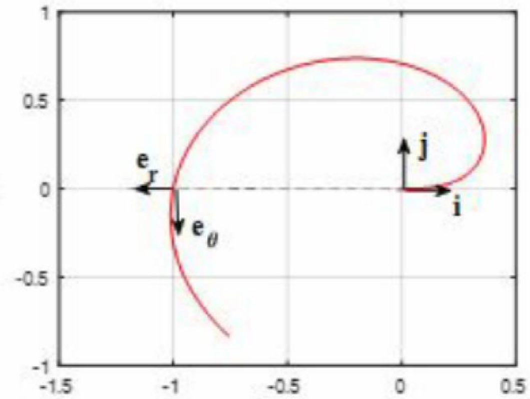
2.16 Example: The particle has polar coordinates

$$\theta = t^2 \qquad r = t/\sqrt{\pi}$$

At the instant when $\theta = \pi$ calculate

(i) the position, velocity and acceleration vectors in the polar basis $\mathbf{e}_r, \mathbf{e}_\theta$

(ii) The normal and tangential components of acceleration



Finally $a_t = \underline{a} \cdot \underline{t} = (-4\pi \underline{e_r} + 6\underline{e_\theta}) \cdot \dfrac{(\underline{e_r} + 2\pi \underline{e_\theta})}{\sqrt{1+4\pi^2}}$

$$\Rightarrow \boxed{a_t = \frac{(-4\pi + 12\pi)}{\sqrt{1+4\pi^2}} = \frac{8\pi}{\sqrt{1+4\pi^2}}}$$

$$\boxed{a_n = \underline{a} \cdot \underline{n} = \frac{(8\pi^2 + 6)}{\sqrt{1+4\pi^2}}}$$

page 34

Summary

- Describing motion of particles: motion along a curved path
- Main concept: using normal-tangential and polar coordinates
  - Review of some aspects of vectors
  - Circular motion
    - Cartesian Coordinates
    - Normal-Tangential Coordinates
  - Motion along an arbitrary planar path: normal/tangential coordinates
  - Motion along an arbitrary planar path: polar coordinates