Foundations of Hardware Verification

John Wickerson

14 December 2023

This lecture

- Temporal logic
- Binary decision diagrams

Temporal logic

- SymbiYosys can verify *immediate* assertions like:

```
assert (count > 0);
```

- It can also handle *concurrent* assertions like:

```
assert (wen > 0 |=> waddr > 0);
```

- These can be generalised to formulas in *temporal logic*.

Linear Temporal Logic

ϕ ::= true
| false
| p // atom
| $\neg\,\phi$ // negation
| $\phi \wedge \phi$ // conjunction
| $\phi \vee \phi$ // disjunction
| $\phi \rightarrow \phi$ // implication
| **X** ϕ // ne**X**t state
| **F** ϕ // some **F**uture state
| **G** ϕ // all future states (**G**lobally)
| ϕ **U** ϕ // **U**ntil
| ϕ **W** ϕ // **W**eak-until

count > 0

count > 0 $\rightarrow$ waddr $\neq$ raddr

en $\rightarrow$ **X** (count > 0)

go $\rightarrow$ **F** (heat)

dead $\rightarrow$ **G** (dead)

G (req $\rightarrow$ **F** (ack))

liftmoving **U** (floor = 5)

motor **W** targetreached

Amir Pnueli
(1941–2009)

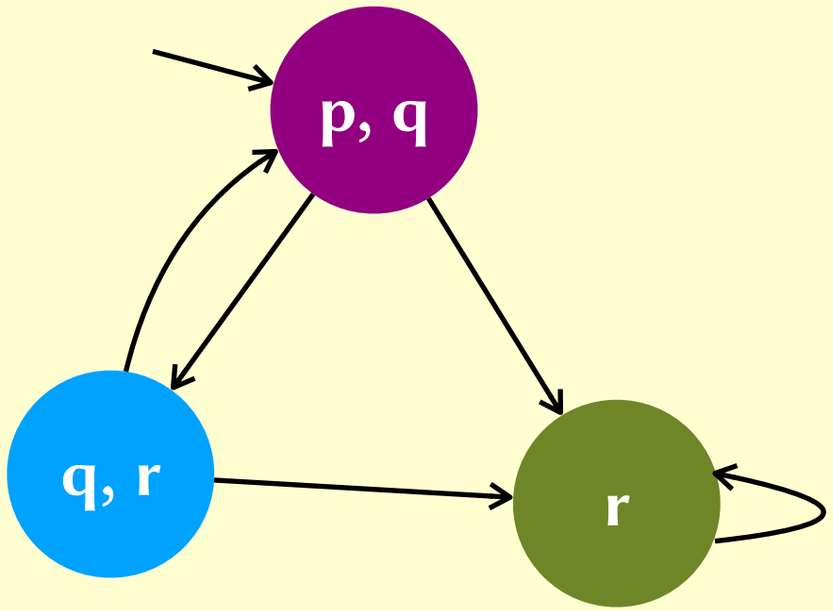
Paths

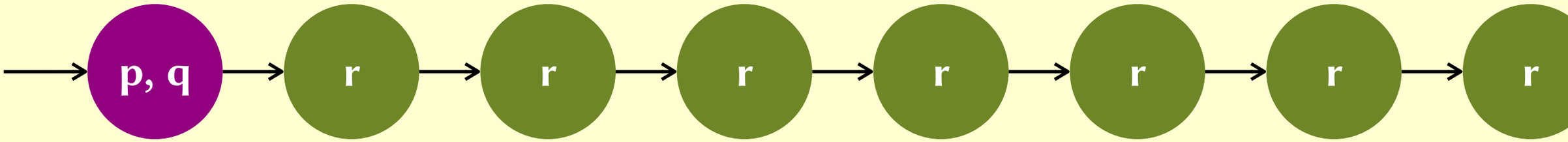
$p \rightarrow q$ ✓

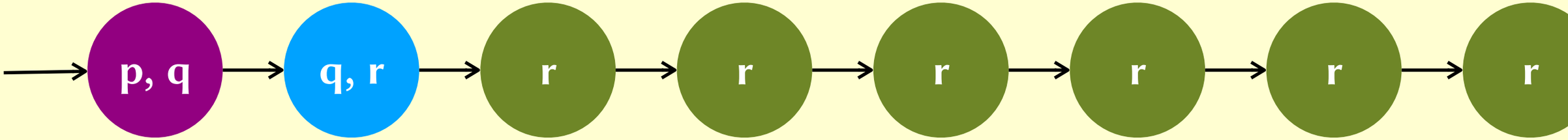
$\mathbf{G}\ (p \rightarrow q)$ ✓

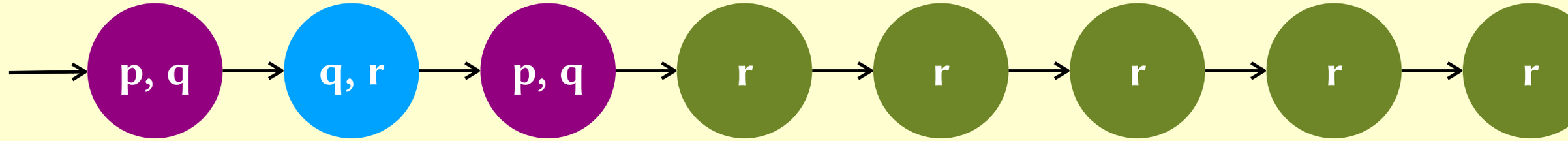
$\mathbf{X}\ r$ ✓ $\mathbf{F}\ r$ ✓ $\mathbf{G}\ \mathbf{F}\ r$ ✓

$\mathbf{F}\ (\neg q)$ ✗ $q\ \mathbf{U}\ r$ ✓ $q\ \mathbf{U}\ \neg q$ ✗ $q\ \mathbf{W}\ \neg q$ ✓



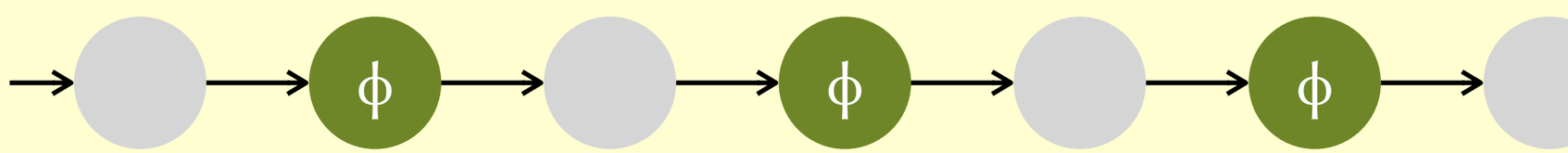






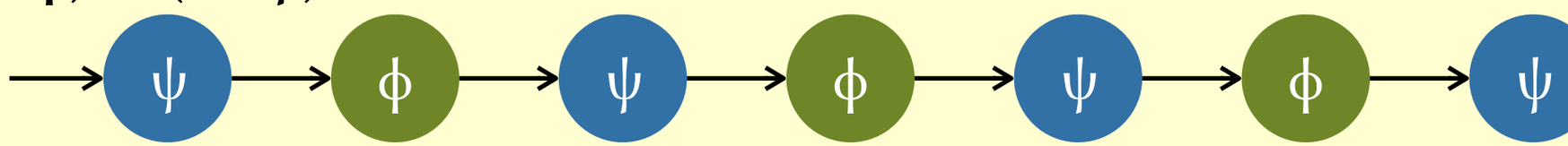
Some identities

- $\mathbf{G}\,\phi \overset{\checkmark}{\equiv} \neg\,\mathbf{F}\,(\neg\,\phi)$
- $\phi\,\mathbf{U}\,\psi \overset{\checkmark}{\equiv} (\phi\,\mathbf{W}\,\psi) \wedge (\mathbf{F}\,\psi)$
- $\phi\,\mathbf{W}\,\psi \overset{\checkmark}{\equiv} (\phi\,\mathbf{U}\,\psi) \vee (\mathbf{G}\,\phi)$
- $\mathbf{F}\,\mathbf{F}\,\phi \overset{\checkmark}{\equiv} \mathbf{F}\,\phi$
- $\mathbf{G}\,\mathbf{G}\,\phi \overset{\checkmark}{\equiv} \mathbf{G}\,\phi$
- $\mathbf{G}\,\mathbf{F}\,\phi \overset{\times}{\equiv} \mathbf{F}\,\mathbf{G}\,\phi$

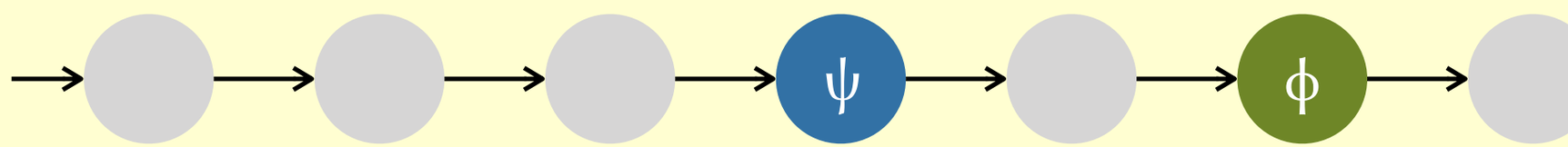


Some more identities

- $\mathbf{F}\,(\phi \vee \psi) \overset{\checkmark}{\equiv} (\mathbf{F}\,\phi) \vee (\mathbf{F}\,\psi)$
- $\mathbf{G}\,(\phi \vee \psi) \overset{\times}{\equiv} (\mathbf{G}\,\phi) \vee (\mathbf{G}\,\psi)$



- $\mathbf{F}\,(\phi \wedge \psi) \overset{\times}{\equiv} (\mathbf{F}\,\phi) \wedge (\mathbf{F}\,\psi)$



- $\mathbf{G}\,(\phi \wedge \psi) \overset{\checkmark}{\equiv} (\mathbf{G}\,\phi) \wedge (\mathbf{G}\,\psi)$

Temporal logic

- SymbiYosys can verify *immediate* assertions like:

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assert (count > 0);
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$\mathbf{G}\ (\text{count} > 0)$

- It can also handle *concurrent* assertions like:

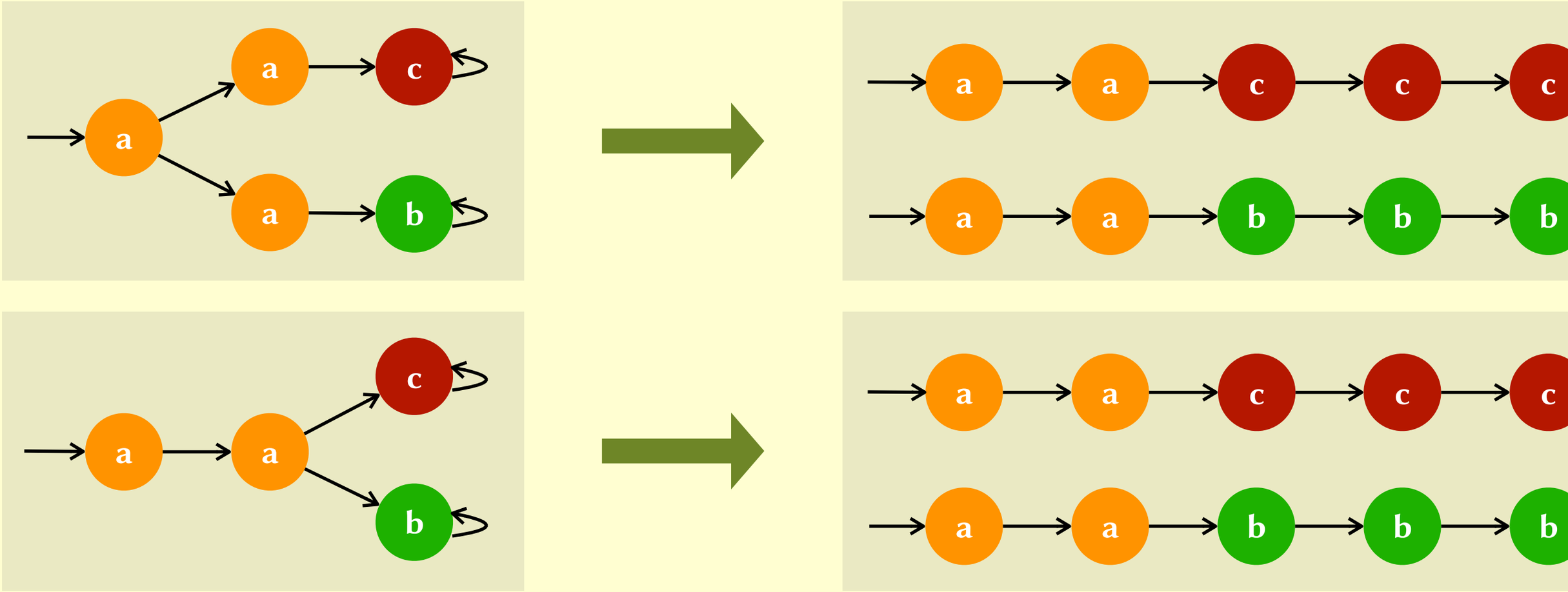
```
assert (wen > 0 |=> waddr > 0);
```

$\mathbf{G}\ (\text{wen} > 0 \longrightarrow \mathbf{X}\ (\text{waddr} > 0))$

- These can be generalised to formulas in *temporal logic*.

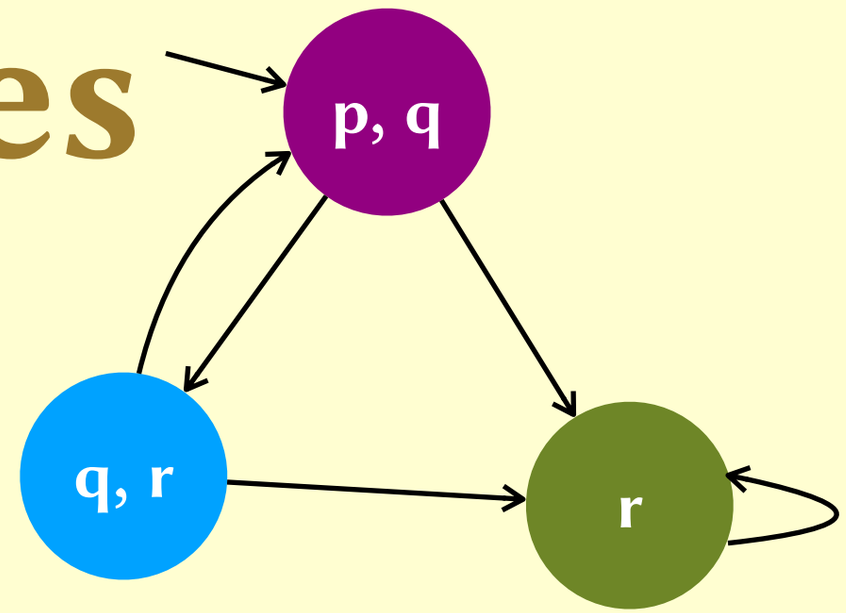
LTL's downfall...

- "From any a state, we can reach a c state."

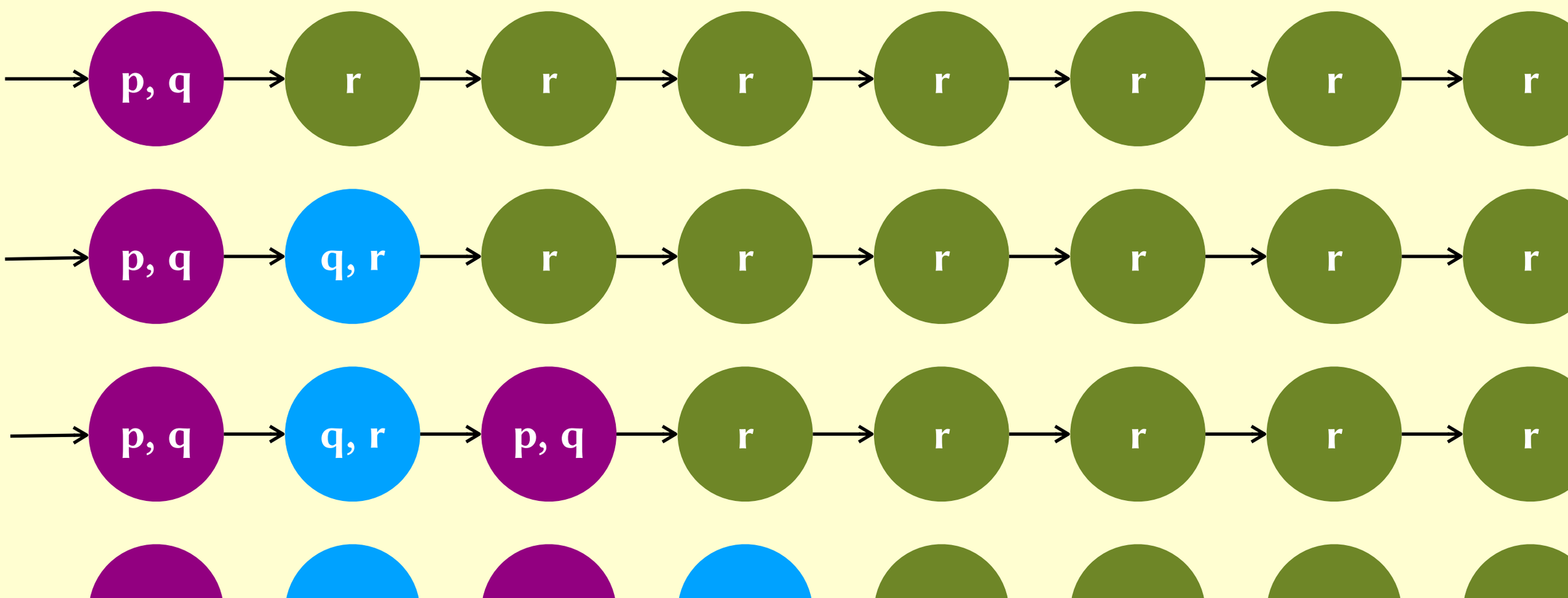


- Not expressible in LTL.

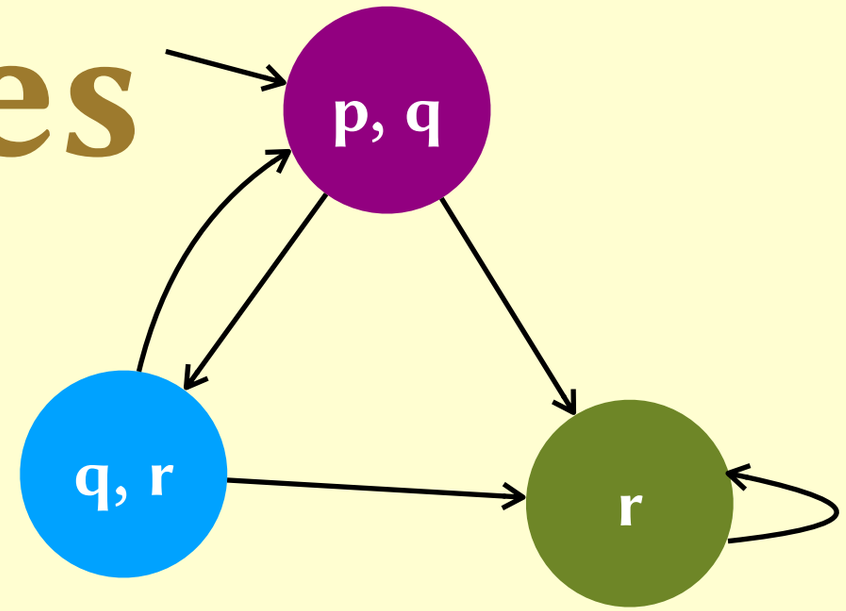
From paths to trees



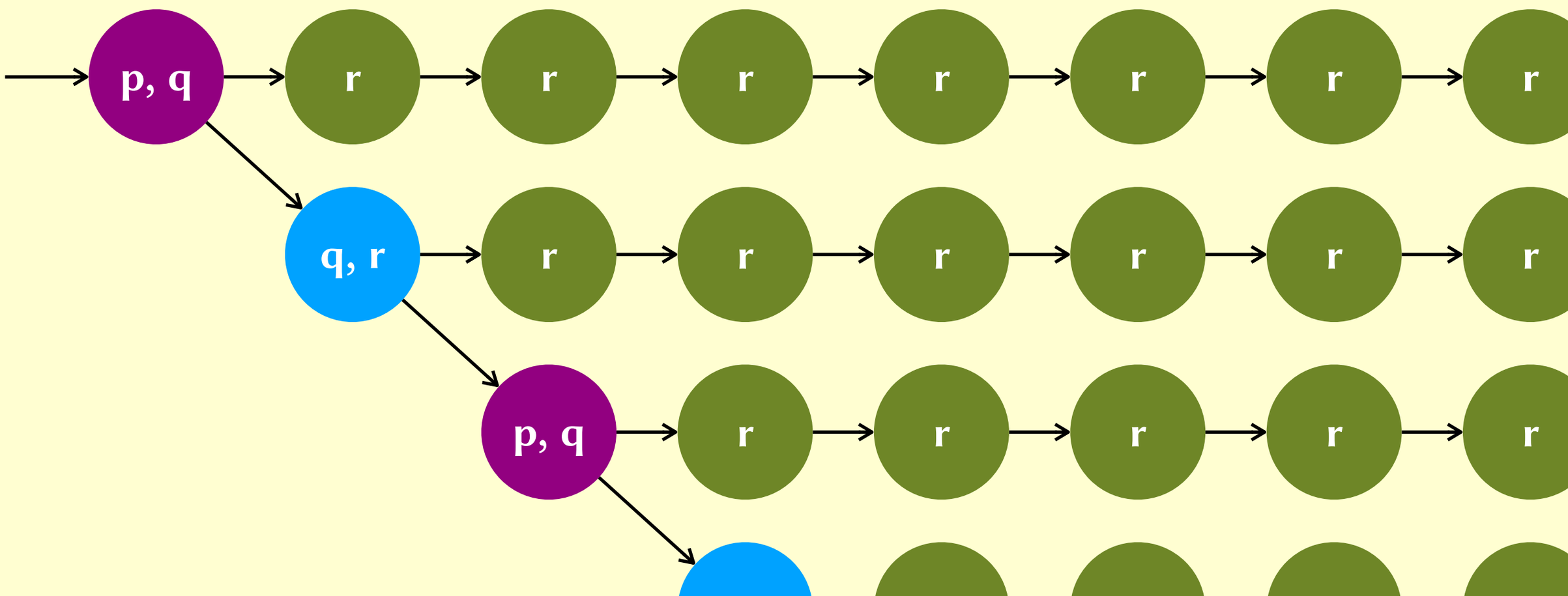
- Rather than unfolding the transition system into a set of paths...



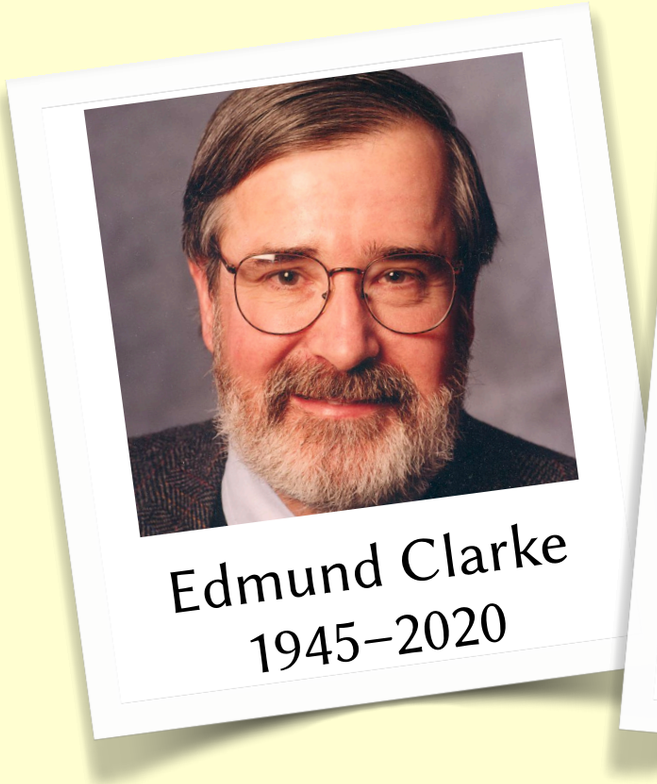
From paths to trees



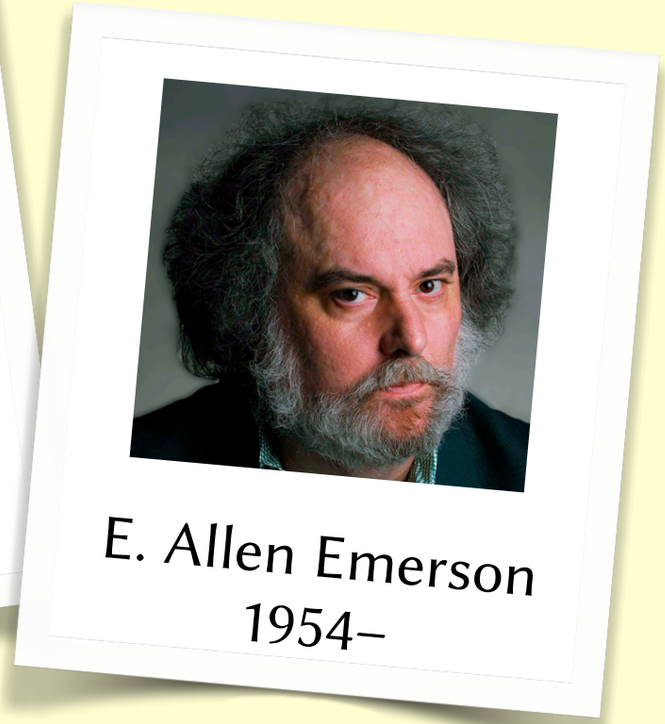
- ...unfold it into a single tree.
- Now formulas talk about this tree, rather than individual paths.



Computational Tree Logic



Edmund Clarke
1945–2020



E. Allen Emerson
1954–

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| false
| p // atom
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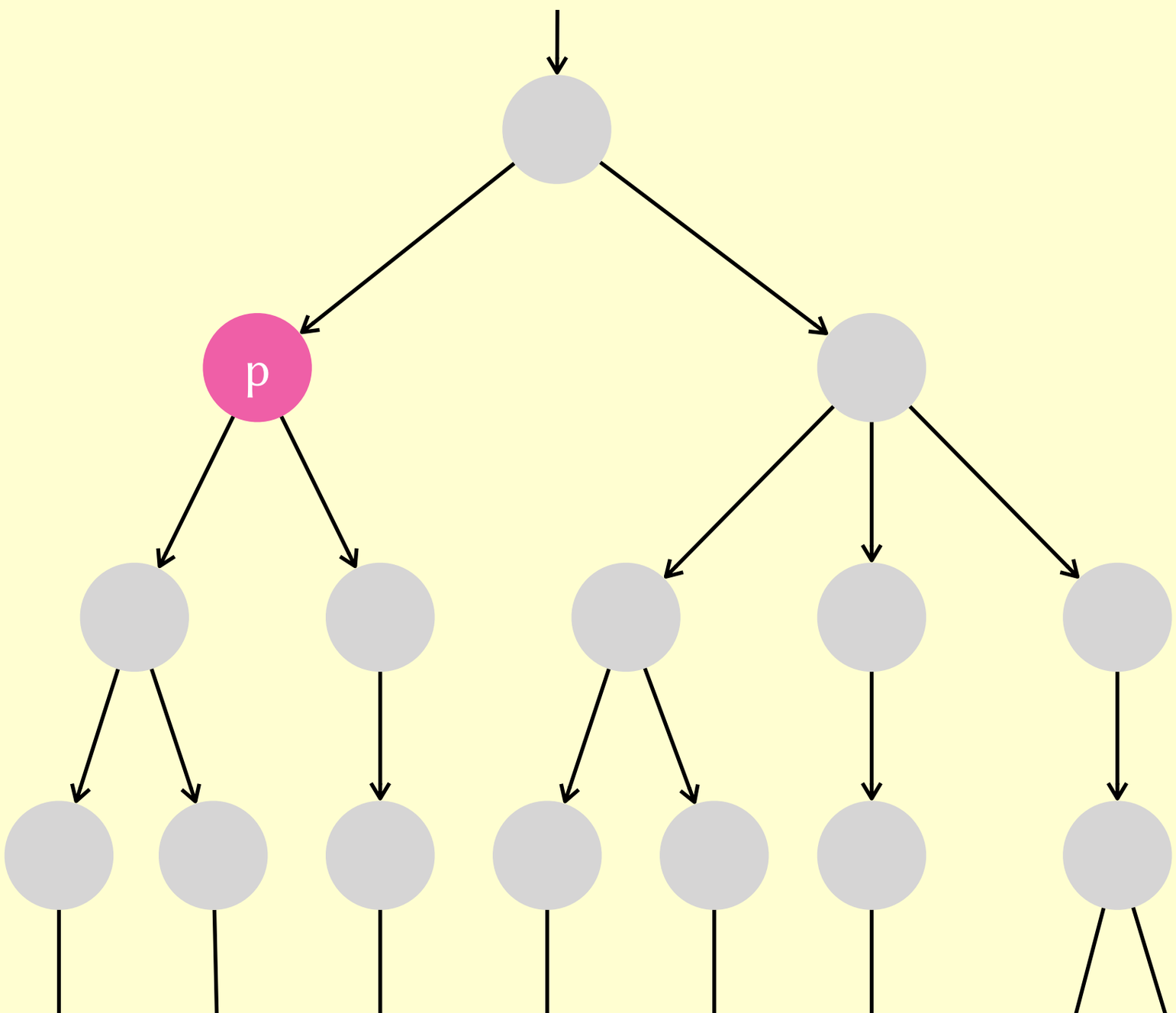
| **AX** ϕ | **EX** ϕ // ne**X**t state (**A**ll paths / there **E**xists a path)
| **AF** ϕ | **EF** ϕ // some **F**uture state
| **AG** ϕ | **EG** ϕ // all future states (**G**lobally)
| **A**[ϕ **U** ϕ] | **E**[ϕ **U** ϕ] // **U**ntil

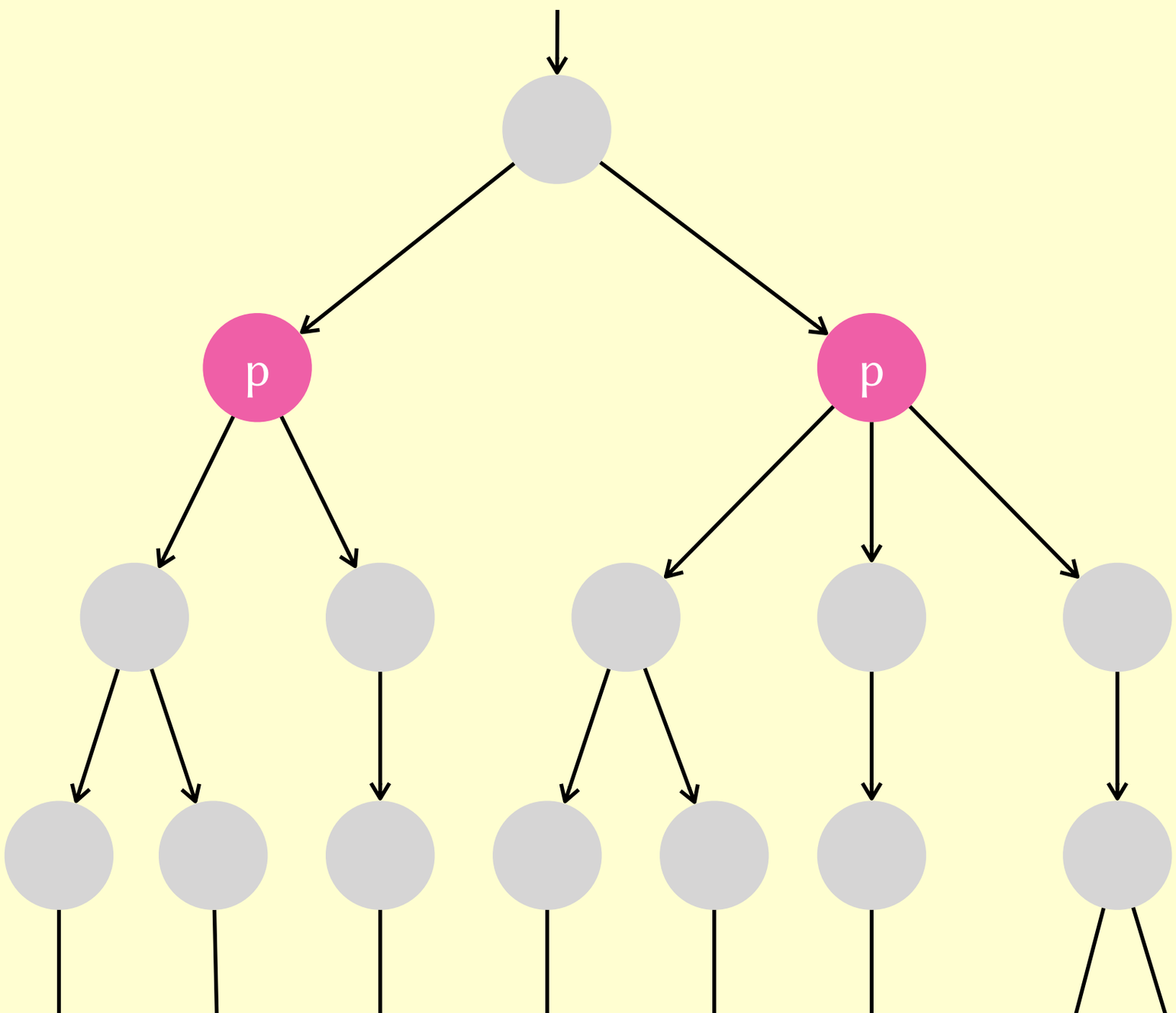
Computational Tree Logic

EX p

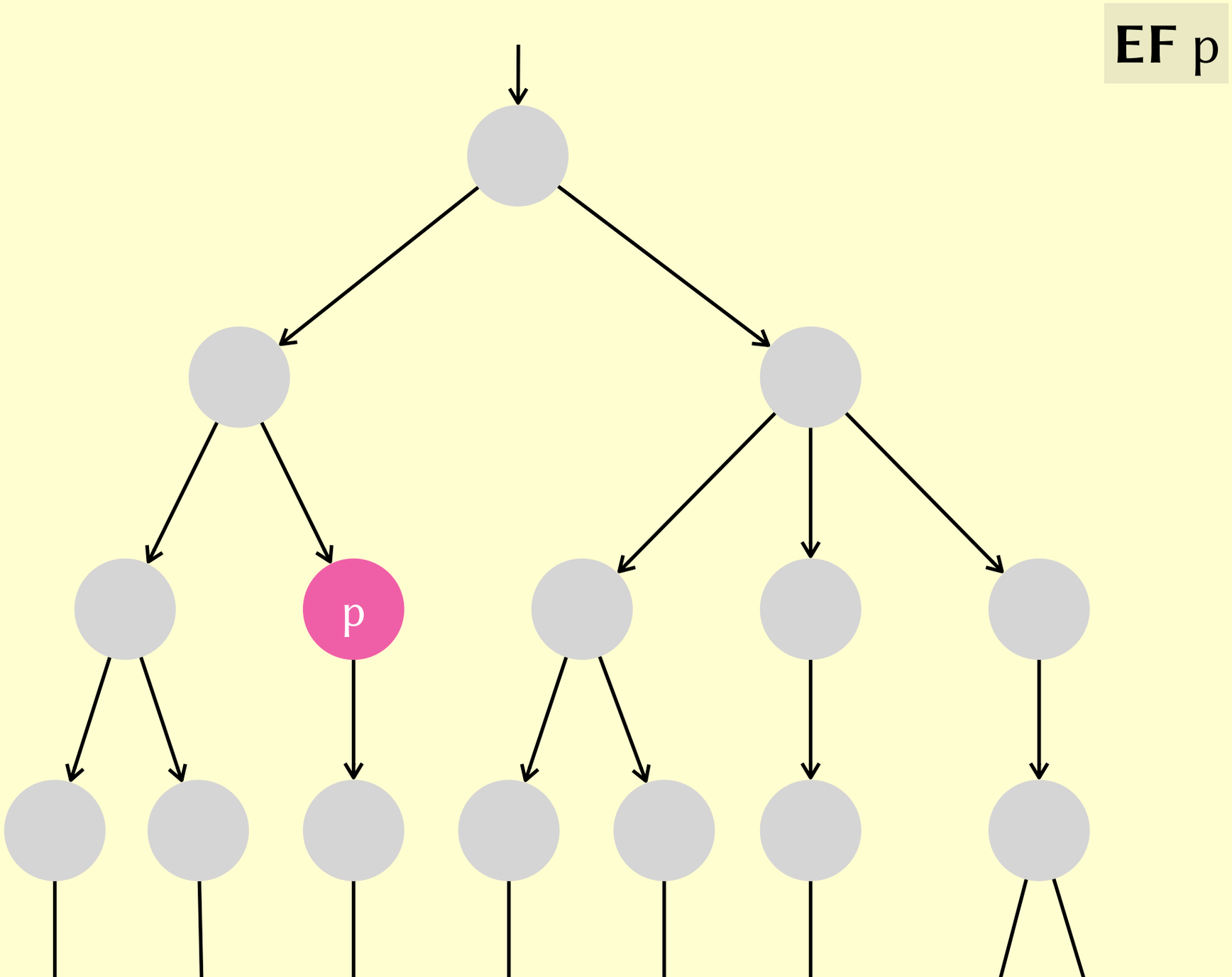


Computational Tree Logic

AX p

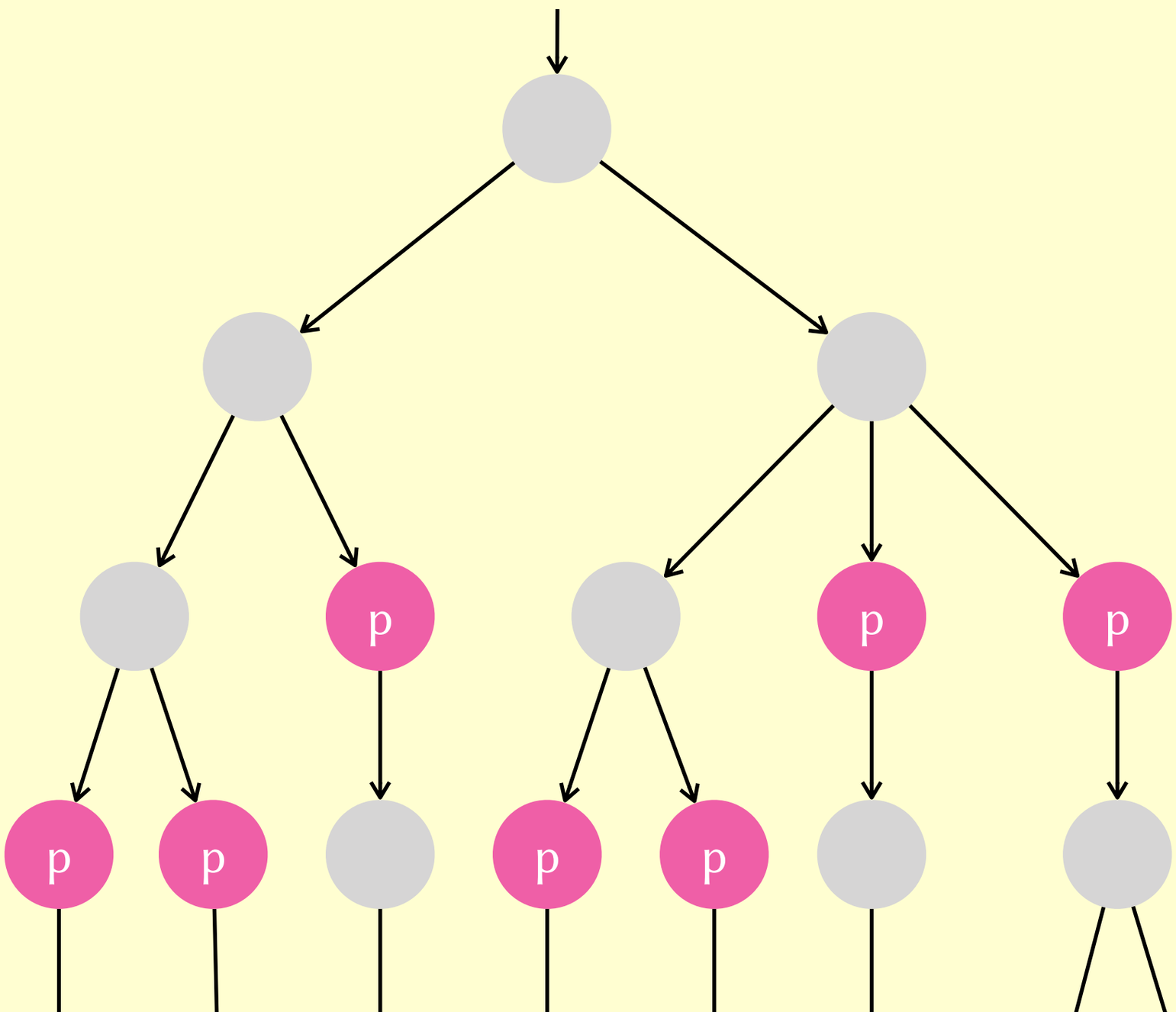


Computational Tree Logic



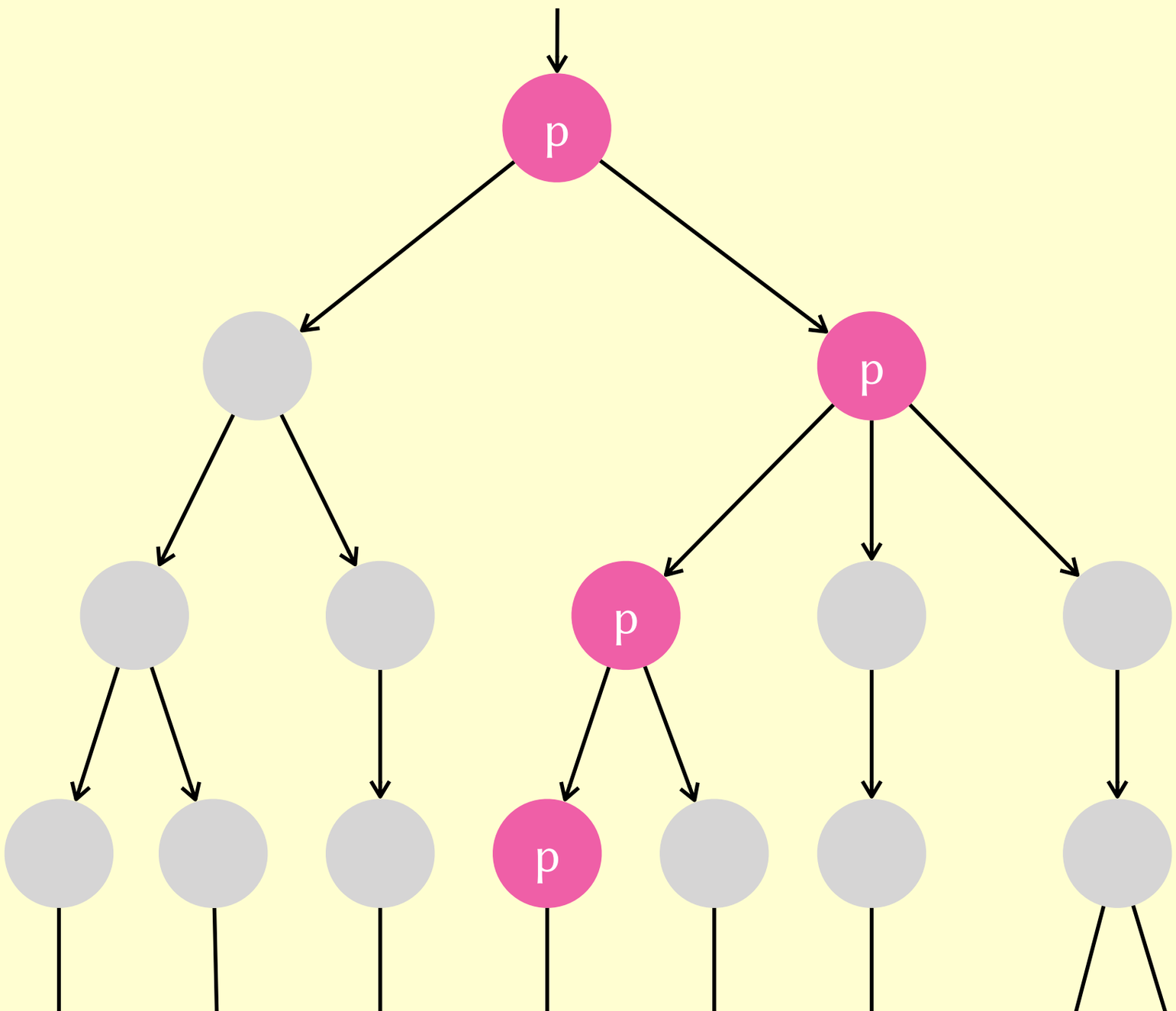
Computational Tree Logic

AF p



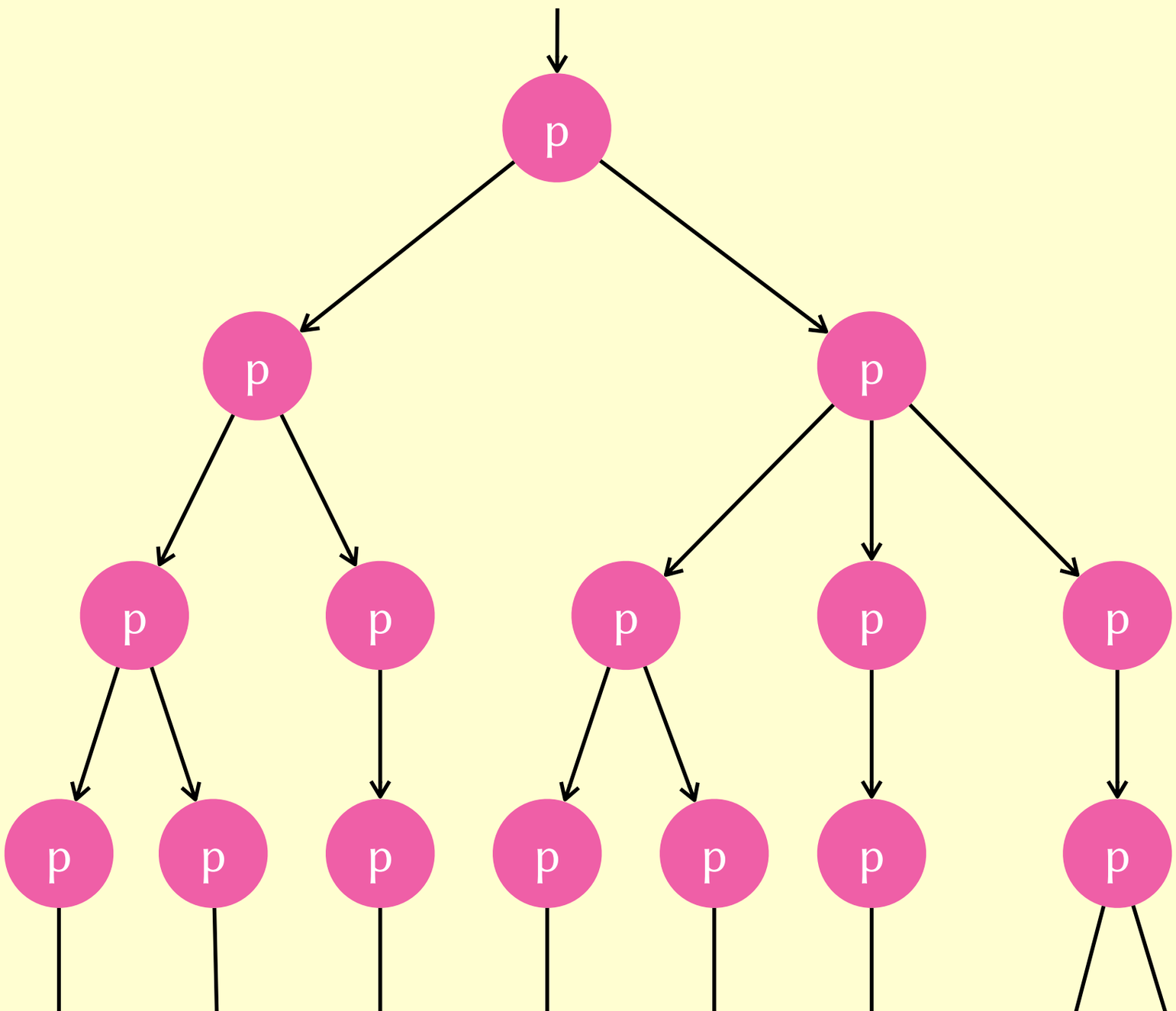
Computational Tree Logic

EG p



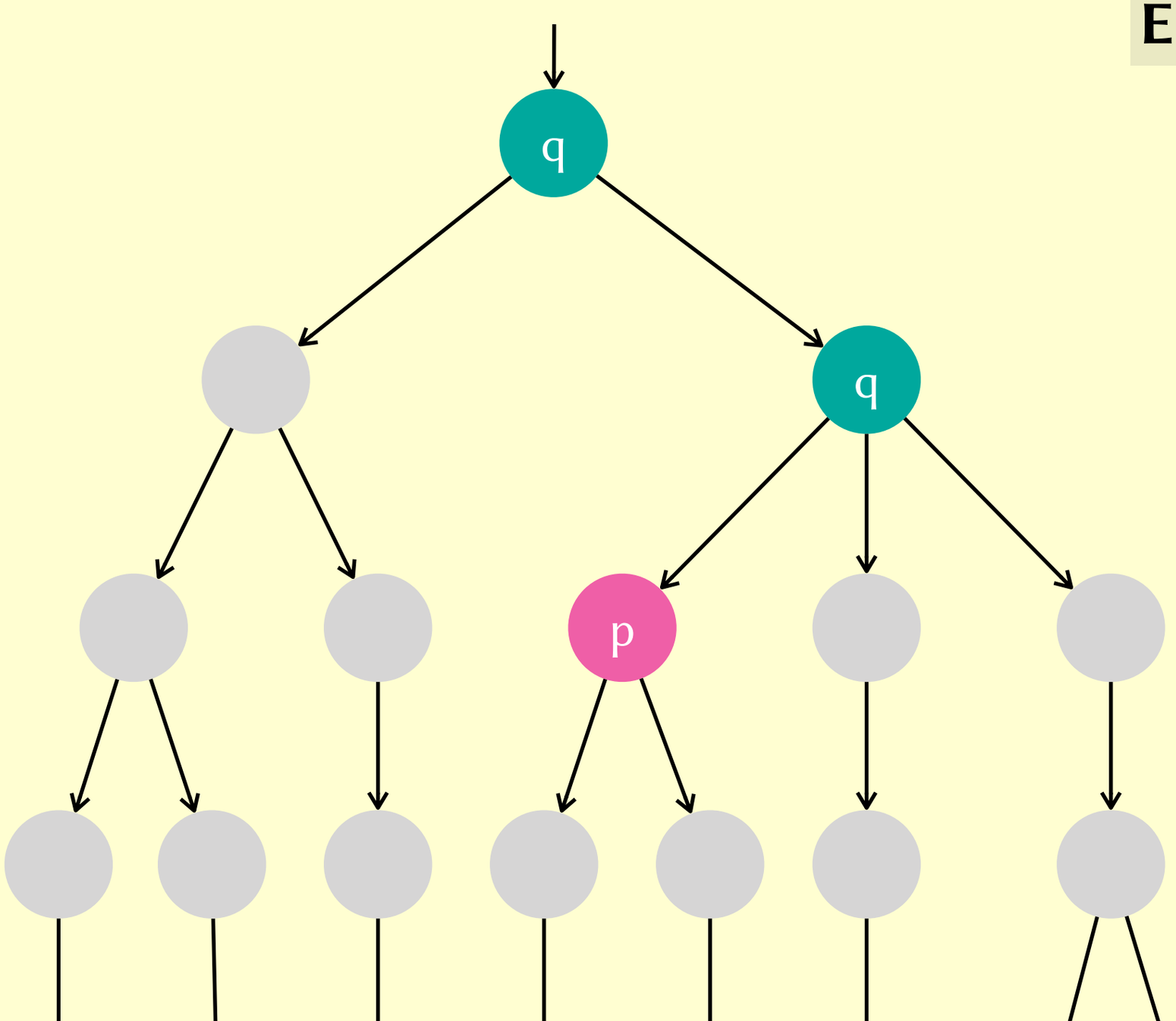
Computational Tree Logic

AG p



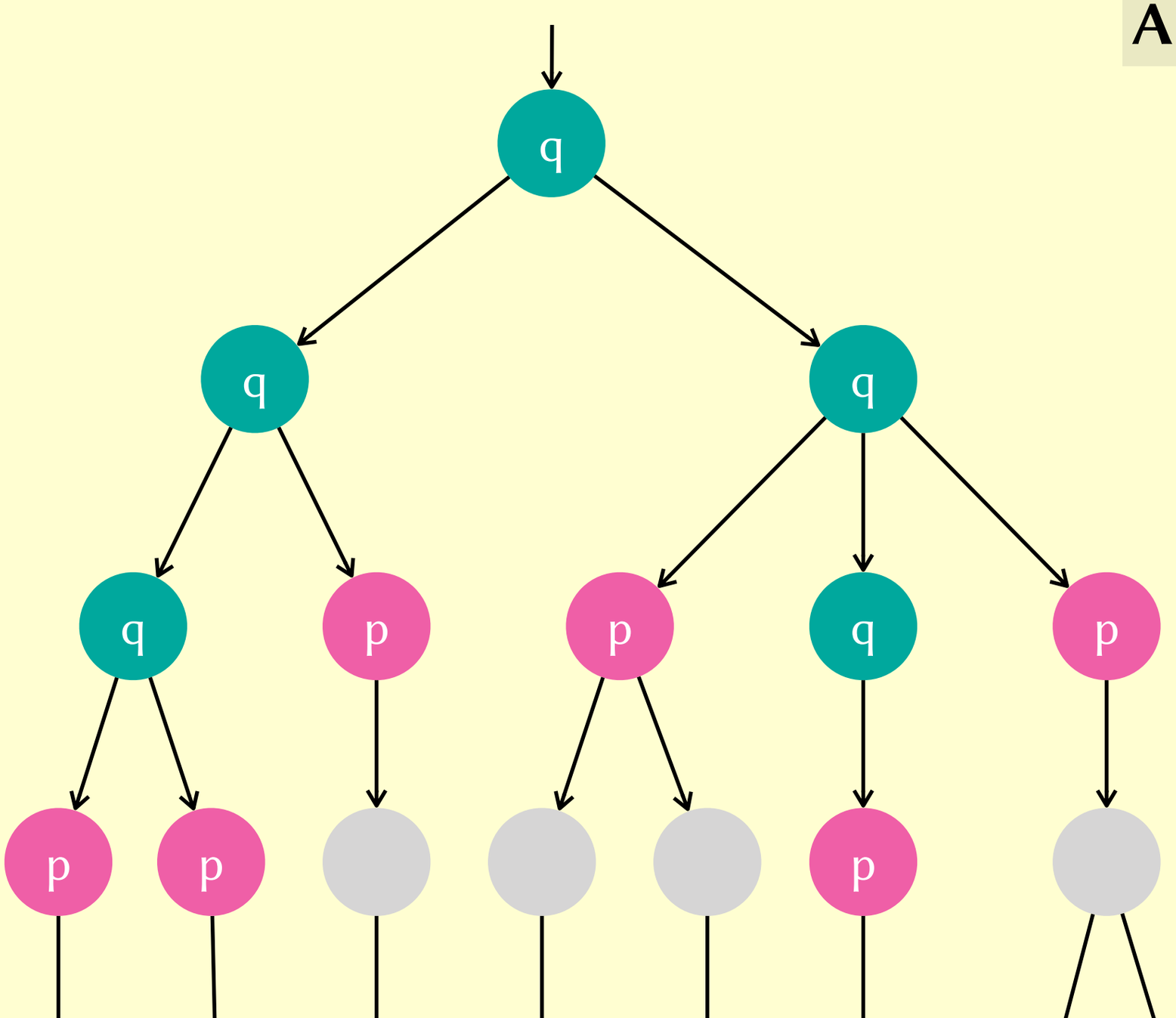
Computational Tree Logic

E[q **U** p]



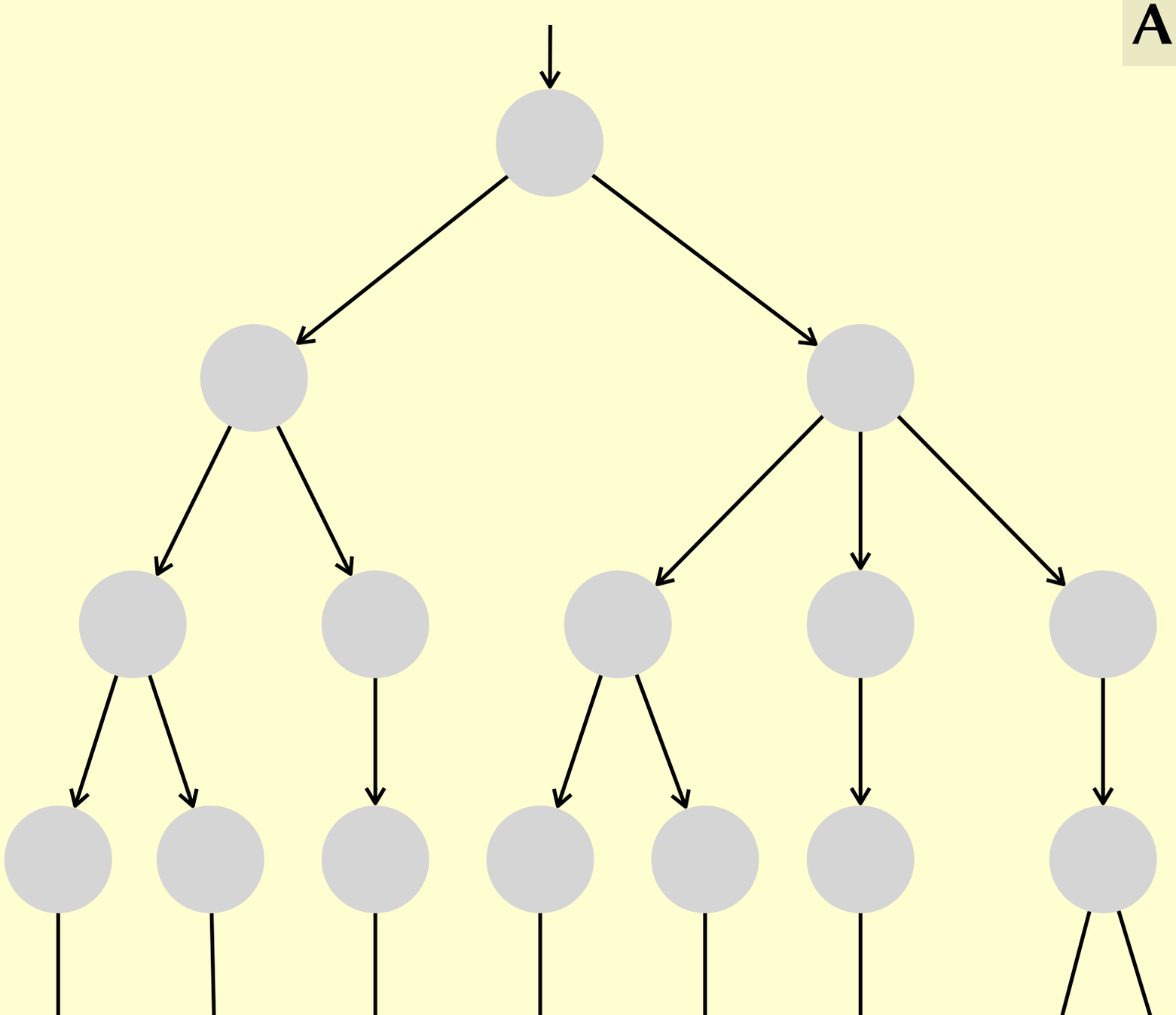
Computational Tree Logic

A[q **U** p]



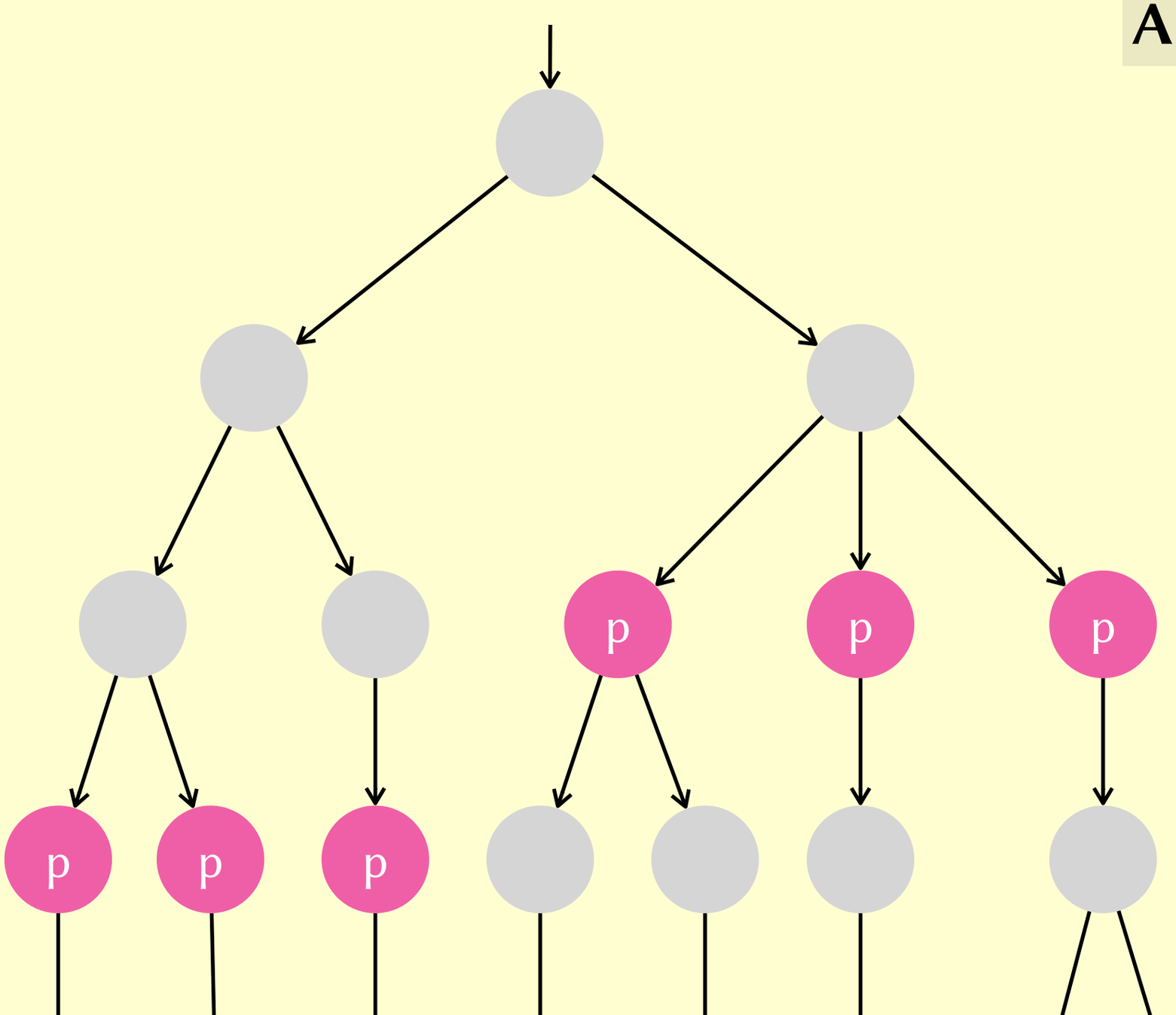
Computational Tree Logic

AF AX p



Computational Tree Logic

AF AX p



Computational Tree Logic

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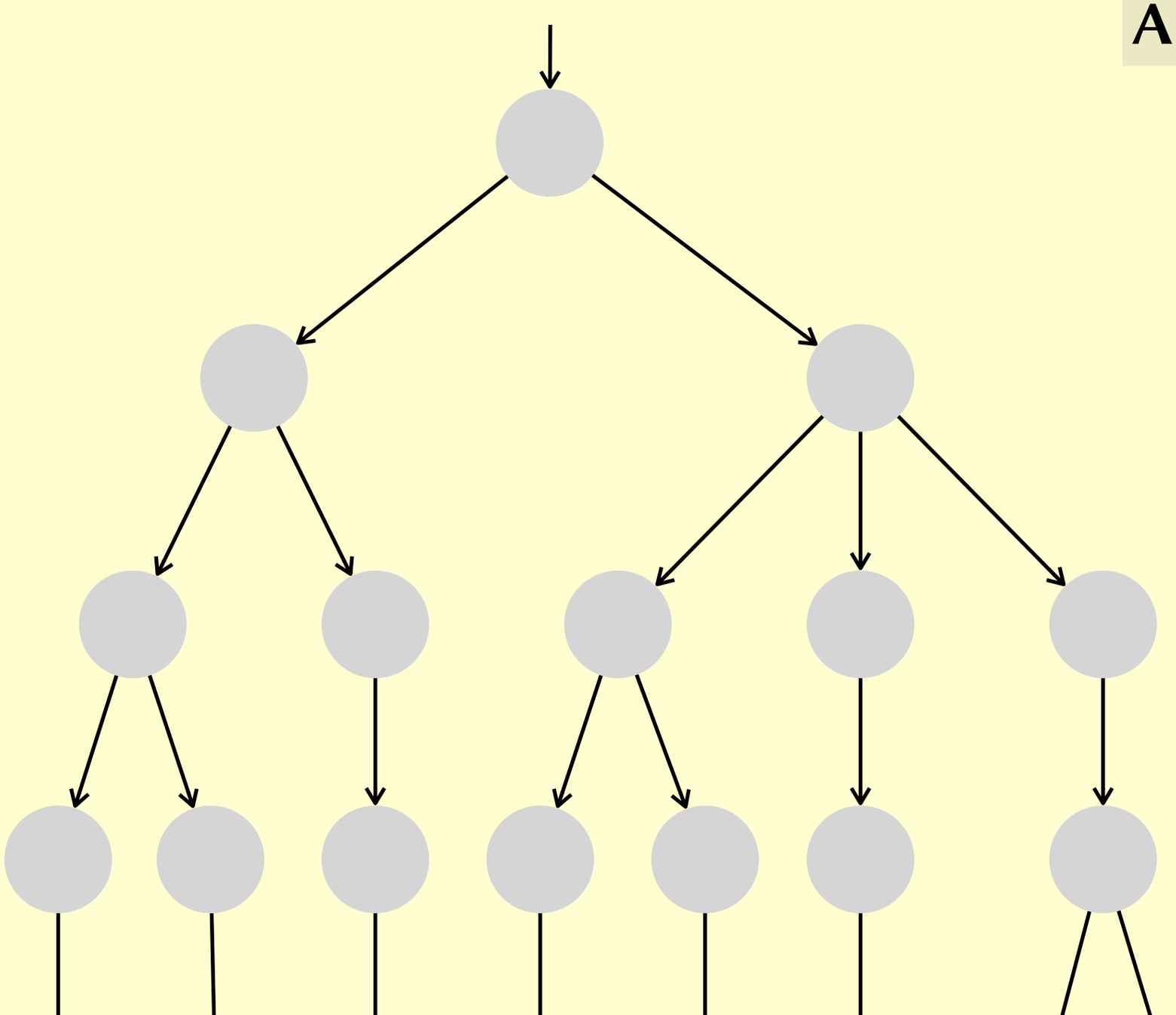
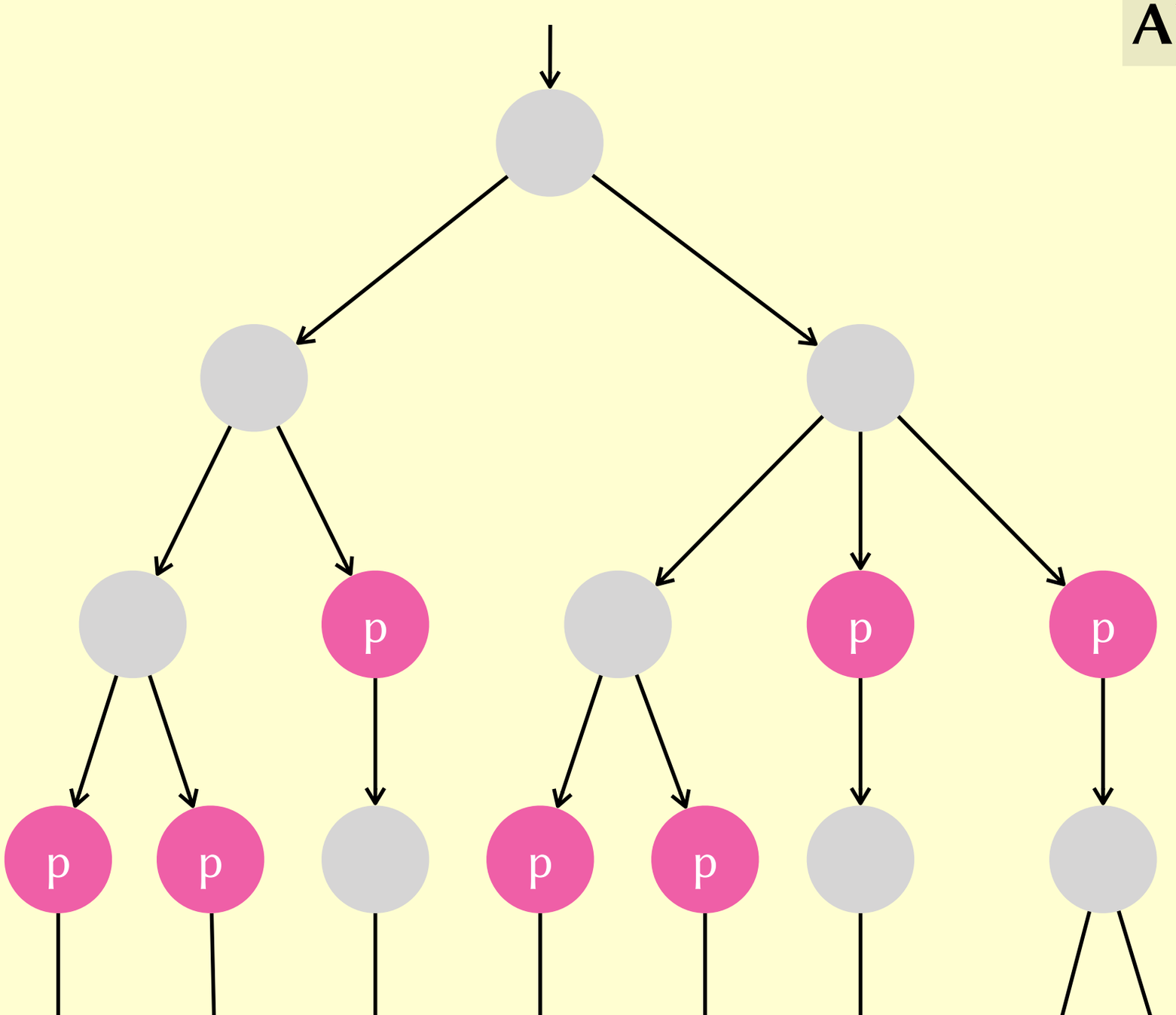


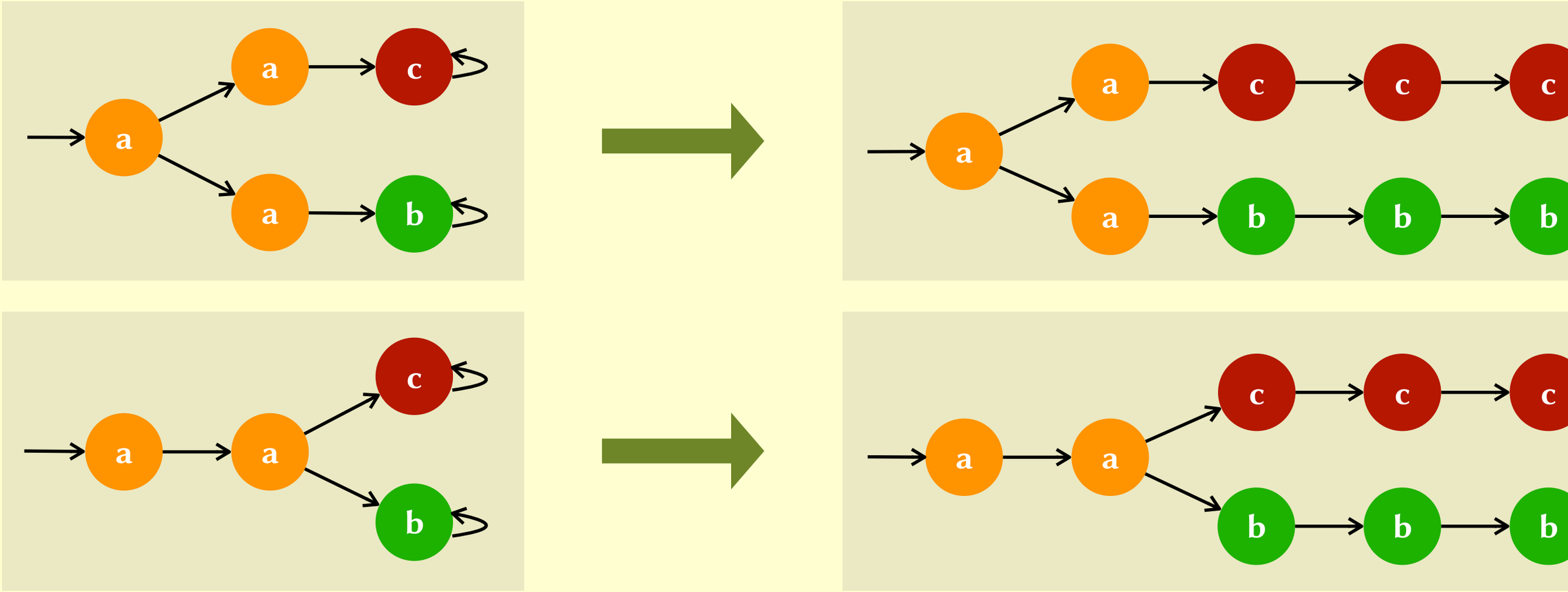
Computational Tree Logic

AX AF p



CTL to the rescue...

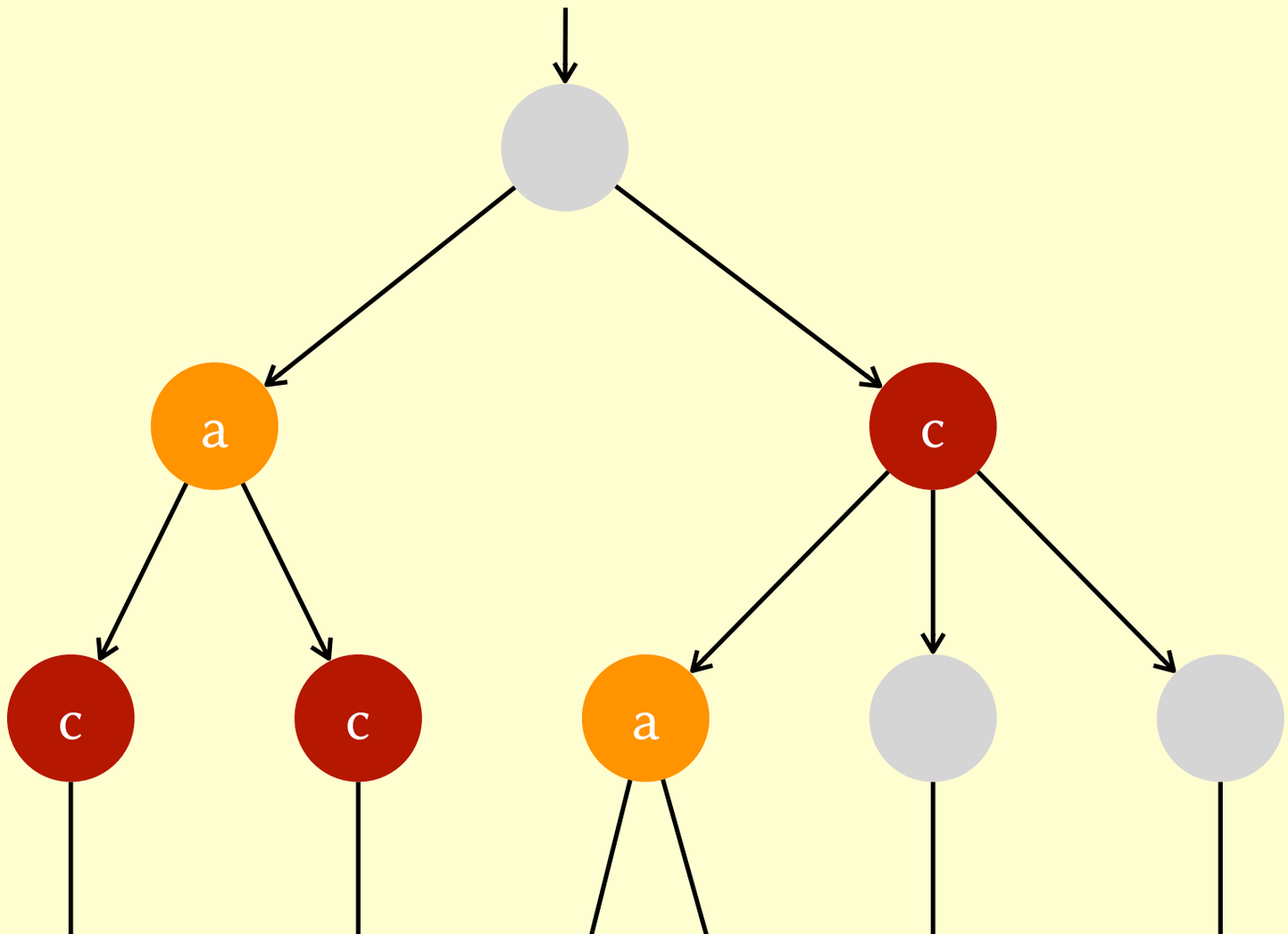
- "From any a state, we can reach a c state."



- **AG**(a $\longrightarrow$ **EF** c)

CTL's downfall...

- "All paths that contain an a also contain a c."
- (**AF** a) ⟶ (**AF** c) ✘
- **AF** (a ⟶ **EF** c) ✘
- **F** a ⟶ **F** c ✔



LTL vs CTL

- Some properties can be expressed in CTL but not LTL, e.g. "from every state, it is possible to reach an accepting state".
- Some properties can be expressed in LTL but not CTL, e.g. "every path that has a foo also has a baz".
- LTL formulas may be simpler to understand.
- CTL formulas may be more efficient to model-check.

CTL*

$$
\begin{aligned}
\phi ::= {} & \text{true} \\
\mid {} & \text{false} \\
\mid {} & p \\
\mid {} & \neg\, \phi \\
\mid {} & \phi \wedge \phi \\
\mid {} & \phi \vee \phi \\
\mid {} & \phi \rightarrow \phi \\
\mid {} & \mathbf{A}\ \alpha \\
\mid {} & \mathbf{E}\ \alpha
\end{aligned}
\qquad
\begin{aligned}
\alpha ::= {} & \phi \\
\mid {} & \neg\, \alpha \\
\mid {} & \alpha \wedge \alpha \\
\mid {} & \alpha \vee \alpha \\
\mid {} & \alpha \rightarrow \alpha \\
\mid {} & \mathbf{X}\ \alpha \\
\mid {} & \mathbf{F}\ \alpha \\
\mid {} & \mathbf{G}\ \alpha \\
\mid {} & \alpha\ \mathbf{U}\ \alpha
\end{aligned}
$$

EGF p

"There exists a path with infinitely many ps."

- Not actually used; just nice to know that LTL and CTL are both special cases of a more general temporal logic.

This lecture

- Temporal logic
- Binary decision diagrams

Boolean functions

- Could represent Boolean functions via their truth tables.
- The time taken to check f≡g is exponential in the number of variables.

A	B	C	f
0	0	0	**0**
0	0	1	**1**
0	1	0	**1**
0	1	1	**0**
1	0	0	**1**
1	0	1	**0**
1	1	0	**0**
1	1	1	**0**

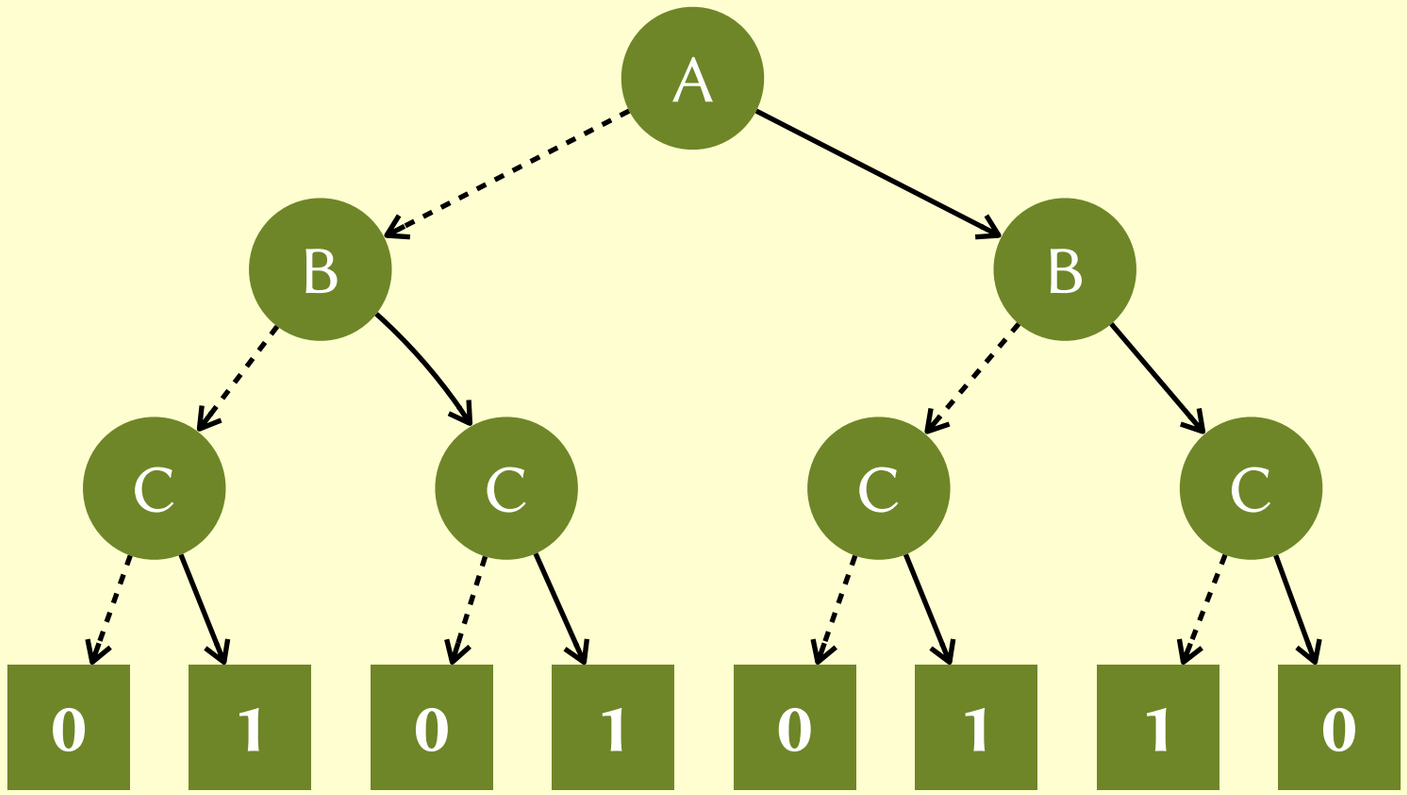
A	B	C	g
0	0	0	**0**
0	0	1	**1**
0	1	0	**0**
0	1	1	**1**
1	0	0	**0**
1	0	1	**1**
1	1	0	**1**
1	1	1	**0**

Boolean functions

- Could represent Boolean functions as syntactic formulas.
- E.g. $f = (A \wedge \neg B \wedge \neg C) \vee (\neg A \wedge (B \oplus C))$
  $g = (\neg A \wedge C) \vee (A \wedge (B \oplus C))$
- The formulas may be compact, but the time taken to check $f \equiv g$ is still exponential in the number of variables.

Boolean functions

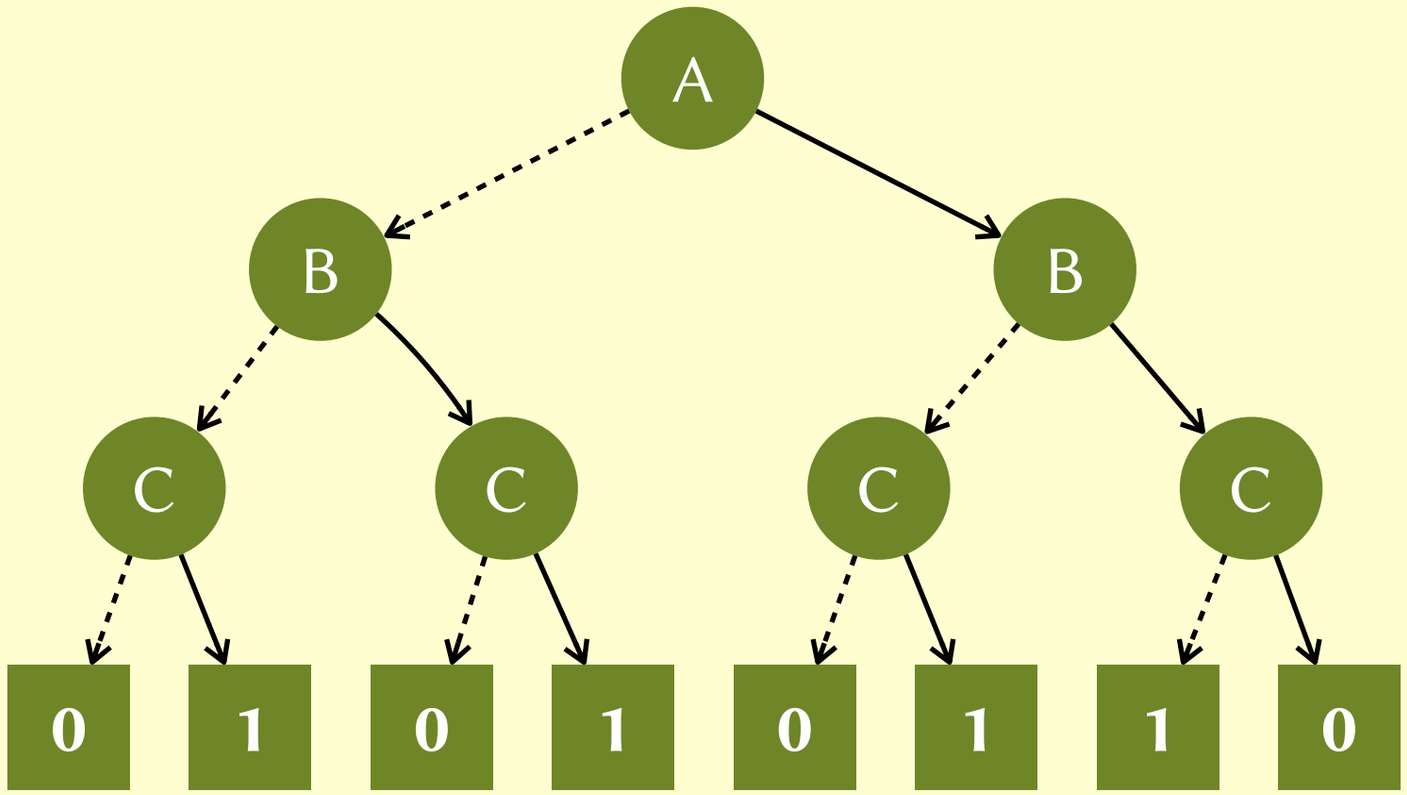
- Could represent Boolean functions as a binary decision tree.



- This is no more space efficient than the truth table, but…

Boolean functions

1. Share all the 0s and 1s.



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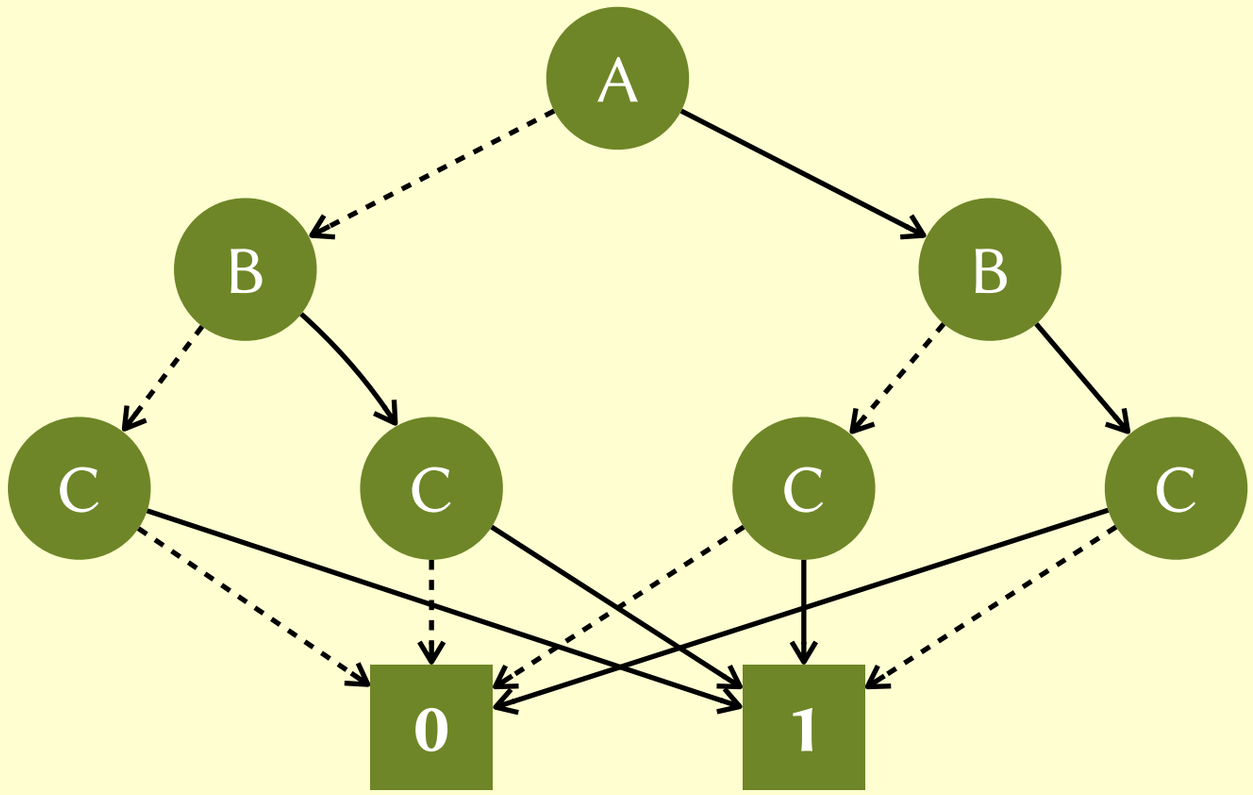
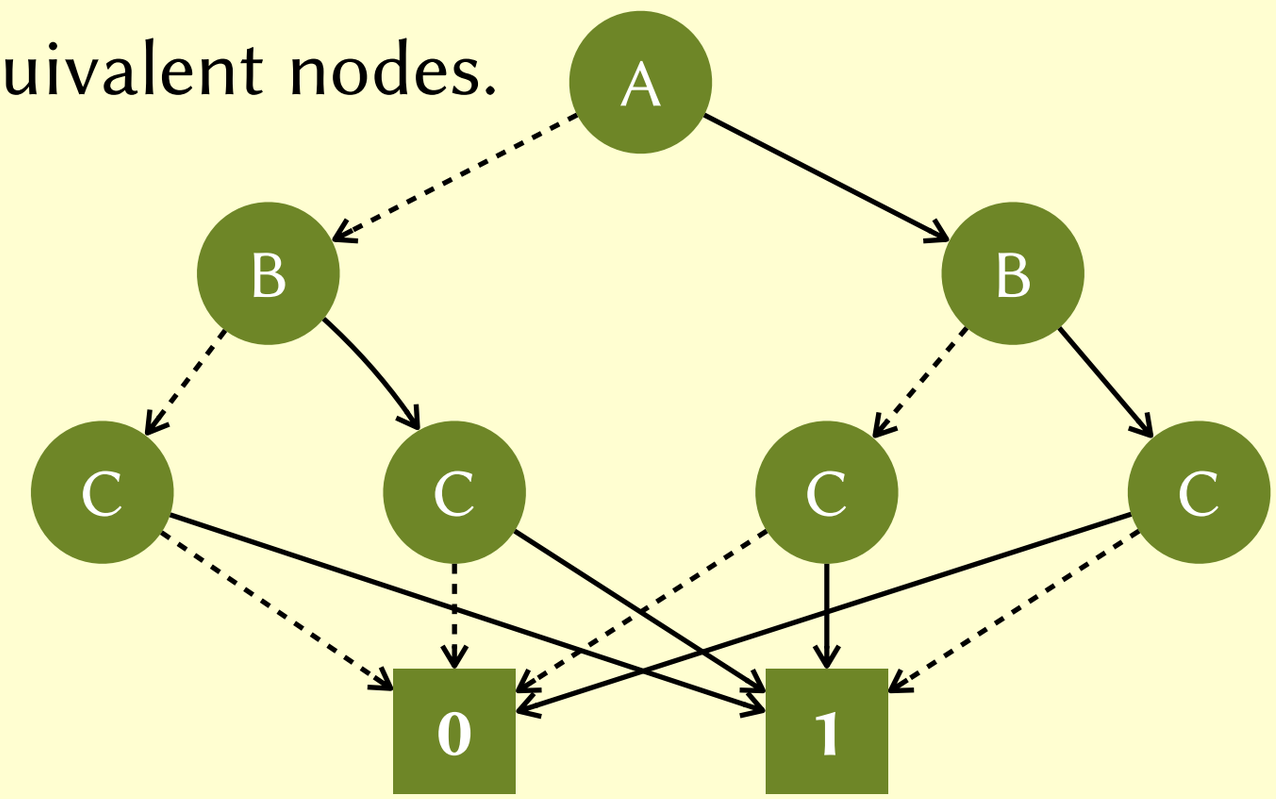


Boolean functions

1. Share all the 0s and 1s.

2. Merge equivalent nodes.



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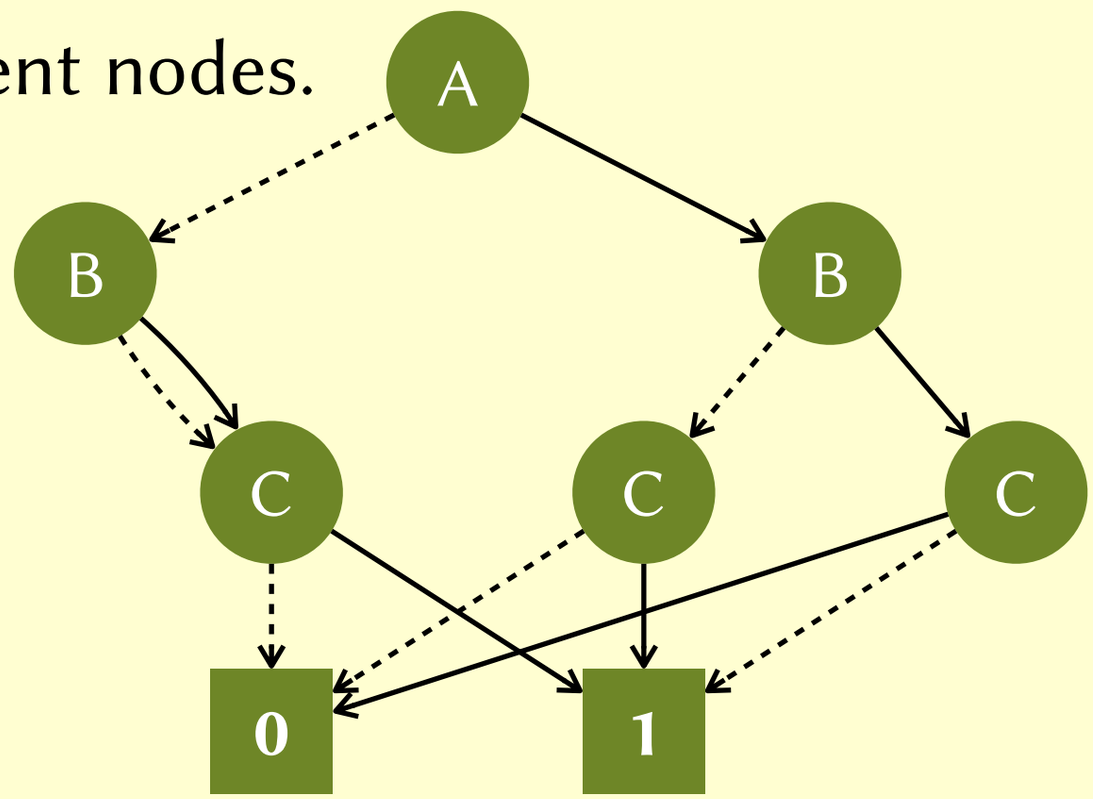

Boolean functions

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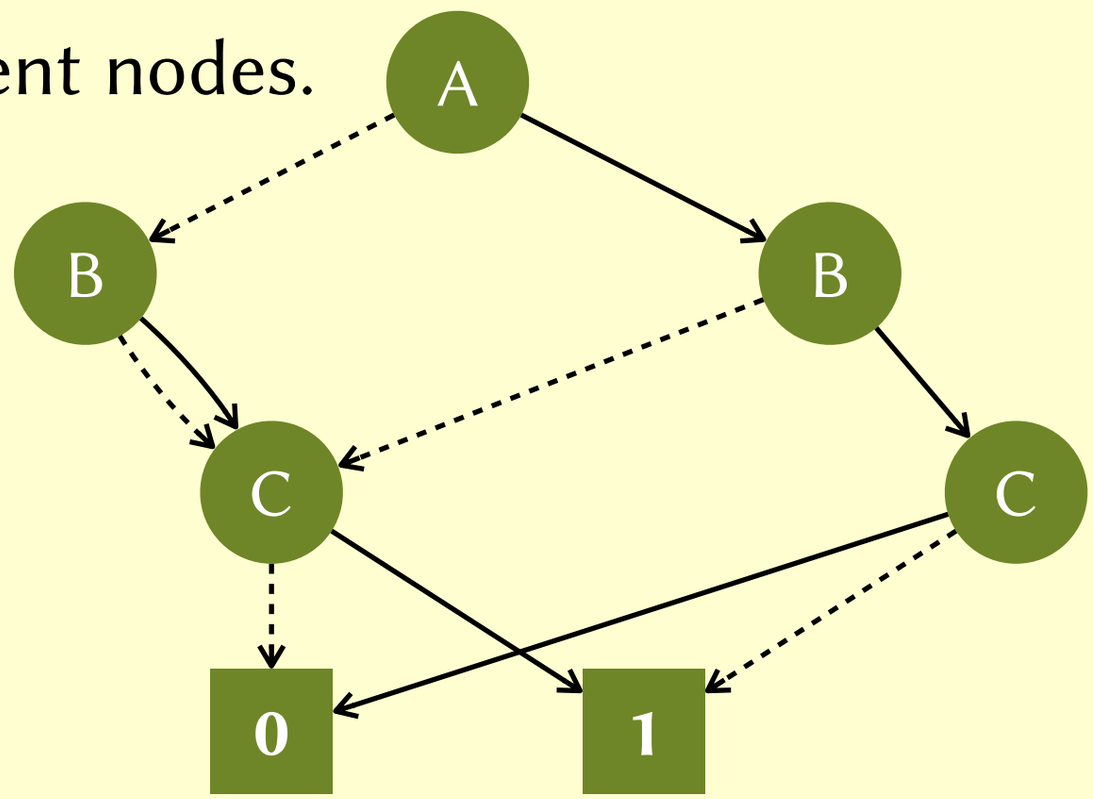
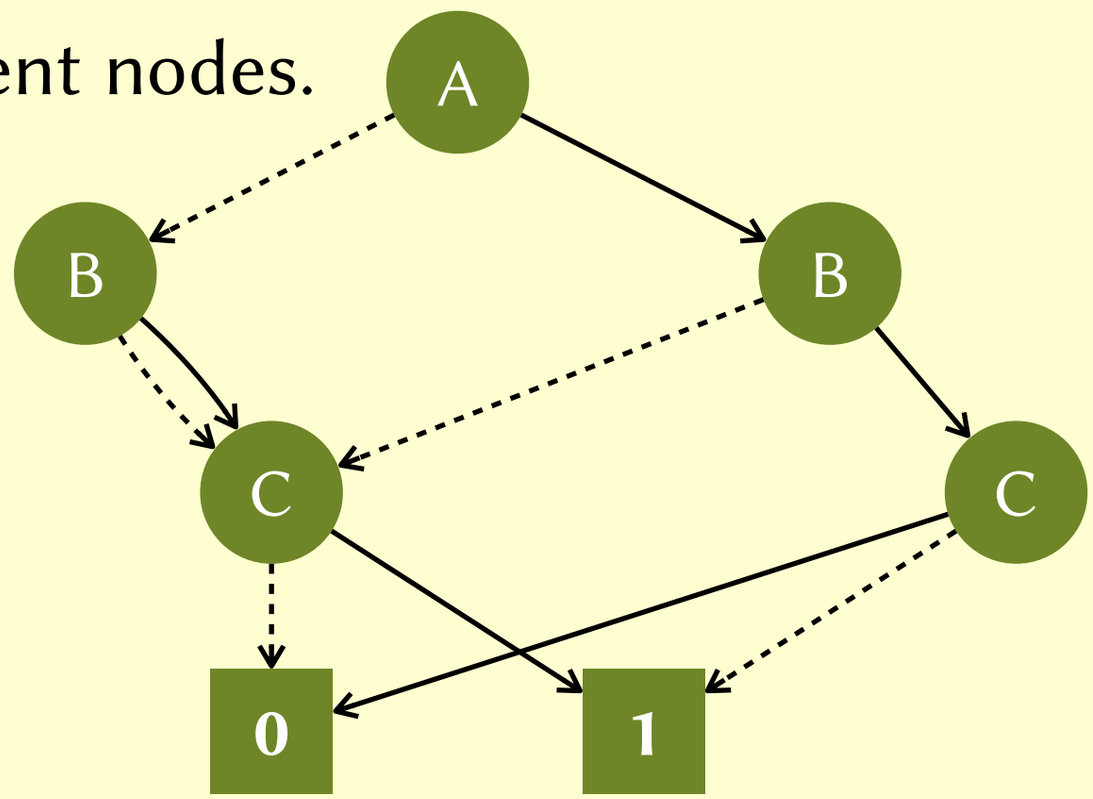


Boolean functions

1. Share all the 0s and 1s.
2. Merge equivalent nodes.
3. If both arrows from a node point to the same place, remove the node.



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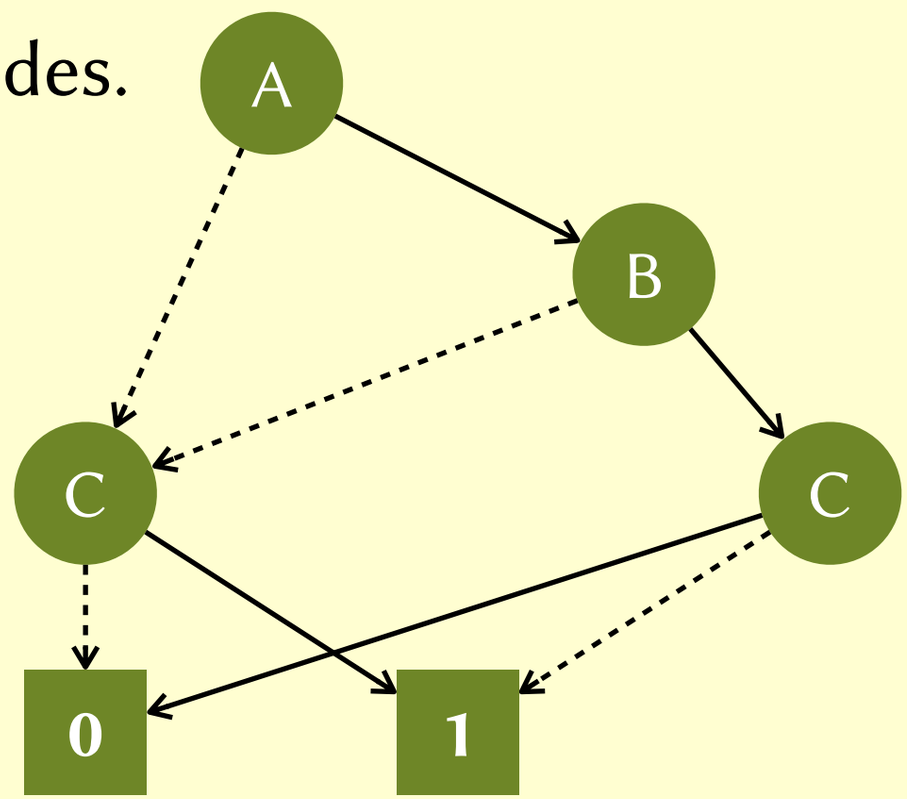


- We have produced a **binary decision diagram** (BDD).

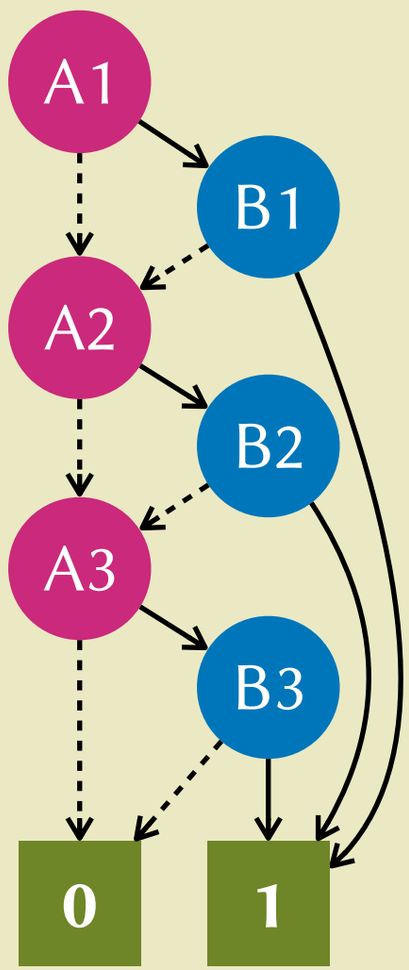
BDDs

- BDDs can be *compact*.
- BDDs are a *canonical representation*: if two formulas are equivalent, their BDDs will be the same. (Assuming BDDs are fully reduced and the order of variables is fixed.)
- Choosing a good order of variables is critical!

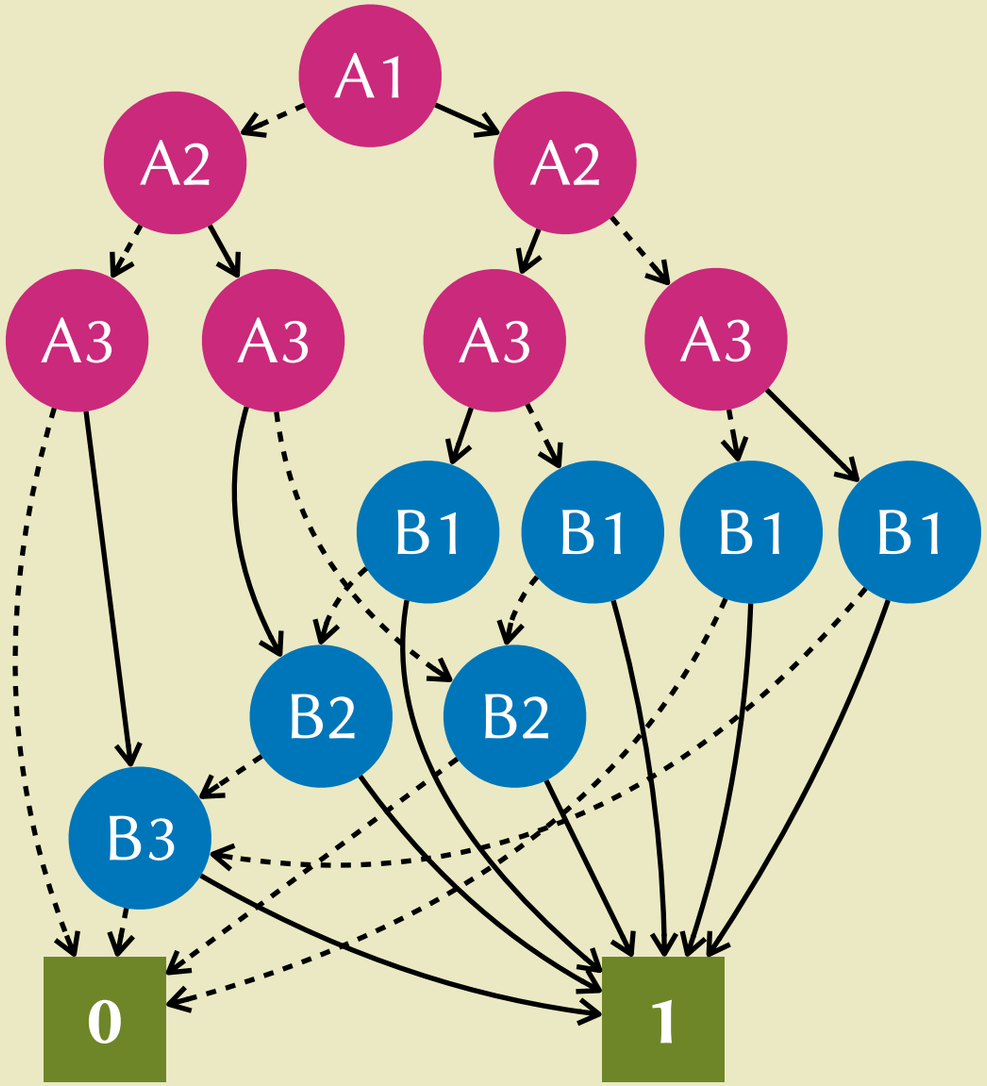
BDDs

- Suppose our formula is (A1 ∧ B1) ∨ (A2 ∧ B2) ∨ (A3 ∧ B3).

Order A1, B1, A2, B2, A3, B3:



Order A1, A2, A3, B1, B2, B3:



BDDs

- BDDs can be *compact*.
- BDDs are a *canonical representation*: if two formulas are equivalent, their BDDs will be the same. (Assuming BDDs are fully reduced and the order of variables is fixed.)
- Choosing a good order of variables is critical!
- BDDs aren't a silver bullet, but are often quite efficient.

Thank you!

- Please fill out the MEQ feedback form!