Hardware & Software Verification

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Lecture 2: Isabelle
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Lecture Outline

- Proving simple theorems by hand.
- Proving simple theorems using Isabelle.
- **Next lecture:** proving the correctness of a logic synthesiser.

Isabelle

- Invented by Lawrence Paulson around 1986. Developed ever since at the University of Cambridge and at TU München.

- Has been used for large mathematical proofs, such as the Kepler conjecture.

- Has been used to build a verified operating system! The OS implementation is about 7.5k lines of C, the proof has about 200k steps, and it uncovered hundreds of bugs in the initial implementation.

Observations

- Use `sorry` to skip a proof.
- Use `find_theorems` to search Isabelle's database of theorems.
- CTRL+click (or CMD+click) on a name to jump to its definition.
- Use `thm` to print out a theorem. Use `thm[of x]` or `thm[OF f]` to print out an instantiated theorem.
- Refer to facts using `` `backticks` `` or by naming them.
- Use `try` to invoke the Sledgehammer.

A simple proof

- **Theorem.** There is no greatest even number.
- **Proof.** To show that the greatest number does *not* exist, we shall assume that it *does*, and deduce a contradiction. To this end, suppose there *is* a greatest even number, and call it *n*. But if n is even, then so is *n*+2, which is greater than *n*. This contradicts the assumption that *n* is the greatest even number. Therefore, the greatest even number does not exist.


DEMO

Observations

- Use `moreover..ultimately` to avoid labelling each fact.
- Isabelle proofs can use the "structured" style or the "procedural" style.
- The procedural style offers various low-level commands like `defer` and `prefer`, and low-level methods like `thin_tac` and `rename_tac`.
- There are a range of automated methods: `auto`, `simp`, `clarify`, `clarsimp`, `blast`, etc.

Some constructions

- `fix` *<variable name>*
- `assume` *<new fact>*
- `have` *<new fact>* `by` *<method>*
- `hence` ~~`from this have`~~ *<new fact>* `by` *<method>*
- `with` *<name of old fact>* `have` *<new fact>* `by` *<method>*
- `have` *<new fact>* `using` *<name of old fact>* `by` *<method>*
- `show` *<thesis>* `by` *<method>*
- `thus` ~~`from this show`~~ *<thesis>* `by` *<method>*
- `moreover..ultimately`

Meta vs Object logic

- This is the difference between making a judgement about a logical statement and the logical statement itself.

- Examples:

  - For every x, if it is the case that $\mathrm{even}(x)$ holds and it is the case that $\mathrm{odd}(x)$ holds then it is the case that $x=0$ holds.

  - For every x, if it is the case that $\mathrm{even}(x) \wedge \mathrm{odd}(x)$ holds then it is the case that $x=0$ holds.

  - For every x, it is the case that $(\mathrm{even}(x) \wedge \mathrm{odd}(x)) \longrightarrow x=0$ holds.

  - It is the case that $\forall x.\ (\mathrm{even}(x) \wedge \mathrm{odd}(x)) \longrightarrow x=0$ holds.

Meta vs Object logic

- This is the difference between making a judgement about a logical statement and the logical statement itself.

- Examples:

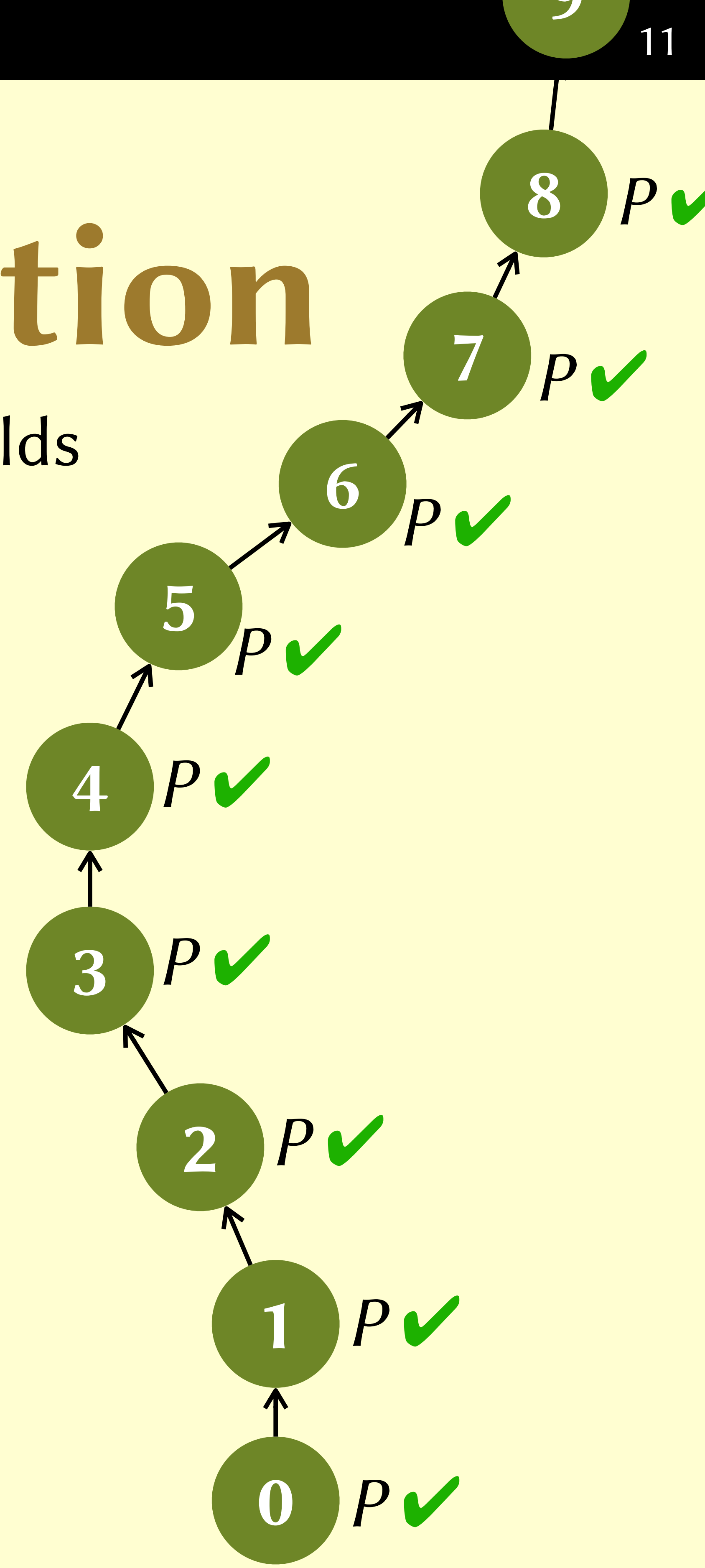
    - $\bigwedge x.\ [\![\mathrm{even}(x);\ \mathrm{odd}(x)]\!] \Longrightarrow x{=}0$

    - $\bigwedge x.\ \mathrm{even}(x) \wedge \mathrm{odd}(x) \Longrightarrow x{=}0$

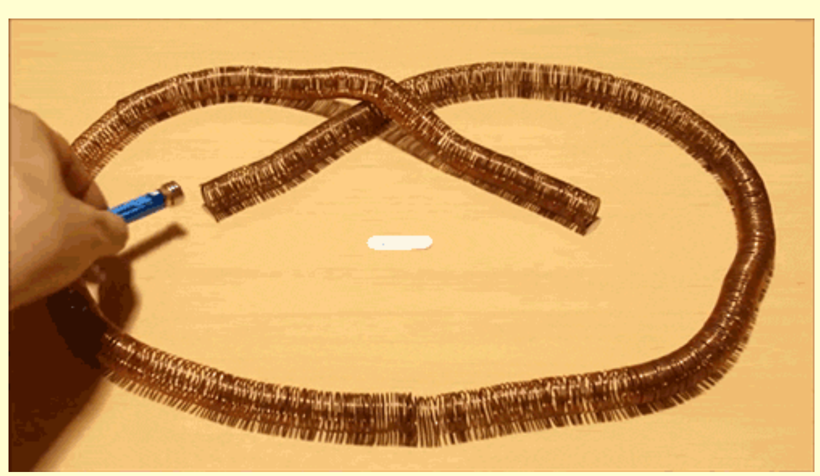
    - $\bigwedge x.\ \mathrm{even}(x) \wedge \mathrm{odd}(x) \longrightarrow x{=}0$

    - $\forall x.\ (\mathrm{even}(x) \wedge \mathrm{odd}(x)) \longrightarrow x{=}0$

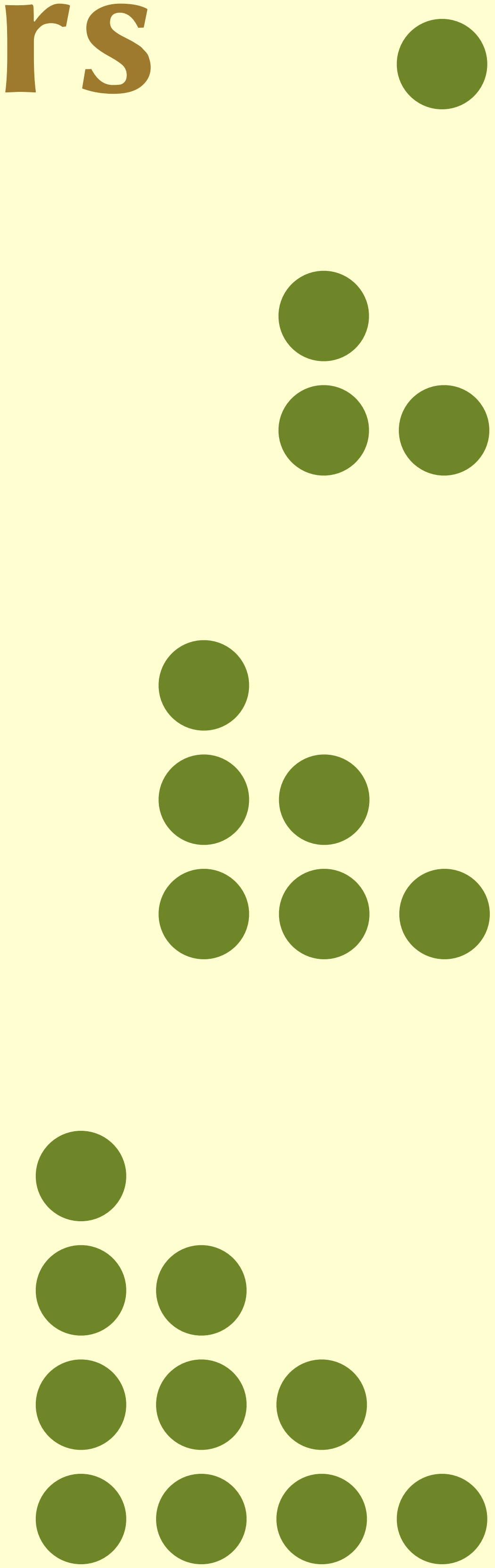
Proof by induction



- Suppose we want to show that property *P* holds for all natural numbers.
- To do this, it suffices to prove two things:
  - *P* holds for 0 (this is called the **base case**), and
  - for all k, if *P* holds for k, then *P* also holds for k+1 (this is called the **inductive step**).



Triangle numbers



Define: triangle(n) = **if** n=0 **then** 0 **else** n + triangle(n-1)

Theorem 1. triangle(n) = (n+1)n/2.

Proof. We proceed by mathematical induction.

Base case. triangle(0) = 0 = (0+1)0/2

definition of triangle | algebraic manipulation

Thus triangle(0) = (0+1)0/2.

Inductive step. Pick arbitrary k.

Assume triangle(k) = (k+1)k/2 as the *induction hypothesis* (IH).

triangle(k+1) = k + 1 + triangle(k) = k + 1 + (k+1)k/2 = (k+2)(k+1)/2

definition of triangle | IH | algebraic manipulation

Thus triangle(k+1) = (k+2)(k+1)/2.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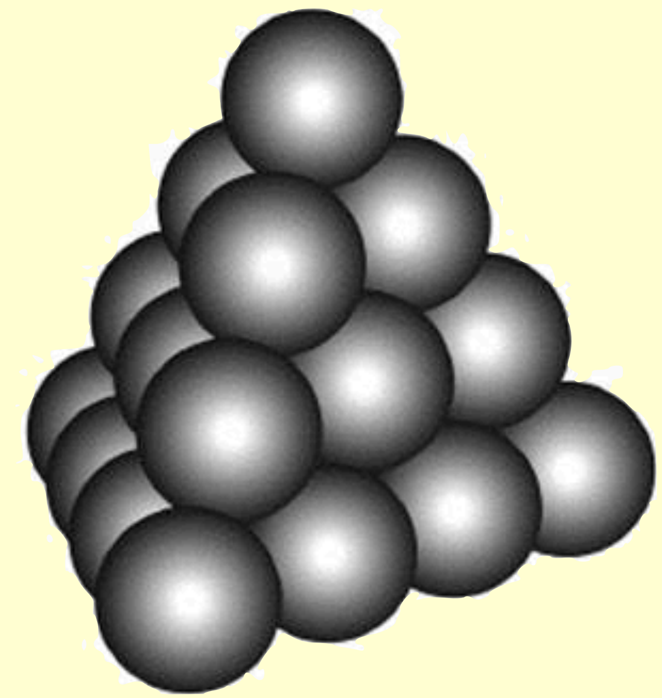
Tetrahedral numbers



Define: tet(n) = **if** n=0 **then** 0 **else** triangle(n) + tet(n-1)

Theorem 2. tet(n) = (n+2)(n+1)n/6.

Proof. We proceed by mathematical induction.

Base case. tet(0) = 0 = (0+2)(0+1)0/6

definition of tet

algebraic manipulation

Thus tet(0) = (0+2)(0+1)0/6.

Inductive step. Pick arbitrary k and assume tet(k) = (k+2)(k+1)k/6 as the induction hypothesis (IH).

tet(k+1) = triangle(k+1) + tet(k) = (k+2)(k+1)/2 + tet(k) = (k+2)(k+1)/2 + (k+2)(k+1)k/6

= (k+3)(k+2)(k+1)/6

definition of tet

Theorem 1

IH

algebraic manipulation

Thus tet(k+1) = (k+3)(k+2)(k+1)/6.


DEMO

Observations

- Use `also..finally` for chains of equational reasoning.
- Isabelle will provide a bare-bones induction proof for you when you type `proof (induct ...)`.
- Use `{` braces `}` to delimit the scope of a local assumption.

Summary

- **This lecture:** how to conduct some basic proofs in Isabelle.
- **Next lecture:** How to implement a (small) logic synthesiser in Isabelle and verify that it is correct.