Foundations of Software Verification

John Wickerson

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Today's lecture

- Some underlying principles of software verification.
- A brief history of the field from around 1969 (Hoare Logic) to around 2001 (Separation Logic).

Hoare Logic

C.A.R. (Tony) Hoare (1934–)

- Invented by Tony Hoare in 1969.
- A mathematical system based on annotating program code with **assertions** that must hold whenever execution reaches that point.
- The basic unit is the Hoare triple, written $\{ p \}$ C $\{ q \}$.

Hoare triples

- What does { p } C { q } mean?

  - It's more-or-less equivalent to the following Dafny code:

    ```
    assume p;
    C;
    assert q;
    ```

  - If C begins execution in a state satisfying p, then any final state it reaches will satisfy q.

  - C can be an entire program, or just a single instruction.

Rules of Hoare logic

$$\frac{\{\,p\,\}\ C_1\ \{\,q\,\} \qquad \{\,q\,\}\ C_2\ \{\,r\,\}}{\{\,p\,\}\ C_1\,;\,C_2\ \{\,r\,\}}\ \text{SEQ}$$

$$\frac{\{\,p \wedge b\,\}\ C_1\ \{\,q\,\} \qquad \{\,p \wedge \neg b\,\}\ C_2\ \{\,q\,\}}{\{\,p\,\}\ \text{if } b \text{ then } C_1 \text{ else } C_2\ \{\,q\,\}}\ \text{IF}$$

$$\frac{p \Longrightarrow q[e/x]}{\{\,p\,\}\ x := e\ \{\,q\,\}}\ \text{ASS}$$

$$\frac{p \Longrightarrow I \qquad \{\,I \wedge b\,\}\ C\ \{\,I\,\} \qquad I \wedge \neg b \Longrightarrow q}{\{\,p\,\}\ \text{while } b \text{ do } C\ \{\,q\,\}}\ \text{WHILE}$$

DEMO

$$\dfrac{\dfrac{\dfrac{}{\{\,\mathrm{true}\,\}\ x := 0\ \{x{=}0\}}\,\mathrm{ASS} \quad \dfrac{}{\{x{=}0\}\ y := 0\ \{\,x{=}0 \wedge y{=}0\,\}}\,\mathrm{ASS}}{\{\,\mathrm{true}\,\}\ x := 0;\ y := 0\ \{x{=}0 \wedge y{=}0\}}\,\mathrm{SEQ} \quad \dfrac{\dfrac{\dfrac{}{\{\,2x{=}y \wedge x{\le}10 \wedge x{<}10\,\}\ x := x{+}1\ \{2(x{-}1){=}y \wedge x{-}1{<}10\}}\,\mathrm{ASS} \quad \dfrac{}{\{2(x{-}1){=}y \wedge x{-}1{<}10\}\ y := y{+}2\ \{\,2x{=}y \wedge x{\le}10\,\}}\,\mathrm{ASS}}{\{\,2x{=}y \wedge x{\le}10 \wedge x{<}10\,\}\ x := x{+}1;\ y := y{+}2\ \{\,2x{=}y \wedge x{\le}10\,\}}\,\mathrm{SEQ}}{\{x{=}0 \wedge y{=}0\}\ \mathrm{while}\ x{<}10\ \mathrm{do}\ (x := x{+}1;\ y := y{+}2)\ \{\,x{=}10 \wedge y{=}20\,\}}\,\mathrm{WHILE}}{\{\,\mathrm{true}\,\}\ x := 0;\ y := 0;\ \mathrm{while}\ x{<}10\ \mathrm{do}\ (x := x{+}1;\ y := y{+}2)\ \{\,x{=}10 \wedge y{=}20\,\}}\,\mathrm{SEQ}$$

Verification conditions:

$$\mathrm{true} \Longrightarrow 0{=}0$$

$$x{=}0 \Longrightarrow x{=}0 \wedge 0{=}0$$

$$x{=}0 \wedge y{=}0 \Longrightarrow 2x{=}y \wedge x{\le}10$$

$$2x{=}y \wedge x{\le}10 \wedge \neg(x{<}10) \Longrightarrow x{=}10 \wedge y{=}20$$

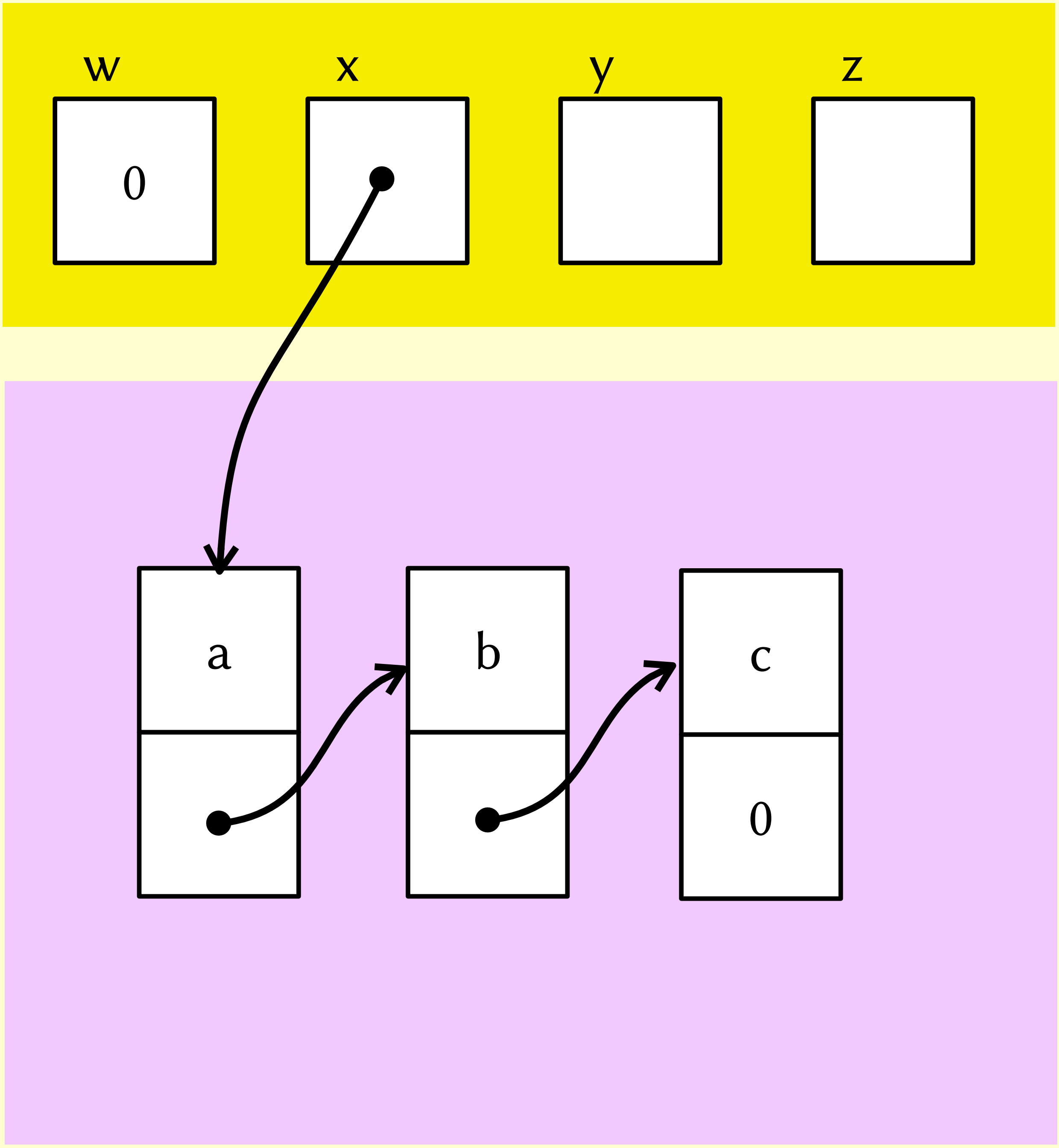
$$2x{=}y \wedge x{\le}10 \wedge x{<}10 \Longrightarrow 2((x{+}1){-}1){=}y \wedge (x{+}1){-}1{<}10$$

$$2(x{-}1){=}y \wedge x{-}1{<}10 \Longrightarrow 2x{=}(y{+}2) \wedge x{\le}10$$

A challenge for Hoare

- Hoare logic struggles to reason about heap-allocated data structures like linked lists.

List reverse




```
{list δ x}
w := 0;
while (x≠0) do {
    z := [x+1];
    [x+1] := w;
    w := x;
    x := z;
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{list -δ w}
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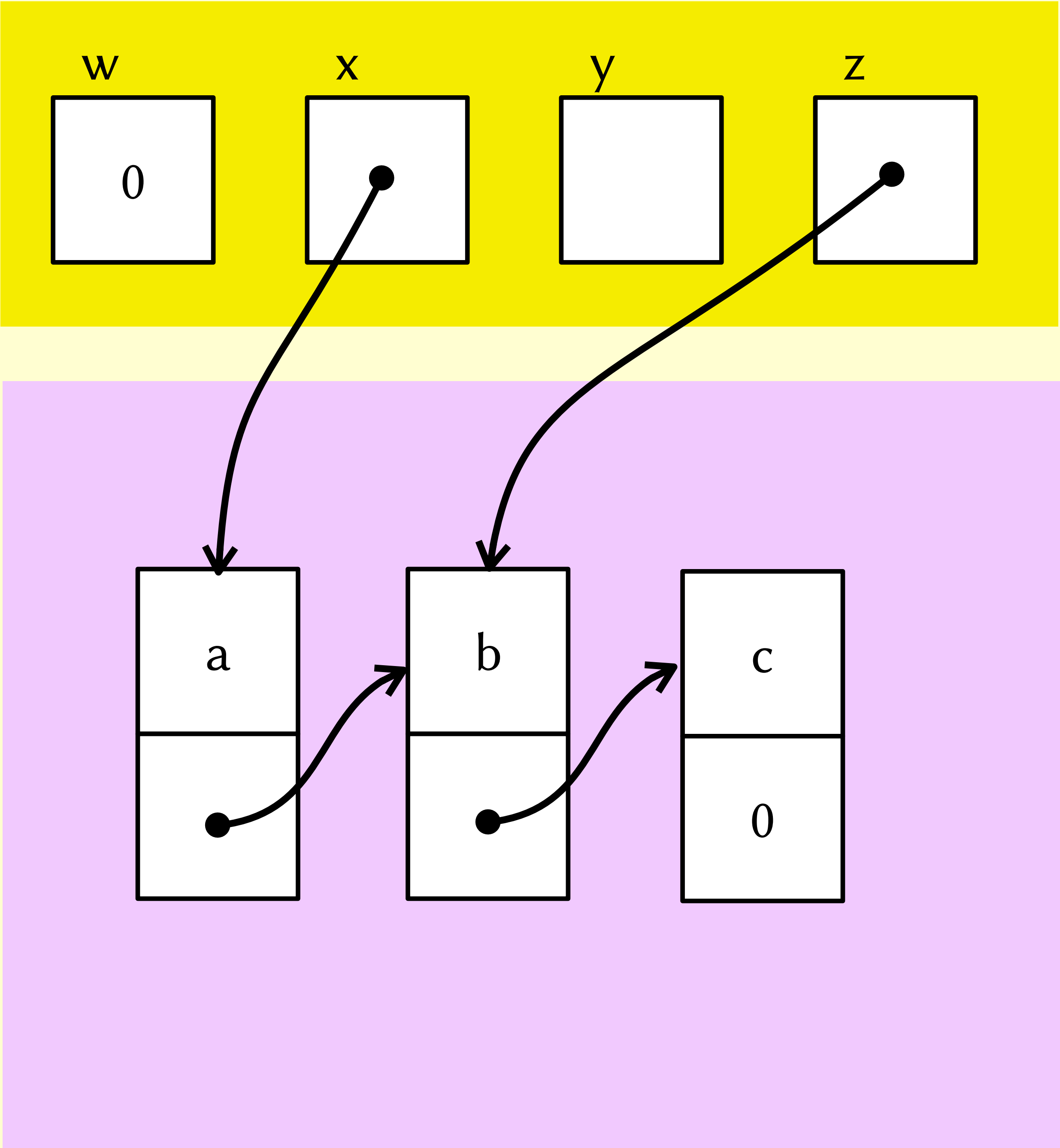

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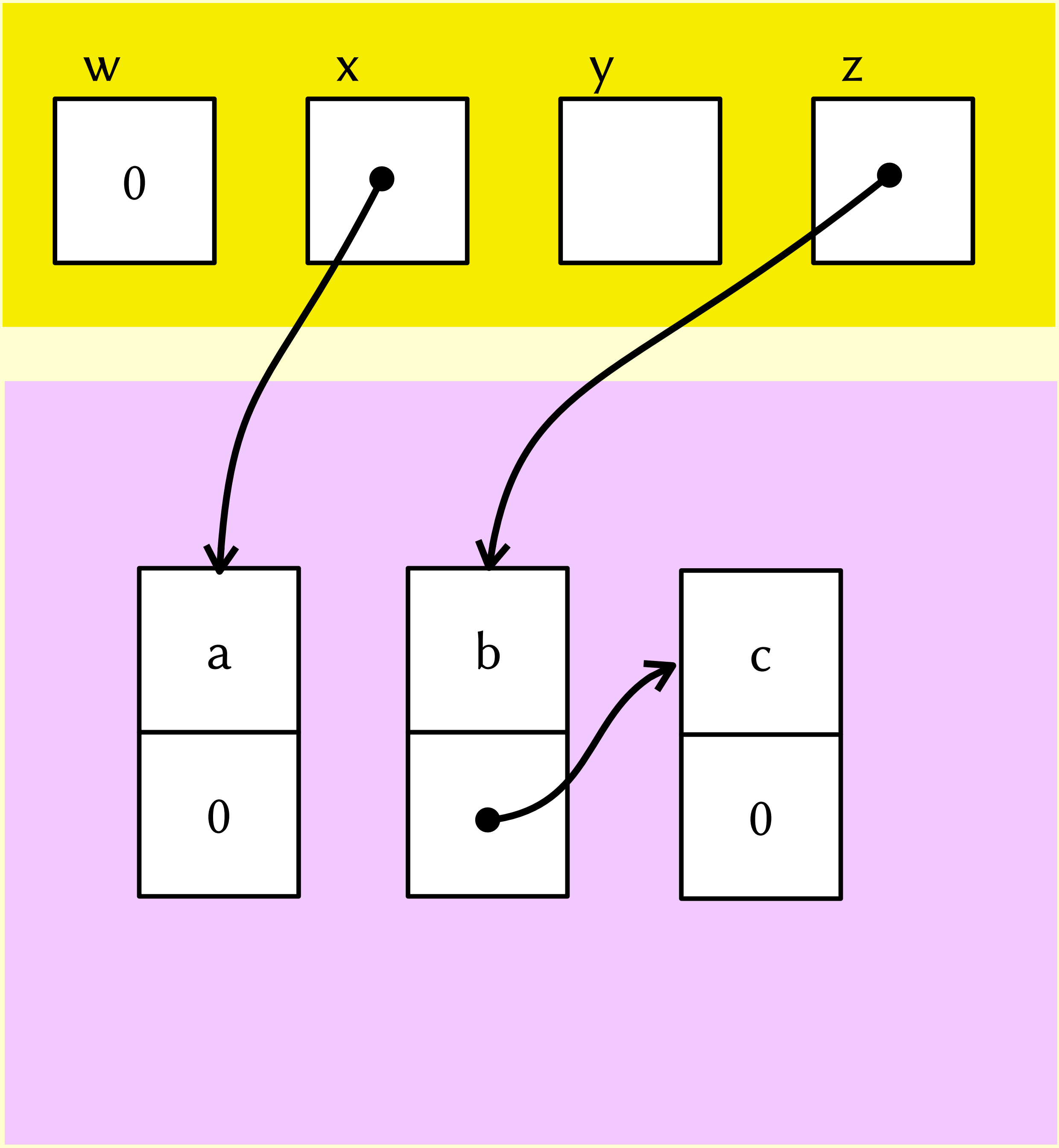

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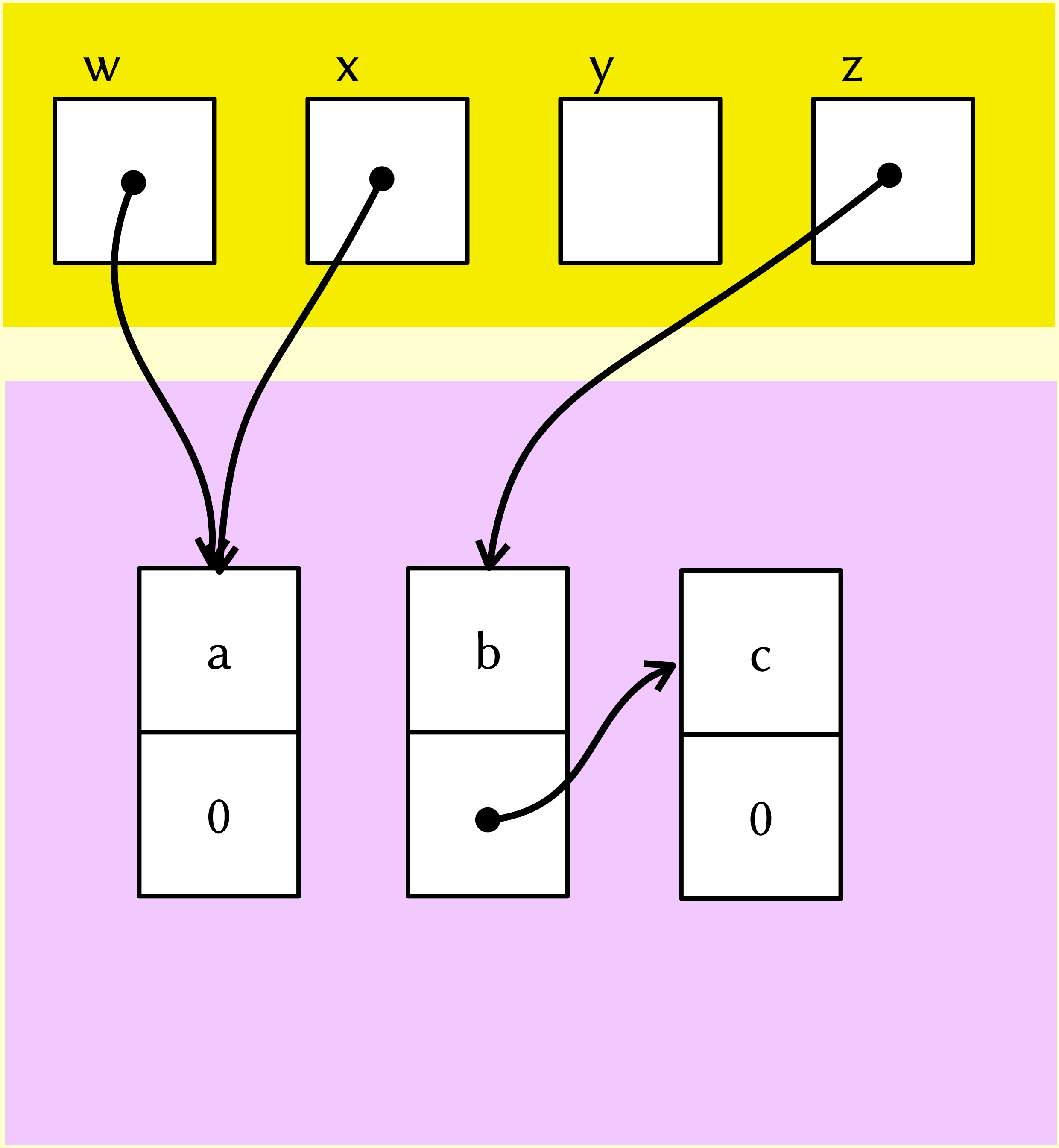
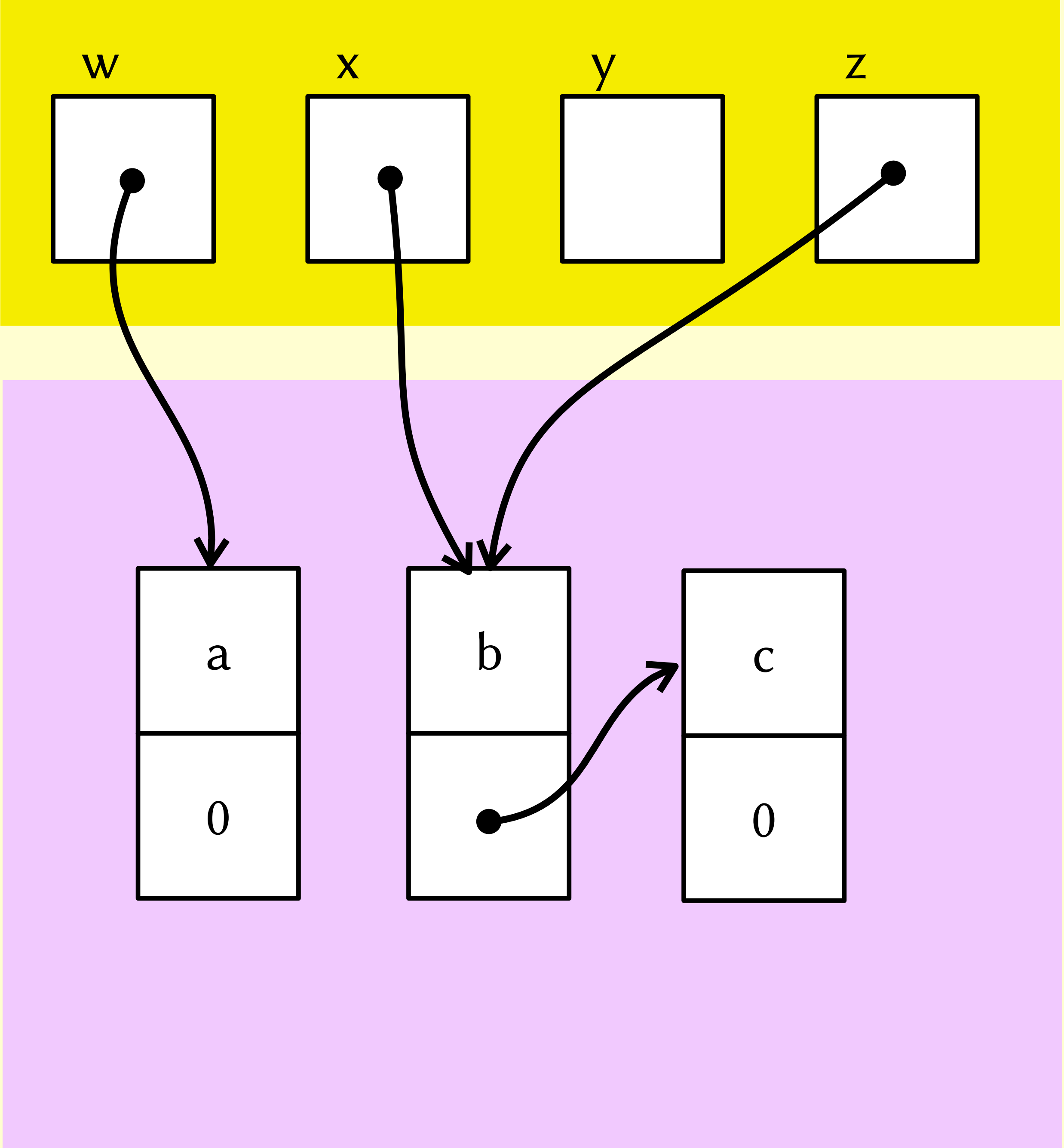


List reverse

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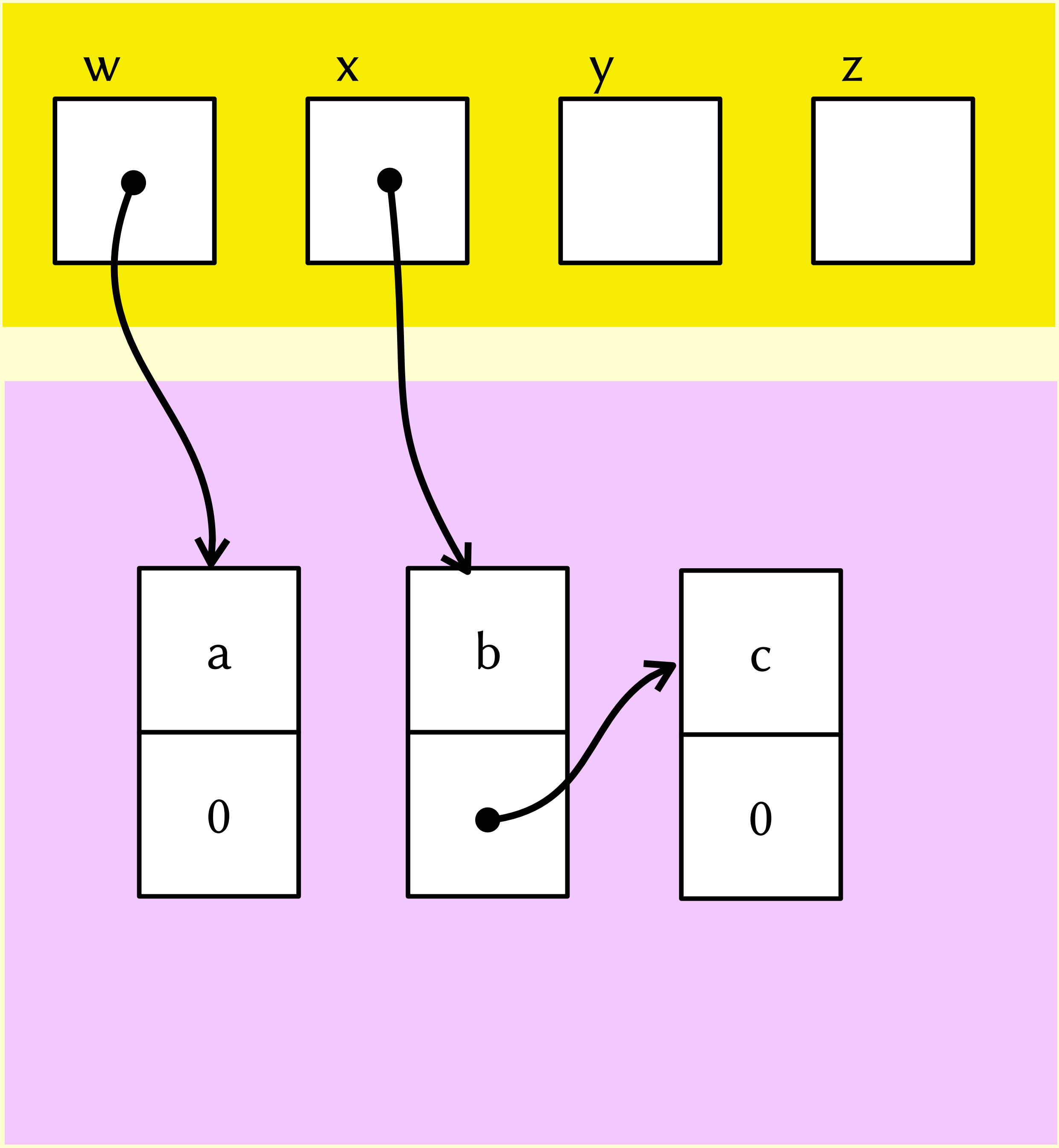
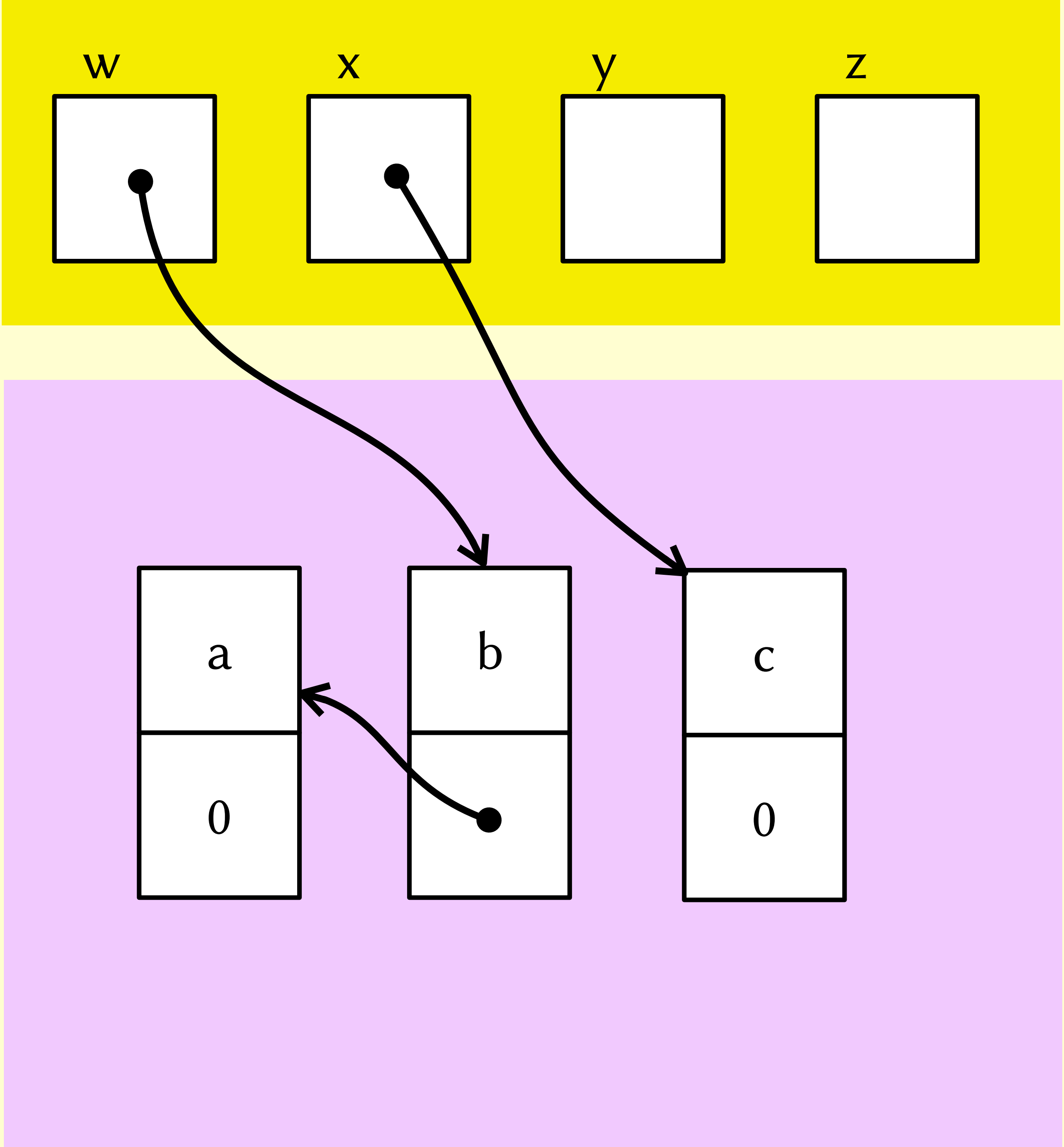


List reverse



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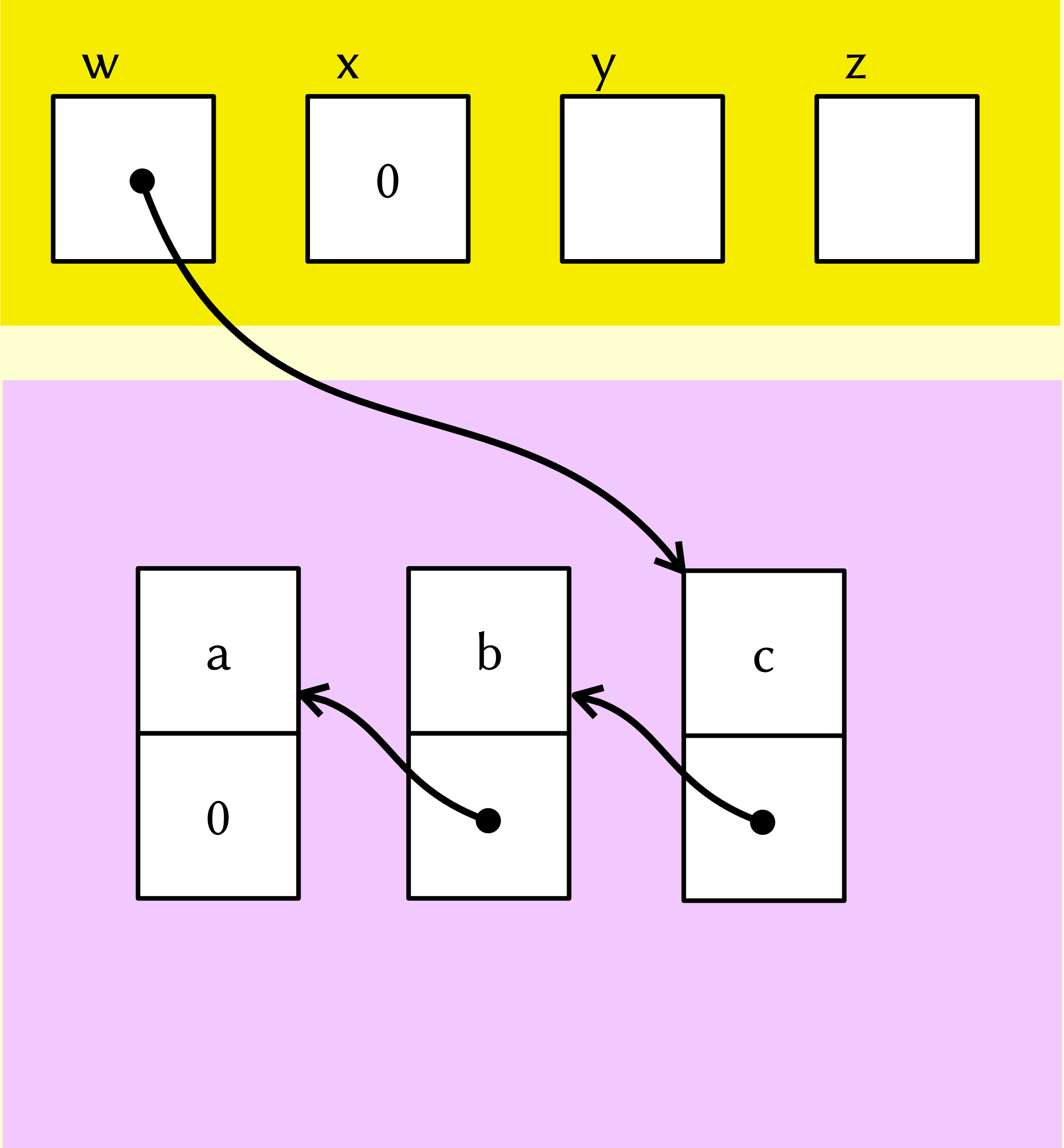

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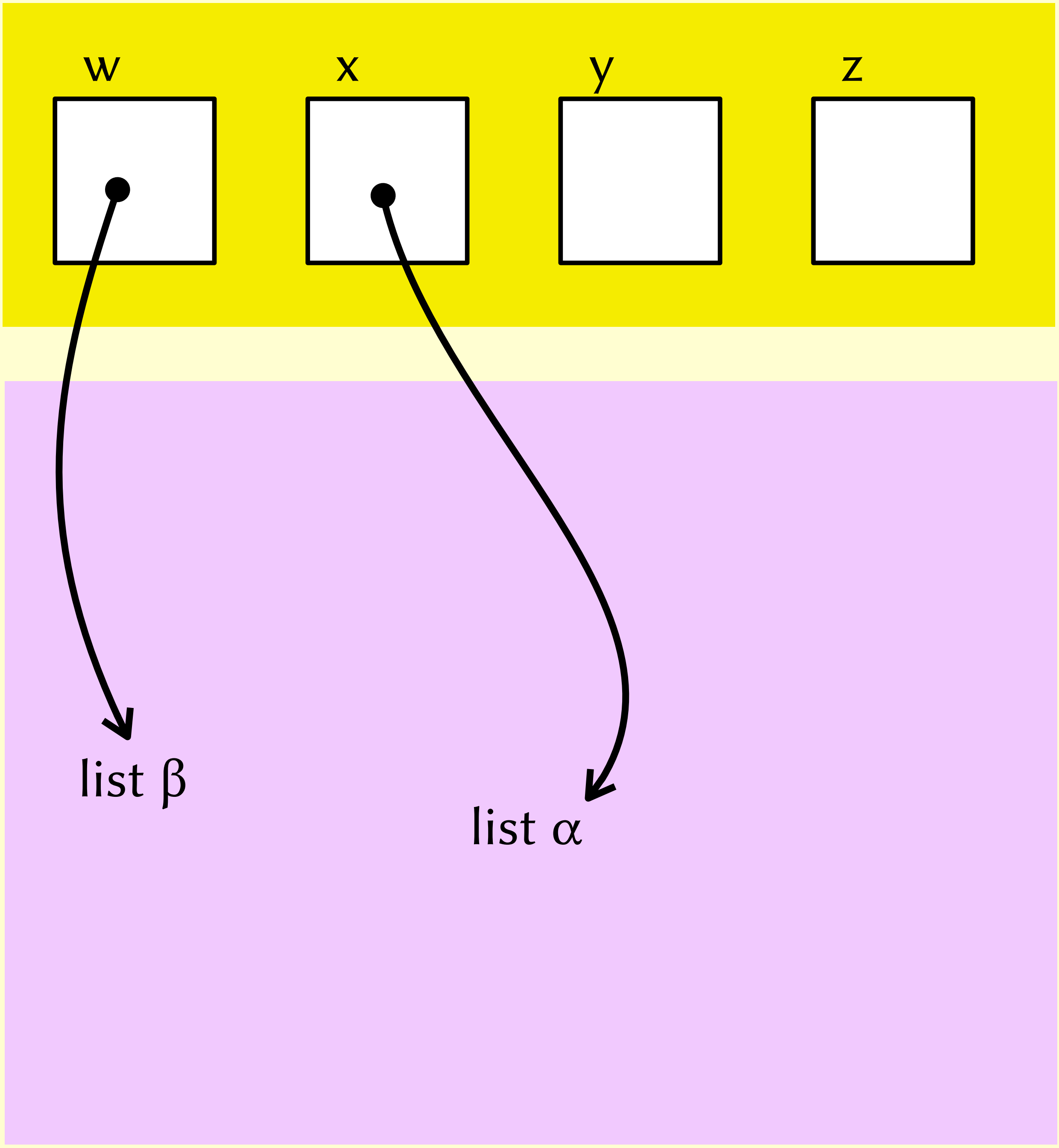
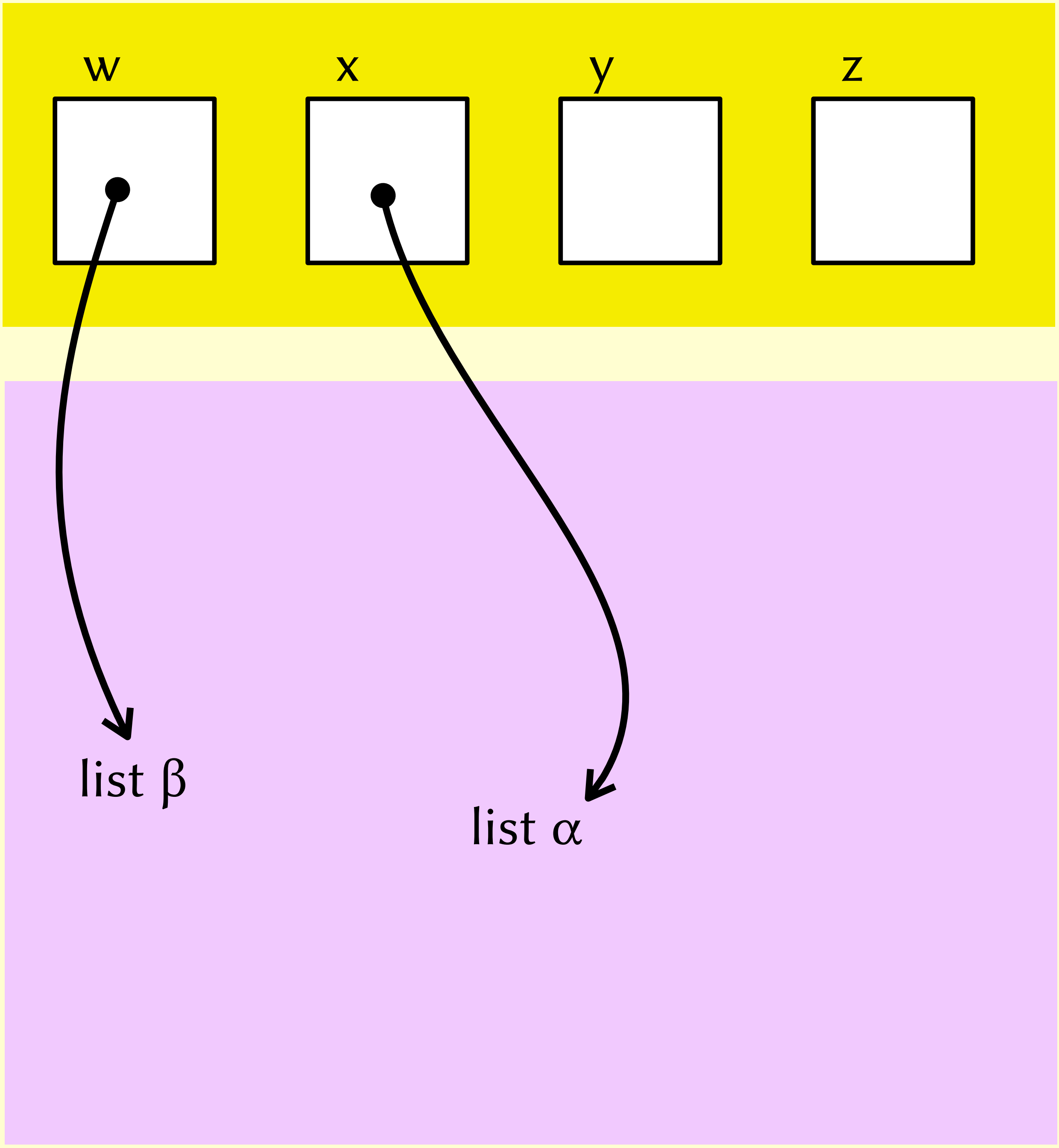


$\delta = -\beta \cdot \alpha$

List reverse



δ = -β · α

{list δ x}
w := 0;
{∃α,β. list α x ∧ list β w ∧ δ = -β·α}
while (x≠0) do {
    z := [x+1];
    [x+1] := w;
    w := x;
    x := z;
}
{list -δ w}

List reverse

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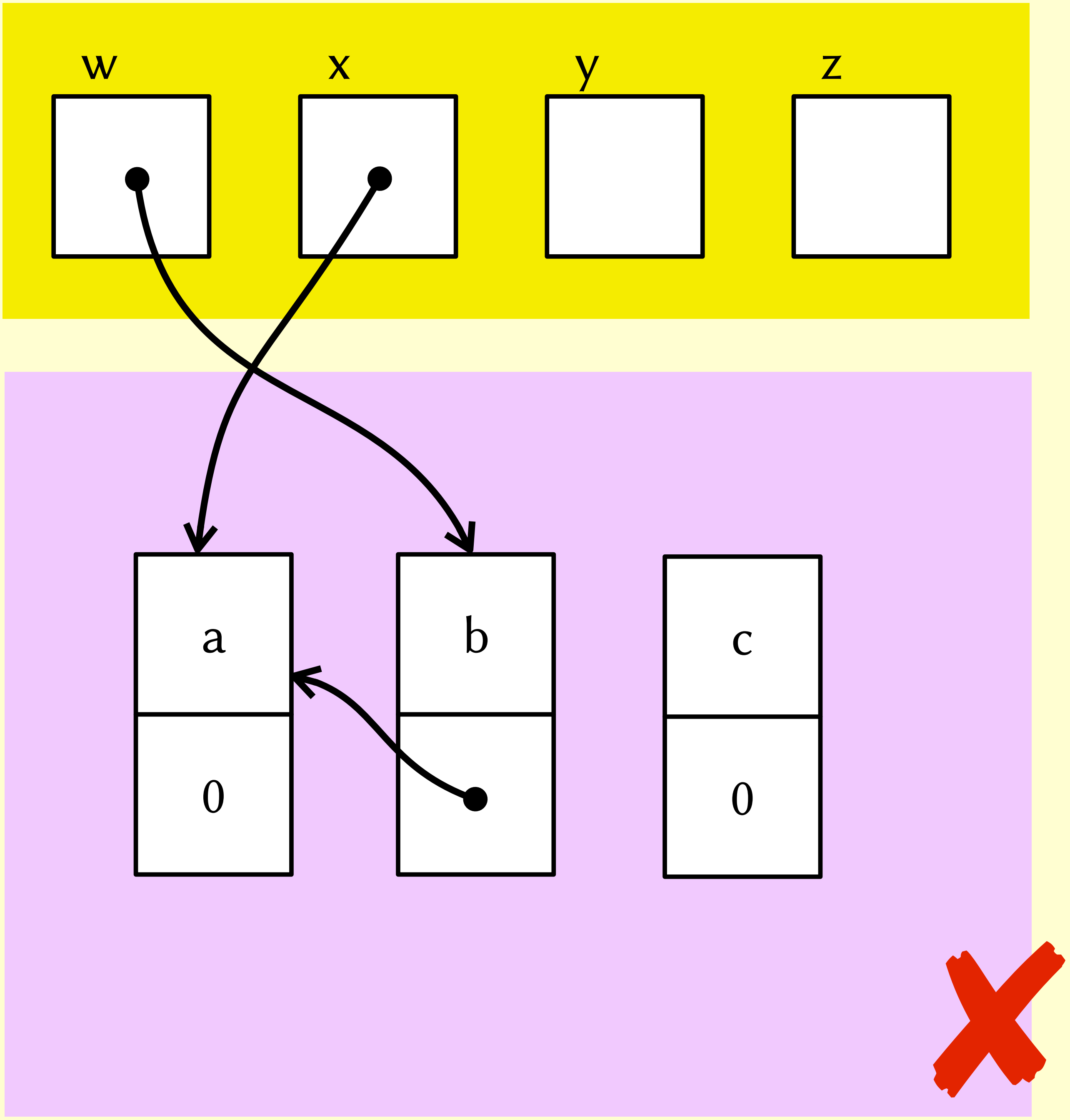
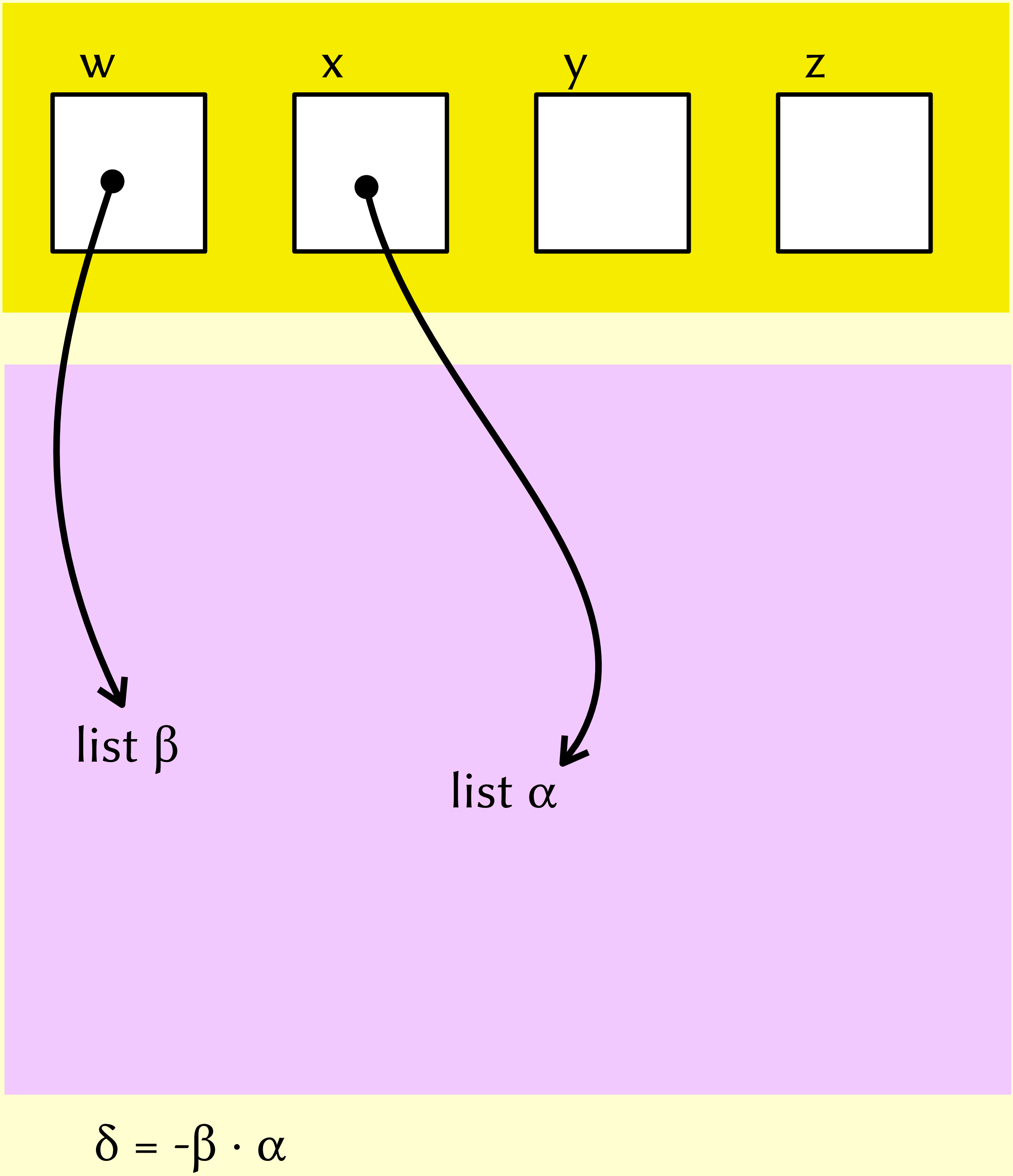


List reverse

{list δ x}
w := 0;
{∃α,β. list α x ∧ list β w ∧ δ = -β·α
∧ (∀z. reach(x,z) ∧ reach(w,z) ⟹ z=0)}
while (x≠0) do {
    z := [x+1];
    [x+1] := w;
    w := x;
    x := z;
}
{list -δ w}



List reverse

{list δ x}
list_reverse(x,w)
{list -δ w}

List reverse

{list δ x ∧ list ε y}
list_reverse(x,w)
{list -δ w}

List reverse

{list δ x ∧ list ε y}
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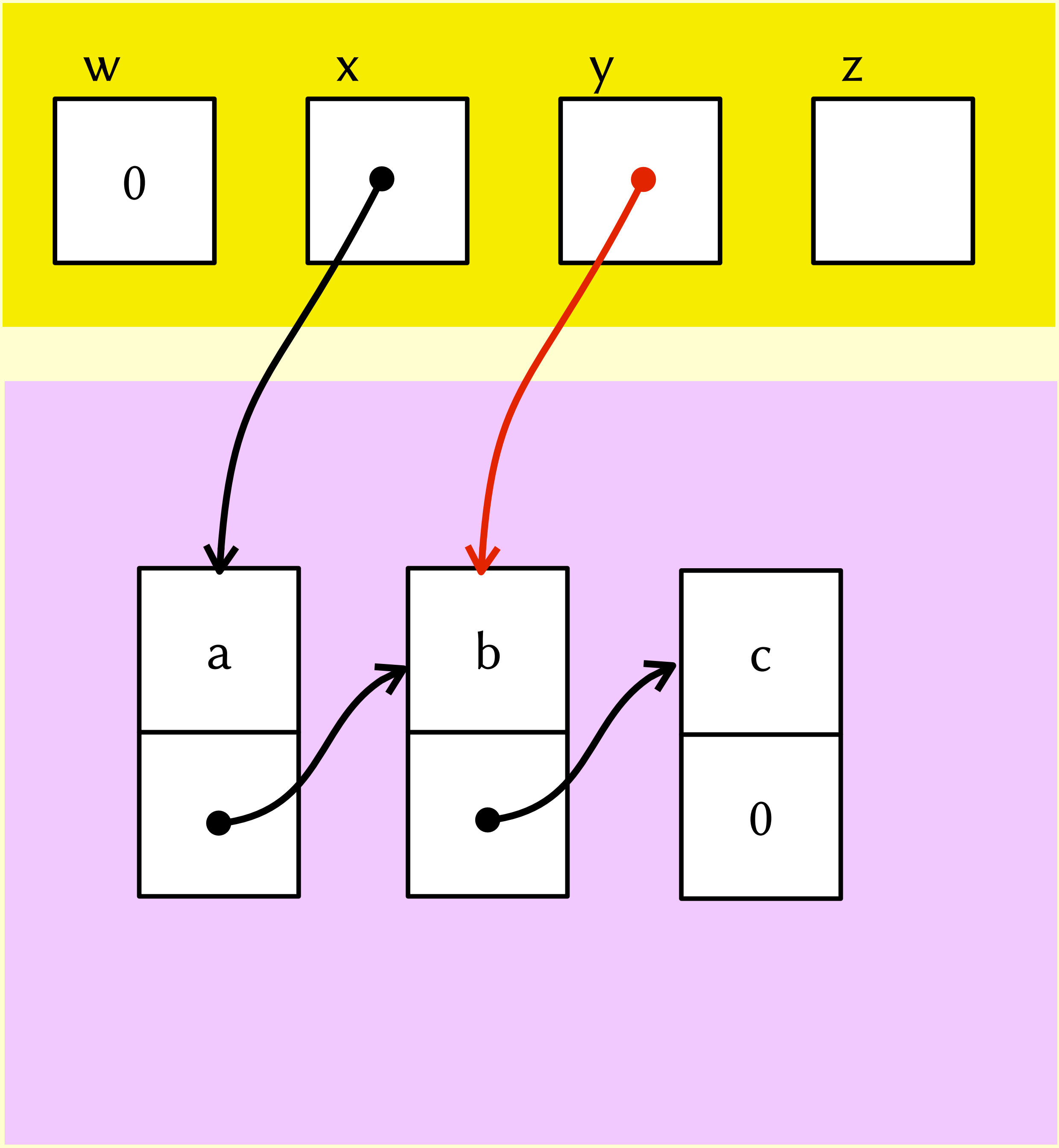

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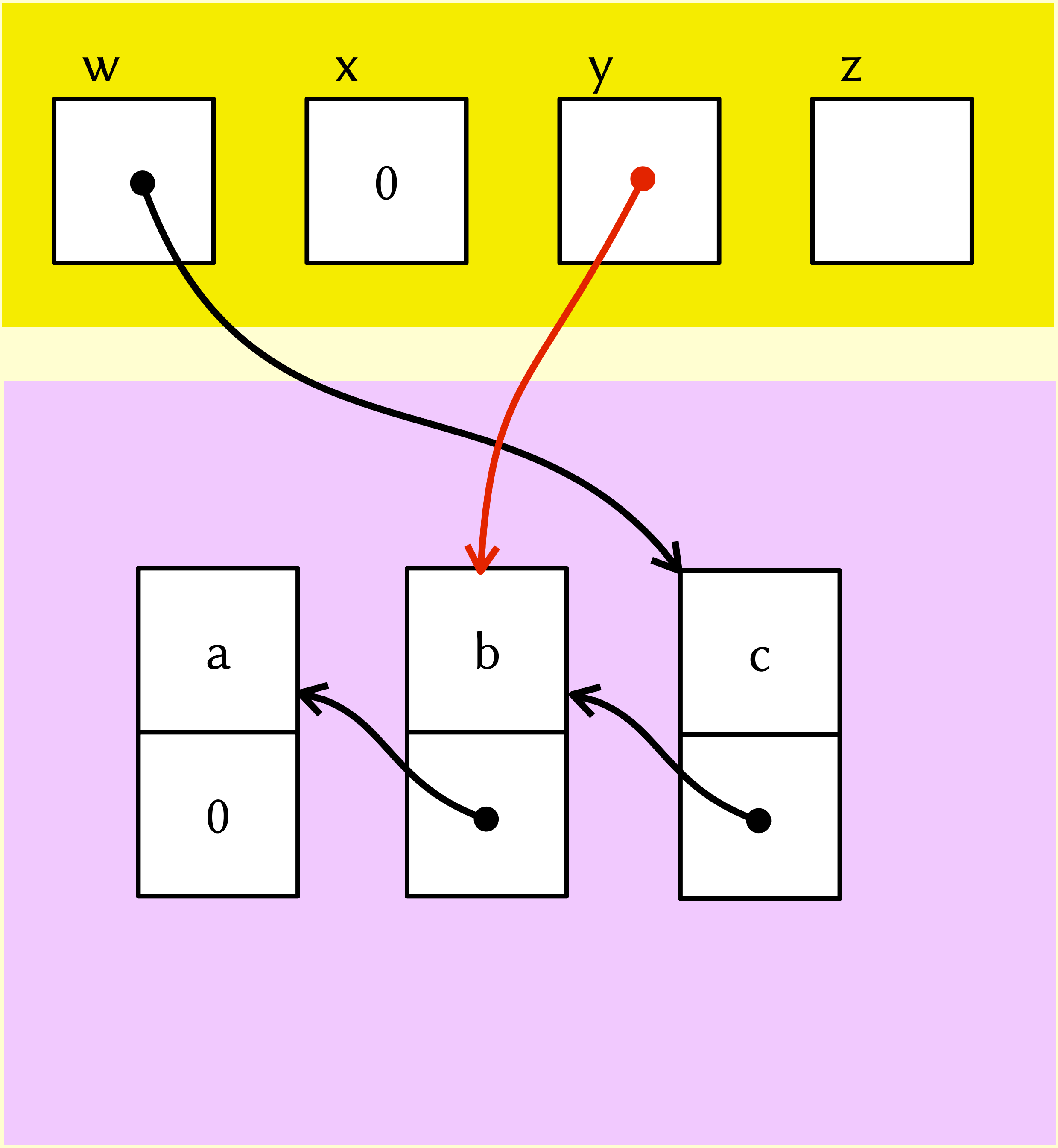

List reverse

{list δ x ∧ list ε y
∧ (∀z. reach(x,z) ∧ reach(y,z) ⟹ z=0)}
list_reverse(x,w)
{list -δ w}

List reverse

$\{\text{list } \delta \ x \wedge \text{list } \varepsilon \ y$
$\wedge (\forall z.\ \text{reach}(x,z) \wedge \text{reach}(y,z) \Longrightarrow z{=}0)\}$

```
w := 0;
```

$\{\exists\alpha,\beta.\ \text{list } \alpha \ x \wedge \text{list } \beta \ w \wedge \delta = \text{-}\beta{\cdot}\alpha$
$\wedge (\forall z.\ \text{reach}(x,z) \wedge \text{reach}(w,z) \Longrightarrow z{=}0)\}$

```
while (x≠0) do {
    z := [x+1];
    [x+1] := w;
    w := x;
    x := z;
}
```

$\{\text{list } \text{-}\delta \ w\}$

List reverse

```
{list δ x ∧ list ε y
∧ (∀z. reach(x,z) ∧ reach(y,z) ⇒ z=0)}
w := 0;
{∃α,β. list α x ∧ list β w ∧ δ = -β·α
∧ (∀z. reach(x,z) ∧ reach(w,z) ⇒ z=0)
∧ list ε y
∧ (∀z. (reach(x,z) ∨ reach(w,z))
  ∧ reach(y,z) ⇒ z=0)}
while (x≠0) do {
    z := [x+1];
    [x+1] := w;
    w := x;
    x := z;
}
{list -δ w}
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List reverse

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while (x≠0) do {
    z := [x+1];
    [x+1] := w;
    w := x;
    x := z;
}
{list -δ w ∧ list ε y
∧ (∀z. reach(w,z) ∧ reach(y,z) ⟹ z=0)}
```

List reverse

- Summary: the proof is fiddly and not modular, but it can be done.

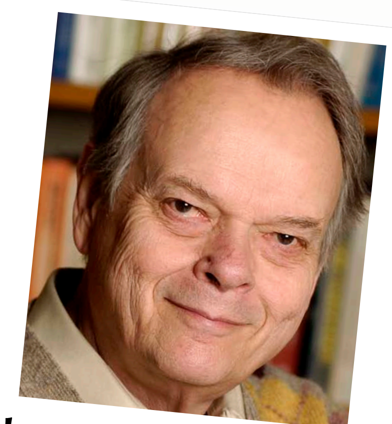
$\{\text{list } \delta\ x \wedge \text{list } \varepsilon\ y$
$\wedge (\forall z.\ \text{reach}(x,z) \wedge \text{reach}(y,z) \implies z=0)\}$

list_reverse(x,w)

$\{\text{list } {-\delta}\ w \wedge \text{list } \varepsilon\ y$
$\wedge (\forall z.\ \text{reach}(w,z) \wedge \text{reach}(y,z) \implies z=0)\}$

Cigarettes and Alcohol

Peter O'Hearn
(1963–)



John C. Reynolds
(1935–2013)

Cigarettes and Alcohol

$(P * Q)\ s \;=\; \exists s_1,s_2.\ s = s_1 + s_2 \text{ and } (P\ s_1) \text{ and } (Q\ s_2)$

$(P \wedge Q)\ s \;=\; (P\ s) \text{ and } (Q\ s)$

$(P \vee Q)\ s \;=\; (P\ s) \text{ or } (Q\ s)$

$(\neg P)\ s \;=\; \text{not } (P\ s)$

 $s \;=\; s \geq £5$

 $s \;=\; s \geq £20$

( $\wedge$ ) $s \;=\;$ ( s) and ( s)

$\;=\; s \geq £5 \text{ and } s \geq £20$

$\;=\; s \geq £20$

Cigarettes and Alcohol

$(P * Q)\ s \;=\; \exists s_1,s_2.\ s = s_1 + s_2 \text{ and } (P\ s_1) \text{ and } (Q\ s_2)$

$(P \wedge Q)\ s \;=\; (P\ s) \text{ and } (Q\ s)$

$(P \vee Q)\ s \;=\; (P\ s) \text{ or } (Q\ s)$

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 $s \;=\; s \geq £5$

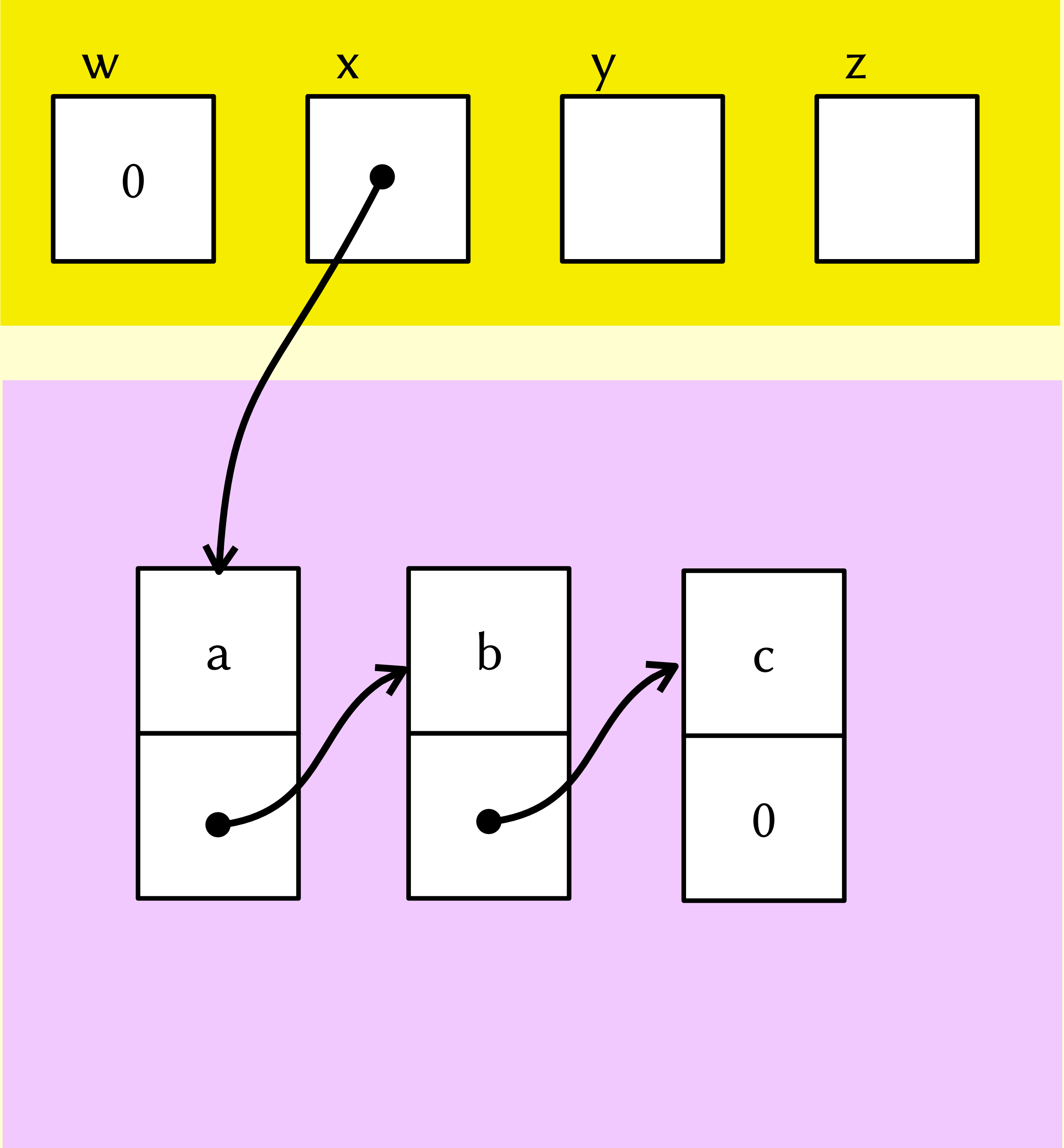
 $s \;=\; s \geq £20$

( $*$ ) $s \;=\; \exists s_1,s_2.\ s = s_1{+}s_2$ and ( s_1) and ( s_2)

$= \exists s_1,s_2.\ s = s_1{+}s_2 \text{ and } s_1 \geq £5 \text{ and } s_2 \geq £20$

$= s \geq £25$

List reverse

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while (x≠0) do {
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    [x+1] := w;
    w := x;
    x := z;
}
{list -δ w}
```
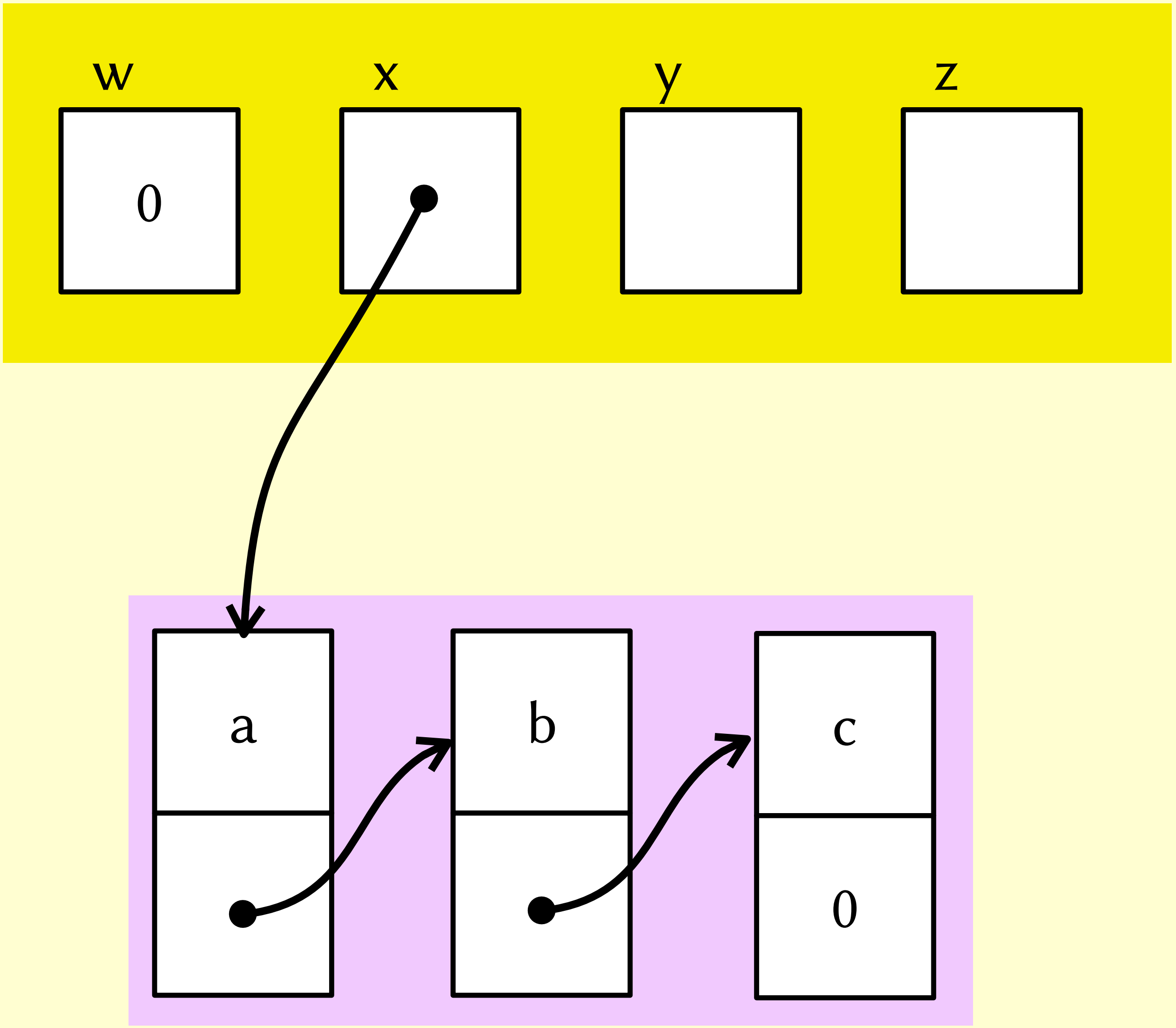

List reverse



{list δ x}
```
w := 0;
while (x≠0) do {
    z := [x+1];
    [x+1] := w;
    w := x;
    x := z;
}
```
{list -δ w}

List reverse

{list δ x}
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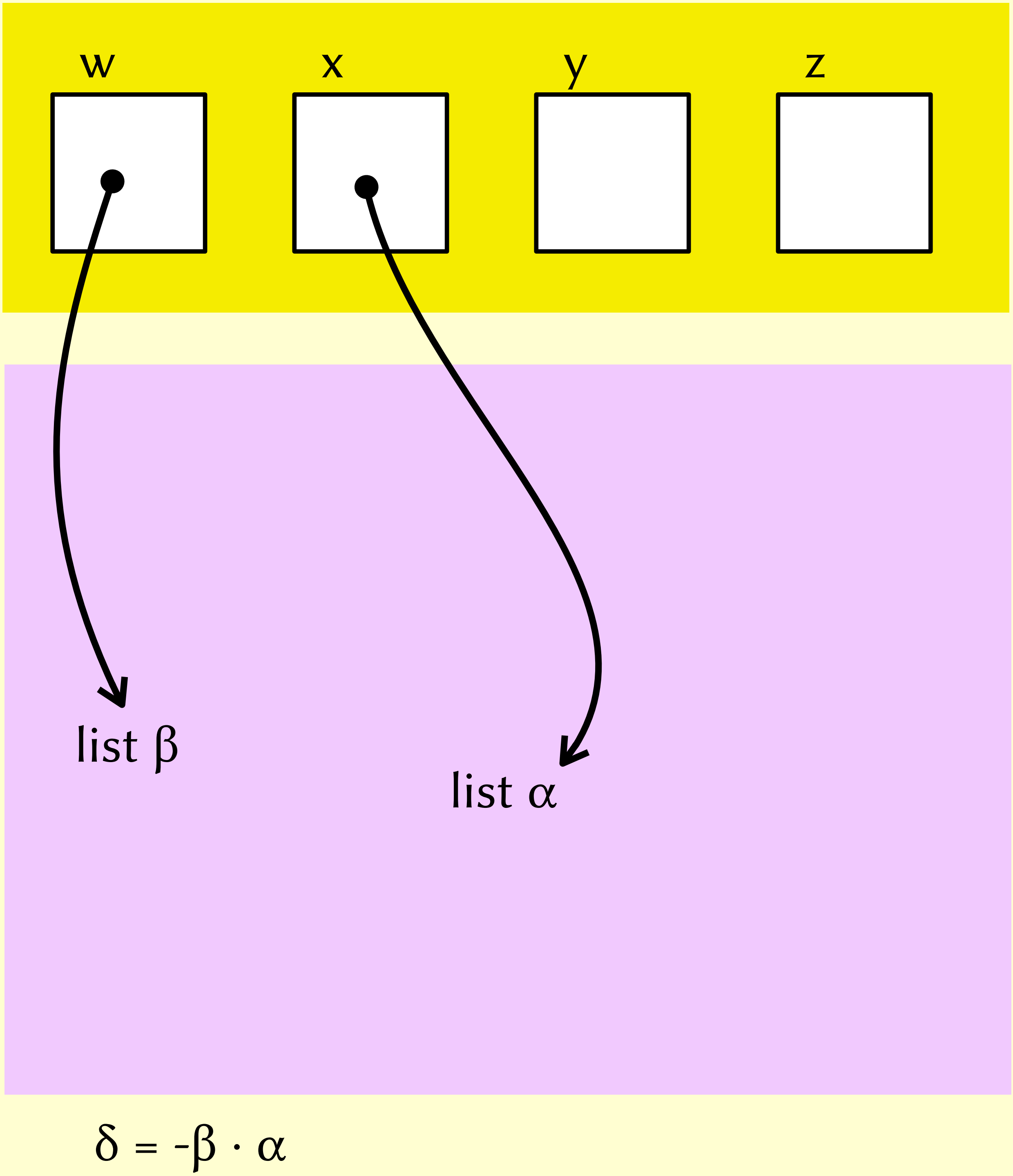
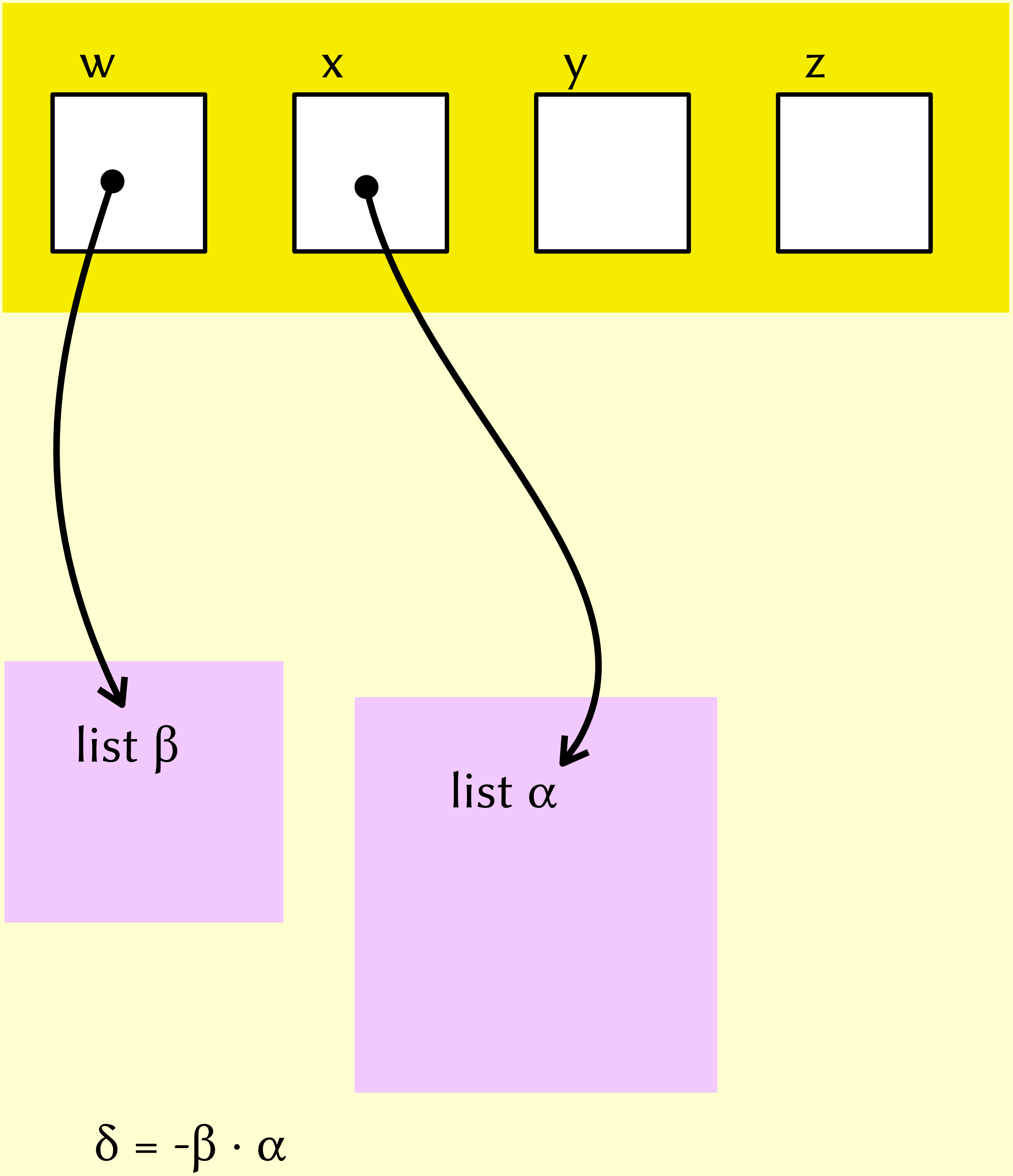


List reverse

$\{\text{list } \delta \text{ x}\}$
```
w := 0;
```
$\{\exists\alpha,\beta.\ \text{list } \alpha \text{ x} * \text{list } \beta \text{ w} * \delta = \text{-}\beta\cdot\alpha\}$
```
while (x≠0) do {
    z := [x+1];
    [x+1] := w;
    w := x;
    x := z;
}
```
$\{\text{list -}\delta \text{ w}\}$



List reverse

{list δ x}
list_reverse(x,w)
{list -δ w}

List reverse

{list δ x * list ε y * tree t}
list_reverse(x,w)
{list -δ w}

List reverse

{list δ x ∗ list ε y ∗ tree t}
list_reverse(x,w)
{list -δ w ∗ list ε y ∗ tree t}

$$\frac{\{\ p\ \}\ C\ \{\ q\ \}\quad (\dagger)}{\{\ p * r\ \}\ C\ \{\ q * r\ \}}$$

†provided r doesn't mention any variable modified by C

Conclusion

- Hoare Logic kicked off the field of software verification around 1969.
- Hoare Logic always struggled to reason about heap-allocated data structures.
- Separation Logic provided a solution around 2001.
- This has powered a lot of software verification tools since then, such as Facebook Infer.
- Dafny uses a variant of separation logic called dynamic frames.