HU, Wenbin 20760747 whuak@connect.ust.hk

Problem 1.

(a)

$$C[i,j,k] = \begin{cases} 0 & , i=0 \text{ or } j=0 \text{ or } k=0 \\ C[i-1,j-1,k-1]+1, & \text{if } W[i]=X[j]=Y[k] \\ \max\{C[i-1,j,k], C[i,j-1,k], C[i,j,k-1]\}, & \text{otherwise.} \end{cases}$$

(b)

case 1 (initial conditions): if $i=0$ or $j=0$ or $k=0$, there is no common sequence

$\therefore C[i,j,k]=0$

case 2: if $W[i]=X[j]=Y[k]$, match them.

the Longest Common Subsequence of these 3 sequence can contain them

$\therefore$ Longest Common Subsequence of $W[1\cdots i], X[1\cdots j], Y[1\cdots k]$ is one more greater than $W[1\cdots i-1], X[1\cdots j-1], Y[1\cdots k-1]$

$\therefore$ so length $C[i,j,k] = C[i-1,j-1,k-1]+1$

case 3: "otherwise" means if it is not the initial case and they don't match.

we need to find the maximum of Longest Common Sequence in $\{W[1\cdots i-1], X[1\cdots j], Y[1\cdots k]\}$ or $\{W[1\cdots i], X[1\cdots j-1], Y[1\cdots k]\}$ or $\{W[1\cdots i], X[1\cdots j], Y[1\cdots k-1]\}$

assign the maximum to $C[i,j,k]$.

$\therefore C[i,j,k] = \max\{C[i-1,j,k], C[i,j-1,k], C[i,j,k-1]\}$.

(c) Alg:

```
let C[0...p, 0...m, 0...n] be an new array of all 0.        // initialization
for i←1 to p:
    for j←1 to m:
        for k←1 to n:
            if W[i] = X[j] = Y[k]:                          // case for matching W[i], X[j], Y[k]
                C[i,j,k] = C[i-1,j-1,k-1] + 1
            else                                            // case for not matching.
                C[i,j,k] ← max{C[i-1,j,k], C[i,j-1,k], C[i,j,k-1]}
```

(d) inside "k=1 to n" loop, we use $O(1)$ time.

3 nested loop: $O(pmn)$ time.

total: $O(pmn)$ time.

Problem 2:

(a) denote: $C[i,j]$ is the money we get or pay when we are at (i,j)

$$C[i,j] = \begin{cases} P[1,1] & \text{, if } i=j=1 \\ P[i,j] + C[i,j-1] & \text{, if } i=1,\ j>1. \\ P[i,j] + C[i-1,j] & \text{, if } j=1,\ i>1 \\ \max\left\{\left(C[i-1,j]+P[i,j]\right), \left(C[i,j-1]+P[i,j]\right), \left(C[i-1,j-1]+\frac{3}{2}P[i,j]\right)\right\} & \text{, otherwise} \end{cases}$$

(b) The solution is a particular table entry. Every entry contain the maximum value when walking to the corresponding place. i.e. maximum prize we get when approaching (i,j). is $C[i,j]$.

(c) proof:

base case ($i=j=1$): when walking in $(1,1)$, the maximum prize we get is just $P[1,1]$

case for $i=1, j>1$: when we are at $(*,1)$ column, we can only move down, so every $(i,1)$ is from $(i-1,1)$

$\therefore C[i,j] = P[i,j] + C[i-1,j]$

case for $i>1, j=1$: when we are at $(1,*)$ row, every (i,j) is from $(i,j-1)$

$\therefore C[i,j] = P[i,j] + C[i,j-1]$

case for "otherwise": as showed below, (i,j) is from $(i-1,j)$ or $(i,j-1)$ or $(i-1,j-1)$

so $C[i,j] = \max\left\{\left(C[i-1,j]+P[i,j]\right), \left(C[i,j-1]+P[i,j]\right), \left(C[i-1,j-1]+\frac{3}{2}P[i,j]\right)\right\}$

since we need to find the maximum value of $C[i,j]$.

from all above, we can get all maximum prize when we walk from $(1,1)$ to (i,j)

so we will get maximum prize when walking from $(1,1)$ to (m,n)

(d) Algorithm:

create $C[1\cdots m, 1\cdots n]$, all of elements are 0.

$C[1,1] \leftarrow P[1,1]$ // initialization

① for $i \leftarrow 2$ to m: // when walking on $(x,1)$ column

$C[i,1] \leftarrow C[i-1,1] + P[i,1]$

② for $j \leftarrow 2$ to n: // when walking on $(1,x)$ row.

$C[1,j] \leftarrow C[1,j-1] + P[1,j]$

③ for $i \leftarrow 2$ to m: // find maximum at (i,j)

for $j \leftarrow 2$ to n:

$C[i,j] \leftarrow \max\left\{\left(C[i-1,j]+P[i,j]\right), \left(C[i,j-1]+P[i,j]\right), \left(C[i-1,j-1]+\frac{3}{2}P[i,j]\right)\right\}$

(e) $O(1)$ for initialization.

loop ① use $O(m-1)$, loop ② use $(n-1)$, loop ③ use $O((m-1)(n-1))$

∴ in total: $O(m-1) + O(n-1) + O((m-1)(n-1))$

$= O(m) + O(n) + O(mn - n - m + 1)$

$= O(mn)$

∴ my algorithm runs in $O(mn)$.

Problem 3 :

(a) if n is even :

(i) edges : $((i,0),(i+1,0))$, $i \in [1 \dots n-2]$

$((n-1,i-1),(n-1,i))$, $i \in [1 \dots n-1]$

$((i,n-1),(i-1,n-1))$, $i \in [1 \dots n-1]$

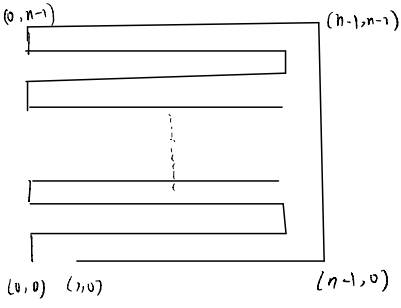
$((0,i),(0,i-1))$, $i \in [1,3,5,7 \dots, n-1]$

$((n-2,i),(n-2,i-1))$, $i \in [2,4,6 \dots, n-2]$

$((i-1,j),(i,j))$, $i \in [1 \dots n-2]$, $j \in [2,4,6 \dots n-2]$

$((i,j),(i-1,j))$, $i \in [1 \dots n-2]$, $j \in [1,3,5, \dots, n-3]$

(ii)



if n is odd :

(i) edges : $((0,1),(0,0))$

$((i,0),(i+1,0))$, $i \in [1 \dots n-2]$

$((n-1,i-1),(n-1,i))$, $i \in [1 \dots n-1]$

$((i,n-1),(i-1,n-1))$, $i \in [1 \dots n-1]$

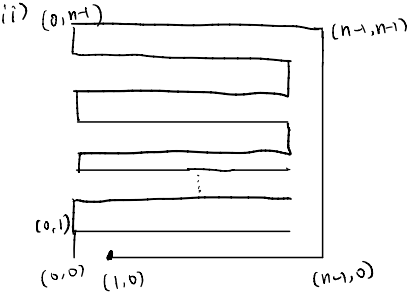
$((0,i),(0,i-1))$, $i \in [2,4,6,8 \dots, n-1]$

$((n-2,i),(n-2,i-1))$, $i \in [3,5 \dots , n-2]$

$((i-1,j),(i,j))$, $i \in [1 \dots n-2]$, $j \in [1,3,5 \dots, n-2]$

$((i,j),(i-1,j))$, $i \in [1 \dots n-2]$, $j \in [2,4,6 \dots , n-3]$

(ii)



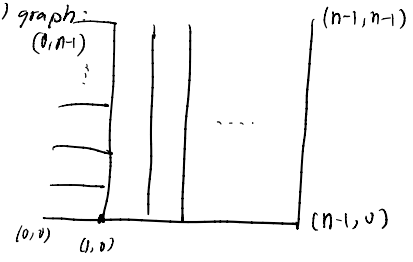
(b) (i) edges:

$((i,i),(0,i))$, $i \in [0 \dots n-1]$

$((i,0),(i+1,0))$, $i \in [1, \dots n-2]$

$((i,j),(i,j+1))$, $i \in [1 \dots n-1]$, $j \in [0 \dots n-2]$

(ii) graph:



Problem 4:

(A) $(8,0,0) \to (3,5,0) \to (0,5,3)$

$(3,5,0) \to (3,2,3) \to (6,2,0) \to (6,0,2) \to (1,5,2) \to (1,4,3) \to (4,4,0)$

$(3,5,0) \to (5,0,3) \to (5,3,0) \to (2,3,3) \to (2,5,1) \to (7,0,1) \to (7,1,0) \to (4,1,3)$

(B) $(8,0,0) \to (3,5,0) \to (0,5,3) \to (5,0,3) \to (5,3,0) \to (2,3,3) \to (2,5,1)$

$(7,0,1) \leftarrow (7,1,0) \leftarrow (4,1,3) \leftarrow (4,4,0) \leftarrow (1,4,3) \leftarrow (1,5,2) \leftarrow (6,0,2) \leftarrow (6,2,0) \leftarrow (3,2,3)$

(C) $(8,0,0) \to (3,5,0) \to (3,2,3) \to (6,2,0) \to (6,0,2) \to (1,5,2) \to (1,4,3) \to (4,4,0)$

Method: Do the BFS at v_0, count the steps until we reach $(4,4,0)$

once reach $(4,4,0)$, finish it, and we get the least number of pouring steps.

(D) correctness: first, we know $(4,4,0)$ is reachable from $(8,0,0)$, from (A)

Do the BFS at v_0, assume there is an S, the set of nodes we have already visited.

we always reach the adjacency of nodes in S in a round, put them into S.

∴ depth increases by one when visiting the adjacency of S. we count the depth

So once we reach $(4,4,0)$, we can get the depth of $(4,4,0)$, whose root is $(8,0,0)$

∴ we get the shortest path from $(8,0,0)$ to $(4,4,0)$

∴ the Method of using BFS to find least number of pouring steps is correct.

(E) there is no sequence from $(8,0,0)$ to $(6,1,1)$

(F) From (A) and (B), we know there is no path from $(8,0,0)$ to $(6,1,1)$, ie. $(6,1,1)$ is not reachable from $(8,0,0)$, so we can't get 6 Liters in first jug and 1 liter in second and third jug.

(G) Proof: $\forall a,b,c,\ a\in[0,8],\ b\in[0,5],\ c\in[0,3],\ a+b+c=8$, exist a path such that:

$(a,b,c) \to (a+c,b,0) \to (a+b+c,0,0)$, so every node v in $V-\{v_0\}$ can reach $v_0(8,0,0)$

∴ for all $v\in V-\{v_0\}$, if v is reachable from v_0 then v_0 is reachable from v

(H) the statement is wrong:

from proof in (G) we know there exists a path for $(6,1,1)$ to $(8,0,0)$

but $(6,1,1)$ is not reachable from $(8,0,0)$,

which is a contradictary for the statement

Problem 5:

i	1	2	3	4	5	6	7	8
a_i	A	B	C	D	E	F	G	H
$f(a_i)$	5	15	5	5	10	10	20	5

a) Fill in the two tables below. As in the example powerpoint only the entries with $i \leq j$ need to be filled in. We have started you off by filling in the $[i, i]$ entries.

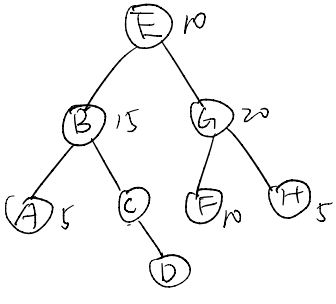
i/j	1	2	3	4	5	6	7	8
1	5	25	35	50	80	110	160	175
2	0	15	25	40	70	95	145	160
3	0	0	5	15	35	55	105	115
4	0	0	0	5	20	40	85	95
5	0	0	0	0	10	30	70	80
6	0	0	0	0	0	10	40	50
7	0	0	0	0	0	0	20	30
8	0	0	0	0	0	0	0	5

Table 1: Left matrix is $e[i, j]$.

i/j	1	2	3	4	5	6	7	8
1	1	2	2	2	2	2	5	5
2	0	2	2	2	2	5	5	5
3	0	0	3	3	4	5	5	7
4	0	0	0	4	5	5	6	7
5	0	0	0	0	5	5	6	7
6	0	0	0	0	0	6	7	7
7	0	0	0	0	0	0	7	7
8	0	0	0	0	0	0	0	8

Right matrix is $root[i, j]$.

(b)



$\therefore$ cost $= 10 \times 1 + (15+20) \times 2 + (5+5+10+5) \times 3$
$+ 5 \times 4$
$= 175$.