COMP 3711 HW2.

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Problem 1.

first: we prove Hint 2: prove the subarray $A[i \dots j]$ cannot contain more than 6 heavy items.

proof: Let $S = Heavy(i, j)$, $n = j - i + 1$, $|S|$ denotes the number of heavy items in $A[i \dots j]$

I. if $20 \nmid 3n$, by definition: a heavy element should appear no less than $\lceil \frac{3}{20} \cdot n \rceil$ times.

$$|S| \leq \left\lfloor \frac{n}{\lceil \frac{3}{20} \cdot n \rceil} \right\rfloor \leq \left\lfloor \frac{n}{\frac{3}{20} \cdot n} \right\rfloor = \left\lfloor \frac{20}{3} \right\rfloor = 6$$

II. if $20 \mid 3n$, by definition: a heavy element should appear no less than $\frac{3n}{20} + 1$ times

$$|S| \leq \left\lfloor \frac{n}{\frac{3n}{20} + 1} \right\rfloor \leq \left\lfloor \frac{n}{\frac{3n}{20}} \right\rfloor = \left\lfloor \frac{20}{3} \right\rfloor = 6$$

$\therefore$ $A[i \dots j]$ cannot contain more than 6 items.

$\therefore$ Hint 2 proved.

(a) Let S be the set of heavy items in $A[i \dots j]$, $S = Heavy(i, j)$

(i) Heavy(i, j):

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create set S, S ← S ∩ φ                      // create a set S to get heavy items in A[i...j]
mid ← ⌊(i+j)/2⌋ , n ← j-i+1                  // find the mid index, n is length
if i = j then ; S = S ∪ A[j] , return S      // base case
else                                          // recursion: find heavy items in 2 Subarray
    S1 ← Heavy(i, mid)                        namely, A[i... mid], A[mid+1 ... j]
    S2 ← Heavy(mid+1, j)
count ← 0
∀ x ∈ S1 ∪ S2 do :                            // find {x | x ∈ S1 ∪ S2} and process each x
    for k ← i to j :                          // for loop: to check each x in S1∪S2,
        if x = A[k] then :                    whether x is a heavy item.
            Count ← count + 1
        if (20∤3n and count ≥ ⌈(3/20)·n⌉) or (20|3n and count ≥ 3n/20 + 1)
            S ← S ∪ {x}
    Count ← 0
return S.
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finally, if $S \neq \phi$. we find the set of heavyitems in $A[i \dots j]$, if $S = \phi$. we say "no heavy items in $A[i \dots j]$"

(ii) description: if A contains heavy items, then every heavy item must be a heavy item in at least one of $A[i \dots \lfloor \frac{i+j}{2} \rfloor]$. $A[\lfloor \frac{i+j}{2} \rfloor + 1 \dots j]$

I. if we find heavy items in $A[i \dots \lfloor \frac{i+j}{2} \rfloor]$, let S_1 be the set of them

II. if we find heavy items in $A[\lfloor \frac{i+j}{2} \rfloor + 1 \dots j]$, let S_2 be the set of them

III. go through every item x in $S_1 \cup S_2$, check whether x is a heavy item of $A[i \dots j]$

let S be the set of them. return S

if $S \neq \phi$, we find heavy items. else there is no heavy items.

(b) Prove this alg by induction:

proof: denote $J(k)$: the alg is correct, when $A[i \dots j]$ have k elements

base case (when $k=1$): $k = 1 \geq \lceil \frac{3}{20} \cdot k \rceil = 1$, which means $J(1)$ is correct.

when $k \geq 2$: induction hypothesis: $J(\lfloor \frac{k}{2} \rfloor)$ is correct. $J(\lceil \frac{k}{2} \rceil)$ is correct

so we find heavy items in $A[i \dots \lfloor \frac{i+j}{2} \rfloor]$, let S_1 be the set of them (by $J(\lceil \frac{k}{2} \rceil)$)

we find heavy items in $A[\lfloor \frac{i+j}{2} \rfloor + 1 \dots j]$, let S_1 be the set of them (by $J(\lfloor \frac{k}{2} \rfloor)$)

$\forall x \in S_1 \cup S_2$, we check whether x is a heavy item in $A[i \dots j]$

put them into S.

since if A contains heavy items, then every heavy item must be a heavy item in at least one of $A[i \dots \lfloor \frac{i+j}{2} \rfloor]$. $A[\lfloor \frac{i+j}{2} \rfloor + 1 \dots j]$, we must have already find all the heavy items in A, which is the set of S

$\therefore J(k)$ is correct.

$\therefore$ alg's correctness is proved.

(c) we analyse line by line. denote $T(n)$ as the running time of the alg with $A[i \dots j]$, $n = j - i + 1$

line 1.2 : we use $O(1)$ time to create set S

line 3 : one comparison (equality) use 1 time, $S = S \cup A[i]$ use 1 time (because S is ϕ before), which is $O(1)$ time in total

line 4 to 6: recurrsion: use $T(\lceil n/2 \rceil) + T(\lfloor n/2 \rfloor)$ time

line 8 to 14: process each element $x \in S_1 \cup S_2$, $O(|S_1 \cup S_2| + 1)$ times.

for each element x:

for loop do n times iteratively.

inside the loop: use $O(1)$ time

so the for loop use $O(n)$ time

outside the for loop: one comparison use 1 time

if the comparison is true, $S \leftarrow S \cup \{x\}$ use another 1 time.

so outside the for loop it use 1 or 2 time, which is $O(1)$ time

so line 8 to 14 use $O((|S_1 \cup S_2| + 1) \cdot (O(n) + O(1)))$ time in total

from Hint 2. We proved there is no more than 6 heavy items in any $A[i \dots j]$.

$\therefore |S_1| \leq 6$. $|S_2| \leq 6$,

$\therefore |S_1 \cup S_2| \leq |S_1| + |S_2| = 12$

$\therefore O((|S_1 \cup S_2| + 1) \cdot (O(n) + O(1))) \leq 13 \cdot [O(n) + O(1)] \leq O(n)$,

$\therefore$ line 8 to 14 use $O(n)$ time

$\therefore$ Totally, we use $T(n) \leq T(\lceil n/2 \rceil) + T(\lfloor n/2 \rfloor) + O(n) + O(1)$ $(n > 1)$ time, when $T(1) = O(1)$

$\therefore \quad \forall n > 1, T(n) \leq T(\lceil n/2 \rceil) + T(\lfloor n/2 \rfloor) + c_1 n, \quad T(1) = c_2$. $(\exists c_1, c_2 \in N)$

$\therefore$ by Master Thm : $T(n) = O(n \log n)$

the case is : $T(n) \leq 2T(n/2) + n$

$\because \quad n = O(n^{\log_2 n}) = O(n)$

$\therefore \quad T(n) = O(n \log n)$

Problem 2:

(a) (i) let X_i be the indicator random variable for the event that $A[i]$ is a local minimum ($i \in [1, n]$). $X = \sum_{1 \le i \le n} X_i$

$$X_i = \begin{cases} 1, & \text{if } x_i \text{ is a local minimum} \\ 0, & \text{if } x_i \text{ is not a local minimum} \end{cases}, \quad E[x_i] = P[x_i]$$

for $i = 1$ or $i = n$:

$$E(X_1) = P(X_1) = \frac{\binom{n}{2}(n-2)!}{n!} = \frac{1}{2}. \quad E(X_n) = P(X_n) = \frac{\binom{n}{2}(n-2)!}{n!} = \frac{1}{2}$$

for $i \ge 2$ and $i \le n-1$:

$$E(X_i) = \frac{2 \cdot \binom{n}{3}(n-3)!}{n!} = \frac{1}{3}$$

$$\therefore E(X) = E(X_1) + \cdots E(X_n) = 1 + \frac{1}{3}(n-2) = \frac{n+1}{3}$$

(ii) let X_i be the indicator random variable for the event that $T[i]$ is a local minimum ($i \in [1, n]$). i.e. $T[i] < \min_{v \in N(i)} T[v]$, $X = \sum_{1 \le i \le n} X_i$

$$X_i = \begin{cases} 1, & \text{if } x_i \text{ is a local minimum} \\ 0, & \text{if } x_i \text{ is not a local minimum} \end{cases}, \quad E[x_i] = P[x_i]$$

in this case: $n = 2^k - 1$ $(k \ge 0)$, $k = \log_2(n+1)$

for $i = 1$: $E(X_1) = \dfrac{2 \cdot \binom{n}{3}(n-3)!}{n!} = \dfrac{1}{3}$

for $i \in [2^{k-1}, 2^k - 1]$:

$$E(X_i) = \frac{\binom{n}{2}(n-2)!}{n!} = \frac{1}{2}$$

$$\therefore E(X_{2^{k-1}}) + E(X_{2^{k-1}+1}) + \cdots + E(X_{2^k-1}) = \frac{1}{2} \cdot 2^{k-1} = \frac{n+1}{4}$$

for $i \in [2, 2^{k-1} - 1]$:

$$E(X_i) = \frac{3! \binom{n}{4}(n-4)!}{n!} = \frac{1}{4}$$

$$\therefore E(X_2) + E(X_3) + \cdots + E(X_{2^{k-1}-1}) = \frac{1}{4}(2^{k-1} - 2) = \frac{n-3}{8}$$

$$\therefore E(X) = E(X_1) + E(X_2) + \cdots + E(X_n) = \frac{1}{3} + \frac{n+1}{4} + \frac{n-3}{8} = \frac{3}{8}n + \frac{5}{24}$$

(iii) let $X_{i,j}$ be the indicator random variable for the event that $M(i,j)$ is a local minimum

$(i \in [1,m],\ j \in [1,m])$, $X = \sum_{1 \le i \le m,\, 1 \le j \le m} X_{i,j}$ $(m = \sqrt{n})$

$X_{i,j} = \begin{cases} 1, & \text{if } X_{i,j} \text{ is a local minimum} \\ 0, & \text{if } X_{i,j} \text{ is not a local minimum} \end{cases}$, $E[X_{i,j}] = P[X_{i,j}]$

in this case $n = m^2$, $m = \sqrt{n}$

for ($i=1$ or $i=m$) and ($j=1$ or $j=m$):

$$E(X_{i,j}) = P(X_{i,j}) = \frac{2!\binom{n}{3}(n-3)!}{n!} = \frac{1}{3}$$

$$\therefore E(X_{1,1}) + E(X_{1,m}) + E(X_{m,1}) + E(X_{m,m}) = \frac{1}{3} \cdot 4 = \frac{4}{3}$$

for $((i=1 \text{ or } i=m) \text{ and } j \in [2,m-1])$ or $(i \in [2,m-1] \text{ and } (j=1 \text{ or } j=m))$:

denote this situation as M.

we have $4(m-2)$ items.

$$E(X_{i,j}) = P(X_{i,j}) = \frac{3!\binom{n}{4}(n-4)!}{n!} = \frac{1}{4}$$

$$\therefore E(M) = \frac{1}{4} \cdot 4(m-2) = m-2 = \sqrt{n}-2$$

for $i \in [2, m-1]$ and $j \in [2, m-1]$: denote as Y

we have $(m-2)^2$ items

$$E(X_{i,j}) = \frac{4!\binom{n}{5}(n-5)!}{n!} = \frac{4!}{5!} = \frac{1}{5}$$

$$\therefore E(Y) = (m-2)^2 \cdot \frac{1}{5} = \frac{n - 4\sqrt{n} + 4}{5}$$

$$\therefore E(X) = \frac{4}{3} + \sqrt{n} - 2 + \frac{n - 4\sqrt{n} + 4}{5} = \frac{n}{5} + \frac{\sqrt{n}}{5} + \frac{2}{15}$$

(b) (i) let X_i be the indicator random variable for the event that $A[i]$ is a saddle point.

$(i \in [2, n-1])$, $X = \sum_{2 \le i \le n-1} X_i$

$X_i = \begin{cases} 1, & \text{if } X_i \text{ is a saddle point} \\ 0, & \text{if } X_i \text{ is not saddle point} \end{cases}$, $E[X_i] = P[X_i]$

$$E(X_i) = \frac{2!\binom{n}{3}(n-3)!}{n!} = \frac{1}{3}$$

$$\therefore E(X) = E(X_2) + \cdots + E(X_{n-1}) = \frac{n-2}{3}$$

(ii)

let X_i be the indicator random variable for the event that $T[i]$ is a saddle point.

$(i \in [2, \frac{n-1}{2}])$, $X = \sum_{2 \le i \le \frac{n-1}{2}} X_i$

$$E(X_i) = \frac{2!\binom{n}{4}(n-4)!}{n!} = \frac{2}{4!} = \frac{1}{12}$$

$$\therefore E(X) = \frac{1}{12} \cdot \left(\frac{n-1}{2} - 2 + 1\right) = \frac{n-3}{24}$$

(iii) let $X_{i,j}$ be the indicator random variable for the event that $M(i,j)$ is a saddle point

$(i \in [1,m],\ j \in [1,m])$, $X = \sum_{1 \le i \le m,\ 1 \le j \le m} X_{i,j}$ $\quad (m = \sqrt{n})$

$$X_{i,j} = \begin{cases} 1 , & \text{if } X_{i,j} \text{ is a local minimum} \\ 0 , & \text{if } X_{i,j} \text{ is not a local minimum} \end{cases} , \quad E[X_{i,j}] = P[X_{i,j}]$$

in this case $n = m^2$, $m = \sqrt{n}$

for ($i=1$ or $i=m$) and ($j=1$ or $j=m$):

$$E(X_{i,j}) = P(X_{i,j}) = \frac{\binom{n}{3}(n-3)!}{n!} = \frac{1}{3!} = \frac{1}{6}$$

$$\therefore E(X_{1,1}) + E(X_{1,m}) + E(X_{m,1}) + E(X_{m,m}) = \frac{1}{6} \cdot 4 = \frac{2}{3}$$

for (($i=1$ or $i=m$) and $j \in [2, m-1]$) or ($i \in [2, m-1]$ and ($j=1$ or $j=m$)):

denote this situation as $\mathcal{M}$.

we have $4(m-2)$ items.

$$E(X_{i,j}) = P(X_{i,j}) = \frac{2!\binom{n}{4}(n-4)!}{n!} = \frac{2!}{4!} = \frac{1}{12}$$

$$\therefore E(\mathcal{M}) = \frac{1}{12} \cdot 4(m-2) = \frac{\sqrt{n}-2}{3}$$

for $i \in [2, m-1]$ and $j \in [2, m-1]$: denote as Y

we have $(m-2)^2$ items

$$E(X_{i,j}) = \frac{2 \cdot 2 \cdot \binom{n}{5}(n-5)!}{n!} = \frac{4}{5!} = \frac{1}{30}$$

$$\therefore E(Y) = (m-2)^2 \cdot \frac{1}{30} = \frac{n - 4\sqrt{n} + 4}{30}$$

$$\therefore E(X) = \frac{2}{3} + \frac{\sqrt{n}-2}{3} + \frac{n - 4\sqrt{n} + 4}{30} = \frac{n}{30} + \frac{\sqrt{n}}{5} + \frac{2}{15}$$

Problem 3:

(a) let $S=\{x_1,\cdots,x_n\}$, assume that there is no pair i,j, s.t. $x_i=x_j$ or $y_i=y_j$

put $x_1,x_2,\cdots,x_n$ onto x-axis, and from left to right on the x-axis,

we denote these points as: $y_1,y_2,\cdots y_n$, let $m=\lceil n/2\rceil$

I. when n is odd and $x=y_m$

$$\sum_{i=1}^{n}|x-y_i|=|y_n-y_1|+|y_{n-1}-y_2|+\cdots+|y_{m+1}-y_{m-1}|$$, which is a constant.

when n is even and $x\in[y_m,y_{m+1}]$:

$$\sum_{i=1}^{n}|x-y_i|=|y_n-y_1|+|y_{n-1}-y_2|+\cdots+|y_{m+1}-y_m|$$, which is a constant.

II. when $x\in[-\infty,y_m]$:

when $x\le y_1$: $f(x)=\sum_{i=1}^{n}|x-x_i|=\sum_{i=1}^{n}x_i-n\cdot x$, which is a decreasing function

when $x\in(y_k,y_{k+1}]$, $(1\le k\le m-1)$:

$$f(x)=\sum_{i=1}^{n}|x-y_i|=\sum_{i=2k+1}^{n}|x-y_i|+\sum_{i=1}^{k}|y_{k+i}-y_{k+1-i}|$$

$$=\sum_{i=2k+1}^{n}y_i-(n-2k)x+\sum_{i=1}^{k}|y_{k+i}-y_{k+1-i}|$$

$$=-(n-2k)x+\sum_{i=2k+1}^{n}y_i+\sum_{i=1}^{k}|y_{k+i}-y_{k+1-i}|$$

let $d_1=-(n-2k)x+\sum_{i=2k+1}^{n}y_i+\sum_{i=1}^{k}|y_{k+i}-y_{k+1-i}|$

since $n-2k\ge n-2(m-1)=n-2\lceil n/2\rceil+2>n-n=0$

and $\sum_{i=2k+1}^{n}y_i+\sum_{i=1}^{k}|y_{k+i}-y_{k+1-i}|$ is constant.

So. when $x\in(y_k,y_{k+1}]$ $(1\le k\le m-1)$, $f(x)$ is decreasing ①

let's compare $f(x)$ when $x\in(y_k,y_{k+1}]$ and $x\in(y_{k+1},y_{k+2}]$ $(k\in[1,m-2])$

$$d_1=-(n-2k)x'+\sum_{i=2k+1}^{n}y_i+\sum_{i=1}^{k}|y_{k+i}-y_{k+1-i}|,\quad x'\in(y_k,y_{k+1}]$$

$$d_2=-(n-2(k+1))x''+\sum_{i=2k+3}^{n}y_i+\sum_{i=1}^{k+1}|y_{k+1+i}-y_{k+2-i}|,\quad x''\in(y_{k+1},y_{k+2}]$$

$$d_2-d_1=[(n-2k)x'-(n-2(k+1))x'']-y_{2k+1}-y_{2k+2}+y_{2k+2}+y_{2k+1}-y_{k+1}-y_{k+1}$$

$$=[(n-2k)x'-(n-2(k+1))x'']-2y_{k+1}$$

$$<[(n-2k)y_{k+1}-(n-2(k+1))y_{k+1}]-2y_{k+1}$$

$$=2y_{k+1}-2y_{k+1}$$

$$=0$$

$\therefore f(x') > f(x'')$ when $x' \in (y_k, y_{k+1}]$, $x'' \in (y_{k+1}, y_{k+2}]$. $k \in [1, m-2]$

$\therefore f(x)$ is always smaller when x goes to $(y_{k+1}, y_{k+2}]$ from $(y_k, y_{k+1}]$ iteratively ② (when $k \in [1, m-2]$)

$\therefore$ Considering ① ②, $f(x)$ is decreasing when $x \in [-\infty, y_m]$

III. when $x \in [y_{m+1}, +\infty)$

when $x \geq y_n$: $f(x) = \sum_{i=1}^{n} |x - x_i| = n \cdot x - \sum_{i=1}^{n} x_i$, which is an increasing function.

when $x \in (y_k, y_{k+1}]$ $(k \in [m+1, +\infty)$

$$f(x) = \sum_{i=1}^{n} |x - y_i| = \sum_{i=1}^{2k-n} (x - y_i) + \sum_{i=k+1}^{n} (y_i - y_{2k+1-i})$$

$$= (2k-n) \cdot x - \sum_{i=1}^{2k-n} y_i + \sum_{i=k+1}^{n} (y_i - y_{2k+1-i})$$

$\because k \in [\lceil n/2 \rceil + 1, n-1]$

$\therefore 2k - n \geq 2\lceil n/2 \rceil + 2 - n > n - n = 0$

and $-\sum_{i=1}^{2k-n} y_i + \sum_{i=k+1}^{n} (y_i - y_{2k+1-i})$ is constant when $x \in [y_k, y_{k+1}]$

$\therefore f(x)$ is increasing on $(y_k, y_{k+1}]$ ③

let's compare $f(x')$ and $f(x'')$ with $x' \in (y_k, y_{k+1}]$, $x'' \in (y_{k+1}, y_{k+2}]$, $k \in [m+1, n-2]$

$$f(x') = d' = (2k-n) \cdot x' - \sum_{i=1}^{2k-n} y_i + \sum_{i=k+1}^{n} (y_i - y_{2k+1-i}), \quad x' \in (y_k, y_{k+1}]$$

$$f(x'') = d'' = (2k+2-n) \cdot x'' - \sum_{i=1}^{2k+2-n} y_i + \sum_{i=k+2}^{n} (y_i - y_{2k+3-i}), \quad x'' \in (y_{k+1}, y_{k+2}]$$

$$d'' - d' = (2k+2-n) \cdot x'' - (2k-n) \cdot x' - y_{2k+1-n} - y_{2k+2-n} + y_{2k+1-n} + y_{2k+2-n} - 2y_{k+1}$$

$$= (2k+2-n) \cdot x'' - (2k-n) x' - 2y_{k+1}$$

$$> (2k+2-n) \cdot y_{k+1} - (2k-n) y_{k+1} - 2y_{k+1}$$

$$= 2y_{k+1} - 2y_{k+1} = 0$$

$\therefore f(x'') > f(x')$, when $x'' \in (y_{k+1}, y_{k+2}]$, $x' \in (y_k, y_{k+1}]$ ④

consider ③ ④, $f(x)$ is increasing in $[y_{m+1}, +\infty)$

IV. as for $x \in [y_m, y_{m+1}]$:

when n is even. we've know $f(y_m) = f(x) = f(y_{m+1})$, $\forall x \in [y_m, y_{m+1}]$ from I.

when n is odd. $f(x) = x - y_1 + \sum_{i=m+1}^{n} (y_i - y_{n-i+2})$, which is increasing.

$\therefore$ considering I. II. III. IV:

when $|\{x_1 \ldots x_n\}|$ is odd, $f(x)$ is dreasing on $(-\infty, y_m)$, increasing on $[y_m, +\infty)$

when $|\{x_1 \ldots x_n\}|$ is even, $f(x)$ is dreasing on $(-\infty, y_m)$, increasing on $(y_{m+1}, +\infty)$, $f(y_m) = f(x) = f(y_{m+1})$ for $\forall x \in [y_m, y_{m+1}]$

∴ draw a conclusion:

when n is even, let $z_1 = y_m$, $z_2 = y_{m+1}$. ($m = \lceil n/2 \rceil$, $z_1, z_2 \in S = \{x_1, x_2 \dots x_n\}$)

∴ $\exists z_1 \le z_2 \in S$, s.t. (i) $f(x)$ is monotonically decreasing for $x \in [-\infty, z_1]$

(ii) $f(z_1) = f(x) = f(z_2)$ for $\forall x \in [z_1, z_2]$

(iii) $f(x)$ is monotonically increasing for $x \in [z_2, +\infty)$

when n is odd, let $z_1 = z_2 = y_m$, ($m = \lceil n/2 \rceil$, $z_1, z_2 \in S = \{x_1, x_2 \dots x_n\}$)

∴ $\exists z_1 \le z_2 \in S$, s.t. (i) $f(x)$ is monotonically decreasing for $x \in (-\infty, z_1)$

(ii) $f(z_1) = f(x) = f(z_2)$ for $x = z_1 = z_2$

(iii) $f(x)$ is monotonically increasing for $x \in (z_2, +\infty)$

∴ proved.

so we get a lemma A:

for $S = \{x_1 \dots x_n\}$. define $f(x) = \sum_{i=1}^{n} |x - x_i|$

$\exists z_1 \le z_2 \in S$, s.t. (i) $f(x)$ is monotonically decreasing for $x \in [-\infty, z_1]$

(ii) $f(z_1) = f(x) = f(z_2)$ for $\forall x \in [z_1, z_2]$

(iii) $f(x)$ is monotonically increasing for $x \in [z_2, +\infty)$

and z_1 is the $\lceil n/2 \rceil$-th smallest point in $\{x_1, \dots, x_n\}$

(b)

<u>Alg1($\{x_1, x_2, \dots x_n\}$):</u>

```
1   create A[1...n]                  // create an Array to save x1..xn from A[1] to A[n]
2   for i ← 1 to n:                  // assign xi to A[i] iteratively
3       A[i] ← xi
4   x̄ ← Select(A, 1, n, ⌈n/2⌉)       // get the center.
5   return x̄
```

from (a), we know the center of $\{x_1 \dots x_n\}$ is $y_{\lceil n/2 \rceil}$ or in $[y_{\lceil n/2 \rceil}, y_{\lceil n/2 \rceil + 1}]$

that means to find the $\lceil n/2 \rceil$-th smallest point in $\{x_1 \dots x_n\}$ ($m = \lceil n/2 \rceil$),

so this alg find the $\lceil n/2 \rceil$-th smallest point, so we find the Center.

(c) from lemma A: we know $f(z_1) \le \min_{x \in S} \{f(x)\}$.

∴ z_1 is a center $\bar{x}$

and z_1 is the $\lceil n/2 \rceil$-th smallest point

the we use Select($A[1 \dots n]$, 1, n, $\lceil n/2 \rceil$) to find the $\lceil n/2 \rceil$-th smallest number in $A[1 \dots n]$.

which is the center of $\{x_1, \dots, x_n\}$

(d) we analyze line by line:

line 1: we create an Array, using $O(1)$ time

line 2.3: we assign every single element in $\{x_1, \dots x_n\}$ to $A[1 \dots n]$ using $O(n)$ time

line 4: we use Select($A, 1, n, \lceil n/2 \rceil$) as a subroutine, using $O(n)$ time.

∴ we use $O(n)$ time totally.

(B) (a)

```
alg2({p1, p2 ... pn}):
    create A1[1...n]            // create 2 n-size Array to save coordinates
    create A2[1...n]            //   of pi(xi, yi)
    for i ← 1 to n:             // assign value to Array iteratively.
        A1[i] ← xi
        A2[i] ← yi
    x̄ ← Select(A1, 1, n, ⌈n/2⌉)   // use Select() subroutine to find ⌈n/2⌉-th smallest
    ȳ ← Select(A2, 1, n, ⌈n/2⌉)   //   element in A1 and A2
    return p̄(x̄, ȳ)
```

this alg uses 2 Select() subroutine to find center $\bar{x}$ of $\{x_1, \ldots, x_n\}$ and center $\bar{y}$ of $\{x_1, \ldots x_n\}$ respectively.

and $\bar{p}(\bar{x}, \bar{y})$ is the center we need to find.

(b) $d'(p, p') = |x - x'| + |y - y'|$ by definition.

we find the $\lceil n/2 \rceil$-th smallest number in A_1, from (A), we know $\bar{x}$ is a center of A_1

we find the $\lceil n/2 \rceil$-th smallest number in A_2, from (A), we know $\bar{y}$ is a center of A_2

That is: $\sum_{i=1}^{n} |\bar{x} - x_i| = \min_{x \in R} \sum_{i=1}^{n} |x - x_i|$

$$\sum_{i=1}^{n} |\bar{y} - y_i| = \min_{y \in R} \sum_{i=1}^{n} |y - y_i|$$

let $S_1 = \{x_1 \ldots x_n\}$, $S_2 = \{y_1 \ldots y_n\}$, $S = \{(x_1, y_1), \ldots, (x_n, y_n)\}$

$$\sum_{i=1}^{n} d_1(\bar{p}, p_i) = \sum_{i=1}^{n} \left(|\bar{x} - x_i| + |\bar{y} - y_i| \right) = \sum_{i=1}^{n} |\bar{x} - x_i| + \sum_{i=1}^{n} |\bar{y} - y_i|$$

$$= \min_{x \in R} \sum_{i=1}^{n} |x - x_i| + \min_{y \in R} \sum_{i=1}^{n} |y - y_i|$$

$$= \min_{(x,y) \in R^2} \sum_{i=1}^{n} \left(|x - x_i| + |y - y_i| \right)$$

$$= \min_{p \in R^2} \sum_{i=1}^{n} d_1(p, p_i)$$

which means $\bar{p}$ is a center in $\{p_1 \ldots p_n\}$

(c) we analyze the alg line by line.

line 1, 2 ; we create 2 Array, using $O(1)$ time

line 3, 4, 5: assign the value to A_1, A_2, using $O(n)$ time

line 6, 7 : use Selete( ) as a subroutine to find the $\lceil n/2 \rceil$-th smallest element of A_1, A_2, which use $O(n)$ time.

$\therefore$ we use $O(n)$ time totally.

Problem 4:

(a) $T(n) = O(n^{\frac{5}{2}})$

(b) $T(n) = O(n^{\log_2 9})$

(c) $T(n) = O(n^2 \log n)$

(d) $T(n) = O(n^{\log_4 3})$

(e) $T(n) = O(n^{\log_7 2} \cdot \log n)$

(f) $T(n) = O(n^2)$