总结：

① 变量可分离： $\frac{dy}{dx} = f(x)g(y) \Rightarrow \int g(y)dy = \int f(x)dx$

② 一阶线性方程： I. 齐次方程： $y' + p(x)y = 0 \Rightarrow \int \frac{1}{y}dy = \int -p(x)\,dx \Rightarrow y = ce^{\int -p(x)dx}$

II. 非齐次方程： $y' + p(x)y = g(x)$, $\Rightarrow \mu(x)\cdot y'(x) + \mu(x)\cdot p(x)\cdot y(x) = g(x)\cdot\mu(x)$

(that is $\mu y' + \mu p y = g\mu$)

find a μ, that : $\mu p = \mu'$ ，符合 $\frac{dy\mu}{dx} = \mu y' + \mu p y$.

$\Rightarrow \mu = e^{\int p(t)dt + C} = e^{\int p(t)dt}$ (let $c=0$) $\quad \therefore y = \frac{1}{\mu(x)}\left(\int \mu(t)g(t)dt + C\right)$

③ 二阶常数方程：

I 齐次： $y'' + py' + qy = 0$ （p, q 为常数）

$$\Rightarrow \begin{cases} \Delta > 0, & y = C_1e^{r_1x} + C_2e^{r_2x} \\ \Delta = 0, & y = (C_1 + C_2x)e^{r_1x} \\ \Delta < 0, & y = e^{\alpha x}(C_1\cos\beta x + C_2\sin\beta x) \end{cases}$$

II 非齐次： $y'' + py' + qy = f(x) \Rightarrow y = Y(x) + y^*(x)$

I. $f(x) = P_n(x)e^{ax}$ 型：
$$\begin{cases} a \text{ 与 } r_1, r_2 \text{ 不相等} &: \text{设 } y^* = H_n(x)e^{ax} \\ a \text{ 与 } r_1, r_2 \text{ 其一相等} &: \text{设 } y^* = xH_n(x)e^{ax} \\ a = r_1 = r_2 &: \text{设 } y^* = x^2H_n(x)e^{ax} \end{cases}$$

II. $f(x) = e^{\alpha x}[P_n(x)\sin\beta x + Q_m(x)\cos\beta x]$ 型：

$$\Rightarrow \begin{cases} \alpha \pm \beta i \text{ 不是特征根：} & \text{设 } y^* = e^{\alpha x}(R_l(x)\sin\beta x + S_l(x)\cos\beta x) \\ \alpha \pm \beta i \text{ 是特征根：} & \text{设 } y^* = xe^{\alpha x}(R_l(x)\sin\beta x + S_l(x)\cos\beta x), \; l = \max\{n, m\}. \end{cases}$$

III. 给出一解，求另一解（齐次）

$y'' + p(x)y' + q(x)y = 0$ ，已知 y_1

设 $y_2 = uy_1$ ，则 $y_2' = u'y_1 + uy_1'$ ， $y_2'' = u''y_1 + 2u'y_1' + y_1''u$

$\Rightarrow u''y_1 + u'(2y_1' + py_1) + u(y_1'' + py_1' + qy_1) = 0$

$\Rightarrow u''y_1 + u'(2y_1' + py_1) = 0 \quad \Rightarrow$ 求 u.

③ Power Series : $P(x)y'' + Q(x)y' + R(x)y = 0$

we assume that : $y = \sum_{n=0}^{\infty} a_n(x - x_0)^n \quad (P(x_0) \neq 0)$ $\quad$ ($P(x_0) \neq 0$, x_0 is an ordinary point. $P(x_0) = 0$, x_0 is an singular point)

then $y' = \sum_{n=1}^{\infty} a_n \cdot n(x - x_0)^{n-1} = \sum_{n=0}^{\infty} a_{n+1}(n+1)(x - x_0)^n$

$y'' = \sum_{n=2}^{\infty} a_n \cdot n(n-1)(x - x_0)^{n-2} = \sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)(x - x_0)^n$

put them into $P(x)y'' + Q(x)y' + R(x)y = 0$, get a_n.

④ Eular function : $P(x)y'' + Q(x)y' + R(x)y = 0 \Rightarrow y'' + \frac{Q(x)}{P(x)}y' + \frac{R(x)}{P(x)}y = 0$

for ordinary points : it can get certain solutions.

for singular points :

we get: $(x-x_0)^2 y + \left[\frac{Q(x)}{P(x)}(x-x_0)\right](x-x_0)y' + \left[\frac{R(x)}{P(x)}(x-x_0)^2\right]y = 0$

Ⅰ、regular singular point:

if $\lim_{x\to x_0}\left(\frac{Q(x)}{P(x)}(x-x_0)\right) = \alpha$, $\lim_{x\to x_0}\frac{R(x)}{P(x)}(x-x_0)^2 = \beta$. α, β exist.

we can call x_0 is a regular singular point.

Ⅱ. Eular's equation:

thus, we can consider: $(x-x_0)^2 y'' + \alpha(x-x_0)y' + \beta y = 0$ we assume $y=(x-x_0)^r$

⇒ characteristic equation: $r^2+(\alpha-1)r+\beta = 0$

$$\Rightarrow \begin{cases} r_1 \neq r_2 : & y = C_1|x-x_0|^{r_1} + C_2|x-x_0|^{r_2} \\ r = \lambda \pm \mu i : & y = C_1|x-x_0|^{\lambda}\cos(\mu\ln|x-x_0|) + C_2|x-x_0|^{\lambda}\sin(\mu\ln|x-x_0|) \\ r_1 \neq r_2 : & y = (C_1 + C_2\ln|x-x_0|)\cdot|x-x_0|^{r} \end{cases}$$

⑤ Laplace transform.

$$F(s) = \int_0^\infty e^{-st} f(t)\,dt = \lim_{A\to\infty}\int_0^A e^{-st} f(t)\,dt.$$

denote it as: $F(s)=\mathcal{L}\{f(t)\}$. $f(t)=\mathcal{L}^{-1}\{F(s)\}$.

重要的: $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$, $\mathcal{L}\{f+g\} = \mathcal{L}\{f\}+\mathcal{L}\{g\}$

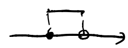
$\mathcal{L}\{y'\} = sY(s) - y(0)$

$\mathcal{L}\{y''\} = s^2Y(s) - sy(0) - y'(0)$, $\mathcal{L}\{y^{(n)}(t)\} = s^nY(s) - s^{n-1}f(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0)$

⑥ other functions:

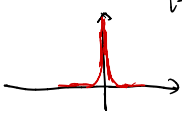
Ⅰ. step function: $u_c(t) = \begin{cases} 0, & t<c \\ 1, & t\geq c \end{cases}$, 用于截取一段函数: $(u_n - u_m)f(x)$.

$\mathcal{L}\{u_c(t)\} = \frac{e^{-cs}}{s}$, $\mathcal{L}\{u_c(t)\cdot f(t-c)\} = e^{-sc}\mathcal{L}\{f(t)\}$



Ⅱ. impluse function: 符合① $t\neq 0$ 时, $\delta(t)=0$

记作 $\delta(t)$ ② $\int_{-\infty}^{\infty}\delta(t)\,dt = 1$



而 $\mathcal{L}\{\delta(t-c)\} = e^{-cs}$

⑦ 补充: $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$, if $W=0$, y_1, y_2 不是 set of fundamental solutions.

otherwise. … 是 ….

(判断 y_1, y_2 linearly independent)

⑧ system of equations: