ELEC 2600H: Probability and Random Processes in Engineering

Part IV: Stochastic Process

- **Lecture 21: Definition of a Random Process**
- Lecture 22: Sum Processes and Independent Stationary Increment Processes
- Lecture 23: Mean and Autocorrelation of Random Process
- Lecture 24: Stationary Random Process

Elec2600H: Lecture 21

- **Definition of a Random Process**
- Specification of a Random Process

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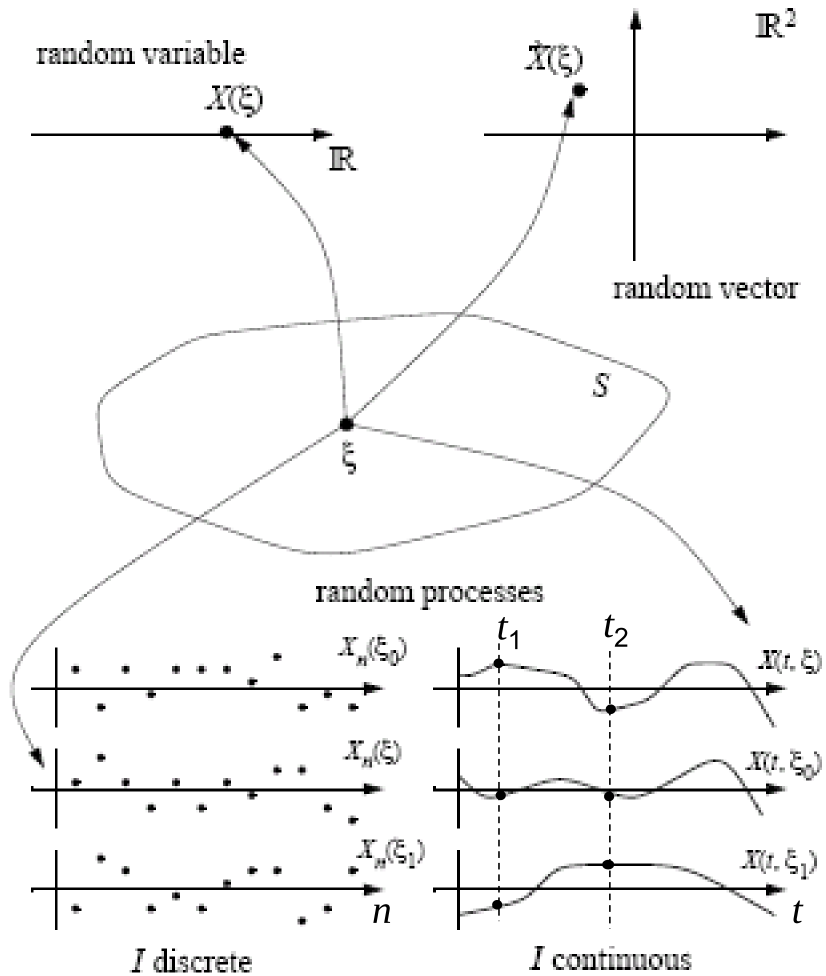
Definition of a Random Process

- Definition: A random process or stochastic process maps a probability space S to a set of functions, $X(t,\xi)$

- It assigns to every outcome $\xi \in S$ a time function $X(t,\xi)$ for $t \in I$ where I is a discrete or continuous index set.
  - If I is discrete (e.g. integer valued), $X(n,\xi)$ is a discrete-time random process.
  - If I is continuous, $X(t,\xi)$ is a continuous-time random process.

- We often simplify notation by not indicating ξ explicitly, e.g. X_n or $X(t)$.



Interpretations of a Random Process

- When both t and ξ vary, $X(t, \xi)$ is an ensemble or family of functions.
- For a fixed ξ, $X(t, \xi)$ is a function of t
  - We call this a **realization**, or **sample path**, or **sample function**, of the random process.
  - Think of this as an "example" of the functions that might occur.
- For a fixed t, $X(t, \xi)$ is a random variable.
- For a fixed t and fixed ξ, $X(t, \xi)$ is a number.
- **Intuitive example:**
  - Suppose that you have 100 songs stored on your phone.
  - Number them from 1 to 100.
  - Pick a number ξ randomly between 1 and 100.
  - Play song ξ. Let

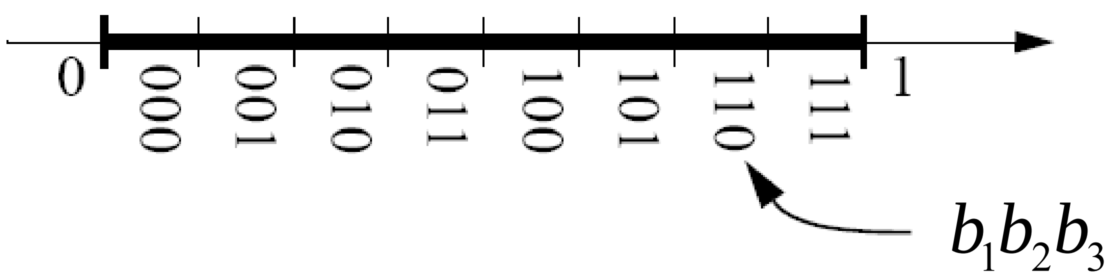
$$X(t,\xi) = \text{sound pressure waveform from your left earphone}$$
$$Y(t,\xi) = \text{sound pressure waveform from your right earphone}$$

Example 9.1 Random Binary Sequence

- A random process is said to be *discrete-time* if the time index set I is a countable set.
- Let ξ be a number selected at random from the interval $S = [0,1]$ and let $b_1, b_2, b_3, \dots$ be the binary expansion of ξ:

$$\xi = \sum_{n=1}^{\infty} b_i 2^{-n} = b_1 \frac{1}{2} + b_2 \frac{1}{4} + b_3 \frac{1}{8} + \dots$$

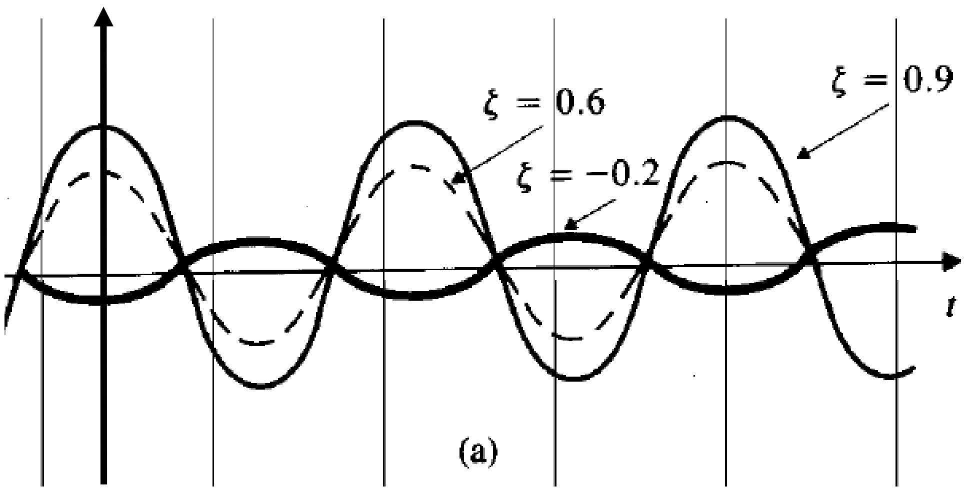
- Define the discrete time random process $X(n,\xi) = b_n \quad n = 1,2,\dots$



- This random process is a Bernoulli random process with $p = 0.5$ (introduced later).

Example 9.2a Random Amplitude Sinusoid

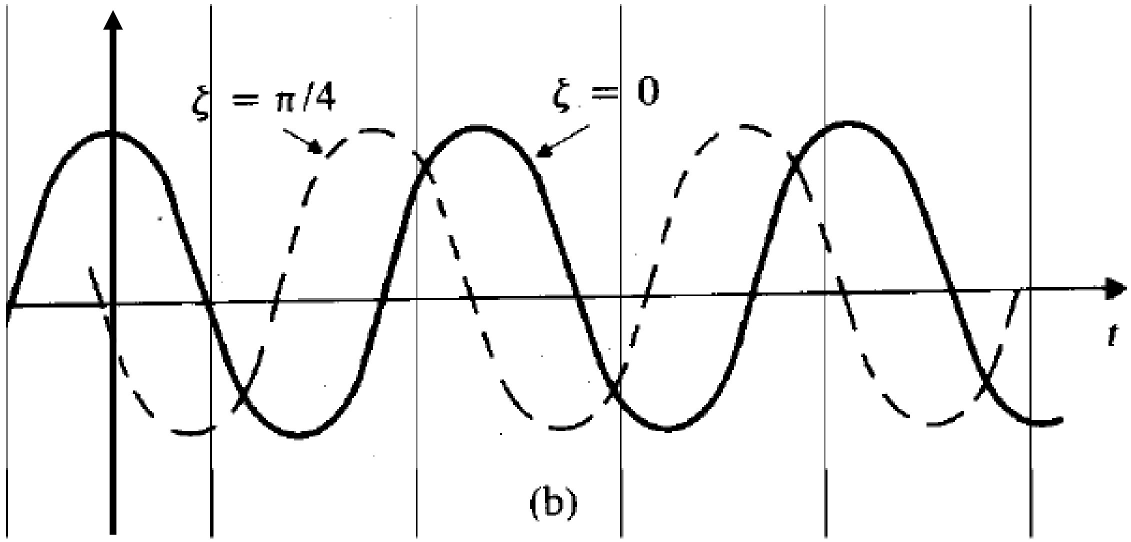
- A continuous-time stochastic process is one in which I is continuous.
- Let ξ be selected at random from the interval [-1, 1].
- Define the continuous time random process: $X(t,\xi) = \xi \cos(2\pi t)$
- The realizations of this random process are sinusoids with amplitude ξ.



Amplitude modulation

Example 9.2b – Random Phase Sinusoid

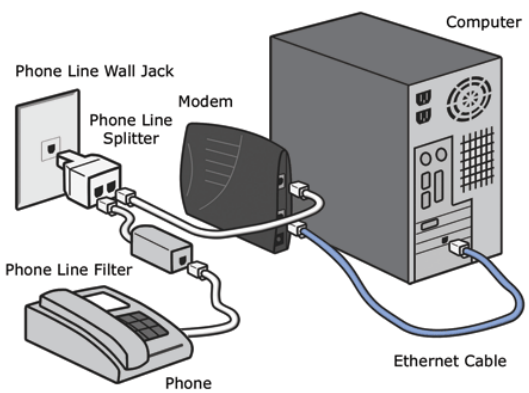
- Let ξ be selected uniformly from $[-\pi, \pi]$.
- Define the continuous time random process $X(t, \xi) = \cos(2\pi t + \xi)$ for $t \in (-\infty, \infty)$
- This is called a *random phase process.*
- All the waveforms are phase shifted versions of each other.



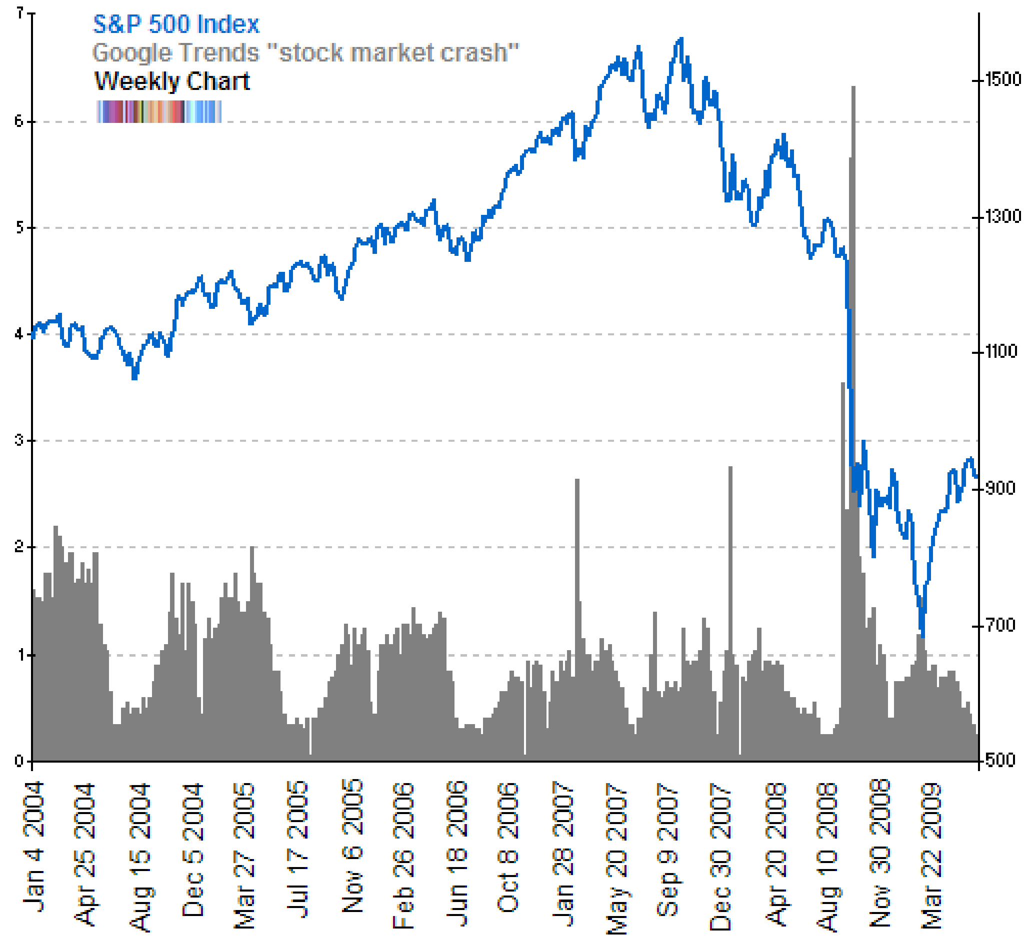
Phase modulation

Quadrature Amplitude Modulation (QAM)

- Modulates the **amplitude and phase** of a waveform



What about EVENTS....?



Events and Equivalent-Events

- In general, events of interest for a random process concern the value of the random process at specific instants in time.
- For example:

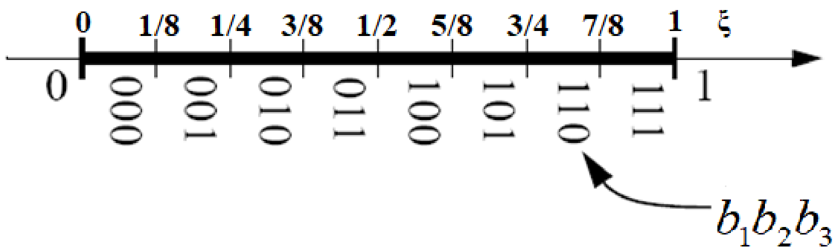
$$A = \{X(0,\xi) < 1\}$$
$$B = \{X(0,\xi) > 0, 1 < X(1,\xi) < 2\}$$

- We can find, in principle, the probability of any such event, by finding the probability of the equivalent-event in terms of the original sample space.

Example 9.3

Consider the random binary sequence process in Example 9.1.

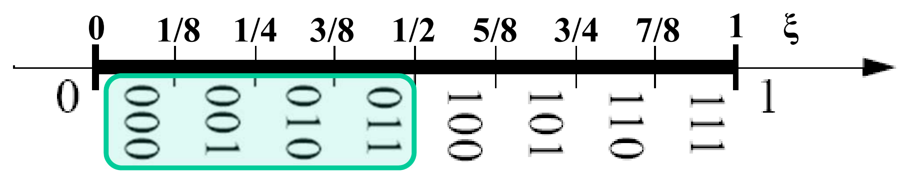
Find the following probabilities: $\boxed{P[X(1,\xi)=0]}$ and $\boxed{P[X(1,\xi)=0 \text{ and } X(2,\xi)=1]}$



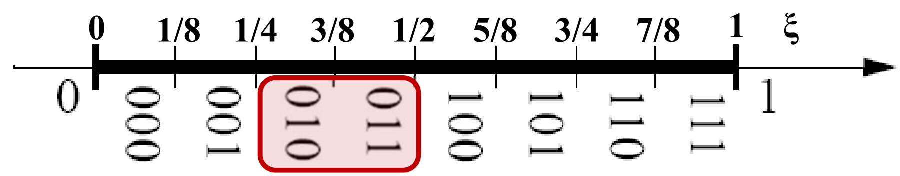
$$X(n,\xi) = b_n \qquad n = 1,2,\ldots$$

Solution

- $P[X(1,\xi)=0] = P[0 \le \xi < \frac{1}{2}] = \boxed{\frac{1}{2}}$



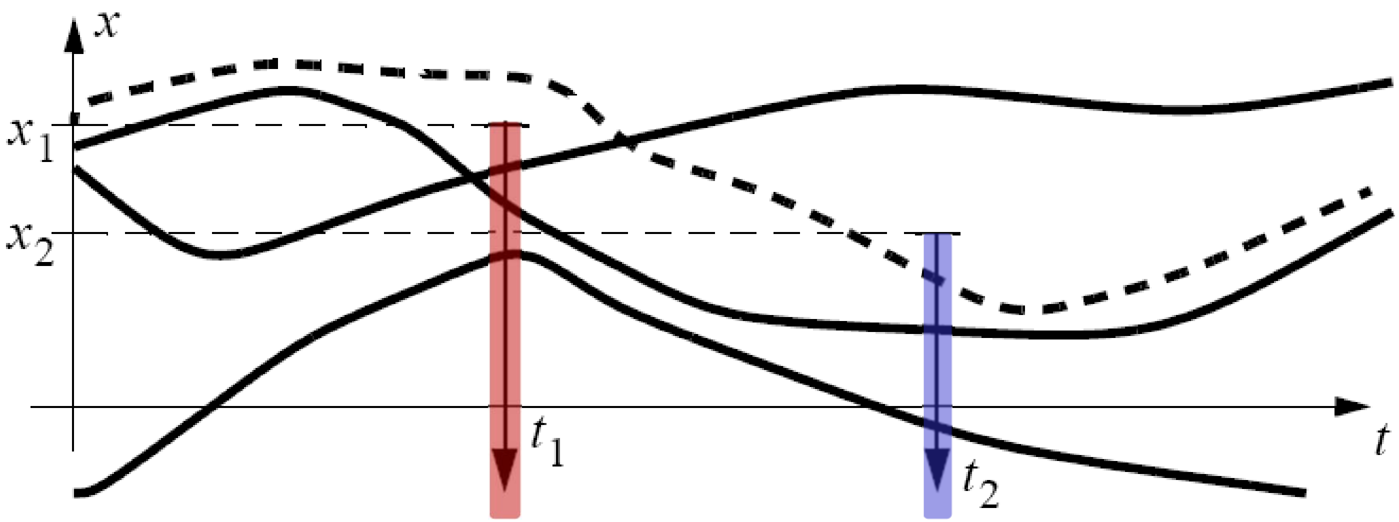
- $P[X(1,\xi)=0 \text{ and } X(2,\xi)=1] = P[\frac{1}{4} \le \xi < \frac{1}{2}] = \boxed{\frac{1}{4}}$



In general, any sequence of k bits corresponds to an interval of length 2^{-k}. Thus, its probability is 2^{-k}.

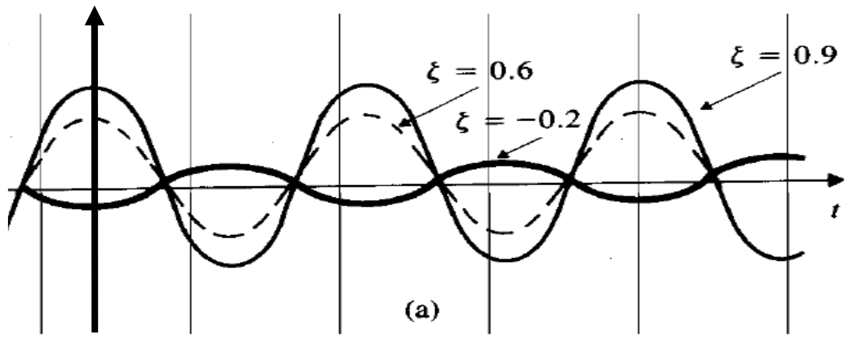
Joint Distributions of Time Samples

- A random process is uniquely specified by the collection of all n-th order distribution or density functions.
- The first order distribution of $X(t,\xi)$ is $F_{X(t)}(x) = P[X(t,\xi) \le x]$
  - This is also sometimes written as $F_X(x;t)$
- The first order density of $X(t,\xi)$ is $f_{X(t)}(x) = \frac{d}{dx} F_{X(t)}(x)$
  - This is sometimes written as $f_X(x;t)$



Example 9.4a Random Amplitude Sinusoid

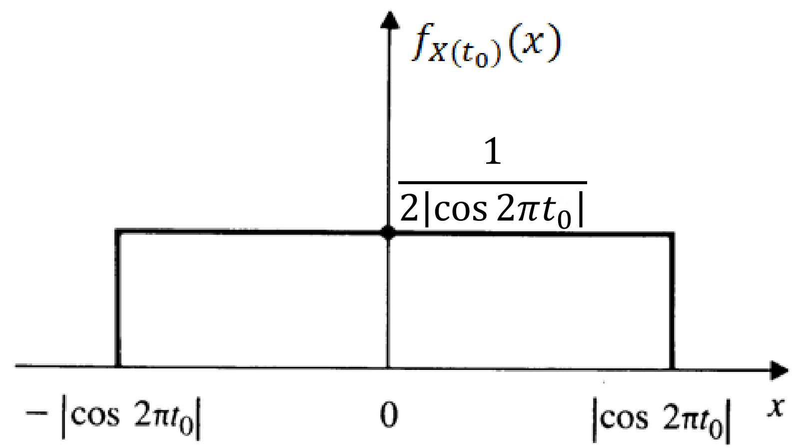
Find the first order pdf of $X(t,\xi)$, where $X(t,\zeta) = \xi\cos(2\pi t)$ and ξ is selected at random from [-1, 1].



Solution

For most t, $X(t)$ is uniform over $[-|\cos(2\pi t)|, |\cos(2\pi t)|]$

Thus, $f_X(x;t) = \begin{cases} \frac{1}{2|\cos(2\pi t)|} & -|\cos(2\pi t)| \le x \le |\cos(2\pi t)| \\ 0 & \text{otherwise} \end{cases}$

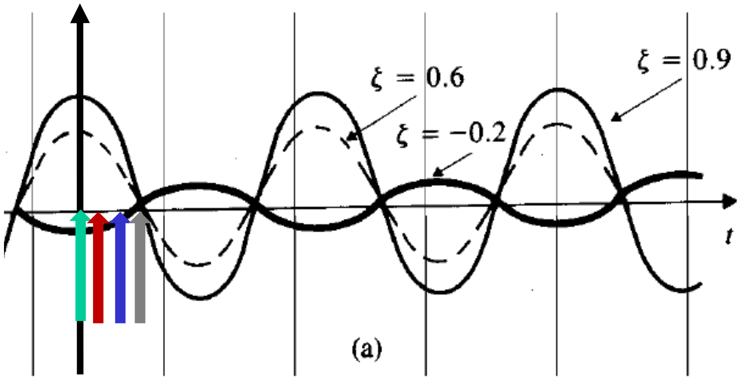


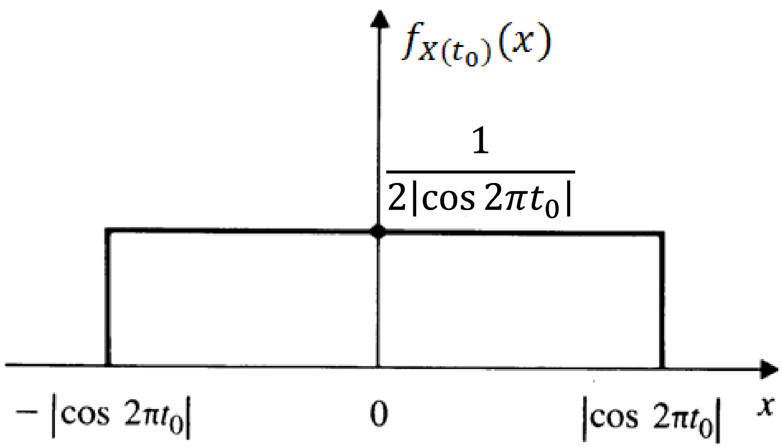
However, for $t \in \{0.25, 0.75, \ldots, 0.25 + 0.5k, \ldots\}$ where $k \in \mathbb{Z}$,

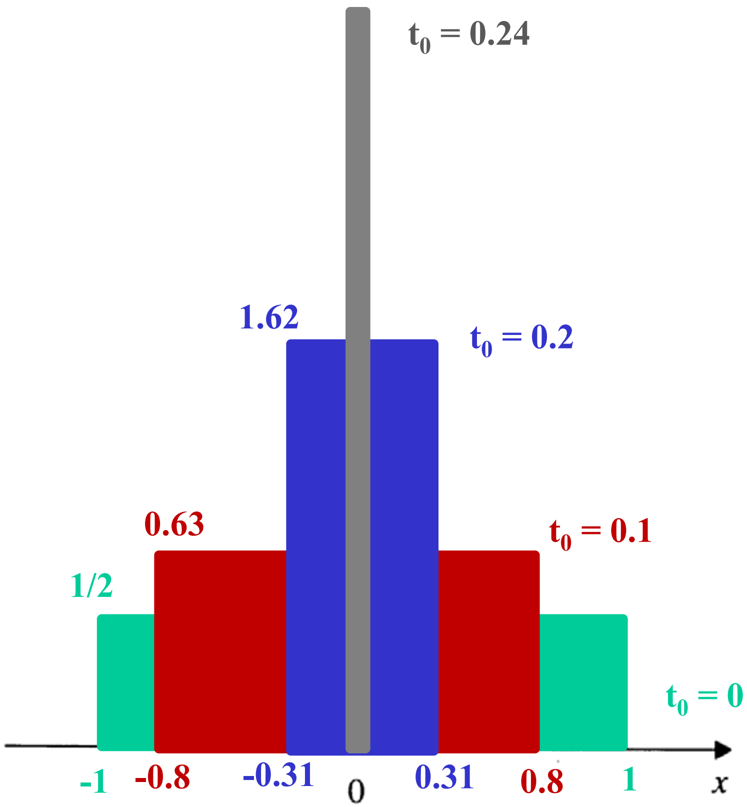
$\cos(2\pi t) = 0 \quad \Rightarrow \quad f_X(x;t) = \delta(x)$

Example 9.4a Random Amplitude Sinusoid

$X(t,\xi) = \xi\cos(2\pi t)$

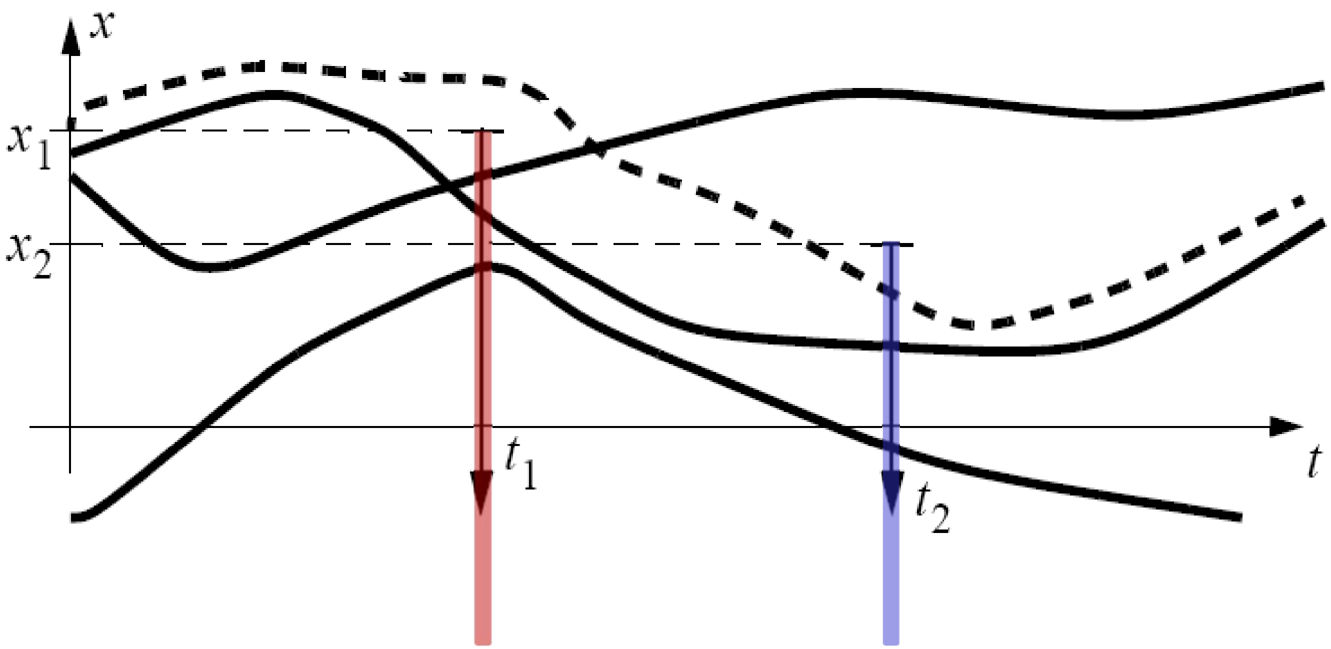






Higher Order Distributions

- The second order distribution of $X(t,\xi)$ is $F_{X(t_1)X(t_2)}(x_1,x_2) = P[\{X(t_1) \le x_1\} \cap \{X(t_2) \le x_2\}]$
  - This is sometimes written as $F_X(x_1,x_2;t_1,t_2)$
- Similarly, the n-th order distribution is $F_{X(t_1)X(t_2)\ldots X(t_n)}(x_1,x_2,\ldots,x_n) = P[\cap_{i=1}^{n}\{X(t_i) \le x_i\}]$
  - This is sometimes written as $F_X(x_1,x_2,\ldots,x_n;t_1,t_2,\ldots,t_n)$



Higher Order Densities

- The second order density of $X(t,\xi)$ is

$$f_{X(t_1)X(t_2)}(x_1,x_2) = \frac{d^2}{dx_1 dx_2} F_{X(t_1)X(t_2)}(x_1,x_2)$$

  - This is sometimes written as $f_X(x_1,x_2;t_1,t_2)$

- Similarly, the n-th order density is

$$f_{X(t_1)X(t_2)\dots X(t_n)}(x_1,x_2,\dots,x_n) = \frac{d^n}{dx_1 dx_2 \dots dx_n} F_{X(t_1)X(t_2)\dots X(t_n)}(x_1,x_2,\dots,x_n)$$

- For discrete valued processes, we typically specify the joint pmf:

$$p_{X(t_1)X(t_2)\dots X(t_n)}(k_1,k_2,\dots,k_n) = P\left[\bigcap_{i=1}^{n}\{X(t_i)=k_i\}\right]$$

Independent and Identically Distributed (I.I.D.) Process

- **Definition:** A discrete time process X_n is said to be *independent and identically distributed* or *i.i.d.* if all vectors formed by a finite number of samples of the process are i.i.d.

- Equivalently, there exists a marginal distribution $F(x)$ such that for all $k < \infty$, the k-th order distribution is given by

$$F_{X_1, X_2, \ldots X_k}(x_1, \ldots, x_k) = \prod_{i=1}^{k} F(x_i)$$

- Thus, an i.i.d. process is completely specified by a **single marginal distribution** (density or mass) function.

Bernoulli Random Process

- **Definition:** A *Bernoulli random process* is a binary alphabet i.i.d. random process. At each time n it assumes values 1 or 0 with probability p or $q = 1\text{-}p$.

- Intuitively, we can think of it as an **infinite sequence of independent coin flips**, each with probability p of heads (1).

- This seems to suggest an infinite number of separate experiments, but fundamentally there is **always just a single experiment**. Once this experiment is performed, the outcomes of all coin flips is determined.

- Example 9.1 gives a way of generating a Bernoulli random process with $p = 0.5$ from a single experiment (picking a number between 0 and 1.)

Example 9.5

Given a Bernoulli random process X_n, what is the probability $P[X_0 = 1, X_1 = 0, X_5 = 0, X_6 = 1]$ if $p = ½$? What is the probability if $p = ¼$?

Solution

For both cases, $P[X_0 = 1, X_1 = 0, X_5 = 0, X_6 = 1] = p(1-p)(1-p)p = p^2(1-p)^2$

If $p = ½$, $P[X_0 = 1, X_1 = 0, X_5 = 0, X_6 = 1] = p^2(1-p)^2 = \frac{1}{2^2} \cdot \frac{1}{2^2} = \frac{1}{16} \approx 0.0625$

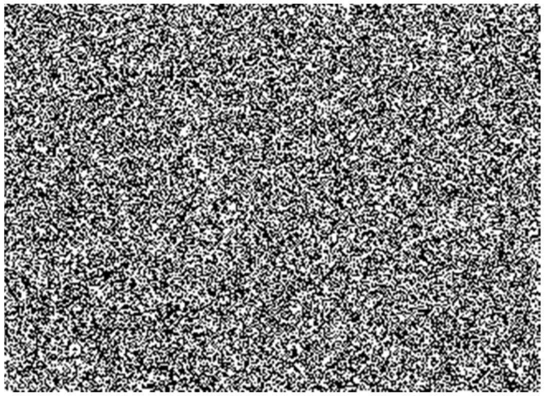
If $p = ½$, the joint pmf of any k time samples, e.g., $P[X_0 = x_0, X_1 = x_1, \cdots, X_k = x_k]$, is $\frac{1}{2^k}$.

If $p = ¼$, $P[X_0 = 1, X_1 = 0, X_5 = 0, X_6 = 1] = p^2(1-p)^2 = \frac{1}{4^2} \cdot \frac{3^2}{4^2} = \frac{9}{256} \approx 0.0352$

If $p \neq ½$, the joint pmf varies depending on the values of the samples.

Example 9.6 (White Gaussian Noise)

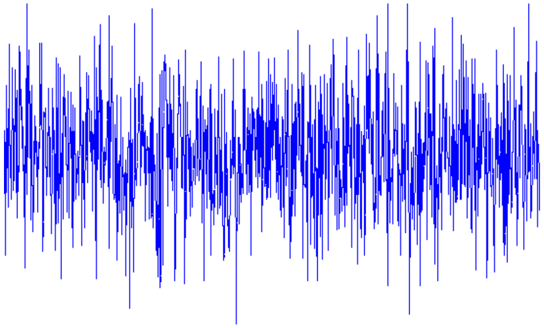
Let X_n be a sequence of i.i.d. Gaussian random variables with zero mean and variance σ^2.



The marginal pdf for any single sample is $f_{X_k}(x_k) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{{x_k}^2}{2\sigma^2}}$

The joint pdf of X_1 and X_2 is

$$f_{X_1 X_2}(x_1, x_2) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{{x_1}^2}{2\sigma^2}} \times \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{{x_2}^2}{2\sigma^2}} = \frac{1}{\left(\sqrt{2\pi}\sigma\right)^2} e^{-\frac{{x_1}^2+{x_2}^2}{2\sigma^2}}$$



The joint pdf for the first k samples is

$$f_{X_1, X_2, \ldots, X_k}(x_1, x_2, \ldots, x_k) = \frac{1}{\left(\sqrt{2\pi}\sigma\right)^k} e^{-\frac{{x_1}^2+{x_2}^2+\cdots+{x_k}^2}{2\sigma^2}}$$

White Noise Generator

Major Points from this Lecture:

- Definition of a Random Process
  - Definition
  - Interpretations
  - Events and Equivalent-Events

- Specification of a Random Process
  - Joint Distributions of Time Samples
  - I.I.D. Random Processes

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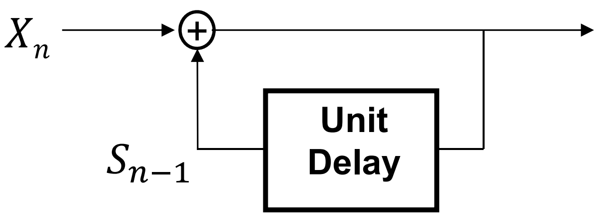
Sum Random Processes

- **Definition**: A sum process S_n is a discrete-time random process obtained by taking the sum of all past values of an i.i.d. random process X_n:

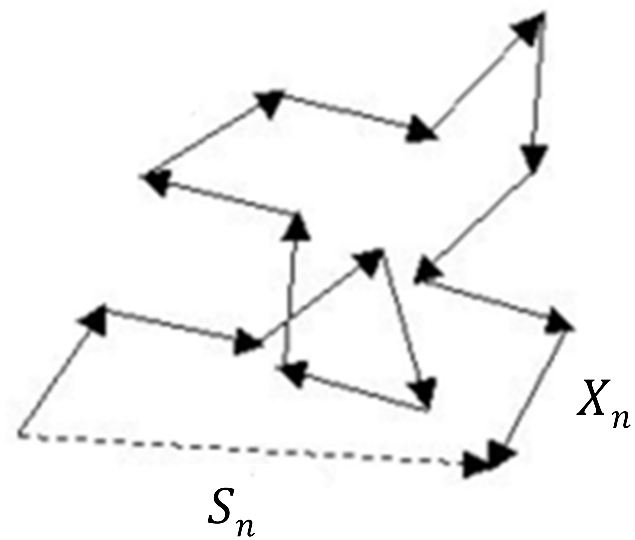
$$S_n = \sum_{i=1}^{n} X_i = X_1 + X_2 + \ldots + X_n$$

- S_n can be generated recursively:



$$S_n = S_{n-1} + X_n$$

- A sum process is like a **random walk**
  - S_n is your current position
  - X_n is the size of your last step



- We assume a sum process always starts at zero: $S_0 = 0$

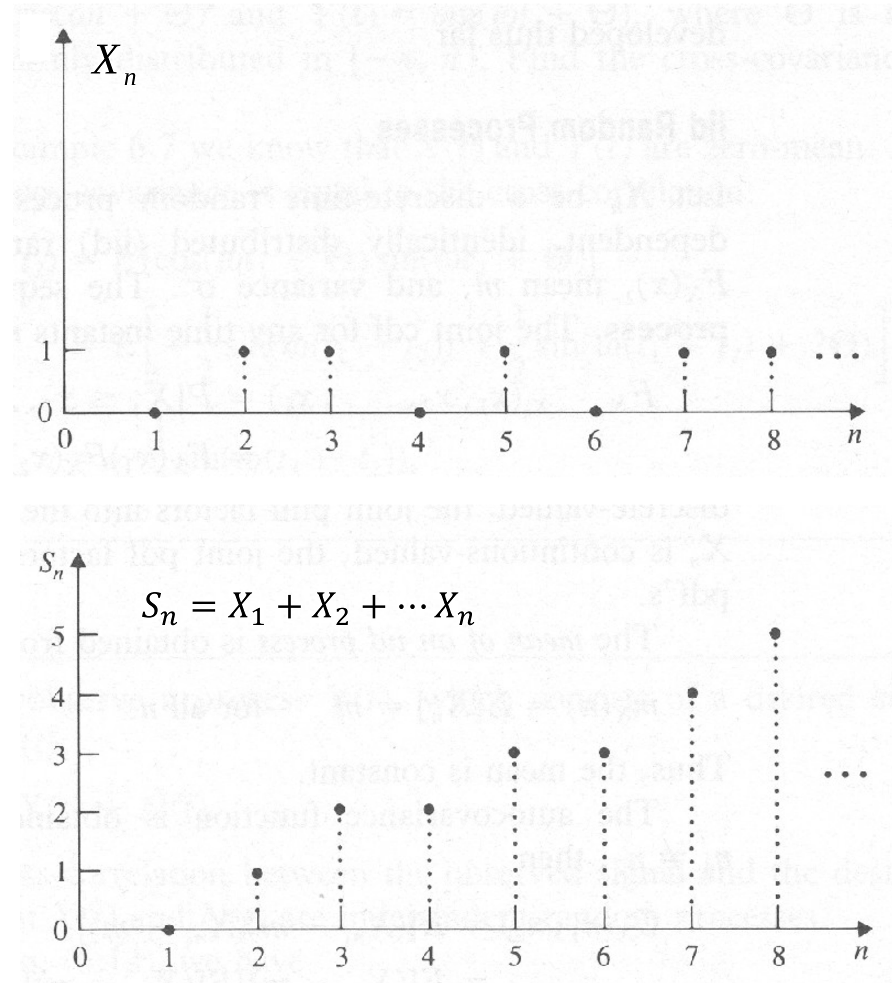
Example 9.7 Binomial Counting Process

Let X_n be a sequence of independent, identically distributed Bernoulli random variables with parameter p.

Let S_n be the number of 1's in the first n trials.

- S_n is an integer-valued non-decreasing function of n that grows by unit steps
- The first order pmf of S_n is a binomial random variable with parameters n and p.

$$p_{S_n}(k) = \binom{n}{k} p^k (1-p)^{n-k} \quad \text{for } k = 0,1,\dots,n$$



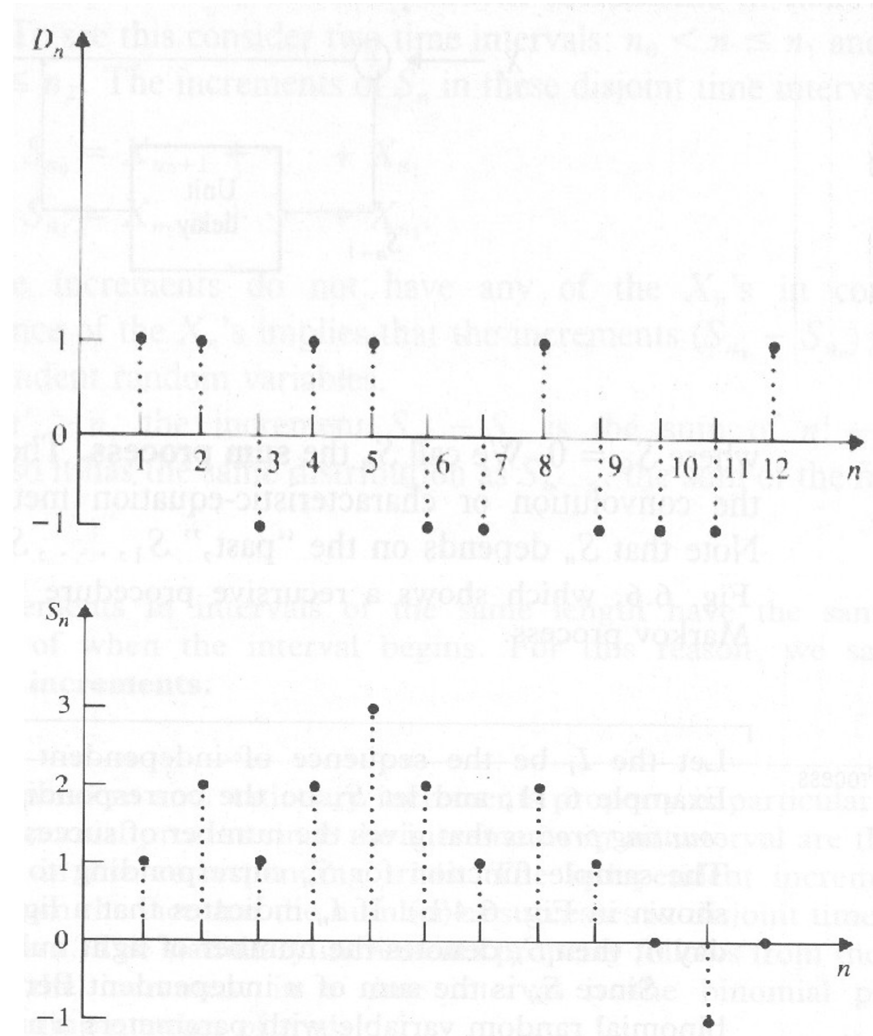
Example 9.16 – 1D Random Walk

- Let $D_n = 2I_n - 1$, where I_n is a Bernoulli random process. The corresponding sum process is called a one-dimensional random walk.

- Properties
  - At any time n, $S_n \in \{-n, \dots, n\}$, since there can be at most n "+1" or "-1" steps.
  - If n is even, then S_n is even.
  - If n is odd, then S_n is odd.
  - If there are k "+1" steps, there must be $(n-k)$ "-1" steps.

- Thus, $S_n = 2k - n$ for $0 \le k \le n$, and

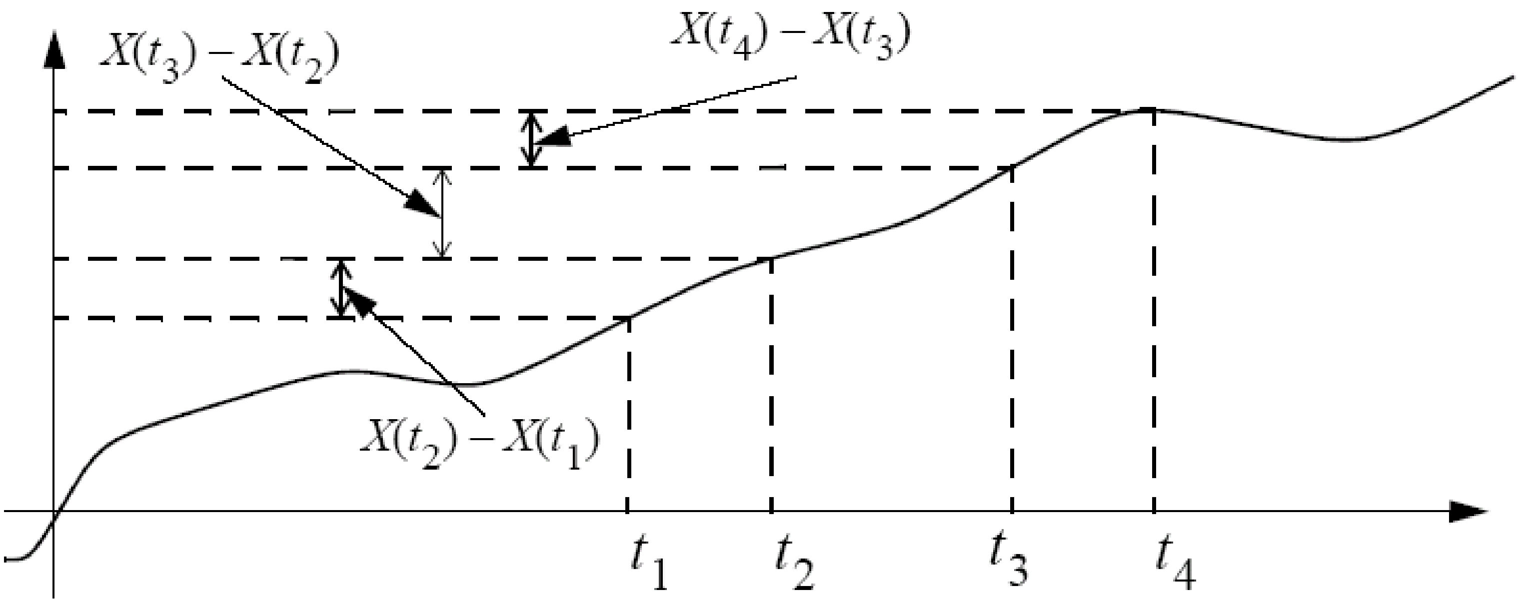
$$P[S_n = 2k - n] = \binom{n}{k} p^k (1-p)^{n-k}$$



Independent and Stationary Increment (I.S.I.) processes

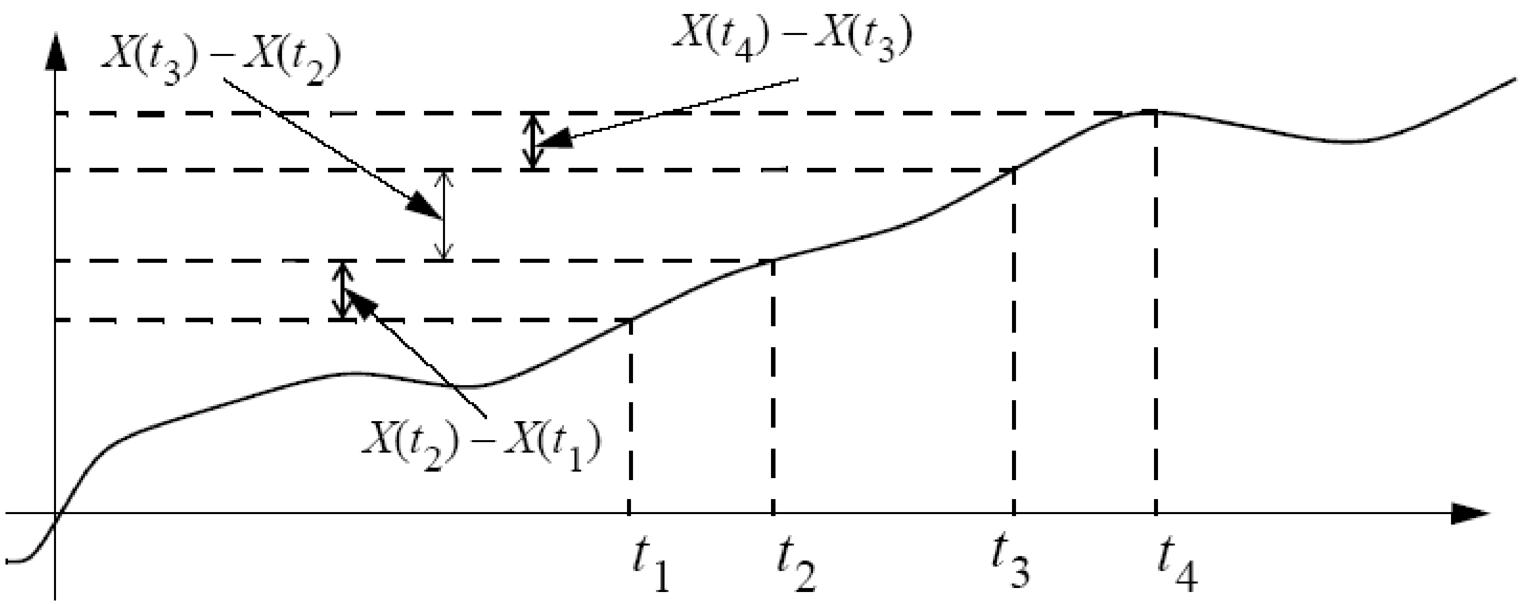
- Definition: A process is said to be an independent stationary increment (i.s.i.) process if its **non-overlapping** increments are both independent and stationary.

- Definition: An increment of a random process is the difference between the values of the random process at two different points in time.



Independent Increments

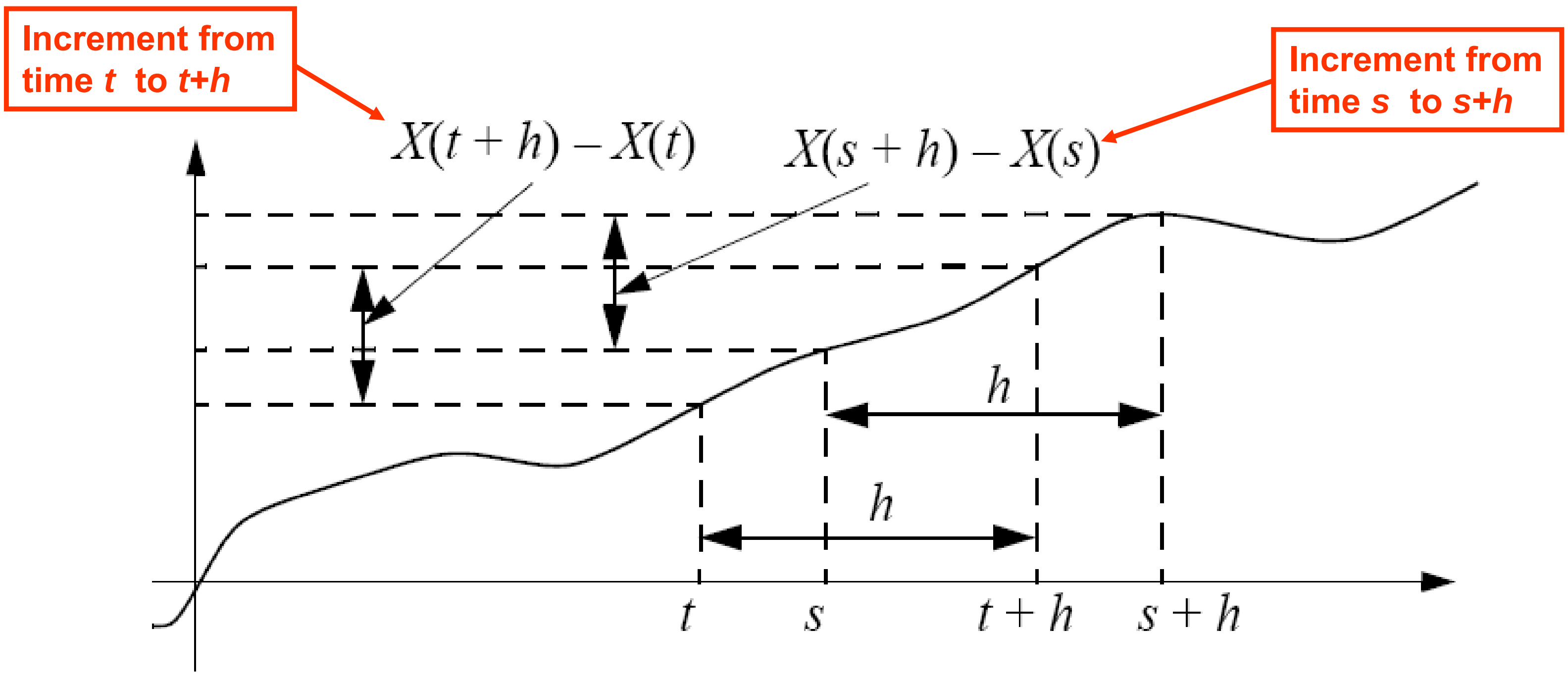
- Definition: A process $X(t)$ has independent increments if for any $k \geq 3$ time points, $t_1 < t_2 < \ldots < t_k$, the increment random variables $X(t_2) - X(t_1)$, $X(t_3) - X(t_2)$,… $X(t_k) - X(t_{k-1})$ are all independent.



- Note: Only **non-overlapping** intervals are independent.

Stationary Increments

- Definition: A process $X(t)$ has stationary increments if the increments over any two intervals with the same length have the same distribution.
- Mathematically, for all t and s in I (the index set) and h such that $t+h$ and $s+h$ are in I, the increments $X(t+h)-X(t)$ and $X(s+h)-X(s)$ have the same distribution.



Note the increments may overlap!

Theorem 1: Sum Processes are I.S.I.

Proof

- Consider a discrete time sum process $S_n = \sum_{i=1}^{n} X_i$ where the X_i are i.i.d.
- S_n has **independent increments** since for all $n_1 < n_2 < \cdots < n_m$

$$\left.\begin{aligned} S_{n_2} - S_{n_1} &= \sum_{i=n_1+1}^{n_2} X_i \\ S_{n_3} - S_{n_2} &= \sum_{i=n_2+1}^{n_3} X_i \\ &\vdots \\ S_{n_m} - S_{n_{m-1}} &= \sum_{i=n_{m-1}+1}^{n_m} X_i \end{aligned}\right\}$$

each increment is the sum of different independent RVs

- S_n has **stationary increments**, since for any k_1, k_2 and $h > 0$

$$\left.\begin{aligned} S_{n_1+h} - S_{n_1} &= \sum_{i=n_1+1}^{n_1+h} X_i \\ S_{n_2+h} - S_{n_2} &= \sum_{i=n_2+1}^{n_2+h} X_i \end{aligned}\right\}$$

Both increments are the sum of h i.i.d. RVs. Therefore, they have the same distribution

- It is possible to prove that any discrete-time i.s.i. process can be expressed as a sum process.

Working with I.S.I. Processes

- When considering the joint density or distribution of multiple samples of an i.s.i. process, it is helpful to consider the *increments* between samples, rather than the samples themselves.

- For **any** integer-valued discrete-time process and for any $m < n$, it is true that

$$P[S_m = j, S_n = k] = P[(S_m = j) \cap (S_n - S_m = k - j)]$$

- The independent increments property implies that

$$P[S_m = j, S_n = k] = P[S_m = j] \times P[S_n - S_m = k - j]$$

- The stationary increments property and $S_0 = 0$ implies that

$$P[S_n - S_m = k - j] = P[S_{n-m} - S_0 = k - j] = P[S_{n-m} = k - j]$$

- Thus, $P[S_m = j \cap S_n = k] = P[S_m = j] \times P[S_{n-m} = k - j]$

- Equivalently, $p_{S_m S_n}(j, k) = p_{S_m}(j) \times p_{S_{n-m}}(k - j)$

Specifying an i.s.i. process

- Any i.s.i. process X_n ($X(t)$) can be specified by
  - Indicating that X_n ($X(t)$) is i.s.i.
  - Specifying the first order pmf $p_{X_n}(k)$ or pdf $f_X(x;t)$ for all n or t

- Any higher order density can be obtained by using the manipulations on the previous page.
- For example, for an integer valued discrete time process,
  - For any $n_1 < n_2 < n_3$,

$$P[S_{n_1} = k_1, S_{n_2} = k_2, S_{n_3} = k_3] = P[S_{n_1} = k_1] \times P[S_{n_2-n_1} = k_2 - k_1] \times P[S_{n_3-n_2} = k_3 - k_2]$$

$$p_{S_{n_1}S_{n_2}S_{n_3}}(k_1, k_2, k_3) = p_{S_{n_1}}(k_1) \times p_{S_{n_2-n_1}}(k_2 - k_1) \times p_{S_{n_3-n_2}}(k_3 - k_2)$$

  - For any $n_1 < n_2 < \cdots < n_m$,

$$P[S_{n_1} = k_1, S_{n_2} = k_2, \cdots, S_{n_m} = k_m] = P[S_{n_1} = k_1] \times \prod_{i=2}^{m} P[S_{n_i-n_{i-1}} = k_i - k_{i-1}]$$

$$p_{S_{n_1}S_{n_2}\cdots S_{n_m}}(k_1, k_2, \cdots, k_m) = p_{S_{n_1}}(k_1) \times \prod_{i=2}^{m} p_{S_{n_i-n_{i-1}}}(k_i - k_{i-1})$$

Example 9.18: Binomial Counting Process

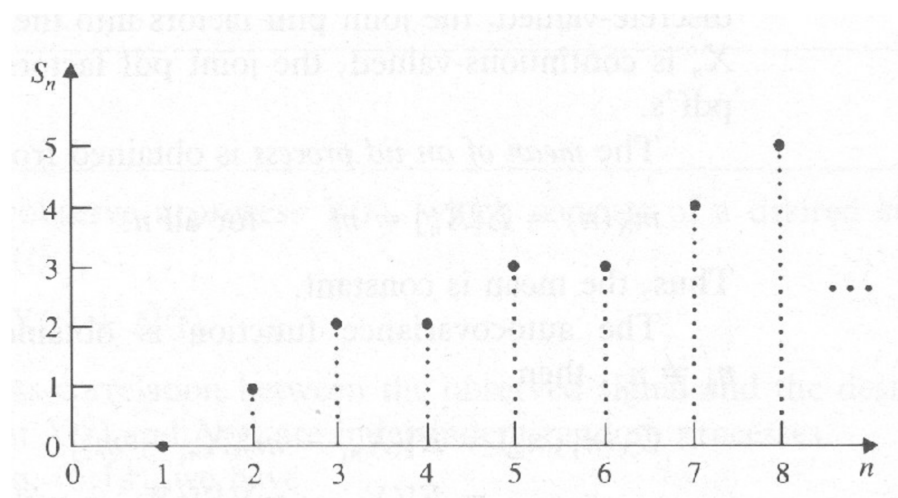
Find the joint pmf for the binomial counting process at times n_1 and n_2, where $n_2 > n_1$.

Solution

If $k_2 \geq k_1$,

$$
\begin{aligned}
p_{S_{n_1} S_{n_2}}(k_1, k_2) &= P[S_{n_1} = k_1, S_{n_2} = k_2] \\
&= P[S_{n_1} = k_1] \cdot P[S_{n_2 - n_1} = k_2 - k_1] \\
&= \binom{n_1}{k_1} p^{k_1} (1-p)^{n_1 - k_1} \cdot \binom{n_2 - n_1}{k_2 - k_1} p^{k_2 - k_1} (1-p)^{(n_2 - n_1) - (k_2 - k_1)} \\
&= \binom{n_1}{k_1} \binom{n_2 - n_1}{k_2 - k_1} p^{k_2} (1-p)^{n_2 - k_2}
\end{aligned}
$$

If $k_2 < k_1$, $p_{S_{n_1} S_{n_2}}(k_1, k_2) = 0$



Example 9.19: Sum of i.i.d. Gaussian Sequence

Let X_i be a sequence of i.i.d. Gaussian RVs with zero mean and variance σ^2. Let $S_n = \sum_{i=1}^{n} X_i$ be the corresponding sum process. Find the joint pdf of S_n at times n_1 and n_2, where $n_2 > n_1$.

Solution

The sum S_n is a Gaussian random variable with mean zero and variance $n\sigma^2$, $f_{S_n}(y) = \frac{1}{\sqrt{2\pi n\sigma^2}} e^{-\frac{y^2}{2n\sigma^2}}$

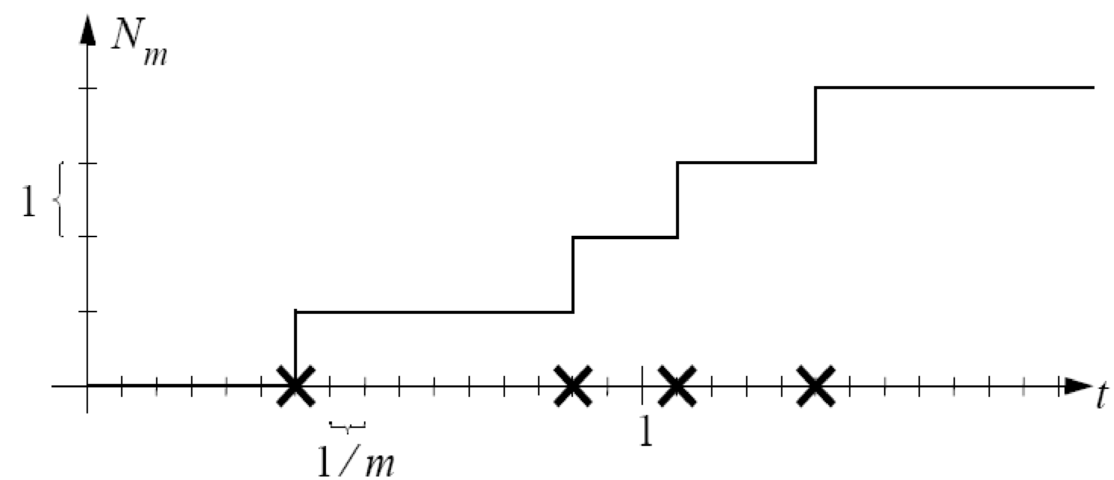
The joint pdf of S_n at times n_1 and n_2 is given by:

$$f_{S_{n_1}S_{n_2}}(y_1, y_2) = f_{S_{n_1}}(y_1) \cdot f_{S_{n_2-n_1}}(y_2 - y_1)$$

$$= \frac{1}{\sqrt{2\pi n_1\sigma^2}} \exp\left\{-\frac{{y_1}^2}{2n_1\sigma^2}\right\} \cdot \frac{1}{\sqrt{2\pi(n_2 - n_1)\sigma^2}} \exp\left\{-\frac{(y_2 - y_1)^2}{2(n_2 - n_1)\sigma^2}\right\}$$

The Poisson Process

- Consider the following sequence of random processes conditioned on m.
- Divide each unit interval of the real line into m equal sub intervals. At each sub interval, we toss a coin with probability of heads $p = \lambda/m$ (The average number of heads in each unit interval is λ.)
  - If heads appears, step forward by 1.
  - If tails appears, stay put.
- Our position at time t is $N_m(t) = \sum_{i=1}^{\lfloor mt \rfloor} X_i$ where
  - X_i is a Bernoulli random process with parameter p
  - $\lfloor x \rfloor$ is the largest integer less than or equal to x.



Properties of $N_m(t)$

- The underlying discrete-time process $N_m(n) = \sum_{i=0}^{n} X_i$ is i.s.i.

- For fixed t and m, $N_m(t)$ is binomial with parameters $n = \lfloor mt \rfloor$ and $p = \lambda/m$, i.e.,

$$P(N_m(t) = k) = \binom{\lfloor mt \rfloor}{k} p^k (1-p)^{\lfloor mt \rfloor - k}$$

- By the Poisson Theorem, the distribution of $N_m(t)$ approaches a Poisson distribution with mean $np = \lfloor mt \rfloor \frac{\lambda}{m} \longrightarrow \lambda t$ i.e.,

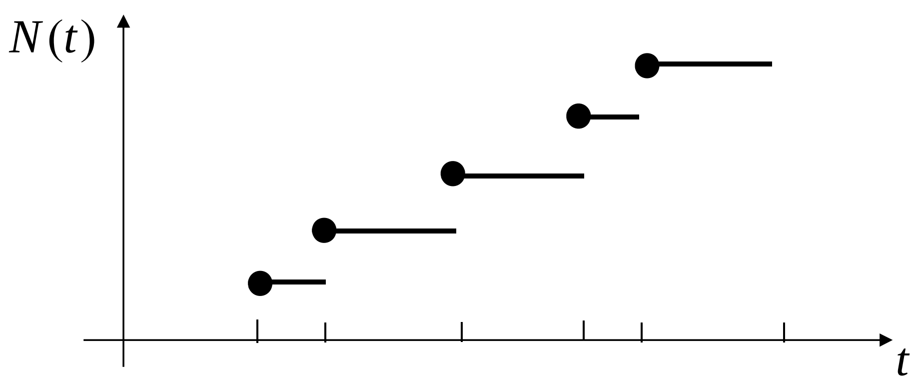
$$P[N_m(t) = k] \xrightarrow[n \to \infty]{} \frac{(\lambda t)^k}{k!} e^{-\lambda t}$$

Definition of the Poisson Counting Process

- The Poisson counting process $N(t)$ is the continuous time non-negative integer valued i.s.i. process whose first order density is Poisson:

$$P[N(t) = k] = \frac{(\lambda t)^k}{k!} e^{-\lambda t} \quad \text{for } k = 0,1,2,\ldots$$

- Interpretation: The number of events that have occurred up to time t, where events randomly in time at an average rate of λ events per unit time.
- Applications: arrivals of customers at a service station, breakdowns of a component, requests to a website server.



Example 9.21: Message Arrivals

- Inquiries arrive at a recorded message device according to a Poisson process with rate 15 inquiries per minute. Find the probability that in a 1-minute period,
  - 3 inquiries arrive during the first 10 seconds
  - 2 inquiries arrive during the last 15 seconds.

Solution

The arrival rate in seconds is $\lambda = 15/60 = ¼$ inquiries per second.

Writing time in seconds, the desired probability is

$$
\begin{aligned}
&P[N(10) = 3 \text{ and } N(60) - N(45) = 2] \\
&\quad = P[N(10) = 3] \times P[N(60) - N(45) = 2] \\
&\quad = P[N(10) = 3] \times P[N(60-45) = 2] \\
&\quad = \frac{(10/4)^3 e^{-10/4}}{3!} \times \frac{(15/4)^2 e^{-15/4}}{2!}
\end{aligned}
$$

independent increments

stationary increments

Second Order Density

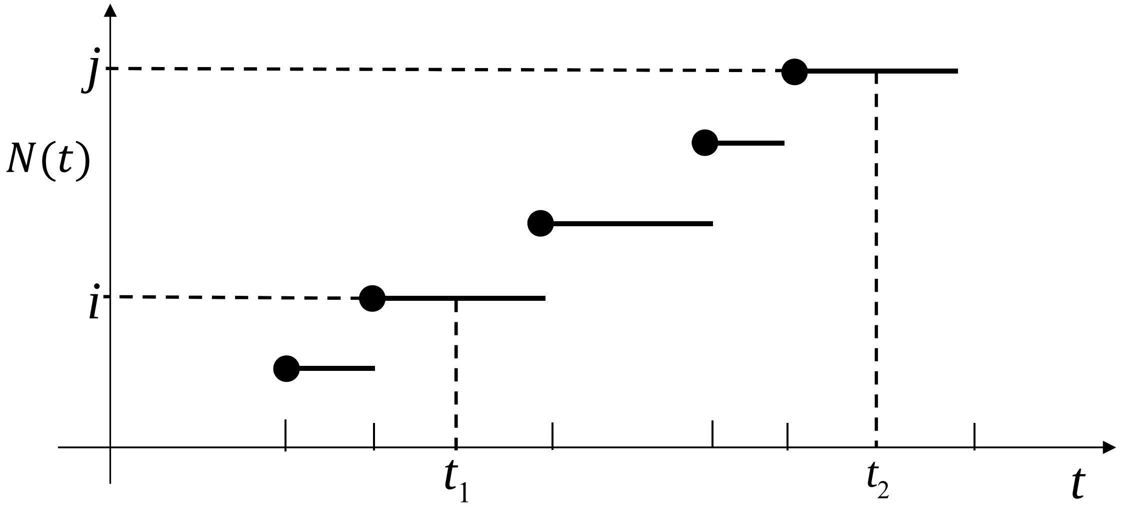
If $t_1 < t_2$ and $j \geq i$,

$$P[N(t_1) = i, N(t_2) = j] = P[(N(t_1) = i) \cap (N(t_2) - N(t_1) = j - i)]$$

$$= P[N(t_1) = i] \times P[N(t_2) - N(t_1) = j - i]$$

$$= P[N(t_1) = i] \times P[N(t_2 - t_1) = j - i]$$

$$= \frac{(\lambda t_1)^i e^{-\lambda t_1}}{i!} \times \frac{(\lambda(t_2 - t_1))^{(j-i)} e^{-\lambda(t_2 - t_1)}}{(j-i)!}$$



Example: Customer Service Calls

Telephone calls arrive at a customer service center according to a Poisson process with rate 10 per minute.

Find the probability that 2 calls arrive in the first 10 seconds and 6 calls arrive in the first 30 seconds.

Solution: $\lambda = 10\,\frac{\text{calls}}{\text{minute}} = \frac{10\text{ calls}}{1\text{ minute}} \times \frac{1\text{ minute}}{60\text{ seconds}} = \frac{1}{6}\frac{\text{calls}}{\text{second}}$

$$\begin{aligned}
P[N(10) = 2 \cap N(30) = 6] &= P[(N(10) = 2) \cap (N(30) - N(10) = 4)] \\
&= P[N(10) = 2] \times P[N(30) - N(10) = 4] \\
&= P[N(10) = 2] \times P[N(20) = 4] \\
&= \frac{(10\lambda)^2 e^{-10\lambda}}{2!} \times \frac{(20\lambda)^4 e^{-20\lambda}}{4!} \\
&= \frac{\left(\frac{10}{6}\right)^2 e^{-\frac{10}{6}}}{2!} \times \frac{\left(\frac{20}{6}\right)^4 e^{-\frac{20}{6}}}{4!}
\end{aligned}$$

Properties of the Poisson Process

- The time interval T between adjacent events (the inter-arrival time) is exponentially distributed with parameter λ.

Proof:

$$
\begin{aligned}
F_T(t) &= P[T \leq t] \\
&= P[\text{at least one arrival in an interval of length } t] \\
&= P[N(t) > 0] = 1 - P[N(t) = 0] = 1 - e^{-\lambda t}
\end{aligned}
$$

Thus, the cumulative distribution function of T is the same as that of an exponential RV.

- The time to the first event is exponentially distributed with parameter λ. (same proof as above)

- The inter-arrival times are independent. (because increments are independent)

Properties of the Poisson Random Process (cont.)

- The time to the mth event is an m-Erlang random variable (the sum of m independent exponential random variables)

Exponential Random Variable

$S_X = [0, \infty)$

$f_X(x) = \lambda e^{-\lambda x} \qquad x \geq 0 \qquad \text{and } \lambda > 0$

$E[X] = \frac{1}{\lambda} \qquad \text{VAR}[X] = \frac{1}{\lambda^2}$

Gamma Random Variable

$S_X = (0, +\infty)$

$$f_X(x) = \frac{\lambda(\lambda x)^{\alpha-1}e^{-\lambda x}}{\Gamma(\alpha)} \qquad x > 0 \quad \text{and} \quad \alpha > 0,\ \lambda > 0$$

where $\Gamma(z)$ is the gamma function (Eq. 3.46).

$E[X] = \alpha/\lambda \qquad \text{VAR}[X] = \alpha/\lambda^2$

$$\Phi_X(\omega) = \frac{1}{(1 - j\omega/\lambda)^\alpha}$$

Special Cases of Gamma Random Variable

m-Erlang Random Variable: $\alpha = m$, a positive integer

$$f_X(x) = \frac{\lambda e^{-\lambda x}(\lambda x)^{m-1}}{(m-1)!} \qquad x > 0$$

$$\Phi_X(\omega) = \left(\frac{\lambda}{\lambda - j\omega}\right)^m$$

Remarks: An m-Erlang random variable is obtained by adding m independent exponentially distributed random variables with parameter λ.

Properties of the Poisson Process (cont.)

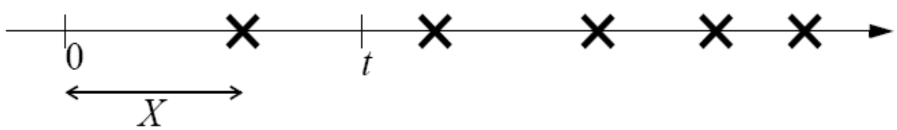
- If exactly one event occurs in $[0, t]$, the time it occurs X is uniformly distributed on $[0, t]$.

Proof: Assume $0 \le x \le t$,

$$F_X(x) = P[N(x) = 1 | N(t) = 1] = \frac{P[N(x) = 1 \cap N(t) = 1]}{P[N(t) = 1]}$$

$$= \frac{P[N(x) = 1]P[N(t) - N(x) = 0]}{P[N(t) = 1]} = \frac{P[N(x) = 1]P[N(t - x) = 0]}{P[N(t) = 1]}$$

$$= \frac{\lambda x e^{-\lambda x} e^{-\lambda(t-x)}}{\lambda t e^{-\lambda t}} = \frac{x}{t}$$



Thus, $f_X(x) = \frac{1}{t}$ for $x \in [0, t]$ and zero otherwise.

- If exactly n events occur in an interval, the arrival times are independent and uniformly distributed.
  - This justifies the statement that events occur "at random."

Example 9.22: Message Recordings

Inquiries arrive at a recorded message device according to a Poisson process with rate 15 inquiries per minute. Find the mean and variance of the time until the arrival of the 10^{th} inquiry.

Solution

The arrival rate in seconds is $\lambda = 15/60 = ¼$ inquiries per second.

Method 1:

The inter-arrival times are independent and exponential with parameter $\lambda = ¼$.

The mean and variance of an exponential random variable are $1/\lambda = 4$ and $1/\lambda^2 = 16$.

The arrival-time of the 10^{th} event is the sum of 10 i.i.d. random variables.

Thus, $E[S_{10}] = 10E[T] = 40$ sec and $\mathrm{Var}[S_{10}] = 10\mathrm{Var}[T] = 160 \text{ sec}^2$.

Method 2:

The arrival-time of the 10^{th} event is a 10-Erlang random variable

From the previous table, the mean and variance of a m-Erlang random variable are $m/\lambda = 4m$ and $m/\lambda^2 = 16m$.

Thus, $E[S_{10}] = 40$ sec and $\mathrm{Var}[S_{10}] = 160$ sec^2.

Example 9.23: Customer Arrivals

Customers arrive at a shop according to a Poisson process. If 2 customers arrive at the shop within a 2-minute period, what is the probability that both arrived during the first minute.

Solution

Method 1:

The arrival times are independent and uniformly distributed during the 2-minutes.

Thus, each customer arrives during the first minute with probability ½.

The probability that both customers arrive during the first minute is $(½)^2 = ¼$.

Method 2:

The probability that both customers arrive during the first minute is

$$P[N(1)=2|N(2)=2] = \frac{P[N(1)=2 \cap N(2)=2]}{P[N(2)=2]} = \frac{P[N(1)=2 \cap N(2)-N(1)=0]}{P[N(2)=2]}$$

$$= \frac{P[N(1)=2] \times P[N(2)-N(1)=0]}{P[N(2)=2]} = \frac{P[N(1)=2] \times P[N(2-1)=0]}{P[N(2)=2]}$$

$$= \frac{\frac{(\lambda\cdot 1)^2}{2!}e^{-\lambda\cdot 1} \times \frac{(\lambda\cdot 1)^0}{0!}e^{-\lambda\cdot 1}}{\frac{(\lambda\times 1)^2}{2!}e^{-\lambda\cdot 2}} = \frac{1}{4}$$

Second Order Density revisited

If $t_1 < t_2$ and $j \geq i$,

$$
\begin{aligned}
P[N(t_1)=k, N(t_2)=n] &= P[N(t_1)=k] \times P[N(t_2-t_1)=n-k] \\
&= \frac{(\lambda t_1)^k e^{-\lambda t_1}}{k!} \times \frac{(\lambda(t_2-t_1))^{(n-k)} e^{-\lambda(t_2-t_1)}}{(n-k)!} \\
&= \frac{(\lambda t_1)^k (\lambda(t_2-t_1))^{(n-k)}}{k!\,(n-k)!} \times e^{-\lambda t_2} \\
&= \frac{n!}{(\lambda t_2)^n} \frac{(\lambda t_1)^k (\lambda(t_2-t_1))^{(n-k)}}{k!\,(n-k)!} \times \frac{(\lambda t_2)^n}{n!} e^{-\lambda t_2} \\
&= \frac{n!}{k!\,(n-k)!} \frac{(\lambda t_1)^k (\lambda(t_2-t_1))^{(n-k)}}{(\lambda t_2)^k (\lambda t_2)^{n-k}} \times \frac{(\lambda t_2)^n}{n!} e^{-\lambda t_2} \\
&= \underbrace{\frac{n!}{k!\,(n-k)!} p^k (1-p)^{n-k}} \times \underbrace{\frac{(\lambda t_2)^n}{n!} e^{-\lambda t_2}} \qquad \text{where } p = \frac{t_1}{t_2}
\end{aligned}
$$

Do these distributions look familiar?

Can you explain why the second order pmf can be expressed like this?

Major Points from this Lecture:

- Sum Processes
- ISI Processes
  - Independent Increments
  - Stationary Increments
- Important Processes
  - Binomial Counting Process
  - Poisson Process

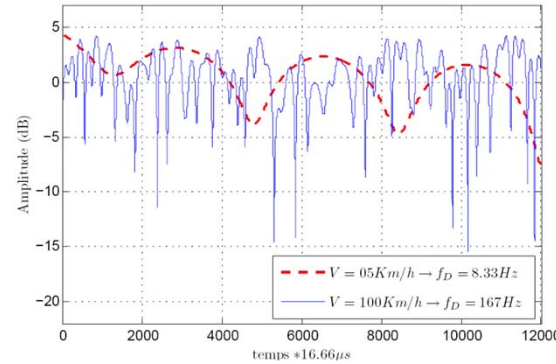
ELEC 2600: Probability and Random Processes in Engineering

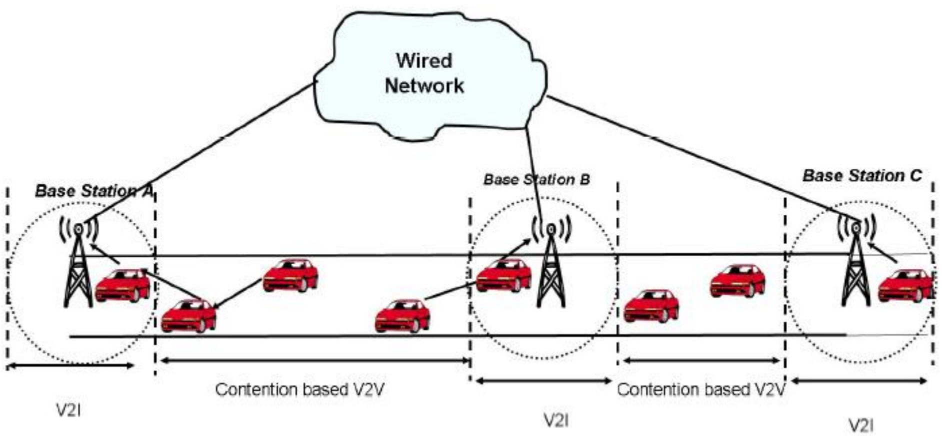
Part IV: Stochastic Process

- Lecture 21: Definition of a Random Process
- Lecture 22: Sum Processes and Independent Stationary Increment Processes
- **Lecture 23: Mean and Autocorrelation of Random Process**
- Lecture 24: Stationary Random Process

Elec2600H: Lecture 23

- **Mean and Variance Functions**
- Correlation and Covariance Functions





Mean and Variance Functions

- **Mean function**

$$
\begin{aligned}
m_X(t) &= E[X(t)] \\
&= \int x f_{X(t)}(x)dx
\end{aligned}
$$

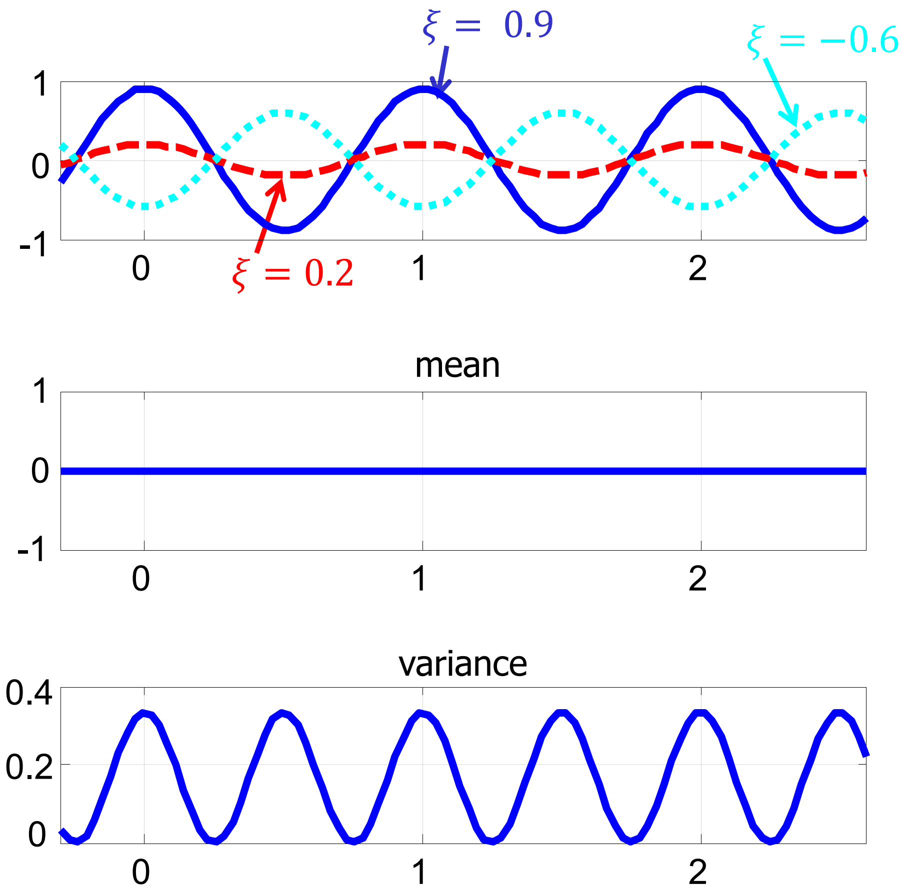
- **Variance function**

$$
\begin{aligned}
\mathrm{Var}[X(t)] &= E\left[\left(X(t) - m_X(t)\right)^2\right] \\
&= \int (x - m_X(t))^2 f_{X(t)}(x)dx \\
&= E[X(t)^2] - m_X(t)^2
\end{aligned}
$$

- Note that the mean and variances are functions of time.

Example: Random Amplitude Sinusoid

Find the mean and variance functions of $X(t,\xi) = \xi\cos(2\pi t)$ where ξ is selected at random from $[-1,1]$.



Solution

$$\begin{aligned} m_X(t) &= E[X(t)] \\ &= E[\xi\cos(2\pi t)] \\ &= E[\xi]\cos(2\pi t) = 0 \end{aligned}$$

$$\begin{aligned} \mathrm{Var}[X(t)] &= E[X(t)^2] = E[\xi^2\cos^2(2\pi t)] \\ &= E[\xi^2]\cos^2(2\pi t) = \frac{1}{3}\cos^2(2\pi t) \end{aligned}$$

$$E[\xi^2] = \frac{\left(1-(-1)\right)^2}{12} = \frac{1}{3}$$

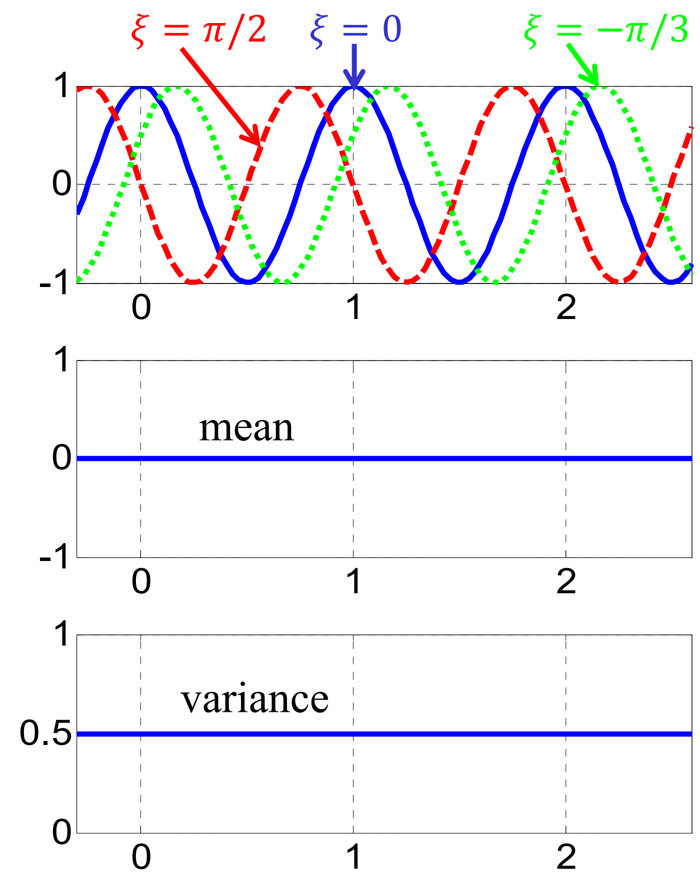
Example: Random Phase Sinusoid

Find the mean and variance functions of $X(t,\xi) = \cos(2\pi t + \xi)$ where ξ is random from $[-\pi, \pi]$.

Solution

$$
\begin{aligned}
m_X(t) &= E[X(t)] \\
&= \int_{-\infty}^{\infty} \cos(2\pi t + \xi)\, f(\xi) d\xi \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos(2\pi t + \xi)\, d\xi \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Var}[X(t)] &= E[X(t)^2] \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos^2(2\pi t + \xi) d\xi \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left\{\frac{1}{2} + \frac{1}{2}\cos(4\pi t + 2\xi)\right\} d\xi \\
&= \frac{1}{2}
\end{aligned}
$$



Mean and Variance of an **I.I.D.** Process

- The **mean** of an i.i.d. process is **constant**: $m_X(n) = m$

Proof:

$$
\begin{aligned}
m_X(n) &= E[X(n)] \\
&= \int x f_{X(n)}(x) dx \\
&= \int x f(x) dx \quad \text{since } f_{X(n)}(x) = f(x) \text{ for all } n. \\
&= m
\end{aligned}
$$

- The **variance** of an i.i.d. process is also **constant**: $\mathrm{Var}[X_n] = \sigma^2$

Proof: Similar to above

The mean and variance of ISI processes grow linearly

Proof:

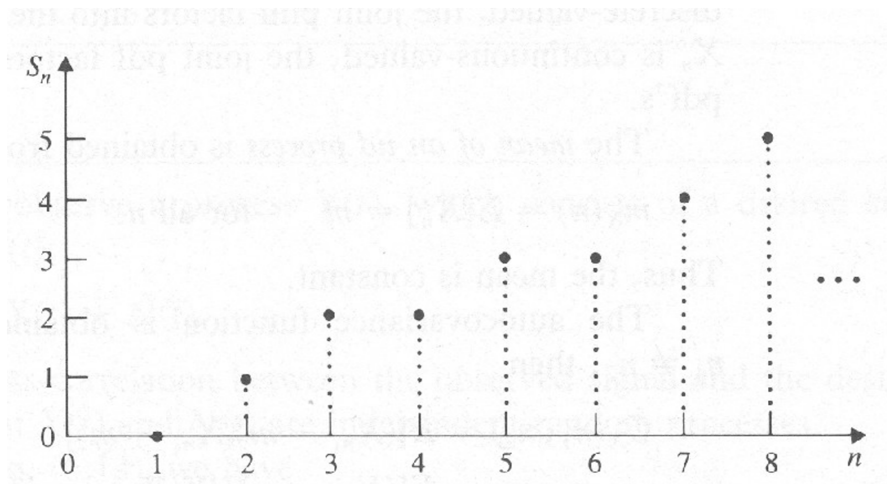
Suppose that S_n is a discrete time i.s.i. process with $S_0 = 0$, i.e. there exists an i.i.d. process X_n mean μ and variance σ^2 such that

$$S_n = \sum_{i=1}^{n} X_i$$

Using our results for sums of RV's, it is easy to prove that

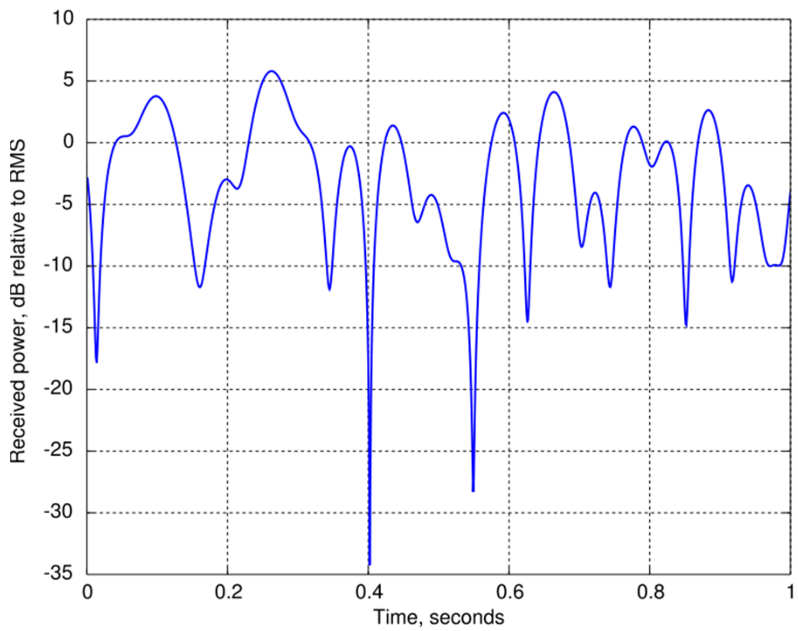
$$m_S(n) = E[S_n] \\ = n\mu$$

$$\mathrm{Var}[S_n] = E[(S_n - m_S(n))^2] \\ = n\sigma^2$$



Elec2600H: Lecture 23

- Mean and Variance Functions
- **Autocorrelation and Covariance Functions**



Autocorrelation and Covariance

- **Autocorrelation** : $R_X(t_1, t_2) = E[X(t_1)X(t_2)] = \int\int xy f_{X(t_1),X(t_2)}(x, y)dx$

- **Covariance** :

$$\mathrm{Var}[X(t)] = C_X(t, t)$$

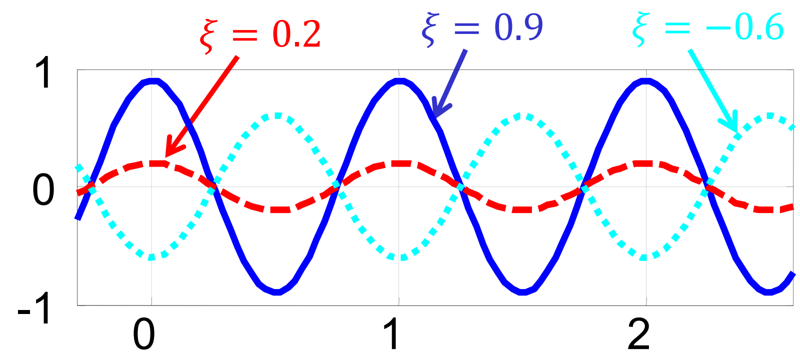
$$\begin{aligned} C_X(t_1, t_2) &= \mathrm{Cov}(X(t_1), X(t_2)) \\ &= E[(X(t_1) - m_X(t_1))(X(t_2) - m_X(t_2)] \\ &= R_X(t_1, t_2) - m_X(t_1)m_X(t_2) \end{aligned}$$

- **Correlation coefficient** : $\rho_X(t_1, t_2) = \frac{C_X(t_1,t_2)}{\sqrt{C_X(t_1,t_1)}\sqrt{C_X(t_2,t_2)}}$

- Note: It is possible for two *different* random processes to have the *same* mean, autocorrelation and covariance functions.
- Note: The covariance function is sometimes also called the **autocovariance** function.

Example 6.6: Random Amplitude Sinusoid

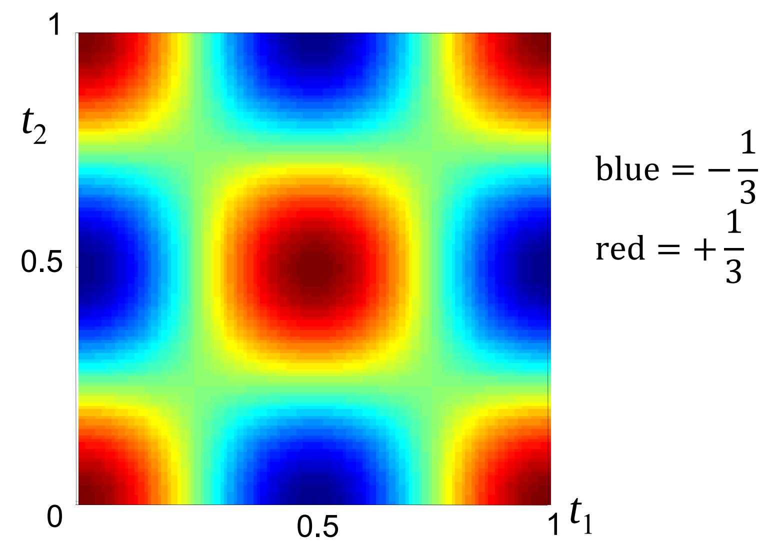
Find the autocorrelation and covariance functions of $X(t,\xi) = \xi\cos(2\pi t)$ where ξ is selected at random from the interval [-1, 1].



Solution

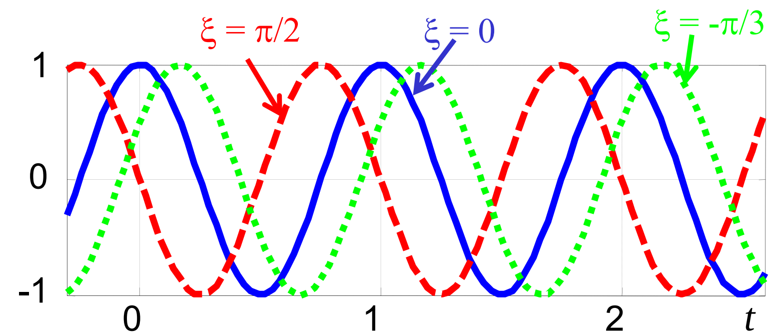
Since $m_X(t) = 0$, $R_X(t_1,t_2) = C_X(t_1,t_2)$.

$$
\begin{aligned}
R_X(t_1,t_2) &= E[X(t_1)X(t_2)] \\
&= E[\xi\cos(2\pi t_1)\xi\cos(2\pi t_2)] \\
&= E[\xi^2]\cos(2\pi t_1)\cos(2\pi t_2) \\
&= \frac{1}{3}\cos(2\pi t_1)\cos(2\pi t_2)
\end{aligned}
$$



Example 6.7: Random Phase Process

Find the autocorrelation and covariance functions of $X(t,\xi) = \cos(2\pi t + \xi)$ where ζ is uniformly distributed over $[-\pi, \pi]$.

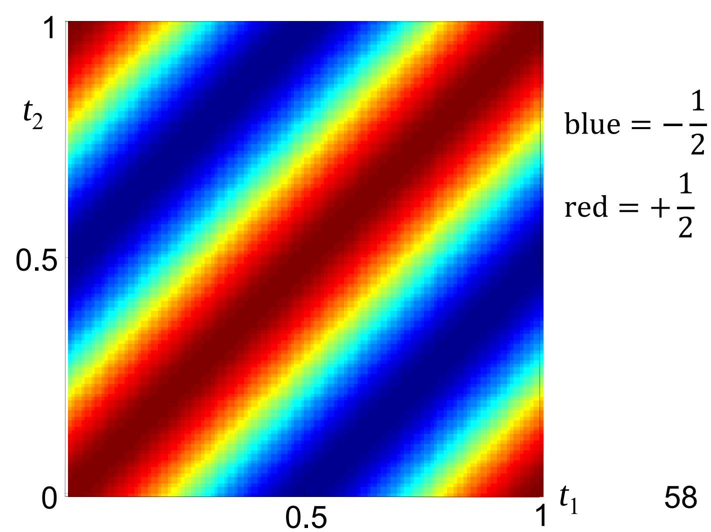


Solution

Since $m_X(t) = 0$, $R_X(t_1,t_2) = C_X(t_1,t_2)$

$$\begin{aligned}
R_X(t_1,t_2) &= E[X(t_1)X(t_2)] \\
&= E[\cos(2\pi t_1 + \xi)\cos(2\pi t_2 + \xi)] \\
&= E\left[\tfrac{1}{2}\cos(2\pi(t_1 - t_2)) + \tfrac{1}{2}\cos(2\pi(t_1 + t_2) + 2\xi)\right] \\
&= \tfrac{1}{2}E[\cos(2\pi(t_1 - t_2))] + \underbrace{\tfrac{1}{2}E[\cos(2\pi(t_1 + t_2) + 2\xi)]}_{=0} \\
&= \tfrac{1}{2}\cos(2\pi(t_1 - t_2))
\end{aligned}$$

$$\boxed{\cos(a)\cos(b) = \frac{1}{2}(\cos(a - b) + \cos(a + b))}$$



Covariance Function of an **I.I.D.** Process

- Suppose that X_n is an i.i.d. process, then $C_X(m,n) = \sigma^2 \delta(m,n)$, where $\delta(m,n) = \begin{cases} 1 & m = n \\ 0 & m \neq n \end{cases}$

- In other words, the covariance function of an i.i.d. process is a **delta function**.

Proof

$$C_X(n_1, n_2) = \mathrm{Cov}\left(X_{n_1}, X_{n_2}\right)$$

$$= \begin{cases} \mathrm{Var}\left[X_{n_1}\right] & \text{if } n_1 = n_2 \\ 0 & \text{if } n_1 \neq n_2 \end{cases} \quad \text{(by independence)}$$

$$= \begin{cases} \sigma^2 & \text{if } n_1 = n_2 \\ 0 & \text{if } n_1 \neq n_2 \end{cases}$$

Correlation Function of an **I.I.D.** Process

Suppose that X_n is an i.i.d. process with mean $\mu = E[X_n]$, then $R_X(m,n) = \sigma^2\delta(m,n) + \mu^2$.

Proof

Since $m_X(n) = \mu$ for an i.i.d. random process,

$$
\begin{aligned}
R_X(m,n) &= C_X(m,n) + m_X(m) \cdot m_X(n) \\
&= \sigma^2\delta(m,n) + \mu^2 \\
&= \begin{cases} \sigma^2 + \mu^2 & \text{if } m = n \\ \mu^2 & \text{if } m \neq n \end{cases}
\end{aligned}
$$

Covariance of i.s.i. process

- For an independent stationary increment (i.s.i.) process, $C_S(m,n) = \sigma^2 \min(m,n)$

Proof

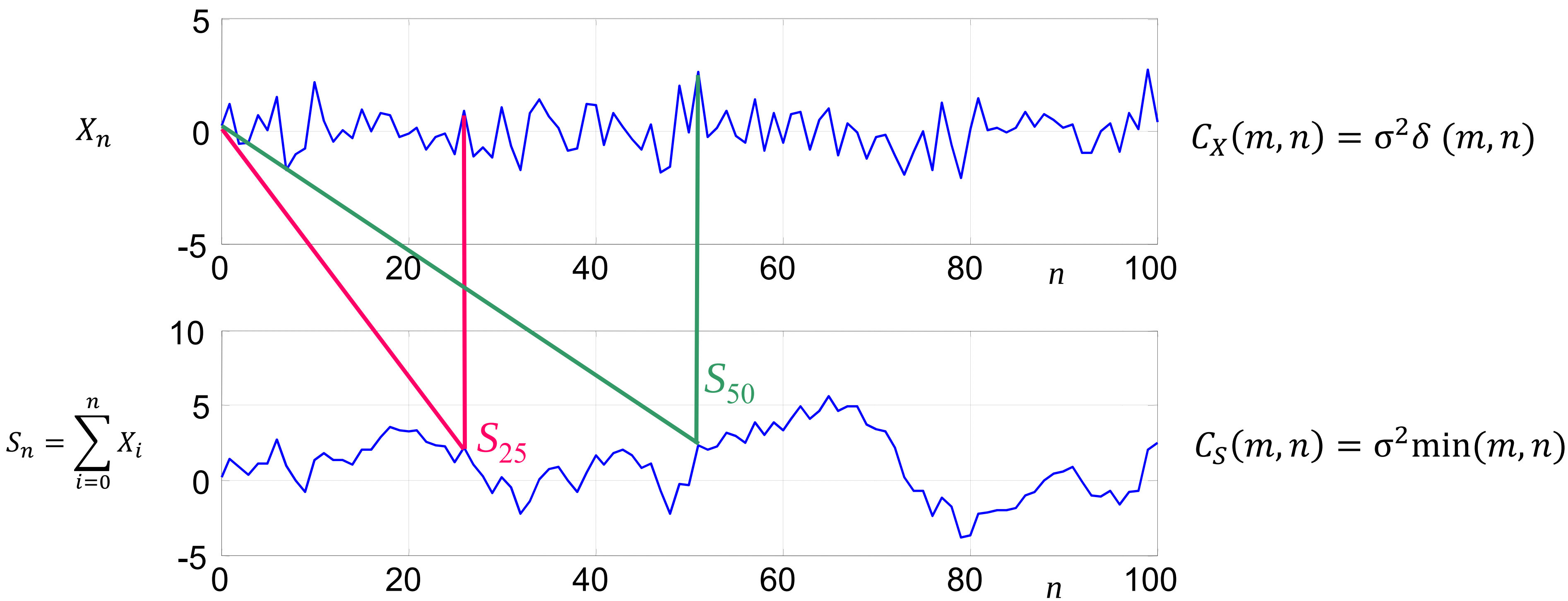
Assume first that $m \geq n$, then

$$
\begin{aligned}
C_S(m,n) &= E[S_m S_n] - E[S_m] \cdot E[S_n] \\
&= E[(S_n + S_m - S_n)S_n] - E[S_m] \cdot E[S_n] \\
&= E[S_n^2 + (S_m - S_n)S_n] - E[S_m] \cdot E[S_n] \\
&= E[S_n^2] + E[S_m - S_n] \cdot E[S_n] - E[S_m] \cdot E[S_n] \\
&= E[S_n^2] + E[S_m]E[S_n] - E[S_n] \cdot E[S_n] - E[S_m] \cdot E[S_n] \\
&= E[S_n^2] - E[S_n] \cdot E[S_n] \\
&= \mathrm{VAR}[S_n] = \sigma^2 n
\end{aligned}
$$

independent increments

A similar argument for $n \geq m$ yields $C_S(m,n) = \sigma^2 m$

The covariance function of an i.s.i. process depends upon the shared history



- S_{25} and S_{50} both contain X_1 to X_{25}.
- Since X_{26} to X_{50} are independent of X_1 to X_{25}, $\mathrm{COV}(S_{25}, S_{50}) = \mathrm{Var}(S_{25}) = 25\sigma^2$

Examples

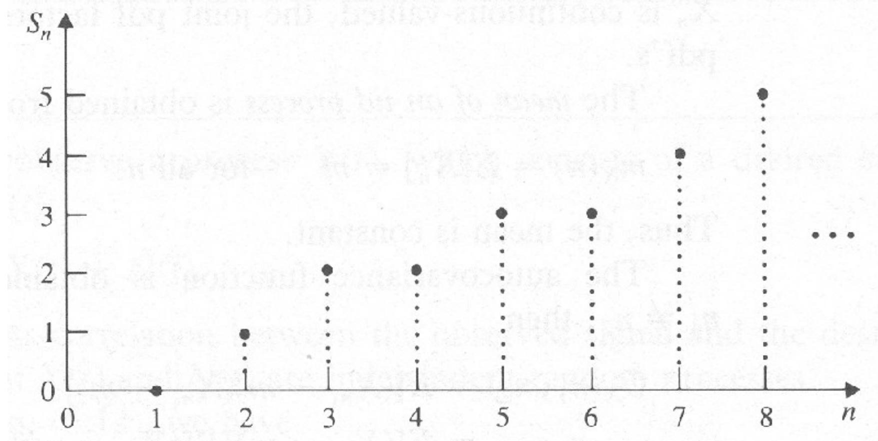
- Binomial Counting Process

$$p_{S_n}(k) = \binom{n}{k} p^k (1-p)^{n-k} \quad \text{for } k = 0,1,\dots,n$$

$$m_s(n) = np$$

$$\mathrm{Var}(S_n) = np(1-p)$$

$$C_S(n_1, n_2) = p(1-p) \cdot \min(n_1, n_2)$$



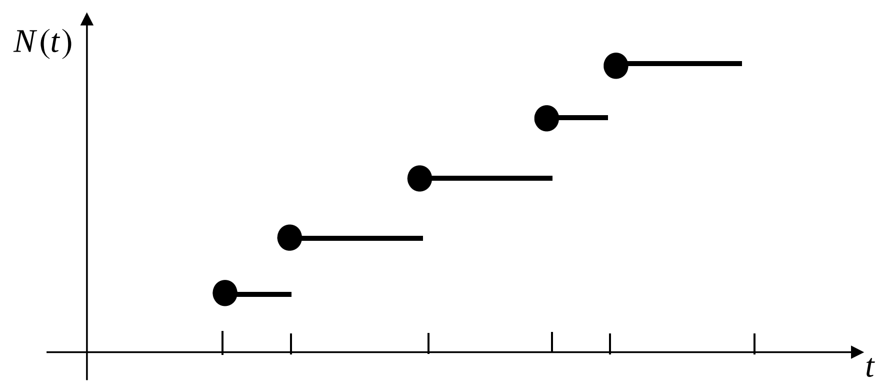
- Poisson Counting Process

$$p_{N(t)}(k) = \frac{(\lambda t)^k}{k!} e^{-\lambda t} \quad \text{for } k = 0,1,2,\dots$$

$$m_N(t) = \lambda t$$

$$\mathrm{Var}(N(t)) = \lambda t$$

$$C_N(t_1, t_2) = \lambda \cdot \min(t_1, t_2)$$



Gaussian Random Process

- **Definition:** A random process is said to be Gaussian if all finite order distributions are jointly Gaussian distributed, i.e., for any $k < \infty$ and any set of k sample times $t_1, t_2, \dots, t_k$,

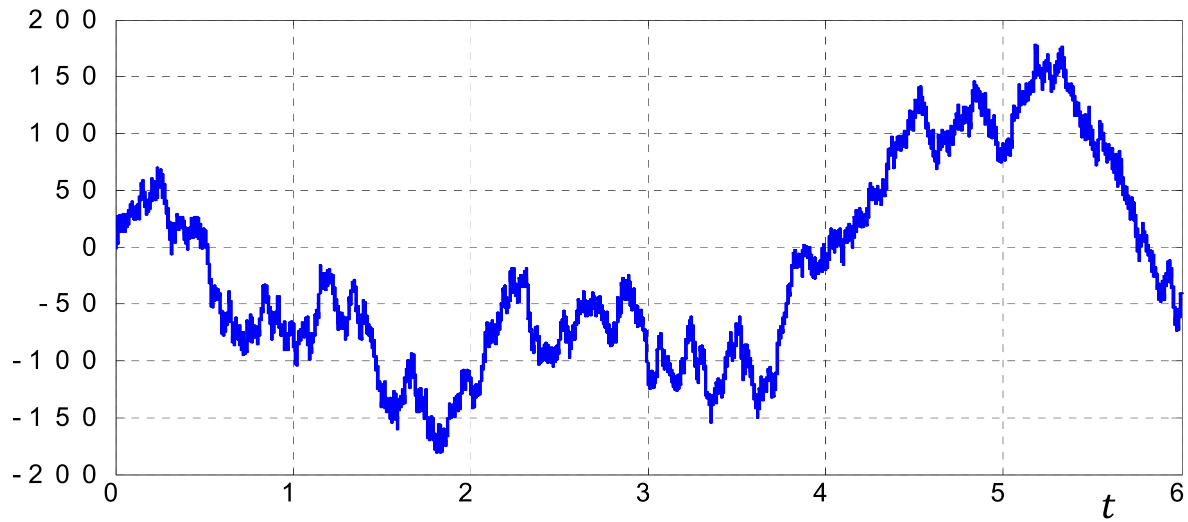
$$f_{X(t_1)X(t_2)\dots X(t_k)}(x_1, \dots, x_k) = \frac{1}{(2\pi)^{\frac{k}{2}}|C|^{\frac{1}{2}}} e^{-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{m})^T \boldsymbol{C}^{-\mathbf{1}}(\boldsymbol{x}-\boldsymbol{m})}$$

where $\boldsymbol{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_k \end{bmatrix}, \boldsymbol{m} = \begin{bmatrix} E[X(t_1)] \\ \vdots \\ E[X(t_k)] \end{bmatrix}$ and $C = \begin{bmatrix} C_X(t_1, t_1) & \cdots & C_X(t_1, t_k) \\ \vdots & \ddots & \vdots \\ C_X(t_k, t_1) & \cdots & C_X(t_k, t_k) \end{bmatrix}$

- Thus, Gaussian random processes are completely specified by their mean and autocovariance functions.

Example: Wiener Process

- The Wiener process $W(t)$ is a continuous time Gaussian i.s.i. random process with
  - Mean function: $m_W(t) = 0$
  - Covariance function: $C_W(t_1, t_2) = \sigma^2 \cdot \min(t_1, t_2)$
- It is commonly used to model quantities that change slowly over time, e.g.
  - Brownian motion, the motion of particle suspended in a fluid that moves due to rapid and random impacts from neighboring particles
  - Stock prices



Example

Suppose that $W(t)$ is a Wiener process with $\sigma^2 = 2$. Find the marginal density of $W(2)$ and the joint density of $W(2)$ and $W(5)$.

Solution

Since the Wiener process is a Gaussian process, all marginal/joint distributions are Gaussian. Thus, we only need to compute means and variances/covariance matrices.

Marginal density of $W(2)$: $f_{W(2)}(w) = \frac{1}{\sqrt{2\pi \cdot 2}} e^{-\frac{1}{2} \cdot \frac{w^2}{4}}$

$$E[W(2)] = 0$$
$$\mathrm{Var}[W(2)] = \sigma^2 2 = 4$$

Joint density of $\vec{X} = \begin{bmatrix} W(2) \\ W(5) \end{bmatrix}$: $f_{\vec{X}}(\vec{x}) = \frac{1}{2\pi \cdot |\boldsymbol{C}|^{0.5}} e^{-\frac{1}{2}\vec{x}^T \boldsymbol{C}^{-1} \vec{x}}$

$$E[\vec{X}] = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\boldsymbol{C} = \begin{bmatrix} C_W(2,2) & C_W(2,5) \\ C_W(2,5) & C_W(5,5) \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ 4 & 10 \end{bmatrix}$$
$$C_W(t_1, t_2) = \sigma^2 \min(t_1, t_2)$$

Major Points from this Lecture:

- Mean and variance of a random process
- Autocorrelation and covariance
- Important random processes to remember
  - I.I.D. random processes
  - I.S.I. random processes
  - Gaussian random processes

ELEC 2600: Probability and Random Processes in Engineering

Part IV: Stochastic Process

- Lecture 21: Definition of a Random Process
- Lecture 22: Sum Processes and Independent Stationary Increment Processes
- Lecture 23: Mean and Autocorrelation of Random Process
- **Lecture 24: Stationary Random Process**

Elec2600H: Lecture 24

- **Stationary Random Processes**
- Wide Sense Stationary (WSS) Random Processes
- Ergodic Process

Stationary Random Processes

Definition: A process is *stationary* if the joint distribution of any set of samples does not depend on the placement of the time origin, i.e.,

$$F_{X(t_1),X(t_2),\ldots X(t_k)}(x_1, x_2, \ldots, x_k) = F_{X(t_1+\Delta),X(t_2+\Delta),\ldots X(t_k+\Delta)}(x_1, x_2, \ldots, x_k)$$

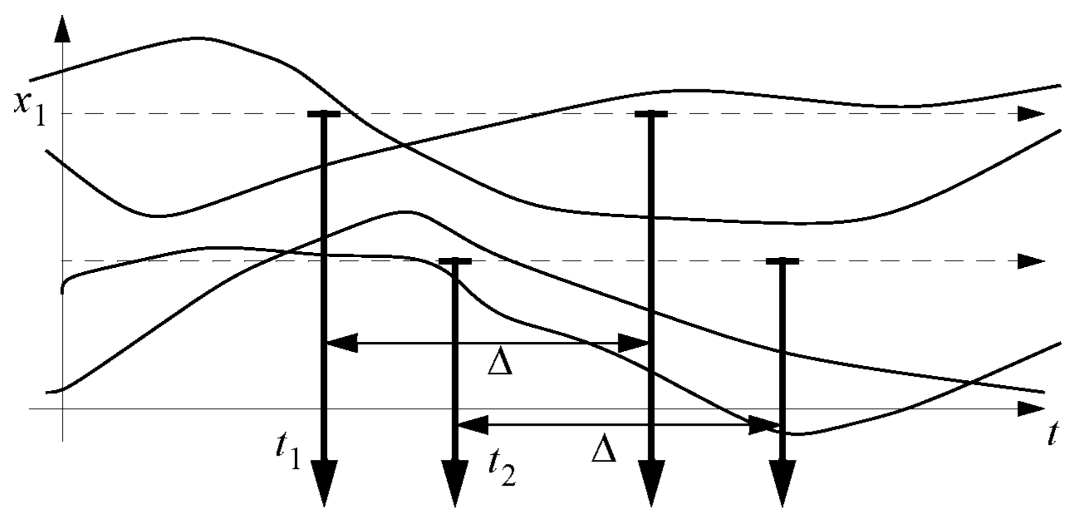
for all $x_1, x_2, \ldots, x_k, t_1, t_2, \ldots, t_k$ and Δ.

- For example,

  $F_{X(t_1)}(x_1) = F_{X(t_1+\Delta)}(x_1)$

  $F_{X(t_1),X(t_2)}(x_1, x_2) = F_{X(t_1+\Delta),X(t_2+\Delta)}(x_1, x_2)$

- Stationarity is like time invariance.



Example 9.31: I.I.D. Processes

The i.i.d. random process is stationary.

Proof

Since any set of samples from an i.i.d. random process are independent and identically distributed, then for all k and Δ and for all $x_1, x_2, \dots, x_k$ and $t_1, t_2, \dots, t_k$,

$$F_{X(t_1),X(t_2),\dots X(t_k)}(x_1, x_2, \dots, x_k) = F_X(x_1) \times F_X(x_2) \times \dots \times F_X(x_k)$$
$$F_{X(t_1+\Delta),X(t_2+\Delta),\dots X(t_k+\Delta)}(x_1, x_2, \dots, x_k) = F_X(x_1) \times F_X(x_2) \times \dots \times F_X(x_k)$$

where $F_X(x)$ is the marginal distribution (which is the same for all samples).

Thus, $F_{X(t_1),X(t_2),\dots X(t_k)}(x_1, x_2, \dots, x_k) = F_{X(t_1+\Delta),X(t_2+\Delta),\dots X(t_k+\Delta)}(x_1, x_2, \dots, x_k)$.

Mean and Variance

The mean and variance of stationary processes are constant.

Proof:

If $X(t)$ is stationary, then $f_{X(t)}(x) = f(x)$.

Thus,

$$
\begin{aligned}
m_X(t) &= \int x \cdot f_{X(t)}(x)dx \\
&= \int x \cdot f(x)dx = m
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Var}[X(t)] &= \int (x-m)^2 f_{X(t)}(x)dx \\
&= \int (x-m)^2 f(x)dx = \sigma^2
\end{aligned}
$$

Example 9.32: Sum Process

Is the sum process a discrete-time stationary process?

Solution

The sum process is defined by $S_n = X_1 + X_2 + \cdots + X_n$ where the X_n are i.i.d. with mean m and variance σ^2.

The mean and variance of S_n increase linearly with time: $m_S(n) = nm$ and $\mathrm{Var}[S_n] = n\sigma^2$

Thus, it is **not** stationary.

Autocorrelation/covariance of a stationary process

The autocorrelation and covariance functions of a stationary process depend only upon the **time difference** $t_1 - t_2$.

We often write them functions of $t_1 - t_2$: $R_X(t_1, t_2) = R_X(t_1 - t_2)$ and $C_X(t_1, t_2) = C_X(t_1 - t_2)$

Proof:

Suppose $X(t)$ is a stationary random process.

$$
\begin{aligned}
R_X(t_1, t_2) &= \int\int x_1 x_2 f_{X(t_1),X(t_2)}(x_1, x_2) dx_1\, dx_2 \\
&= \int\int x_1 x_2 f_{X(t_1+\Delta),X(t_2+\Delta)}(x_1, x_2) dx_1\, dx_2 \\
&= R_X(t_1 + \Delta, t_2 + \Delta)
\end{aligned}
$$

Since the mean of a stationary process is constant,

$$
\begin{aligned}
C_X(t_1, t_2) &= R_X(t_1, t_2) - m_X(t_1) m_X(t_2) \\
&= R_X(t_1 - t_2) - m_X^2
\end{aligned}
$$

Covariance Function of an **I.I.D.** Process

- The **covariance** of an i.i.d. process is a **delta function**:

$$C_X(n_1, n_2) = \sigma^2 \delta(n_1 - n_2)$$

where $\delta(k) = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{if } k \neq 0 \end{cases}$

Proof

$$\begin{aligned} C_X(n_1, n_2) &= \text{Cov}\left(X_{n_1}, X_{n_2}\right) \\ &= \begin{cases} \text{Var}\left[X_{n_1}\right] & \text{if } n_1 = n_2 \\ 0 & \text{if } n_1 \neq n_2 \end{cases} \\ &= \begin{cases} \sigma^2 & \text{if } n_1 = n_2 \\ 0 & \text{if } n_1 \neq n_2 \end{cases} \\ &= \begin{cases} \sigma^2 & \text{if } n_1 - n_2 = 0 \\ 0 & \text{if } n_1 - n_2 \neq 0 \end{cases} \end{aligned}$$

Elec2600H: Lecture 24

- Stationary Random Processes
- **Wide Sense Stationary (WSS) Random Processes**
- Ergodic Process

Wide sense stationarity

- In many situations, it is difficult to determine whether or not a random process is stationary, but we can determine the mean and autocorrelation functions.

- **Definition:** $X(t)$ is *wide sense stationary* (WSS) if and only if its mean is constant and its correlation function $R_X(t_1, t_2)$ (or its covariance function) depends only on the time difference $t_1 - t_2$.

$$m_X(t) = m \text{ for all } t$$

$$R_X(t_1, t_2) = R_X(t_1 - t_2) \text{ for all } t_1, t_2$$

$$C_X(t_1, t_2) = C_X(t_1 - t_2) \text{ for all } t_1, t_2$$

- All stationary processes are wide sense stationary.
- The converse is not true (see next example).

Example 9.34: Interleaved Processes

Let X_n consist of two interleaved i.i.d. sequences of random variables.

- For n even, X_n assumes values +1 or -1 with

$$P[X_n = k] = \begin{cases} \frac{1}{2} & \text{if } k = +1 \\ \frac{1}{2} & \text{if } k = -1 \end{cases}$$

- For n odd, X_n assumes values +1/3 or -3 with $P[X_n = +1/3] = 9/10$ and $P[X_n = -3] = 1/10$

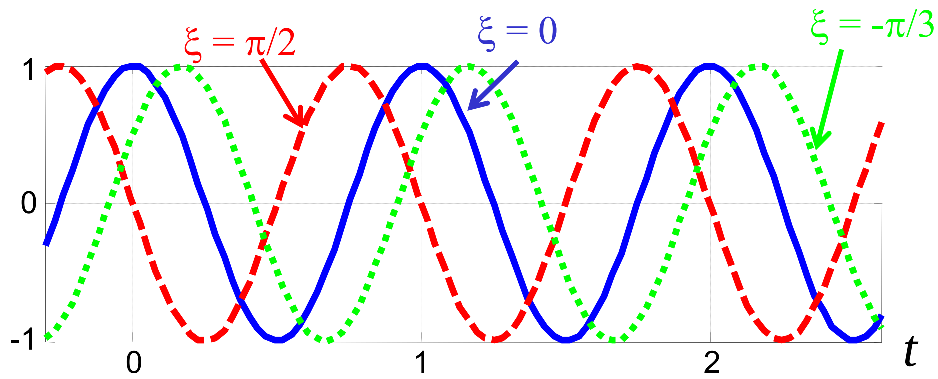
$$P[X_n = k] = \begin{cases} \frac{9}{10} & \text{if } k = +\frac{1}{3} \\ \frac{1}{10} & \text{if } k = -3 \end{cases}$$

X_n is **not stationary** since its pmf varies with n. However, X_n is **wide sense stationary** since its mean is constant and its covariance depends only on the time difference:

$$m_X(n) = 0 \qquad\qquad C_X(i,j) = \begin{cases} E[X_i]E[X_j] = 0 & \text{if } i \neq j \\ E[{X_i}^2] = 1 & \text{if } i = j \end{cases}$$

Example: Random Phase Process

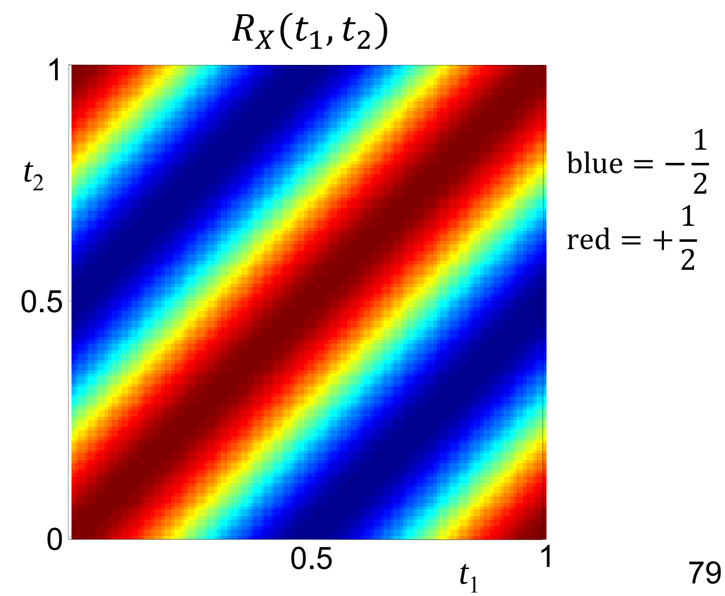
The random phase process is given by $X(t_0, \xi) = \cos(2\pi t + \xi)$ where ξ is random on $[-\pi, \pi]$.



The random phase process is WSS since

- $m_X(t) \;=\; 0$ for all t
- $R_X(t_1, t_2) = \frac{1}{2}\cos(2\pi(t_1 - t_2))$ for all t_1, t_2

However, the random phase process is **not** stationary.



Gaussian Random Processes

- **Definition:** A random process is said to be Gaussian if all finite order distributions are jointly Gaussian distributed.
- Gaussian random processes are completely specified by their **mean and covariance** functions.

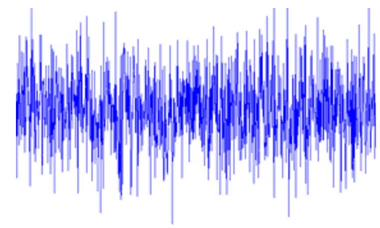
- For example, for a discrete-time Gaussian random process with mean function $m_X(n)$ and covariance function $C_X(m,n)$, the kth order density is given by

$$f_{X(t_1),\ldots,X(t_k)}(x_1,\ldots,x_k) = \frac{1}{(2\pi)^{k/2}|C|^{1/2}} e^{-\frac{1}{2}(\vec{x}-\vec{m})^T C^{-1}(\vec{x}-\vec{m})}$$

$$\text{where } \vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_k \end{bmatrix} \text{ and } \vec{m} = \begin{bmatrix} m_X(t_1) \\ \vdots \\ m_X(t_k) \end{bmatrix} \text{ and } C = \begin{bmatrix} C_X(t_1,t_1) & \cdots & C_X(t_1,t_n) \\ \vdots & \ddots & \vdots \\ C_X(t_n,t_1) & \cdots & C_X(t_n,t_n) \end{bmatrix}$$

Example: I.I.D. Gaussian Process

Let X_n be a discrete time wide sense stationary Gaussian random process with mean and covariances given by $m_X(n) = 0$ and $C_X(n_1, n_2) = \sigma^2 \delta(n_1 - n_2)$.



The joint pdf for X_1 and X_2 is $f_{X_1 X_2}(x_1, x_2) = \frac{1}{2\pi |C|^{1/2}} e^{-\frac{1}{2}(\vec{x}-\vec{m})^T C^{-1} (\vec{x}-\vec{m})}$

where $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $\vec{m} = \begin{bmatrix} m_X(1) \\ m_X(2) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $C = \begin{bmatrix} C_X(1,1) & C_X(1,2) \\ C_X(2,1) & C_X(2,2) \end{bmatrix} = \begin{bmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{bmatrix}$

$$|C| = \sigma^4$$

$$C^{-1} = \begin{bmatrix} \frac{1}{\sigma^2} & 0 \\ 0 & \frac{1}{\sigma^2} \end{bmatrix}$$

Substituting, we obtain $f_{X_1 X_2}(x_1, x_2) = \frac{1}{2\pi\sigma^2} e^{-\frac{x_1{}^2 + x_2{}^2}{2\sigma^2}}$

This is the same Gaussian white noise process we studied earlier, just described differently.

Example: WSS Gaussian Process

Let $X(t)$ be a continuous time wide sense stationary Gaussian random process with mean and covariances given by $m_X(t) = 0$ and $C_X(t_1, t_2) = 4e^{-0.5|t_1 - t_2|}$. Find the joint pdfs for the following samples: $X(10)$ and $X(11)$, $X(10)$ and $X(14)$, and the triple $X(10)$, $X(11)$ and $X(14)$.

Solution: All pdf's have the same form $\frac{1}{2\pi|C|^{1/2}} e^{-\frac{1}{2}(\vec{x}-\vec{m})^T C^{-1}(\vec{x}-\vec{m})}$ just with different $\vec{m}$ and C.

For $X(10)$ and $X(11)$, $\vec{m} = \begin{bmatrix} m_X(10) \\ m_X(11) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $C = \begin{bmatrix} C_X(10,10) & C_X(10,11) \\ C_X(11,10) & C_X(11,11) \end{bmatrix} = \begin{bmatrix} 4 & 4e^{-0.5} \\ 4e^{-0.5} & 4 \end{bmatrix} \approx \begin{bmatrix} 4 & 2.4 \\ 2.4 & 4 \end{bmatrix}$

For $X(10)$ and $X(14)$, $\vec{m} = \begin{bmatrix} m_X(10) \\ m_X(14) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $C = \begin{bmatrix} C_X(10,10) & C_X(10,14) \\ C_X(14,10) & C_X(14,14) \end{bmatrix} = \begin{bmatrix} 4 & 4e^{-2} \\ 4e^{-2} & 4 \end{bmatrix} \approx \begin{bmatrix} 4 & 0.5 \\ 0.5 & 4 \end{bmatrix}$

For $X(10)$, $X(11)$ and $X(14)$,

$$\vec{m} = \begin{bmatrix} m_X(10) \\ m_X(11) \\ m_X(14) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ and } C = \begin{bmatrix} C_X(10,10) & C_X(10,11) & C_X(10,14) \\ C_X(11,10) & C_X(11,11) & C_X(11,14) \\ C_X(14,10) & C_X(14,11) & C_X(14,14) \end{bmatrix} = \begin{bmatrix} 4 & 4e^{-0.5} & 4e^{-2} \\ 4e^{-0.5} & 4 & 4e^{-1.5} \\ 4e^{-2} & 4e^{-1.5} & 4 \end{bmatrix} \approx \begin{bmatrix} 4 & 2.4 & 0.5 \\ 2.4 & 4 & 0.9 \\ 0.5 & 0.9 & 4 \end{bmatrix}$$

Properties of the Autocorrelation of a WSS Process

Suppose $X(t)$ is WSS with autocorrelation $R_X(\tau)$ where $\tau = t_1 - t_2$.

- $R_X(0)$ is the *average power* of the process, $E[X(t)^2]$.

- $R_X(\tau)$ is an even function of τ.

- $|R_X(\tau)| \leq R_X(0)$ (the autocorrelation function is maximum at the origin)

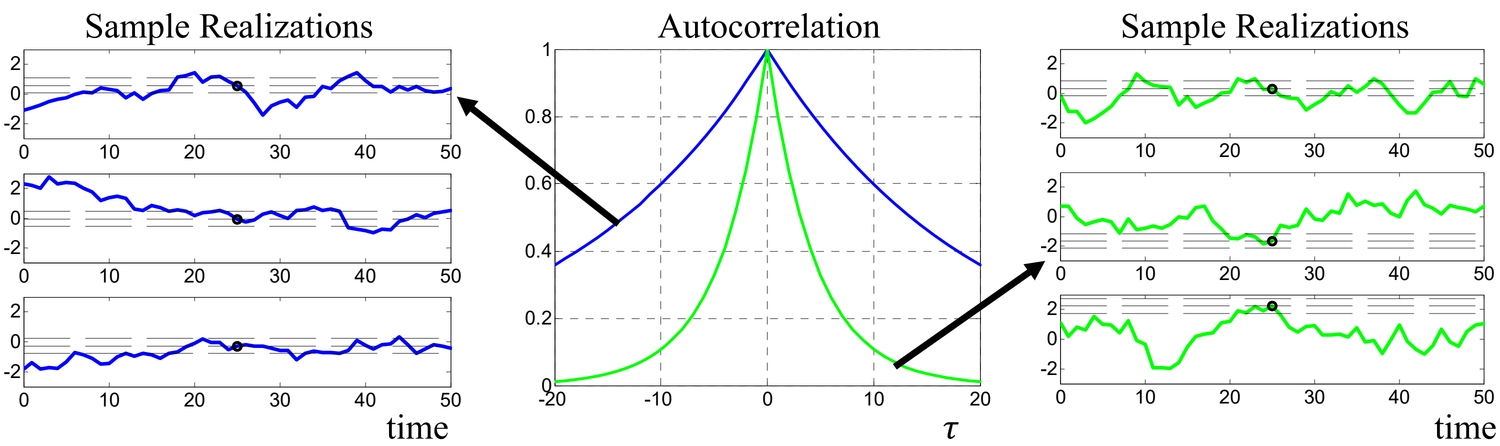
- If $X(t)$ is Gaussian, it is also stationary.

Properties of the Autocorrelation of a WSS Process

- The slower the autocorrelation decays as $\tau \to \infty$, the slower the realizations of the process change.

Proof:

$$
\begin{aligned}
P[|X(t+\tau) - X(t)| > \varepsilon] &= P[|X(t+\tau) - X(t)|^2 > \varepsilon^2] \\
&\le \frac{E[(X(t+\tau) - X(t))^2]}{\varepsilon^2} \\
&= \frac{2(R_X(0) - R_X(\tau))}{\varepsilon^2}
\end{aligned}
$$



Examples (discrete time)

- Let $X_n = x$ for all n, where x is chosen randomly over [0,1].
  - This is a **very** slowly varying process. It does not change at all over time.
  - X_n is wide sense stationary with constant autocorrelation:

$$\begin{aligned} R_X(n_1, n_2) &= E[X(n_1)X(n_2)] \\ &= E[x^2] \\ &= 1/3 \end{aligned}$$

- Let X_n be an i.i.d. random process with zero mean and variance σ^2.
  - This is a **very** quickly varying process.
  - X_n is wide sense stationary with autocorrelation:

$$\begin{aligned} R_X(n_1, n_2) &= E[X(n_1)X(n_2)] = \sigma^2\delta(n_1 - n_2) \\ &= \begin{cases} \sigma^2 & \text{if } n_1 - n_2 = 0 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

  - This autocorrelation decays quickly to zero.

Example 9.36

Let X_n be an i.i.d. Gaussian random process with zero mean and variance σ^2.

Let Y_n be the average of two consecutive values of X_n : $Y_n = \frac{X_n + X_{n-1}}{2}$

Y_n is a Gaussian random process with mean and covariance functions given by

$$m_Y(n) = E[Y_n] = E\left[\frac{X_n + X_{n-1}}{2}\right] = 0$$

$$\begin{aligned} C_Y(i,j) &= E[Y_i Y_j] = \tfrac{1}{4} E\big[(X_i + X_{i-1})(X_j + X_{j-1})\big] \\ &= \tfrac{1}{4}\big(E[X_i X_j] + E[X_i X_{j-1}] + E[X_{i-1} X_j] + E[X_{i-1} X_{j-1}]\big) \\ &= \begin{cases} \tfrac{1}{2}\sigma^2 & \text{if } i = j \\ \tfrac{1}{4}\sigma^2 & \text{if } |i-j| = 1 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Elec2600H: Lecture 24

- Stationary Random Processes
- Wide Sense Stationary (WSS) Random Processes
- **Ergodic Process**

Averages, Autocorrelation and Ergodic

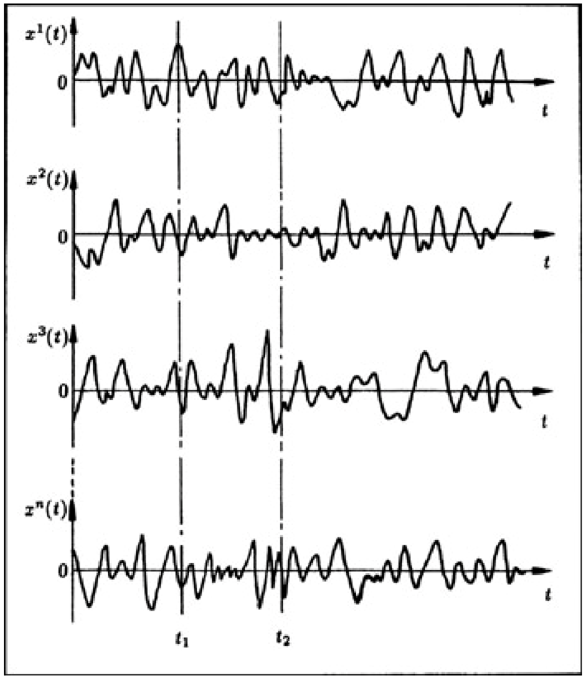
$$X_1 = X(t_1) \quad X_2 = X(t_2)$$

$$E[X(t)] = \int_{-\infty}^{\infty} x f(x,t)\,dx$$

Ensemble Average

$$\begin{aligned} E[X_1 X_2] &\triangleq E[X(t_1)X(t_2)] \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 f(x_1,x_2;t_1,t_2)\,dx_1 dx_2 \\ &= R_{XX}(t_1,t_2) \end{aligned}$$

Autocorrelation



- The physical meaning of $R_X(t, \tau)$ is a **measure** of the **relationship between** the two variables $X(t)$ and $X(t + \tau)$.

- Due to the randomness of $X(t)$, and by fixing the value of t, one should expect that the magnitude of correlation function decreases with increasing τ.

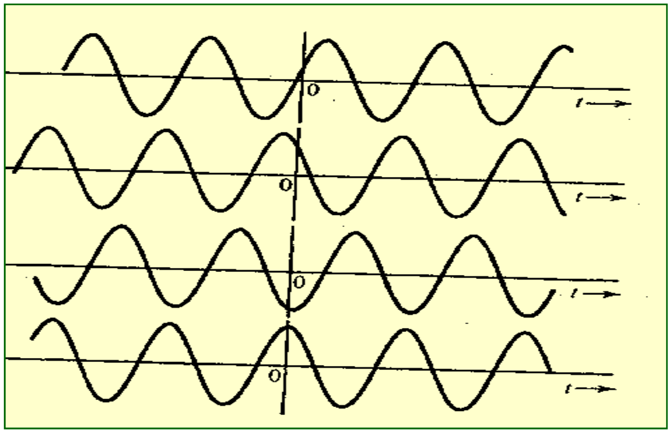
- In general, the correlation function is a function of both t and τ.

We already know

- For a ***stationary process*** all the statistics (e.g., **mean, correlation functions, and all higher order statistics**) are **independent of time** t. ***Else the process is non-stationary***

- A **WSS process** is **characterized** by a ***constant mean*** and a **one-dimensional autocorrelation function** that **depends on only the *time difference*** t. (~ **Weakly Stationary Process**).

Example : $X(t) = A\cos(2\pi f_0 t + \theta)$ where $\theta \sim U(0, 2\pi)$



$$E[X(t)] = E[A\cos(2\pi f_0 t + \theta)]$$
$$= \frac{A}{2\pi}\int_0^{2\pi}\cos(2\pi f_0 t + \theta)d\theta$$
$$= 0$$

$$R_X(t_1, t_2) = E[A\cos(2\pi f_0 t_1 + \theta)\cdot A\cos(2\pi f_0 t_2 + \theta)]$$
$$= A^2 E\left[\frac{1}{2}\cos(2\pi f_0 (t_2 - t_1)) + \frac{1}{2}\cos(2\pi f_0 (t_1 + t_2) + 2\theta)\right]$$
$$= \frac{A^2}{2}\cos(2\pi f_0 (t_2 - t_1))$$

Only depends on time Difference $\tau = t_2 - t_1$.

***X(t)* is thus WSS**

Time Averages

$$< X(t) > \triangleq \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} X(t)\,dt$$

$$\equiv \text{DC component}$$

$$< X^2(t) >$$

$$\triangleq \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} X^2(t)\,dt \equiv \text{Total Power}$$

Ergodic Processes

- For an ergodic process, all possible ensemble averages are equal to the corresponding time averages of one sample function.

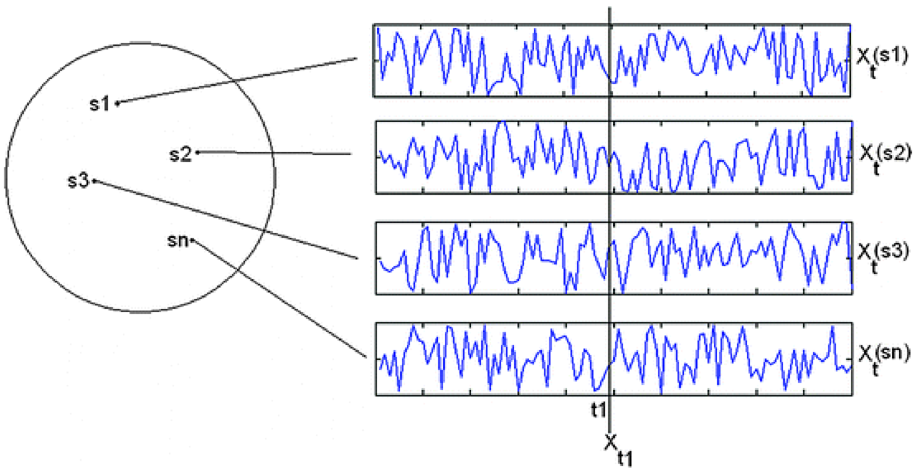
- In general, it is not easy to test if a process is ergodic ~ Must test all possible orders of time and ensemble averages.

- In practice, many of the stationary processes are ergodic with respect at least 2^{nd} order averages (e.g., mean & autocorrelation) ~ *Typically what we need*.

- Ergodicity is extremely important as we do not have a large number of sample functions available in practice from which to compute ensemble averages.

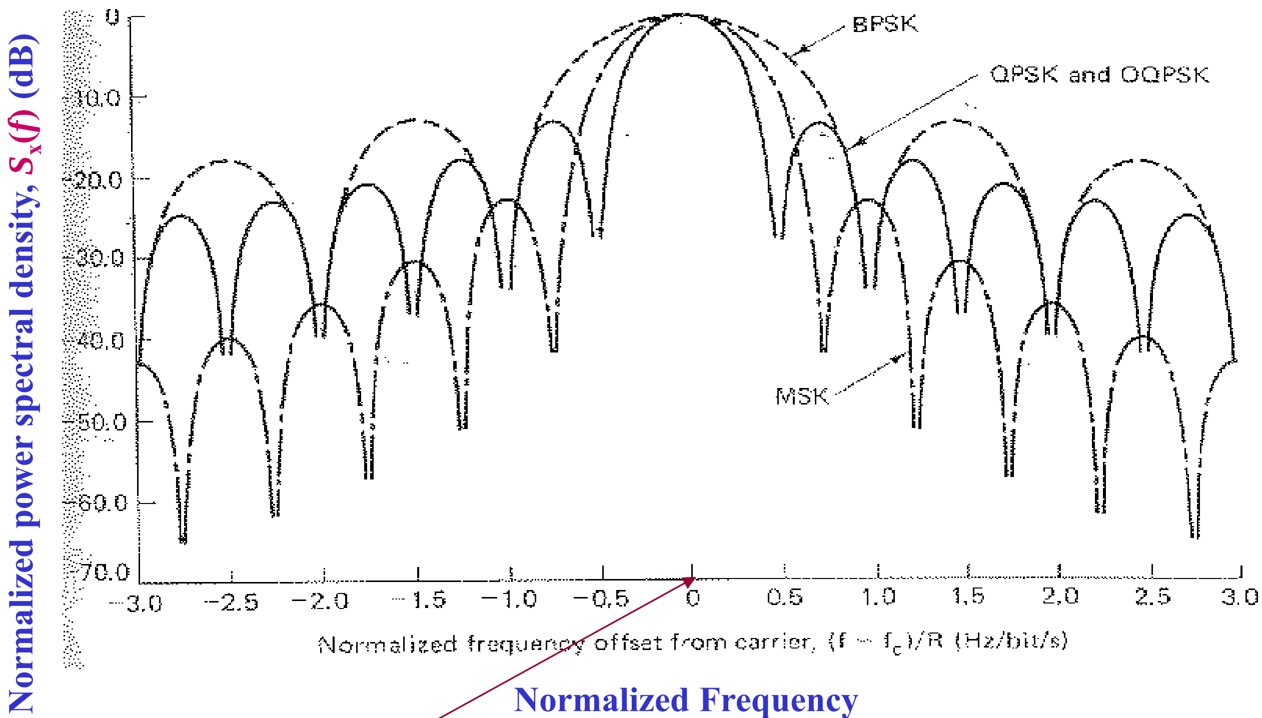
- A r.p. $X(t)$ is said to be **Ergodic** if **all time averages** and **Ensemble averages** are **equal**. I.e.,

$$\overline{X(t)} = \langle X(t) \rangle \triangleq \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} X(t)\,dt \quad \text{and} \quad \overline{X^2(t)} = \langle X^2(t) \rangle \triangleq \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} X^2(t)\,dt$$



Ergodic $\Rightarrow$ Stationary $\Rightarrow$ WSS

Power Spectral Density



Center frequency (e.g., 900 MHz, 2 GHz, etc.) → **Bandwidth**

Power Spectra Density

- It turns out that the **Fourier Transform** of the **autocorrelation** calculated by an ensemble average is equal to the **power spectral density** (PSD) *if the process is WSS*

$$\boxed{S_X(f) \leftrightarrow R_X(\tau)}$$

VIP

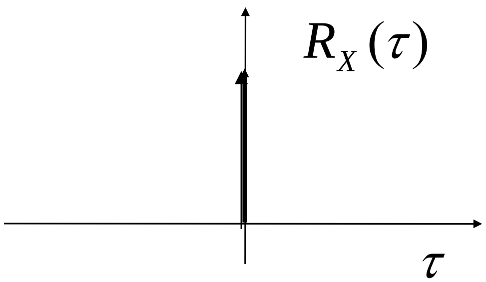
- In addition, if the process is **ergodic**, $R_x(\tau)$ can be calculated as a time average. I.e.,

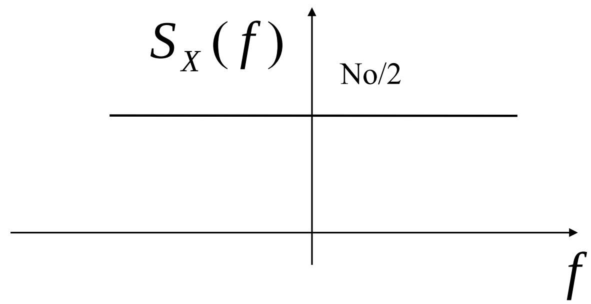
$$R_X(\tau) = E[x(t)x(t+\tau)] = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)x(t+\tau)dt$$

- The **PSD** is a **measure** of the **relative power** in the signal at each frequency component

Example (White Noise)

$$R_X(\tau) = \frac{N_0}{2}\delta(\tau)$$





- The **total power** is then simply calculated as

Always true

$$\boxed{R_X(0) = \int_{-\infty}^{\infty} S_X(f)df} = \infty$$

Application to Noise

- Noise is a critical component in the analysis of the performance of communication receivers
- Generally, noise can be modeled as a Gausssian process (*by the central limit theorem*). In this case, we say that at a particular time, *t*, the noise signal amplitude will be Gaussian distributed
- We *also assume the noise process is ergodic, hence stationary.*
- For Gaussian processes, knowledge of mean and autocorrelation is enough to completely specify them.
  - The mean is taken to be zero while the autocorrelation is usually specified by the power spectral density
- For white noise, all frequency components appear with equal power (white used in white light for similar reason)

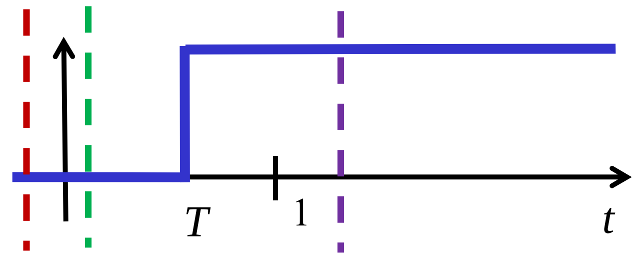
ELEC 2600H Fall 2021

APPENDIX (FYI):

ADDITIONAL RANDOM PROCESSES

Additional Example: Random Step (FYI)

Suppose that $X(t)$ is a unit step that starts at time T, where T is uniform on $[0,1]$. Find the pmf of $X(\mathrm{t})$.



Solution

Note that $X(t)$ is a discrete RV, since it can be either 0 or 1.

$$\text{If } t<0, \text{ then } P[X(t)=k]=\begin{cases}1 & k=0\\ 0 & k=1\end{cases}$$

$$\text{If } t>1, \text{ then } P[X(t)=k]=\begin{cases}0 & k=0\\ 1 & k=1\end{cases}$$

Note that the PMF changes with t!!

$$\text{If } 0\le t\le 1, \text{ then } \begin{aligned} P[X(t)=0]&=P[T>t]=(1-t)\\ P[X(t)=1]&=P[T\le t]=t\end{aligned}$$

Example: Random Step (FYI Cont'd)

Find the mean and variance of $X(t)$ from the previous example.

Solution

Since $X(t)$ can be either 0 or 1.

$$m_X(t) = E[X(t)] = 0 \cdot P[X(t) = 0] + 1 \cdot P[X(t) = 1] = P[X(t) = 1]$$

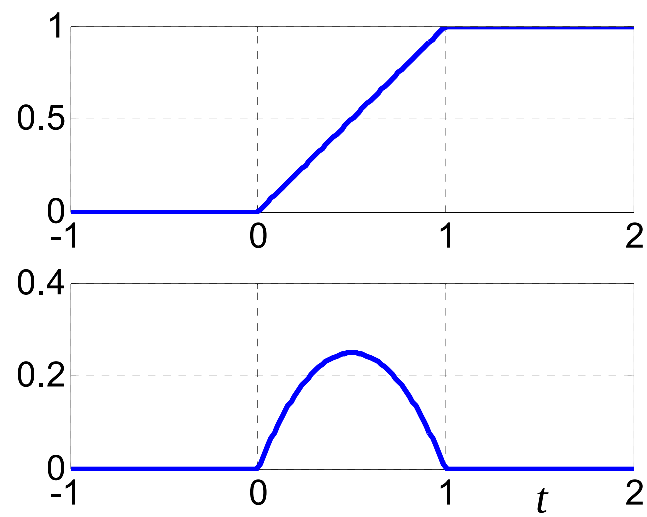
$$E[X(t)^2] = 0^2 \cdot P[X(t) = 0] + 1^2 \cdot P[X(t) = 1] = P[X(t) = 1]$$

$$\mathrm{Var}[X(t)] = E[X(t)^2] - m_X(t)^2 = P[X(t) = 1] - P[X(t) = 1]^2$$

Substituting our previous results.

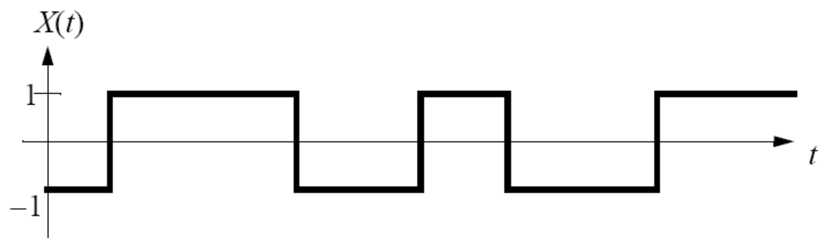
$$m_X(t) = \begin{cases} 0 & t < 0 \\ t & 0 \le t \le 1 \\ 1 & t > 0 \end{cases}$$

$$\mathrm{Var}[X(t)] = \begin{cases} 0 & t < 0 \\ t - t^2 & 0 \le t \le 1 \\ 0 & t > 0 \end{cases}$$



Random Telegraph Signal

- The random telegraph signal $X(t)$ assumes values ± 1. It changes polarity at every discontinuity of a Poisson counting process $N(t)$.



- If $P[X(0) = 1] = P[X(0) = -1] = ½$, then for all t, $P[X(t) = \pm 1] = ½$,

  Proof:

$$P[X(t)=1] = P[X(t)=1 \mid X(0)=1]\cdot P[X(0)=1] + P[X(t)=1 \mid X(0)=-1]\cdot P[X(0)=-1]$$

  where (see next page),

$$P[X(t)=1 \mid X(0)=1] = P[N(t)\text{ is even}] = \tfrac{1}{2}(1+e^{-2\lambda t})$$

$$P[X(t)=1 \mid X(0)=-1] = P[N(t)\text{ is odd}] = \tfrac{1}{2}(1-e^{-2\lambda t})$$

  Thus,

$$P[X(t)=1] = \tfrac{1}{2}(1+e^{-2\lambda t})\cdot\tfrac{1}{2} + \tfrac{1}{2}(1-e^{-2\lambda t})\cdot\tfrac{1}{2} = \tfrac{1}{2}$$

Probability of Even/Odd values of Poisson RV

Let N be a Poisson distributed random variable with parameter α:

$$p_N(n) = \frac{\alpha^n}{n!} e^{-\alpha} \qquad \text{for } \; n \in \{0, 1, 2, \ldots\}$$

Find the probabilities that N is even and N is odd.

Solution

$$P[N \text{ even}] = e^{-\alpha} \sum_{k \text{ even}} \frac{\alpha^k}{k!}$$

$$\boxed{\begin{aligned}
e^{\alpha} &= 1 + \alpha + \frac{\alpha^2}{2!} + \frac{\alpha^3}{3!} + \frac{\alpha^4}{4!} + \frac{\alpha^5}{5!} + \frac{\alpha^6}{6!} \cdots &&= \sum_{k=0}^{\infty} \frac{\alpha^k}{k!} \\
e^{-\alpha} &= 1 - \alpha + \frac{\alpha^2}{2!} - \frac{\alpha^3}{3!} + \frac{\alpha^4}{4!} - \frac{\alpha^5}{5!} + \frac{\alpha^6}{6!} \cdots &&= \sum_{k=0}^{\infty} \frac{(-\alpha)^k}{k!} \\
\frac{e^{\alpha} + e^{-\alpha}}{2} &= 1 - \alpha + \frac{\alpha^2}{2!} - \frac{\alpha^3}{3!} + \frac{\alpha^4}{4!} - \frac{\alpha^5}{5!} + \frac{\alpha^6}{6!} \cdots &&= \sum_{k \text{ even}} \frac{\alpha^k}{k!}
\end{aligned}}$$

$$\text{Thus, } P[N \text{ even}] = e^{-\alpha} \left(\frac{e^{a} + e^{-\alpha}}{2} \right) = \frac{1}{2} + \frac{1}{2} e^{-2\alpha}$$

$$P[N \text{ odd}] = 1 - P[N \text{ even}] = \frac{1}{2} - \frac{1}{2} e^{-2\alpha}$$

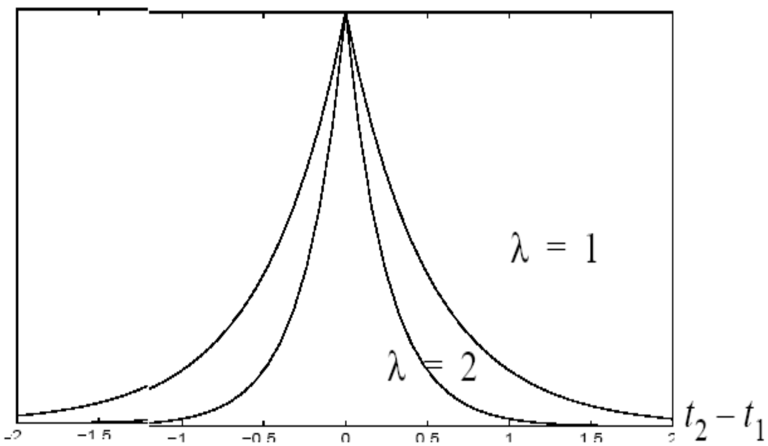
Mean and Autocorrelation of Random Telegraph

- Mean: $m_X(t) = E[X(t)] = (1)\cdot P[X(t)=1] + (-1)\cdot P[X(t)=-1] = 0$

- Autocorrelation: $R_X(t_1,t_2) = e^{-2\lambda|t_2-t_1|}$

  Proof: Assume that $t_2 \geq t_1$. A similar argument holds for $t_1 \geq t_2$.

$$\begin{aligned} R_X(t_1,t_2) &= (1)\cdot P[X(t_1)X(t_2)=1] + (-1)\cdot P[X(t_1)X(t_2)=-1] \\ &= (1)\cdot P[N(t_2-t_1) \text{ is even}] + (-1)\cdot P[N(t_2-t_1) \text{ is odd}] \\ &= (1)\cdot \tfrac{1}{2}\left(1+e^{-2\lambda(t_2-t_1)}\right) + (-1)\cdot \tfrac{1}{2}\left(1-e^{-2\lambda(t_2-t_1)}\right) \\ &= e^{-2\lambda(t_2-t_1)} \end{aligned}$$



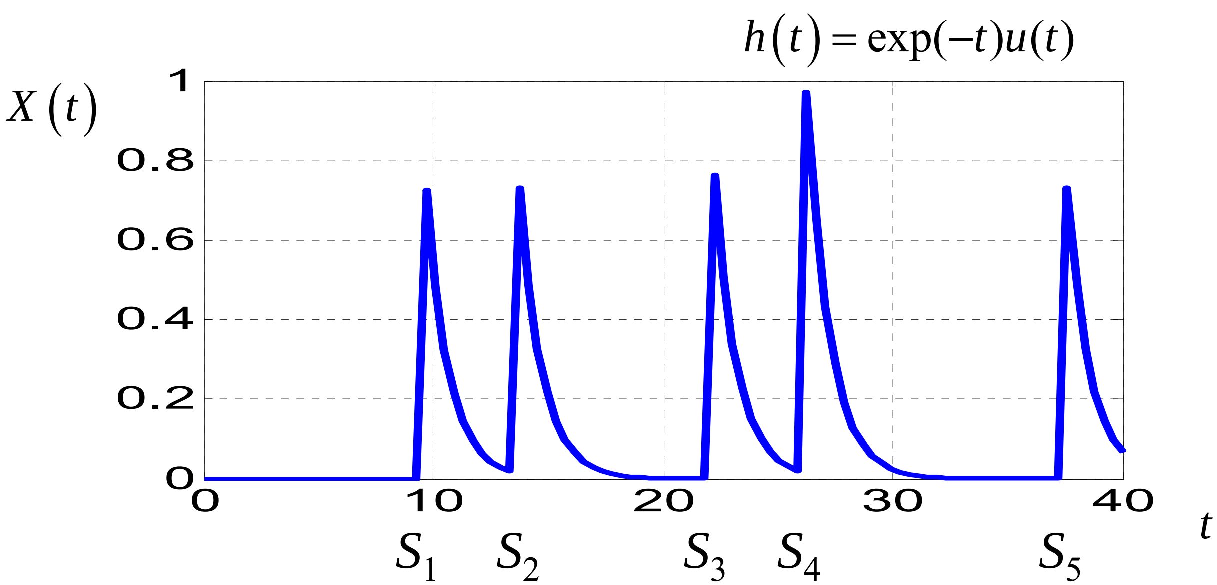
Example 9.25 Shot Noise

Suppose that photons arrive at a photodetector according to a Poisson Process. Let S_i be the arrival time of the ith photon.

Each photon results in a current pulse through the photodetector with shape $h(t)$. The total current flowing at time t is the sum of all current pulses:

$$X\left(t\right)=\sum_{i=1}^{\infty}h(t-S_i)$$

This is called a *shot noise* process.



Example 9.26: Mean of Shot Noise

To find the mean of $X(t)$, we condition on the number of photons that have arrived, $N(t)$, and then remove conditioning by taking the expected value.

$$E[X(t)] = E\left[E[X(t)|N(t)]\right]$$

where

$$E[X(t)|N(t)] = E\left[\sum_{j=1}^{N(t)} h(t - S_j)\right] = \sum_{j=1}^{N(t)} E\left[h(t - S_j)\right]$$

Since the arrival times S_j are independent and uniformly distributed on $[0,t]$

$$E\left[h(t - S_j)\right] = \int_0^t h(t-s) f_{S_j}(s) ds = \int_0^t h(t-s) \cdot \frac{1}{t} \cdot ds = \frac{1}{t}\int_0^t h(u) du \begin{cases} u = t - s \\ du = -ds \end{cases}$$

Thus,

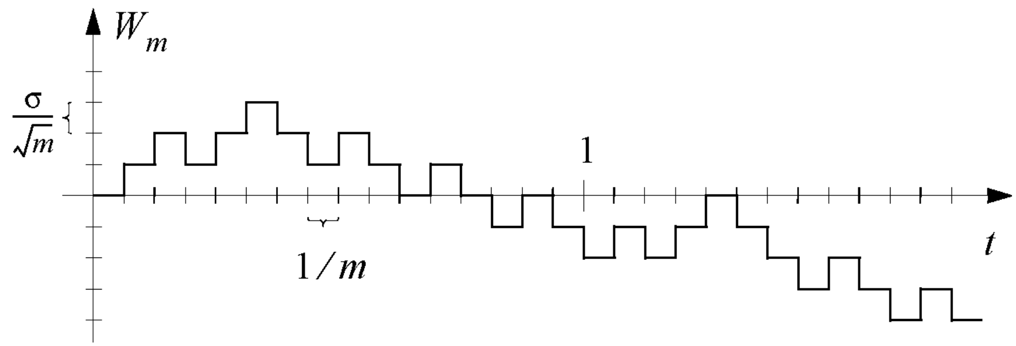
$$E[X(t)|N(t)] = \frac{N(t)}{t}\int_0^t h(u) du$$

$$E[X(t)] = \frac{E[N(t)]}{t}\int_0^t h(u) du = \frac{\lambda t}{t}\int_0^t h(u) du = \lambda \int_0^t h(u) du$$

as $t \to \infty$, this approaches the arrival rate λ times the total charge in one current pulse.

The Wiener Process

- Consider the following sequence of random processes conditioned on m.
- Divide each unit interval of the real line into m equal sub intervals. At each sub interval, we toss a coin with probability of heads $p = ½$.
  - If head appears, step forward by $\sigma / \sqrt{m}$
  - If tail appears, step backwards by $\sigma / \sqrt{m}$



- Our position at time t is

$$W_m(t) = \sum_{i=1}^{\lfloor mt \rfloor} \frac{\sigma}{\sqrt{m}} X_i$$

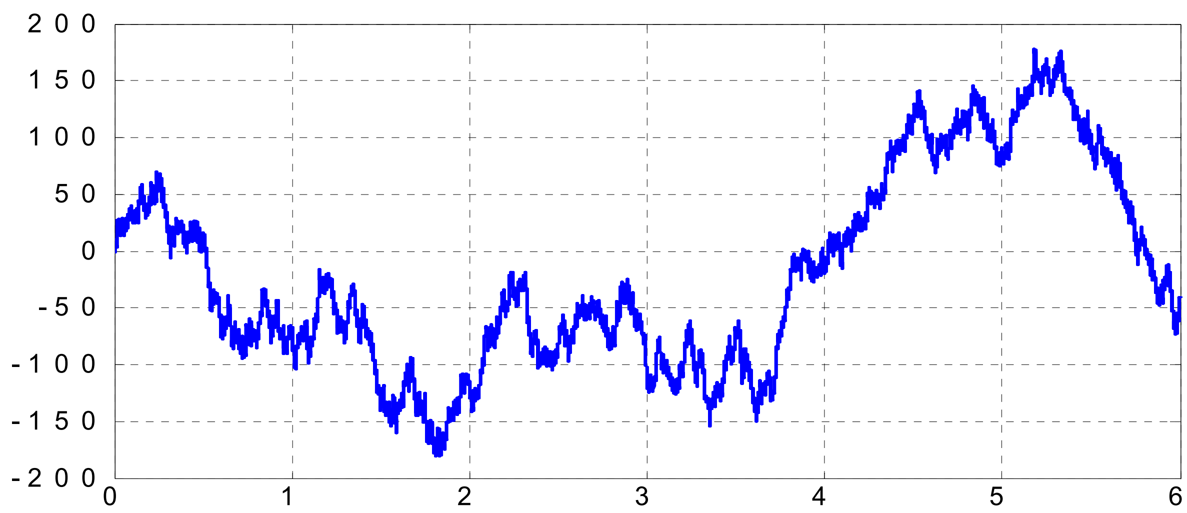
where $X(i)$ are i.i.d. binary random variables with equal probability of ± 1 and $\lfloor x \rfloor$ is the largest integer less than or equal to x.

Properties of $W_m(t)$

- The underlying discrete time process $W_m(t) = \sum_{i=1}^{\lfloor mt \rfloor} \frac{\sigma}{\sqrt{m}} X(i)$ is i.s.i.
- $$E[W_m(t)] = E\left[\sum_{i=1}^{\lfloor mt \rfloor} \frac{\sigma}{\sqrt{m}} X(i)\right] = 0$$
- $$VAR[W_m(t)] = \sum_{i=1}^{\lfloor mt \rfloor} \frac{\sigma^2}{m} = \frac{\lfloor mt \rfloor}{m}\sigma^2 \to \sigma^2 t \quad as \quad m \to \infty$$
- By the Central Limit Theorem, the distribution of $W_m(t)$ approaches a Gaussian distribution with mean 0 and variance $\sigma^2 t$ as $m \to \infty$.

Wiener Process Definition

- The *Wiener process* $W(\mathrm{t})$ is the continuous time Gaussian i.s.i. random process with zero mean and variance $\sigma^2 t$.
- The Wiener process is commonly used to model Brownian motion, the motion of particle suspended in a fluid that moves due to rapid and random impacts from neighboring particles.



$C_W(t_1,t_2) = R_W(t_1,t_2) = \sigma^2 \min(t_1,t_2)$

<u>Proof</u>:

Note that $C_W(t_1,t_2) = R_W(t_1,t_2)$ since $W(t)$ is zero mean.

Assume that $t_1 \geq t_2$. Then

$$\begin{aligned} R_W(t_1,t_2) &= E[W(t_1)\cdot W(t_2)] \\ &= E[\big(W(t_2)+W(t_1)-W(t_2)\big)\cdot W(t_2)] \\ &= E[W(t_2)^2]+E[\big(W(t_1)-W(t_2)\big)\cdot W(t_2)] \\ &= E[W(t_2)^2]+E[W(t_1)-W(t_2)]\cdot E[W(t_2)] \\ &= \sigma^2 t_2 \end{aligned}$$

by independent increments

zero mean

A similar proof for $t_2 \geq t_1$ can be used to show that

$$R_W(t_1,t_2) = \sigma^2 t_1$$

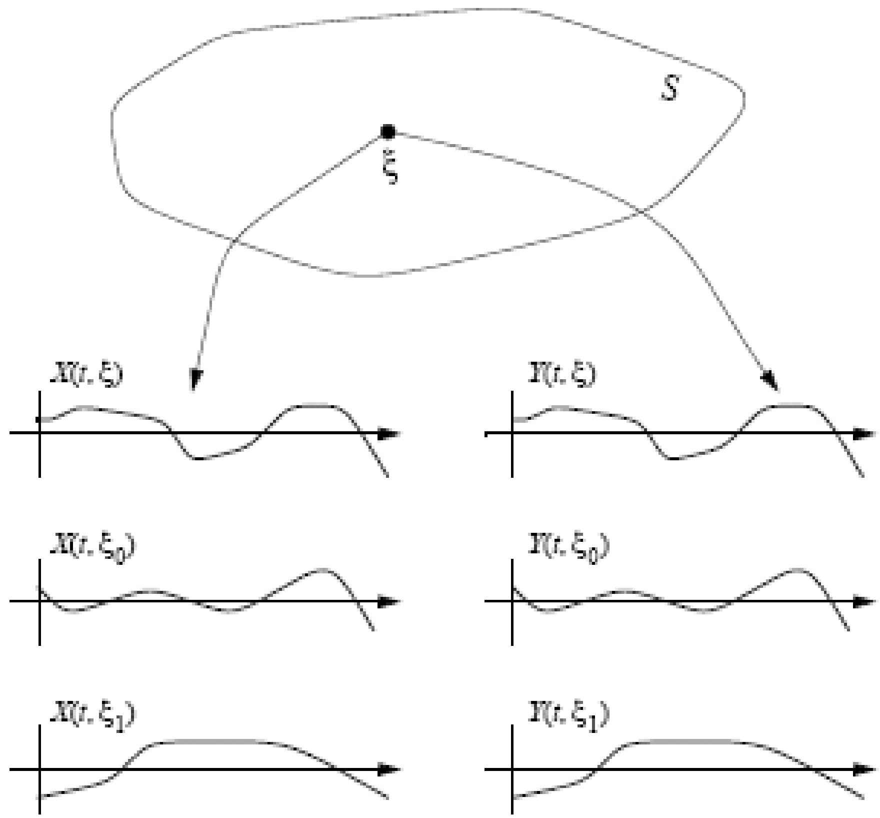
Thus,

$$C_W(t_1,t_2) = R_W(t_1,t_2) = \sigma^2 \min(t_1,t_2)$$

The Wiener Process and the Poisson process have the same autocovariance functions, despite the fact that their realizations look completely different!

Multiple Random Processes

- When discussing more than one random process, e.g. $X(t)$ and $Y(t)$, remember that both are determined by the *same* outcome in the original sample space.



joint random processes

Definitions of Terms

- Two random processes $X(t)$ and $Y(t)$ are said to be ***independent*** if the vectors $\begin{bmatrix} X(t_1) \\ X(t_2) \\ \vdots \\ X(t_k) \end{bmatrix}$ and $\begin{bmatrix} Y(s_1) \\ Y(s_2) \\ \vdots \\ Y(s_j) \end{bmatrix}$ are independent for all k, j, and all choices of times $t_1, t_2, \ldots, t_k$ and $s_1, s_2, \ldots, s_j$.

- The ***cross-correlation*** of $X(t)$ and $Y(t)$ is $R_{XY}(t_1, t_2) = E[X(t_1)Y(t_2)]$

- The ***cross-covariance*** of $X(t)$ and $Y(t)$ is $C_{XY}(t_1, t_2) = E[(X(t_1) - m_X(t_1))(Y(t_2) - m_Y(t_2))]$
$$= R_{XY}(t_1, t_2) - m_X(t_1)m_Y(t_2)$$

- $X(t)$ and $Y(t)$ are ***uncorrelated*** if $C_{XY}(t_1, t_2) = 0$ for all t_1, t_2

Example 9.11: Random Phase Sinusoid

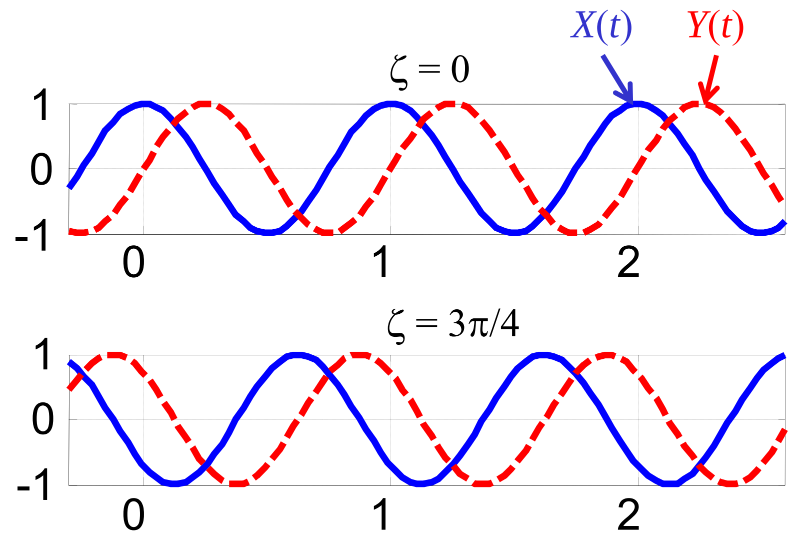
Find cross-covariance of $X(t)$ and $Y(t)$
where $X(t) = \cos(2\pi t + \zeta)$

$Y(t) = \sin(2\pi t + \zeta)$

and ζ is selected at random from $[-\pi, \pi]$.

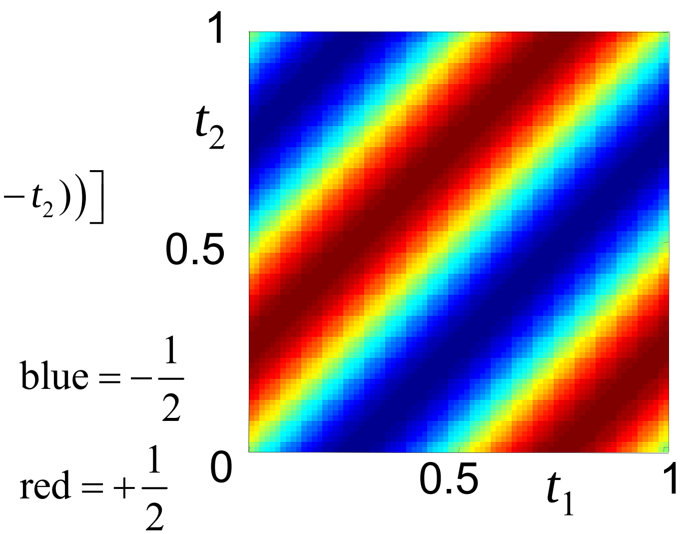
Solution

From a previous example, $m_X(t) = 0$ and $m_Y(t) = 0$. Thus, $C_{XY}(t_1,t_2) = R_{XY}(t_1,t_2)$.



$$
\begin{aligned}
R_X(t_1,t_2) &= E[X(t_1)Y(t_2)] \\
&= E[\cos(2\pi t_1 + \zeta)\sin(2\pi t_2 + \zeta)] \\
&= \frac{1}{2}E\left[\sin\left(2\pi(t_1+t_2) + 2\zeta\right) - \sin\left(2\pi(t_1 - t_2)\right)\right] \\
&= -\frac{1}{2}\sin\left(2\pi(t_1 - t_2)\right)
\end{aligned}
$$

$$\boxed{\cos(a)\sin(b) = \frac{1}{2}(\sin(a+b) - \sin(a-b))}$$



Example 9.12: Communication System

Suppose we observe a process $Y(t)$ which consists of a desired signal $X(t)$ plus noise $N(t)$, where $X(t)$ and $N(t)$ are independent:

$$Y(t) = X(t) + N(t)$$

Find the cross-correlation between the desired and observed signals

Solution:

$$\begin{aligned} R_{XY}(t_1,t_2) &= E[(X(t_1)Y(t_2)] \\ &= E[X(t_1)(X(t_2)+N(t_2))] \\ &= E[X(t_1)X(t_2)] + E[X(t_1)N(t_2)] \\ &= R_X(t_1,t_2) + E[X(t_1)]E[N(t_2)] \\ &= R_X(t_1,t_2) + m_X(t_1)m_N(t_2) \end{aligned}$$

since independent