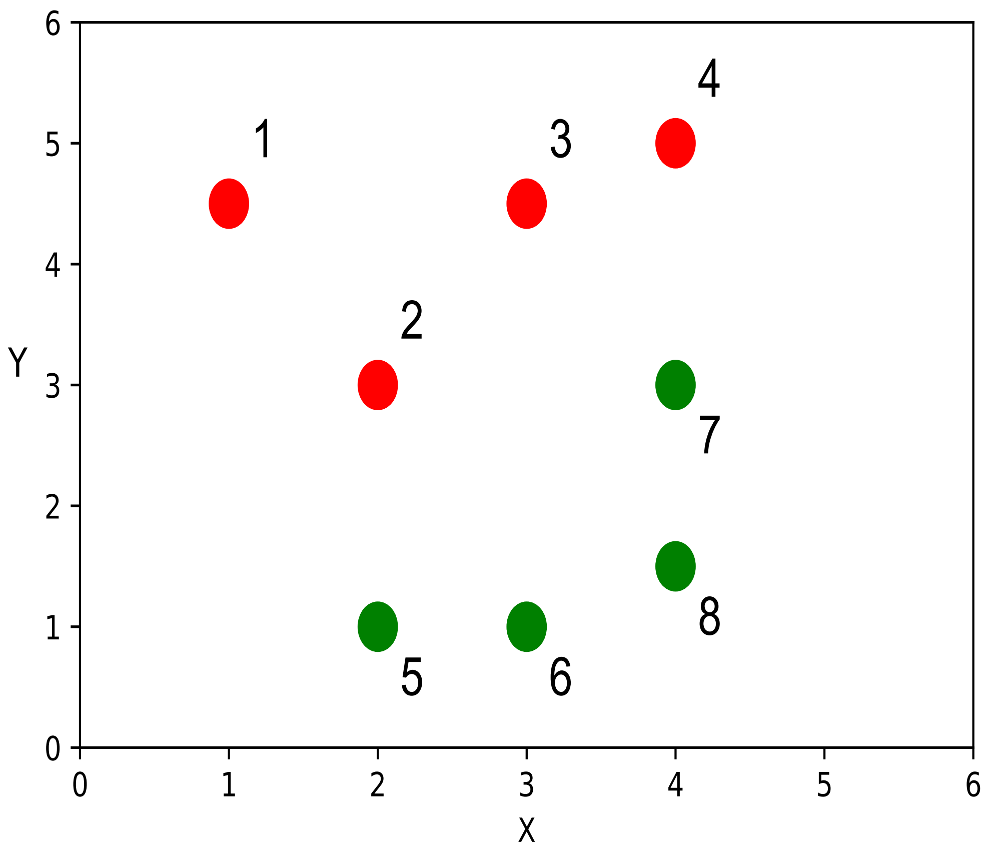
LOGISTIC

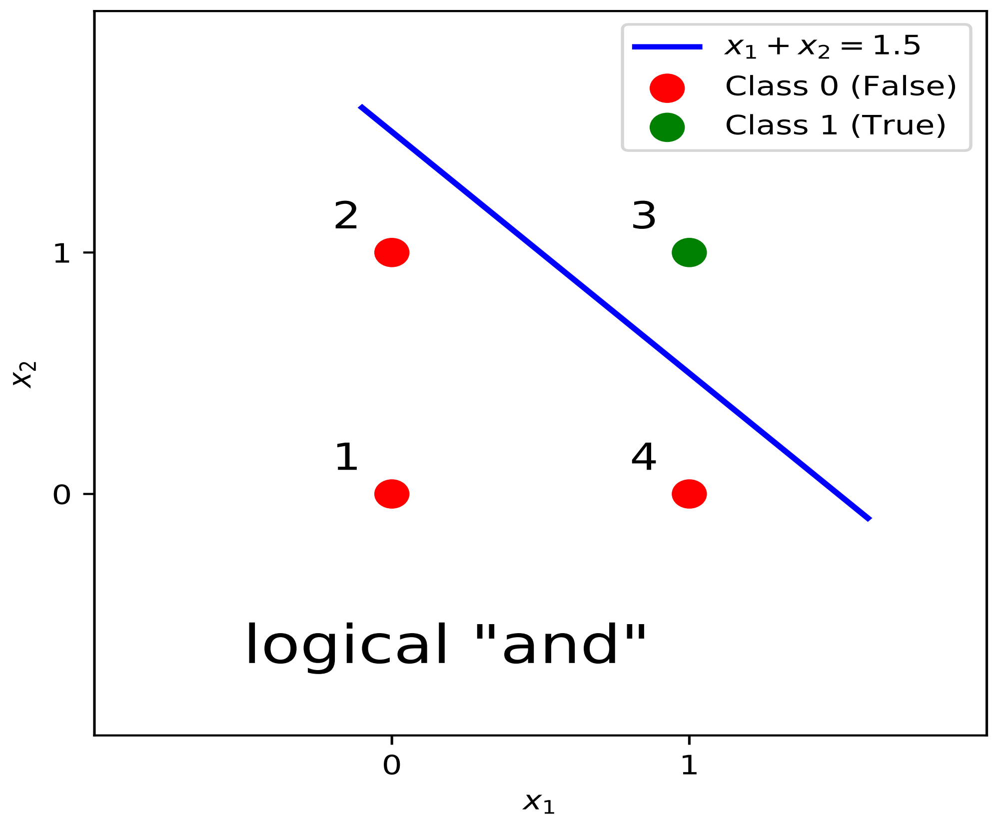
REGRESSION

Overview



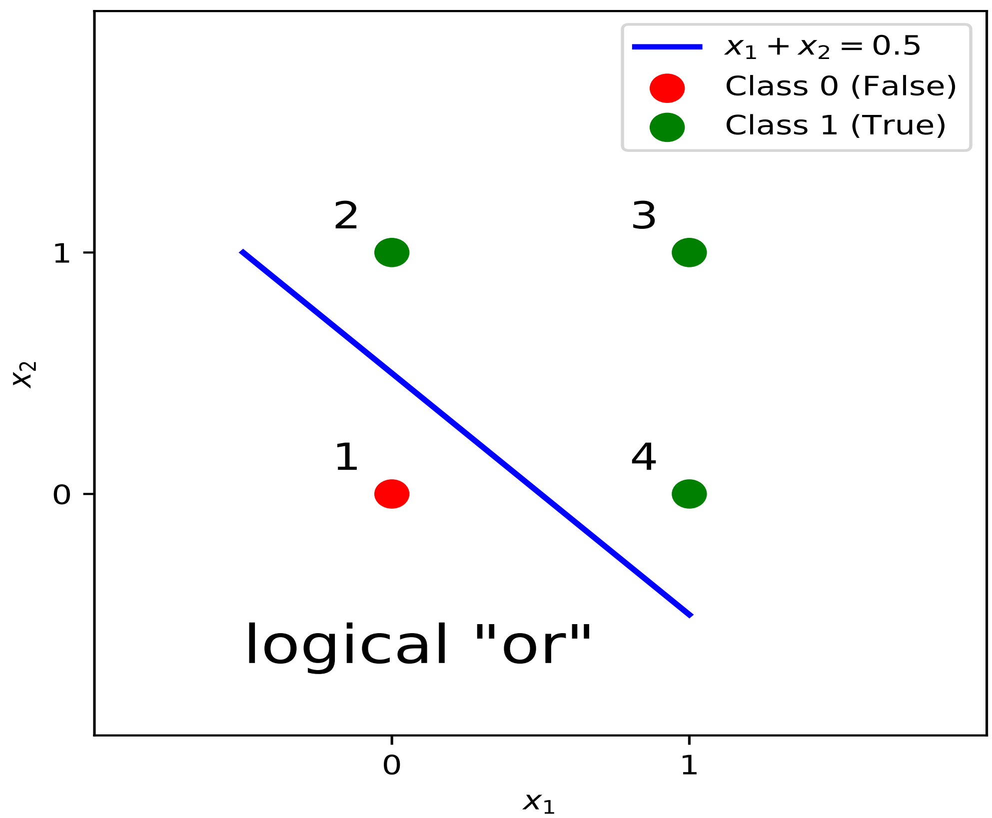
- want to separate classes
- smooth decision function

Example: "AND" Gate



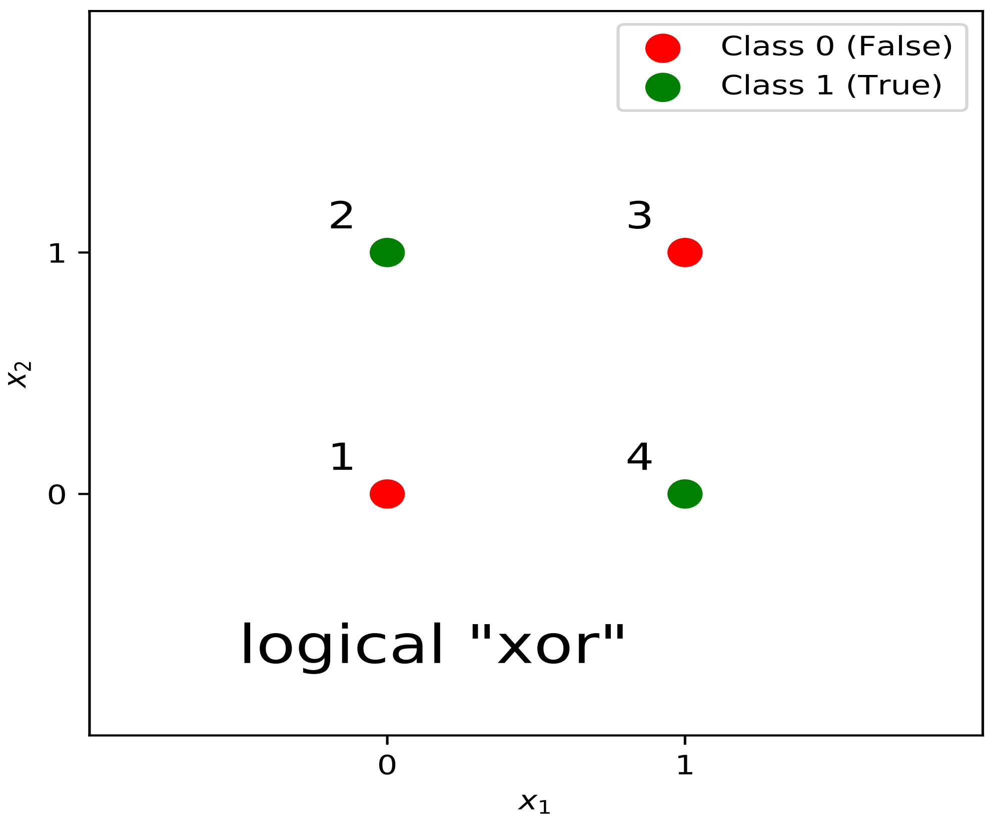
- linearly separable

Example: ”OR” Gate



- linearly separable

Example: "XOR" Gate



- not linearly separable

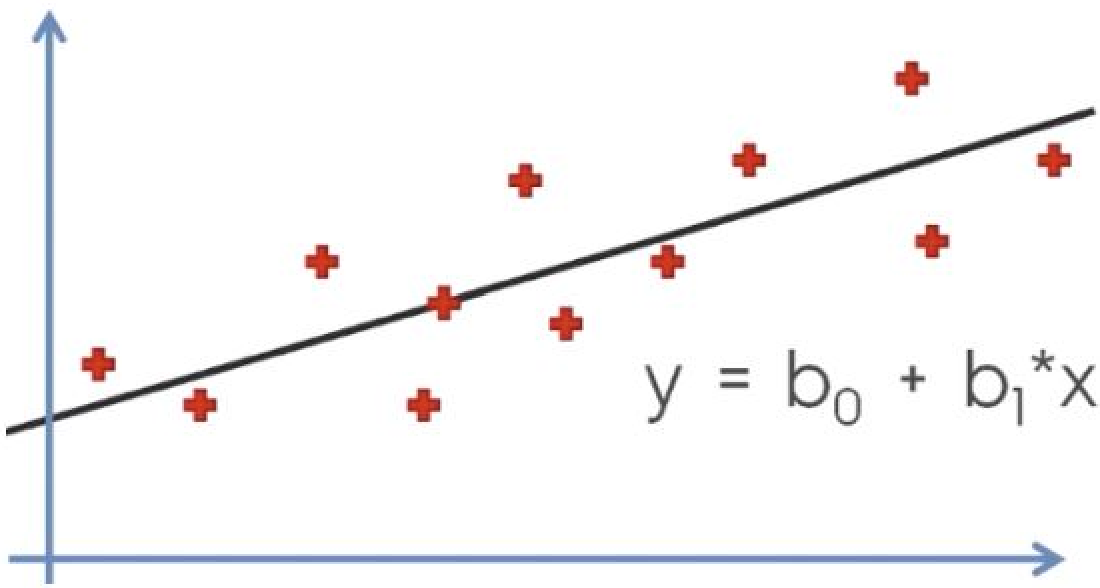
Binary Classification

- training set S with labels $\{0,1\}$
- find a classifier H

$$H : X \mapsto \{0,1\}$$

- low generalization error
- linear classification (based on logistic regression)
- dividing (hyper)plane is called linear discriminant

Background: Linear Regression



- example of a Generalized Linear Model (GLM)

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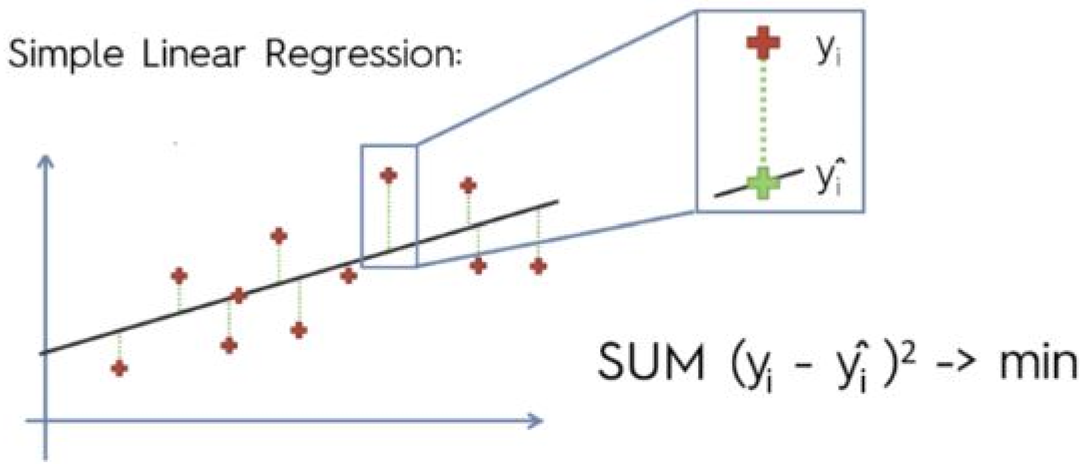
Simple Linear Regression



- choose line to minimize loss

$$\text{Loss} = \sum_{i=1}^{N} (y_i - \hat{y_i})^2$$

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Classification Problem

- how do we transform a linear prediction model to classification problem?

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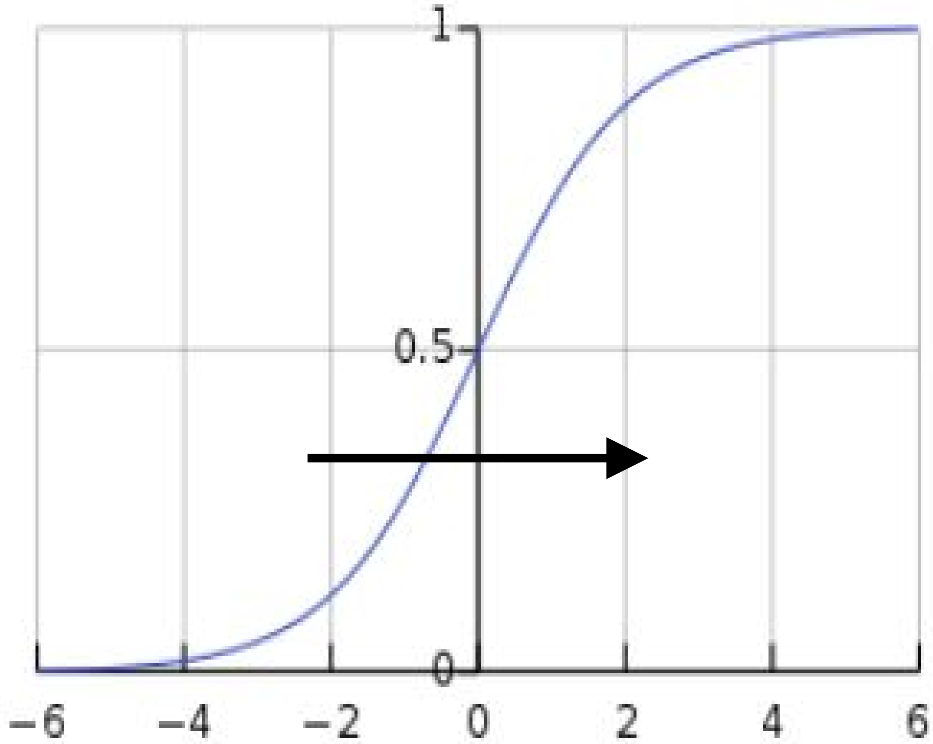
Issues

- linear regression: continuous variables
- classification: discrete
- probabilities must be in $[0, 1]$
- solution:

  linear regression $\mapsto$ classification
- how: use *logit* function

logit Function



$$\frac{1}{1+\exp(-x)} = \frac{\exp(x)}{1+\exp(x)}$$

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Probability and *Odds*

- assume probability P
- define *odds* as

$$\text{odds} = \frac{P}{1-P}$$

- ex.1: $P = 0.25 \mapsto \text{odds} = 1/3$
- ex.2: $P = 0.50 \mapsto \text{odds} = 1$
- ex.3: $P = 0.75 \mapsto \text{odds} = 3/1$

Main Idea:

- estimate logit(odds)
- use regression

$$\log\left(\frac{P}{1-P}\right) = b_0 + b_1 x$$

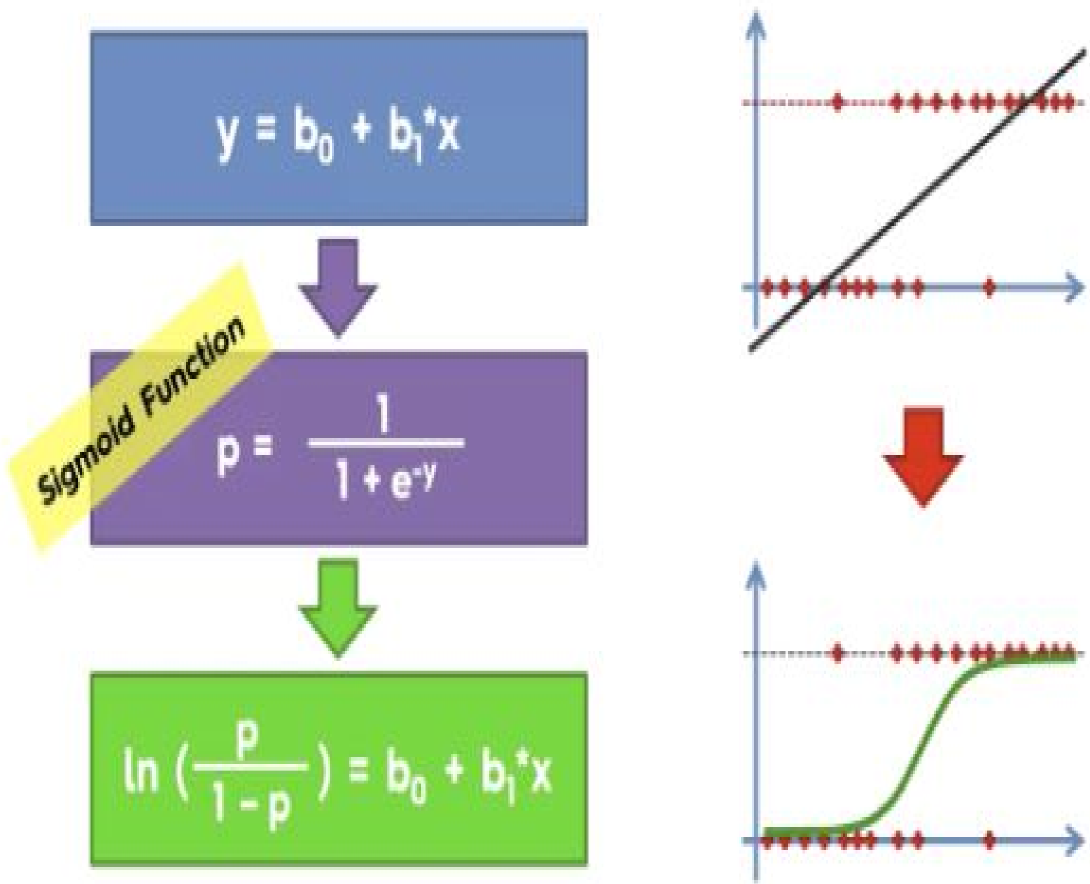
$$\frac{P}{1-P} = \exp(b_0 + b_1 x)$$

$$P = \frac{\exp(b_0 + b_1 x)}{1 + \exp(b_0 + b_1 x}$$

- note:

$$\frac{\exp(b_0 + b_1 x)}{1 + \exp(b_0 + b_1 x)} = \frac{1}{1 + \exp(-(b_0 + b_1 x))}$$

Illustration



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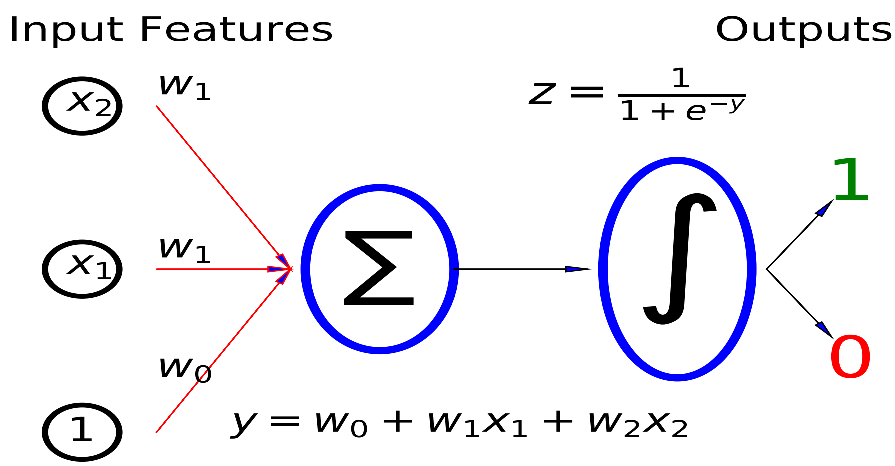
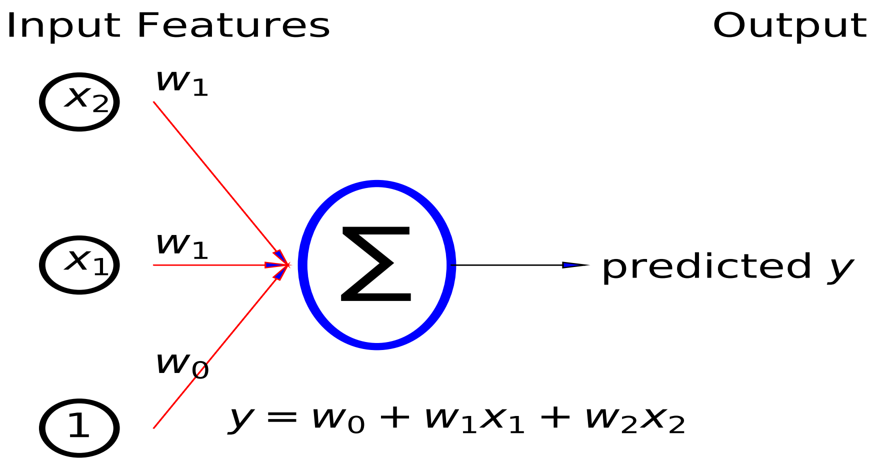
Logistic Regression



- supervised learning
- estimate label probabilities by sigmoid function

Linear Regression

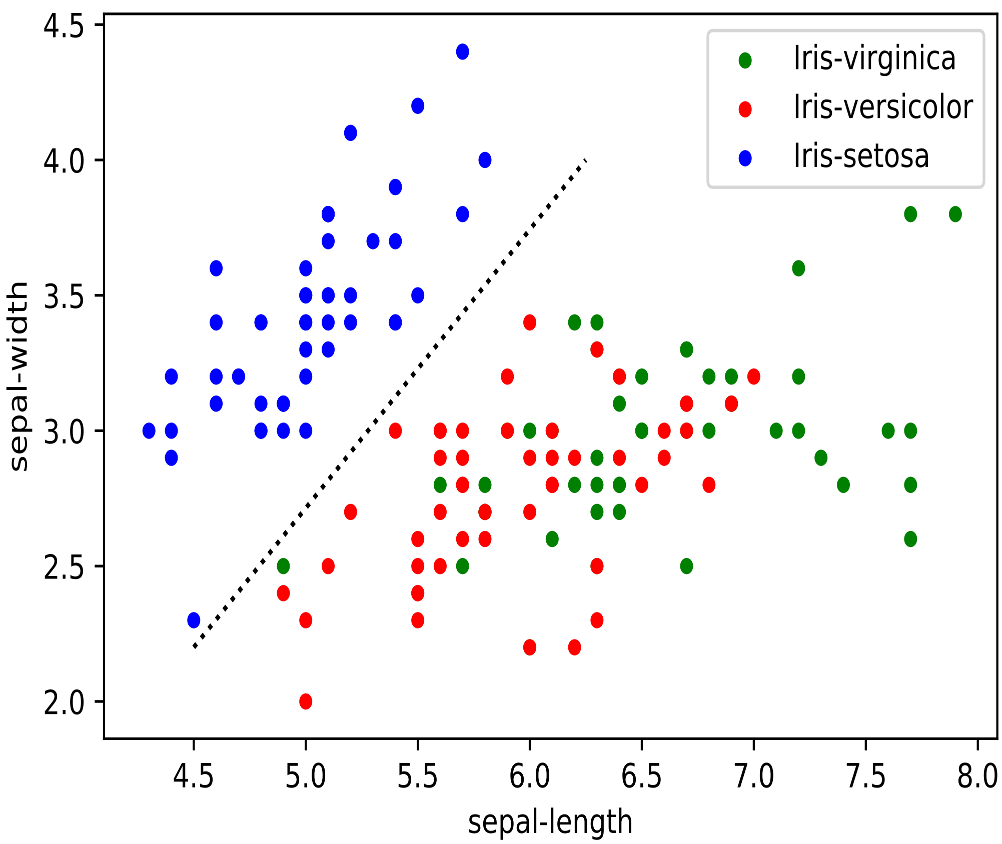


- real-valued output from weighted sum of inputs

Linear vs. Logistic

- linear regression:
  1. estimate $w_0, w_1, \ldots, w_n$ using min squared error
  2. predict $y = w_0 + w_1 x_x + \cdots + w_n x_n$
- logistic regression:
  1. estimate $w_0, w_1, \ldots, w_n$ using min squared error
  2. compute $y = w_0 + w_1 x_x + \cdots + w_n x_n$
  3. apply the signoid function $z(y)$ to compute label probabilities

Linear Separability



- draw a hyperplane
- difficult in many cases

Logistic Regression

- dependent variable is class label - categorical
- output is a weighted sum of inputs

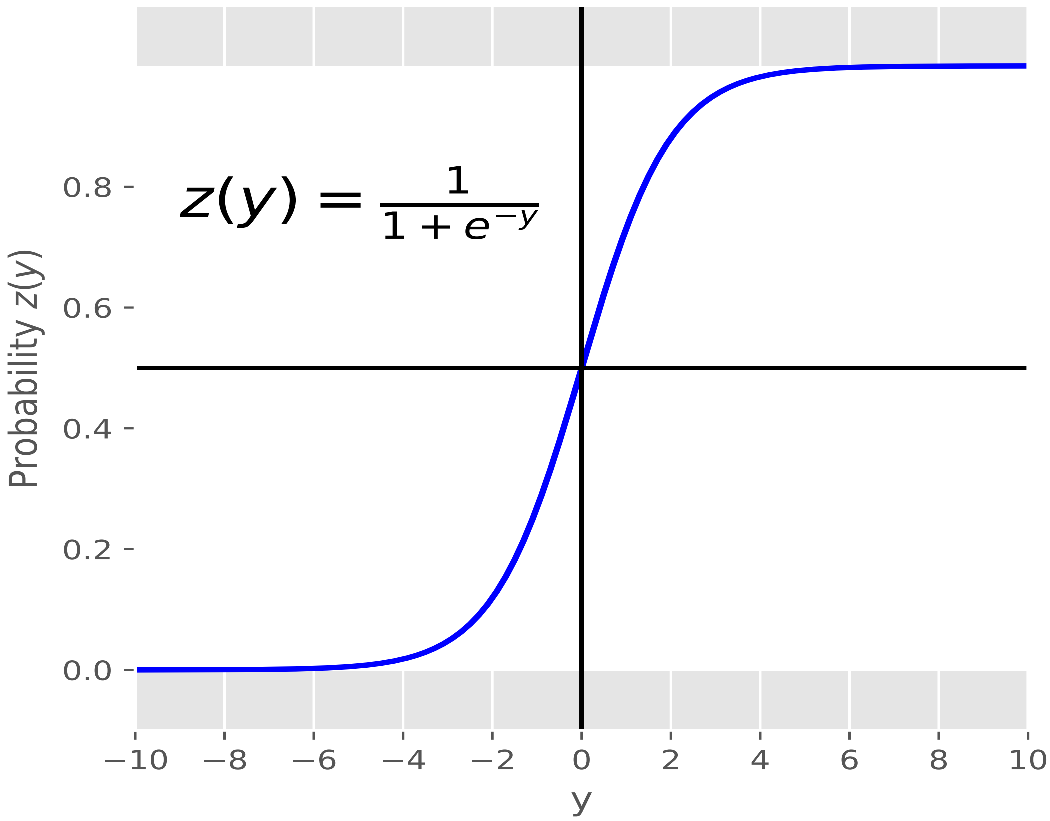
$$y = w_0 + w_1 x_1 + \cdots + w_m x_m$$

- weighted sum is passed through a sigmoid function

$$z(y) = \frac{1}{1 + e^{-y}}$$

- assign labels based on $z(y)$

Sigmoid Function $z(y)$



- $z(y) > 0.5$ if $y > 0$ (class 1)
- $z(y) < 0.5$ if $y < 0$ (class 0)

A Numerical Dataset

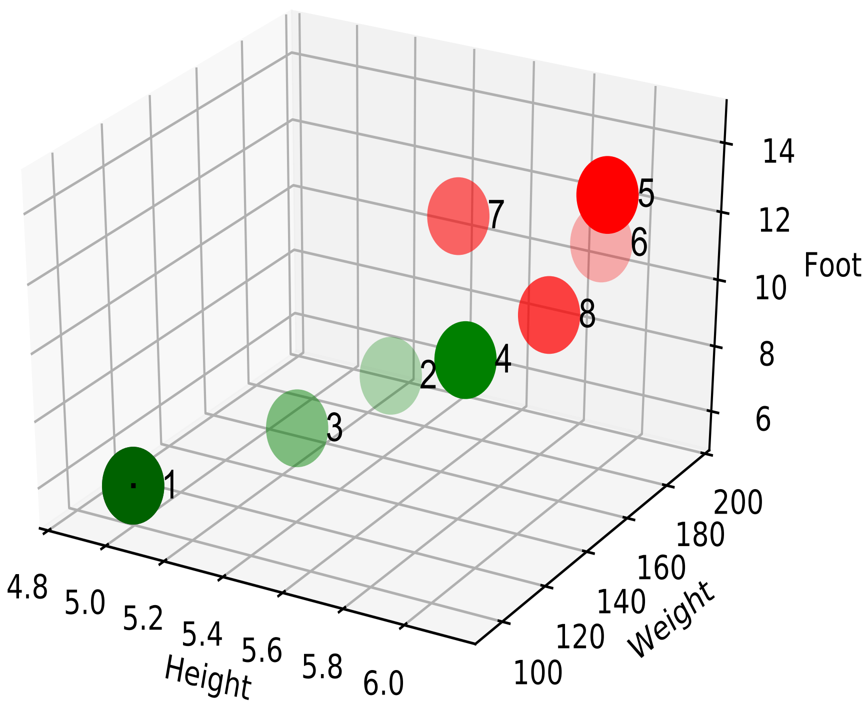
object x_i	Height (H)	Weight (W)	Foot (F)	Label (L)
x_1	5.00	100	6	green
x_2	5.50	150	8	green
x_3	5.33	130	7	green
x_4	5.75	150	9	green
x_5	6.00	180	13	red
x_6	5.92	190	11	red
x_7	5.58	170	12	red
x_8	5.92	165	10	red

Code for the Dataset

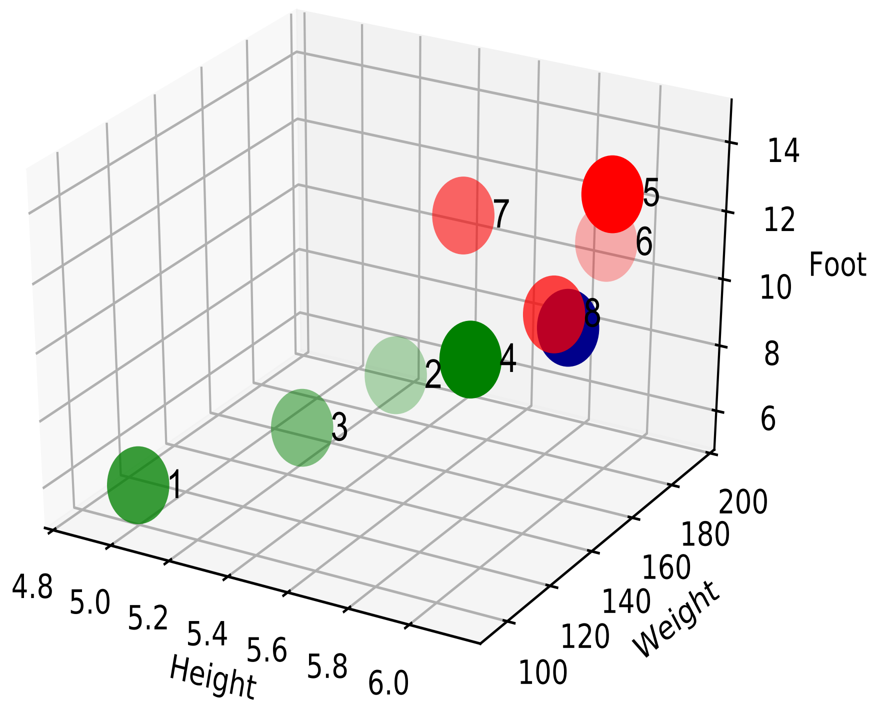
```
import pandas as pd
data = pd.DataFrame(
        {'id': [ 1,2,3,4,5,6,7,8],
        'Label': ['green','green','green','green',
                   'red','red','red','red'],
        'Height': [5, 5.5, 5.33, 5.75,
                   6.00, 5.92,  5.58, 5.92],
        'Weight': [100, 150, 130, 150,
                   180, 190, 170, 165],
        'Foot': [6, 8, 7, 9, 13, 11, 12, 10]},
         columns = ['id', 'Height', 'Weight',
                      'Foot', 'Label'] )
```

```
ipdb> data
  id Height Weight Foot Label
0  1   5.00    100    6 green
1  2   5.50    150    8 green
2  3   5.33    130    7 green
3  4   5.75    150    9 green
4  5   6.00    180   13   red
5  6   5.92    190   11   red
6  7   5.58    170   12   red
7  8   5.92    165   10   red
```

A Dataset Illustration

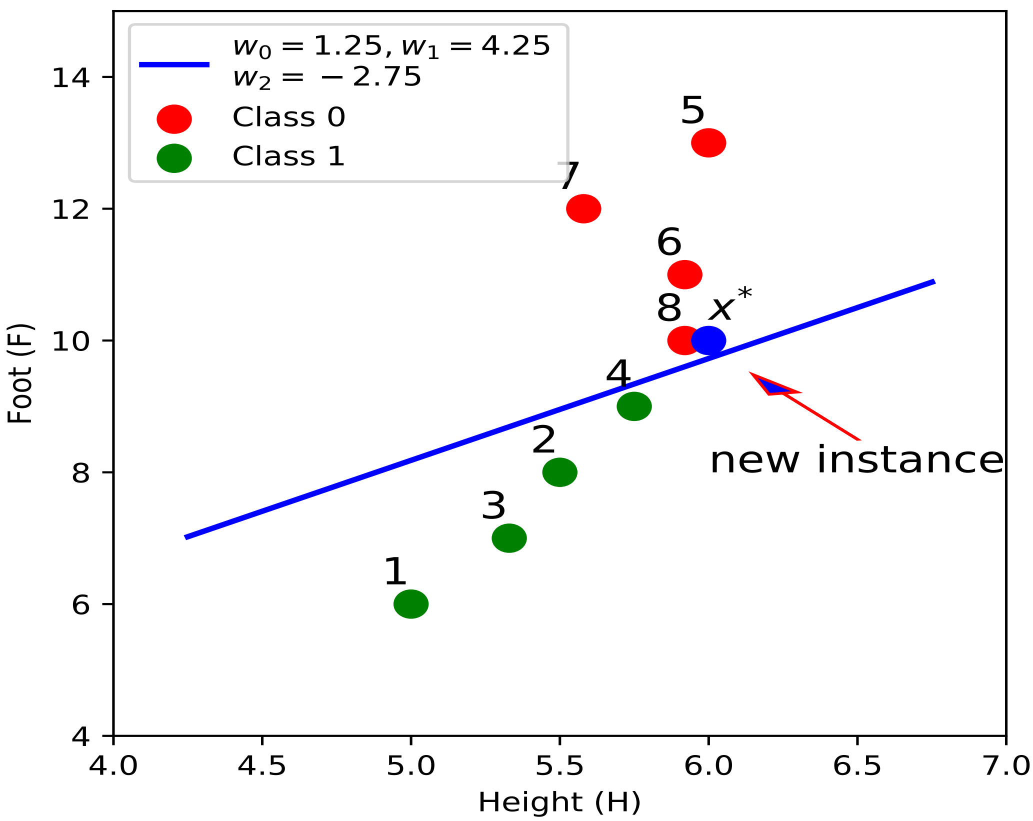


A New Instance

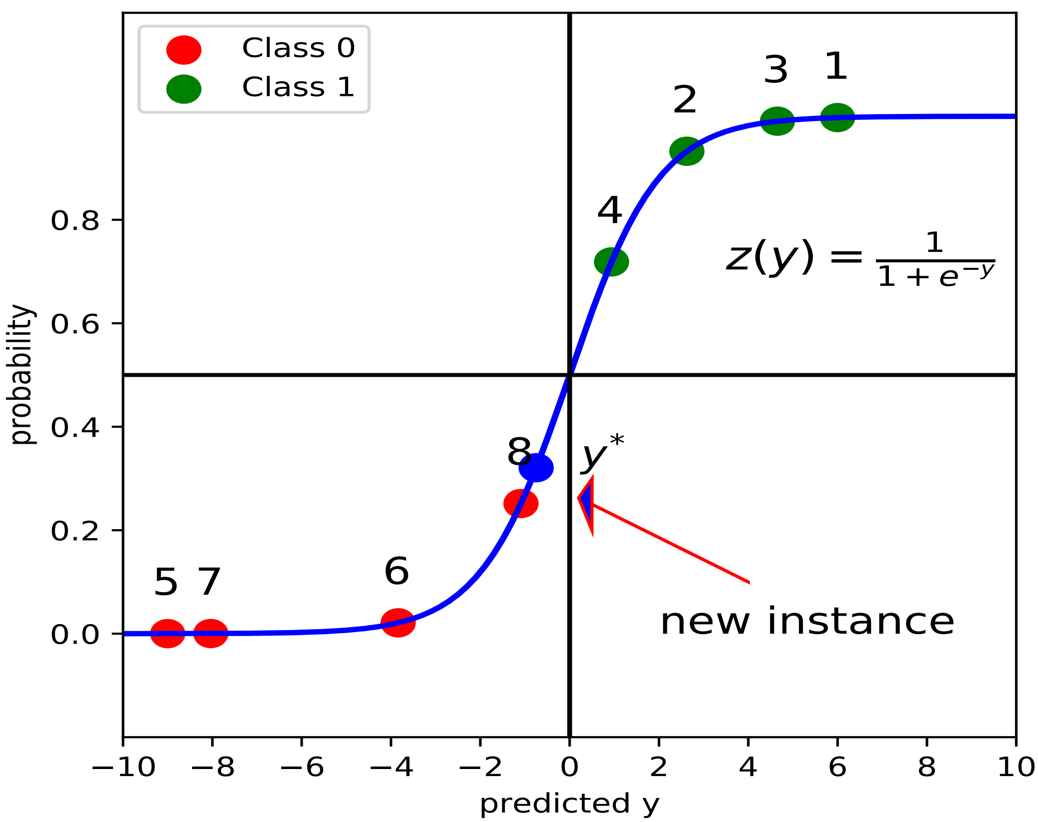


$$(\mathrm{H}=6,\ \mathrm{W}=160,\ \mathrm{F}=10) \mapsto ?$$

Separability in Detail



Computing Class Labels



- $z(y^*) < 0.5$ - "red" (class 0)

Summary of Logistic Regression

- feature vector: $X = (1, x_1, \ldots, x_m)$
- weights $W = (w_0, w_1, \ldots, w_m)$
- compute

$$y = W \cdot X = w_0 + w_1 x_1 + \cdots + w_m x_m$$

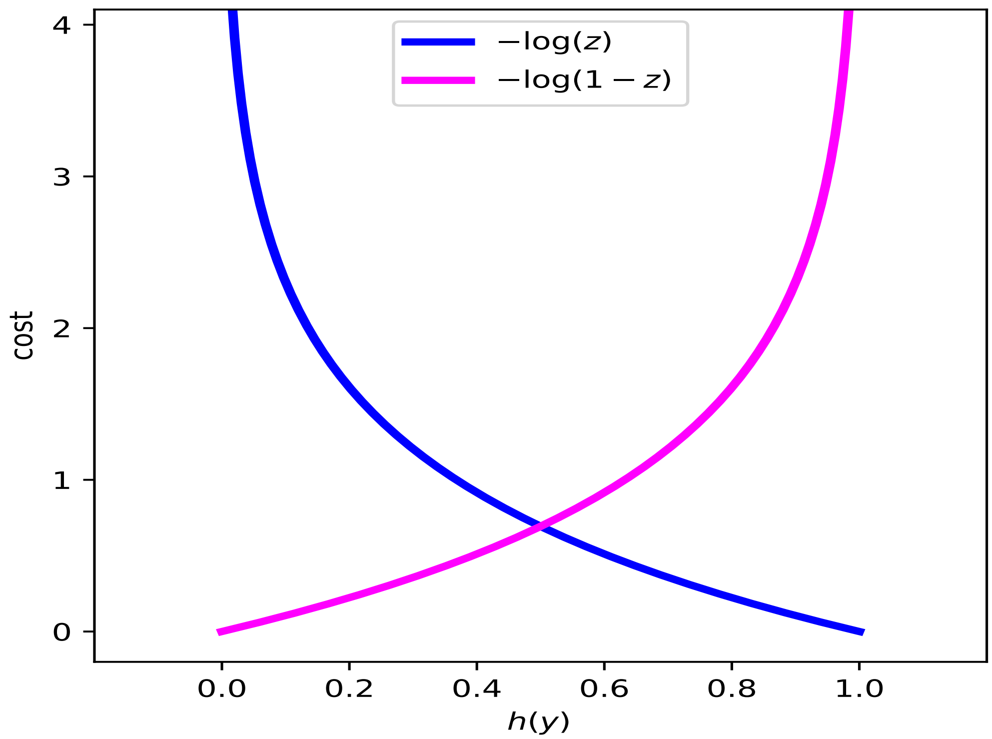
- compute probability $h(x)$:

$$h(x) = \frac{1}{1 + e^{-W \cdot X}}$$

- assign label $C(X)$:

$$C(X) = \begin{cases} 1, & \text{if } h(x) > 0.5 \\ 0, & \text{if } h(x) < 0.5 \end{cases}$$

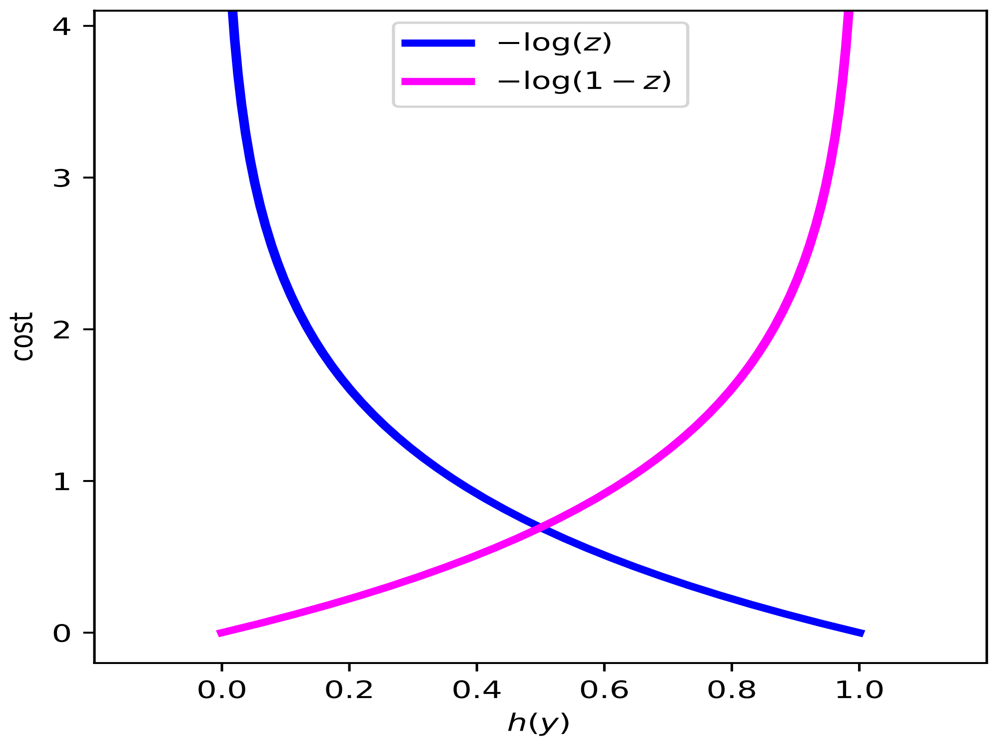
How to Compute w?



- maximize likelihood

$$L = \prod_X h(X)^{C(X)} \cdot [1 - h(X)]^{1-C(X)}$$

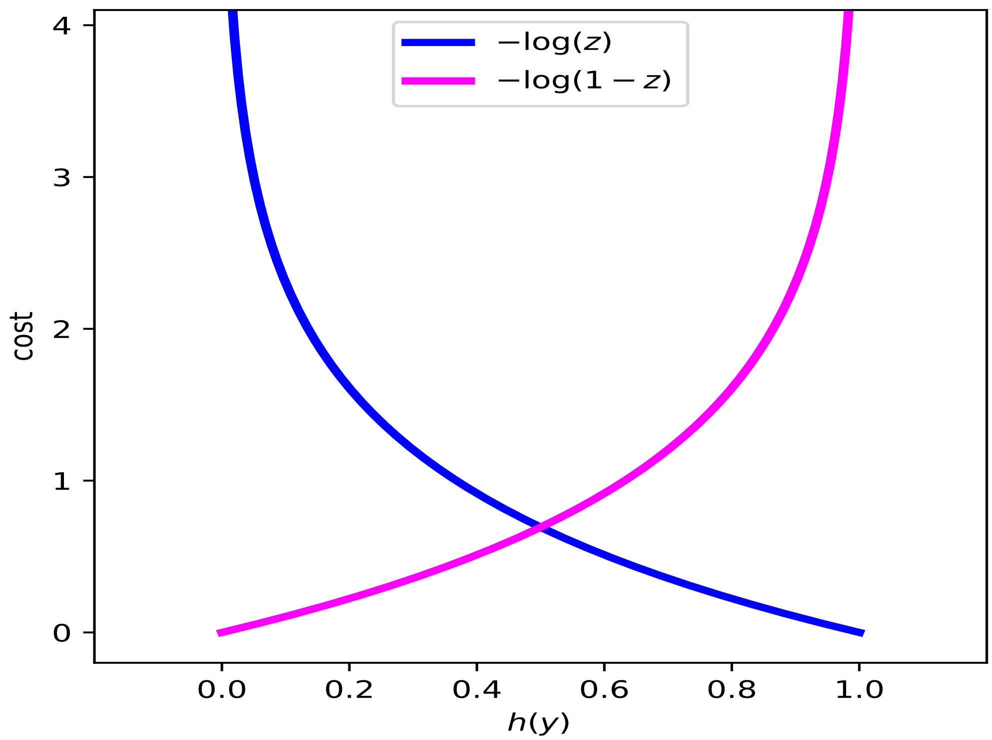
Cost Function



- minimize cost ("loss"):

$$Q = -\sum_X [C(X)\log(h(X)) + (1-C)\log(1-h(X))]$$

Cost Intuition



- correct classification cost: 0
- misclassification cost: $\mapsto \infty$

Computing Gradient

- gradient (with respect to w_i):

$$\frac{\partial Q}{\partial w_i} = \sum_X \Big[h\big(X\big) - C\big(X\big)\Big] \cdot x_i$$

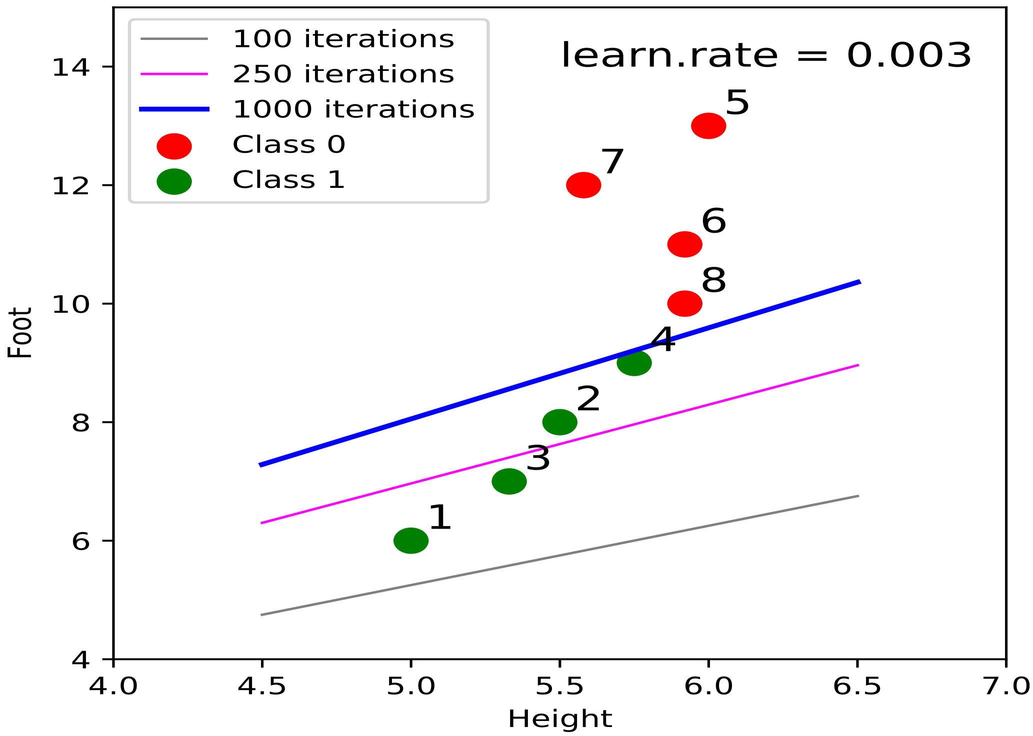
- computation of weights:

1. initialize weights
2. (simultaneously) update

$$w_i = w_i - \alpha \sum_X \left[h(X) - C(X)\right] x_i$$

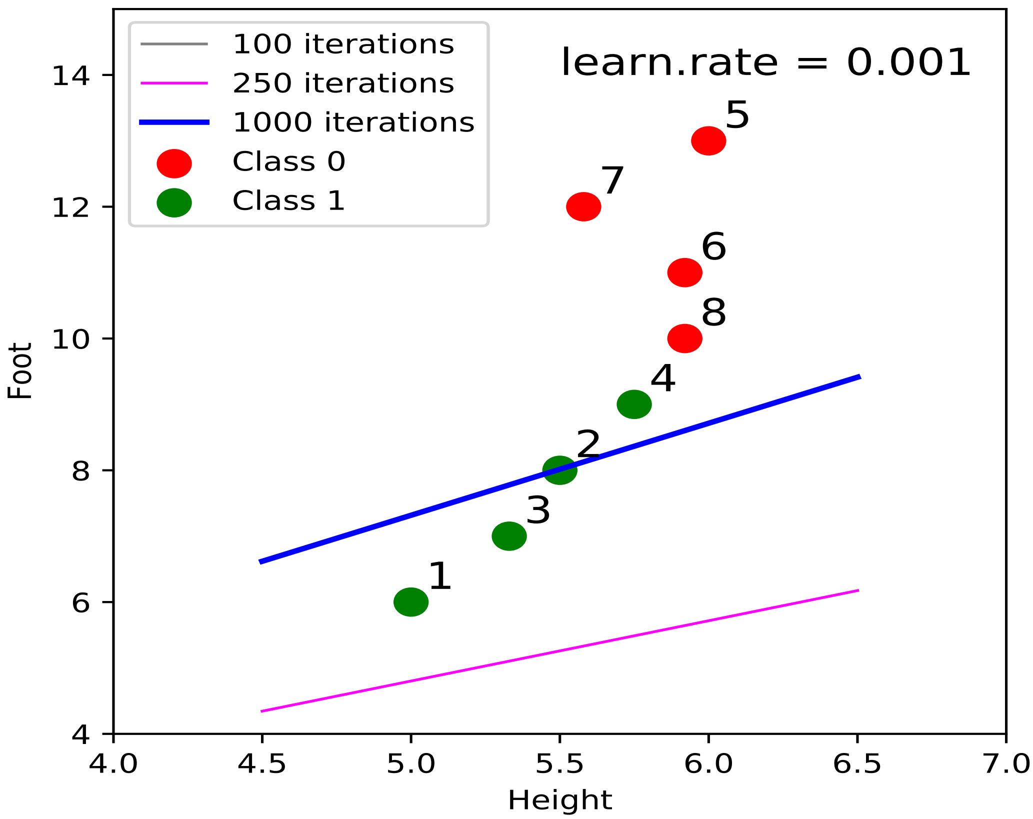
3. α is the learning rate

Computing Weights



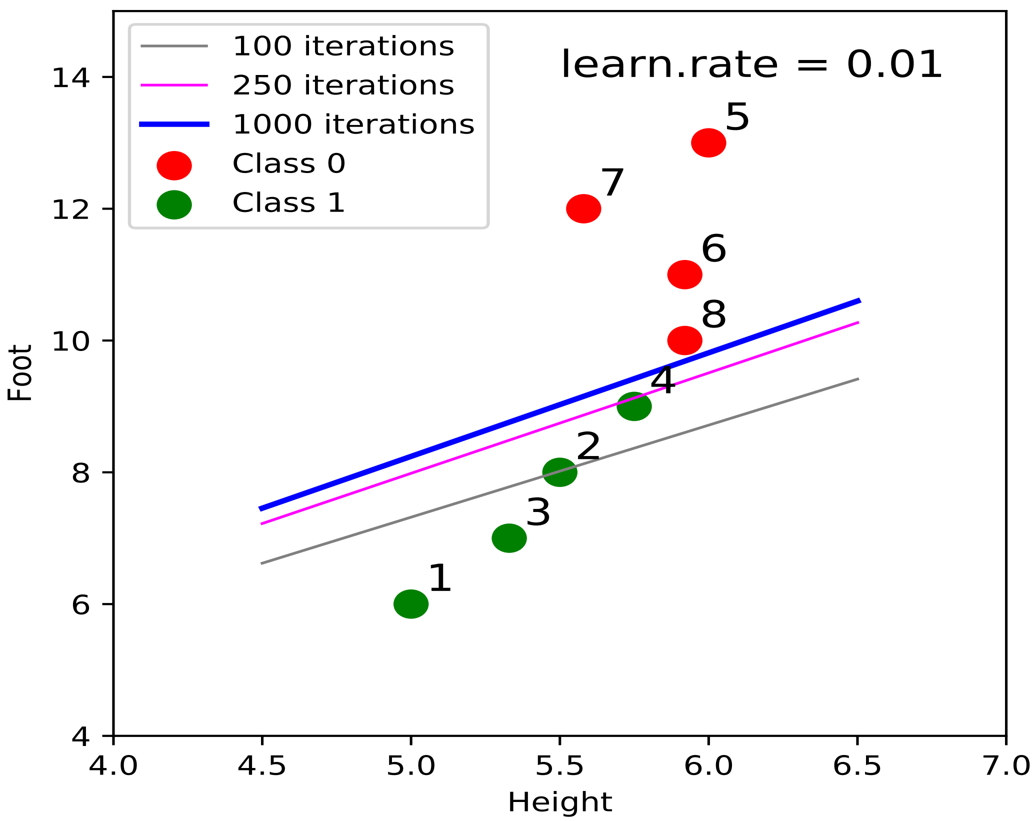
iterations	w_0	w_1	w_2	accuracy
100	0.021	0.084	-0.084	50%
250	0.053	0.221	-0.166	75%
1000	0.177	0.741	-0.482	100%

Effect of Lower Rate



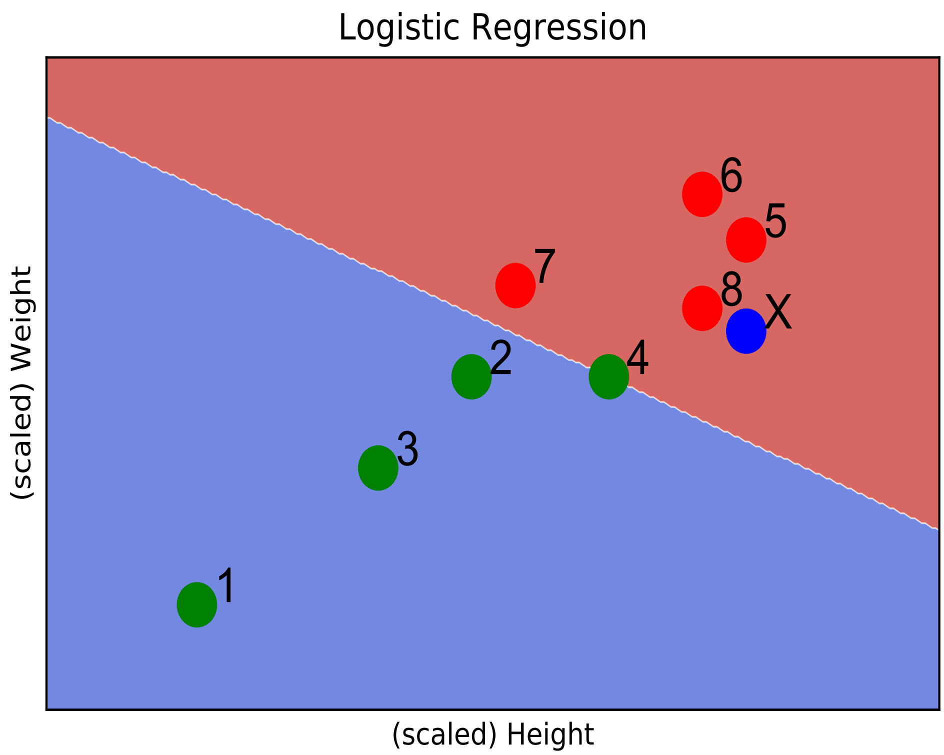
- need more iterations

Effect of Higher Rate



- get higher accuracy for the same number of iterations

Logistic Regression (original dataset)



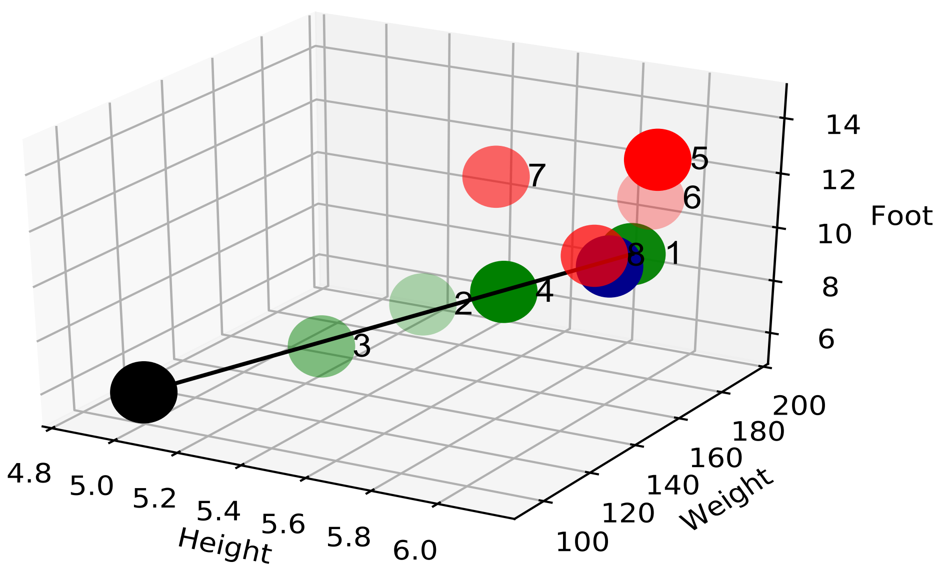
- predict(x^*)=red
- accuracy = 100%

Code: Log. Regression

```
import pandas as pd
import numpy as np
from sklearn.linear_model import LogisticRegression
from sklearn.preprocessing import StandardScaler, LabelEncoder

data = pd.DataFrame( {'id': [ 1,2,3,4,5,6,7,8],
        'Label': ['green', 'green', 'green', 'green',
                        'red', 'red', 'red', 'red'],
        'Height': [5, 5.5, 5.33, 5.75, 6.00, 5.92,  5.58, 5.92],
        'Weight': [100, 150, 130, 150, 180, 190, 170, 165],
        'Foot': [6, 8, 7, 9, 13, 11, 12, 10]},
         columns = ['id', 'Height', 'Weight', 'Foot', 'Label'] )

X = data[['Height', 'Weight']].values
scaler = StandardScaler()
scaler.fit(X)
X = scaler.transform(X)
Y = data['Label'].values

log_reg_classifier = LogisticRegression()
log_reg_classifier.fit(X,Y)

new_x = scaler.transform(np.asmatrix([6, 160]))
predicted = log_reg_classifier.predict(new_x)
accuracy = log_reg_classifier.score(X, Y)
```

```
ipdb> predicted[0]
red
ipdb> accuracy
0.875
```

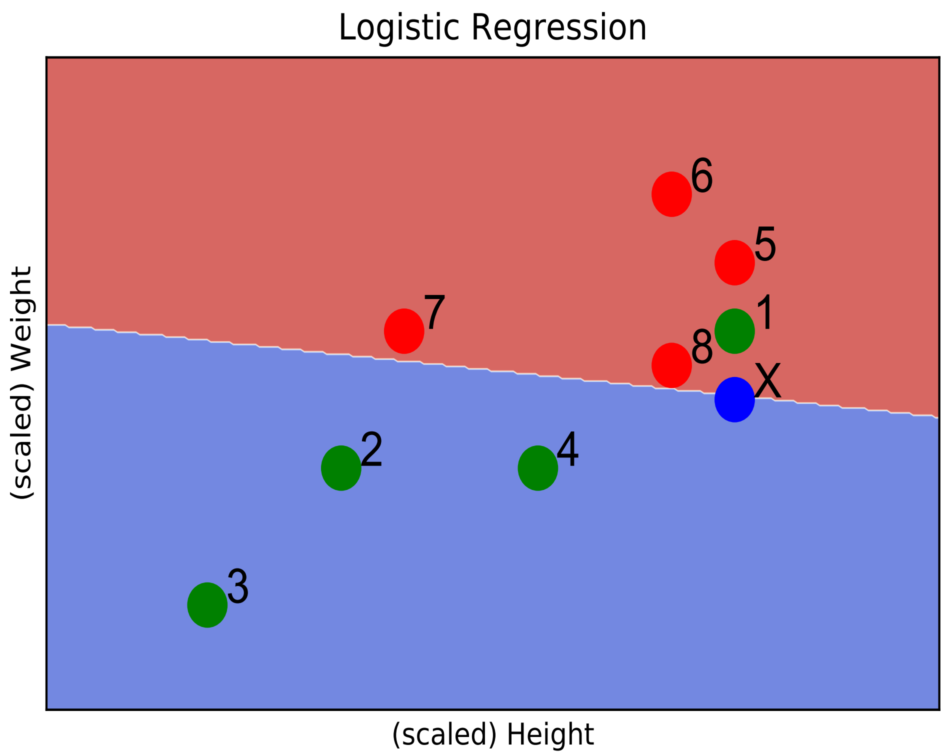
F/W/H Change



id	Height	Weight	Foot	Label
1	$5 \mapsto 6$	$100 \mapsto 170$	$6 \mapsto 10$	green

$$(\mathrm{H}{=}6,\ \mathrm{W}{=}160,\ \mathrm{F}{=}10) \mapsto ?$$

Logistic Regression (modified dataset)



- predict(x^*)=green
- accuracy = 87.5%

Code: Log. Regression (modified dataset)

```
import pandas as pd
import numpy as np
from sklearn.linear_model import LogisticRegression
from sklearn.preprocessing import StandardScaler, LabelEncoder

data = pd.DataFrame( {'id': [ 1,2,3,4,5,6,7,8],
        'Label': ['green', 'green', 'green', 'green',
                        'red', 'red', 'red', 'red'],
        'Height': [5, 5.5, 5.33, 5.75, 6.00, 5.92,  5.58, 5.92],
        'Weight': [100, 150, 130, 150, 180, 190, 170, 165],
        'Foot': [6, 8, 7, 9, 13, 11, 12, 10]},
         columns = ['id', 'Height', 'Weight', 'Foot', 'Label'] )

data['Height'].iloc[0] = 6;
data['Weight'].iloc[0] = 170;
data['Foot'].iloc[0] = 10
X = data[['Height', 'Weight']].values
scaler = StandardScaler().fit(X)

X = scaler.transform(X)
Y = data['Label'].values
log_reg_classifier = LogisticRegression()
log_reg_classifier.fit(X,Y)
new_x = scaler.transform(np.asmatrix([6, 160]))
predicted = log_reg_classifier.predict(new_x)
accuracy = log_reg_classifier.score(X, Y)
```

```
ipdb> predicted[0]
green
ipdb> accuracy
0.875
```

Categorical Dataset

Day	Weather	Temperature	Wind	Play
1	sunny	hot	low	**no**
2	rainy	mild	high	**yes**
3	sunny	cold	low	**yes**
4	rainy	cold	high	**no**
5	sunny	cold	high	**yes**
6	overcast	mild	low	**yes**
7	sunny	hot	low	**yes**
8	overcast	hot	high	**yes**
9	rainy	hot	high	**no**
10	rainy	mild	low	**yes**

- $x^* = (\text{sunny, cold, low}) \mapsto ?$
- need numeric values for attributes

Change to Dummy Variables

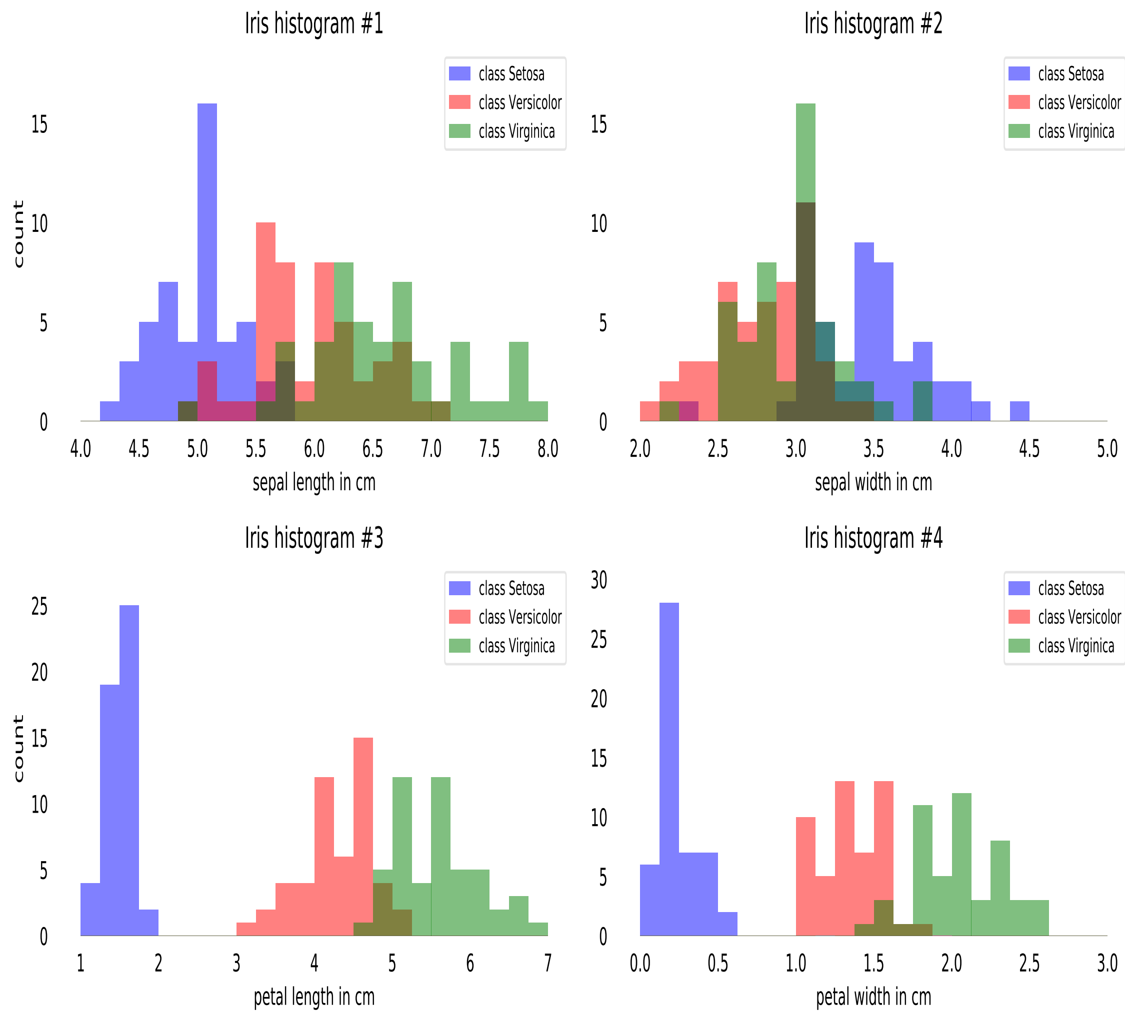
Day	Weather			Temp.			Wind	
	overcast	rainy	sunny	cold	hot	mild	high	low
1	0	0	1	0	1	0	0	1
2	0	1	0	0	0	1	1	0
3	0	0	1	1	0	0	0	1
4	0	1	0	1	0	0	1	0
5	0	0	1	1	0	0	1	0
6	1	0	0	0	0	1	0	1
7	0	0	1	0	1	0	0	1
8	1	0	0	0	1	0	1	0
9	0	1	0	0	1	0	1	0
10	0	1	0	0	0	1	0	1

Python Code

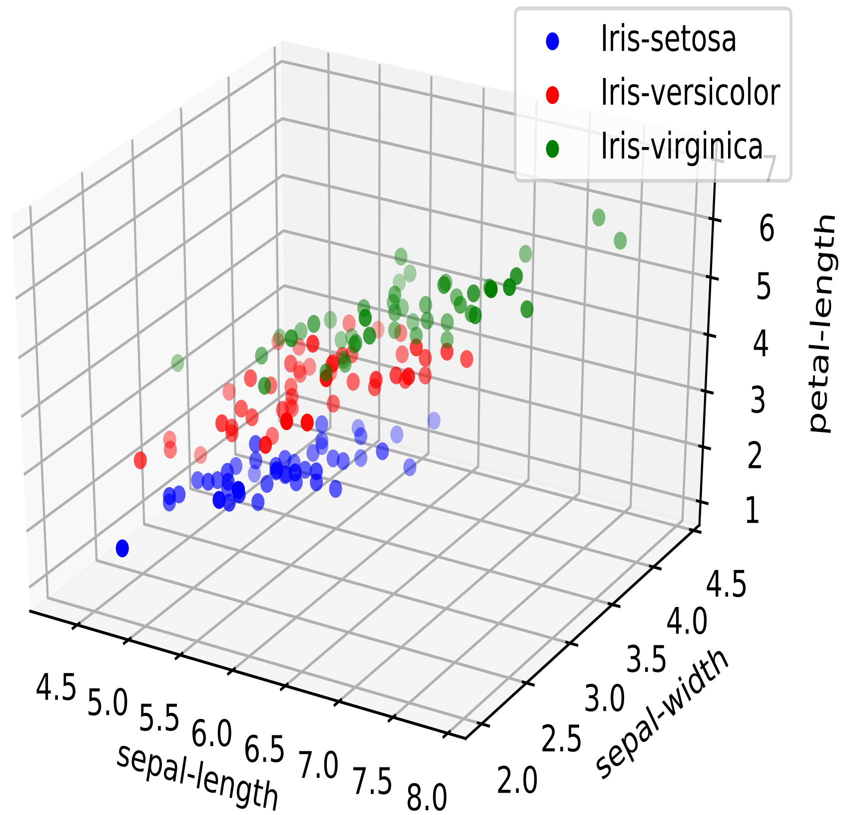
```
import numpy as np
import pandas as pd
from sklearn.linear_model import LogisticRegression
from sklearn.preprocessing import LabelEncoder
data = pd.DataFrame(
        {'Day':          [1,2,3,4,5,6,7,8,9,10],
        'Weather':       ['sunny','rainy','sunny','rainy',
                          'sunny','overcast','sunny','overcast',
                          'rainy','rainy'],
        'Temperature': ['hot', 'mild', 'cold','cold','cold',
                          'mild','hot','hot', 'hot','mild'],
        'Wind':          ['low','high','low','high','high',
                          'low','low', 'high','high','low'],
        'Play':          ['no', 'yes','yes','no','yes',
                          'yes','yes','yes','no','yes']},
        columns = ['Day','Weather','Temperature','Wind','Play'])
input_data = data[['Weather', 'Temperature', 'Wind']]
dummies = [pd.get_dummies(data[c]) for c in input_data.columns]
binary_data = pd.concat(dummies, axis=1)
X = binary_data[0:10].values
le = LabelEncoder()
Y = le.fit_transform(data['Play'].values)
log_reg_classifier = LogisticRegression()
log_reg_classifier.fit(X,Y)

# sunny -> (0,0,1), cold-> (0,1,0), low -> (0,1)
new_instance = np.asmatrix([0,0,1,1,0,0,0,1])
prediction = log_reg_classifier.predict(new_instance)
accuracy = log_reg_classifier.score(X, Y)
```

```
ipdb> prediction[0]
1
ipdb> accuracy
0.8
```

Iris Histograms



Iris Dataset:

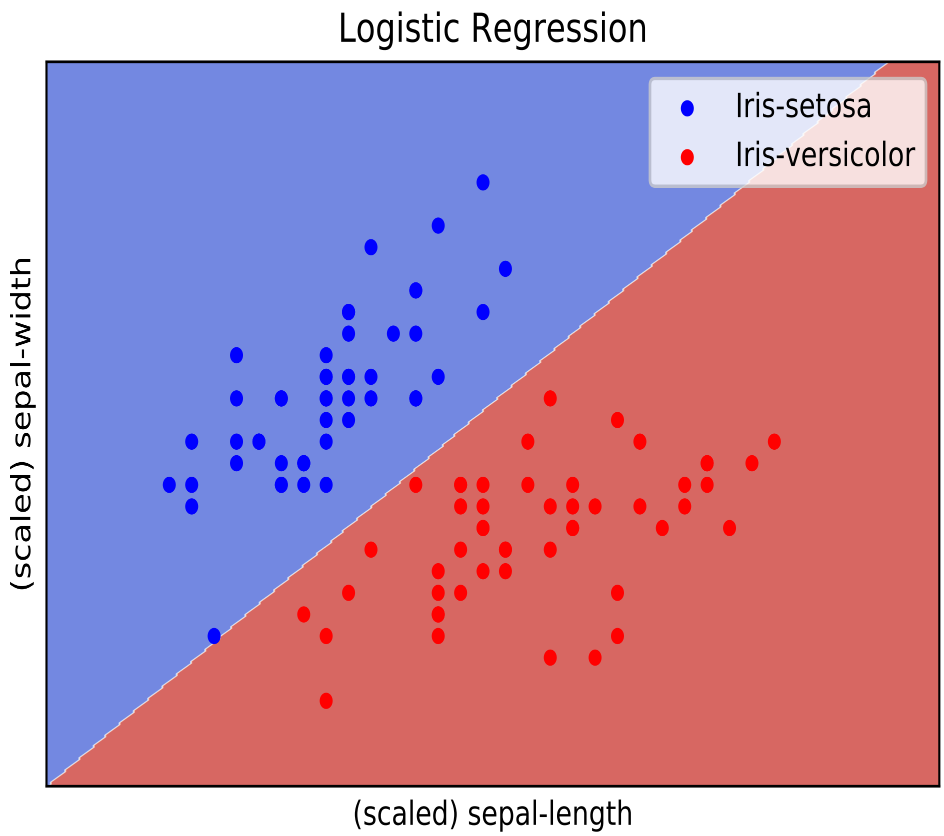


Iris: Python Code

```
import pandas as pd
from sklearn.linear_model import LogisticRegression
from sklearn.preprocessing import LabelEncoder
from sklearn.model_selection import train_test_split

url = r'https://archive.ics.uci.edu/ml/' + \
          r'machine-learning-databases/iris/iris.data'

data = pd.read_csv(url, names=['sepal-length', 'sepal-width',
                     'petal-length', 'petal-width', 'Class'])

features = ['sepal-length', 'sepal-width']
class_labels = ['Iris-setosa', 'Iris-versicolor']

X = data[features].values

le = LabelEncoder()
Y = le.fit_transform(data['Class'].values)

X_train,X_test,Y_train,Y_test = train_test_split(X, Y,
                       test_size=0.5, random_state=3)
log_reg_classifier = LogisticRegression()
log_reg_classifier.fit(X_train,Y_train)

prediction = log_reg_classifier.predict(X_test)
accuracy = np.mean(prediction == Y_test)
```

```
ipdb> accuracy
1.0
```

Iris: Logistic Regression



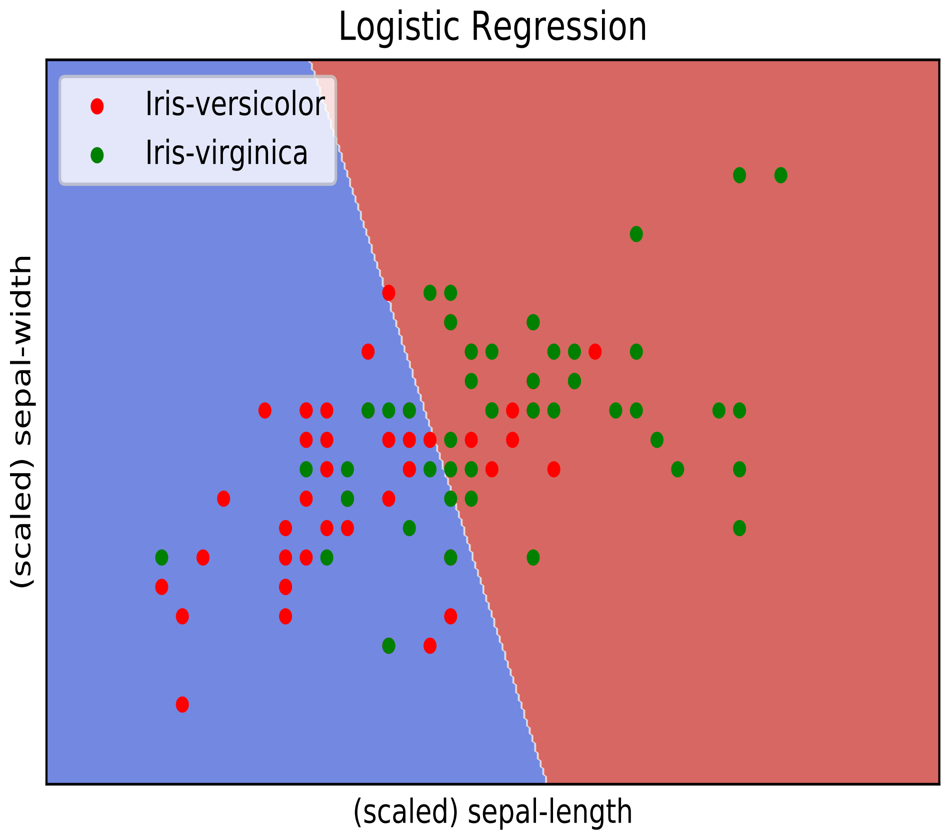
- accuracy = 100%
- easy to separate

Iris: Python Code

```
import pandas as pd
from sklearn.linear_model import LogisticRegression
from sklearn.preprocessing import LabelEncoder
from sklearn.model_selection import train_test_split

url = r'https://archive.ics.uci.edu/ml/' + \
          r'machine-learning-databases/iris/iris.data'

data = pd.read_csv(url, names=['sepal-length', 'sepal-width',
                     'petal-length', 'petal-width', 'Class'])

features = ['sepal-length', 'sepal-width']
class_labels = ['Iris-versicolor', 'Iris-virginica']

X = data[features].values

le = LabelEncoder()
Y = le.fit_transform(data['Class'].values)

X_train,X_test,Y_train,Y_test = train_test_split(X, Y,
                      test_size=0.5, random_state=3)
log_reg_classifier = LogisticRegression()
log_reg_classifier.fit(X_train,Y_train)

prediction = log_reg_classifier.predict(X_test)
accuracy = np.mean(prediction == Y_test)
```

```
ipdb> accuracy
0.68
```

Iris: Logistic Regression



- accuracy = 68%
- difficult to separate

Concepts Check:

(a) linear separability
(b) logistic vs. linear regression
(c) odds and logit function
(d) computing weights
(e) analysis of categorical data