Linear Algebra in Image Processing

Examples of Static Images Compression and Sequence of Images Motion Estimation

Khang Vo Huynh, Soua Yang

Haar Wavelet

- 1D grayscale image example
- 8x8 matrix
- Linear Algebra Approach

1D grayscale images example

[1 , 2 , 3 , 4 , 20 , 22 , 24 , 28]

$\frac{x_i + x_{i+1}}{2}$ for $i \in \{1,3,5,7\}$ $\qquad$ $x_i - \frac{x_i + x_{i+1}}{2}$ for $i \in \{1,3,5,7\}$

[[1.5 , 3.5 , 21 , 25] , [-0.5 , -0.5 , -1 , -2]]

Approximation Coefficient

Detail Coefficient

$\frac{x_i + x_{i+1}}{2}$ for $i \in \{1,3\}$ $\qquad$ $x_i - \frac{x_i + x_{i+1}}{2}$ for $i \in \{1,3\}$

[[2.5, 23.5] , [-1 , -2.5] , [-0.5 , -0.5 , -1 , -2]]

$\frac{x_1 + x_2}{2}$ $\qquad$ $x_1 - \frac{x_1 + x_2}{2}$

[13 , -10.5 , -1 , -2.5 , -0.5 , -0.5 , -1 , -2]

8x8 matrix

$$
\begin{bmatrix}
90 & 68 & 72 & 92 & 35 & 40 & 42 & 49 \\
92 & 87 & 72 & 63 & 21 & 15 & 29 & 45 \\
29 & 83 & 88 & 32 & 65 & 43 & 22 & 44 \\
12 & 28 & 39 & 55 & 92 & 78 & 54 & 85 \\
26 & 63 & 45 & 58 & 98 & 65 & 56 & 65 \\
76 & 65 & 89 & 45 & 38 & 54 & 86 & 92 \\
92 & 65 & 66 & 34 & 54 & 77 & 88 & 33 \\
32 & 92 & 83 & 88 & 45 & 37 & 32 & 65
\end{bmatrix}
\xrightarrow[\text{2. Column Transform}]{\text{1. Row Transform}}
\begin{bmatrix}
\frac{3765}{64} & \frac{277}{64} & -\frac{21}{32} & -\frac{15}{16} & -\frac{51}{8} & \frac{87}{16} & \frac{39}{16} & -\frac{69}{16} \\
-\frac{243}{64} & \frac{209}{64} & -\frac{27}{32} & \frac{17}{8} & 1 & -\frac{29}{16} & \frac{35}{16} & -\frac{83}{16} \\
\frac{63}{32} & \frac{477}{32} & \frac{25}{4} & -\frac{127}{16} & \frac{97}{8} & -\frac{51}{8} & -\frac{35}{8} & \frac{15}{4} \\
\frac{19}{16} & -\frac{13}{2} & -\frac{17}{16} & -\frac{39}{16} & \frac{7}{8} & \frac{1}{2} & 4 & -\frac{37}{8} \\
4 & -3 & -\frac{25}{4} & \frac{11}{4} & \frac{17}{4} & -\frac{29}{4} & -\frac{11}{4} & \frac{9}{4} \\
-\frac{37}{16} & \frac{233}{16} & \frac{23}{4} & \frac{11}{8} & -\frac{19}{2} & 18 & 2 & \frac{9}{4} \\
-\frac{69}{16} & -\frac{97}{16} & -\frac{21}{8} & 16 & -12 & -\frac{57}{4} & \frac{49}{4} & -\frac{3}{4} \\
\frac{35}{16} & -\frac{111}{16} & 13 & \frac{25}{8} & \frac{87}{4} & \frac{37}{4} & -\frac{31}{4} & 22
\end{bmatrix}
$$

Linear Algebra Approach

$$W_1 = \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & -\frac{1}{2} \end{bmatrix}$$

$$W_3 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$W_2 = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Linear Algebra Approach

$$W = W_1 W_2 W_3 = \begin{bmatrix} \frac{1}{8} & \frac{1}{8} & \frac{1}{4} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{8} & \frac{1}{8} & \frac{1}{4} & 0 & -\frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{8} & \frac{1}{8} & -\frac{1}{4} & 0 & 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{8} & \frac{1}{8} & -\frac{1}{4} & 0 & 0 & -\frac{1}{2} & 0 & 0 \\ \frac{1}{8} & -\frac{1}{8} & 0 & \frac{1}{4} & 0 & 0 & \frac{1}{2} & 0 \\ \frac{1}{8} & -\frac{1}{8} & 0 & \frac{1}{4} & 0 & 0 & -\frac{1}{2} & 0 \\ \frac{1}{8} & -\frac{1}{8} & 0 & -\frac{1}{4} & 0 & 0 & 0 & \frac{1}{2} \\ \frac{1}{8} & -\frac{1}{8} & 0 & -\frac{1}{4} & 0 & 0 & 0 & -\frac{1}{2} \end{bmatrix}$$

$$B = W^T A W = \begin{bmatrix} \frac{3765}{64} & \frac{277}{64} & -\frac{21}{32} & -\frac{15}{16} & -\frac{51}{8} & \frac{87}{16} & \frac{39}{16} & -\frac{69}{16} \\ -\frac{243}{64} & \frac{209}{64} & -\frac{27}{32} & \frac{17}{8} & 1 & -\frac{29}{16} & \frac{35}{16} & -\frac{83}{16} \\ \frac{63}{32} & \frac{477}{32} & \frac{25}{4} & -\frac{127}{16} & \frac{97}{8} & -\frac{51}{8} & -\frac{35}{8} & \frac{15}{4} \\ \frac{19}{16} & -\frac{13}{2} & -\frac{17}{16} & -\frac{39}{16} & \frac{7}{8} & \frac{1}{2} & 4 & -\frac{37}{8} \\ 4 & -3 & -\frac{25}{4} & \frac{11}{4} & \frac{17}{4} & -\frac{29}{4} & -\frac{11}{4} & \frac{9}{4} \\ -\frac{37}{16} & \frac{233}{16} & \frac{23}{4} & \frac{11}{8} & -\frac{19}{2} & 18 & 2 & \frac{9}{4} \\ -\frac{69}{16} & -\frac{97}{16} & -\frac{21}{8} & 16 & -12 & -\frac{57}{4} & \frac{49}{4} & -\frac{3}{4} \\ \frac{35}{16} & -\frac{111}{16} & 13 & \frac{25}{8} & \frac{87}{4} & \frac{37}{4} & -\frac{31}{4} & 22 \end{bmatrix}$$

Larger Matrix - No problem

Step-by-step to Haar Wavelet transform a square matrix (2^r x 2^r)

- There will exists exactly r matrices W_i. Find these matrices and compute W. The algorithm to find these matrices will be provided inside the final paper
- Compute the transpose of W
- Finally Compute the transform matrix

$$W = \prod_{i=1}^{r} W_i$$

$$B = W^T A W$$

Properties of Haar Wavelet Transform

- W_i is orthogonal
- W is orthogonal
- W is invertible which means the transform is reversible
- Normalizing W result an unitary matrix

$$B = W^T A W = ((AW)^T W)^T \text{ and } A = (W^{-1})^T B W^{-1} = ((B^T) W^{-1})^T W^{-1}$$

Threshold (Image compression)

$$B = \begin{bmatrix} \frac{3765}{64} & \frac{277}{64} & -\frac{21}{32} & -\frac{15}{16} & -\frac{51}{8} & \frac{87}{16} & \frac{39}{16} & -\frac{69}{16} \\ -\frac{243}{64} & \frac{209}{64} & -\frac{27}{32} & \frac{17}{8} & 1 & -\frac{29}{16} & \frac{35}{16} & -\frac{83}{16} \\ \frac{63}{32} & \frac{477}{32} & \frac{25}{4} & -\frac{127}{16} & \frac{97}{8} & -\frac{51}{8} & -\frac{35}{8} & \frac{15}{4} \\ \frac{19}{16} & -\frac{13}{2} & -\frac{17}{16} & -\frac{39}{16} & \frac{7}{8} & \frac{1}{2} & 4 & -\frac{37}{8} \\ 4 & -3 & -\frac{25}{4} & \frac{11}{4} & \frac{17}{4} & -\frac{29}{4} & -\frac{11}{4} & \frac{9}{4} \\ -\frac{37}{16} & \frac{233}{16} & \frac{23}{4} & \frac{11}{8} & -\frac{19}{2} & 18 & 2 & \frac{9}{4} \\ -\frac{69}{16} & -\frac{97}{16} & -\frac{21}{8} & 16 & -12 & -\frac{57}{4} & \frac{49}{4} & -\frac{3}{4} \\ \frac{35}{16} & -\frac{111}{16} & 13 & \frac{25}{8} & \frac{87}{4} & \frac{37}{4} & -\frac{31}{4} & 22 \end{bmatrix}$$

Threshold = 0

$$\text{Compression Ratio} = \frac{64}{58} = 1.103$$

$$B_l = \begin{bmatrix} \frac{3765}{64} & \frac{277}{64} & -\frac{21}{32} & 0 & -\frac{51}{8} & \frac{87}{16} & \frac{39}{16} & -\frac{69}{16} \\ -\frac{243}{64} & \frac{209}{64} & 0 & \frac{17}{8} & 0 & -\frac{29}{16} & \frac{35}{16} & -\frac{83}{16} \\ \frac{63}{32} & \frac{477}{32} & \frac{25}{4} & -\frac{127}{16} & \frac{97}{8} & -\frac{51}{8} & -\frac{35}{8} & \frac{15}{4} \\ \frac{19}{16} & -\frac{13}{2} & -\frac{17}{16} & -\frac{39}{16} & 0 & 0 & 4 & -\frac{37}{8} \\ 4 & -3 & -\frac{25}{4} & \frac{11}{4} & \frac{17}{4} & -\frac{29}{4} & -\frac{11}{4} & \frac{9}{4} \\ -\frac{37}{16} & \frac{233}{16} & \frac{23}{4} & \frac{11}{8} & -\frac{19}{2} & 18 & 2 & \frac{9}{4} \\ -\frac{69}{16} & -\frac{97}{16} & -\frac{21}{8} & 16 & -12 & -\frac{57}{4} & \frac{49}{4} & 0 \\ \frac{35}{16} & -\frac{111}{16} & 13 & \frac{25}{8} & \frac{87}{4} & \frac{37}{4} & -\frac{31}{4} & 22 \end{bmatrix}$$

Threshold = 1

Threshold (Image compression)

$$\begin{bmatrix} 90 & 68 & 72 & 92 & 35 & 40 & 42 & 49 \\ 92 & 87 & 72 & 63 & 21 & 15 & 29 & 45 \\ 29 & 83 & 88 & 32 & 65 & 43 & 22 & 44 \\ 12 & 28 & 39 & 55 & 92 & 78 & 54 & 85 \\ 26 & 63 & 45 & 58 & 98 & 65 & 56 & 65 \\ 76 & 65 & 89 & 45 & 38 & 54 & 86 & 92 \\ 92 & 65 & 66 & 34 & 54 & 77 & 88 & 33 \\ 32 & 92 & 83 & 88 & 45 & 37 & 32 & 65 \end{bmatrix}$$

Original Reconstruction

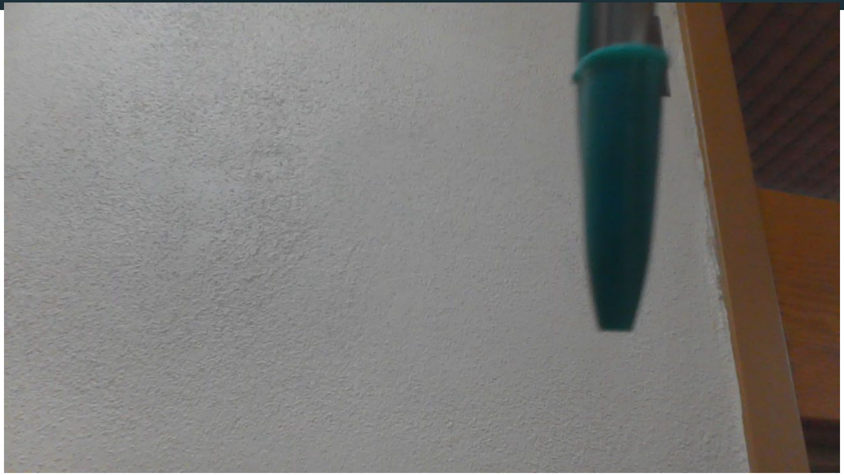
$$A_a = (W^T)^{-1} B_l W^{-1} = \begin{bmatrix} 89 & 69 & 72 & 92 & \frac{575}{16} & \frac{655}{16} & \frac{657}{16} & \frac{769}{16} \\ 91 & 88 & 72 & 63 & \frac{351}{16} & \frac{255}{16} & \frac{449}{16} & \frac{705}{16} \\ 28 & 84 & 88 & 32 & \frac{1055}{16} & \frac{703}{16} & \frac{337}{16} & \frac{689}{16} \\ 11 & 29 & 39 & 55 & \frac{1487}{16} & \frac{1263}{16} & \frac{849}{16} & \frac{1345}{16} \\ \frac{209}{8} & \frac{503}{8} & \frac{89}{2} & \frac{117}{2} & \frac{1583}{16} & \frac{1055}{16} & \frac{893}{16} & \frac{1013}{16} \\ \frac{609}{8} & \frac{519}{8} & \frac{177}{2} & \frac{91}{2} & \frac{623}{16} & \frac{879}{16} & \frac{1349}{16} & \frac{1469}{16} \\ \frac{751}{8} & \frac{505}{8} & \frac{133}{2} & \frac{67}{2} & \frac{879}{16} & \frac{1247}{16} & \frac{1393}{16} & \frac{513}{16} \\ \frac{271}{8} & \frac{721}{8} & \frac{167}{2} & \frac{175}{2} & \frac{735}{16} & \frac{607}{16} & \frac{497}{16} & \frac{1025}{16} \end{bmatrix}$$

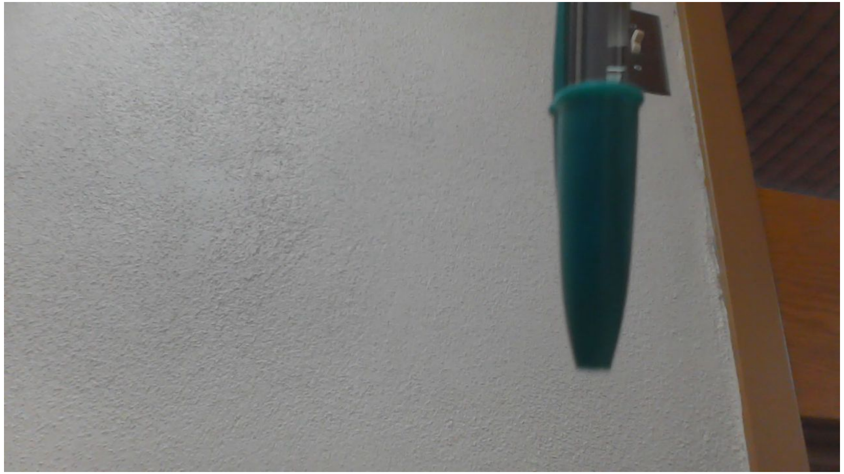
Compressed Reconstruction

Motion Estimation

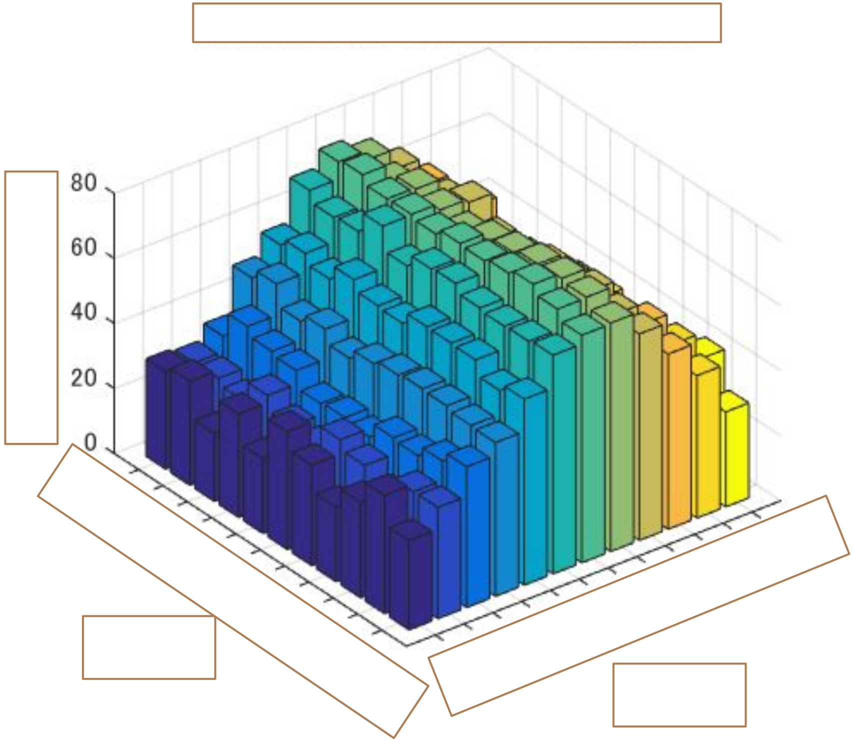
1

2





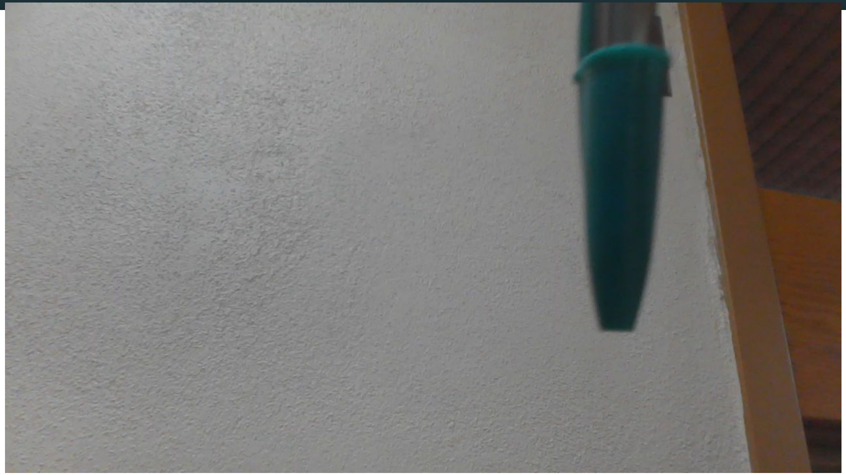
f(x, y)

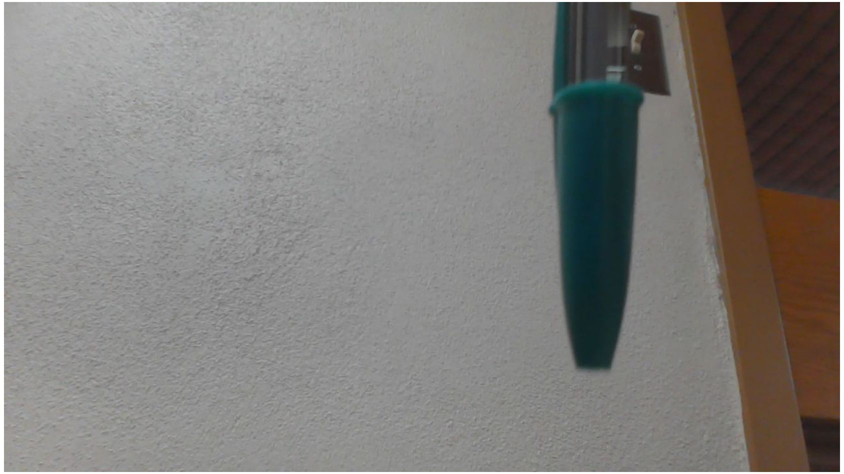


$f(x(t), y(t), t)$

1

2





Brightness Constancy Assumption

$$\frac{\partial f(x(t), y(t), t)}{\partial t} = 0$$

$$\frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial t} = 0$$

$$\frac{\partial f}{\partial x}v_x + \frac{\partial f}{\partial y}v_y + \frac{\partial f}{\partial t} = 0$$

$$f_x v_x + f_y v_y + f_t = 0$$

$$\begin{bmatrix} f_x & f_y \end{bmatrix} \begin{bmatrix} v_x \\ v_y \end{bmatrix} + f_t = 0$$

$$\begin{bmatrix} f_x(x_1, y_1) & f_y(x_1, y_1) \\ f_x(x_2, y_2) & f_y(x_2, y_2) \\ \vdots & \vdots \\ f_x(x_9, y_9) & f_y(x_9, y_9) \end{bmatrix} \begin{bmatrix} v_x \\ v_y \end{bmatrix} + \begin{bmatrix} f_t(x_1, y_1) \\ f_t(x_2, y_2) \\ \vdots \\ f_t(x_9, y_9) \end{bmatrix} = 0$$

$$Av + b = 0$$

$$Av + b = 0$$

$$\boxed{E(v) = ||Av + b||^2}$$

$$\frac{dE(v)}{dv} = 2A^T(Av + b) = 0$$

$$2A^TAv + 2A^Tb = 0$$

$$A^TAv + A^Tb = 0$$

$$A^TAv = -A^Tb$$

$$v = -(A^TA)^{-1}A^Tb$$

$$v = -(A^T A)^{-1} A^T b$$

$$A^T A = \begin{bmatrix} f_x(x_1, y_1) & f_x(x_2, y_2) & \dots & f_x(x_9, y_9) \\ f_y(x_1, y_1) & f_y(x_2, y_2) & \dots & f_x(x_9, y_9) \end{bmatrix} \begin{bmatrix} f_x(x_1, y_1) & f_y(x_1, y_1) \\ f_x(x_2, y_2) & f_y(x_2, y_2) \\ \vdots & \vdots \\ f_x(x_9, y_9) & f_y(x_9, y_9) \end{bmatrix} = \begin{bmatrix} \sum f_x^2 & \sum f_y f_x \\ \sum f_x f_y & \sum f_y^2 \end{bmatrix}$$

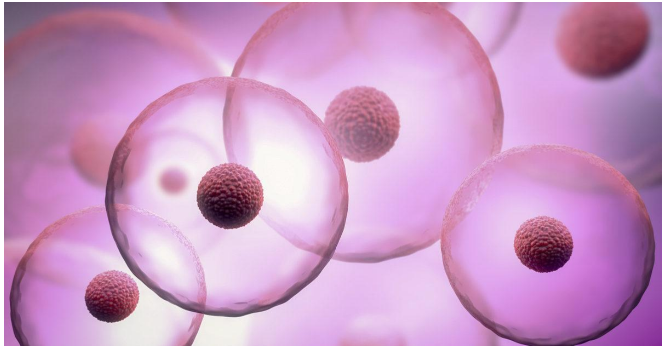
Conclusion

Haar Wavelet Transform

- Lossy Image Compression (Threshold > 0)
- Lossless Image Compression (Threshold = 0)
- Progressive Image Transmission (Multiple transforms)

Motion Estimation

- Autonomous vehicles
- Cell tracking



Reference

Dazhi, Yang, et al. “Block Matching Algorithms: Their Applications and Limitations in Solar Irradiance Forecasting.” *Energy Procedia*, Elsevier, 21 June 2013, https://www.sciencedirect.com/science/article/pii/S1876610213000842.

Mulcahy, Colm. *Image Compression Using the Haar Wavelet Transform*. Jan. 1997, https://www.researchgate.net/publication/265217188_Image_compression_using_the_Haar_wavelet_transform.

Haar Wavelet Image Compression - Ohio State University. https://people.math.osu.edu/husen.1/teaching/572/image_comp.pdf.

Fundamentals of Image Processing - UC Berkeley. https://farid.berkeley.edu/downloads/tutorials/fip.pdf.

Chao, Haiyang, et al. “A Survey of Optical Flow Techniques for Robotics Navigation Applications.” *Journal of Intelligent & Robotic Systems*, vol. 73, no. 1-4, 2013, pp. 361–372., https://doi.org/10.1007/s10846-013-9923-6.

Hayashida, Junya, and Ryoma Bise. “Cell Tracking with Deep Learning for Cell Detection and Motion Estimation in Low-Frame-Rate.” *Lecture Notes in Computer Science*, 2019, pp. 397–405., https://doi.org/10.1007/978-3-030-32239-7_44.

Thank you