RiemannZetaFct_and_RiemannHypothesis

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1 Riemann's Zeta-Function and Riemann's Hypothesis

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1.1 Item1: Riemann's Zeta-Function

See: https://en.wikipedia.org/wiki/Riemann_zeta_function

or the following YouTube Video: https://www.youtube.com/watch?v=sZhl6PyTflw&vl=en

The Riemann zeta function or Euler–Riemann zeta function, (s), is a function of a complex variable s that analytically continues the sum of the Dirichlet serie which converges when the real part of s is greater than 1.

More general representations of (s) for all s are given below. The Riemann zeta function plays a pivotal role in analytic number theory and has applications in physics, probability theory, and applied statistics. As a function of a real variable, Leonhard Euler first introduced and studied it in the first half of the eighteenth century without using complex analysis, which was not available at the time. Bernhard Riemann's 1859 article "On the Number of Primes Less Than a Given Magnitude" extended the Euler definition to a complex variable, proved its meromorphic continuation and functional equation, and established a relation between its zeros and the distribution of prime numbers.[2]

The values of the Riemann zeta function at even positive integers were computed by Euler. The first of them, (2), provides a solution to the Basel problem. In 1979 Roger Apéry proved the irrationality of (3). The values at negative integer points, also found by Euler, are rational numbers and play an important role in the theory of modular forms. Many generalizations of the Riemann zeta function, such as Dirichlet series, Dirichlet L-functions and L-functions, are known.

[1]:
```
print ("*** DirichletForm of the Riemann Zeta-Fuction (Euler**** ")
print ("*** LATEX syntax of zeta-fct for re(z)>1: '$ displaystyle␣
 ↪\\zeta(s)=\sum_{n=1}^\infty 1/n^s $' ***")

from IPython.display import Image

Image('Images/DirichletForm4Riem-ZetaFct.jpg')
```

```
*** DirichletForm of the Riemann Zeta-Fuction (Euler****
*** LATEX syntax of zeta-fct for re(z)>1: '$ displaystyle
\zeta(s)=\sum_{n=1}^\infty 1/n^s $' ***
```

[1]:

Definition und elementare Darstellungsformen [Bearbeiten | Quelltext bearbeiten]

Dirichlet-Reihe [Bearbeiten | Quelltext bearbeiten]

Die Zeta-Funktion wird in der Literatur oft über ihre Darstellung als Dirichlet-Reihe definiert.

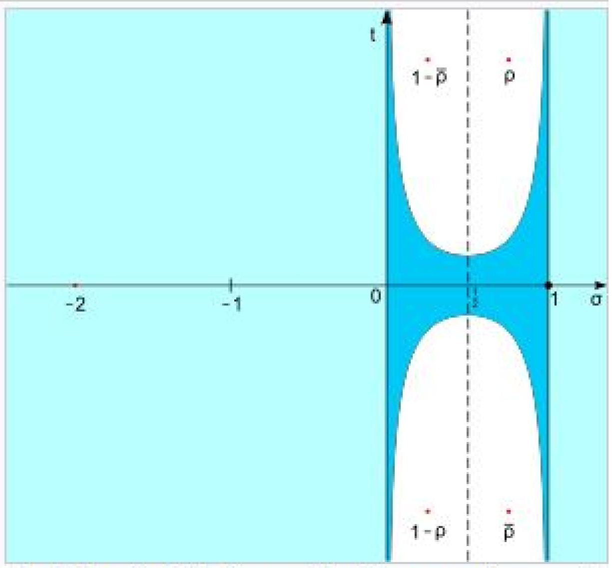
Für komplexe Zahlen s, deren Realteil größer als 1 ist, ist die Zeta-Funktion definiert durch die Dirichlet-Reihe[63]

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \frac{1}{6^s} + \frac{1}{7^s} + \cdots, \qquad n^s := \exp(s \log(n)).$$

[2]:
```
print ("*** Zero-free_region_for_the_Riemann_zeta-function**** ")
from IPython.display import Image

Image('Images/Zero-free_region_for_the_Riemann_zeta-function.jpg')
```

```
*** Zero-free_region_for_the_Riemann_zeta-function****
```

[2]:



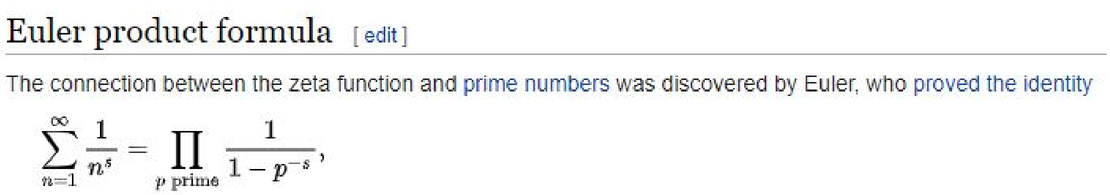
Apart from the trivial zeros, the Riemann zeta function has no zeros to the right of $\sigma = 1$ and to the left of $\sigma = 0$ (neither can the zeros lie too close to those lines). Furthermore, the non-trivial zeros are symmetric about the real axis and the line $\sigma = \frac{1}{2}$ and, according to the Riemann hypothesis, they all lie on the line $\sigma = \frac{1}{2}$.

[3]:
```
print␣
↪("**************************************************************************")
```

```python
print ("****** The bridge between zeta-fct in 'Complex Analysis' and prim-␣
 ↪*****")
print ("****** numbers in 'Number Theory' is given by EulerProduct formula␣
 ↪*****")
print␣
 ↪("************************************************************************")

from IPython.display import Image

Image('Images/EulerProduct.jpg')
```

```
************************************************************************
****** The bridge between zeta-fct in 'Complex Analysis' and prim- *****
****** numbers in 'Number Theory' is given by EulerProduct formula *****
************************************************************************
```

[3]:


Euler product formula [edit]

The connection between the zeta function and prime numbers was discovered by Euler, who proved the identity

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \frac{1}{1-p^{-s}},$$

1.2 Item2: Riemann's Hypothesis

See: https://en.wikipedia.org/wiki/Riemann_hypothesis

In mathematics, the Riemann hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part = 1/2. Many consider it to be the most important unsolved problem in pure mathematics.[1] It is of great interest in number theory because it implies results about the distribution of prime numbers. It was proposed by Bernhard Riemann (1859), after whom it is named. The Riemann hypothesis and some of its generalizations, along with Goldbach's conjecture and the twin prime conjecture, comprise Hilbert's eighth problem in David Hilbert's list of 23 unsolved problems; it is also one of the Clay Mathematics Institute's Millennium Prize Problems. The name is also used for some closely related analogues, such as the Riemann hypothesis for curves over finite fields.

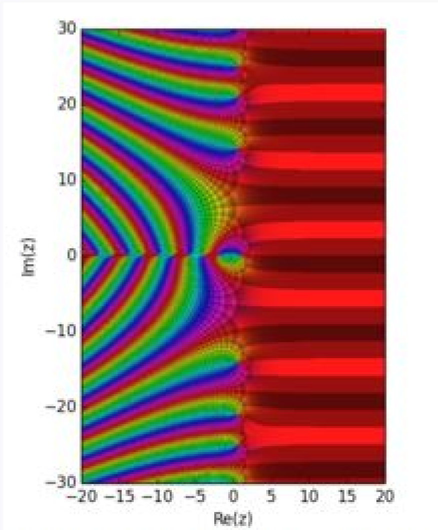
[4]:
```python
print (" ************************************************************** ")
print (" **** Here is an example-plot of the riemann zeta-function **** ")
print (" **** See non-trival zeros at 'critical' line real(z)=0.5 ***** ")
print (" **** This is a visualization of the Riemann-Hypothesis ******* ")
print (" ************************************************************** ")

from IPython.display import Image
```

```python
Image('Images/riemann-zeta1.jpg')
```

```
************************************************************
**** Here is an example-plot of the riemann zeta-function ****
**** See non-trival zeros at 'critical' line real(z)=0.5 *****
**** This is a visualization of the Riemann-Hypothesis *******
************************************************************
```

[4]:

Riemann zeta function



The Riemann zeta function $\zeta(z)$ plotted with domain coloring.[1]

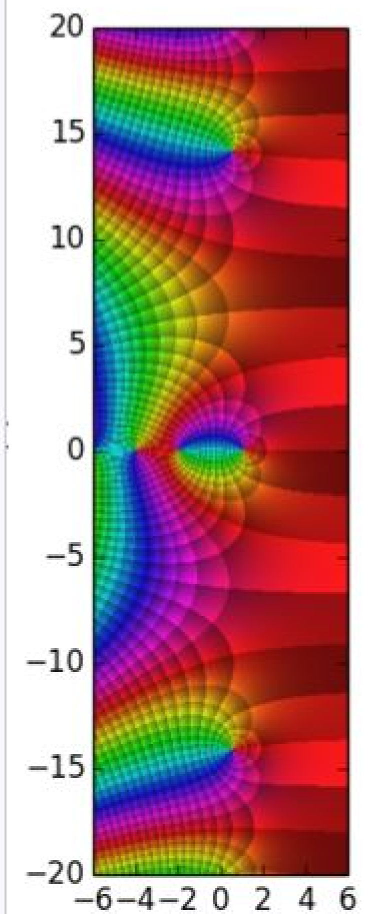
Basic features	
Domain	$\mathbb{C} \setminus \{1\}$
Codomain	$\mathbb{C}$

Specific values	
At zero	$-\frac{1}{2}$
Limit to +∞	1
Value at 2	$\frac{\pi^2}{6}$
Value at -1	$-\frac{1}{12}$
Value at -2	0

[5]:
```python
print ("*** Here is the example of a plot of the zeta function in more detail***")
print ("*** See two zeros at at the points z=0.5 + 14,12...z=0.5-14,12...***")
print ("*********************************************************************")

from IPython.display import Image

Image('Images/riemann-zeta2.jpg')
```

```
*** Here is the example of a plot of the zeta function in more detail***
*** See two zeros at at the points z=0.5 + 14,12…z=0.5-14,12…***
*********************************************************************
```

[5]:



The pole at $z = 1$, and two zeros on the critical line.

[6]:
```python
# Program/Source Code
# Here is the source code of a Python program to calculate the zeta function values
```

```
# The program output is shown below.

def zeta(s, t = 100):
    if s == 1:
        return float("inf")
    term = (1 / 2 ** (n + 1)
              * sum((-1) ** k * binom(n, k) * (k +1 ) ** -s
                    for k in range (n + 1))
           for n in count(0))
    return sum(islice(term, t)) / (1 - 2 ** (1- s))
```

[7]:
```
# Import libaries

from itertools import count, islice
from scipy.special import binom
```

[8]:
```
print ("*************************************")
print ("*** List of zeta-function values ***")
print ("*************************************")

# 1. zeta(-2)=0
print ("1. check zeta(-2)= 0:")
print ("zeta(-2) =",zeta(-2))

# 2. zeta(-1)=-1/12=-0,08333...
print ("*************************************")
print ("2. check zeta(-1)=-1/12=-0,08333...:")
print ("zeta(-1) =",zeta(-1))

# 3. zeta(0)=-1/2
print ("*************************************")
print ("3. check zeta(0)=-1/2:")
print ("zeta(0) =",zeta(0))

# 4. zeta(1)=unendlich
print ("*************************************")
print ("4. check zeta(1)=unendlich(inf):")
print ("zeta(1) =",zeta(1))

# 5. zeta(2)=pi²/6=1,64493...
print ("*************************************")
print ("5. check zeta(2)=pi²/6=1,644934...:")
print ("zeta(2) =",zeta(2))

# 6. zeta(3)=1,2020...
print ("*************************************")
print ("6. check zeta(3)= 1,202056...:")
```

```python
print ("zeta(3) =",zeta(3))

# 7. zeta(4)=(pi²)²/90
print ("**********************************")
print ("7. zeta(4)=((pi²))²/90 ~ 1,082323...:")
print ("zeta(4) =",zeta(4))
```

```
************************************
*** List of zeta-function values ***
************************************
1. check zeta(-2)= 0:
zeta(-2) = 1.5603186562147366e-13
************************************
2. check zeta(-1)=-1/12=-0,08333…:
zeta(-1) = -0.08333333333332381
************************************
3. check zeta(0)=-1/2:
zeta(0) = -0.49999999999999906
************************************
4. check zeta(1)=unendlich(inf):
zeta(1) = inf
************************************
5. check zeta(2)=pi²/6=1,644934…:
zeta(2) = 1.6449340668482266
************************************
6. check zeta(3)= 1,202056…:
zeta(3) = 1.2020569031595942
************************************
7. zeta(4)=((pi²))²/90 ~ 1,082323…:
zeta(4) = 1.0823232337111381
```

[9]:

```python
print("***********************************************")
print("** 'Trival' zeros are for z=-2,-4,-6,-8,etc. **")
print("***********************************************")
# 1. zeta(-2)=0
print ("1. check zeta(-2)=0:")
print ("zeta(-2) =",zeta(-2))

# 2. zeta(-4)=0
print ("**********************************")
print ("2. check zeta(-4)=0:")
print ("zeta(-4) =",zeta(-4))

# 3. zeta(-6)=0
print ("**********************************")
print ("3. check zeta(-6)=0:")
print ("zeta(-6) =",zeta(-6))
```

```python
# 4. zeta(-8)=0
print ("************************************")
print ("4. check zeta(-8)=0:")
print ("zeta(-8) =",zeta(-8))
```

```
***********************************************
** 'Trival' zeros are for z=-2,-4,-6,-8,etc. **
***********************************************
1. check zeta(-2)=0:
zeta(-2) = 1.5603186562147366e-13
************************************
2. check zeta(-4)=0:
zeta(-4) = 6.429216196053237e-11
************************************
3. check zeta(-6)=0:
zeta(-6) = 2.8347851868673592e-08
************************************
4. check zeta(-8)=0:
zeta(-8) = 1.3859169880942308e-05
```

[10]:
```python
import time
print("****current date and time **************")
print("Date and Time:",time.strftime("%d.%m.%Y %H:%M:%S"))
print("end")
```

```
****current date and time **************
Date and Time: 16.05.2021 15:37:49
end
```