RiemannZetaFct_and_RiemannHypothesis

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1 Riemann's Zeta-Function and Riemann's Hypothesis

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Prereq.'s: you need to extract the zip-file 'Images.zip' in a directory with name 'Images'

1.1 Item1: Riemann's Zeta-Function

See: https://en.wikipedia.org/wiki/Riemann_zeta_function

or the following YouTube Video: https://www.youtube.com/watch?v=sZhl6PyTflw&vl=en

The Riemann zeta function or Euler–Riemann zeta function, (s), is a function of a complex variable s that analytically continues the sum of the Dirichlet serie which converges when the real part of s is greater than 1.

More general representations of (s) for all s are given below. The Riemann zeta function plays a pivotal role in analytic number theory and has applications in physics, probability theory, and applied statistics. As a function of a real variable, Leonhard Euler first introduced and studied it in the first half of the eighteenth century without using complex analysis, which was not available at the time. Bernhard Riemann's 1859 article "On the Number of Primes Less Than a Given Magnitude" extended the Euler definition to a complex variable, proved its meromorphic continuation and functional equation, and established a relation between its zeros and the distribution of prime numbers.[2]

The values of the Riemann zeta function at even positive integers were computed by Euler. The first of them, (2), provides a solution to the Basel problem. In 1979 Roger Apéry proved the irrationality of (3). The values at negative integer points, also found by Euler, are rational numbers and play an important role in the theory of modular forms. Many generalizations of the Riemann zeta function, such as Dirichlet series, Dirichlet L-functions and L-functions, are known.

1.1.1 Item1.1: Dirichlet Series of Zeta-Function

https://en.wikipedia.org/wiki/Dirichlet_series In mathematics, a Dirichlet series is any series of the form of the following picture (see below).
The Dirichlet series of Riemann's Zeta-Function is a complex sequence. It is a special case of general Dirichlet series. Dirichlet series play a variety of important roles in analytic number theory. The most usually seen definition of the Riemann zeta function is a Dirichlet series, as are the Dirichlet L-functions

```python
print("*** DirichletForm of the Riemann Zeta-Fuction (Euler-Function)****")
# print("*** LATEX syntax of zeta-fct for re(z)>1: '$ displaystyle
→\zeta(s)=\sum_{n=1}^\infty 1/n^s $' ***")
from IPython.display import Image

Image('Images/DirichletForm4Riem-ZetaFct.jpg')
```

[1]:

*** DirichletForm of the Riemann Zeta-Fuction (Euler-Function)****

Definition und elementare Darstellungsformen [Bearbeiten | Quelltext bearbeiten]

Dirichlet-Reihe [Bearbeiten | Quelltext bearbeiten]

Die Zeta-Funktion wird in der Literatur oft über ihre Darstellung als Dirichlet-Reihe definiert.

Für komplexe Zahlen s, deren Realteil größer als 1 ist, ist die Zeta-Funktion definiert durch die Dirichlet-Reihe[63]

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \frac{1}{6^s} + \frac{1}{7^s} + \cdots, \quad n^s := \exp(s \log(n)).$$

1.1.2 Item1.2: The Basel Problem

The basel problem is a problem in mathematical analysis with relevance to number theory, first posed by Pietro Mengoli in 1650 and solved by Leonhard Euler in 1734,[1] and read on 5 December 1735 in The Saint Petersburg Academy of Sciences.[2] Since the problem had withstood the attacks of the leading mathematicians of the day, Euler's solution brought him immediate fame when he was twenty-eight. Euler generalised the problem considerably, and his ideas were taken up years later by Bernhard Riemann in his seminal 1859 paper ``On the Number of Primes Less Than a Given Magnitude'', in which he defined his zeta function and proved its basic properties. The problem is named after Basel, hometown of Euler as well as of the Bernoulli family who unsuccessfully attacked the problem.

The Basel problem asks for the precise summation of the reciprocals of the squares of the natural numbers, i.e. the precise sum of the infinite series:

The sum of the series is approximately equal to 1.644934.[3] The Basel problem asks for the exact sum of this series (in closed form), as well as a proof that this sum is correct. Euler found the exact sum to be pi²/6 and announced this discovery in 1735. His arguments were based on manipulations that were not justified at the time, although he was later proven correct. He produced a truly rigorous proof in 1741.

```python
print("*** Consider the special case s = 2 + i*0 so we get the follw. series:
→***")
print("*** This is the famous 'Basel-Problem' solved by L. Euler in 1735
→*******")
```

```python
from IPython.display import Image

Image('Images/Basel_Problem.jpg')
```

*** Consider the special case s = 2 + i*0 so we get the follw. series: ***
*** This is the famous 'Basel-Problem' solved by L. Euler in 1735 *******

[2]:

The Basel problem asks for the precise summation of the reciprocals of the squares of the natural numbers, i.e. the precise sum of the infinite series:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots.$$

1.1.3 Item1.3: Euler Product Formula

https://en.wikipedia.org/wiki/Proof_of_the_Euler_product_formula_for_the_Riemann_zeta_function

Leonhard Euler proved the Euler product formula for the Riemann zeta function in his thesis Variae observationes circa series infinitas (Various Observations about Infinite Series), published by St Petersburg Academy in 1737.[1][2]

[3]:
```python
print␣
  ↪("***********************************************************************")
print ("****** The bridge between zeta-fct in 'Complex Analysis' and prim-␣
  ↪*****")
print ("****** numbers in 'Number Theory' is given by EulerProduct formula␣
  ↪*****")
print␣
  ↪("***********************************************************************")

from IPython.display import Image

Image('Images/EulerProduct.jpg')
```

```
***********************************************************************
****** The bridge between zeta-fct in 'Complex Analysis' and prim- *****
****** numbers in 'Number Theory' is given by EulerProduct formula *****
***********************************************************************
```

[3]:

Euler product formula [edit]

The connection between the zeta function and prime numbers was discovered by Euler, who proved the identity

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \frac{1}{1-p^{-s}},$$

1.2 Item2: Riemann's Hypothesis

See: https://en.wikipedia.org/wiki/Riemann_hypothesis

In mathematics, the Riemann hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part = 1/2. Many consider it to be the most important unsolved problem in pure mathematics.[1] It is of great interest in number theory because it implies results about the distribution of prime numbers. It was proposed by Bernhard Riemann (1859), after whom it is named. The Riemann hypothesis and some of its generalizations, along with Goldbach's conjecture and the twin prime conjecture, comprise Hilbert's eighth problem in David Hilbert's list of 23 unsolved problems; it is also one of the Clay Mathematics Institute's Millennium Prize Problems. The name is also used for some closely related analogues, such as the Riemann hypothesis for curves over finite fields.

1.2.1 Item2.1: Zero-free region of Zeta-Function

Apart from the trivial zeros, the Riemann zeta function has no zeros to the right of = 1 and to the left of = 0 (neither can the zeros lie too close to those lines). Furthermore, the non-trivial zeros are symmetric about the real axis and the line = 1/2 and, according to the Riemann hypothesis, they all lie on the line = 1/2.

```python
print ("*** Zero-free_region_for_the_Riemann_zeta-function**** ")
from IPython.display import Image

Image('Images/Zero-free_region_for_the_Riemann_zeta-function.jpg')
```

```
*** Zero-free_region_for_the_Riemann_zeta-function****
```
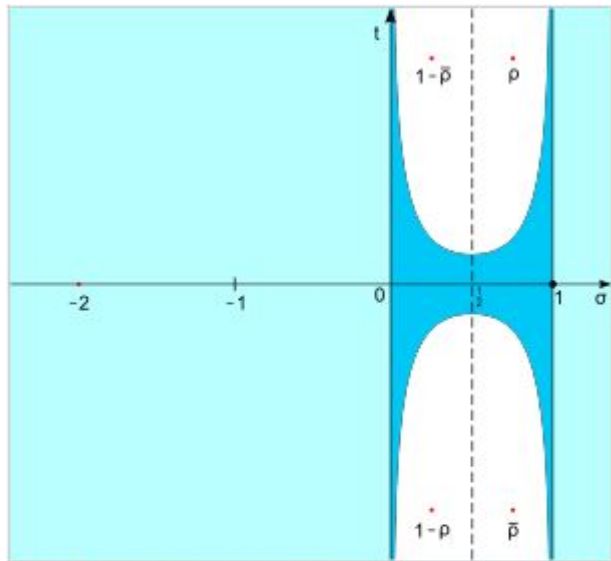
[4]:

Apart from the trivial zeros, the Riemann zeta function has no zeros to the right of $\sigma = 1$ and to the left of $\sigma = 0$ (neither can the zeros lie too close to those lines). Furthermore, the non-trivial zeros are symmetric about the real axis and the line $\sigma = \frac{1}{2}$ and, according to the Riemann hypothesis, they all lie on the line $\sigma = \frac{1}{2}$.

[5]:
```python
print (" ************************************************************* ")
print (" **** Here is an example-plot of the riemann zeta-function **** ")
print (" **** See non-trival zeros at 'critical' line real(z)=0.5 ***** ")
print (" **** This is a visualization of the Riemann-Hypothesis ******* ")
print (" ************************************************************* ")

from IPython.display import Image

Image('Images/riemann-zeta1.jpg')
```

```
*************************************************************
**** Here is an example-plot of the riemann zeta-function ****
**** See non-trival zeros at 'critical' line real(z)=0.5 *****
**** This is a visualization of the Riemann-Hypothesis *******
*************************************************************
```
[5]:

Riemann zeta function

The Riemann zeta function $\zeta(z)$ plotted with domain coloring.[1]

| Basic features	
Domain	$\mathbb{C} \setminus \{1\}$
Codomain	$\mathbb{C}$
Specific values	
At zero	$-\dfrac{1}{2}$
Limit to $+\infty$	1
Value at 2	$\dfrac{\pi^2}{6}$
Value at -1	$-\dfrac{1}{12}$
Value at -2	0
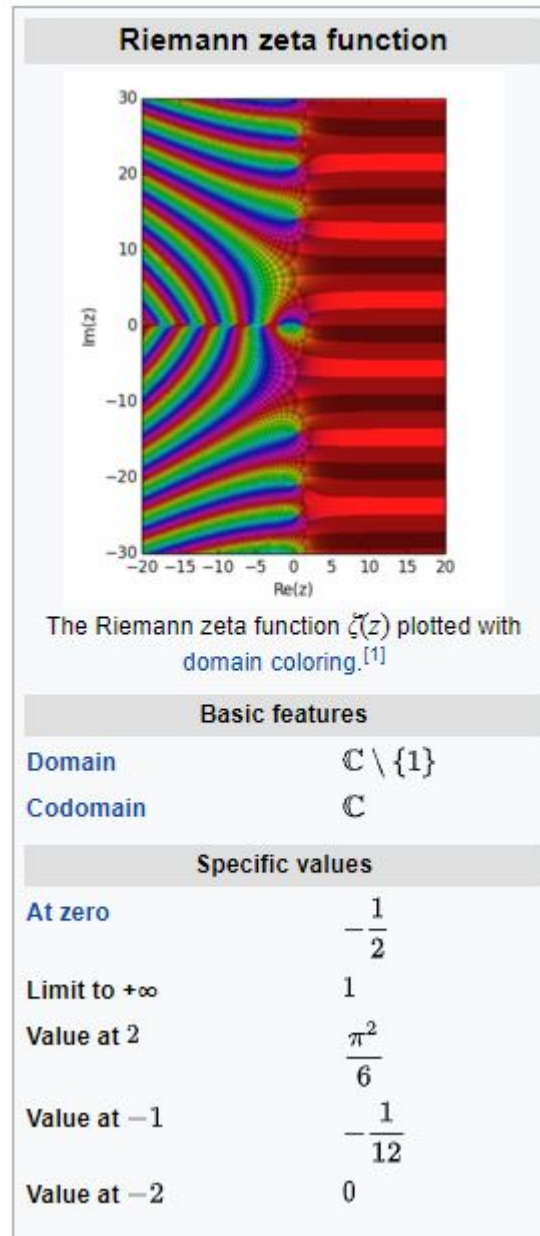
```
[6]: print ("*** Here is the example of a plot of the zeta function in more␣
     ↪detail***")
     print ("*** See two zeros at at the points z=0.5 + 14,12...z=0.5-14,12...***")
     print ("******************************************************************")

     from IPython.display import Image

     Image('Images/riemann-zeta2.jpg')
```

```
        *** Here is the example of a plot of the zeta function in more detail***
        *** See two zeros at at the points z=0.5 + 14,12…z=0.5-14,12…***
        ********************************************************************
```
[6]:

The pole at $z = 1$, and two
zeros on the critical line.
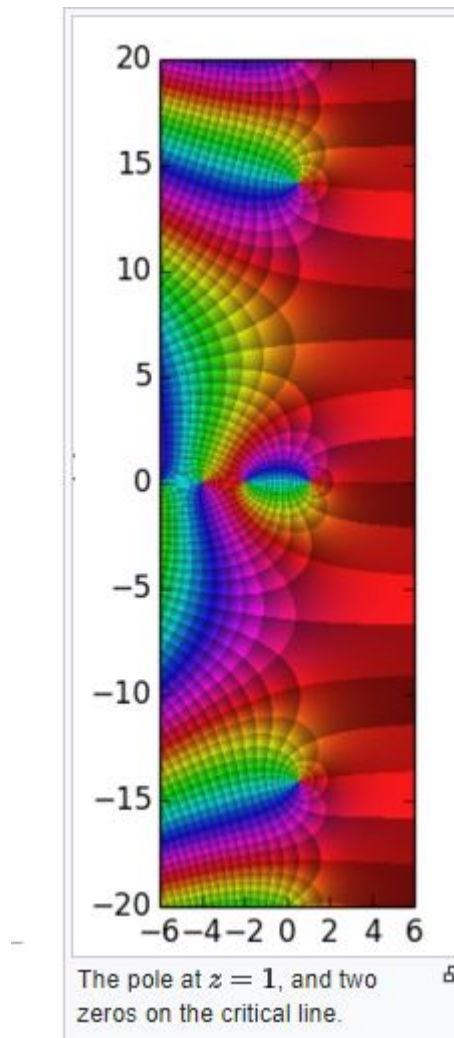
[7]:
```python
# Program/Source Code
# Here is the source code of a Python program to calculate the zeta function
 ↪values
# The program output is shown below.

def zeta(s, t = 100):
    if s == 1:
        return float("inf")
    term = (1 / 2 ** (n + 1)
            * sum((-1) ** k * binom(n, k) * (k +1 ) ** -s
                  for k in range (n + 1))
          for n in count(0))
```

```
        return sum(islice(term, t)) / (1 - 2 ** (1- s))
```

[8]:
```
# Import libaries

from itertools import count, islice
from scipy.special import binom
```

1.2.2 Item2.2: List of Special values of Zeta-Fct.

We calcualate here some special values of the Riemann Zeta function Zeta(s), where s is a complex number, with Im(s)=0. So we list up the values of Zeta(s) with s = {-2, -1, 0, 1, 2, 3, 4}. For s=2 we see the famous Basel-problem (see Item1.2 above)

[9]:
```
print ("***********************************")
print ("*** List of zeta-function values ***")
print ("***********************************")

# 1. zeta(-2)=0
print ("1. check zeta(-2)= 0:")
print ("zeta(-2) =",zeta(-2))

# 2. zeta(-1)=-1/12=-0,08333...
print ("***********************************")
print ("2. check zeta(-1)=-1/12=-0,08333...:")
print ("zeta(-1) =",zeta(-1))

# 3. zeta(0)=-1/2
print ("***********************************")
print ("3. check zeta(0)=-1/2:")
print ("zeta(0) =",zeta(0))

# 4. zeta(1)=unendlich
print ("***********************************")
print ("4. check zeta(1)=unendlich(inf):")
print ("zeta(1) =",zeta(1))

# 5. zeta(2)=pi²/6=1,64493...
print ("***********************************")
print ("5. check zeta(2)=pi²/6=1,644934...:")
print ("zeta(2) =",zeta(2))

# 6. zeta(3)=1,2020...
print ("***********************************")
print ("6. check zeta(3)= 1,202056...:")
print ("zeta(3) =",zeta(3))
```

```
# 7. zeta(4)=(pi²)²/90
print ("**********************************")
print ("7. zeta(4)=((pi²))²/90 ~ 1,082323...:")
print ("zeta(4) =",zeta(4))
```

```
**********************************
*** List of zeta-function values ***
**********************************
1. check zeta(-2)= 0:
zeta(-2) = 1.5603186562147366e-13
**********************************
2. check zeta(-1)=-1/12=-0,08333…:
zeta(-1) = -0.08333333333332381
**********************************
3. check zeta(0)=-1/2:
zeta(0) = -0.49999999999999906
**********************************
4. check zeta(1)=unendlich(inf):
zeta(1) = inf
**********************************
5. check zeta(2)=pi²/6=1,644934…:
zeta(2) = 1.6449340668482266
**********************************
6. check zeta(3)= 1,202056…:
zeta(3) = 1.2020569031595942
**********************************
7. zeta(4)=((pi²))²/90 ~ 1,082323…:
zeta(4) = 1.0823232337111381
```

1.2.3 Item2.3: Riem. Funct. Equation & trival zeros of Zeta-Fct

We calcualate here some special values with trival zeros of the Riemann Zeta function Zeta(s), where s is a complex number, with Im(s)=0. So we list up the values of Zeta(s) with s = {-8, -6, -4, -2}.

The zeta function satisfies the 'Riemann's Functional Equation" - see image below:

This is an equality of meromorphic functions valid on the whole complex plane. The equation relates values of the Riemann zeta function at the points s and 1 − s, in particular relating even positive integers with odd negative integers. Owing to the zeros of the sine function, the functional equation implies that ζ(s) has a simple zero at each even negative integer's = -2n, known as the trivial zeros of Zeta(s). When s is an even positive integer, the product sin(s/2)*Γ(1 − s) on the right is non-zero because Γ(1 − s) has a simple pole, which cancels the simple zero of the sine factor.

```
[10]: print ("**** The zeta function satisfies the 'Riemann's Functional Equation␣
      ↪****")

      from IPython.display import Image

      Image('Images/Riemann_functional_equation.JPG')
```

**** The zeta function satisfies the 'Riemann's Functional Equation ****

[10]:

Riemann's functional equation [edit]

The zeta function satisfies the functional equation

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\, \zeta(1-s),$$

where $\Gamma(s)$ is the gamma function. This is an equality of meromorphic functions valid on the whole complex plane.

```
[11]: print("*********************************************")
      print("** 'Trival' zeros are for z=-2,-4,-6,-8,etc. **")
      print("*********************************************")
      # 1. zeta(-2)=0
      print ("1. check zeta(-2)=0:")
      print ("zeta(-2) =",zeta(-2))

      # 2. zeta(-4)=0
      print ("*********************************")
      print ("2. check zeta(-4)=0:")
      print ("zeta(-4) =",zeta(-4))

      # 3. zeta(-6)=0
      print ("*********************************")
      print ("3. check zeta(-6)=0:")
      print ("zeta(-6) =",zeta(-6))

      # 4. zeta(-8)=0
      print ("*********************************")
      print ("4. check zeta(-8)=0:")
      print ("zeta(-8) =",zeta(-8))
```

```
*********************************************
** 'Trival' zeros are for z=-2,-4,-6,-8,etc. **
*********************************************
1. check zeta(-2)=0:
zeta(-2) = 1.5603186562147366e-13
*********************************
2. check zeta(-4)=0:
zeta(-4) = 6.429216196053237e-11
```

```
**********************************
3. check zeta(-6)=0:
zeta(-6) = 2.8347851868673592e-08
**********************************
4. check zeta(-8)=0:
zeta(-8) = 1.3859169880942308e-05
```

```python
import time
print("****current date and time *************")
print("Date and Time:",time.strftime("%d.%m.%Y %H:%M:%S"))
print("end")
```

```
****current date and time **************
Date and Time: 16.05.2021 23:42:37
end
```