RiemannZetaFct

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1 # Python Program to check values of Riemann's Zeta-Function

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See https://en.wikipedia.org/wiki/Riemann_zeta_function

YouTube Video: https://www.youtube.com/watch?v=sZhl6PyTflw&vl=en

The Riemann zeta function or Euler–Riemann zeta function, (s), is a function of a complex variable s that analytically continues the sum of the Dirichlet serie which converges when the real part of s is greater than 1.

More general representations of (s) for all s are given below. The Riemann zeta function plays a pivotal role in analytic number theory and has applications in physics, probability theory, and applied statistics. As a function of a real variable, Leonhard Euler first introduced and studied it in the first half of the eighteenth century without using complex analysis, which was not available at the time. Bernhard Riemann's 1859 article "On the Number of Primes Less Than a Given Magnitude" extended the Euler definition to a complex variable, proved its meromorphic continuation and functional equation, and established a relation between its zeros and the distribution of prime numbers.[2]

The values of the Riemann zeta function at even positive integers were computed by Euler. The first of them, (2), provides a solution to the Basel problem. In 1979 Roger Apéry proved the irrationality of (3). The values at negative integer points, also found by Euler, are rational numbers and play an important role in the theory of modular forms. Many generalizations of the Riemann zeta function, such as Dirichlet series, Dirichlet L-functions and L-functions, are known.

[1]:
```
print("***LATEX syntax of zeta-fct for re(z)>1: '$ displaystyle␣
 ↪\\zeta(s)=\sum_{n=1}^\infty 1/n^s $'***")
```

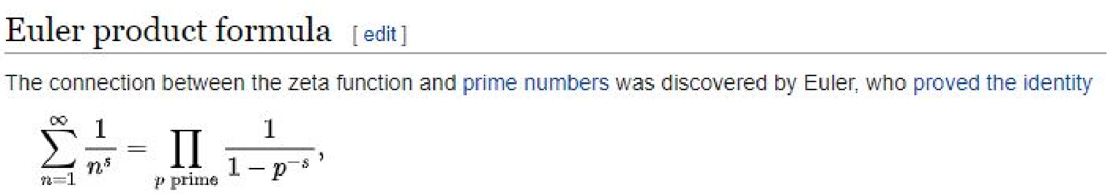
```
***LATEX syntax of zeta-fct for re(z)>1: '$ displaystyle
\zeta(s)=\sum_{n=1}^\infty 1/n^s $'***
```

[2]:
```
print␣
 ↪("*************************************************************************")
print ("****** The bridge between zeta-fct in 'Complex Analysis' and prim-␣
 ↪*****")
print ("****** numbers in 'Number Theory' is given by EulerProduct formula␣
 ↪*****")
```

```
print␣
↪("************************************************************************")

from IPython.display import Image

Image('EulerProduct.jpg')
```

```
************************************************************************
****** The bridge between zeta-fct in 'Complex Analysis' and prim- *****
****** numbers in 'Number Theory' is given by EulerProduct formula *****
************************************************************************
```

[2]:



Euler product formula [edit]

The connection between the zeta function and prime numbers was discovered by Euler, who proved the identity

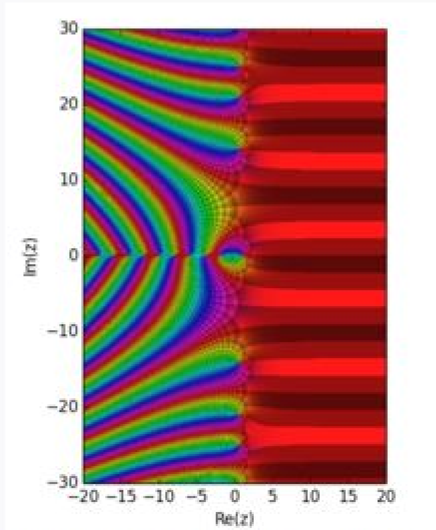
$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \frac{1}{1-p^{-s}},$$

[3]:
```
print ("***** Here is the example of a plot of the zeta function ******")
print ("*** See the non-trival zeros at 'critical' line real(z)=0.5 ***")
print ("***************************************************************")

from IPython.display import Image

Image('riemann-zeta1.jpg')
```

```
***** Here is the example of a plot of the zeta function ******
*** See the non-trival zeros at 'critical' line real(z)=0.5 ***
***************************************************************
```

[3]:

Riemann zeta function



The Riemann zeta function $\zeta(z)$ plotted with domain coloring.[1]

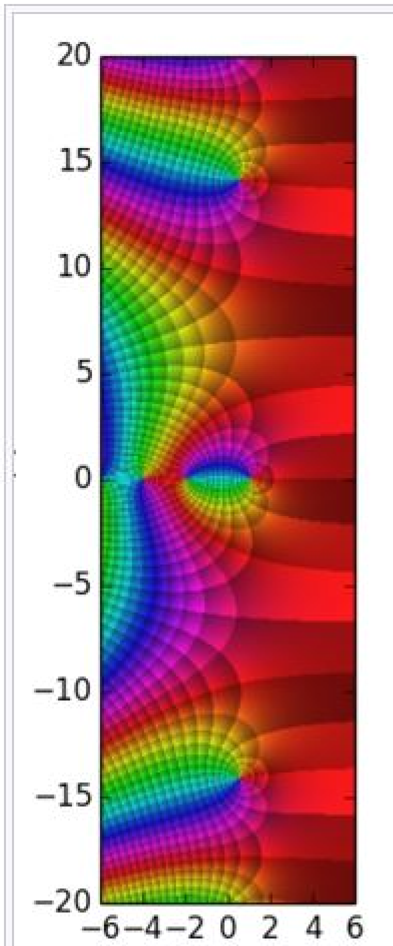
Basic features	
Domain	$\mathbb{C} \setminus \{1\}$
Codomain	$\mathbb{C}$

Specific values	
At zero	$-\frac{1}{2}$
Limit to +∞	1
Value at 2	$\frac{\pi^2}{6}$
Value at -1	$-\frac{1}{12}$
Value at -2	0

[4]:
```
print ("*** Here is the example of a plot of the zeta function in more␣
↪detail***")
print ("*** See two zeros at at the points z=0.5 + 14,12...z=0.5-14,12...***")
print ("*******************************************************************")

from IPython.display import Image

Image('riemann-zeta2.jpg')
```

```
*** Here is the example of a plot of the zeta function in more detail***
*** See two zeros at at the points z=0.5 + 14,12…z=0.5-14,12…***
*********************************************************************
```

[4]:



The pole at $z = 1$, and two zeros on the critical line.

[5]:

```python
# Import libaries

from itertools import count, islice
from scipy.special import binom
```

[6]:

```python
# Program/Source Code
# Here is the source code of a Python program to calculate the zeta function␣
↪values
# The program output is shown below.

def zeta(s, t = 100):
    if s == 1:
```

```
        return float("inf")
    term = (1 / 2 ** (n + 1)
              * sum((-1) ** k * binom(n, k) * (k +1 ) ** -s
                    for k in range (n + 1))
            for n in count(0))
    return sum(islice(term, t)) / (1 - 2 ** (1- s))
```

[7]:
```
print ("value of zeta(2)=pi²/6 ~ 1,644934")
zeta(2)
```

```
value of zeta(2)=pi²/6 ~ 1,644934
```

[7]: 1.6449340668482266

[8]:
```
#pi * pi / 6
```

[9]:
```
print ("value of zeta(4)=(pi²)*(pi²)/90 ~ 1,0823236")
zeta(4)
```

```
value of zeta(4)=(pi²)*(pi²)/90 ~ 1,0823236
```

[9]: 1.0823232337111381

[10]:
```
zeta(1)
```

[10]: inf

[11]:
```
zeta(0)
```

[11]: -0.49999999999999906

[12]:
```
print("zeta(-1)= -1/12 ~ -0,0833333333")
zeta(-1)
```

```
zeta(-1)= -1/12 ~ -0,0833333333
```

[12]: -0.08333333333332381

[13]:
```
print("****'Trival' zeros are for z=-2,-4,-6,...****")
```

```
****'Trival' zeros are for z=-2,-4,-6,…****
```

[14]:
```
zeta(-2)
```

[14]: 1.5603186562147366e-13

[15]:
```
zeta(-4)
```

[15]: 6.429216196053237e-11

[16]:
```
zeta(-6)
```

[16]: 2.8347851868673592e-08

[17]:
```
import time
print("****current date and time **************")
print("Date and Time:",time.strftime("%d.%m.%Y %H:%M:%S"))
print("end")
```

```
****current date and time **************
Date and Time: 03.09.2020 21:53:39
end
```