Forecasting and stress testing with quantile vector autoregression

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Table of contents

Introduction

Backdrop

- Original paper is [Chavleishvili and Manganelli, 2024]
- Three novel contributions:
  - Link to traditional VAR and identification literature
  - Constructs structural quantile impulse response function
  - Perform multi-step forecast with QVAR
- I will focus on Structural identification (and links to VAR).
- **Reason**: The forecasting part builds heavily on [White et al., 2015].

Multivariate Quantiles

Multivariate Quantiles: The literature

- No widely accepted definition of multivariate quantiles
    - Active area of statistical research
    - [Hallin et al., 2017] offers an review of the current state of research
- **1:** [Chaudhuri, 1996] was the first contribution to the multivariate quantile literature
    - Based on a geometric configuration of multivariate data
    - proposes Spatial (geometric) quantiles
    - **Problem**: Probability content of a quantile region of order p is **not** p

Multivariate Quantiles: The literature

- **2:** Multivariate directional quantiles linked to half-space depth [Hallin et al., 2010]
  - Decompose **Y** into direction Y_u and an orthogonal direction $Y_u^{\perp}$
  - Minimise the usual pinball loss function along a direction u.
  - **Problem**: Probability mass is unknown $\rightarrow$ Probability content of a quantile region of order p is **not** p
- **3:** Transport function from the joint distribution to a hypercube [Carlier et al., 2014]
  - Use Monge-Kantorovich theorem for the multivariate quantile case
  - The optimal transport problem boils down to minimax on the wasserstein distance
  - So we want to maximise similarity of joint density while minimising ”effort” to move portions of a uniform density.

Multivariate Quantiles: The literature

- 4: Use [Rosenblatt, 1952] transformation (PAPERS CHOICE)
    - Retain coordinatewise definition of quantiles where quantiles are d-tuples $(\tau_1, ..., \tau_d) \in (0, 1)^d$
    - As long as there is a triangular structure in the data, you can create valid quantile biplots!
    - **Advantage**: Simple to do: Just specify the triangular structure and run normal QR on each equation
    - **Problem**: Order can influence results

Essentially: all multivariate quantile definitions try and capture ”direction” and ”quantiles” jointly without specifying a parametric form (copula). One way to think about it is redefining Sklar’s theorem in a semiparametric way.

Quantile VAR

QVAR

Definition 2.2. (Recursive QVAR)—$\{Y_t\}$ follows a recursive QVAR(1) process if for any $\theta_i \in (0,1)$, $i \in \{1, \ldots, n\}$, the θ_i-quantile of Y_{it} can be written as follows:

$$
\begin{aligned}
Q_{\theta_1}(Y_{1t}|\Omega_{1t}) &= \omega_1(\theta_1) + a_{11}(\theta_1)Y_{1,t-1} + a_{12}(\theta_1)Y_{2,t-1} + \ldots + a_{1n}(\theta_1)Y_{n,t-1} \\
Q_{\theta_2}(Y_{2t}|\Omega_{2t}) &= \omega_2(\theta_2) + a_{021}(\theta_2)Y_{1t} + \\
&\quad + a_{21}(\theta_2)Y_{1,t-1} + a_{22}(\theta_2)Y_{2,t-1} + \ldots + a_{2n}(\theta_2)Y_{n,t-1} \\
&\vdots \\
Q_{\theta_n}(Y_{nt}|\Omega_{nt}) &= \omega_n(\theta_n) + a_{0n1}(\theta_n)Y_{1t} + \ldots + a_{0n,n-1}(\theta_n)Y_{n-1,t} + \\
&\quad + a_{n1}(\theta_n)Y_{1,t-1} + a_{n2}(\theta_n)Y_{2,t-1} + \ldots + a_{nn}(\theta_n)Y_{n,t-1}
\end{aligned}
$$

$$Y_t = \omega(U_t) + A_0(U_t)Y_t + A_1(U_t)Y_{t-1} \tag{2}$$

Definition 2.3. (Reduced form QVAR)—Consider the recursive QVAR model (2). The QVAR process can be written in its reduced form as follows:

$$Y_t = \nu(U_t) + B(U_t)Y_{t-1} \tag{3}$$

where $\nu(U_t) \equiv [I_n - A_0(U_t)]^{-1}\omega(U_t)$, $B(U_t) \equiv [I_n - A_0(U_t)]^{-1}A_1(U_t)$ and I_n is the n-dimensional identity matrix.

QVAR: Definition implications

1. The standard defintion of quantile continues to hold element wise!
2. The matrix A_0 is a lower triangular $n \times n$ coefficientr matrix with zeros along the main diagonal. Switching ordering provides different estimates: there are $n!$ possible ordering
3. If the quantile model is correctly specified, then the population quantiles should be monotonic.

QVAR Stationarity

Proposition 2.1. *(Stationarity of the QVAR process)— Assume that:*

1. *$v(U_t) - E[v(U_t)]$ is a vector of i.i.d. random variables with zero mean, finite variance, and continuous density function;*
2. *The matrix $E[B(U_t) \otimes B(U_t)]$ has the largest eigenvalue less than one in absolute value.*

Then the QVAR process (2) is covariance stationary and satisfies

$$\frac{1}{\sqrt{T}} \sum_{t=1}^{T} (Y_t - \mu_Y) \sim N\left(0, \lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} E\,(Y_t - \mu_Y)\,(Y_t - \mu_Y)'\right)$$

where $\mu_Y = [I_n - E[B(U_t)]]^{-1} E[v(U_t)]$.

Structural Quantile Impulse Response Functions

- Think of the QSVAR as a random coefficients SVAR model:

$$Y_t = v + B(U_t)Y_{t-1} + e(U_t) \tag{1}$$

- where $U_t \sim U(0,1)$, and $e(U_t) = v(U_t) - v$, and $v = E[v(U_t)]$
- The identities related to v can be estimated via simulation: Take a random draw of U_t and estimate the corresponding recursive QVAR.
- The above QVAR specification reduces to a standard VAR model when $B(U_t) = B \; \forall \; U_t$.
- Can define $e(U_t)$ as in the standard OLS VAR:

$$C\varepsilon(U_t) = e(U_t), \; \varepsilon(U_t) \sim (0, I_n) \tag{2}$$

- where C is a matrix of unknown structural parameters

Structural Quantile Impulse Response Functions

- How to measure the treatment effect of Y_i on Y_j?
- Consider that we alter the whole distribution of Y_i:

$$\varepsilon_i^*(U_t) = \varepsilon(U_t) + \delta\iota \tag{3}$$

- where $\delta > 0$ is a scalar and ι is a vector of zeros with 1 at the i position
- Denote with Y_i^* the shocked variable, the impulse response at the t is:

$$\Delta_t^i(U_t) = Y_t^* - Y_t = C\delta\iota \tag{4}$$

- With this in mind, for the h following periods the quantile impulse response is quantile-path dependent:

$$\Delta_{t+h}^i(U_{t+h}|U_t, ..., U_{t+h-1}) = B(U_{t+h})...B(U_{t+1})C\delta\iota \tag{5}$$

- **Identification in this setup is exactly the same as in the OLS VAR setting!**

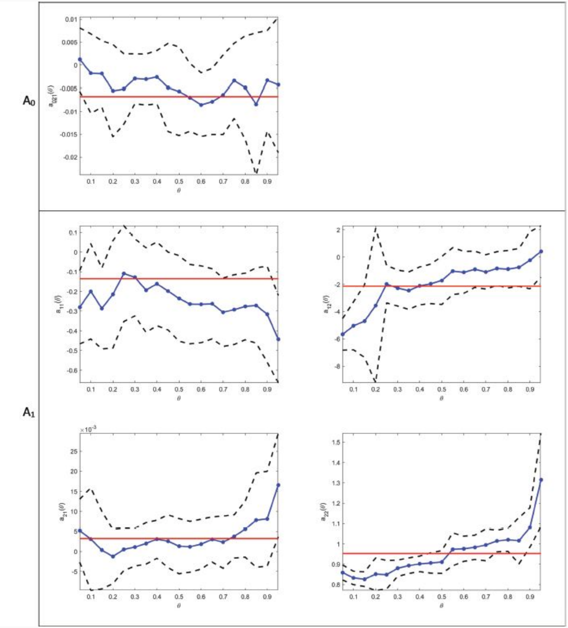
Vulnerable growth

Vulnerable Growth as QVAR

- Cast [Adrian et al., 2019] into the QVAR framework: Assume that financial conditions react more freely
- Financial conditions measured with the Composite index of systemic stress (CISS) of [Hollo et al., 2012]
- Real Economy measured by Industrial Production.
- By ordering CISS after IP, we impose the structural identification assumption that financial variables can react contemporaneously to real variables, but real variables react to financial developments only with a lag:

$$\begin{aligned} IP_t &= \omega_1 + a_{1,1} IP_{t-1} + a_{1,2} CISS_{t-1} \\ CISS_t &= \omega_2 + \alpha IP_t + a_{2,1} IP_{t-1} + a_{2,2} CISS_{t-1} \end{aligned} \tag{6}$$

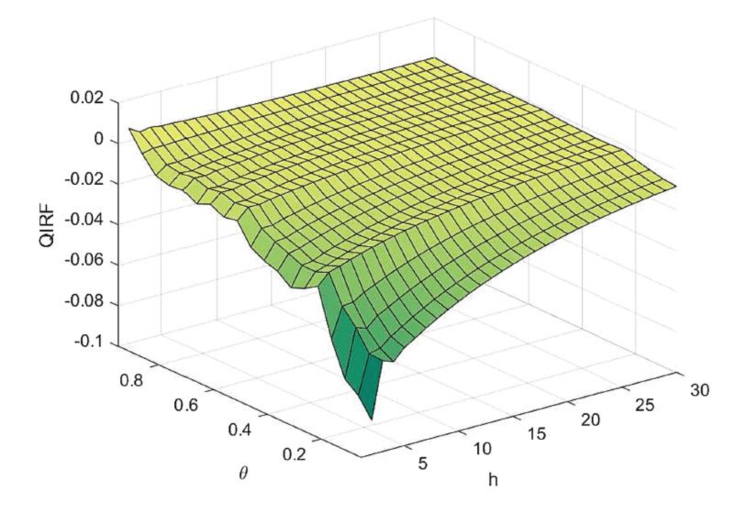
Coefficients



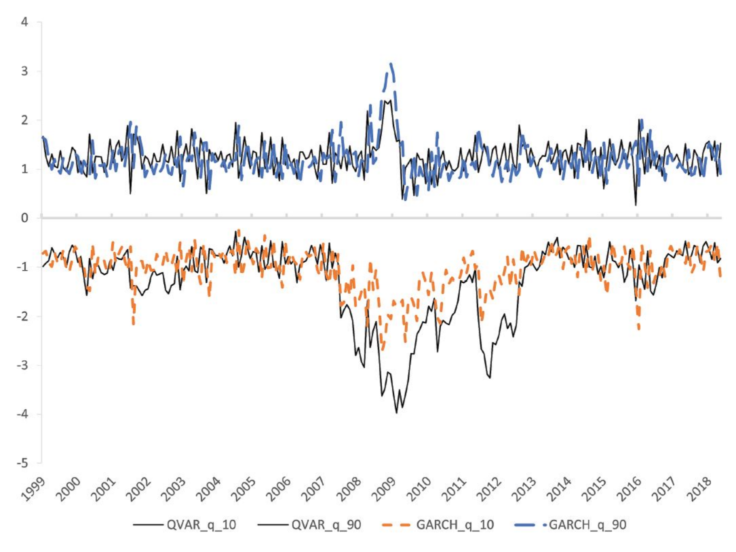
Coefficients

- Interaction between real and financial variables can be tested by checking whether the off-diagonal coefficients are statistically different from zero!
- In this case they are as $CISS_{t-1} \rightarrow CISS_t$ is significantly different from 0
- The impact of $CISS_{t-1} \rightarrow CISS_t$ follows the pattern of ABG
- The coefficients also highlight that tail behaviour would be missed by a normal VAR (red line)

QIRF of IP to 1 s.d. CISS shock



Comparison to MGARCH: QVAR captures asymmetries better

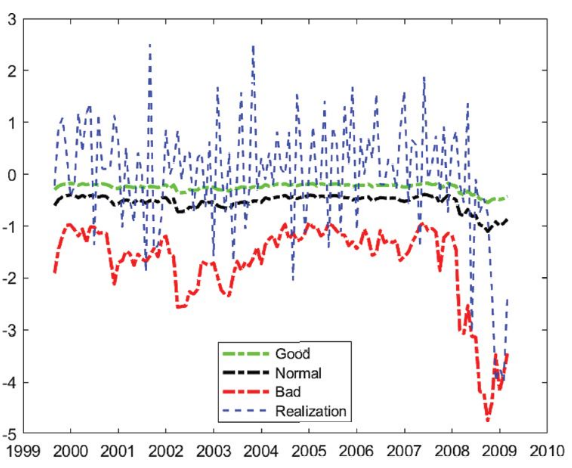


Stress Testing

Good finance: 6 periods of 10 quantile in CISS and IP

Normal finance: 6 periods of 10 quantile of IP and median of CISS

Bad finance: 6 periods of 10 quantile of IP and 90 of CISS



Why JAE?

QVAR positives

- Takes the incredibly diverse and complicated topic of multivarite quantiles
- Offers a simple approach to estimate with tools commonly used in the VAR literature: Cholesky decomposition
- Links quantile IRFs to VAR IRFs
- The QVAR allows for simple way to create principled stress testing
- Likely to be popular in applied research

QVAR questions

- The ordering likely becomes increasingly problematic as we increase dimensionality
  - n! potential ordering...
  - Is it possible to have quantile specific ordering? i.e. something akin to quantile specific sparsity
- How well does the method fair in ”banana” shaped data?
  - There might be further nonlinearities beyond quantiles: See ”Momentum informed Inflation-at-Risk”.
- Ordering vs data
  - Need to motivate ordering over **whole distribution**
  - Variable ordering for SVAR’s is #1 cause of unhappy referees. The recursive QVAR is an even stronger assumption.

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