Doubling the bet game with coin tossing

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Game Setting

- There are two players: me and dealer
- We start the game with $1 bet. The total pot is $2.
- Each round, we flip the fair coin. When head, point increase by 1 point. When tail, point decrease by 1 point.
- Game ends when the point become either +2 or -2. When the point reaches 2, I take the pot. When the point reaches -2, dealer takes the pot.
- Each round, I am asked whether to double the pot or not.
- There are some possible strategies
  1. Never raise the pot size
  2. Double the pot size whenever the point reaches to 1.
  3. Double the pot size whenever the point reaches to 0 or 1.
  4. Double the pot size every time.
- What are the expected payoff for each strategy?

Strategy 0 and 4: "Never raise the pot size" and "Double the pot size every time"

- The expected payoff for this strategy would be 0.

Strategy 1 and 2: "Double the pot size whenever the point reaches" to 1 and "Double the pot size whenever the point reaches to 0 or 1."

1 Expected Payoff for Two Strategies

- N=2
  - HH : $\underbrace{4 \to +2}_{\text{Strategy 1}}$ $\underbrace{4 \to +2}_{\text{Strategy 2}}$
  - TT : $2 \to -1$ $2 \to -1$

$$\text{Strategy1: } E[\text{Payoff} \mid N=2] = 1 = \frac{3^0}{2^1}$$

$$\text{Strategy2: } E[\text{Payoff} \mid N=2] = 1 = 2^0 \cdot \frac{3^0}{2^1}$$

$$P(N=2) = \frac{1}{2^1}$$

- N=4
  - HTHH : $\underbrace{8 \to +4}_{\text{Strategy 1}}$ $\underbrace{16 \to +8}_{\text{Strategy 2}}$

- THHH : $4 \to +2$ $\quad 8 \to +4$

- HTTT : $4 \to -2$ $\quad 8 \to -4$
- THTT : $2 \to -1$ $\quad 4 \to -2$

$$\text{Strategy1: } E[\text{Payoff} \mid N = 4] = \frac{4+2-2-1}{4} = \frac{3}{4} = \frac{3^1}{2^2}$$
$$\text{Strategy2: } E[\text{Payoff} \mid N = 4] = \frac{8+4-4-2}{4} = \frac{2 \cdot 3}{4} = 2^1 \cdot \frac{3^1}{2^2}$$
$$P(N = 4) = \frac{1}{2^2}$$

- N=6
  - HTHTHH : $16 \to +8$ $\quad 64 \to +32$
  - THHTHH : $8 \to +4$ $\quad 32 \to +16$

  - HTTHHH : $8 \to +4$ $\quad 32 \to +16$
  - THTHHH : $4 \to +2$ $\quad 16 \to +8$

  - HTHTTT : $8 \to -4$ $\quad 32 \to -16$
  - THHTTT : $4 \to -2$ $\quad 16 \to -8$

  - HTTHTT : $4 \to -2$ $\quad 16 \to -8$
  - THTHTT : $2 \to -1$ $\quad 8 \to -4$

$$\text{Strategy1: } E[\text{Payoff} \mid N = 6] = \frac{8+4-2-1}{8} = \frac{9}{8} = \frac{3^2}{2^3}$$
$$\text{Strategy2: } E[\text{Payoff} \mid N = 6] = \frac{32+16-8-4}{8} = \frac{2^2 \cdot 9}{8} = 2^2 \cdot \frac{3^2}{2^3}$$
$$P(N = 6) = \frac{1}{2^3}$$

- N=8 and so on....
  - Similar

$$\text{Strategy1: } E[\text{Payoff} \mid N = 8] = \frac{16+8+8+4-4-2-2-1}{16} = \frac{27}{16} = \frac{3^3}{2^4}$$
$$\text{Strategy2: } E[\text{Payoff} \mid N = 8] = \frac{2^3 \cdot (16+8+8+4-4-2-2-1)}{16} = 2^3 \cdot \frac{27}{16} = 2^3 \cdot \frac{3^3}{2^4}$$
$$P(N = 8) = \frac{1}{2^4}$$

To conclude, for strategy 1,

$$
\begin{aligned}
E[\textbf{Payoff}] &= E\left[E[\textbf{Payoff} \mid N]\right] \\
&= \left(\frac{1}{2}\right)^1 \cdot \frac{3^0}{2^1} + \left(\frac{1}{2}\right)^2 \cdot \frac{3^1}{2^2} + \left(\frac{1}{2}\right)^3 \cdot \frac{3^2}{2^3} + \left(\frac{1}{2}\right)^4 \cdot \frac{3^3}{2^4} + \cdots \\
&= \sum_{k=1}^{n} \frac{3^{k-1}}{4^k} = \frac{1}{3} \sum_{k=1}^{n} \left(\frac{3}{4}\right)^k \\
&= \frac{1}{3} \left(\frac{\frac{3}{4}}{1 - \frac{3}{4}} \right) = 1
\end{aligned}
$$

For strategy 2,

$$
\begin{aligned}
E[\textbf{Payoff}] &= E\left[E[\textbf{Payoff} \mid N]\right] \\
&= \left(\frac{1}{2}\right)^1 \cdot 2^0 \cdot \frac{3^0}{2^1} + \left(\frac{1}{2}\right)^2 \cdot 2^1 \cdot \frac{3^1}{2^2} + \left(\frac{1}{2}\right)^3 \cdot 2^2 \cdot \frac{3^2}{2^3} + \left(\frac{1}{2}\right)^4 \cdot 2^3 \cdot \frac{3^3}{2^4} \\
&= \sum_{k=1}^{n} \frac{3^{k-1}}{4^k} \cdot 2^{k-1} = \frac{1}{6} \sum_{k=1}^{n} \left(\frac{3}{2}\right)^k
\end{aligned}
$$

It does not converge to some number. In simulation, it shows large variation from huge negative number to large positive number.

2 Expected Number of Toss

Using the information above,

$$
\begin{aligned}
E[N] &= 2 \sum_{k=1}^{n} k \cdot \left(\frac{1}{2}\right)^k \\
&= 2 \cdot 2 \\
&= 4
\end{aligned}
$$

3 For the simulation result, please check the jupyter notebook.