Martingale Betting Strategy

HW Roh

The simplest of these strategies was designed for a game in which the gambler wins the stake if a coin comes up heads and loses it if the coin comes up tails. The strategy had the gambler double the bet after every loss, so that the first win would recover all previous losses plus win a profit equal to the original stake.

We are going to implement this strategy with the initial bet of \$1. The game is well summarized by the picture below.[1]

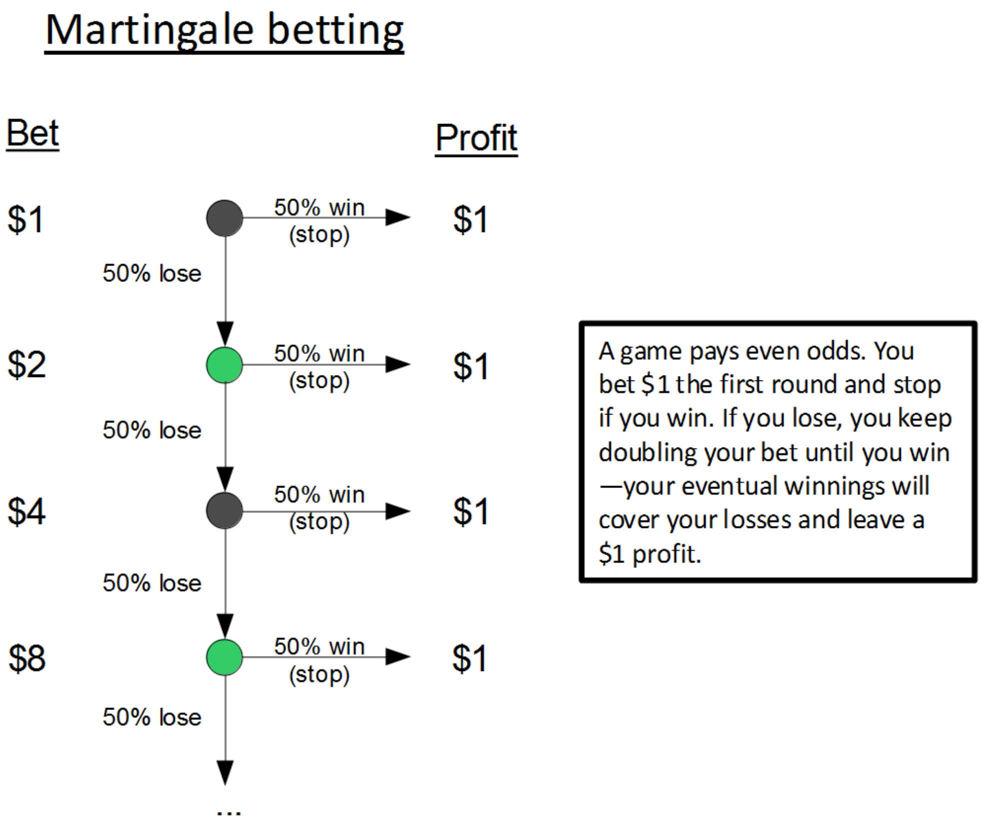

Figure 1:

Before we do simulation, let's consider some interesting fact of this game.

Question. *What is the expected number of the game play when we stop with profit \$1?*

Solution. We know that each coin toss follows the Bernoulli process with $p = \frac{1}{2}$. Let $\{H, T\}^n$ be possible set of events when n games are played and we are interested in expected number of game that will be played. Let X be a random variable

[1] https://github.com/AndyAn0724/Martingale_Betting_Strategy

that represents the number of game played. Then we can write

$$
\begin{aligned}
P[X=1] &= P[H] = 1/2 \\
P[X=2] &= P[TH] = (1/2)^2 \\
&\vdots \\
P[X=n] &= P[\underbrace{T\cdots T}_{n-1}H] = (1/2)^n
\end{aligned}
$$

Therefore,

$$E[X] = 1\cdot\frac{1}{2} + 2\cdot\left(\frac{1}{2}\right)^2 + 3\cdot\left(\frac{1}{2}\right)^3 + \cdots + n\cdot\left(\frac{1}{2}\right)^n$$

To calculate this, let's go back to middle school calculus.

$$
\begin{aligned}
S &= 1\cdot\frac{1}{2} + 2\cdot\left(\frac{1}{2}\right)^2 + 3\cdot\left(\frac{1}{2}\right)^3 + \cdots + n\cdot\left(\frac{1}{2}\right)^n \\
\frac{1}{2}S &= \qquad +1\cdot\left(\frac{1}{2}\right)^2 + 2\cdot\left(\frac{1}{2}\right)^3 + \cdots + (n-1)\cdot\left(\frac{1}{2}\right)^n + \frac{n}{2}\left(\frac{1}{2}\right)^n
\end{aligned}
$$

We subtract to get

$$
\begin{aligned}
\frac{1}{2}S &= \frac{1}{2}\left[\sum_{k=1}^{n}\left(\frac{1}{2}\right)^k\right] + \frac{n}{2^{n+1}} \\
&\text{as } n\to\infty \\
\frac{1}{2}S &= \frac{\frac{1}{2}}{1-\frac{1}{2}} + 0 \\
&= 1
\end{aligned}
$$

Therefore we get $E[X]=2$

Question. *What is the expected number of the game play when we stop with profit \$2?*

Solution. Similarly,

$$
\begin{aligned}
P[X=1] &= 0 \\
P[X=2] &= P[HH] = 1\cdot(1/2)^2 \\
P[X=3] &= P[THH] = 1\cdot(1/2)^3 \\
P[X=4] &= P[TTHH, HTHH] = 2\cdot(1/2)^4 \\
P[X=5] &= P[TTTHH, HTTHH, THTHH] = 3\cdot(1/2)^5 \\
P[X=6] &= P[TTTTHH, HTTTHH, THTTHH, TTHTHH, HTHTHH] = 5\cdot(1/2)^6 \\
P[X=7] &= P[TTTTTHH, HTTTTHH, THTTTHH, TTHTTHH, TTTHTHH \\
&\quad , HTHTTHH, HTTHTHH, THTHTHH] = 8\cdot(1/2)^6 \\
&\vdots \\
P[X=n] &= P[\underbrace{T\cdots T}_{n-2}HH] = F_{n-1}\cdot(1/2)^n
\end{aligned}
$$

where the coefficients are Fibonacci numbers, i.e., the sequence

$$1, 1, 2, 3, 5, 8, \cdots$$

Therefore, we can calculate the expected value of X

$$\begin{aligned} E[X] &= 1 \cdot 0 \cdot (1/2)^1 + 2 \cdot 1 \cdot (1/2)^2 + 3 \cdot 1 \cdot (1/2)^3 + 4 \cdot 2 \cdot (1/2)^4 + 5 \cdot 3 \cdot (1/2)^5 \cdots \\ &= \sum_{k=1}^{n} kF_{k-1} \left(\frac{1}{2}\right)^k \end{aligned}$$

Through the simulation, we can see this number converges to 4.

Question. *What is the expected number of the game play when we stop with profit $n?*

Solution. Through the simulation, we can see this number converges to $2n$. This means that on average, we need to do betting around $2n$ times to get n profit. This is intuitive because we flip a coin to choose winner. We check with the simulation.

Question. *What is the variance of the number of the game play when we stop with profit $1?*

Solution. To answer this, we calculate

$$\begin{aligned} Var[X] &= E[X^2] - E[X]^2 \\ &= E[X^2] - 4 \end{aligned}$$

From above, we can calculate

$$E[X^2] = 1 \cdot \frac{1}{2} + 2^2 \cdot \left(\frac{1}{2}\right)^2 + 3^2 \cdot \left(\frac{1}{2}\right)^3 + \cdots + n^2 \cdot \left(\frac{1}{2}\right)^n$$

This is not easy to solve. However, there is a formula as follows[2]:

$$\begin{aligned} \sum_{k=1}^{n} k^2 \left(\frac{1}{2}\right)^k &= \left(\frac{1}{2}\right)^n \left[6 \cdot 2^n - n^2 - 4n - 6\right] \\ &\text{as } n \to \infty \\ &= 6 \end{aligned}$$

Therefore, we have

$$\begin{aligned} Var[X] &= 6 - 4 \\ &= 2 \end{aligned}$$

Question. *What is the variance of the number of the game play when we stop with profit $n?*

Solution. It is $2n$. We check with the simulation.

Will be continued!

[2] https://math.stackexchange.com/questions/1504529/evaluate-the-sum-sum-k-1nk2-2-k