Econometrics (ECON 21020) – Winter 2020
Problem Set #1
Date Due: January 15, 2019

1. Stock and Watson, Exercises 2.6, 2.8, 2.9, 2.22

2. Let $X \sim \text{Bernoulli}(p)$. Define $Z = 3^X - 1$.

   (a) Show that $E[Z] = 2p$.

   (b) Show that $E[Z^2] = 4p$.

   (c) What is the variance of Z?

3. Let X denote the annual salary of scientists in the United States, measured in thousands of dollars. Suppose that $E[X] = 48.8$ and that the standard deviation of X is 12.1. Let Y denote the annual salary of scientists in the United States measured in dollars. What is $E[Y]$ and what is the standard deviation of Y?

4. Let GPA denote a random variable for the college student's grade point average, and SAT denote a random variable for the college student's SAT score. Suppose that there is the following relationship between GPA and SAT: $E[\text{GPA}|\text{SAT}] = .70 + .002\text{SAT}$.

   (a) What is the expected GPA when SAT$= 750$? What is the expected GPA when SAT$= 1500$?

   (b) If $E[\text{SAT}] = 1000$, what is $E[\text{GPA}]$?

5. Consider the problem of predicting Y using another variable X, so that the prediction of Y is some function of X, say $g(X)$. Suppose that the quality of the prediction is measured by its mean squared error:

$$E[(Y - g(X))^2] \ .$$

   (a) Let $U = Y - E[Y|X]$. Show that $E[U|X] = 0$.

   (b) Show that $E[Uh(X)] = 0$ for any function $h(X)$.

   (c) Show that

$$E[(Y - g(X))^2] \geq E[(Y - E[Y|X])^2] \ ,$$

   for any predictor $g(X)$.
   (Hint: let $V = Y - g(X)$, let $h(X) = g(X) - E[Y|X]$, then $V = (Y - E[Y|X]) - h(X)$. Use this representation to show that $E[V^2] \geq E[U^2]$).

(d) Conclude that the best predictor of Y given X, under mean squared error-loss, is the conditional mean $E[Y|X]$.

6. Suppose $X \sim \text{Uniform}[-1, 1]$, and let $Y = X^2$. Show that $\text{Cov}[X, Y] = 0$, but that X and Y are not independent.

7. Let $Y = a + bX + U$, where X and U are random variables and a and b are constants. Assume that $E[U|X] = 0$ and $\text{Var}[U|X] = X^2$.

   (a) Is U mean independent of X? Why? Is U independent of X? Why?

   (b) Show that $E[U] = 0$ and $\text{Var}[U] = E[X^2]$.

   (c) Show that $E[Y|X] = a + bX$, and that $E[Y] = a + bE[X]$.

   (d) Show that $\text{Var}[Y|X] = X^2$, and that $\text{Var}[Y] = b^2\text{Var}[X] + E[X^2]$.