Econometrics (ECON 21020) – Winter 2020
Problem Set #1 Solutions

SW 2.6

(a)

$$\mathbf{E}[Y] = \sum_{y=0,1} y \Pr\{Y = y\} = 1 \cdot 0.965 + 0 \cdot 0.035 = 0.965$$

(b) The unemployment rate is the probability of being unemployed, i.e. $Y = 0$.

$$\Pr\{Y = 0\} = \mathbf{E}[\mathbf{I}\{Y = 0\}] = \mathbf{E}[(1 - Y)] = 1 - \mathbf{E}[Y].$$

(c)

$$\mathbf{E}[Y|X = 1] = \sum_{y=0,1} y \Pr\{Y = y|X = 1\} = 1 \cdot \frac{389}{398} + 0 \cdot \frac{9}{398} \approx 0.9774$$
$$\mathbf{E}[Y|X = 0] = \sum_{y=0,1} y \Pr\{Y = y|X = 0\} = 1 \cdot \frac{576}{602} + 0 \cdot \frac{26}{602} \approx 0.9568$$

(d) Since the unemployment rate is $\mathbf{E}[1 - Y]$, the unemployment rate for college graduates and non-college graduates are conditional expectations of $1 - Y$.

$$\mathbf{E}[1 - Y|X = 1] = 1 - \mathbf{E}[Y|X = 1] = \frac{9}{398} \approx 0.0226$$
$$\mathbf{E}[1 - Y|X = 0] = 1 - \mathbf{E}[Y|X = 0] = \frac{26}{602} \approx 0.0432$$

(e) The probability of a randomly selected unemployed person being a college graduate is

$$\Pr\{X = 1|Y = 0\} = \frac{9}{35} \approx 0.2571$$

and likewise the probability of a randomly selected unemployed person being a non-college graduate is

$$\Pr\{X = 0|Y = 0\} = \frac{26}{35} \approx 0.7429.$$

(f) Educational achievement (Y) and employment status (X) is not independent since Y and X are not mean independent as shown in **c**.

SW 2.8

We have

$$Z = \frac{1}{2}(Y - 1)$$
$$\mu_Y = \mathbf{E}[Y] = 1$$
$$\sigma_Y^2 = \mathbf{E}[(Y - \mathbf{E}[Y])^2] = \mathbf{E}[(Y - 1)^2] = 4.$$

Then, by plugging in moments for Y, we have

$$\begin{aligned}
\mu_Z &= \mathbf{E}[Z] = \mathbf{E}\left[\frac{1}{2}(Y-1)\right] = \frac{1}{2}\mathbf{E}[Y] - \frac{1}{2} \\
&= \frac{1}{2} - \frac{1}{2} = 0 \\
\sigma_Z^2 &= \mathbf{E}\left[(Z - \mathbf{E}[Z])^2\right] = \mathbf{E}[Z^2] \\
&= \mathbf{E}\left[\frac{1}{4}(Y-1)^2\right] = \frac{1}{4}\mathbf{E}[(Y-1)^2] \\
&= \frac{1}{4} \cdot 4 = 1.
\end{aligned}$$

SW 2.9

(a) The probability distribution of Y is

$$F_Y(y) = \begin{cases} 0, & \text{if } y < 14 \\ 0.21, & \text{if } 14 \le y < 22 \\ 0.44, & \text{if } 22 \le y < 30 \\ 0.74, & \text{if } 30 \le y < 40 \\ 0.89, & \text{if } 40 \le y < 65 \\ 1, & \text{if } y \ge 65 \end{cases}$$

and its mean and variance are

$$\begin{aligned}
\mathbf{E}[Y] &= 30.15 \\
\mathrm{Var}(Y) &= 218.2075.
\end{aligned}$$

(b) The conditional probability distribution of Y given $X = 8$ is

$$F_{Y|X=8}(y) = \begin{cases} 0, & \text{if } y < 14 \\ \frac{2}{39} \approx 0.0513, & \text{if } 14 \le y < 22 \\ \frac{5}{39} \approx 0.1282, & \text{if } 22 \le y < 30 \\ \frac{20}{39} \approx 0.5128, & \text{if } 30 \le y < 40 \\ \frac{30}{39} \approx 0.7692, & \text{if } 40 \le y < 65 \\ 1, & \text{if } y \ge 65 \end{cases}$$

and its conditional mean and conditional variance given $X = 8$ are

$$\begin{aligned}
\mathbf{E}[Y|X=8] &\approx 39.2051 \\
\mathrm{Var}[Y|X=8] &\approx 241.6502.
\end{aligned}$$

(c)

$$\begin{aligned}
\mathbf{E}[X] &= 1 \cdot 0.21 + 5 \cdot 0.4 + 8 \cdot 0.39 = 5.33 \\
\mathrm{Var}(X) &= 6.7611 \\
\mathrm{Cov}(X,Y) &= \mathbf{E}[XY] - \mathbf{E}[X]\mathbf{E}[Y] \\
&= 11.0005 \\
\mathrm{Corr}(X,Y) &= \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\mathrm{Var}(Y)}} = \frac{11.0005}{\sqrt{6.7611 \cdot 218.2075}} \approx \frac{11.0005}{38.4099} \approx 0.2864
\end{aligned}$$

SW 2.22

We have

$$\begin{aligned}
\mathbf{E}[R_s] &= 0.08 \\
\mathrm{Var}(R_s) &= 0.07^2 \\
\mathbf{E}[R_b] &= 0.05 \\
\mathrm{Var}(R_b) &= 0.04^2 \\
\mathrm{Cov}(R_s, R_b) &= 0.0007
\end{aligned}$$

and

$$R(w) = wR_s + (1-w)R_b.$$

In the generalized case,

$$\begin{aligned}
\mathbf{E}[R(w)] &= \mathbf{E}[wR_s + (1-w)R_b] \\
&= w\mathbf{E}[R_s] + (1-w)\mathbf{E}[R_b] = 0.08w + 0.05(1-w) \\
&= 0.03w + 0.05 \\
\mathrm{Var}(R(w)) &= \mathrm{Var}(wR_s + (1-w)R_b) \\
&= w^2\mathrm{Var}(R_s) + (1-w)^2\mathrm{Var}(R_b) + 2w(1-w)\mathrm{Cov}(R_s, R_b) \\
&= 0.0051w^2 - 0.0018w + 0.0016.
\end{aligned}$$

(a) $w = 0.5$. Then,

$$\begin{aligned}
\mathbf{E}[R(0.5)] &= 0.03 \cdot 0.5 + 0.05 = 0.065 \\
\mathrm{Var}(R(0.5)) &= 0.0051 \cdot 0.25 - 0.0018 \cdot 0.5 + 0.0016 = 0.001975 \\
\mathrm{Sd}(R(0.5)) &\approx 0.0444
\end{aligned}$$

(b) $w = 0.75$. Then,

$$\begin{aligned}
\mathbf{E}[R(0.75)] &= 0.03 \cdot 0.75 + 0.05 = 0.0725 \\
\mathrm{Var}(R(0.75)) &= 0.0051 \cdot 0.5625 - 0.0018 \cdot 0.75 + 0.0016 = 0.00311875 \\
\mathrm{Sd}(R(0.75)) &\approx 0.0558
\end{aligned}$$

(c) Note that $\mathbf{E}[R(w)]$ is an increasing function of w. Thus, $w = 1$ makes the mean of R as large as possible, using all of the money for buying stock market mutual funds. The standard deviation in that case is 0.07.

(d) Note that minimizing standard deviation is equal to minimizing variance. Let us find the first and the second derivatives of $\mathrm{Var}(R(w))$.

$$\begin{aligned}
\frac{\partial}{\partial w}\mathrm{Var}(R(w)) &= 0.0102w - 0.0018 \\
\frac{\partial^2}{\partial w^2}\mathrm{Var}(R(w)) &= 0.0102 > 0.
\end{aligned}$$

We can see that the $w^* = 9/51 \approx 0.1765$ satisfying the first order condition is the global minimizer.

Problem 2

(a) Note that $Z = 2 \Leftrightarrow X = 1$ and $Z = 0 \Leftrightarrow X = 0$.

$$\mathbf{E}[Z] = 2\Pr\{Z = 2\} + 0\Pr\{Z = 0\} = 2\Pr\{X = 1\} + 0\Pr\{X = 0\} = 2p.$$

(b)

$$\mathbf{E}\left[Z^2\right] = 4\Pr\{Z = 2\} + 0\Pr\{Z = 0\} = 4\Pr\{X = 1\} + 0\Pr\{X = 0\} = 4p.$$

(c)

$$\mathrm{Var}(Z) = \mathbf{E}\left[Z^2\right] - \mathbf{E}[Z]^2 = 4p - 4p^2 = 4p(1-p)$$

Problem 3

Note that we have

$$Y = 1000X.$$

Thus,

$$\begin{aligned}
\mathbf{E}[Y] &= \mathbf{E}[1000X] = 1000\mathbf{E}[X] = 48800 \\
\mathrm{Var}(Y) &= \mathrm{Var}(1000X) = 1000^2\mathrm{Var}(X) = 1000^2 \cdot 12.1^2 \\
\mathrm{Sd}(Y) &= 1000 \cdot 12.1 = 12100.
\end{aligned}$$

Problem 4

(a)

$$\begin{aligned}
\mathbf{E}[GPA|SAT = 750] &= 0.7 + 0.002 \cdot 750 = 2.2 \\
\mathbf{E}[GPA|SAT = 1500] &= 0.7 + 0.002 \cdot 1500 = 3.7
\end{aligned}$$

(b) By the law of iterated expectations,

$$\mathbf{E}[GPA] = \mathbf{E}[\mathbf{E}[GPA|SAT]] = \mathbf{E}[0.7 + 0.002SAT] = 0.7 + 0.002\mathbf{E}[SAT] = 2.7$$

Problem 5

(a) By the definition of U,

$$\mathbf{E}[U|X] = E[Y - E[Y|X]|X] = E[Y|X] - E[Y|X] = 0\ .$$

(b) Using the law of iterated expectations,

$$\mathbf{E}[Uh(X)] = E[E[Uh(X)|X]] = E[E[U|X]h(X)] = 0\ ,$$

since $E[U|X] = 0$ from part (a).

(c) Using the hint, we evaluate $E[V^2]$:

$$\begin{aligned}
E[V^2] &= E[(U - h(X))^2] \\
&= E[U^2] - 2E[Uh(X)] + E[h(X)^2] \\
&= E[U^2] + E[h(X)^2] \\
&\geq E[U^2]\ ,
\end{aligned}$$

where the second equality comes from part (b), and the inequality comes from the fact that $h(X)^2 \geq 0$ (with equality if $g(X) = E[Y|X]$).

(d) From part (c) we can conclude immediately that $E[Y|X]$ minimizes mean-squared error among all predictors of Y given X.

Problem 6

$$\begin{aligned}
\mathrm{Cov}(X,Y) &= \mathbf{E}[XY] - \mathbf{E}[X]\mathbf{E}[Y] \\
&= \mathbf{E}\left[X^3\right] - \mathbf{E}[X]\mathbf{E}\left[X^2\right] \\
&= \int_{-1}^{1} x^3 \frac{1}{2} dx - \left(\int_{-1}^{1} x \frac{1}{2} dx\right)\left(\int_{-1}^{1} x^2 \frac{1}{2} dx\right) \\
&= \left[\frac{x^4}{8}\right]_{-1}^{1} - \left[\frac{x^2}{4}\right]_{-1}^{1}\left[\frac{x^3}{6}\right]_{-1}^{1} \\
&= 0 - 0 \cdot \frac{1}{3} = 0
\end{aligned}$$

But X and Y are not independent since

$$\begin{aligned}
\mathbf{E}[Y|X = 0.1] = 0.01 \\
\mathbf{E}[Y|X = 0.2] = 0.04 \ .
\end{aligned}$$

Problem 7

(a) U is mean independent of X by definition. However, U is not independent of X. If U and X are independent, we would have

$$\begin{aligned}
\mathrm{Var}(U|X) &= \mathbf{E}\left[(U - \mathbf{E}[U|X])^2 \,|X\right] \\
&= \mathbf{E}\left[U^2|X\right] \\
&= \int u^2 f_{U|X}(u|X) du \\
&= \int u^2 f_U(u) du = \mathbf{E}\left[U^2\right] \ .
\end{aligned}$$

We are given that $\mathrm{Var}(U|X) = X^2$. Unless X is a degenerate random variable (which means X is a constant), U cannot be independent of X.

(b) By the law of iterated expectations,

$$\begin{aligned}
\mathbf{E}[U] &= \mathbf{E}[\mathbf{E}[U|X]] = \mathbf{E}[0] = 0 \\
\mathrm{Var}(U) &= \mathbf{E}\left[(U - \mathbf{E}[U])^2\right] \\
&= \mathbf{E}\left[U^2\right] = \mathbf{E}\left[\mathbf{E}\left[U^2|X\right]\right] \\
&= \mathbf{E}\left[\mathrm{Var}\left(U|X\right)\right] = \mathbf{E}\left[X^2\right]
\end{aligned}$$

(c)

$$
\begin{aligned}
\mathbf{E}[Y|X] &= \mathbf{E}[a + bX + U|X] \\
&= a + bX + 0 = a + bX \\
\mathbf{E}[Y] &= \mathbf{E}[\mathbf{E}[Y|X]] \\
&= \mathbf{E}[a + bX] = a + b\mathbf{E}[X]
\end{aligned}
$$

(d)

$$
\begin{aligned}
\mathrm{Var}(Y|X) &= \mathbf{E}\left[(Y - \mathbf{E}[Y|X])^2|X\right] \\
&= \mathbf{E}\left[(a + bX + U - a - bX)^2|X\right] \\
&= \mathbf{E}\left[U^2|X\right] = X^2
\end{aligned}
$$

and

$$
\begin{aligned}
\mathrm{Var}(Y) &= \mathbf{E}\left[(Y - \mathbf{E}[Y])^2\right] \\
&= \mathbf{E}\left[(a + bX + U - a - b\mathbf{E}[X])^2\right] \\
&= \mathbf{E}\left[(b(X - \mathbf{E}[X]) + U)^2\right] \\
&= b^2\mathbf{E}\left[(X - \mathbf{E}[X])^2\right] + \mathbf{E}\left[U^2\right] + 2b\mathbf{E}\left[(X - \mathbf{E}[X])U\right] \\
&= b^2\mathrm{Var}(X) + \mathrm{Var}(U) + 2b\mathbf{E}\left[\mathbf{E}\left[(X - \mathbf{E}[X])U|X\right]\right] && \because \text{law of iterated expectations} \\
&= b^2\mathrm{Var}(X) + \mathbf{E}\left[X^2\right] + 2b\mathbf{E}\left[\mathbf{E}\left[(X - \mathbf{E}[X])U|X\right]\right] && \because \mathrm{Var}(U) = \mathbf{E}\left[X^2\right] \\
&= b^2\mathrm{Var}(X) + \mathbf{E}\left[X^2\right] + 2b\mathbf{E}\left[(X - \mathbf{E}[X])\mathbf{E}[U|X]\right] && \because X - \mathbf{E}[X] \text{ is fixed given } X \\
&= b^2\mathrm{Var}(X) + \mathbf{E}\left[X^2\right] + 2b\mathbf{E}\left[(X - \mathbf{E}[X]) \cdot 0\right] \\
&= b^2\mathrm{Var}(X) + \mathbf{E}\left[X^2\right]
\end{aligned}
$$