# Problem Set 2

## Question 5

Let $(X, Y)$ denote a random vector, and let $(X_1, Y_1), \cdots, (X_n, Y_n)$ denote an i.i.d. sample of size $n$ from $(X, Y)$. Suppose there exists variance for both $X$ and $Y$. Consider estimating $E[X]E[Y]$ using the estimator $\bar{X}_n\bar{Y}_n$.

**Problem.** Can you prove that $\bar{X}_n\bar{Y}_n$ is an unbiased estimator of $E[X]E[Y]$? Why or why not?

**Solution.** To begin,

$$\begin{aligned}
E\left[\bar{X}_n\bar{Y}_n\right] &= E\left[\frac{1}{n}\sum_{i=1}^{n} X_i \frac{1}{n}\sum_{i=1}^{n} Y_i\right] \\
&= E\left[\frac{1}{n^2}\sum_{i=1}^{n} X_i \sum_{i=1}^{n} Y_i\right] \\
&= \frac{1}{n^2}E\left[\sum_{i=1}^{n} X_iY_i + \sum_{i=1}^{n}\sum_{j\neq i}^{n} X_iY_j\right] \\
&= \frac{1}{n^2}E\left[\sum_{i=1}^{n} X_iY_i\right] + \frac{1}{n^2}E\left[\sum_{i=1}^{n}\sum_{j\neq i}^{n} X_iY_j\right] \\
&(\because X_i \perp Y_j \text{ for } i \neq j, \quad E[X_iY_j] = E[X_i]E[Y_j] = E[X]E[Y]) \\
&= \frac{1}{n}E[XY] + \frac{n(n-1)}{n^2}E[X]E[Y]
\end{aligned}$$

Therefore,

$$E\left[\bar{X}_n\bar{Y}_n\right] \longrightarrow E[X]E[Y] \text{ as } n \to \infty$$

If we had an information about $X \perp Y$, then we can continue as follows:

$$\begin{aligned}
&= \frac{1}{n}E[XY] + \frac{n(n-1)}{n^2}E[X]E[Y] \\
&= \frac{1}{n}E[X]E[Y] + \frac{n-1}{n}E[X]E[Y] \\
&= E[X]E[Y]
\end{aligned}$$

Therefore, without the information about the independence, we can conclude that the estimator is not unbiased but **asymptotically** unbiased.

**Problem.** Show that $\bar{X}_n\bar{Y}_n$ is a consistent estimator.

**Solution.** By the WLNN, we have the following.

$$\begin{aligned}
\bar{X}_n &\xrightarrow{p} E[X] \\
\bar{Y}_n &\xrightarrow{p} E[Y]
\end{aligned}$$

Consider the continuous function $g(t, s) = ts$.
By the CMT,

$$\bar{X}_n\bar{Y}_n = g\left(\bar{X}_n\bar{Y}_n\right) \xrightarrow{p} g\left(E[X], E[Y]\right) = E[X]E[Y]$$