APM 462 Lecture Notes

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Contents

1 Preface

Question 1.1. *What is optimisation?*

Our goal here is to find a max or min of a function

$$f : \Omega \to \mathbb{R}$$

That is, find $x_0 \in \Omega$ such that $f(x_0) \leq f(x)$ $\forall x \in \Omega$ or at least in a neighbourhood of x_0 (we will further explain what this means).

Ω can be a range of values

- $\Omega \in \mathbb{R}$
- $\Omega \in \mathbb{R}^n$
- $\Omega \in$ Function space (e.g. $C^0(\mathbb{R}^n)$)

Example 1.2 (Utility Maximisation Problem)**.** *Suppose we have n products, x_j is the number of product j, p_j is the price per unit of product j. We have a utility function*

$$u : \mathbb{R}^n_{\geq 0} \to \mathbb{R}$$

In this problem, we want to find

$$\max_{x\in\Omega} F(x)$$

where

$$F(x) = u(x) - x\cdot p = u(x_1,\ldots,x_n) - x_1p_1 - \ldots x_np_n$$

we can set $\Omega = \mathbb{R}^n_{\geq 0}$

Example 1.3 (Geodesic Distance)**.** *Suppose* $D \subseteq \mathbb{R}^2$*, consider a function* $L : D\times\mathbb{R}^2 \to \mathbb{R}_{\geq 0}$
Fix $x, y \in D$*, we want to find*

$$\min_{\alpha\in\Omega} F(\alpha)$$

where

$$\Omega = \{\alpha \in C^1([0,1], D) : \alpha(0) = x, \alpha(1) = y\}$$

and

$$F(\alpha) = \int_0^1 L(\alpha(t), \alpha(t))dt$$

In high school, optimisation is to find the minimum of a function $f : \mathbb{R} \to \mathbb{R}$. We do this by finding x_0 such that $f'(x_0) = 0$, we then check $f''(x_0) > 0$ to check if this is indeed a minimum. We will learn how to extend and generalise those ideas to $\Omega = \mathbb{R}^n$ or even more general Ω in this course.

In this course, we want to solve the general question involves a function $f : \Omega \to \mathbb{R}$, we want to find

$$\min_{x\in\Omega} f(x)$$

We want to find

1. Does the minimum exist
2. How to find a minimum
3. How to distinguish minimum and maximum

We also need to consider different properties of function f, such as differentiability, convexity, as well as different properties of Ω such as convexity, boundedness, compactness.

To discuss this, this course would be split into three chapters.

1. Unconstrained Optimisation ($f : \mathbb{R}^n \to \mathbb{R}$ where $\Omega \subseteq \mathbb{R}^n$)
2. Constrained Optimisation ($f : \mathbb{R}^n \to \mathbb{R}$ where Ω is given through inequality constraints)
3. Calculus of variations ($f : \mathbb{R}^n \to \mathbb{R}$, where Ω is some function space, e.g. $C^1(\mathbb{R}^n)$.)

2 Uncontstrained Optimisation

The basic problem in this chapter is that we want to find

$$\min_{x\in\Omega} f(x)$$

That is, we want to minimise $f(x)$ subject to $x \in \Omega$. The following conditions must hold

- $f : \Omega \to \mathbb{R}$ function
- $\Omega \subseteq \mathbb{R}^n$ (non-empty), very often $\Omega = \mathbb{R}^n$

Definition 2.1. *$x_0 \in \Omega$ is a relative/local minimum point of f over Ω if there exists $\epsilon > 0$ such that*

$$f(x) \geq f(x_0) \ \forall x \in \Omega, \ s.t. \ |x - x_0| \leq \epsilon$$

if

$$f(x) > f(x_0) \ \forall x_0 \neq x \in \Omega, \ s.t. \ |x - x_0| \leq \epsilon$$

then x_0 is a strict relative/local minimum of f over Ω

Definition 2.2. *$x_0 \in \Omega$ is a global minimum point of f over Ω if*

$$f(x) \geq f(x_0) \ \forall x \in \Omega$$

If $f(x) > f(x_0) \ \forall x_0 \neq x \in \Omega$ then x_0 is a strict global minimum point of f over Ω

Remark 2.3. *If we want to solve*

$$\max_{x \in \Omega} f(x)$$

we can solve

$$-\min_{x \in \Omega} -f(x)$$

this shows that the definition from minimisation extend to maximisation

Example 2.4.

1. *Let $f(x) = 1, \Omega = \mathbb{R}$, all $x_0 \in \Omega$ are local & global minimum and maximum but not strict*
2. *Let $f(x) = x, \Omega = \mathbb{R}$, no local/global maximum or minimum*
3. *$f(x) = x^2, \Omega = \mathbb{R}$, $x_0 = 0$ is the strict local & global minimum. There is no local/global maximum.*
4. *$f(x) = x^2, \Omega = [-1, 2]$, $x_0 = 0$ is strict local/global minimum. Local & global maximum at $x_0 = 2$ since $= f(2) > f(x) \ \forall x \in \Omega \backslash \{2\}$.*
5. *$f(x) = x^2, \Omega = [1, 2]$, strict local & global minimum at $x_0 = 1$.*

Remark 2.5. *Existence + Uniqueness of local/global maximum/minimum is not trivial. It heavily depends on Ω and f.*

Theorem 2.6 (Weierstrass/Extreme Value Theorem)**.** *If f is continuous and Ω is compact then f admits a global minimum point on Ω; i.e.*

$$\exists x_0 \in \Omega \ s.t. \ f(x_0) \leq f(x) \ \forall x \in \Omega$$

This also holds for maximum.

Example 2.7 (Example 2.4)**.**

1. *Ω not compact*
2. *Ω not compact*
3. *Ω not compact*
4. *Ω compact, Weierstrass Theorem applies*
5. *Ω compact, Weierstrass Theorem applies*

Idea for characterising local minimum: assume x_0 is a local minimum, take a direction $d \in \mathbb{R}^n$ and analyse behaviour of $f(x_0 + \alpha d)$ where $\alpha \geq 0$ and small.

If x_0 is a local minimum then $f(x_0 + \alpha d) \geq f(x_0) \ \forall d, \forall \alpha$ sufficiently small.

We can use knowledge of 1-dimensional optimisation to analyse $f(x_0 + \alpha d)$ as a function of α.

Definition 2.8. *Let $x_0 \in \Omega$ then $d \in \mathbb{R}^n$ is a feasible direction at x_0 if $\exists \overline{\alpha} > 0$ such that $x_0 + \alpha d \in \Omega$ $\forall \alpha$ where $0 \leq \alpha \leq \overline{\alpha}$.*

Theorem 2.9 (Taylor/Mean Value Theorem)**.**
- *Let $f \in C^1(\Omega), x, y \in \Omega$ s.t. $[x, y] \in \Omega$ ($[x, y]$ is the line segment between x and y)*

  Then $\exists \theta \in [0, 1]$ such that $f(y) = f(x) + \nabla f(\theta x + (1 - \theta)y) \cdot (y - x)$.
- *Let $f \in C^2(\Omega), x, y \in \Omega$ s.t. $[x, y] \subseteq \Omega$*

  Then $\exists \theta \in [0, 1]$ s.t. $f(y) = f(x) + \nabla f(x) \cdot (y - x) + \frac{1}{2}(y - x)^\top \nabla^2 f(\theta x + (1 - \theta)y) \cdot (y - x)$

Proposition 2.10 (First Order Necessary Condition)**.** *Let $f \in C^1(\Omega)$, if $x_0 \in \Omega$ is a local minimum point of f over Ω then $\forall d$ is a feasible direction at x_0 we have*

$$\nabla f(x_0) \cdot d \geq 0$$

Proof. **Idea:** x_0 local minimum, d feasiable direction at $x_0, y = x_0 + \alpha d$ for $\alpha \in [0, \overline{\alpha}]$ then

$$\begin{aligned} f(x_0) \leq & f(x_0 + \alpha d) \\ = & f(x_0) + \nabla f(\theta x_0 + (1 - \theta)(x_0 + \alpha d)) \cdot (\alpha d) \qquad \text{[By Taylor]} \\ = & f(x_0) + \alpha \nabla f(x_0 + \alpha(1 - \theta)d) \cdot d \\ \implies & 0 \leq \alpha \nabla f(x_0 + \alpha(1 - \theta)d) \cdot d \\ \implies & 0 \leq \nabla f(x_0 + \alpha(1 - \theta)d) \cdot d \\ \implies & 0 \leq \nabla f(x_0) \cdot d \qquad \text{[As } \alpha \to 0] \end{aligned}$$

Proof: $\forall 0 \leq \alpha \leq \overline{\alpha}$, we define $x : [0, \overline{\alpha}] \to \Omega$ as $x(\alpha) = x_0 + \alpha d$.
We also define $g : [0, \overline{\alpha}] \to \mathbb{R}$ as $g(\alpha) = f(x(\alpha))$.
Now g has a local minimum at $\alpha = 0$.
By Taylors Theorem (first order Taylor expansion), $g(\alpha) - g(0) = g'(0) \cdot \alpha + o(\alpha)$
If $g'(0) < 0$ then we can take $\alpha > 0$ sufficiently small such that $g'(0)\alpha + o(\alpha) < 0 \implies g(\alpha) - g(0) < 0$, a contradiction that 0 is the local minimum of g.
Hence, $0 \leq g'(0) = \nabla f(x_0) \cdot d$. □

Remark 2.11. *$f(x)$ is $o(x) \iff \lim_{x \to 0} |\frac{f(x)}{x}| = 0$*
$f(x)$ is $O(x) \iff \lim_{x \to 0} |\frac{f(x)}{x}| \leq k$ where $k > 0$

Example 2.12. *Let $\Omega \subseteq \mathbb{R}^2$ be $\Omega = \{(x_1, x_2) \in \mathbb{R}^2 : x_1, x_2 \geq 0\}$. Let $f : \Omega \to \mathbb{R}$ be $f(x_1, x_2) = x_1^2 - x_1 + x_2 + x_1 x_2$ has a local/global minimum at $(\frac{1}{2}, 0)$*
We have $\nabla f(x_1, x_2) = (2x_1 + x_2 - 1, 1 + x_1) \implies \nabla f(\frac{1}{2}, 0) = (0, \frac{3}{2})$.
If d is a feasible direction at $(\frac{1}{2}, 0), d = (d_1, d_2)^\top$ then $d_2 \geq 0$.
Hence $\nabla f(\frac{1}{2}, 0) \cdot d = \frac{3}{2} d_2 \geq 0$.

Corollary 2.13 (First Order Necessary Condition, Unconstrained)**.** *Let $f \in C^1(\Omega)$, if x_0 is a local minimum of f over Ω and $x_0 \in int(\Omega)$ then $\nabla f(x_0) = 0$.*

Proof. If $x_0 \in int(\Omega)$ then $\forall d \in \mathbb{R}^n$ are feasible direction.
Therefore, we have $\nabla f(x_0) \cdot d \geq 0$ for all $d \in \mathbb{R}^n$.
Hence, $\nabla f(x_0) = 0$. □

Example 2.14. *Take $\Omega = \mathbb{R}^2$ then every point is an interior point.*
Take $f : \Omega \to \mathbb{R}$ by $f(x_1, x_2) = x_1^2 - x_1 x_2 + x_2^2 - 3x_2$.
We have

$$\nabla f(x_1, x_2) = (2x_1 - x_2, -x_1 + 2x_2 - 3)$$

.

Set it to 0 we have

$$0 = \nabla f(x_1, x_2) = (2x_1 - x_2, -x_1 + 2x_2 - 3)$$

This gives us

$$\begin{cases} 2x_1 = x_2 \\ 2x_2 - x_1 = 3 \end{cases}$$

The solution is

$$\begin{cases} x_1 = 1 \\ x_2 = 2 \end{cases}$$

Hence, $(1, 2)$ is only a candidate for the local minimum (we don't know if it actually is a local/global minimum or not since FONC is a necessary condition)

Weierstrass Theorem does not apply as Ω is not compact.

Proposition 2.15 (Second Order Necessary Condition)**.** *Let $f \in C^2(\Omega)$ if x_0 is a local minimum of f over Ω then for all feasible direction $d \in \mathbb{R}^n$ at x_0 we get*

1. $\nabla f(x_0) \cdot d \geq 0$
2. *If* $\nabla f(x_0) \cdot d = 0$ *then* $d^\top \nabla^2 f(x_0) \cdot d \geq 0$

Proof. (i) Proved

(ii) If $\nabla f(x_0) \cdot d = 0$, define $x : [0, \overline{\alpha}] \to \Omega$ by $x(\alpha) = x_0 + \alpha d$. Also define $g : [0, \overline{\alpha}] \to \mathbb{R}$ by $g(\alpha) = f(x(\alpha))$.

Then $g'(0) = \nabla f(x_0) \cdot d = 0$.

Using Taylor expansion we have

$$g(\alpha) - g(0) = \frac{1}{2} g''(0)\alpha^2 + o(\alpha^2)$$

If $g''(0) < 0$ then $\frac{1}{2} g''(0)\alpha^2 + o(\alpha^2) < 0$ when α is small. But this implies $g(\alpha) - g(0) < 0$ contradiction that g has a local minimum at 0.

Hence, we have $0 \leq g''(0) = d^\top \nabla^2 f(x_0) d$ □

Example 2.16. *Recall from Example 2.12, we have $\Omega = \{(x_1, x_2) \in \mathbb{R}^2 : x_1, x_2 \geq 0\}$, let $f : \Omega \to \mathbb{R}$ by $f(x_1, x_2) = x_1^2 - x_1 + x_2 + x_1 x_2$. We have the local/global minimum is at $x_0 = (\frac{1}{2}, 0)$.*

The first condition of the SONC is satisfied.

For all feasible direction $d = (d_1, d_2)$, if we have

$$\nabla f(x_0) \cdot d = \frac{3}{2} d_2 = 0$$

we must have $d_2 = 0$

We then have

$$d^\top \nabla f(x_0) d = (d_1, 0) \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} d_1 \\ 0 \end{pmatrix} = 2d_1^2 \geq 0$$

Proposition 2.17 (Second Order Necessary Condition, Unconstrained)**.** *Let $f \in C^2(\Omega), x_0 \in int(\Omega)$ and x_0 is a local minimum of f over Ω then*

1. $\nabla f(x_0) = 0$

2. $\forall d \in \mathbb{R}^n, d^\top \nabla^2 f(x_0) d \geq 0$

Definition 2.18. *Let $M \in \mathbb{R}^{n\times n}$ be a symmetric matrix, then M is positive semi-definite (alternate expressions, M is PSD, $M \geq 0$) if*

$$\forall d \in \mathbb{R}^n, d^\top M d \geq 0$$

M is positive definite (alternate expressions, M is PD, $M > 0$) if

$$\forall d \in \mathbb{R}^n, d^\top M d > 0$$

Remark 2.19. *In SONC, the condition 2 means that $\nabla^2 f(x_0)$ is positive semi-definite*

Theorem 2.20 (Sylvester's Criterion for PD)**.** *Symmetric $M \in \mathbb{R}^{n\times n}$ is PD if and only if*

- *The upper left 1×1 matrix in M have positive determinant*
- *The upper left 2×2 matrix in M have positive determinant*
- *...*
- *The upper left $n \times n$ matrix in M have positive determinant*

Example 2.21. *The following is a PD matrix*

$$M = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

as $det(2) = 2 > 0, det(M) = 3 > 0$

Theorem 2.22 (Sylvester's Criterion for PSD)**.** *Symmetric 2×2 matrix*

$$M = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

is PSD if and only if

- *$det(a) \geq 0$*
- *$det(c) \geq 0$*
- *$det(M) \geq 0$*

Example 2.23. *The following is a PD matrix*

$$M = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

as $det(2) = 2 > 0, det(M) = 3 > 0$

Proposition 2.24 (Second Order Sufficient Condition, unconstrained)**.** *Let $f \in C^2(\Omega), x_0 \in int(\Omega)$. If*

1. *$\nabla f(x_0) = 0$*
2. *$\nabla^2 f(x_0)$ is PD*

then x_0 is a strict local minimum of f on Ω

Proof. If $\nabla^2 f(x_0)$ is PD then there exist $a > 0$ such that $d^\top f(x_0)d \geq a|d|^2$. By Taylor's theorem

$$f(x_0 + d) - f(x_0) = \frac{1}{2}d^\top \nabla^2 f(x_0)d + o(|d|^2) \geq \frac{1}{2}a|d|^2 + o(|d|^2)$$

If $|d|$ is small then $\frac{1}{2}a|d|^2 > o(|d|^2)$, hence $f(x_0 + d) - f(x_0) > 0 \implies f(x_0 + d) > f(x_0)$. Hence, x_0 is the strict minimum point. □

Definition 2.25. *A set $S \subseteq \mathbb{R}^n$ is convex if $\forall x, y \in S, t \in [0, 1]$ then $tx + (1 - t)y \in S$.*

Definition 2.26. *Let $S \subseteq \mathbb{R}^n$ be convex, then the function $f : S \to \mathbb{R}$ is convex if*

$$f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y)$$

f is concave if $-f$ is convex.

Proposition 2.27. *Let $S \subseteq \mathbb{R}^n$ be a convex set.*

1. *f is convex $\implies \Gamma_c = \{x \in S : f(x) \leq c\}$ is convex for all $c \in \mathbb{R}$*
2. *If $f \in C^1(S)$ then f is convex $\iff f(y) - f(x) \geq \nabla f(x) \cdot (y - x), \forall x, y \in S$*
3. *$f \in C^2(S)$ then f is convex $\iff \nabla^2 f(x)$ is PSD $\forall x \in S$*

Proof. (i) Fix $c \in \mathbb{R}, x, y \in \Gamma_c, t \in [0, 1]$

$$f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y) \leq tc + (1 - t)c = c$$

showing $tx + (1 - t)y \in \Gamma_c$.

(ii) $(\Rightarrow)$: If f is convex, for all $x, y \in S, t \in [0, 1]$ we have

$$f(ty + (1 - t)x) \leq tf(y) + (1 - t)f(x)$$

For $t > 0$ we have

$$\frac{f(x + t(y - x)) - f(x)}{t} \leq f(y) - f(x)$$

Let $t \to 0$, we have

$$\nabla f(x) \cdot (y - x) \leq f(y) - f(x)$$

$(\Leftarrow)$: Let $f(y) \geq f(x) + \nabla f(x) \cdot (y - x), \forall x, y \in S$.
Fix $x_1, x_2 \in S, t \in [0, 1]$, let $y = x_1$ or $y = x_2$ we have $\forall x \in S$

$$\begin{cases} f(x_1) \geq f(x) + \nabla f(x) \cdot (x_1 - x) \\ f(x_2) \geq f(x) + \nabla f(x) \cdot (x_2 - x) \end{cases}$$

Multiply the first equation by t, the second by $(1 - t)$ and add them up we get $\forall x \in S$

$$tf(x_1) + (1 - t)f(x_2) \geq f(x) + \nabla f(x) \cdot (tx_1 + (1 - t)x_2 - x)$$

Take $x = tx_1 + (1 - t)x_2$ would get the required inequality

$$f(tx_1 + (1 - t)x_2) \leq tf(x_1) + (1 - t)f(x_2)$$

3. $(\Leftarrow)$: If $\nabla^2 f(x)$ is PSD, $\forall x \in S$, take $x, y \in S$ by MVT $\exists t \in [0, 1]$ s.t.

$$f(y) - f(x) = \nabla f(x) \cdot (y - x) + \frac{1}{2}(y - x)^\top \cdot \nabla^2 f(x + t(y - x)) \cdot (y - x) \geq \nabla f(x) \cdot (y - x)$$

($\Rightarrow$) : Suppose f is convex, assume $\nabla^2 f(x_0)$ is not PSD for some $x_0 \in S$. This means there is $\nabla^2 f(x_0)$ is not PSD for some $x_0 \in int(S)$ (If $x_0 \in \delta S$, we can use continuity of $\nabla^2 f$ such that $x_0 \in int(S)$)

Since $\nabla^2 f(x_0)$ is not PSD, there is also a $y_0 \in S$ such that

$$(y_0 - x_0)^\top \nabla^2 f(x_0)(y_0 - x_0) < 0$$

Using the continuity of $\nabla^2 f$ we can take y_0 arbitrary close to x_0 while fixing the feasible direction such that $\forall t \in [0,1]$ we have $(y_0 - x_0)^\top \nabla^2 f(x_0 + t(y_0 - x_0))(y_0 - x_0) < 0$

By MVT, this gives

$$f(y_0) - f(x_0) = \nabla f(x_0) \cdot (y_0 - x_0) + \frac{1}{2}(y_0 - x_0)^\top \nabla^2 f(x_0 + t(y_0 - x_0))(y_0 - x_0) < \nabla f(x_0) \cdot (y_0 - x_0)$$

This shows f is not convex on S. □

Theorem 2.28. *Let Ω be a convex set, $f : \Omega \to \mathbb{R}$ be a convex function, then*

1. *$\Gamma =$ The set of minimisers of f on Ω is convex*

2. *Every local minimiser of f on Ω is a global minimiser.*

Proof. If f has no local minimum then theorem is automatically valid.

Let f have local minimiser x_0 such that $f(x_0) = c_0$.

2. Let x_0 be the local minimiser but not a global minimiser. This means $\exists y_0 \in \Omega$ s.t. $f(y_0) < f(x_0)$.

Hence, $\forall t \in (0,1)$ we have

$$f(ty_0 + (1-t)x_0) \le tf(y_0) + (1-t)f(x_0) < tf(x_0) + (1-t)f(x_0) = f(x_0)$$

By setting t extremely small, this shows x_0 is not a local minimiser.

1. $\Gamma = \{x \in \Omega : f(x) \le c_0\}$ is convex by Proposition 2.27 □

Remark 2.29. *This theorem does not tell us anything about the existence of minimisers.*

Example 2.30.

1. *Let $\Omega = \mathbb{R}, f(x) = e^x$ is convex as $f''(x) = e^x > 0$. But this doesn't have a minimiser*

2. *Let $\Omega = (0,1), f(x) = x^2$ is convex as $f''(x) = 2 > 0$. But this doesn't have a minimiser.*

Theorem 2.31. *Let Ω be convex, $f : \Omega \to \mathbb{R}$ convex, $f \in C^1(\Omega)$. If $\exists x_0 \in \Omega$ s.t. $\forall x \in \Omega$*

$$\nabla f(x_0) \cdot (x - x_0) \ge 0$$

Then x_0 is a global minimum of f over Ω

Proof. By Proposition 2.27, we have $f(x) - f(x_0) \ge \nabla f(x_0) \cdot (x - x_0) \ge 0$

This gives $f(x) \ge f(x_0), \forall x \in \Omega$.

Hence, x_0 is the global minimum. □

Remark 2.32. *In this theorem, note that $x - x_0$ is feasible direction at x_0 as $x_0 + t(x - x_0) \in \Omega, \forall t \in [0,1]$.*

Another way to write Theorem 2.31 is

Let Ω be convex, $f : \Omega \to \mathbb{R}$ convex, $f \in C^1(\Omega)$. If $\exists x_0 \in \Omega$ s.t.

$$\nabla f(x_0) \cdot d \ge 0$$

for all feasible direction d at x_0, then x_0 is a global minimum of f over Ω

This is commonly known as the First Order Sufficient Condition.

Remark 2.33. *Proposition 2.27, Theorem 2.28, Theorem* **??** *hold for concave functions with opposite inequalities and max instead of min.*

In other words, convex functions are good for minimisation. Concave functions are good for maximisation.

Question 2.34. *How can we find the uniqueness of minimum for convex functions*

Example 2.35. *Let* $\Omega = \mathbb{R}, f(x) = c$*. We have* f *is convex but minimum not unique.*

Definition 2.36. *Let* $S \subseteq \mathbb{R}$ *be convex,* $f : S \to \mathbb{R}$*, then we say* f *is strictly convex if* $\forall t \in (0,1), \forall x, y \in S$ *s.t.* $x \neq y$*. It holds that*

$$f(tx + (1-t)y) < tf(x) + (1-t)f(y)$$

Proposition 2.37. *Let* Ω *be a convex set,* $f : \Omega \to \mathbb{R}$ *be a strictly convex function. Assume* f *has a local/global minimiser (they are equivalent as* f *is convex given Theorem 2.28). Then this minimiser is unique.*

Proof. Let $x \neq y \in \Omega$ be local/global minimisers. This gives $f(x) = f(y)$.

Then $\forall t \in (0,1)$ we have

$$f(tx + (1-t)y) < tf(x) + (1-t)f(y) = tf(x) + (1-t)f(x) = f(x)$$

This shows that x is not a local/global minimiser. □

Proposition 2.38. *Let* $f \in C^2(S)$ *is strictly convex if* $\nabla^2 f(x) > 0$ *is PD* $\forall x \in S$*.*

Remark 2.39. *The only if direction is wrong:* $f(x) = x^4$ *is strictly convex but* $0 = f''(0) = 12x^2$*.*

2.1 Descent algorithms

So far we have talked about analytical properties of minimisers and convex functions. This involves inverting matrices which are costly in the real world when computing those. Instead we want to numerically (iteratively) compute solutions to the problem

$$\min_{x \in \Omega} f(x)$$

Definition 2.40. *An algorithm is a function* $A : \Omega \to \Omega$

The idea is pick $x_0 \in \Omega$, the algorithm A would define iteratively sequence $\{x_k\}_{k \in \mathbb{N}}$ by $x_{k+1} = A(x_k)$ and then (hopefully) $x_k \to \overline{x}$ which is the minimum of f over Ω

Convergence depends on the existence of the minimiser (obviously), uniqueness, locality/globality, properties of A, choice of x_0

Definition 2.41. *A solution set is* $\emptyset \neq \Gamma \subseteq \Omega$

Let Γ *be a solution set,* A *an algorithm,* $Z : \Omega \to \mathbb{R}$ *which is continuous, we say that* Z *is a descent function for* Γ *and* A *if*

1. $x \notin \Gamma, y = A(x) \implies Z(y) < Z(x)$
2. $x \in \Gamma, y = A(x) \implies Z(y) \leq Z(x)$

Remark 2.42. *For* (1) *pick* $\Gamma = \{$*minimisers of* f *over* $\Omega\}$*, pick* A *s.t.* f *is a descent function for* Γ *and* A*.*

Theorem 2.43 (Global convergence)**.** *Let* A *be an algorithm,* $x_0 \in \Omega, \Gamma \subseteq \Omega$ *the solution set,* Z *a descent function, define iteratively*

$$x_{k+1} = A(x_k)$$

If we have

- *All points x_k are contained in compact $S \subseteq \Omega$*
- *A is continuous*

then the limit of any convergent subsequence of $\{x_k\}$ is a solution, i.e. is in Γ.

Proof. Let $\{x_{k_j}\}$ be a convergent subsequence of $\{x_k\}$ with $x_{k_j} \to \overline{x}$. We want to show $\overline{x} \in \Gamma$.
Since Z is continuous, $Z(x_{k_j}) \to Z(\overline{x})$ if $j \to \infty$.
We first prove a lemma

Lemma 2.44. $Z(x_k) \to Z(\overline{x})$ *(Note that this does not mean $x_k \to \overline{x}$)*

Proof. For all k_0, we can find a j_0 such that $k_{j_0} > k_0 \implies Z(x_{k_0}) \geq Z(x_{k_{j_0}})$.
Since $\overline{x}$ is the limit, we have $Z(x_{k_0}) \geq Z(\overline{x}),\ \forall k_0$.
This means, we have $Z(x_k) - Z(\overline{x}) \geq 0$ for all k.
Since we also have

$$Z(x_{k_j}) \to Z(\overline{x})$$

There is some $\epsilon > 0$, there is some j_0 s.t.

$$Z(x_{k_j}) - Z(\overline{x}) < \epsilon,\ \forall j \geq j_0$$

This shows for all $k \geq k_{j_0}$ we have

$$Z(x_k) - Z(\overline{x}) = Z(x_k) - Z(x_{k_{j_0}}) + Z(x_{k_{j_0}}) - Z(\overline{x}) < Z(x_{k_{j_0}}) - Z(\overline{x}) < \epsilon$$

.
Hence, $Z(x_k) \to Z(\overline{x})$ □

Now, we show $\overline{x} \in \Gamma$, assume $\overline{x} \notin \Gamma$.
Consider the sequence $\{x_{k_j+1}\}_j$ as a sequence in S. Thus, we can find another subsequence $\{x_{k_{j_l}+1}\}_l$ such that

$$x_{k_{j_l}+1} \to \overline{y}$$

We also have

$$x_{k_{j_l}} \to \overline{x}$$

Since $x_{k_{j_l}+1} = A(x_{k_{j_l}})$ for all l, we have

$$x_{k_{j_l}+1} = A(x_{k_{j_l}}) \to A(\overline{x}) = \overline{y}$$

By the lemma, $Z(x_k)$ both approach to $Z(\overline{x})$ and $Z(\overline{y})$. Hence, we must have $Z(\overline{x}) = Z(\overline{y})$. Since $\overline{x} \notin \Gamma$, and $A(\overline{x}) = \overline{y}$ where we have $Z(\overline{x}) = Z(\overline{y})$, this contradicts that Z is a descent algorithm. □

Remark 2.45. *Since S is compact, $\{x_k\}$ always has a convergent subsequence.*

Corollary 2.46. *Under the conditions of the Theorem 2.43, if $\Gamma = \{\overline{x}\}$ then $x_k \to \overline{x}$.*

Proof. Assume the sequence does not converge, there exists a subsequence $\{x_{k_j}\}$ such that

$$|x_{k_j} - \overline{x}| > \epsilon, \forall j$$

Since S is compact, there is a further subsequence $\{x_{k_{j_l}}\}$ such that $x_{k_{j_l}} \to \overline{y}$ where $|\overline{y} - \overline{x}| \geq \epsilon$.
But by Theorem 2.43, since $\overline{y}$ is a limit of some subsequence of $\{x_k\}$, we have $\overline{y} \in \Gamma$. Contradiction that $|\overline{y} - \overline{x}| \geq \epsilon$. □

2.2 Method of Steepest Descent

Definition 2.47. *Let $g : \mathbb{R} \to \mathbb{R}$ then $\overline{\alpha} = \operatorname*{argmin}_{\alpha \in \mathbb{R}} g(\alpha)$ if $g(\overline{\alpha}) \leq g(\alpha)$ for all $\alpha \in \mathbb{R}$.*

Setting: We are considering $f \in C^1(\mathbb{R}^n)$.
In this case, $\nabla f(x) \in \mathbb{R}^{1\times n}$ which will be a row vector.
The algorithm works as follows: denote $g(x) = \nabla f(x)^\top$, fix x_0 we compute

$$x_{k+1} = x_k - \alpha_k g(x_k) = x_k - \alpha_k \nabla f(x_k)^\top \tag{1}$$

where $\alpha_k \in [0, \infty)$

Remark 2.48. *The idea is that $\nabla f^\top$ points in the direction of steepest ascent. Hence, $-\nabla f^\top$ points in the direction of steepest descent.*

α_k is some learning rate where

- *Fixed*
- *Variable*
- *Optimal* $(\arg \min_{\alpha \in [0,\infty)} f(x_k - \alpha g(x_k)))$

In Global Convergence Theorem, let $\Gamma = \{x \in \mathbb{R}^n : \nabla f(x) = 0\}$ then $Z(x) = f(x)$ is the descent function for Γ and Algorithm (1) with $\alpha_k = \arg \min_{\alpha \in [0,\infty)} f(x_k - \alpha g(x_k))$

We have two different cases for x_k

1. *We have $x_k \in \Gamma \implies \nabla f(x_k) = 0 \implies x_{k+1} = x_k \implies f(x_{k+1}) = f(x_k)$*
2. *We have $x_k \notin \Gamma \implies \nabla f(x_k) \neq 0 \implies f(x_{k+1}) < f(x_k)$*

By Global Convergence Theorem, if the sequence $\{x_k\}$ is bounded, we get a subsequence whose limit is in Γ.

One possible issue is that not all points is in Γ might be minimisers (since the zero gradient is a necessary condition, not sufficient condition).

We now look at an example

Example 2.49 (Quadratic Case)**.** *Let $f : \Omega \to \mathbb{R}$ where $\Omega = \mathbb{R}^n$ and $f(x) = \frac{1}{2} x^\top Q x - x^\top b$ where $Q \in \mathbb{R}^{n\times n}$ is PD, $b \in \mathbb{R}^n$.*

We have f is strictly convex, as $\nabla^2 f = Q$ which is PD.

Proposition 2.50. *The minimiser of f on $\mathbb{R}^n$ is given by $\overline{x} = Q^{-1} b$*

Proof. $0 = \nabla f(x) = Qx - b \implies x = Q^{-1} b$

Moreover, f is convex hence by the First Order Sufficient Condition $\overline{x}$ is the global minimum of f on $\mathbb{R}^n$. □

Remark 2.51. *Using the Method of Steepest Descent, note that $g(x) = \nabla f(x)^\top = Qx - b$. We now want to find the optimal a_k.*

Since $x_{k+1} = x_k - \alpha_k g_k = x_k - \alpha_k(Qx_k - b)$ (by setting $g_k = Qx_k - b$). To determine the optimal α_k, we define $h(\alpha) = f(x_k - \alpha g_k) = \frac{1}{2}(x_k - \alpha g_k)^\top Q (x_k - \alpha g_k) - (x_k - \alpha g_k)^\top b$.

First order condition: *The optimal α should satisfy the first order condition*

$$\begin{aligned} & 0 = h'(\alpha) = -g_k^\top Q x_k + \alpha g_k^\top Q g_k + g_k^\top b \\ \implies & 0 = \alpha g_k^\top Q g_k - g_k^\top g_k \\ \implies & \alpha = \frac{g_k^\top g_k}{g_k^\top Q g_k} \end{aligned}$$

where $g_k^\top Q g_k > 0$ if $g_k \neq 0$ (since Q is PD). If $g_k = 0$ then $Qx_k - b = 0 \implies x_k$ is the minimiser already.

Second order condition: *We have*

$$h''(\alpha) = g_k^\top Q g_k > 0$$

hence α is the minimum.

In summary, the Method of Steepest Descent gives

$$x_{k+1} = x_k - \frac{g_k^\top g_k}{g_k^\top Q g_k} g_k$$

where $g_k = Qx_k - b$

2.3 Conjugate Direction Method

Setting: We are considering

$$f(x) = \frac{1}{2} x^\top Q x - b^\top x \tag{2}$$

where $b \in \mathbb{R}^n, Q \in \mathbb{R}^{n \times n}, \Omega = \mathbb{R}^n$

We already know the minimiser is when $x = Q^{-1}b$, but inverting matrices can be really costly. We will now find a method that solve this quadratic functions.

Remark 2.52. *The reason why this is useful is that around the minimiser we have SOSC, which implies $\nabla^2 f$ is PD. Hence, we can assume many functions to be approximately quadratic.*

Definition 2.53. *Given $Q \in \mathbb{R}^{n \times n}$ symmetric, we say that $d_1, d_2 \in \mathbb{R}^n$ are Q-orthogonal (Q-conjugate) if $d_1^\top Q d_2 = 0$.*

A set $\{d_0, \ldots, d_k\} \subseteq \mathbb{R}^n$ is Q-orthogonal if $d_i^\top Q d_j = 0, \forall i \neq j$

Proposition 2.54. *Let $Q \in \mathbb{R}^{n \times n}$ be PD, $\{d_0, \ldots, d_k\} \subseteq \mathbb{R}^n$ be Q-orthogonal where all $d_i \neq 0$. Then $\{d_0, \ldots, d_k\}$ are linearly independent.*

Proof. Assume there exists $\alpha_0, \ldots, \alpha_k$ such that $\alpha_0 d_0 + \ldots + \alpha_k d_k = 0$. We will show $\alpha_i = 0$ for all i.

Wwe have

$$\begin{aligned}
\alpha_0 d_0 + \ldots + \alpha_k d_k = 0 &\implies Q(\alpha_0 d_0 + \ldots + \alpha_k d_k = 0) = 0 \\
&\implies \alpha_0 Q d_0 + \ldots \alpha_k Q d_k = 0 \\
&\implies \alpha_0 d_i^\top Q d_0 + \ldots \alpha_k d_i^\top Q d_k = 0, \ \forall i \\
&\implies \alpha_i d_i^\top Q d_i = 0 \qquad [\{d_0, \ldots, d_k\} \text{ is } Q\text{-orthogonal}]
\end{aligned}$$

since $d_i \neq 0$ and Q is PD this shows $d_i^\top Q d_i > 0$ and hence $\alpha_i = 0$. Hence, $\{d_0, \ldots, d_k\}$ are linearly independent. □

Idea of CD Method: Let $\overline{x}$ be a minimiser to (2) (this means $\overline{x} = Q^{-1}b$).

Let $\{d_0, \ldots, d_{n-1}\}$ to be Q-orthogonal where $d_i \neq 0$ for all i. Then there exists $\alpha_0, \ldots, \alpha_{n-1}$ such that $\overline{x} = \alpha_0 d_0 + \ldots + \alpha_{n-1} d_{n-1}$

We have

$$\begin{aligned}
\overline{x} &= \alpha_0 d_0 + \ldots + \alpha_{n-1} d_{n-1} \\
\implies Q\overline{x} &= \alpha_0 Q d_0 + \ldots + \alpha_{n-1} Q d_{n-1} \\
\implies d_i^\top Q\overline{x} &= \alpha_0 d_i^\top Q d_0 + \ldots + \alpha_{n-1} d_i^\top Q d_{n-1} \\
\implies d_i^\top b &= \alpha_i d_i^\top Q d_i \\
\implies \alpha_i &= \frac{d_i^\top b}{d_i^\top Q d_i} \\
\implies \overline{x} &= \sum_{i=0}^{n-1} \frac{d_i^\top b}{d_i^\top Q d_i} d_i
\end{aligned}$$

We will built minimiser $\overline{x}$ iteratively with some starting point x_0 and adding $\alpha_i d_i$ for each step. This gives us the CD-Algorithm

CD-Algorithm: Let $\{d_0, \ldots, d_{n-1}\}$ be Q-orthogonal with $d_i \neq 0$ for all i. Fix x_0 define

$$x_{k+1} = x_k + \alpha_k d_k$$

where $\alpha_k = -\frac{g_k d_k^\top}{d_k^\top Q d_k}, g_k = Qx_k - b$

Theorem 2.55. *The CD-Algorithm converges to the unique minimum $\overline{x} = Q^{-1}b$ of (2) after n steps. I.e. $x_n = \overline{x}$.*

Proof. We have $\overline{x} - x_0 = \beta_0 d_0 + \ldots \beta_{n-1} d_{n-1}$ for some $\{\beta_i\}$ because they are linearly independent.

Hence, $Q(\overline{x} - x_0) = \beta_0 Q d_0 + \ldots \beta_{n-1} Q d_{n-1}$.

And $d_k^\top Q(\overline{x} - x_0) = \beta_0 d_k^\top Q d_0 + \ldots \beta_{n-1} d_k^\top Q d_{n-1} = \beta_k d_k^\top Q d_k$ for any k.

This gives us $\beta_k = \frac{d_k^\top Q(\overline{x} - x_0)}{d_k^\top Q d_k}$.

Now we can write $x_k - x_0 = \beta_0 d_0 + \ldots \beta_{k-1} d_{k-1}$. This implies $d_k^\top Q(x_k - x_0) = d_k^\top Q(\beta_0 d_0 + \ldots \beta_{k-1} d_{k-1}) = 0$.

Therefore, we have $\beta_k = \frac{d_k^\top Q(\overline{x} - x_k + x_k - x_0)}{d_k^\top Q d_k} = \frac{d_k^\top Q(\overline{x} - x_k)}{d_k^\top Q d_k} = \frac{d_k^\top (b - Qx_k)}{d_k^\top Q d_k} = -\frac{d_k^\top g_k}{d_k^\top Q d_k} = \alpha_k$.

This is true for all β_k which follows inductively. □

Definition 2.56. *$B_k \subseteq \mathbb{R}^n$ is defined by $B_k = span\{d_0, \ldots, d_{k-1}\}$.*

Moreover, $x_0 + B_k = \{x \in \mathbb{R}^n : \exists \alpha_0, \ldots, \alpha_{k-1}$ s.t. $x = x_0 + \alpha_0 d_0 + \ldots + \alpha_{k-1} d_{k-1}\}$

Theorem 2.57 (Expanding Subspace Theorem)**.** *Let $\{d_0, \ldots, d_{n-1}\}$ where $\forall i, d_i \neq 0$ be Q-orthogonal, define $x_{k+1} = x_k + \alpha_k d_k, \alpha_k = -\frac{g_k^\top d_k}{d_k^\top Q d_k}$ where $g_k = Qx_k - b$ as the CD-Algorithm.*

Then x_k minimises $f(x) = \frac{1}{2} x^\top Q x - x^\top b$ on the line $x = x_{k-1} + \alpha d_{k-1}$ for $\alpha \in \mathbb{R}$ and on the set $x_0 + B_k$.

Proof. Note that the line is in $x_0 + B_k$. Therefore, it is enough to prove x_k is minimum over $x_0 + B_k$.

Also note that f is strictly convex. Therefore, by FOSC, we only need to show that $g_k \cdot d = \nabla f(x_k) \cdot d = 0$ for all $d \in B_k$ to show x_k is the global minimum of f over $x_0 + B_k$ as $x_0 + B_k$ is convex.

We will prove this by induction

Base case: $B_0 = \emptyset \implies$ true for $k = 0$ that x_0 minimises on the set $\{x_0\}$.

Inductive step: Assume $g_k \cdot d = 0$ for all $d \in B_k$, we now show $g_{k+1} \cdot d = 0$ for all $d \in B_{k+1}$.

We have

$$g_{k+1} = Qx_{k+1} - b = Q(x_k + \alpha_k d_k) - b = Qx_k + \alpha_k Q d_k - b = g_k + \alpha_k Q d_k$$

For $d_k^\top g_{k+1}$, we have $d_k^\top g_{k+1} = d_k^\top g_k + \alpha_k d_k^\top Q d_k = 0$ by the definition of α_k.

For $d_i^\top g_{k+1}$ where $i < k$, we have $d_i^\top g_{k+1} = d_i^\top g_k + \alpha_k d_i^\top Q d_k = 0$ where the previous part is by Inductive Hypothesis, the latter part is by definition of Q-orthogonal.

By linearity, we have $g_{k+1} \cdot d = 0$ for all $d \in B_{k+1}$. □

Remark 2.58. *In CD-Algorithm, $B_k \subset B_{k+1}$ with $\mathbb{R}^n = x_0 + B_n$. Hence x_n is the minimum over $x_0 + B_n = \mathbb{R}^n$.*

Question 2.59. *Now, how to find $\{d_0, \ldots, d_{n-1}\}$*

This leads to the Conjugate Gradient Algorithm (CG-Algorithm).

CG-Algorithm: Fix x_0, define $d_0 = -g_0 = b - Qx_0$. Define inductively $x_{k+1} = x_k + \alpha_k d_k$ where $\alpha_k = \frac{g_k^\top d_k}{d_k^\top Q d_k}$ and $d_{k+1} = -g_{k+1} + \beta_k d_k$ where $\beta_k = \frac{g_{k+1}^\top Q d_k}{d_k^\top Q d_k}$. Note again that $g_k = Qx_k - b$ and if $g_k = 0$ then $x_{k+1} = x_k = \overline{x} = Q^{-1}b$ which is the global minimum of f over $\mathbb{R}^n$

Theorem 2.60 (Conjugate Gradient Theorem)**.** *The CG-Algorithm is a CD-Algorithm. If $g_k \neq 0$ then*

1. $span\{g_0, \ldots, g_k\} = span\{g_0, Qg_0, \ldots, Q^k g_0\}$
2. $span\{d_0, \ldots, d_k\} = span\{g_0, Qg_0, \ldots, Q^k g_0\}$
3. $d_k^\top Q d_i = 0, \forall i < k$
4. $\alpha_k = \frac{g_k^\top g_k}{d_k^\top Q d_k}$
5. $\beta_k = \frac{g_{k+1}^\top g_{k+1}}{g_k^\top g_k}$

Proof. We will prove $(1), (2), (3)$ simultaneously by induction over k.

Base case ($k = 0$): We have $span\{g_0\} = span\{g_0\}$, $span\{d_0\} = span\{-g_0\} = span\{g_0\}$ and $d_k^\top Q d_i = 0$ for all $i < 0$.

Inductive step: Assume $(1), (2), (3)$ are true for k. We now show this is true for $k + 1$.

(1) We have

$$g_{k+1} = Qx_{k+1} - b = Qx_k + \alpha_k Q d_k - b = g_k + \alpha_k Q d_k$$

By Inductive Hypothesis, we have $g_k, d_k \in span\{g_0, \ldots, Q^k g_0\}$. This gives $Qd_k \in span\{g_0, \ldots, Q^{k+1} g_0\}$. Hence, $g_{k+1} \in span\{g_0, \ldots, Q^{k+1} g_0\}$.

Moreover, by the Expanding Subspace Theorem, $g_{k+1} = \nabla f(x_{k+1})$ is orthogonal to B_{k+1}, which is all feasible direction in order to become a minimiser, unless $g_{k+1} = 0$ but this is excluded by $g_k \neq 0$. Note that by Inductive Hypothesis, $B_{k+1} = span\{g_0, \ldots, Q^k g_0\}$ Hence, we know that $g_{k+1} \notin span\{g_0, \ldots, Q^k g_0\} = span\{d_0, \ldots, d_k\}$,

Hence, $span\{g_0, \ldots, g_{k+1}\} = span\{g_0, \ldots, Q^{k+1} g_0\}$

(2) We have $d_{k+1} = g_{k+1} + \beta_k d_k \in span\{g_0, Qg_0, \ldots, Q^{k+1} g_0\}$. This is true as g_{k+1} is in the span by (1), and d_k is in the span by Induction Hypothesis.

This gives $span\{d_0, \ldots, d_{k+1}\} \subseteq span\{g_0, \ldots, Q^{k+1} g_0\}$.

We also have $g_{k+1} = \beta_k d_k - d_{k+1} \implies g_{k+1} \in span\{d_k, d_{k+1}\} \implies span\{d_0, \ldots, d_{k+1}\} \supseteq span\{g_0, \ldots, Q^{k+1} g_0\}$.

(3) We have for any $i < k + 1$

$$d_{k+1}^\top Q d_i = -g_{k+1}^\top Q d_i + \beta_k d_k^\top Q d_i$$

If $i = k$ then $-g_{k+1}^\top Q d_i + \beta_k d_k^\top Q d_i = 0$ by the definition of β_k.

If $i < k$ then $Qd_i \in Q span\{g_0, \ldots, Q^i g_0\} = span\{Qg_0, \ldots, Q^{i+1} g_0\} = span\{d_1, \ldots, d_{i+1}\}$.

By Expanding Subspace Theorem, g_{k+1} is orthogonal to $B_{k+1} \supseteq span\{d_1, \ldots, d_{i+1}\}$.

Hence, $g_{k+1} Q d_i = 0$. We also have $d_k^\top Q d_i = 0$ by the Induction Hypothesis. Hence $d_{k+1}^\top Q d_i = 0$ for all $i < k$.

(4) We have $-g_k^\top d_k = g_k^\top g_k - \beta_{k-1} g_k^\top d_{k-1}$ by definition of d_k. By Expanding Subspace Theorem, $-g_k^\top d_k = g_k^\top g_k - \beta_{k-1} g_k^\top d_{k-1} = g_k^\top g_k$.

Hence, $\alpha = \frac{g_k^\top g_k}{d_k^\top Q d_k}$.

(5) We have by (2) $g_k \in span\{d_0, \dots, d_k\}$ where the latter is orthogonal to g_{k+1} by Expanding Subspace Theorem. Hence $g_k^\top g_{k+1} = 0$.

We have $Qd_k = \frac{1}{\alpha_k}(g_{k+1} - g_k)$ as $g_{k+1} = Qx_{k+1} - b = Qx_k + \alpha_k Qd_k - b = g_k + \alpha_k Qd_k$.

Therefore, we have $g_{k+1}^\top Qd_k = \frac{1}{\alpha_k} g_{k+1}^\top g_{k+1}$.

Hence, $\beta_k = \frac{g_{k+1}^\top Qd_k}{d_k^\top Qd_k} = \frac{g_{k+1}^\top g_{k+1}}{\alpha_k d_k^\top Qd_k} = \frac{g_{k+1}^\top g_{k+1}}{g_k^\top g_k}$ by (4). □

Remark 2.61. *(3) verifies that the CG-Algorithm is CD-Algorithm.*

(4), (5) might be easier to compute than the original formulas for α_k, β_k.

3 Constrained Optimisation

In Chapter 1, constraints are given in a general form $x \in \Omega \subseteq \mathbb{R}^n$. Now, we want more specific constraints, where Ω is given through equality/inequality constraints.

The reason having this kind of problem is that feasible direction are often hard to find. A special form of Ω in this chapter allows for a more constructive and convenient theory and computations.

Setting: We are now looking for

$$
\begin{aligned}
&\min f(x) \\
\text{s.t. } &h_1(x) = 0, g_1(x) \geq 0 \\
&\vdots, \quad \vdots \\
&h_m(x) = 0, g_p(x) \geq 0 \\
&x \in \Omega \subseteq \mathbb{R}^n
\end{aligned}
$$

We typically look at $x \in \mathbb{R}^n$ instead of $x \in \Omega \subseteq \mathbb{R}^n$.

We also assume

- $m \leq n$, that is less constraints than the variables
- $f, h_i, g_j : \mathbb{R}^n \to \mathbb{R}$, and are usually assumed C^1
- Vector valued functions $h : \mathbb{R}^n \to \mathbb{R}^m$ and $g : \mathbb{R}^n \to \mathbb{R}^p$ by

$$
h(x) = \begin{pmatrix} h_1(x) \\ h_2(x) \\ \vdots \\ h_m(x) \end{pmatrix} \qquad g(x) = \begin{pmatrix} g_1(x) \\ g_2(x) \\ \vdots \\ g_p(x) \end{pmatrix}
$$

We often write

$$
\min_{\text{s.t. } h(x)=0, g(x)\geq 0, x\in\Omega} f(x)
$$

Definition 3.1. *Constraints $h(x) = 0, g(x) \geq 0$ are called functional constraints and constraint $x \in \Omega$ is called set constraint.*

$x \in \Omega$ satisfying functional constraint is called feasible.

Remark 3.2. *Here we will de-emphasize the set constraint, usually $\Omega = \mathbb{R}^n$ or $x \in int(\Omega)$. Here we study*

$$
\min_{s.t.\ h(x)=0, g(x)\geq 0} f(x)
$$

Definition 3.3. *Inequality constraint $g_i(x) \geq 0$ is*

- *Active at feasible x if $g_i(x) = 0$*
- *Inactive at feasible x if $g_i(x) > 0$*

We call every inequality constraint $h_i(x) = 0$ active at feasible x.

Remark 3.4. *The idea for setting this definition is to study equality constraints. At a feasible point, inequality constraints only matter (locally) if they are active, which become equality constraint.*

3.1 Equality Constraints

Setting:

$$\min_{\text{s.t. } h(x)=0} f(x) \tag{3}$$

where $f : \mathbb{R}^n \to \mathbb{R}, h : \mathbb{R}^n \to \mathbb{R}^m$ where $n \geq m$. Moreover, f, h are C^1.

Remark 3.5. *$\{x \in \mathbb{R}^n : h(x) = 0\}$ is a hypersurface of dimension $n - m$ in $\mathbb{R}^n$.*

Definition 3.6. *Let $S \subseteq \mathbb{R}^n$ be a hypersurface. A curve on S is a function $\alpha \in C([a, b], S)$ for some $[a, b] \in \mathbb{R}$.*

A curve is differentiable if $\dot{\alpha}(t) = \frac{d}{dt}\alpha(t)$ exists $\forall t \in [a, b]$ and is twice differentiable if $\ddot{\alpha}(t) = \frac{d^2}{dt^2}\alpha(t)$ exists $\forall t \in [a, b]$

Let $x_0 \in S$, the tangent plane/space at x_0 is

$$\{\dot{\alpha}(t_0) : \alpha \in C([a, b], S), \exists t_0 \in [a, b] \text{ s.t. } \alpha(t_0) = x_0 \text{ and s.t. } \dot{\alpha}(t_0) \text{ exists}\} \subseteq \mathbb{R}^n$$

Remark 3.7. *The previous definition of tangent plane is not useful for computations/applications.*

Definition 3.8. *A point $x_0 \in \mathbb{R}^n$ satisfying $h(x_0) = 0$ is said to be a regular point of the constraints if $\nabla h_1(x_0), \nabla h_2(x_0), \ldots, \nabla h_m(x_0)$ are linearly independent.*

Theorem 3.9. *Let $x_0 \in \mathbb{R}^n$ be a regular point of $S = \{x \in \mathbb{R}^n : h(x) = 0\}$. Then the tangent plane at x_0 is equal to*

$$M = \{d \in \mathbb{R}^n : \nabla h(x_0) \cdot d = 0\} = \{d \in \mathbb{R}^n : \nabla h_1(x_0) \cdot d = 0, \ldots, \nabla h_m(x_0) \cdot d = 0\}$$

Proof. See the book □

Lemma 3.10. *Let $x_0 \in \mathbb{R}^n$ be a regular point of the constraint $h(x) = 0$ and a local minimum (or a local maximum) of f subject to the constraints.*

Then we have $\forall d \in \mathbb{R}^n$ that $\nabla h(x_0) \cdot d = 0 \implies \nabla f(x_0) \cdot d = 0$

Proof. Let $d \in \mathbb{R}^n$ such that $\nabla d(x_0) \cdot d = 0$ by Theorem 3.9 d is in the tangent plane at x_0.

This means there exists a curve $\alpha : [-a, a] \to \mathbb{R}^n, (a > 0)$ such that $\alpha(0) = x_0, \dot{\alpha}(0) = d, h(\alpha(t)) = 0$ for $t \in [-a, a]$.

Because x_0 is a local minimum, we have

$$0 = \frac{d}{dt} f(\alpha(0)) = \nabla f(\alpha(0)) \cdot \dot{\alpha}(0) = \nabla f(x_0) \cdot d$$

□

Remark 3.11. *The above lemma uses (again) the ideas of 1-dimensional optimisation through the use of a curve.*

d is equivalent to feasible direction in Chapter 1.

Lemma states that $\nabla f(x_0)$ is orthogonal to the tangent plane.

Theorem 3.9 states that $\nabla h(x_0)$ is orthogonal to the tangent plane.

Theorem 3.12 (Farkas Lemma)**.** *Let $A \in \mathbb{R}^{m\times n}, b \in \mathbb{R}^m$, then either the linear system $Ax = b$ admits a solution $x \in \mathbb{R}^n$ or the linear system $A^\top y = 0$ admits a solution $y \in \mathbb{R}^m$ such that $b^\top y \neq 0$.*

Theorem 3.13 (Lagrange multiplier)**.** *Let $x_0 \in \mathbb{R}^n$ be a local minimiser of f subject to constraints $h(x) = 0$ and assume that x_0 is a regular point with respect to the constraint $h(x) = 0$. Then there exists $\lambda \in \mathbb{R}^m$ such that $\nabla f(x_0) - \lambda^\top \nabla h(x_0) = 0$.*

Proof. From Lemma 3.10 we conclude that the linear system

$$\begin{cases} \nabla h(x_0) \cdot d = 0 \\ \nabla f(x_0) \cdot d \neq 0 \end{cases}$$

has no solution d.

Using the Farkas Lemma with $y = d, b^\top = \nabla f(x_0), A^\top = \nabla h(x_0), x = \lambda$ we get $Ax = b$ admits the solution λ to

$$\nabla h(x_0)^\top \lambda = \nabla f(x_0)^\top \implies \lambda^\top \nabla h(x_0) = \nabla f(x_0)$$

$\square$

Remark 3.14.

- *The conclusion of the Lagrange Multiplier can be written as*

$$\nabla f(x_0) = \lambda_1 \nabla h_1(x_0) + \lambda_2 \nabla h_2(x_0) + \ldots + \lambda_m \nabla h_m(x_0)$$

  That is, $\nabla f(x_0)$ is a linear combination of $\{\nabla h_1(x_0), \ldots, \nabla h_m(x_0)\}$

- *The Lagrant Multiplier is the FONC for Equation (3)*

- *Practically: To find candidates for minimisers we solve*

$$\begin{cases} \nabla f(x) - \lambda^\top \nabla h(x) = 0 \\ h(x) = 0 \end{cases}$$

  The first has n equations, the second has m equations, where x has n variables, λ has m variables.

  In total we solve $n + m$ equations (often non-linear) in $n + m$ variables.

Example 3.15 (Maximum Volume)**.** *Given a fixed area of cardboard $c > 0$. We want to construct a box of maximum volume.*

The variables are the length of the sides x, y, z.

We want to solve

$$\max_{s.t.\ h(x,y,z)=2xy+2yz+2xz-c=0} f(x, y, z) = xyz$$

We first have

$$\nabla h(x, y, z) = (y + z, x + z, x + y)$$

Assume $0 = \nabla h(x, y, z)$. This gives $x = y = z = 0$. But $h(0, 0, 0) = -c < 0$. Hence, $(0, 0, 0)$ is not feasible.

We now want to solve $\nabla f(x, y, z) - \lambda \nabla h(x, y, z) = 0$ under the condition that $h(x, y, z) = 0$.

We get

$$\begin{cases} (yz, xz, xy) - 2\lambda(y + z, x + z, x + y) = (0, 0, 0) \\ 2xy + 2yz + 2xz - c = 0 \end{cases}$$

Case 1: *If $\lambda = 0$ then $(xy = xz = yz = 0 \implies 2xz + 2yz + 2xz - c < 0$. Hence, there is no feasible solution.*

Case 2: *If $\lambda \neq 0$. If $x = 0$, then $0z - 2\lambda(0 + z) = 0 \implies z = 0$. Also similar $y = 0$. Hence, $x \neq 0$. Similarly, $y \neq 0, z \neq 0$.*

We now solve the equation, we get

$$\begin{cases} yz - 2\lambda(y+z) = 0 \\ xz - 2\lambda(x+z) = 0 \end{cases} \implies -2\lambda(xy + zy) + 2\lambda(xy + zx) = 0 \implies z(y - x) = 0 \implies y = x$$

Similarly, we get $x = y = z$, where $0 = h(x, x, x) = 6x^2 - c \implies (x, y, z) = \left(\sqrt{\frac{c}{6}}, \sqrt{\frac{c}{6}}, \sqrt{\frac{c}{6}}\right)$ is a candidate for the maximiser.

Theorem 3.16. *Let $f \in C^2(\mathbb{R}^n), h \in C^2(\mathbb{R}^n, \mathbb{R}^m)$, suppose x_0 is a local minimum of f s.t. $h(x) = 0$, suppose x_0 is a regular point of the constraints.*

Denote the tangent plane at x_0 is

$$M = \{d \in \mathbb{R}^n : \nabla h(x_0) \cdot d = 0\}$$

Then

1. *There exists $\lambda \in \mathbb{R}^m$ s.t. $\nabla f(x_0) - \lambda^\top \nabla h(x_0) = 0$*
2. *The matrix $L(x_0) = \nabla^2 f(x_0) - \lambda^\top \nabla^2 h(x_0)$ is PSD on M. That is $d^\top L(x_0) d \geq 0, \forall d \in M$*

Proof. (1) is proved.

For (2), let $d \in M$, define $a > 0$, $\alpha \in C^1([-a, a], \mathbb{R}^n)$ s.t.

- $\alpha(0) = x_0$
- $\dot\alpha(0) = d$
- $h(\alpha(t)) = 0, \forall t \in [-a, a]$

Then, $\frac{d^2}{dt^2} f(\alpha(0)) \geq 0$ as x_0 is the local minimum.

And by chain rule

$$\frac{d^2}{dt^2} f(\alpha(0)) = \dot\alpha(0)^\top \nabla^2 f(\alpha(0)) \dot\alpha(0) + \nabla f(\alpha(0)) \ddot\alpha(0) \geq 0$$

We also have

$$\frac{d^2}{dt^2} \lambda^\top h(\alpha(0)) = \dot\alpha(0)^\top \lambda^\top \nabla^2 h(\alpha(0)) \dot\alpha(0) + \lambda^\top \nabla h(\alpha(0)) \dot\alpha(0) = 0$$

since $h(\alpha(t)) = 0, \forall t \in [-a, a]$.

We get by subtracting $\frac{d^2}{dt^2} \lambda^\top h(\alpha(0))$ from $\frac{d^2}{dt^2} f(\alpha(0))$ that

$$\begin{aligned} 0 &\leq d^\top \nabla^2 f(x_0) d + \nabla f(x_0) \ddot\alpha(0) - d^\top \lambda^\top \nabla^2 h(x_0) d - \lambda^\top \nabla h(x_0) \ddot\alpha(0) \\ &= d^\top (\nabla^2 f(x_0) - \lambda^\top \nabla^2 h(x_0))) \cdot d + (\nabla f(x_0) - \lambda^\top \nabla h(x_0)) \ddot\alpha(0) \\ &= d^\top L(x_0) d \qquad \text{[By FONC]} \end{aligned}$$

Hence $L(x_0)$ is PSD on M. $\square$

Remark 3.17. *This is the SONC for Equation (3)*

Theorem 3.18. *Let $f \in C^2(\mathbb{R}^n), h \in C^2(\mathbb{R}^n, \mathbb{R}^m), x_0$ a regular point for constraints such that $h(x_0) = 0$, let $\lambda \in \mathbb{R}^m$ fulfill*

$$\nabla f(x_0) - \lambda^\top \nabla h(x_0) = 0$$

and assume the matrix

$$L(x_0)) = \nabla^2 f(x_0) - \lambda \nabla^2 h(x_0)$$

is PD on $M = \{d \in \mathbb{R}^n : \nabla h(x_0) \cdot d = 0\}$.

Then x_0 is a strict local minimum of f s.t. $h(x) = 0$.

Remark 3.19. *This is the SOSC for Equation (3)*

Example 3.20. *We want to find*

$$\min_{h(x_1,x_2)=x_1^2+x_2^2-1=0} f(x_1, x_2) = -(x_1 - 1)^2 - (x_2 - 1)^2$$

First we have $\nabla h(x_1, x_2) = (2x_1, 2x_2)$.
This gives $\nabla h(x_1, x_2) \neq 0 \iff (x_1, x_2) \neq (0, 0)$.
For non-regular point $(0, 0)$ we have $h(0, 0) = -1 \neq 0 \implies (0, 0)$ is not feasible.
Hence, all feasible points are regular.
By FONC

$$\begin{aligned}
&\nabla f(x_1, x_2) - \lambda \nabla h(x_1, x_2) = 0 \\
\implies & -2(x_1 - 1, x_2 - 1) - \lambda(2x_1, 2x_2) = (0, 0) \\
\implies & \begin{cases} x_1 - 1 + \lambda x_1 = 0 \\ x_2 - 1 + \lambda x_2 = 0 \end{cases} \\
\implies & \begin{cases} 1 = (1 + \lambda)x_1 \\ 1 = (1 + \lambda)x_2 \end{cases}
\end{aligned}$$

Case 1: *If $\lambda = -1$, the FONC reads $(-1, -1) = (0, 0)$, hence there is no solution.*
Case 2: *If $\lambda \neq -1$, then $x_1 = x_2 = \frac{1}{1+\lambda}$.*
Take $0 = h(x_1, x_2) = \frac{2}{(1+\lambda)^2} - 1 \implies \lambda^2 + 2\lambda + 1 = 2 \implies \lambda = \frac{-2\pm\sqrt{8}}{2} = -1 \pm \sqrt{2}$.
Hence, $x_1 = x_2 = \pm\frac{1}{\sqrt{2}}$.
SONC, SOSC: $L(x_1, x_2) = \nabla^2 f(x_1, x_2) - \lambda \nabla h(x_1, x_2) = -2I - 2\lambda I = -2(1 + \lambda)I$
Hence $L(x_1, x_2) = \pm 2\sqrt{2}I$.
If $x_1 = x_2 = -\frac{1}{\sqrt{2}} \implies L(x_1, x_2) = 2\sqrt{2}I$ which is positive definite.
If $x_1 = x_2 = \frac{1}{\sqrt{2}} \implies L(x_1, x_2) = -2\sqrt{2}$ which is negative definite.
Hence by SOSC, $x_1 = x_2 = -\frac{1}{\sqrt{2}}$ is strict local minimum of f under constraint h.
Also by SOSC, $x_1 = x_2 = \frac{1}{\sqrt{2}}$ is strict local maximum of f under constraint h.

3.2 Inequality Constraints

Setting:

$$\min_{h(x)=0, g(x)\geq 0} f(x) \tag{4}$$

assume $g : \mathbb{R}^n \to \mathbb{R}^p$ with $g \in C^1(\mathbb{R}^n, \mathbb{R}^p)$.

Definition 3.21. *Let $x_0 \in \mathbb{R}^n$ s.t. $h(x_0) = 0, g(x_0) \geq 0$ and $J \subseteq \{1, \ldots, p\}$ s.t. $g_j(x_0) = 0, \forall j \in J$ and $g_j(x_0) > 0, \forall j \notin J$ (J is the set of indices of active constraints g_j).*

Then x_0 is called a regular point of the constraints $h(x) = 0, g(x) \geq 0$ if the vectors $\nabla h_i(x_0), \nabla g_j(x_0), \forall 1 \leq i \leq j, j \in J$ are linearly independent.

Remark 3.22. *This definition follows the strategy of treating active inequality constraints similar to equality constraints.*

Theorem 3.23 (Karush-Kuhn-Tucker (KKT) Conditions)**.** *Let $x_0 \in \mathbb{R}^n$ be a local minimum of Equation (4) and assume that x_0 is a regular point for the constraints. Then there exists $\lambda \in \mathbb{R}^m, \mu \in \mathbb{R}^p$ with $\mu \geq 0$ such that*

$$\nabla f(x_0) - \lambda^\top \nabla h(x_0) - \mu^\top \nabla g(x_0) = 0$$

and

$$\mu^\top g(x_0) = 0$$

Remark 3.24. *Because $\mu \geq 0, g(x_0) \geq 0$, the condition $\mu^\top g(x_0) = \sum_{j=1}^p \mu_j g_j(x_0) = 0$ implies that*

$$\begin{aligned} \mu_j > 0 &\implies g_j(x_0) = 0 \\ g_j(x_0) > 0 &\implies \mu_j = 0 \end{aligned}$$

and this is called complimentary slackness condition

The KKT conditions are the FONC for Equation (4).

Proof. Let x_0 is a local minimum over the constraint set. Therefore, it is also a local minimum over $S = \{x \in \mathbb{R}^n : h(x) = 0, g_j(x) = 0, j \in J\}$

Therefore, in a neighbourhood of x_0 we get this inequality constraint is just an equality constraint.

Therefore, we can find Lagrange multipliers $\lambda_1, \ldots, \lambda_m, \mu_j$ for all $j \in J$. We can also define $\mu_j = 0$ if $j \neq J$ $(g_j(x_0) > 0)$.

Therefore

$$\begin{aligned} \nabla f(x_0) - \lambda^\top \nabla h(x_0) - \sum_{j \in J} \mu_j \nabla g_j(x_0) - \sum_{j \notin J} \mu_j \nabla g_j(x_0) &= 0 - \sum_{j \notin J} \mu_j \nabla g_j(x_0) \qquad \text{[By FONC for equality constraint]} \\ &= 0 \qquad [\text{By } \mu_j = 0, \forall j \notin J] \end{aligned}$$

We also have

$$\begin{aligned} \mu^\top g(x_0) &= \sum_{j \in J} \mu_j g_j(x_0) + \sum_{j \notin J} \mu_j g_j(x_0) \\ &= 0 + 0 \\ &= 0 \end{aligned}$$

Lastly, we need to show that $\mu_j \geq 0, \forall j$. We have $\mu_j = 0$, if $j \notin J$. Suppose now there exists $k \in J, \mu_k < 0$.

Define the hypersurface $S' = \{x \in \mathbb{R}^n : h(x) = 0, g_j(x) = 0, \forall j \in J \backslash \{k\}\}$, also M' to be the tangent plane to S' at x_0.

Since x_0 is regular, we have the vectors $\nabla h_i(x_0)$ for all i, $\nabla g_j(x_0)$ for all $j \in J$ are linearly independent.

Therefore, there exists $d \in M'$ such that $\nabla g_k(x_0) \cdot d > 0$. If not, we have $\forall d \in M', \nabla g_k(x_0) \cdot d = 0$.

Take $(M')^\top = \{v \in \mathbb{R}^n : v \cdot d = 0, \forall d \in M'\} = span\{\nabla h_i(x_0), \nabla g_j(x_0) : i = 1, \ldots, m, j \in J \backslash \{k\}\}$. We have $\nabla g_j(x_0) \in (M')^\top$, contradiction to the definition of linearly independent.

If we compute

$$\begin{aligned} 0 &= (\nabla f(x_0) - \lambda^\top \nabla h(x_0) - \mu^\top \nabla g(x_0)) \cdot d \\ &= \nabla f(x_0) \cdot d - \lambda^\top \nabla h(x_0)) \cdot d - \mu^\top \nabla g(x_0) \cdot d \\ &= \nabla f(x_0) \cdot d - 0 - \mu^\top \nabla g(x_0) \cdot d \qquad [d \in M' \implies \nabla h(x_0) \cdot d = 0] \\ &= \nabla f(x_0) \cdot d - \mu_k \nabla g_k(x_0) \cdot d \qquad [j \notin J \implies \mu_j = 0, j \in J \backslash \{k\} \implies \nabla g_j(x_0) \cdot d = 0] \\ &\implies \nabla f(x_0) \cdot d = \mu_k \nabla g_k(x_0) \cdot d < 0 \qquad [\text{By assumption that } \mu_k < 0] \end{aligned}$$

This shows d is a descent direction for f (i.e. there exists $\bar{t} > 0$ s.t. $f(x_0 + td) < f(x_0)$ for all $0 \le t \le \bar{t}$) in the unconstrainted case.

We can now take $\alpha \in C^1([-a,a], S')$ s.t. $\alpha(0) = x_0, \dot{\alpha}(0) = d$. For small t, $\alpha(t)$ is feasible and $g_k(\alpha(t)) > 0$ but then $\frac{d}{dt} f(\alpha(0)) = \nabla f(\alpha(0)) \cdot \dot{\alpha}(0) = \nabla f(x_0) \cdot d < 0$, this contradicts to that x_0 is a local minimal. □

Theorem 3.25. *Let $x_0 \in \mathbb{R}^n$ be a regular point of the constraints $h(x) = 0, g(x) \ge 0$. Let x_0 be a local minimum of Equation (4).*

Assume that $f \in C^2(\mathbb{R}^n), g \in C^2(\mathbb{R}^n, \mathbb{R}^p), h \in C^2(\mathbb{R}^n, \mathbb{R}^m)$. Then there exists $\lambda \in \mathbb{R}^m, \mu \in \mathbb{R}^p_{\ge 0}$ s.t.

$$\begin{aligned} \nabla f(x_0) - \lambda^\top \nabla h(x_0) - \mu^\top \nabla g(x_0) &= 0 \\ \mu^\top g(x_0) &= 0 \end{aligned}$$

hold and such that

$$L(x_0) = \nabla^2 f(x_0) - \lambda^\top \nabla^2 h(x_0) - \mu^\top \nabla^2 g(x_0)$$

is PSD on the tangent space of the active constraints at x_0

Remark 3.26. *This is the SONC for (4).*

Proof. If x_0 is a local minimum subject to the constraints, it is also a local minimum for the problem with the active constraints taken as equality constraints.

By the SONC for equality constraints implies the statement. □

Theorem 3.27. *Let $f \in C^2(\mathbb{R}^n), g \in C^2(\mathbb{R}^n, \mathbb{R}^p), h \in C^2(\mathbb{R}^n, \mathbb{R}^m), x_0 \in \mathbb{R}^n$ be a regular point. If there exists $\lambda \in \mathbb{R}^m, \mu \in \mathbb{R}^p_{\ge 0}$ s.t.*

$$\begin{aligned} \nabla f(x_0) - \lambda^\top \nabla h(x_0) - \mu^\top \nabla g(x_0) &= 0 \\ \mu^\top g(x_0) &= 0 \end{aligned}$$

and such that

$$L(x_0) = \nabla^2 f(x_0) - \lambda^\top \nabla^2 h(x_0) - \mu^\top \nabla^2 g(x_0)$$

is PD on $M = \{d \in \mathbb{R}^n : \nabla h(x_0) \cdot d = 0, \nabla g_j(x_0) \cdot d = 0, \forall j \in J\}$ where $J = \{j \in \{1, \ldots, p\} : g_j(x_0) = 0 \wedge \mu_j > 0\}$

Then x_0 is a strict local minimum of f subject to the constraints.

Remark 3.28. *This is the SOSC for (4).*

Proof. Assume x_0 is not a strict local minimum, let $\{y_k\}_k$ be a sequence such that

- y_k is feasible for all k.
- $y_k \to x_0$ as $k \to \infty$
- $f(y_k) \le f(x_0)$
- $y_k = x_0 + \delta_k s_k$ where $\delta_k > 0$ and $\delta_k \to 0$ and $|s_k| = 1$ and $s_k \to \bar{s}$. This is possible as $|s_k| = 1 \implies \{s_k\}$ is bounded, hence it has a convergent subsequence.

We have $h(y_k) - h(x_0) = 0$. Hence, $\frac{h(y_k) - h(x_0)}{\delta_k} = 0$. As $k \to \infty, \nabla h(x_0) \cdot \bar{s} = 0$.

We also have $f(y_k) - f(x_0) \le 0 \implies \frac{f(y_k) - f(x_0)}{\delta_k} \le 0$. As $k \to \infty, \nabla f(x_0) \cdot \bar{s} \le 0$.

We also have for j active $g_j(y_k) - g_j(x_0) \ge 0 \implies \nabla g_j(x_0) \cdot \bar{s} \ge 0$.

Now, we have two cases

Case 1: If $\nabla g_j(x_0) \cdot \bar{s} = 0$ for all $j \in J$. By second order MVT, we can write $0 = h_i(y_k) = h_i(x_0) + \delta_k \nabla h_i(x_0) s_k + \frac{\delta_k^2}{2} s_k^\top \nabla^2 h_i(\eta_i) s_k$ for $i = 1, \ldots, m, \eta_i \in [x_0, y_k]$.

Similarly, $0 \leq g_j(y_k) = g_j(x_0) + \delta_k \nabla g_j(x_0) s_k + \frac{\delta_k^2}{2} s_k^\top \nabla^2 g_j(\overline{\eta_j}) s_k$ where $\overline{\eta_j} \in [x_0, y_k]$, for all $j = 1, \dots, p$.

Similarly, $0 \geq f(y_k) - f(x_0) = \delta_k \nabla f(x_0) s_k + \frac{\delta_k^2}{2} s_k^\top \nabla^2 f(\eta_0) s_k$.

Multiply the first, the second by $-\lambda_i, -\mu_j$ and sum over i, j

$$0 \geq \delta_k(\nabla f(x_0) - \sum_{i=1}^m \lambda_i \nabla h_i(x_0) - \sum_{i=1}^p \mu_j \nabla g_j(x_0)) s_k + \frac{\delta_k^2}{2} s_k (\nabla^2 f(\eta_0) - \sum_{i=1}^m \lambda_i \nabla^2 h(\eta_i) - \sum_{j=1}^p \mu_j \nabla^2 g_j(\overline{\eta_j}) s_k$$

divide the equation by δ_k^2, multiply with 2 and $k \to \infty$ and we get

$$\begin{aligned} & 0 \geq \overline{s}^\top (\nabla^2 f(x_0) - \lambda^\top \nabla^2 h(x_0) - \mu^\top \nabla^2 g(x_0)) \overline{s} \\ \implies & 0 \geq \overline{s} L(x_0) \overline{s} \end{aligned}$$

recall that $\overline{s} \in M \implies$ contradiction to $L(x_0)$ being PD on M.

Case 2: If there exists $j \in J$ such that $\nabla g_j(x_0) \cdot \overline{s} > 0$ then

$$\begin{aligned} 0 \geq \nabla f(x_0) \cdot \overline{s} &= (\lambda^\top \nabla h(x_0) + \mu^\top \nabla g(x_0)) \cdot \overline{s} \\ &= \lambda^\top \nabla h(x_0) \cdot \overline{s} + \mu^\top \nabla g(x_0) \cdot \overline{s} \end{aligned}$$

We know that $\lambda^\top \nabla h(x_0) \cdot \overline{s} = 0$. We also have

$$\begin{aligned} \mu^\top \nabla g(x_0) \cdot \overline{s} &= \sum_{j=1}^p \mu_j \nabla g_j(x_0) \cdot \overline{s} \\ &= \sum_{j \text{ inactive}} \mu_j \nabla g_j(x_0) \overline{s} + \sum_{j \in J} \mu_j \nabla g_j(x_0) \overline{s} + \sum_{j \text{ active, } j \notin J} \mu_j \nabla g_j(x_0) \overline{s} \\ &= \sum_{j \in J} \mu_j \nabla g_j(x_0) \overline{s} > 0 \end{aligned}$$

This is a contradiction. □

Example 3.29. *Consider we want to solve*

$$\min_{s.t.\ g(x,y)=1-x^2-y^2 \geq 0} f(x,y) = -(x-1)^2 - (y-1)^2$$

We have that

$$\nabla g(x,y) = (-2x, -2y)$$

and this is only equal to 0 *at* $(0,0)$, *where* $g(0,0) = 1 \geq 0$, *which is feasible. And we have* $g(0,0) = -2$

Using the KKT conditions, we have

$$\begin{aligned} & -2(x-1, y-1) - \mu(-2x, -2y) = (0,0) \\ \implies & \mu(1 - x^2 - y^2) \geq 0 \\ \implies & \mu \geq 0 \end{aligned}$$

So

$$\begin{aligned} 1 &= (1-\mu)x \\ 1 &= (1-\mu)y \end{aligned}$$

Case 1: *If* $\mu = 0$. *Then* $x = y = 1$, *but* $g(1,1) = -1 < 0$, *so* $(-1,-1)$ *is not feasible.*

Case 2: *If $\mu = 1$. Then the solution is not defined.*
Case 3: *If $\mu \in (0,1) \cap (1, \infty)$, then $x = y = \frac{1}{1-\mu}$ and g must be active.*
So, $0 = 1 - \frac{2}{(1-\mu)^2} \implies \mu = 1 \pm \sqrt{2}$. Since $\mu \geq 0$, we must have $\mu = 1 + \sqrt{2}$.
This gives us one candidate $x = y = -\frac{1}{\sqrt{2}}$.
Now, we apply SONC/SOSC, we have

$$L(x_0) = -2I + 2\mu I = 2(\mu - 1)I$$

which is positive definite for $\mu = 1 + \sqrt{2}$.
Moreover, $g(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}) = -3 - 2\sqrt{2} < g(0,0)$
So, $x = y = -\frac{1}{\sqrt{2}}$ is the strict local minimiser.

3.3 Lagrangian, convexity and special cases

Settings: We now look back to the Equality constraints

$$\min_{\text{s.t. } h(x)=0} f(x) \tag{5}$$

Definition 3.30. *The lagrangian/lagrangian function associated to Equation 5 is*

$$L : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$$

defined by $L(x, \lambda) = f(x) - \lambda^\top h(x)$

Proposition 3.31. *The FONC are equivalent to the problem $\nabla L(x, \lambda) = 0$*

Proof. We have

$$\frac{\partial}{\partial x} L(x, \lambda) = \nabla f(x) - \lambda^\top h(x)$$

and

$$\frac{\partial}{\partial \lambda} L(x, \lambda) = -h(x)$$

□

Remark 3.32. *The Lagrangian converts a constrained problem into an unconstrained problem*

Theorem 3.33. *Let f be convex and h be affine function ($h(x) = Ax - b$ where $A \in \mathbb{R}^{m\times n}, b \in \mathbb{R}^n$). Then, the FONC is a FOSC.*

Proof. We know that $L(\cdot, \lambda)$ is convex for all $\lambda \in \mathbb{R}$ since $L(x, \lambda) = f(x) - \lambda^\top Ax + \lambda^\top b$.
So, for a fixed $\overline{\lambda}$, if $\frac{\partial}{\partial x} L(x_0, \overline{\lambda}) = 0$, then x_0 is the global unconstrained minimiser of $L(\cdot, \overline{\lambda})$.
If, also, $h(x_0) = 0 \implies L(x_0, \lambda) = f(x_0)$. For any feasible x such that $h(x) = 0$, then we have $f(x) = L(x, \overline{\lambda}) \geq L(x_0, \overline{\lambda}) = f(x_0)$. This shows x_0 is the global minimiser of f subject to $h(x) = 0$. □

Remark 3.34. *We have $L(x) = \nabla^2 f(x) - \lambda^\top \nabla^2 h(x) = \nabla_x^2 l(x, \lambda)$*

Settings: Now we will solve

$$\min_{\text{s.t. } h(x)=0, g(x)\geq 0} f(x) \tag{6}$$

The lagrangian associated to Equation 6 is $L : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p_{\geq 0} \to \mathbb{R}$ by $L(X, \lambda, \mu) = f(x) - \lambda^\top h(x) - \mu^\top g(x)$

Remark 3.35. *Using the Lagrangian, we can write the KKT conditions as follows*

- *Original variable constraints (OVC),* $h(x) = 0, g(x) \geq 0$
- *Multiplier sign constraints (MSC),* λ *is free,* $\mu \geq 0$
- *Lagrangion derivative condition (LDC),* $0 = \frac{\partial}{\partial x} L(x, \lambda, \mu) = \nabla f(x) - \lambda^\top \nabla h(x) - \mu \nabla g(x)$

Theorem 3.36. *Let* f *be convex,* $h(x) = Ax - b$ *be affine, individual constraints* g_j *be concave for* $j \in \{1, \ldots, p\}$*. Then the FONC is a FOSC.*

Proof. We have $L(x, \lambda, \mu) = f(x) - \lambda^\top Ax + \lambda^\top b - \sum_{j=1}^p \mu_j g_j(x)$ is convex in x for all λ and $\mu \geq 0$ since $-\mu_j g_j$ is convex. Assume that x is a feasible solution to Equation 6. Let (x_0, λ_0, μ_0) be a solution to the FONC (then x_0 is a global solution of $L(\cdot, \lambda_0, \mu_0)$ and $\mu^\top g(x_0) = 0$) and let J be the set of active inequality constraints.

Then

$$\begin{aligned}
0 &\leq L(x, \lambda_0, \mu_0) - L(x_0, \lambda_0, \mu_0) \\
&= f(x) - f(x_0) - \lambda_0^\top (h(x) - h(x_0)) - \mu_0^\top (g(x) - g(x_0)) \\
&= f(x) - f(x_0) - \mu_0^\top (g(x) - g(x_0)) \\
&= f(x) - f(x_0) - \sum_{j \in J} (\mu_0)_j (g_j(x) - g_j(x_0)) \\
&= f(x) - f(x_0) - \sum_{j \in J} (\mu_0)_j g_j(x) \\
&\leq f(x) - f(x_0)
\end{aligned}$$

□

3.4 Lagrangian Duality

Definition 3.37. *Let* $S \subset \mathbb{R}$*, then* $x_0 \in \mathbb{R}$ *is a lower bound of* S *if* $x_0 \leq x$ *for all* $x \in S$

$x_0 \in \mathbb{R}$ *is an upper bound of* S *if* $x_0 \geq x$ *for all* $x \in S$

If $x_0 \in \mathbb{R}$ *is a lower bound of* S*, then it is called an infimum of* S *if* $y \leq x_0$ *for all* $y \in \mathbb{R}$ *that are lower bounds of* S*.*

If $x_0 \in \mathbb{R}$ *is a upper bound of* S*, then it is called an supremum of* S *if* $y \geq x_0$ *for all* $y \in \mathbb{R}$ *that are upper bounds of* S*.*

Example 3.38. *Let* $S = [0, 1]$*. Any negative number and* 0 *are lower bounds. Any number greater than or equal to* 1 *is an upper bound.*

Remark 3.39.

- *An infimum is the greatest lower bound. A supremum is the smallest upper bound.*
- *Moreover, infima and suprema are not always in* S*. Take* $S = (0, 1)$*, we have* $\inf(S) = 0, sup(S) = 1$*.*
- *Infima and suprema always exist, if we allow* $\pm\infty$ *as values.*
- *The infimum of* $\emptyset$ *is* ∞*, the supremum of* $\emptyset$ *is* $-\infty$*.*

Settings: We consider

$$\min_{\text{s.t. } g(x) \geq 0} f(x) \tag{7}$$

and assume that there is a feasible point

Definition 3.40. *The parametric primal function of Equation 7 is $\omega : \mathbb{R}^p \to \mathbb{R} \cup \{\pm\infty\}$ by $\omega(z) = \inf_{x\in\mathbb{R}^p}\{f(x) : g(x) \geq z\}$*

Remark 3.41. *If Equation 7 has a solution x_0, then $\omega(0) = f(x_0) = f_0$ is the point on the x_{p+1} axis where the graph of ω passes through the x_{p+1} axis.*

If Equation 7 does not have a solution then $\omega(0) = \inf_{x\in\mathbb{R}^n}\{f(x) : g(x) \geq 0\}$ is the intersection of the graph with the x_{p+1} axis. Note that $f_0 \neq \infty$ as we have assumed there is a feasible point.

The general idea is that we will consider hyperplanes in $\mathbb{R}^{p+1}$ that touch the graph of ω from below. The intersection of the hyperplanes with the x_{p+1} axis will lie below $\omega(0) = f_0$

Definition 3.42. *The dual function to Equation 7 is $\phi : \mathbb{R}^p_{\geq 0} \to \mathbb{R} \cup \{\pm\infty\}$ defined by $\phi(\mu) = \inf_{x\in\mathbb{R}^n}\{f(x) - \mu^\top g(x)\}$*

Note: ϕ may not always be finite.

Proposition 3.43. *ϕ is concave in the region where it is finite.*

Proof. Assume $\mu_1, \mu_2 \in \mathbb{R}^p_{\geq 0}$ such that $\phi(\mu_1), \phi(\mu_2)$ are finite. Let $t \in [0, 1]$. Then

$$\begin{aligned}
\phi(t\mu_1 + (1-t)\mu_2) &= \inf_{x\in\mathbb{R}^n}\{f(x) - (t\mu_1 + (1-t)\mu 2)^\top g(x)\} \\
&\geq \inf_{x\in\mathbb{R}^n}\{tf(x) - t\mu_1^\top g(x)\} + \inf_{x\in\mathbb{R}^n}\{(1-t)f(x) - (1-t)\mu_2^\top g(x)\} \\
&= t\inf_{x\in\mathbb{R}^n}\{f(x) - \mu_1^\top g(x)\} + (1-t)\inf_{x\in\mathbb{R}^n}\{f(x) - \mu_2^\top g(x)\} \\
&= t\phi(\mu_1) + (1-t)\phi(\mu_2)
\end{aligned}$$

□

Definition 3.44. $f_0 = \inf_{x\in\mathbb{R}^n}\{f(x) : g(x) \geq 0\} \in \mathbb{R} \cup \{\pm\infty\}$.

$\phi_0 = \sup_{\mu\in\mathbb{R}^p_{\geq 0}}\{\phi(\mu)\} \in \mathbb{R} \cup \{\pm\infty\}$

Theorem 3.45 (Weak Duality)**.** $\phi_0 \leq f_0$

Proof. Let $\mu \in \mathbb{R}^p_{\geq 0}$, then

$$\begin{aligned}
\phi(\mu) &= \inf_{x\in\mathbb{R}^n}\{f(x) - \mu^\top g(x)\} \\
&\leq \inf_{x\in\mathbb{R}^n}\{f(x) - \mu^\top g(x) : g(x) \geq 0\} \\
&\leq \inf_{x\in\mathbb{R}^n}\{f(x) : g(x) \geq 0\} \\
&= f_0
\end{aligned}$$

Hence, $\phi_0 = \sup_{\mu\in\mathbb{R}^p_{\geq 0}}\phi(\mu) \leq f_0$ □

Remark 3.46 (The intepretation of the dual function)**.** *Let $(\mu, 1) \in \mathbb{R}^{p+1}$ with $\mu \in \mathbb{R}^p, c > 0$. Then $\{(-z, r) \in \mathbb{R}^{p+1} : (\mu, 1)^\top(-z, r) = r - \mu^\top z = c\}$ is a hyperplane in $\mathbb{R}^{p+1}$.*

For a fixed μ, we consider the highest (largest value of c) hyperplane that touches the graph of ω from below. Assuming the touching point of ω and the hyperplane is $(-\overline{z}, \overline{r})$, we have $\overline{r} = f(x_1), \overline{z} = g(x_1)$. Then we get $c = f(x_1) - \mu^\top g(x_1) = \phi(\mu)$.

This hyperplane intersects the x_{p+1} axis at some $(0, r_0)$. Then we get $c = (\mu, 1)^\top(0, r_0) = r_0 = \phi(\mu)$.

So, the value at the dual function at μ is the intersection of the unique hyperplane defined by μ that touches the graph of the primal from below.

If we take $\phi_0 = \sup_{\mu\in\mathbb{R}^p_{\geq 0}}\phi(\mu)$ then we are finding the largest intercept. We can see that $\phi_0 \leq f_0$.

Settings: Now consider

$$\min_{\text{s.t. } h(x)=0, g(x)\geq 0} f(x) \tag{8}$$

The dual is $\phi : \mathbb{R}^m \times \mathbb{R}^p_{\geq 0} \to \mathbb{R} \cup \{\pm\infty\}$ such that $\phi(\lambda, \mu) = \inf_{x\in\mathbb{R}^n}\{f(x) - \lambda^\top h(x) - \mu^\top g(x)\}$

The dual problem is to find $\phi_0 = \sup_{(\lambda,\mu)\in\mathbb{R}^m\times\mathbb{R}^p_{\geq 0}} \phi(\lambda, \mu)$

Remark 3.47. *Weak duality also holds for Equation 8.*

Theorem 3.48 (Zero-Order Condition)**.** *If Equation 8 has a feasible solution x_0 and ϕ_0 has a feasible solution λ_0, μ_0 such that $f(x_0) = \phi(\lambda_0, \mu_0)$ then x_0 is a global miniser of Equation 8, and (λ_0, μ_0) is a global maximiser of ϕ.*

Proof. We have $\inf_{x\in\mathbb{R}^n}\{f(x) : h(x) = 0, g(x) \geq 0\} \leq f(x_0) = \phi(\lambda_0, \mu_0) \leq \sup_{\lambda,\mu} \phi(\lambda, \mu) \leq \inf_x\{f(x) : h(x) = 0, g(x) \geq 0\}$, where the last one is by weak duality.

Hence, this gives us the equality of all terms. And x_0 is a global minimiser, λ_0, μ_0 is a global maximiser of ϕ. □

Theorem 3.49 (Strong Duality)**.** *Assume that in Equation 8 we have*

- *$f \in C^1(\mathbb{R}^n)$ is convex*
- *$h(x) = Ax - b$ is affine ($A \in \mathbb{R}^{m\times n}, b \in \mathbb{R}^m$)*
- *$g \in C^1(\mathbb{R}^n, \mathbb{R}^p)$ such that each g_j is concave*

Assume also Equation 8 has a local (= global) minimiser x_0 with $f(x_0) = f_0$ and f_0 solves the KKT conditions for $(\lambda_0, \mu_0) \in \mathbb{R}^m \times \mathbb{R}^p_{\geq 0}$. Then we get

- *$\phi_0 = f_0$*
- *(λ_0, μ_0) is a solution to $\phi_0 = \sup_{\mu\in\mathbb{R}^p_{\geq 0}} \phi(\mu)$*

Proof. Since Lagrangian is convex, by the KKT conditions we have $\mu_0^\top g(x_0) = 0$. This gives $\phi(\lambda_0, \mu_0) = f(x_0) - \lambda_0^\top h(x_0) - \mu_0^\top g(x_0) = f(x_0) = f_0$.The zero-order conditions gives the result. □

Example 3.50. *If we have*

$$\min_{s.t.\ g(x,y)=1-x-y\geq 0} x^2 + y^2$$

Clearly f is convex, g is concave.

We can define $\phi(\mu) = \inf_{(x,y)\in\mathbb{R}^2}\{x^2 + y^2 - \mu(1 - x - y)\}$.

By the FONC, $(0, 0) = (2x + \mu, 2y + \mu) \implies x_0 = y_0 = -\frac{\mu}{2}$.

By the SOSC, since $2I$ is positive definite, (x_0, y_0) is the minimiser.

Now, we need to find the specific μ. We have $\phi(\mu) = 2(-\frac{\mu}{2})^2 - \mu(1 + \mu) = \frac{\mu^2}{2} - \mu - \mu^2 = -\frac{\mu^2}{2} - \mu$.

We can now solve $\phi_0 = \sup_{\mu\geq 0} -\frac{\mu^2}{2} - \mu = 0$ when $\mu = 0$. This gives $x_0 = y_0 = 0 \implies f(x_0, y_0) = 0 = \phi_0$.

By the zero-order condition, we found the global minimum and maximum for Equation 8 and 9 respectively.

Example 3.51. *If we have*

$$\min_{s.t.\ h(x)=x-1=0} f(x) = x^3$$

Clearly the only solution $x_0 = 1$ is a global minimum. Therefore, $f_0 = f(1) = 1$. We also knows that $x_0 = 1$ fulfills the KKT conditions where $\lambda_0 = 3$.

However, by solving this using dual function we have $\phi(\lambda) = \inf_{x\in\mathbb{R}}\{f(x) - \lambda h(x)\} = \inf_{x\in\mathbb{R}}\{x^3 - \lambda x + \lambda\} = -\infty$. Hence, $\phi_0 = \sup_{\lambda\in\mathbb{R}} \phi(\lambda) = -\infty < f_0 = 1$. Since f is not convex, hence the strong duality theorem fails.

4 Calculus of Variations

In Chapter 1, 2 we studied problem

$$\min_{\text{s.t. } x\in\Omega} f(x)$$

where $f : \mathbb{R}^n \to \mathbb{R}, \Omega \subseteq \mathbb{R}^n$. This is a finite dimensional problem.

Settings: In this chapter, we will study an infinite dimensional problem

$$\min_{\text{s.t. } y\in S} J(y)$$

where

- X is a vector space of functions on $\mathbb{R}^n$ (function space)
- $S \subseteq X$
- $\|\cdot\|_X$ is a norm on X
- $y \in S$ is a function $y : \mathbb{R}^n \to \mathbb{R}$
- $J : X \to \mathbb{R}$ is a functional

Remark 4.1. *We do not assume that J is continuous/linear.*

Example 4.2. *We can let*

- $X = C^0(\mathbb{R}^n)$ *(continuous functions)*
- $X = C^\infty(\mathbb{R}^n)$ *(smooth functions)*
- $X = \{f : \mathbb{R} \to \mathbb{R} : f(x) = ax^2 + bx + c \text{ for some } a, b, c \in \mathbb{R}\}$ *(polynomial functions of degree 2 or less)*
- *If* $X = C^2(\mathbb{R})$ *we could have* $S = \{y \in X : y(0) = 1, y(1) = 1\}$
- *If* $X = C^\infty(\mathbb{R})$ *we could have* $S = \{y \in X : y(0) = 0, y'(0) = 1\}$
- *The most common example:* $X = C^2([0, 1]), S = \{y \in X : y(0) = 0, y(1) = 1\}$ *or* $S = \{y \in X : y(0) = 0, y'(0) = 1\}, J(y) = \int_0^1 (y'(x))^2 dx$ *or* $J(y) = \int_0^1 y(x) \cdot y'(x) - 1dx$.

Definition 4.3. *Let $(X, \|\cdot\|_X)$ be a normed function space, $J : X \to \mathbb{R}$ a functional and $S \subseteq X$. Then J has a local minimum in S at $y \in S$ if*

$$\exists \epsilon > 0, \text{ s.t. } J(y) \le J(\tilde{y}) \ \forall \tilde{y} \in S \text{ s.t. } \|y - \tilde{y}\|_X < \epsilon$$

J has a local maximum in S at $y \in S$ if $-J$ has a local minimum at $y \in S$.

Remark 4.4. *In $\mathbb{R}^n$, all norms are equivalent, i.e. if we have two norms $\|\cdot\|_1, \|\cdot\|_2$, there exists $0 < c_1 \le c_2$ such that $c_1\|x\|_1 \le \|x\|_2 \le c_2\|x\|_1$ for all $x \in \mathbb{R}^n$.*

In infinite dimensional spaces (especially in function spaces), different norms need not be equivalent. Therefore, the choice of x and $\|\cdot\|_X$ is very important.

Remark 4.5. *Let $\tilde{y} \in S$ be in an ϵ-neighbourhood of $y \in S$ (that is $\|\tilde{y} - y\|_X < \epsilon$), then we can define*

$$\eta = \frac{y - \tilde{y}}{\epsilon} \in X$$

and hence

$$\tilde{y} = y + \epsilon\eta$$

we have written $\tilde{y}$ as ϵ-perturbation of y with $\|\eta\|_X \leq 1$

Hence, every $\tilde{y}$ in an ϵ-neighbourhood of y can be generated from a suitable set

$$H_\epsilon = \{\eta \in X : y + \epsilon\eta \in S \wedge \|\eta\|_X < 1\}$$

similar to Chapter 1, we want to study the behaviour of J at a local minimum $y \in S$. Hence, there exists $\epsilon > 0$ such that

$$J(y) \leq J(\tilde{y}), \forall \tilde{y} \in S \wedge \|y - y\|_X^{\sim} < \epsilon$$

We write $\tilde{y} = y + \epsilon\eta$ with $\eta \in H_\epsilon$ and similar to Chapter 1 we use the idea that

$$0 = \frac{d}{d\epsilon} J(y + \epsilon\eta)|_{\epsilon=0}$$

Similar to $0 = \frac{d}{d\alpha} f(x_0 + \alpha d)$ in Chapter 1 for d to be a feasible direction.

As we are taking $\epsilon \to 0$, we can instead use as the set for ϵ-perturbations ("= feasible directions") the set

$$H = \{\eta \in X : y + \epsilon\eta \in S\}$$

instead. We call $\eta \in H$ a test function.

4.1 Euler - Lagrange Equation

Settings: Here we consider

$$\min_{\text{s.t. } y \in S} J(y)$$

where $X = C^2([x_0, x_1])$ with $x_0, x_1 \in \mathbb{R}, x_0 < x_1$ fixed and $S = \{y \in X : y(x_0) = y_0, y(x_1) = y_1\}$ with $y_0, y_1 \in \mathbb{R}$ fixed. And

$$J(y) = \int_{x_0}^{x_1} f(x, y, y')dx = \int_{x_0}^{x_1} f(x, y(x), y'(x))dx$$

where $f \in C^2([x_0, x_1] \times \mathbb{R} \times \mathbb{R})$

Hence, we get for $\epsilon > 0$ and $y \in S$ the set of ϵ-perturbations as $H = \{\eta \in X : y + \epsilon\eta \in S\} = \{\eta \in X : \eta(x_0) = \eta(x_1) = 0\}$

We assume that J has a local minimum at $y \in S$. That means, there exists $\epsilon > 0$ such that $J(y) \leq J(\tilde{y}), \forall y \in S$ such that $\|y - \tilde{y}\| < \epsilon$. We write $\tilde{y} = y + \epsilon\eta$, and we have

$$\begin{aligned}
f(x, \tilde{y}, \tilde{y}') &= f(x, \tilde{y}(x), \tilde{y}'(x)) \\
&= f(x, y + \epsilon\eta, y' + \epsilon\eta') \\
&= f(x, y, y') + \epsilon\frac{\partial f}{\partial y}(x, y, y')\eta + \epsilon\frac{\partial f}{\partial y'}(x, y, y')\eta' + \mathcal{O}(\epsilon^2) \qquad \text{[Taylor Expansion if } \epsilon \text{ sufficiently small]}
\end{aligned}$$

This gives

$$\begin{aligned}
0 \leq J(\tilde{y}) - J(y) &= \int_{x_0}^{x_1} f(x, \tilde{y}, \tilde{y}')dx - \int_{x_0}^{x_1} f(x, y, y')dx \\
&= \int_{x_0}^{x_1} f(x, y, y') + \epsilon\left(\frac{\partial f}{\partial y}(x, y, y')\eta + \frac{\partial f}{\partial y'}(x, y, y')\eta'\right) + \mathcal{O}(\epsilon^2) - f(x, y, y')dx \\
&= \epsilon\int_{x_0}^{x_1} \frac{\partial f}{\partial y}(x, y, y')\eta + \frac{\partial f}{\partial y'}(x, y, y')\eta' dx + \mathcal{O}(\epsilon^2)
\end{aligned}$$

Definition 4.6. *We call*

$$\delta J(\eta, y) = \int_{x_0}^{x_1} \frac{\partial f}{\partial y}(x, y, y')\eta + \frac{\partial f}{\partial y'}(x, y, y')\eta' dx$$

the first variation of J

We have seen that

$$0 \leq \epsilon \delta J(\eta, y) + \mathcal{O}(\epsilon^2)$$

If $\eta \in H$ then $-\eta \in H$. Using $-\eta$ instead of η in the above reasoning, we get

$$\begin{aligned} 0 &\leq \epsilon \delta J(-\eta, y) + \mathcal{O}(\epsilon^2) \\ &= -\epsilon \delta J(\eta, y) + \mathcal{O}(\epsilon^2) \\ \implies 0 &\geq \epsilon \delta J(\eta, y) + \mathcal{O}(\epsilon^2) \end{aligned}$$

If ϵ is sufficiently small, the sign of $\epsilon \delta J(\eta, y) + \mathcal{O}(\epsilon^2)$ is determined by $\delta J(\eta, y)$. We therefore have $\delta J(\eta, y) = 0, \forall \eta \in H$ if y is a local minimum.

Definition 4.7. *We say that J is stationary at $y \in S$ if*

$$0 = \delta J(\eta, y), \forall \eta \in H$$

In this case, we also call y an extremum for J

Lemma 4.8. *Let $\alpha, \beta \in \mathbb{R}$ such that $\alpha < \beta$. Then there exists $\eta \in C^2(\mathbb{R})$ such that*

- $\eta(x) > 0, \forall x \in (\alpha, \beta)$
- $\eta(x) = 0, \forall x \in (-\infty, \alpha] \cup [\beta, \infty)$

Proof. Define

$$\eta(x) = \begin{cases} (x - \alpha^3)(\beta - x^3) & x \in (\alpha, \beta) \\ 0 & x \in (-\infty, \alpha] \cup [\beta, \infty)] \end{cases}$$

We need to check that η has continuous second derivatives at α, β.
We have

$$\lim_{x \to a+} \frac{\eta(x) - \eta(\alpha)}{x - \alpha} = \lim_{x \to a+} \frac{(x - \alpha)^3(\beta - x)^3}{x - \alpha} = \lim_{x \to a+} (x - \alpha)^2(\beta - x)^3 = 0$$

And

$$\lim_{x \to a-} \frac{\eta(x) - \eta(\alpha)}{x - \alpha} = \lim_{xraa-} \frac{0 - 0}{x - \alpha} = 0$$

This shows that $\eta'(\alpha)$ exists and $\eta'(\alpha) = 0$.
In a similar way, $\eta'(\beta) = 0$.
We also have

$$\lim_{x \to a+} \frac{\eta'(x) - \eta'(\alpha)}{x - \alpha} = \lim_{x \to a+} \frac{3(x - \alpha)^2(\beta - x)^3 - 3(x - \alpha)^3(\beta - x)^2 - 0}{x - \alpha} = \lim_{x \to a+} 3(x - \alpha)(\beta - x)^3 - 3(x - \alpha)^2(\beta - x)^2 = 0$$

And

$$\lim_{x \to a-} \frac{\eta'(x) - \eta'(\alpha)}{x - \alpha} = \lim_{x \to a+} \frac{0 - 0}{x - \alpha} = 0$$

Hence, $\eta''(\alpha)$ exists and $\eta''(\alpha) = 0$.
In a similar way $\eta''(\beta) = 0$

Hence, we have

$$\eta''(x) = \begin{cases} 6(x-\alpha)(\beta-x)^3 - 18(x-\alpha)^2(\beta-x)^2 + 6(x-\alpha)^3(\beta-x) & x \in (\alpha,\beta) \\ 0 & \text{otherwise} \end{cases}$$

and clearly $\lim_{x\to\alpha} \eta''(x) = 0 = \eta''(\alpha)$ and similar for β.

Hence, $\eta \in C^2(\mathbb{R})$. □

Lemma 4.9. *Let $g \in C^0([x_0, x_1])$ and assume that*

$$0 = \int_{x_0}^{x_1} \eta(x)g(x)dx, \forall \eta \in H$$

Then $g(x) = 0, \forall x \in [x_0, x_1]$

Proof. Assume $\exists c \in [x_0, x_1]$ such that $g(c) \neq 0$. WLOG, assume $g(c) > 0$ and that by continuity of g, we have $c \in (x_0, x_1)$.

Since g is continuous, there exists α, β such that $x_0 < \alpha < c < \beta < x_1$, such that $g(x) > 0$ on (α, β).

By Lemma 4.8 there exists $\eta \in C^2(\mathbb{R})$ such that

- $\eta(x) > 0, \forall x \in (\alpha, \beta)$
- $\eta(x) = 0, \forall x \in \mathbb{R}\backslash(\alpha, \beta)$

This means, $\eta \in H$ as $\eta(x_0) = \eta(x_1) = 0$ and

$$0 = \int_{x_0}^{x_1} \eta(x)g(x)dx = \int_{\alpha}^{\beta} \eta(x)g(x)dx > 0$$

A contradiction. Hence, $g(x) = 0, \forall x \in (x_0, x_1)$ and by continuity $g(x) = 0, \forall [x_0, x_1]$ □

Now, we have $\delta J(\eta, y) = 0$, how to get a statement independent of η.

Theorem 4.10. *Let $J : C^2([x_0, x_1]) \to \mathbb{R}$ be given by*

$$J(y) = \int_{x_0}^{x_1} f(x, y, y')dx$$

such that $f \in C^2([x_0, x_1] \times \mathbb{R} \times \mathbb{R}), x_0, x_1 \in R, x_0 < x_1$.

Let $S = \{y \in C^2([x_0, x_1]) : y(x_0) = y_0, y(x_1) = y_1\}$, for $y_0, y_1 \in \mathbb{R}$ fixed.

If $y \in S$ is an extremal for J, then

$$\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) - \frac{\partial f}{\partial y}\bigg|_{x,y(x),y'(x)} = 0, \forall x \in [x_0, x_1]$$

Proof. We have shown that $0 = \delta J(\eta, y), \forall \eta \in H$.

Now,

$$\begin{aligned} 0 &= \delta J(\eta, y) \\ &= \int_{x_0}^{x_1} \eta \frac{\partial f}{\partial y}(x, y(x), y'(x)) + \eta' \frac{\partial f}{\partial y'}(x, y(x), y'(x))dx \\ &= \int_{x_0}^{x_1} \eta \left(\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'}\right)(x, y(x), y'(x))dx + \left(\eta \frac{\partial f}{\partial y'}(x, y(x), y'(x))\right)_{x=x_0}^{x=x_1} \\ &= \int_{x_0}^{x_1} \eta \left(\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'}\right)(x, y(x), y'(x))dx \forall \eta \in H \qquad [\eta(x_0) = \eta(x_1) = 0] \\ \implies 0 &= \frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\bigg|_{x,y(x),y'(x)} \qquad \text{[By Lemma 4.9]} \end{aligned}$$

$\square$

Definition 4.11. *The equation*

$$\begin{cases} \left(\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right)\Big|_{x,y(x),y'(x)} = 0, \forall x \in [x_0, x_1] \\ y(x_0) = y_0, y(x_1) = y_1 \end{cases}$$

is called Euler Lagrange equation

Remark 4.12.

- *The EL equation is the infinite dimensional analog to $\nabla f(x_0) = 0$ for x_0 a local miminum of $f : \mathbb{R}^n \to \mathbb{R}$*
- *The EL equation is a second order ODE (with boundary values) that can be highly non-linear*
- *Similar to FONC, he previous theorem states if y is an extremum then y solves EL equation. But not every solution to the EL equation is necessarily an extremum (similarly $\nabla f(x_0) = 0 \not\Longrightarrow x_0$ is a local minimum*
- *Now the idea is to minimise J, we compute the solution to the EL equation. Solutions to EL equation are candidates for minimisers of J.*

Example 4.13 (Geodesic problem)**.** *Take $(x_0, y_0) \in \mathbb{R}^2, (x_1, y_1) \in \mathbb{R}^2$, consider the curve $y : [x_0, x_1] \to \mathbb{R}$ such that $y(x_0) = y_0, y(x_1) = y_1$.*

Now, find the curve that has minimal arc length. Meaning we want to solve

$$\min_{s.t.\ y\in S} J(y) = \int_{x_0}^{x_1} \sqrt{1 + (y'(x))^2} dx$$

where

$$S = \{y \in C^2([x_0, x_1]) : y(x_0) = y_0, y(x_1) = y_1\}$$

and $f(x, y, y') = \int_{x_0}^{x_1} \sqrt{1 + (y'(x))^2} dx$

The EL equation gives

$$\begin{aligned} 0 &= \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) - \frac{\partial f}{\partial y} = \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\left(\sqrt{1 + (y')^2}\right)\right) - 0 \\ &= \frac{d}{dx}\left(\frac{y'}{\sqrt{1 + (y')^2}}\right) \end{aligned}$$

We now solve $c = \frac{y'}{\sqrt{1+(y')^2}}$ on $[x_0, x_1]$ such that $y(x_0) = y_0, y(x_1) = y_1$. This gives

$$\begin{aligned} c^2(1 + (y')^2) &= (y')^2 \\ \implies c^2 &= (1 - c^2)(y')^2 \\ \implies y'(x) &\pm c_1 \end{aligned}$$

This shows all minimisers have constant derivative/velocity.

If $x_0 = 0, y_0 = 1, x_1 = 1, y_1 = 1$, we have

$$\begin{aligned} y'(x) &= \pm c_1 x + b \\ \implies 0 &= y'(0) = b \\ \implies 1 &= y'(1) = \pm c_1 \\ \implies y(x) &= x \end{aligned}$$

The straight line between $(0, 0)$ and $(1, 1)$ which (is graphically clear) describes the shortest (with respect to arc length) path between $(0, 0)$ and $(1, 1)$.

Example 4.14. *Let $X = C^2([0,1]), x_0 = y_0 = 0, x_1 = y_1 = 1, S = \{y \in X : y(x_0) = y_0, y(x_1) = y_1\}$. We want to solve*

$$\min_{s.t.\ y \in S} J(y) = \int_0^1 (y')^2 - y^2 + 2xydx$$

The EL equation gives

$$0 = \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) - \frac{\partial f}{\partial y} = \frac{d}{dx}2y' - (-2y + 2x) = 2y'' + 2y - 2x$$

This gives an ODE

$$y'' + y = x$$

on $[0,1]$ where $y(0) = 0, y(1) = 1$.
First, we find a solution to the homogenous problem $y_n'' + y_n = 0 \implies y_n(x) = A\sin(x) + B\cos(x)$.
Second, we find a solution to the particular problem $y_p'' + y = x \implies y_p(x) = x$
This gives $y(x) = y_n(x) + y_p(x) = A\sin(x) + B\cos(x) + x$.
To solve for A and B we have $0 = y(0) = B$ and $1 = y(1) = A\sin(1) + 1 \implies A = 0$.
Hence, the solution is $y(x) = x$.

Example 4.15. *Let a constant $k > 0, x_0 = 0, x_1 = \pi, y_0 = y_1 = 0, X = C^2([0,\pi]), S = \{y \in X : y(0) = y(\pi) = 0\}$. We want to find*

$$\min J(y) = \int_0^\pi (y')^2 - ky^2dx$$

The EL equation gives

$$\begin{aligned} 0 &= \frac{d}{dx}(2y') - (-2ky) = 2y'' + 2ky \\ \implies& y'' + ky = 0 \\ &\implies y(x) = A\sin(\sqrt{k}x) + B\cos(\sqrt{k}x) \end{aligned}$$

To solve for A, B, we have $0 = y(0) = B$ and ***Case 1:*** *For $\sqrt{k}$ is not an integer we have $\sin(\sqrt{k}\pi) \neq 0 \implies A = 0 \implies y = 0$*
Case 2: *For $\sqrt{k}$ is an integer, we have $\sin(\sqrt{k}\pi) = 0 \implies A \in \mathbb{R}$ can be any number. Hence, $y(x) = A\sin(\sqrt{k})$ for any $A \in \mathbb{R}$.*

Settings: Assume now that $J(y) = \int_{x_0}^{x_1} f(x, y')dx$ i.e. that f does not depend on y. Then the EL equation

$$\frac{d}{dx}\left(\frac{\partial f}{\partial y'}(x, y')\right) - \frac{\partial f}{\partial y}(x, y') = 0$$

reduces to

$$\frac{d}{dx}\left(\frac{\partial f}{\partial y'}(x, y')\right) = 0 \implies \frac{\partial f}{\partial y'}(x, y') = c$$

This is a first order ODE for y (while the normal EL equation is a second order ODE).

Example 4.16. *Let $J(y) = \int_{x_0}^{x_1} e^x\sqrt{1 + (y')^2}dx$*

is given by the EL equation

$$\begin{aligned}
0 &= \frac{d}{dx}\left(\frac{e^x y'}{\sqrt{1+(y')^2}}\right) \\
\implies c &= \frac{e^x y'}{\sqrt{1+(y')^2}} \\
\implies c^2(1+(y')^2) &= e^{2x}(y')^2 \\
\implies c^2 &= (y')^2(e^{2x}-c^2) \\
\implies y'(x) &= \frac{c}{\sqrt{e^{2x}-c^2}}
\end{aligned}$$

Note: from EL equation we get $|c| = e^x \frac{|y'|}{\sqrt{1+(y')^2}} < e^x \implies e^{2x} - c^2 > 0$.

This gives $y(x) = \sec^{-1}\left(\frac{e^x}{c}\right) + A$

Settings: Assume now that $J(y) = \int_{x_0}^{x_1} f(y, y')dx$ i.e. that f does not depend on x.

Theorem 4.17. *Let* $J(y) = \int_{x_0}^{x_1} f(y, y')dx$ *with* $x_0 < x_1, f \in C^2(\mathbb{R}\times\mathbb{R})$ *and define*

$$H(y, y') = y'\frac{\partial f}{\partial y'}(y, y') - f(y, y')$$

Then H is constant along any extremum $\overline{y}$, i.e.

$$\frac{d}{dx}H(\overline{y}(x), \overline{y}'(x)) = 0$$

Proof. Assume y is an extremum for J, then

$$\begin{aligned}
\frac{d}{dx}H(y, y') &= \frac{d}{dx}\left(y'\frac{\partial f}{\partial y'}(y, y') - f(y, y')\right) \\
&= y''\frac{\partial f}{\partial y'}(y, y') + y'\frac{d}{dx}\left(\frac{\partial f}{\partial y'}(y, y')\right) - y'\frac{\partial f}{\partial y}(y, y') - y''\frac{\partial f}{\partial y'}(y, y') \\
&= y'\left(\frac{d}{dx}\left(\frac{\partial f}{\partial y'}(y, y')\right) - \frac{\partial f}{\partial y}(y, y')\right) \\
&= 0
\end{aligned}$$

□

Hence, if y is an extremal for J, the equation $c = H(y, y')$ is a first-order ODE for y.

Example 4.18. *Let* $J(y) = \int_{x_0}^{x_1} y\sqrt{1+(y')^2}dx$. *We get*

$$H(y, y') = y'\frac{\partial f}{\partial y'} - f = y'\frac{yy'}{\sqrt{1+(y')^2}} - y\sqrt{1+(y')^2} = y\left((y')^2 - 1 - (y')^2\right)\frac{1}{\sqrt{1+(y')^2}} = -\frac{y}{\sqrt{1+(y')^2}}$$

This gives $c^2 = H(y, y')^2 = \frac{y^2}{\sqrt{1+(y')^2}}$

- *If* $c = 0$, *we have* $y^2 = 0 \implies y = 0$

- *If $c \neq 0$, we have $y' = \sqrt{\frac{y^2}{c^2} - 1}$. Solve the ODE by separating the variables we get*

$$y = \int \sqrt{\frac{y^2}{c^2} - 1} = cln\left(\frac{y + \sqrt{y^2 - c^2}}{c}\right) + A$$

$$\implies ce^{\frac{x-A}{c}} = y + \sqrt{y^2 - c^2}$$

$$\implies ce^{-\frac{x-A}{c}} = \frac{c^2}{y + \sqrt{y^2 - c^2}}$$

$$\begin{aligned}\implies c\left(e^{\frac{x-A}{c}} + e^{-\frac{x-A}{c}}\right) &= y + \sqrt{y^2 - c^2} + \frac{c^2}{y + \sqrt{y^2 - c^2}} \\ &= \frac{y^2 + 2y\sqrt{y^2 - c^2} + (y^2 - c^2) + c^2}{y + \sqrt{y^2 - c^2}} \\ &= 2y\frac{y + \sqrt{y^2 - c^2}}{y + \sqrt{y^2 - c^2}} \\ &= 2y \implies y(x) = c\cosh(\frac{x-A}{c})\end{aligned}$$

4.2 Generalisations of the Euler Lagrange Equation

Settings: We consider

$$\min_{\text{s.t. } y \in S = \{y \in X : y(x_0) = y_0, y'(x_0) = y'_0, y(x_1) = y_1, y'(x_1) = y'_1\}} J(y)$$

where $X = C^4([x_0, x_1]), J(y) = \int_{x_0}^{x_1} f(x, y, y', y'')dx, f \in C^3([x_0, x_1] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R})$ where $x_0, x_1 \in \mathbb{R}, x_0 < x_1, y_0, y'_0, y_1, y'_1 \in \mathbb{R}$. Then

$$H = \{\eta \in X : \eta(x_0) = \eta'(x_0) = \eta(x_1) = \eta'(x_1) = 0\}$$

assuming that J has a local minimum at $y \in S$ analogue to last section, we consider perturbation $\tilde{y} = y + \epsilon\eta \in S, \eta \in H$. We have

$$\begin{aligned}f(x, \tilde{y}, \tilde{y}', \tilde{y}'') &= f(x, y + \epsilon + \eta, y' + \epsilon\eta', y'' + \epsilon\eta'') \\ &= f(x, y, y', y'') + \epsilon\left(\eta\frac{\partial f}{\partial y}(x, y, y', y'') + \eta'\frac{\partial f}{\partial y'}(x, y, y', y'') + \eta''\frac{\partial f}{\partial y''}(x, y, y', y'')\right) + \mathcal{O}(\epsilon^2) \\ \implies 0 \leq J(\tilde{y}) - J(y) &= \epsilon\int_{x_0}^{x_1} \eta\frac{\partial f}{\partial y} + \eta'\frac{\partial f}{\partial y'} + \eta''\frac{\partial f}{\partial y''}dx + \mathcal{O}(\epsilon^2)\end{aligned}$$

and hence the first variation of this function J is

$$\delta J(\eta, y) = \int_{x_0}^{x_1} \eta\frac{\partial f}{\partial y} + \eta'\frac{\partial f}{\partial y'} + \eta''\frac{\partial f}{\partial y''}dx$$

$$\begin{aligned}\int_{x_0}^{x_1} \eta''\frac{\partial f}{\partial y''}dx &= \eta'\frac{\partial f}{\partial y''}\bigg|_{x=x_0}^{x=x_1} - \int_{x_0}^{x_1} \eta'\frac{d}{dx}\frac{\partial f}{\partial y''}dx \\ &= \left(-\eta\frac{d}{dx}\frac{\partial f}{\partial y''}\right)\bigg|_{x=x_0}^{x=x_1} + \int_{x_0}^{x_1} \eta\frac{d^2}{dx^2}\frac{\partial f}{\partial y''}dx \\ &= \int_{x_0}^{x_1} \eta\frac{d^2}{dx^2}\frac{\partial f}{\partial y''}dx\end{aligned}$$

Hence, $\delta J(\eta, y) = \int_{x_0}^{x_1} \eta\left(\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} + \frac{d^2}{dx^2}\frac{\partial f}{\partial y''}\right)dx$ Analogue to before, we use $\eta, -\eta$ we get

$$0 = \delta J(\eta, y) \implies 0 = \frac{d^2}{dx^2}\left(\frac{\partial f}{\partial y''}(x, y, y', y'')\right) - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}(x, y, y', y'')\right) + \frac{\partial f}{\partial y}(x, y, y', y'')$$

Theorem 4.19. *Let $X = C^4([x_0, x_1]), x_0 < x_1, x_0, x_1 \in \mathbb{R}$ and define*

$$J(y) = \int_{x_0}^{x_1} f(x, y, y', y'')dx$$

with $f \in C^3([x_0, x_1] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R})$ and let $S = \{y \in X : y(x_0) = y_0, y'(x_0) = y_0', y(x_1) = y_1, y'(x_1) = y_1'\}$ for y_0, y_0', y_1, y_1'

If $y \in S$ is an extremum for J then

$$0 = \frac{d^2}{dx^2}\left(\frac{\partial f}{\partial y''}(x, y, y', y'')\right) - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}(x, y, y', y'')\right) + \frac{\partial f}{\partial y}(x, y, y', y'')$$

holds for all $x \in [x_0, x_1]$

Remark 4.20. *This equation is also called Euler-Lagrange Equation*

Example 4.21. *Let $x_0 = 0, x_1 = 1, y_0 = y_0' = 0, y_1 = y_1' = 1$ and $J(y) = \int_0^1 (y'')^2 - 2cydx$ for $c \in \mathbb{R}$ fixed.*

The EL Equation gives

$$\frac{d^2}{dx^2}(2y'') - 2c = 0 \implies y^{(4)}(x) = c \ \forall x \in [0, 1] \implies y(x) = \frac{1}{24}cx^4 + Ax^3 + Bx^2 + Cx + D$$

We have $0 = y(0) = D, 0 = y'(0) = C, 1 = y(1) = \frac{1}{24}C + A + B, 1 = y'(1) = \frac{1}{12}C + 3A + 2B \implies A = -1, B = 2$.

Remark 4.22. *The previous theorem can be extended. If we consider $J(y) = \int_{x_0}^{x_1} f(x, y, y', y'', \ldots, y^{(n)})dx$ then the associated EL equation is*

$$0 = (-1)^n \frac{d^n}{dx^n}\left(\frac{\partial f}{\partial y^{(n)}}\right) + (-1)^{n-1}\frac{d^{n-1}}{dx^{n-1}}\left(\frac{\partial f}{\partial y^{(n-1)}}\right) + \ldots + (-1)\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) + \frac{\partial f}{\partial y}$$

Now consider $X = C^2([t_0, t_1], \mathbb{R}^n)$ for $t_0, t_1 \in \mathbb{R}, t_0 < t_1$. We denote $q \in X, i.e.\ q : [t_0, t_1] \to \mathbb{R}^n$ by $q(t) = (q_1(t), \ldots, q_n(t))$ with $q_k \in C^2([t_0, t_1], \mathbb{R})$ for $k = 1, \ldots, n$.

So X is a vector space and we can define (for example)

$$\|q\|_X = \max_{k=1,\ldots,n} \sup_{t \in [t_0, t_1]} |q_k(t)|$$

Example 4.23. *Let $n = 3, t_0 = 0, t_1 = T$ and $q(t) = (x(t), y(t), z(t))$ describes the position of a particle in $\mathbb{R}^3$ at time t.*

Consider now $J(q) = \int_{t_0}^{t_1} L(t, q, \dot{q})dt$ where $\dot{q}(t) = (\dot{q}_1(t), \ldots, \dot{q}_n(t))$. We have $L \in C^2([t_0, t_1] \times \mathbb{R}^n \times \mathbb{R}^n)$. Fix $q_0, q_1 \in \mathbb{R}^n$ and consider

$$\min_{\text{s.t. } q \in S = \{q \in X : q(t_0) = q_0, q(t_1) = q_1\}} J(q)$$

Let q be a local minimum consider the perturbation $\tilde{q} = q + \epsilon\eta$ where $\eta = (\eta_1, \ldots, \eta_n) \in H = \{\eta \in X : \eta(t_0) = \eta(t_1) = (0, \ldots, 0) \in \mathbb{R}^n\}$

Therefore, we can write

$$\begin{aligned} L(t, \tilde{q}, \dot{\tilde{q}}) &= L(t, q + \epsilon\eta, \dot{q} + \epsilon\dot{\eta}) \\ &= L(t, q, \tilde{q}) + \epsilon \sum_{k=1}^{n} \left(\eta_k \frac{\partial L}{\partial q_k} + \dot{\eta}_k \frac{\partial L}{\partial \dot{q}_k}\right) + \mathcal{O}(\epsilon^2) \end{aligned}$$

Hence,

$$0 \leq L(\tilde{q}) - L(q) = \int_{t_0}^{t_1} L(t, \tilde{q}, \dot{\tilde{q}})dt - \int_{t_0}^{t_1} L(t, q, \dot{q})dt$$
$$= \epsilon \int_{t_0}^{t_1} \sum_{k=1}^{n} \left(\eta_k \frac{\partial L}{\partial q_k} + \dot{\eta}_k \frac{\partial L}{\partial \dot{q}_k} \right) dt + \mathcal{O}(\epsilon^2)$$

Therefore,

$$\delta J(\eta, q) = \int_{t_0}^{t_1} \sum_{k=1}^{n} \left(\eta_k \frac{\partial L}{\partial q_k} + \dot{\eta}_k \frac{\partial L}{\partial \dot{q}_k} \right) dt$$

and again using $\eta, -\eta$ we reason
$$0 = \delta J(\eta, q), \forall \eta \in H$$

Now for $\eta = (\eta_1, \ldots, \eta_n) \in H$, we can define

$$H_k = \{(0, \ldots, \eta_k, 0, \ldots, 0) \in H\} \subseteq H$$

for $k = 1, \ldots, n$.
Hence we get $0 = \partial J(\eta, y)$ for all $\eta \in H_k$.
But then

$$0 = \delta J(\eta, y) = \int_{t_0}^{t_1} \eta_k \frac{\partial L}{\partial q_k} + \dot{\eta}_k \frac{\partial L}{\partial \dot{q}_k} dt$$

And by integration by parts

$$0 = \int_{t_0}^{t_1} \eta_k \left(\frac{\partial L}{\partial q_k} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_k} \right) \right) dt \qquad \forall k = 1, \ldots, n$$

This gives that
$$\frac{\partial L}{\partial q_k} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_k} \right) \bigg|_{(t,q,\dot{q})} = 0 \qquad \forall k = 1, \ldots, n$$

Theorem 4.24. *Let $J : X \to \mathbb{R}$ be given by*

$$J(q) = \int_{t_0}^{t_1} L(t, q, \dot{q})dt$$

where $q \in C^2([t_0, t_1], \mathbb{R}^n) = X, L \in C^2([t_0, t_1], \mathbb{R}^n, \mathbb{R}^n), S = \{q \in X : q(q_0) = q_0, q(t_1) = q_1\}$ for $q_0, q_1 \in \mathbb{R}^n$.

If q is an extremal for J in S then

$$\begin{cases} \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_1} - \frac{\partial L}{\partial q_1} \Big|_{(t,q,\dot{q})} = 0 \\ \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_2} - \frac{\partial L}{\partial q_2} \Big|_{(t,q,\dot{q})} = 0 \\ \vdots \\ \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_n} - \frac{\partial L}{\partial q_n} \Big|_{(t,q,\dot{q})} = 0 \end{cases}$$

holds.

Remark 4.25. *The previous equations are also called Euler-Lagrange equations.*

There are n equations.

Definition 4.26. *The function L in J is called Lagrangian.*

Example 4.27. *Take $n = 2, t_0 = 0, t_1 = 1$ and*

$$J(q) = \int_0^1 \dot{q}_1^2 + (\dot{q}_1 - 1)^2 + q_1^2 + q_1 q_2 dt$$

We get the EL-equations

$$0 = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_1} - \frac{\partial L}{\partial q_1} = \frac{d}{dt}(2\dot{q}_1) - (2q_1 + q_2) = 2\ddot{q}_1 - 2q_1 - q_2$$

$$0 = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_2} - \frac{\partial L}{\partial q_2} = \frac{d}{dt}(2(\dot{q}_2 - 1)) - q_1 = 2\ddot{q}_2 - q_1$$

Therefore, we get

$$\begin{cases} 0 = \ddot{q}_1 - q_1 - \frac{q_2}{2} \\ 0 = \ddot{q}_2 - \frac{q_1}{2} \implies q_1 = 2\ddot{q}_2 \end{cases}$$

This gives a linear ODE $0 = q_2^{(4)} - 2\ddot{q}_2 + \frac{q_2}{2}$.

We can thus solve q_1, q_2.

Example 4.28 (Motion of particle)**.** *Let $n = 3, q(t) = (q_1(t), q_2(t), q_3(t))$ denotes the position of particle in $3d$ space at time t. Moreover, assume the particle has mass $m > 0$ fixed.*

We have kinetic energy is $T(q, \dot{q}) = \frac{1}{2}m|\dot{q}|^2 = \frac{1}{2}m(\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2)$.

We also have the potential energy to be $V(t, q)$. We also have Lagrangian $L(t, q, \dot{q}) = T(q, \dot{q}) - V(t, q)$, we want to find a path of particle that minimisers total energy, i.e.

$$\min J(q) = \int_{t_0}^{t_1} L(t, q, \dot{q})dt$$

The EL-equation gives

$$\begin{aligned} \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} = 0 \qquad \forall k = 1, 2, 3 \\ \implies 0 = \frac{d}{dt}(m\dot{q}_k) + \frac{\partial V}{\partial q_k} \\ \implies m\ddot{q}_k = -\frac{\partial V}{\partial q_k} \qquad \forall k = 1, 2, 3 \end{aligned}$$

This equations of motion if we define $a(t) = (\ddot{q}_1(t), \ddot{q}_2(t), \ddot{q}_3(t)) = \ddot{q}(t)$ the acceleration and $f_k = -\frac{\partial V}{\partial q_k}$ the k-th component of force, then we get

$$f = ma \qquad \textit{Newton's Equation}$$

Now we consider $\Omega \subseteq \mathbb{R}^2$ (connected, bounded, open, smooth) with a boundary $\partial\Omega$ and closure $\overline{\Omega} = \partial\Omega \cup \Omega$ and take $X = C^2(\overline{\Omega}, \mathbb{R})$ and consider

$$\min_{\text{s.t. } u \in S = \{u \in X : u(x,y) = u_0(x,y) \forall (x,y) \in \delta\Omega\}} J(u) = \int_\Omega f(x, y, u, u_x, u_y) dxdy$$

with the boundary condition $u_0 : \partial\Omega \to \mathbb{R}$ given and f smooth (with arguments x, y, u, p, q)

Remark 4.29. *Before we have* $x \in [x_0, x_1], y(x_0) = y_0, y(x_1) = y_1$.

Now, we have $(x, y) \in \Omega$ *and* $u|_{\delta\Omega} = u_0$. *There are boundary conditions in both cases*

Hence, $H = \{\eta \in X : \eta|_{\partial\Omega} = 0\}$ *and consider perturbation of minimiser* u *given by*

$$\hat{u}(x, y) = u(x, y) + \epsilon\eta(x, y)$$

Hence,

$$\begin{aligned} f(x, y, \hat{u}, \hat{u}_x, \hat{u}_y) &= f(x, y, u + \epsilon\eta, u_x + \epsilon\eta_x, u_y + \epsilon\eta_y) \\ &= f(x, y, u, u_x, u_y) + \epsilon\left(\eta\frac{\partial f}{\partial u} + \eta_x\frac{\partial f}{\partial p} + \eta_y\frac{\partial f}{\partial q}\right) + \mathcal{O}(\epsilon^2) \end{aligned}$$

Hence

$$0 \geq J(\hat{u}) - J(u) = \epsilon\int_\Omega \eta\frac{\partial f}{\partial u} + \eta_x\frac{\partial f}{\partial p} + \eta_y\frac{\partial f}{\partial q}dxdy + \mathcal{O}(\epsilon^2)$$

and similar to above we see

$$0 = \int_\Omega \eta\frac{\partial f}{\partial u} + \eta_x\frac{\partial f}{\partial p} + \eta_y\frac{\partial f}{\partial q}dxdy$$

Now we went to get rid of the x *and* y *derivatives on* η *we need the following theorem*

Theorem 4.30 (Integration by parts)**.** *Let* $u, v \in C^1(\overline{\Omega})$. *Then*

$$\int_\Omega u_x v dxdy = -\int_\Omega u v_x dxdy + \int_{\delta\Omega} uv\nu^x dS$$

where $\nu = (\nu^x, \nu^y)$ *is the outward pointing unit normal vector field and* dS *denotes the integration along* $\delta\Omega$.

Now, continue with Remark 4.29 we get

$$\begin{aligned} \int_\Omega \eta_x\frac{\partial f}{\partial p}dxdy &= -\int_\Omega \eta\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial p}\right)dxdy + \int_{\delta\Omega}\eta\frac{\partial f}{\partial p}\nu^x dS \\ &= \int -\eta\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial p}\right)dxdy \end{aligned}$$

And hence

$$0 = \int_\Omega \eta\left(\frac{\partial f}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial p}\right) - \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial q}\right)\right)dxdy$$

and because $\eta \in H$ was an arbitrary test function we get

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial p}\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial q}\right) - \frac{\partial f}{\partial u}\bigg|_{(x,y,u,u_x,u_y)} = 0, \forall(x, y) \in \Omega$$

If u is a minimiser.

Definition 4.31. *The previous equation is called Euler-Lagrange equation.*

Remark 4.32. *The EL-equation now is a PDE (with boundary condition* $u|_{\delta\Omega} = u_0$ *for* u.

This makes it way more difficult to solve.

Remark 4.33. *The previous ideas can easily be generalised to*

- $\Omega \subseteq \mathbb{R}^n$
- $u \in C^2(\overline{\Omega}, \mathbb{R}^m)$

- *f depending on higher derivatives of u*

Definition 4.34. *Let $u \in C^2(\overline{\Omega})$ with $\overline{\Omega} \in \mathbb{R}^n$, let $\varphi \in C^2(\overline{\Omega}, \mathbb{R}^n)$. Then*

- *$\nabla u = Du = (u_{x_1}, \ldots, u_{x_n})$ is called gradient of u*
- *$div(\varphi) = \nabla \dot{\varphi} = \varphi^1_{x_1} + \ldots + \varphi^n_{x_n}$ is called divergence of φ*
- *$\triangle u = u_{x_1 x_1} + \ldots u_{x_n x_n}$ is called Laplacian of u*

Remark 4.35. *If holds $\triangle u = div(\nabla u)$*

Example 4.36. *Let $\Omega \subseteq \mathbb{R}^2$ be the unit disc, i.e. $\Omega = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ and let*

$$J(u) = \int_\Omega |\nabla u|^2 dxdy = \int_\Omega u_x^2 + u_y^2 dxdy \qquad [\textit{Dirichlet energy}]$$

and let $S = \{u \in C^2(\overline{\Omega}) : u(x,y) = 2x^2 - 1 \ \forall (x,y) \in \delta\Omega\}$ with $\delta\Omega = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$
We have $f(x,y,u,p,q) = p^2 + q^2$ and hence the EL Equation becomes

$$0 = \frac{\partial}{\partial x}(u_x) + \frac{\partial}{\partial y}(u_y) = u_{xx} + u_{yy} = \triangle u \qquad [\textit{Laplace equation}]$$

hence if u is a minimiser of J it must solve

$$\begin{cases} \triangle u = 0 & \textit{in } \Omega \\ u = u_0 & \textit{on } \delta\Omega \end{cases}$$

A solution for this problem is $u(x,y) = x^2 - y^2$

Example 4.37 (Minimal surface)**.** *Let $\Omega \subseteq \mathbb{R}^2$ and let $r : \Omega \to \mathbb{R}^3$ by $r(x,y) = (x, y, u(x,y)$ describe a surface in $\mathbb{R}^3$ where $u : \Omega \to \mathbb{R}$. This surface has area*

$$J(u) = \int_\Omega \sqrt{1 + |\nabla u|^2} dxdy = \int_\Omega \sqrt{1 + u_x^2 + u_y^2} dxdy$$

Consider the minimal surface problem

$$\min_{s.t.\ u \in S = \{u \in C^2(\overline{\Omega}) : u|_{\delta\Omega} = u_0} J(u)$$

i.e. the surface is enclosed by the curve $r_0 : \delta\Omega \to \mathbb{R}^3$ where $r_0(x,y) = (x, y, u_0(x,y))$
The EL equation is (where $f(x,y,u,p,q) = \sqrt{1 + p^2 + q^2}$)

$$\begin{aligned} 0 &= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial p}\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial q}\right) \\ &= \frac{\partial}{\partial x}\left(\frac{u_x}{\sqrt{1 + u_x^2 + u_y^2}}\right) + \frac{\partial}{\partial y}\left(\frac{u_y}{\sqrt{1 + u_x^2 + u_y^2}}\right) \\ &= \ldots (\textit{ something too complicated}) \end{aligned}$$

4.3 Second Variation

Settings: Consider again $x_0 < x_1, x_0, x_1 \in \mathbb{R}$ and the problem

$$\min_{\text{s.t. } y \in S = \{y \in X : y(x_0) = y_0, y(x_1) = y_1\}} J(y)$$

for $y_0, y_1 \in \mathbb{R}, X = C^2([x_0, x_1])$ and $J(y) = \int_{x_0}^{x_1} f(x, y, y')dx$ where $f \in C^\infty([x_0, x_1])$ (or f at least sufficiently smooth). Assume J has an extremum at y, let $\hat{y} = y + \epsilon\eta$ be a perturbatioin where $\eta \in H = \{\eta \in X : \eta(x_0) = \eta(x_1) = 0\}$ and we compute (this time with a higher order expansion)

$$f(x, \hat{y}, \hat{y}') = f(x, y, y') + \epsilon\left(\eta\frac{\partial f}{\partial y} + \eta'\frac{\partial f}{\partial y'}\right) + \frac{\epsilon^2}{2}\left(\eta^2\frac{\partial^2 f}{\partial y^2} + 2\eta\eta'\frac{\partial^2 f}{\partial y \partial y'} + (\eta')^2\frac{\partial^2 f}{\partial (y')^2}\right) + \mathcal{O}(\epsilon^3)$$

We write $f_{yy} = \frac{\partial^2 f}{\partial y^2}$ similarly for $f_{yy'}$ and more.

Hence,

$$J(\hat{y}) - J(y) = \epsilon\delta J(\eta, y) + \frac{\epsilon^2}{2}\delta^2 J(\eta, y) + \mathcal{O}(\epsilon^3)$$

where $\delta^2 J(\eta, y) = \int_{x_0}^{x_1} \eta^2 f_{yy} + 2\eta\eta' f_{yy'} + (\eta')^2 f_{y'y'}dx$

Because y is an extremal we get $\delta J(\eta, y) = 0$ hence $J(\hat{y}) - J(y) = \frac{\epsilon^2}{2}\delta^2 J(\eta, y) + \mathcal{O}(\epsilon^3)$

Definition 4.38. *$\delta^2 J(\eta, y)$ is called the second variation of J at y*

Theorem 4.39. *Suppose J has a local extremum in S at y*

- *If y is a local minimum then*
$$\delta^2 J(\eta, y) \geq 0 \qquad \forall \eta \in H$$
- *If y is a local maximum then*
$$\delta^2 J(\eta, y) \leq 0 \qquad \forall \eta \in H$$

Remark 4.40. *The above result is of limited use, as the sign condition still depends on η. Hence, we need to study further conditions of local minimum/maximum.*

Idea:

$$\delta^2 J(\eta, y) = \int_{x_0}^{x_1} \eta^2 f_{yy} + 2\eta\eta' f_{yy'} + (\eta')^2 f_{y'y'}dx$$

Note $2\eta\eta' f_{yy'} = (\eta^2)' f_{yy'}$

Hence,

$$\int_{x_0}^{x_1} 2\eta\eta' f_{yy'}dx = [\eta^2 f_{yy'}]_{x_0}^{x_1} - \int_{x_0}^{x_1} \eta^2 \frac{d}{dx}(f_{yy'})\, dx = -\int_{x_0}^{x_1} \eta^2 \frac{d}{dx}(f_{yy'})dx$$

Hence,

$$\delta^2 J(\eta, y) = \int_{x_0}^{x_1} \eta^2\left(f_{yy} - \frac{d}{dx}(f_{yy'})\right) + (\eta')^2 f_{yy'}dx$$

So the sign of $\delta^2 J(\eta, y)$ depends on interplay between

$$A(x) = f_{y'y'}(x, y(x), y'(x))$$

and

$$B(x) = f_{yy}(x, y(x), y'(x)) - \frac{d}{dx}\left(f_{yy'}(x, y(x), y'(x))\right)$$

which are continuous functions.

The intuition is the sin of $\delta^2 J(\eta, y)$ is determined by A. This is because we recall $\eta(x_0) = \eta(x_1) = 0$.

If $|\eta'(x)| << 1 \implies |\eta(x)| << 1$

But if $|\eta(x)| << 1 \not\Longrightarrow |\eta'(x)| << 1$

Theorem 4.41 (Legendre condition)**.** *Assume J has a local minimum in S at y, then*

$$f_{y'y'}(x, y(x), y'(x)) \geq 0 \qquad \forall x \in [x_0, x_1]$$

Proof. Suppose by contradiction, that $\exists c \in (x_0, x_1)$ s.t. $A(c) < 0$.

We know that y is a local minimum for $J \implies \delta^2 J(\eta, y) \geq 0, \forall \eta \in H$.

Therefore, we want to find $\nu \in H$ s.t. $\delta^2 J(\nu, y) < 0$

Since A is continuous, there is some $\gamma > 0$ s.t. $A(x) < \frac{A(c)}{2}, \forall x \in (c-\gamma, c+\gamma)$.

Therefore, the idea is we want to construct ν iwth small value but large (in magnitude) derivative.

Define

$$\nu(x) = \begin{cases} \sin^4\left(\frac{\pi(x-c)}{\gamma}\right) & \text{if } x \in [c-\gamma, c+\gamma] \\ 0 & \text{otherwise} \end{cases}$$

We get

$$\nu'(x) = \begin{cases} \frac{4\pi}{\gamma}\sin^3\left(\frac{\pi(x-c)}{\gamma}\right)\cos\left(\frac{\pi(x-c)}{\gamma}\right) & \text{if } x \in [c-\gamma, c+\gamma] \\ 0 & \text{otherwise} \end{cases}$$

Now we also have

$$\begin{aligned}
\int_{x_0}^{x_1} A(x)(\nu'(x))^2 dx &= \int_{c-\gamma}^{c+\gamma} A(x)(\nu'(x))^2 dx \\
&< \frac{A(c)}{2}\int_{c-\gamma}^{c+\gamma} (\nu'(x))^2 dx \\
&= \frac{A(c)}{2}\int_{c-\gamma}^{c+\gamma} \frac{16\pi^2}{\gamma^2}\sin^6\left(\frac{\pi(x-c)}{\gamma}\right)\cos^2\left(\frac{\pi(x-c)}{\gamma}\right) dx \\
&< \frac{A(c)}{2}\frac{16\pi^2}{\gamma^2}\int_{c-\gamma}^{c+\gamma} dx \\
&= \frac{16\pi^2}{\gamma}A(c)
\end{aligned}$$

Since B is continuous, there exists N such that $|B(x)| < N, \forall x \in [c-\gamma, c+\gamma]$. Hence

$$\begin{aligned}
\int_{x_0}^{x_1} B(x)(\nu(x))^2 dx &= \int_{c-\gamma}^{c+\gamma} B(x)(\nu(x))^2 dx \\
&< N\int_{c-\gamma}^{c+\gamma} (\nu(x))^2 dx \\
&= N\int_{c-\gamma}^{c+\gamma} \sin^8\left(\frac{\pi(x-c)}{\gamma}\right) dx \\
&\leq 2\gamma N
\end{aligned}$$

Hence, $\delta^2 J(\nu, y) < \frac{16\pi^2}{\gamma}A(c) + 2N\gamma$. Hence, if γ is chosen small, we get $\delta^2 J(\nu, y) < 0$, a contradiction that y is a minimum.

Hence, $A(x) \geq 0, \forall \in [x_0, x_1]$. □

Example 4.42. *Suppose $x_0 = -1, x_1 = 1$ and*

$$J(y) = \int_{-1}^{1} x\sqrt{1 + (y')^2} dx$$

with $f(x,y,y') = x\sqrt{1+(y')^2}$.

We get

$$f_{y'} = x\frac{y'}{\sqrt{1+(y')^2}}$$

and

$$f_{y'y'}x\left(\frac{\sqrt{1+(y')^2} - y'\frac{y'}{\sqrt{1+(y')^2}}}{1+(y')^2}\right) = \frac{x}{(1+(y')^2)^{\frac{3}{2}}}$$

If $x \in [-1,0]$ *then* $f_{y'y'} \leq 0$.
If $x \in [1,0]$ *then* $f_{y'y'} \geq 0$.
Hence, there is no local minimum or maximum by Legendre condition.

Remark 4.43. *The Legendre condition is still a necessary (not sufficient) condition. Even the strengthened Legendre condition*

$$f_{y'y'} > 0$$

is not sufficient (see Example in textbook)

To get sufficient condition for $\delta^2 J(\eta,y) \geq 0$, we write

$$\delta^2 J(\eta,y) = \int_{x_0}^{x_1} f_{y'y'}y^2 dx$$

where $Y = Y(x,y,\eta)$ and ideally $Y = 0$ on $[x_0,x_1]$ if and only if $\eta = 0$ on $[x_0,x_1]$ so that the sign of $\delta^2 J(\eta,y)$ depends only on the sign of $f_{y'y'}$.

Recall $\delta^2 J(\eta,y) = \int_{x_0}^{x_1} f_{y'y'}(\eta')^2 + B\eta^2 dx$. If w smooth then

$$\int_{x_0}^{x_1} (w\eta^2)'\triangle x = [w\eta^2]_{x_0}^{x_1} = 0$$

hence, we can add such a term to $\delta^2 J(\eta,y)$ without changing its value.

Goal: Find w such that $f_{y'y'}(\eta')^2 + B\eta^2 + (w\eta^2)' = f_{y'y'}Y^2$, assume $f_{y'y'} > 0$ then

$$f_{y'y'}(\eta')^2 + B\eta^2 + (w\eta^2)' = f_{y'y'}\left((\eta')^2 + 2\frac{w}{f_{y'y'}}\eta\eta' + \frac{B+w'}{f_{y'y'}}\eta^2\right)$$

assume $w^2 = f_{y'y'}(B+w')$, (i.e. w solves an ODE) then

$$f_{y'y'}(\eta')^2 + B\eta^2 + (w\eta^2)' = f_{y'y'}\left(\eta; + \frac{w}{f_{y'y'}}\eta\right)^2$$

Definition 4.44. *We say that the second variation is positive definite (PD) if* $\delta^2 J(\eta,y) > 0, \forall \eta \in H\backslash\{0\}$.

We have $Y = \eta' + \frac{w}{f_{y'y'}}\eta$ where $Y = 0$ if and only if $0 = \eta' + \frac{w}{f_{y'y'}}\eta$ (ODE for η)

Lemma 4.45. $Y = 0$ *if and only if* $\eta = 0$

Proof. $(\Rightarrow) : \eta = 0 \implies \eta' = 0 \implies Y = \eta' + \ldots\eta = 0$.

$(\Leftarrow)$: We have $f_{y'y'}$ is smooth and > 0.

Picard's Theorem gives existence + uniqueness of

$$\begin{cases} \eta' = -\frac{w}{f_{y'y'}}\eta \\ \eta(x_0) = 0 \end{cases}$$

on $[x_0,x_1]$. But $\eta = 0$ is a solution. Hence $\eta = 0$ is the only solution. ∎

Lemma 4.46. *Let y be a smooth extremal for J s.t. $f_{y'y'} > 0$ on $[x_0, x_1]$. If there is a solution w to the ODE $w^2 = f_{y'y'}(B + w')$ defined on the whole interval $[x_0, x_1]$ then the second variation is PD.*

Remark 4.47. *The existence of local solution w is clear, but the existence of a global solution, i.e. defined on the whole interval $[x_0, x_1]$ is not clear.*

The ODE $w^2 = f_{y'y'}(w' + B)$ is an example of a class of equations called Ricati equations.

This is a 1st order nonlinear ODE, so we make it into 2nd order linear ODE by a transformation $w = -\frac{u'}{u} f_{y'y'}$

The ODE becomes

$$\begin{aligned}\left(\frac{u'}{u} f_{y'y'}\right)^2 &= f_{y'y'}\left(\frac{d}{dx}\left(-\frac{u'}{u} f_{y'y'}\right) + B\right)\\ &= f_{y'y'}\left(-\frac{1}{u}\frac{d}{dx}(u' f_{y'y'}) + u' f_{y'y'} \frac{1}{u^2} u' + B\right)\\ \implies& \frac{d}{dx}(u' f_{y'y'}) - B_n = 0\\ \implies& \frac{d}{dx}(f_{y'y'} u') - (f_{yy} - \frac{d}{dx} f_{y'y'})u = 0\end{aligned}$$

Definition 4.48. *The ODE*

$$\frac{d}{dx}(f_{y'y'} u') - (f_{yy} - \frac{d}{dx} f_{yy'})u = 0$$

is called Jacobi accessory equation.

Hence, we need to find a solution to the Jacobi accessory equation defined on $[x_0, x_1]$ s.t. $u(x) \neq 0, \forall x \in [x_0, x_1]$.

Definition 4.49. *Let $\kappa \in [x_0, x_1]$. If there exists a nontrivial solution u to the Jacobi accessory equation (i.e. if there exists $c \in [x_0, x_1]$ s.t. $u(c) \neq 0$) such that $u(x_0) = u(\kappa) = 0$ then κ is called a conjugate point to x_0.*

Lemma 4.50. *Assume there are no conjugate points to x_0 in $(x_0, x_1]$. Then there exists a solution u to the Jacobi accessory equation such that $u(x) \neq 0, \forall x \in [x_0, x_1]$.*

Theorem 4.51. *Let y be a smooth extremal for J such that $f_{y'y'} > 0, \forall x \in [x_0, x_1]$. If there are no points in the interval $[x_0, x_1]$ which are conjugate to x_0 then the second variation is PD.*

Proof. See textbook □

Example 4.52. *Let $J(y) = \int_{x_0}^{x_1} (y')^2 dx$. We get $f_{y'} = 2y' \implies f_{y'y'} = 2 > 0$. This gives the Jacobi accessory equation to be*

$$\frac{d}{dx}(2u') - (0 - \frac{d}{dx}2)u = 0 \implies u''(x) = 0$$

This has a general solution $u(x) = ax + b$. Hence, assume $\kappa \neq x_0$ is a conjugate point to x_0 we have $0 = u(x_0) = u(\kappa) \implies u = 0$ is the trivial solution.

Hence, there are no points conjugate to x_0.

Remark 4.53. *If we consider the second variation as a functional for η with y fixed, i.e.*

$$V(\eta) = \delta^2 J(\eta, y) = \int_{x_0}^{x_1} f_{y'y'}(\eta')^2 + B\eta^2 dx$$

Then the Jacobi accessory equation

$$\frac{d}{dx}(f_{y'y'} u') - Bu = 0$$

is the EL-equation for $V(\eta)$ (with u instead of η) but $\eta \in H \implies \eta(x_0) = \eta(x_1) = 0$ while we seek solutions u to the Jacobi accessory equation such that $u(x) \neq 0, \forall x \in [x_0, x_1]$

Lemma 4.54. *Let u be a solution to the Jacobi accessory equation in $[x_0, x_1]$ such that $u(x_0) = u(x_1) = 0$. Then*

$$\int_{x_0}^{x_1} f_{y'y'}(u')^2 + Bu^2 dx = 0$$

Proof. u solve the Jacobi accessory equation hence

$$0 = \int_{x_0}^{x_1} \left(\frac{d}{dx}(f_{y'y'}u') - Bu \right) u dx$$

and

$$\int_{x_0}^{x_1} \frac{d}{dx}(f_{y'y'}u') u dx = [uu' f_{y'y'}]_{x_0}^{x_1} - \int_{x_0}^{x_1} f_{y'y'}(u')^2 dx = -\int_{x_0}^{x_1} f_{y'y'}(u')^2 dx$$

□

Theorem 4.55. *Let f be a smooth function of x, y, y'. Let y be a smooth extremal for $J(y) = \int_{x_0}^{x_1} f(x, y, y')dx$ such that $f_{y'y'} > 0, \forall x \in [x_0, x_1]$.*

Then

1. *If $\delta^2 J(\eta, y) > 0, \forall \eta \in H\{0\}$ then there is no point conjugate to x_0 in $(x_0, x_1]$.*
2. *If $\delta^2 J(\eta, y) \geq 0, \forall \eta \in H$ then there is no point conjugate to x_0 in (x_0, x_1).*

Proof. Assume $\delta^2 J(\eta, y)$ is PD, we show x_1 cannot be conjugate to x_0.

Assume x_1 is conjugate to x_0, meaning there exists a solution to the Jacobi accessory equation such that $u(x_0) = u(x_1) = 0$ but u is non-trivial. This means $u \neq 0, u \in H$.

Hence $\delta^2 J(\eta, y) = \int_{x_0}^{x_1} f_{y'y'}(u')^2 + Bu^2 dx = 0 \implies \delta^2 J(\eta, y)$ is not PD.

The rest of the proof see textbook. □

Theorem 4.56 (Jacobi necessary condition)**.** *Let y be a smooth extremal for J such that $f_{y'y'} > 0, \forall x \in [x_0, x_1]$.*

If y is a local minimum for J in S then there are no points conjugate to x_0 in (x_0, x_1)

Proof. Since y is a minimum we have $\delta^2 J(\eta, y) \geq 0, \forall \eta \in H$.

By Theorem 4.55, there is no point that is conjugate to x_0 in (x_0, x_1). □

Remark 4.57. *For the second variation, we have proved*

- *y is a local minimum $\implies \delta^2 J(\eta, y) \geq 0, \forall \eta \in H$*
- *y is a local minimum $\implies f_{y'y'}(x, y(x), y'(x) \geq 0, \forall x \in [x_0, x_1]$*
- *y is a local minimum, $f_{y'y'} > 0$, there is a solution w to $w^2 = f_{y'y'}(w' + B)$ defined on $[x_0, x_1]$. This means $\delta^2 J(\eta, y) > 0, \forall \eta \in H \backslash \{0\}$.*
- *y is a local minimum, $f_{y'y'} > 0$, no points in interval $(x_0, x_1]$ are conjugate to $x_0 \implies \delta^2 J(\eta, y) > 0, \forall \eta \in H \backslash \{0\}$*
- *y is a local minimum, $f_{y'y'} > 0 \implies$ no points conjugate to x_0 in (x_0, x_1)*

But those are necessary conditions, not sufficient.

Recall: if y is an extremal, $\hat{y} = y + \epsilon\eta$. We get

$$J(\hat{y}) - J(y) = \frac{\epsilon^2}{2}\delta^2 J(\eta, y) + \mathcal{O}(\epsilon^3)$$

If y is a minimum we have $J(y) \leq J(\hat{y}) \implies \delta^2 J(\eta, y) \geq 0, \forall \eta \in H$.

But if there exists $\eta \neq 0$ such that $\delta^2 J(\eta, y) = 0 \implies \mathcal{O}(\epsilon^3)$ term controls the sign of $J(\hat{y}) - J(y)$

Recall: for finite dimensional optimisation we required the second variation to PD not just PSD to get a sufficient condition. But even if $\delta^2 J(\eta, y) > 0, \forall \eta \in H\{0\}$, there might be $\eta \neq 0$ such that $\delta^2 J(\eta, y)$ has order $\epsilon \implies$ the sign of $J(\hat{y}) - J(y)$ depends also on higher order terms.

Hence we need to control the magnitude of $\delta^2 J(\eta, y)$ relative to the remainder term.

This means $\delta^2 J(\eta, y)$ is PD is not a sufficient condition for a minimum.

Definition 4.58. *Let $X = C^2([x_0, x_1])$ define $\|\cdot\|_1$ by*

$$\|y\|_1 = \sup_{x \in [x_0, x_1]} |y(x)| + \sup_{x \in [x_0, x_1]} |y'(x)|$$

$\|\cdot\|_1$ is a norm on X. We also define $\|\cdot\|_0$ by

$$\|y\|_0 = \sup_{x \in [x_0, x_1]} |y(x)|$$

Definition 4.59. *Let $X = C^2([x_0, x_1])$. We say that $J : X \to \mathbb{R}$ has a weak local minimum at $y \in S$ if $\exists \delta > 0$ such that $J(\hat{y}) \geq J(y), \forall \hat{y} \in S$ such that $\|\hat{y} - y\|_1 < \delta$. We say that $J : X \to \mathbb{R}$ has a strong local minimum at $y \in S$ if $\exists \delta > 0$ such that $J(\hat{y}) \geq J(y), \forall \hat{y} \in S$ such that $\|\hat{y} - y\|_0 < \delta$.*

Remark 4.60. *Clearly y is a weak local minimum $\implies$ y is a strong local minimum as*

$$\|\hat{y} - y\|_0 \leq \|\hat{y} - y\|_1$$

But the converse is not true.

Lemma 4.61. *There exists functions ν, ρ such that for $\hat{y} = y + \epsilon\eta$*

$$J(\hat{y}) - J(y) = \epsilon^2 \left(\frac{1}{2}\delta^2 J(\eta, y) + \int_{x_0}^{x_1} \nu(n')^2 + \rho\eta^2 dx \right)$$

where $\nu, \rho \to 0$ as $\|\eta\|_1 \to 0$

Proof. See textbook Geffand & Fomin □

Theorem 4.62. *Let $y \in S$ be an extremal for J and assume that $f_{y'y'}(x, y(x), y'(x)) > 0, \forall x \in [x_0, x_1]$ and assume that there are no conjugate points to x_0 in $(x_0, x_1]$.*

Then J has a weak local minimum in S at y.

Proof. Let $\mu \in \mathbb{R}$ be a parameter, define

$$\begin{aligned} K(\mu) &= \int_{x_0}^{x_1} f_{y'y'}(\eta')^2 + B\eta^2 dx - \mu^2 \int_{x_0}^{x_1} (\eta')^2 dx \\ &= \int_{x_0}^{x_1} (f_{y'y'} - \mu^2)(\eta')^2 + B\eta^2 dx \end{aligned}$$

where K has Jacobi accessory equation

$$\frac{d}{dx}((f_{y'y'} - \mu^2)u') - Bu = 0$$

Since f is smooth, and $f_{y'y'} > 0$ on $[x_0, x_1] \implies \exists \sigma$ such that $f_{y'y'} \geq \sigma, \forall x \in [x_0, x_1]$.
Hence, $\forall \mu^2 < \sigma$ we have $f_{y'y'} - \mu^2 > 0$.

Now the solutions $u(x, \mu)$ to the JA equation depend continuously on μ if $|\mu|$ is small. We know that for $\mu = 0$ there are no points conjugate to x_0 in $(x_0, x_1]$ by continuity of $u(x, \cdot)$ there are also no points conjugate to x_0 in $(x_0, x_1]$ for $u(\cdot, \mu)$ if $|\mu|$ small.

This means $\exists \mu_1 > 0$ s.t.

1. $f_{y'y'} - \mu^2 > 0$
2. $K(\mu)$ has no points conjugate to x_0 in $(x_0, x_1]$ for all μ s,t, $|\mu| < \mu_1$

By previous theorem, we have $K(\mu)$ is PD, $\forall |\mu| < \mu_1$.
This means $\delta^2 J(\eta, y) - \mu^2 \int_{x_0}^{x_1} (\eta')^2 dx = K(\mu) > 0$, if $|\mu| < \mu_1$ and $\eta \neq 0$.
This means there exists $p > 0$ s.t.

$$\delta^2 J(\eta, y) > p \int_{x_0}^{x_1} (\eta')^2 dx$$

Now remainder:

$$R = \int_{x_0}^{x_1} \nu(\eta')^2 + \rho\eta^2 dx$$

because $\nu, \rho \to 0$ as $\|\eta\|_1 \to 0 \implies \exists q$ s.t. $q \to 0$ as $\|\eta\|_1 \to 0$ and $|R| \leq q \left(\int_{x_0}^{x_1} (\eta')^2 + \eta^2 dx \right)$
Recall the Cauchy Schwarz inequality, if g, h are continuous functions then

$$\langle g, h \rangle^2 = \left(\int_{x_0}^{x_1} g(x)h(x) \right)^2 = \int_{x_0}^{x_1} g(x)^2 dx \int_{x_0}^{x_1} h(x)^2 dx$$

We have

$$\eta(x) = \int_{x_0}^{x} \eta'(t) dt$$

Hence

$$\eta^2(x) = \left(\int_{x_0}^{x} \eta'(t) dt \right)^2$$

Use the Cauchy Schwarz inequality with $g = 1, h = \eta'$ gives

$$\eta^2(x) = \left(\int_{x_0}^{x} \eta'(t) \cdot 1 dt \right)^2 \leq \left(\int_{x_0}^{x} 1^2 dt \right) \left(\int_{x_0}^{x} (\eta')^2 dt \right) = (x - x_0) \int_{x_0}^{x} (\eta')^2 dt \leq (x - x_0) \int_{x_0}^{x_1} (\eta')^2 dt$$

Hence, we get

$$\int_{x_0}^{x_1} \eta^2(x) dx \leq \frac{(x_1 - x_0)^2}{2} \int_{x_0}^{x_1} (\eta')^2(t) dt$$

and therefore

$$|R| \leq |q| \left(1 + \frac{(x_1 - x_0)^2}{2} \right) \int_{x_0}^{x_1} (\eta')^2 dt$$

If $\|\eta\|_1$ sufficiently small we have

$$|q| \left(1 + \frac{(x_1 - x_0)^2}{2} \right) < \frac{p}{2}$$

Therefore,

$$J(\hat{y}) - J(y) > \epsilon^2 \left(\int_{x_0}^{x_1} (\eta')^2 dx - \frac{p}{2} \int_{x_0}^{x_1} (\eta')^2 dx \right) = \epsilon^2 \frac{p}{2} \int_{x_0}^{x_1} (\eta')^2 dx > 0$$

Hence, $J(\hat{y}) - J(y) > 0$ if $\|\eta\|_1$ is sufficiently small and if $\eta \neq 0$.
Hence, y is a weak local minimum. □

Now, how to find conjugate points (besides using the definition). How to relate solutions to the EL-equation with solutions to the JA-equation?

Here, we assume f is smooth, y is a smooth solution to the EL equation

$$\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) - \frac{\partial f}{\partial y} = 0$$

the EL-equation is a second order ODE for y. Hence the solution y depends on two parameters c_1, c_2 (those are determined through the boundary conditions $y(x_0) = y_0, y(x_1) = y_1$.

Consider $y = y(x, c_1, c_2)$ as a smooth function of x, c_1, c_2.
Hence, $f(x, y(x, c_1, c_2), y'(x, c_1, c_2))$ depends smoothly on c_1, c_2.
Hence, differentiate the EL equation with respect to c_1 we get

$$\begin{aligned}
0 &= \frac{\partial}{\partial c_1}\left(\frac{d}{dx}\frac{\partial f}{\partial y'} - \frac{\partial f}{\partial y}\right) \\
&= \frac{d}{dx}\left(\frac{\partial}{\partial y'}\left(\frac{\partial f}{\partial y}\frac{\partial y}{\partial c_1} + \frac{\partial f}{\partial y'}\frac{\partial y'}{\partial c_1}\right)\right) - \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\frac{\partial y}{\partial c_1} + \frac{\partial f}{\partial y'}\frac{\partial y'}{\partial c_1}\right) \\
&= \frac{d}{dx}\left(\frac{\partial^2 f}{\partial y' y}\frac{\partial y}{\partial c_1} + \frac{\partial^2 f}{\partial (y')^2}\frac{\partial y'}{\partial c_1}\right) - \frac{\partial^2 f}{\partial y' y}\frac{\partial y}{\partial c_1} + \frac{\partial^2 f}{\partial (y')^2}\frac{\partial y'}{\partial c_1}
\end{aligned}$$

Now, define

$$u_1 = \frac{\partial y}{\partial c_1}$$

then

$$u_1' = \frac{\partial y'}{\partial c_1}$$

Hence, we get the JA equation to be

$$\begin{aligned}
0 &= \frac{d}{dx}(f_{y'y}u_1 + f_{y'y'}u_1') - (f_{y'y}u_1 + f_{y'y'}u_1') \\
&= \frac{d}{dx}(f_{y'y}u_1) + \frac{d}{dx}(f_{y'y'}u_1') - f_{yy}u_1 - f_{y'y}u_1' \\
&= \frac{d}{dx}(f_{y'y})u_1 + f_{y'y}u_1' + \frac{d}{dx}(f_{y'y'}u_1') - f_{yy}u_1 - f_{y'y}u_1' \\
&= \frac{d}{dx}(f_{y'y'})u_1' - (f_{yy} - \frac{d}{dx}f_{y'y})u_1
\end{aligned}$$

Hence u_1 solves the JA euqation. Similarly, $u_2 = \frac{\partial y}{\partial c_2}$ solves the JA euqation.
Therefore, we get solutions to the JA equation by differentiating solutions to the EL equation

Lemma 4.63. *$\{u_1, u_2\}$ where $u_i = \frac{\partial y}{\partial c_i}$ for $i = 1, 2$ forms a basis for the solution space of the JA euqation.*

Proof. See Brechtken-Mandersheid : Introduction to the Calculus of Variations. □

We now first write $c = (c_1, c_2), k = (k_1, k_2)$ and assume that $y(x, k)$ solves the EL-equation and that $y(x, k) \in S$ (i.e. k is such that $y(x_0, k) = y_0$ and $y(x_1, k) = y_1$) and define

$$u_1(x,k) = \left.\frac{\partial y}{\partial c_1}\right|_{c=k}$$
$$u_2(x,k) = \left.\frac{\partial y}{\partial c_2}\right|_{c=k}$$

This means the gernal solution to the JA-equation is $u(x, k) = \alpha u_1(x, k) + \beta u_2(x, k)$.

If we want u to be nontrivial then $\alpha, \beta \neq 0$.

Assume that κ conjugate to x_0 we have

$$0 = u(x_0, k) = \alpha u_1(x_0, k) + \beta u_2(x_0, k)$$

But we also have

$$0 = u(\kappa, k) = \alpha u_1(\kappa, k) + \beta u_2(\kappa, k)$$

Rewrite this we get

$$u_2(\kappa, k)u_1(x_0, k) = u_2(x_0, k)u_1(\kappa, k)$$

which characterises conjugate points.

Example 4.64. *Consider*

$$J(y) = \int_0^L (y')^2 - y^2 dx$$

with $L > 0$ fixed.

The EL-equation gives

$$0 = \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) - \frac{\partial f}{\partial y} = \frac{d}{dx}(2y') + 2y = 2y'' + 2y$$

Solve $y'' = -y$ we get $y(x) = c_1 \cos(x) + c_2 \sin(x)$.

We also get $u_1(x) = \frac{\partial y}{\partial c_1}(x) = \cos(x), u_2(x) = \frac{\partial y}{\partial c_2}(x) = \sin(x)$ we have $x_0 = 0$. If κ is conjugate point to 0, we get

$$\begin{aligned}
&u_2(\kappa)u_1(x_0) = u_2(x_0)u_1(\kappa) \\
\implies &\sin(\kappa)\cos(0) = \sin(0)\cos(\kappa) \\
\implies &\sin(\kappa) = 0 \\
\implies &\kappa = n\pi, n \in \mathbb{Z}
\end{aligned}$$

Example 4.65. *Consider*

$$J(y) = \int_0^1 \sqrt{1 + (y')^2} dx$$

and $x_0 = 0, x_1 = 1$. The EL equation gives

$$0 = \frac{d}{dx}\left(\frac{y'}{\sqrt{1 + (y')^2}}\right)$$

This gives

$$\begin{aligned}
&c = \frac{y'}{\sqrt{1 + (y')^2}} \\
\implies &\frac{c^2}{1 - c^2} = (y')^2 \\
\implies &y'(x) = c_1 \\
\implies &y(x) = c_1 x + c_2
\end{aligned}$$

This gives $u_1(x) = x, u_2(x) = 1$.
If κ is a conjugate point to $x_0 = 0$. Then

$$u_2(\kappa)u_1(0) = u_2(0) = u_1(\kappa) \implies 0 = \kappa$$

This shows there are no conjugate points to x_0.

4.4 Convexity

What about convexity for infinite-dimensional optimisation problems? Assuming $f \in C^2([x_0, x_1] \times \mathbb{R} \times \mathbb{R})$.

Definition 4.66. *The function $J : X \to \mathbb{R}$ defined by*

$$J(y) = \int_{x_0}^{x_1} f(x, y, y')dx$$

is convex if $\forall y, \overline{y} \in X, t \in [0, 1]$ we have

$$J(ty + (1-t)\overline{y}) \leq tJ(y) + (1-t)J(\overline{y})$$

Remark 4.67. *J is convex means that $\forall x \in [x_0, x_1]$*

$$f(x, t_y(x) + (1-t)\overline{y}(x), ty'(x) + (1-t)\overline{y}'(x)) \leq tf(x, y(x), y'(x)) + (1-t)f(x, \overline{y}(x), \overline{y}(x)')$$

I.e. if $f(x, y, y')$ is convex for all x as a function on $\Omega_X = \{(y, y') \in \mathbb{R}^2\}$ (that is, the function $\Omega_X \to \mathbb{R}$ where $(y, y') \mapsto f(x, y, y'), \forall x \in [x_0, x_1]$ is convex)

Remark 4.68. *If f is only defined on $D_f \subseteq \mathbb{R}^3$ then we define*

$$\Omega_X = \{(y, y') \in \mathbb{R}^2 : (x, y, y') \in D_f\}$$

and then we require Ω_X to be convex in the above definitions in order for f and J to be convex.

Now, how to check if f is convex on Ω_X?

Theorem 4.69. *Let Ω_X be convex, $f \in C^2$ then $\forall x \in [x_0, x_1]$ the function $f(x, \cdot, \cdot)$ is convex on Ω_X if and only if $\forall (y, w) \in \Omega_X$*

$$\begin{aligned} f_{yy}(x, y, w) &\geq 0 \\ f_{ww}(x, y, w) &\geq 0 \\ f_{yy}(x, y, w)f_{ww}(x, y, w) - f_{yw}(x, y, w)^2 &\geq 0 \end{aligned}$$

Proof. This directly follows from Sylvester's Criterion □

Theorem 4.70. *Assume that $\forall x \in [x_0, x_1]$ the set Ω_X is convex and that $\forall x \in [x_0, x_1]$ the function $f(x, \cdot, \cdot)$ is convex on Ω_X. If y is a smooth extremal for J, then y is a minimum of J over S*

Proof. Let $\hat{y} \in S$ be such that $(\hat{y}(x), \hat{y}'(x)) \in \Omega_X$ then

$$f(x, \hat{y}(x), \hat{y}'(x)) - f(x, y(x), y'(x)) \geq f_y(x, y(x), y'(x))(\hat{y}(x) - y(x)) + f_{y'}(x, y(x), y'(x))(\hat{y}'(x) - y'(x))$$

Hence,

$$\begin{aligned} J(\hat{y}) - J(y) &= \int_{x_0}^{x_1} f(x, \hat{y}(x), \hat{y}'(x)) - f(x, y(x), y'(x))dx \\ &\geq \int_{x_0}^{x_1} (\hat{y}(x) - y(x))f_y(x, y(x), y'(x)) + (\hat{y}'(x) - y'(x))f_{y'}(x, y(x), y'(x))dx \\ &= \int_{x_0}^{x_1} (\hat{y} - y)(f_y(x, y, y') - \frac{d}{dx}(f_{y'}(x, y, y')))dx \\ &= 0 \end{aligned}$$

This shows that y is a minimum □

Remark 4.71. *Theorem 4.70 is the FOSC for infinite-dimensional optimisation.*

This theorem shows that J is convex y solves the EL-equation gives that y is the minimum of J.

The previous theorem also holds if $y \in C^2(D, \mathbb{R})$ where $D \subseteq \mathbb{R}^n$, i.e. if $y = y(x_1, \ldots, x_n)$, if f is a convex function of $y, y_{x_1}, \ldots, y_{x_n}$

Example 4.72. *Consider $D \subseteq \mathbb{R}^2$ a domain (open, bounded, smooth boundary). Consider $X = C^2(D, \mathbb{R})$ and $J(u) = \int_D |\nabla_w|^2 dxdy = \int_D u_x^2 + u_y^2 dxdy$ then $f(x, u, u_x, u_y) = u_x^2 + u_y^2$.*

We then have

$$\begin{aligned}
&f((tu + (1-t)\overline{u})_X, (tu + (1-t)\overline{u})_Y) \\
=&f(tu_X + (1-t)\overline{u}_X, tu_Y + (1-t)\overline{u}_Y) \\
=&(tu_X + (1-t)\overline{u}_X)^2 + (tu_Y + (1-t)\overline{u}_Y)^2 \\
\leq&tu_X^2 + (1-t)\overline{u}_X^2 + tu_Y^2 + (1-t)\overline{u}_Y^2 \\
=&tf(u_X, u_Y) + (1-t)f(\overline{u}_X, \overline{u}_Y)
\end{aligned}$$

This shows J is convex as f is convex.

We know that EL equation for J is given by

$$\begin{cases} \triangle u = 0 \\ u|_{\delta D} = u_0 \end{cases}$$

Hence, by theorem, if u minimisers the EL-equation it is a minimiser of J subject to the boundary condition.

Example 4.73. *Let $J(y) = \int_0^1 \sqrt{1 + (y')^2} dx$, clearly $\Omega_X = \mathbb{R}^2, \forall X$.*

NOte that $f_{yy} = 0 = f_{yy'}$. Moreover, $f_{y'y'} = \frac{1}{(1+(y')^2)^{1.5}} > 0$.

This shows f is convex. Hence J is convex.

Hence, the solutions of the EL equations which are straight lines are minimisers between two points.

Remark 4.74. *Similar to the finite-dimensional case, strict convexity of J (and f), i.e. $J(ty + (1-t)\overline{y}) < tJ(y) + (1-t)J(\overline{y}), \forall y \neq \overline{y} \in X, t \in (0,1)$ genereates uniqueness of the minimiser.*

4.5 Isoperimetric problems

Setting: Consider $X = C^2([x_0, x_1]), J : X \to \mathbb{R}$ by

$$J(y) = \int_{x_0}^{x_1} f(x, y(x), y'(x)) dx$$

with f sufficiently smooth.

Now we consider the problem

$$\min_{\text{s.t. } y \in S = \{y \in X : y(x_0) = y_0, y(x_1) = y_1\} \wedge I(y) = L} J(y)$$

where $L \in \mathbb{R}$ is fixed and $I(y) = \int_{x_0}^{x_1} g(x, y(x), y'(x)) dx$ with g given sufficiently smooth.

Definition 4.75. *The condition $I(y) = L$ is called isoperimetric condition.*

The above minimisation problem is called an isoperimetric problem.

Idea: Take y s.t. J has an extremum at y and take $\eta \in H = \{\eta \in X : \eta(x_0) = \eta(x_1) = 0\}$ and consider a perturbation $\hat{y} = y + \epsilon\eta$.

But $\hat{y} \in S$, we also need $I(\hat{y}) = L \implies$ need to determine which $\eta \in H$ also fulfills this condition, we can not just take η arbitrarily hence instead we consider $\hat{y} = y + \epsilon_1\eta_1 + \epsilon_2\eta_2$ s.t. $\eta_1, \eta_2 \in H$, i.e. $\eta_1(x_0) = \eta_1(x_1) = \eta_2(x_0) = \eta_2(x_1) = 0$ and s.t. ϵ_1, ϵ_2 are small.

We pick $\eta_1 \in H$ arbitrarily and then we pick $\eta_2 \in H$ s.t. $I(\hat{y}) = L$, i.e. η_2 corrects so that the isoperimetric condition is satisfied.

But if η_1 is arbitrary, can we always find η_2, ϵ_2 such that $I(\hat{y}) = L$

Example 4.76. *Consider* $I(y) = \int_0^1 \sqrt{1+(y')^2}dx = \sqrt{2}$ *and* $S = \{y \in X = C^2([0,1])$ *s.t.* $y(0) = 0, y(1) = 1\}$*, we know* $g(y') = \sqrt{1+(y')^2}$ *is strictly convex and we have shown that* $\overline{y}(x) = x$ *is a minimum for* I *in* S *with* $I(\overline{y}) = \sqrt{2}$*.*

Hence if $y \in S \implies I(y) > \sqrt{2}$ *unless* $y = \overline{y}$*.*

Therefore, there are no non-trivial $\epsilon_1, \epsilon_2, \eta_1, \eta_2$ *s.t.* $I(\hat{y}) = I(\overline{y} + \epsilon_1\eta_1 + \epsilon_2\eta_2) = \sqrt{2}$*.*

Definition 4.77. *An extremal of* $J, y \in X$ *s.t.* y *cannot be varied subject to the isoperimetric constraints, i.e. s.t. there are no non-trivial* $\epsilon_1, \epsilon_2, \eta_1, \eta_2$ *with* $\hat{y} \in S$ *and* $I(\hat{y}) = L$ *is called rigid extremal.*

We fix $\eta_1, \eta_2 \in H$ and define

$$\Xi(\epsilon_1, \epsilon_2) = I(y + \epsilon_1\eta_1 + \epsilon_2\eta_2)$$

and

$$\Theta(\epsilon_1, \epsilon_2) = J(y + \epsilon_1\eta_1 + \epsilon_2\eta_2)$$

Since g is smooth, Ξ is also a smooth function.

If J has an extremum at y subject to the constraints then

$$\Xi(0,0) = L$$

If $\nabla\phi(0,0) \neq 0$ then by the implicit function theorem if $\max(|\epsilon_1|, |\epsilon_2|))$ is sufficiently small, there exists a curve $\epsilon_2 = \epsilon_2(\epsilon_1$ s.t.

$$\Xi(\epsilon_1, \epsilon_2(\epsilon_1)) = L, \forall \epsilon_1 \text{ small}$$

Hence if y is a rigid extremal then $\nabla\Xi(0,0) = 0$.

Definition 4.78. *We call* $\nabla\Xi(0,0) \neq 0$ *the non-rigidity condition.*

Lemma 4.79. *Assume that* J *has an extremum at* y *subject to the constraints and that the non-rigidity condition holds. Then* $\exists\lambda \in \mathbb{R}$ *s.t.*

$$\nabla(\Theta(\epsilon_1, \epsilon_2) - \lambda(\Xi(\epsilon_1, \epsilon_2) - L))\Big|_{(0,0)} = 0$$

i.e. $\nabla\Theta(0,0) - \lambda\nabla\Xi(0,0) = 0$

Proof. The fact Θ has an extremum at $(0,0)$ subject to the condition $\Xi(\epsilon_1\epsilon_2) - L = 0$ the non-rigidity condition $\nabla\Xi(0,0) \neq 0 \implies (0,0)$ is a regular point.

Hence the theory of finite-dimensional optimisation asserts $\exists\lambda$ langrange multiplier s.t. the statement holds $\square$

We have

$$\frac{\partial}{\partial\epsilon_i}\left(\Theta(\epsilon_1, \epsilon_2) - \lambda\Xi(\epsilon_1, \epsilon_2)\right)\Big|_{(0,0)} = 0 \qquad i = 1, 2$$

We have

$$\begin{aligned}\frac{\partial}{\partial\epsilon_1}\Theta(\epsilon_1, \epsilon_2)\Big|_{(0,0)} &= \int_{x_0}^{x_1} \frac{\partial}{\partial\epsilon_1} f(x, y, y + \epsilon_1\eta_1 + \epsilon_2\eta_2, y' + \epsilon_1\eta_1' + \epsilon_2\eta_2')dx\Big|_{(0,0)} \\ &= \int_{x_0}^{x_1} \eta_1\frac{\partial f}{\partial y}(x, y, y') + \eta_1'\frac{\partial f}{\partial y'}(x, y, y')dx\end{aligned}$$

Using integration by parts gives

$$\frac{\partial}{\partial \epsilon_1}\Theta(\epsilon_1,\epsilon_2)\bigg|_{(0,0)} = \int_{x_0}^{x_1} \eta_1 \left(\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right) dx$$

Similarly,

$$\frac{\partial}{\partial \epsilon_2}\Xi(\epsilon_1,\epsilon_2)\bigg|_{(0,0)} = \int_{x_0}^{X_!} \eta_1 \left(\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) - \frac{\partial f}{\partial y} - \lambda\left(\frac{d}{dx}\left(\frac{\partial g}{\partial y'}\right) - \frac{\partial g}{\partial y}\right)\right) dx$$

Since η_1 is arbitrary,

$$\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) - \frac{\partial f}{\partial y} = 0$$

with $F = f - \lambda g$, we equally get

$$0 = \frac{\partial}{\partial \epsilon_2}\left(\Theta(\epsilon_1,\epsilon_2) - \lambda\Xi(\epsilon_1,\epsilon_2)\right)\bigg|_{(0,0)} = \int_{x_0}^{x_1} \eta_2 \left(\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y}\right) dx$$

so that the condition

$$0 = \frac{\partial}{\partial \epsilon_2}\left(\Theta(\epsilon_1,\epsilon_2) - \lambda\Xi(\epsilon_1,\epsilon_2)\right)\bigg|_{(0,0)}$$

does not overdetermine the problem if y is a rigid extremal then $\nabla\Xi(0,0) = 0$ hence

$$0 = \frac{\partial}{\partial \epsilon_i}\Xi(\epsilon_1,\epsilon_2)\bigg|_{(0,0)} \qquad i = 1,2$$

This gives

$$0 = \int_{x_0}^{x_1} \eta_1 \left(\frac{\partial g}{\partial y} - \frac{d}{dx}\left(\frac{\partial g}{\partial y'}\right)\right) dx = \int_{x_0}^{x_1} \eta_2 \left(\frac{\partial g}{\partial y} - \frac{d}{dx}\left(\frac{\partial g}{\partial y'}\right)\right) dxdx$$

Hence,

$$0 = \frac{d}{dx}\left(\frac{\partial g}{\partial y'}\right) - \frac{\partial g}{\partial y} \implies y \text{ is an extremal for } I$$

Theorem 4.80. *Assume J has an extremum at $y \in X$ subject to the constraints*

$$y \in S$$
$$I(y) = L$$

Suppose that y is not an extremal for I then there exists $\lambda \in \mathbb{R}$ such that

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0$$

with $F = f - \lambda g$ holds

Remark 4.81. *The isoperimetric problem reduces to solving the unconstrainted (i.e. without isoperimetric constraint) problem with fixed end-points (i.e. $y \in S$) for F instead of f.*

The solution to

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0$$

will have 2 constrants and depend on λ.

Boundary conditions $y(x_0) = y_0, y(x_1) = y_1$ and the isoperimetric constraint $I(y) = L$ determine the 3 constants.

The non-rigidity condition $\nabla\Xi(0,0) \neq 0$ is equivalent condition to $(0,0)$ being regular (in constrained optimisation).

We also need to check that the solution to

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0$$

is not an extremal for I, i.e. does not solve

$$\frac{d}{dx}\left(\frac{\partial g}{\partial y'}\right) - \frac{\partial g}{\partial y} = 0$$

to apply the previous theorem

Example 4.82 (Catenary)**.** *Assume we have two poles of height h that are distance 1 apart, we want to span a flexible cable of length L between them*

The cable always hangs down, where $y \in C^2([0,1]) = X$ describes the position (i.e. height) of the cable at point $x \in [0,1]$.

Now, we want to minimiser

$$\min_{s.t.\ y \in S, I(y) = L} J(y) = \int_0^1 y(x)\sqrt{1 + (y'(x))^2}dx$$

where $S = \{y \in X : y(0) = h = y(1)\}, I(y) = \int_0^1 \sqrt{1+(y')^2}dx$ for some $L > 1$.

We know that extremals for I are line segments but if $L > 1$ then no solution to

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0$$

fulfilling $y \in S, I(y) = L$ can be extremal for I. Hence, non-rigidity condition is fulfilled.

Assume y is a minimum of J such that conditions fulfilled. Then

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y}) = 0$$

where $F = f - \lambda g = y\sqrt{1+(y')^2} - \lambda\sqrt{1+(y')^2}$

F does not depend on x hence const $= H = y'F_{y'} - F = \frac{(y-\lambda)(y')^2}{\sqrt{1+(y')^2}} - (y-\lambda)\sqrt{1+(y')^2}$

Now define $u = y - \lambda \implies u' = y'$

We get

$$\begin{aligned}
const &= \frac{u(u')^2}{\sqrt{1+(u')^2}} - u\sqrt{1+(u')^2} \\
\implies c^2 &= \frac{u^2(u')^4}{1+(u')^2} - 2u^2(u')^2 + u^2(1+(u')^2) \\
\implies c^2 &= \frac{1}{1+(u')^2}(u^2(u')^4 + u^2(1-(u')^4)) = \frac{1}{1+(u')^2u^2}
\end{aligned}$$

Since $L > 1 \implies y$ is not constant. Hence, $u = y - \lambda \neq 0 \implies c \neq 0$.

By previous example,

$$u(x) = c_1 \cosh\left(\frac{x - c_2}{c_1}\right) \implies y(x) = \lambda + c_1\cosh\left(\frac{x-c_2}{c_1}\right)$$

Now define $\kappa_1 = c_1 (= c \neq 0), \kappa_2 = -\frac{c_2}{c_1}$.
We get

$$h = y(0) = \lambda + \kappa_1 \cosh(k_2) \implies h - \lambda = \kappa_1 \cosh(\kappa_2)$$
$$h = y(1) = \lambda + \kappa_1 \cosh(\frac{1}{\kappa_1} + \kappa_2) \implies h - \lambda = \kappa_1 \cosh(\frac{1}{\kappa_2} + \kappa_2)$$

This gives

$$\cosh(\kappa_2) = \cosh(\frac{1}{\kappa_1} + \kappa_2) \implies \kappa_2 = -\frac{1}{\kappa_1} - \kappa_2 \implies 2\kappa_2 = -\frac{1}{\kappa_1} \implies \kappa_2 = -\frac{1}{2\kappa_1}$$

This gives

$$y(x) = \lambda + \kappa_1 \cosh(\frac{x}{\kappa_1} - \frac{1}{2\kappa_1})$$

Hence, we get

$$\begin{aligned} L = \int_0^1 \sqrt{1 + (y')^2} dx &= \int_0^1 \left(1 + \sinh^2(\frac{x}{\kappa_1} + \kappa_2)\right)^{\frac{1}{2}} dx \\ &= \int_0^1 \cosh(\frac{x}{\kappa_1} + \kappa_2) dx \\ &= \kappa_1 \sinh\left(\frac{x}{\kappa_1} + \kappa_2\right)\Big|_0^1 \\ &= \kappa_1 2 \sinh(\frac{1}{2\kappa_1}) \end{aligned}$$

This means we want to solve

$$\frac{L}{2\kappa_1} = \sinh\left(\frac{1}{2\kappa_1}\right)$$

define $\xi = \frac{1}{2\kappa_1} \implies L\xi = \sinh(\xi)$
The equation is solved by $\xi = 0 \implies \kappa_1 = \infty \implies y = \infty$ *obviously is not a solution.*
Since $L > 0 \implies L\xi = \sinh(\xi)$ *has two solutions* $\pm\hat{xi} \neq 0$.
This shows $\kappa_1 = \frac{1}{2\hat{\xi}} \implies \kappa_2 = -\hat{\xi}$.
Hence, $h - \lambda = \kappa_1 \cosh(\kappa_2) \implies \lambda = h - \frac{1}{2\hat{\xi}} \cosh(\hat{\xi})$.
Hence, $y(x) = h + \frac{1}{2\hat{\xi}} \left(\cosh(\hat{\xi}(2x-1)) - \cosh(\hat{\xi})\right)$
Hence, $y(x) \leq h$ *which makes physically sense.*

Example 4.83 (Dido's problem)**.** *We want to find* $y \in C^2([-1,1])$ *such that* $y(-1) = y(1) = 0$ *such that the perimeter of the curve described by* y *is equal to* $L = \sqrt{2}$ *and such that the area between* $y([-1,1])$ *and* $[-1,1]$ *is maximal.*

The solution will be a circle.

We specifically want to find

$$\max_{s.t.\ y \in S, I(y) = \sqrt{2}} J(y)$$

where $J(y) = \int_{-1}^1 y(x)dx, S = \{y \in C^2([-1,1]) : y(-1) = 0 = y(1)\}, I(y) = \int_{-1}^1 \sqrt{1 + (y')^2}dx$

Since $L > 2$, *recall that extremals of* I *have form* $y(x) = c_1 x + c_2 \implies y'(x) = c_1$ *but also* $0 = y(-1) = -c_1 + c_2 \implies c_1 = c_2$. *Moreover,* $0 = y(1) = c_1 + c_2 \implies 2c_1 \implies c_1 = 0 \implies y = c_2 \implies y'(0) \implies I(y) = 2 < L$.

Hence, the extremals of J *subject to conditions cannot be extremals of* I. *Hence, non-rigidity condition holds.*

Now, $F = f - \lambda g = y - \lambda\sqrt{1+(y')^2} \implies$

$$0 = \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = \frac{d}{dx}\left(-\lambda\frac{y'}{\sqrt{1+(y')^2}}\right) - 1 \implies \frac{y''}{(1+(y')^2)^{\frac{3}{2}}} + \frac{1}{\lambda} = 0$$

Note: F *does not depend on* x*, hence*

$$c_1 = H = y'F_{y'} - F = y'\left(-\lambda\frac{y'}{\sqrt{1+(y')^2}}\right) - (y - \lambda\sqrt{1+(y')^2})$$

$$\implies \lambda = (y+c_1)\sqrt{1+(y')^2}$$

$$\implies y'(x) = \sqrt{\frac{\lambda^2}{(y+c_1)^2} - 1}$$

$$\implies \int \frac{y+c_1}{\sqrt{\lambda^2 - (y+c_1)^2}}dy = x + c_2$$

Take $y + c_1 = \lambda\sin(\phi) \implies \frac{dy}{d\phi} = \lambda\cos(\phi) \implies x + c_2 = \int \frac{\lambda\sin(\phi)}{\sqrt{\lambda^2 - \lambda^2\sin^2(\phi)}}\lambda\cos(\phi)d\phi = \int \frac{\sin(\phi)}{\cos(\phi)}\lambda\cos(\phi)d\phi = \lambda\int\sin(\phi)d\phi = -\lambda\cos(\phi)$

We get

$$y = \lambda\sin(\phi) - c_1$$
$$x = -\lambda\cos(\phi) - c_2$$

This describes a circle centered at $(-c_2, -c_1)$*, specifically, we get*

$$(x+c_2)^2 + (y+c_1)^2 = \lambda^2$$

Since $y(1) = 0 = y(-1)$*, hence we get*

$$(1+c_2)^2 + c_1^2 = \lambda^2$$
$$(-1+c_2)^2 + c_1^2 = \lambda^2$$

This gives $c_2 = 0$*.*
Hence $y = \lambda\sin(\phi) - c_1 mx = -\lambda\cos(\phi)$*.*

Now we want to discuss the role/meaning/interpretation of λ, we write

$$\begin{aligned} J(y) &= \int_{x_0}^{x_1} f(x,y,y') + \lambda\left(\frac{L}{x-x_0} - g(x,y,y')\right)dx \\ &= J(y) + \lambda\left(\int_{x_0}^{x_1}\frac{L}{x-x_0}dx - I(y)\right) \\ &= J(y) + \lambda(L - I(y)) \end{aligned}$$

Assume that J has an extremum at y, and solution to the EL equation depends ono $x_0, x_1, y_0, y_1, L \implies \lambda$ depends on x_0, x_1, y_0, y_1, L. Fix boundary conditions and we consider J as a function of L and we compute

$$\begin{aligned} \frac{\partial J}{\partial L} &= \int_{x_0}^{x_1}\frac{\partial}{\partial L}(f(x,y,y') - \lambda g(x,y,y') + \frac{\lambda L}{x_1 - x_0})dx \\ &= \int_{x_0}^{x_1}\frac{\partial F}{\partial y}\frac{\partial y}{\partial L} + \frac{\partial F}{\partial y'}\frac{\partial y'}{\partial L} + \frac{\partial\lambda}{\partial L}\left(\frac{L}{x_1-x_0} - g(x,y,y')\right) + \frac{\lambda}{x_1-x_0}dx \\ &= \int_{x_0}^{x_1}\left(\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right)\frac{\partial y}{\partial L}dx + \frac{\partial\lambda}{\partial L}(L - I(y)) + \lambda \end{aligned}$$

Since y is an extremal, we get

$$\frac{\partial J}{\partial L} = \lambda$$

This means λ gives the rate of change of the extremum $J(y)$ w.r.t. L.

This gives an interpretation of λ, if we assume $\lambda \neq 0, y$ to be extremal, i.e. solves

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0 \qquad F = f - \lambda g$$

Then, y also solves

$$\frac{d}{dx}\left(\frac{d}{dx}\right)\left(\frac{\partial G}{\partial y'}\right) - \frac{\partial G}{\partial y} = 0$$

with $G = g - \hat{\lambda} f = \frac{1}{\lambda} f$

Asume that y minimisers J subject to $y \in S, I(y) = L$. Define $k = J(y)$ we get

$$k = J(y) - \lambda(I(y) - L) = J(y) - \lambda I(y) + \lambda L$$

We can rewrite to

$$k - J(y) = -\lambda I(y) + \lambda L \implies \frac{1}{\lambda}(k - J(y)) = -I(y) + L \implies L = I(y) - \hat{\lambda}(J(y) - k)$$

Hence, we get

$$\begin{aligned} J - \lambda I &= -\lambda(I - \hat{\lambda} J) \\ \implies \min \int_{x_0}^{x_1} F(x, y, y') &= \max \int_{x_0}^{x_1} G(x, y, y')dx \end{aligned}$$

This gives us the theorem

Theorem 4.84. *Assume $\overline{y}$ produces a minimum (maximum) for J subject to $I(y) = L, y \in S$ and such that $\lambda \neq 0$.*

Define $k = J(\overline{y})$, then $\overline{y}$ produces a maximum (minimum) for I subject to $y \in S, J(y) = k$.

Especially, $I(\overline{y}) = L$.

Remark 4.85. *The Theorem has consequences for the non-rigidity of y:*

If y is an extremal for I, y solves $\frac{d}{dx}\left(\frac{\partial g}{\partial y'}\right) - \frac{\partial g}{\partial y} = 0$.

Then we can pick $\hat{\lambda}$ as

$$\frac{d}{dx}\left(\frac{\partial G}{\partial y'}\right) - \frac{\partial G}{\partial y} = \frac{d}{dx}\left(\frac{\partial g}{\partial y'}\right) - \frac{\partial g}{\partial y} - \hat{\lambda}\left(\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) - \frac{\partial f}{\partial y}\right)$$

Hence, $\hat{\lambda}$ is a solution. Hence $I(y) = L$ is true independent of $J(y) = k$.

Hence, we can prescribe k without changing the extremum for I.

If $\lambda = 0 \implies J(y) = k$ independent if $I(y) = L$. The problem does not depend on I.

Hence, if $\lambda = 0$ the problem is dependent.

What about generalisations for the isoperimetric problem?

Setting: Consider

$$\min_{\text{s.t. } y \in S, I(y) = L} J(y)$$

where $J(y) = \int_{x_0}^{x_1} f(x, y, y', y'')dx, I(y) = \int_{x_0}^{x_1} g(x, y, y', y'')dx$ and $S = \{y \in X : y(x_0) = y_0, y'(x_0) = y_0', y(x_1) = y_1, y'(x_1) = y_1'$ and $X = C^4([x_0, x_1])$.

Suppose f, g are smooth, then if y is a smooth extremal for J subject to the constraints we get

$$\frac{d^2}{dx^2}\left(\frac{\partial F}{\partial f''}\right) - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + \frac{\partial F}{\partial y} = 0 \qquad F = f - \lambda g$$

for some $\lambda \in \mathbb{R}$ (as long as y is not an extremal for I) details are similar to previous discussions.

Example 4.86. *Let $J(y) = \int_{x_0}^{x_1} (y'')^2, I(y) = \int_{x_0}^{x_1} y dx$ where $x_0 = 0, x_1 = 1, y_0 = y_0' = 0 = y_1 = y_1'$.*
We get $f = (y'')^2, g = y$. Then extremals for I satisfies

$$0 = \frac{d}{dx}\left(\frac{\partial g}{\partial y'}\right) - \frac{\partial g}{\partial y} = -1$$

Hence, I has no extremals.
Hence, $F = y'' - \lambda y \implies 0 = \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right) + \frac{\partial F}{\partial y} = \frac{d^2}{dx^2}(2y'') - \lambda = 2y^{(4)} - \lambda \implies y(x) = \frac{\lambda}{2}\frac{x^4}{4!} + c_3x^3 + c_2x^2 + c_1x + c_0$.
We get

$$\begin{aligned}
0 &= y(0) = c_0 \\
0 &= y'(0) = c_1 \\
0 &= y(1) = \frac{\lambda}{2}\frac{1}{4!} + c_3 + c_2 \\
0 &= y'(1) = \frac{\lambda}{2}\frac{1}{3!} + 3c_3 + 2c_2
\end{aligned}$$

For the isoperimetric constraint we get

$$1 = I(y) = \int_0^1 y(x)dx = \left[\frac{\lambda}{2}\frac{x^5}{5!} + \frac{c_3}{4}x^4 + \frac{c_2}{3}x^3\right]_0^1 = \frac{\lambda}{2}\frac{1}{5!} + \frac{1}{4}c_3 + \frac{1}{3}c_2$$

This gives us

$$\begin{aligned}
\lambda &= 2 - 6! \\
c_2 &= 30 \\
c_3 &= -60
\end{aligned}$$

Hence, $y(x) = 30x^4 - 60x^3 + 30x^2$.

Setting: Consider now

$$\min_{\text{s.t.} y \in S = \{y \in X = C^2([x_0,x_1]), y(x_0)=y_0, y(x_1)=y_1, I_1(y)=L_1,\ldots,I_m(y)=L_m} J(y)$$

with $J(y) = \int_{x_0}^{x_1} f(x, y, y')dx$ and $I_k(y) = \int_{x_0}^{x_1} g_k(x, y, y')dx$ for all $k = 1, \ldots, m$, with for all $k = 1, \ldots, m, L_k \in \mathbb{R}$ fixed

We want to see how to adapt the Lagrange multiplier technique. Let's consider $m = 2$.

Let y be an extremum for $J, y \in S$ s.t. $I_1(y) = L_1, I_2(y) = L_2$. To meet the constraints and still have arbitrary perturbation we use 3 test functions

$$\hat{y} = y + \epsilon_1\eta_1 + \epsilon_2\eta_2 + \epsilon_3\eta_3 = y + \langle \epsilon, \eta \rangle$$

where $\epsilon = (\epsilon_1, \epsilon_2, \epsilon_3), \eta = (\eta_1, \eta_2, \eta_3)$. with $\eta_k \in C^2([x_0, x_1]), \eta_k(x_0) = \eta_k(x_1) = 0, k = 1, 2, 3$.

Again consider $J(\hat{y}), I_1(\hat{y}), I_2(\hat{y})$ as functions of ϵ and define

$$\Theta(\epsilon) = \int_{x_0}^{x_1} f(x, y, +\langle\epsilon, \eta\rangle, y' + \langle\epsilon, \eta'\rangle)dx$$

$$\Xi_k(\epsilon) = \int_{x_0}^{x_1} g_k(x, y, +\langle\epsilon, \eta\rangle, y' + \langle\epsilon, \eta'\rangle dx \qquad k = 1, 2$$

Assume y is an extremum for J subject to the constraints, we know that 0 is an extremum for θ s.t. $\Xi_k(\epsilon) = L_k, k = 1, 2$. Therefore, there exists lagrange multipliers λ_1, λ_2 such that (under the assumption that 0 is a regular point)

We get

$$\nabla(\Theta - \lambda_1\Xi_1 - \lambda_2\Xi_2)\Big|_{\epsilon=0} = 0$$

regularity of 0 will be discussed below.

Hence, we compute

$$\frac{\partial\Theta}{\partial\epsilon_j}\Big|_{\epsilon=0} = \int_{x_0}^{x_1} \eta_j \left(\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right) dx$$

and

$$\frac{\partial\Xi}{\partial\epsilon_j}\Big|_{\epsilon=0} \int_{x_0}^{x_1} \eta_j \left(\frac{\partial g_k}{\partial y} - \frac{d}{dx}\left(\frac{\partial g_k}{\partial y'}\right)\right) dx \qquad j = 1, 2, 3$$

Hence, we get

$$0 = \int_{x_0}^{x_1} \eta_j \left(\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right) dx \qquad j = 1, 2, 3$$

where $F = f - \lambda_1 g_1 - \lambda_2 g_2$.

If we regard η_1 as arbitrary and η_2, η_3 as corrections then

$$0 = \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y}$$

Remark 4.87. *What about regularity?*

We impose the same conditions as in the finite-dimensional case, i.e. the above reasoning holds under the condition that

$$\nabla\Xi_1(0, 0, 0), \nabla\Xi_2(0, 0, 0)$$

are linearly independent.

This guarantees that y is not a rigid extremal through the implicit function theorem, and $\exists\lambda_1, \lambda_2$ s.t. $\nabla\Theta(0, 0, 0) = \lambda_1\nabla\Xi_1(0, 0, 0) + \lambda_2\nabla\Xi_2(0, 0, 0)$.

The condition in the textbook is given that

$$rank \begin{pmatrix} \nabla\Xi_1(0, 0, 0) \\ \nabla\Xi_2(0, 0, 0) \\ \nabla\Theta(0, 0, 0) \end{pmatrix} \leq rank \begin{pmatrix} \nabla\Xi_1(0, 0, 0) \\ \nabla\Xi_2(0, 0, 0) \end{pmatrix}$$

This condition is weaker as linearly independence implies $\nabla\Theta(0) \in span\{\nabla\Xi_1(0), \nabla\Xi_2(0)\}$ but does not guarantee non-rigidity of y.

Example 4.88. *Consider*

$$J(y) = \int_0^1 (y')^2 dx, I_1(y) = \int_0^1 y dx, I_2(y) = \int_0^1 xy dx$$

with $L_1 = 2, L_2 = \frac{1}{2}, S = \{y \in C^2([0,1]) : y(0) = y(1) = 0\}$

We have $F = (y')^2 - \lambda_1 y - \lambda_2 yx$

We also have

$$\nabla\Xi_1(0,0,0) = \left(\int_0^1 \eta_1 dx, \int_0^1 \eta_2 dx, \int_0^1 \eta_3 dx\right)$$
$$\nabla\Xi_2(0,0,0) = \left(\int_0^1 \eta_1 x dx, \int_0^1 \eta_2 x dx, \int_0^1 \eta_3 x dx\right)$$

are linearly independent $\implies$ *non-rigidity condition holds.*

We have

$$0 = \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = \frac{d}{dx}2y' + (\lambda_1 + \lambda_2 x) = 2y'' + \lambda_1 + \lambda_2 x$$

This gives

$$y''(x) = -\frac{\lambda_2}{2}x - \frac{\lambda_1}{2} \implies y(x) = -\frac{\lambda_2}{12}x^3 - \frac{\lambda_1}{4}x^2 + c_1 x + c_2$$

We have

$$0 = y(0) = c_2$$
$$0 = y(1) = -\frac{\lambda_2}{12} - \frac{\lambda_1}{4} + c_1 \implies c_1 = \frac{\lambda_1}{4} + \frac{\lambda_2}{12}$$

We also have

$$2 = I_1(y) = \int_0^1 y(x)dx = [-\frac{\lambda_2}{48}x^4 - \frac{\lambda_1}{12}x^3 + \frac{c_1}{2}x^2]_0^1 = -\frac{\lambda_2}{48} - \frac{\lambda_1}{12} + \frac{c_1}{2}$$

$$\frac{1}{2} = I_2(y) = \int_0^1 y(x)xdx = [-\frac{\lambda_2}{5\cdot 12}x^5 - \frac{\lambda_1}{4\cdot 4}x^4 + \frac{c_1}{3}x^3]_0^1 = -\frac{\lambda_2}{60} - \frac{\lambda_1}{16} + \frac{c_1}{3}$$

Solve this linear systems we get

$$\lambda_1 = 408, \lambda_2 = -720, c_1 = 42$$

Hence

$$y(x) = 60x^3 - 102x^2 + 42x$$

Consider now several dependent variables, i.e. $X = C^2([t_0,t_1], \mathbb{R}^n)$ with $t_0 < t_1 \in \mathbb{R}$ and if $q \in X$ then $q = (q_1, \ldots, q_n)$ with $q_j \in C^2([t_0,t_1])$ and we use the notation

$$\dot{q} = \frac{d}{dt}q = \left(\frac{d}{dt}q_1, \ldots, \frac{d}{dt}q_n\right)$$

Setting: We want to find

$$\min_{\text{s.t. } q\in S, I(q)=k} J(q)$$

with $J(q) = \int_{t_0}^{t_1} g(t,q,\dot{q})dt, I(q) = \int_{t_0}^{t_1} g(t,q,\dot{q})dt$ and $S = \{q \in X : q(t_0) = \overline{q_0}, q(t_1) = \overline{q_1}$ where $\overline{q_0}, \overline{q_1} \in \mathbb{R}^n$ fixed. Let $q \in S$ be a smooth extremum for J subject to $I(q) = k$ and such that q is not an extremal for I.

Then $\exists \lambda \in \mathbb{R}$ s.t. q solves

$$0 = \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{q}_j}\right) - \frac{\partial F}{\partial q_j} \qquad j = 1, 2, \ldots, n$$

where $F = L - \lambda g$ (n EL-equation and one Lagrange Multiplier)

Example 4.89 (Dido's Problem). *We want to find a curve $J : [t_0, t_1] \to \mathbb{R}^2$ with length $l > 2$ s.t. $\gamma(t_0) = (-1, 0), \gamma(t_1) = (1, 0)$ such that the curve formed by γ and the linear segment between $(-1, 0)$ and $(1, 0)$ encloses a maximal area.*

Describe $J(t) = (x(t), y(t)) = q(t)$ then solve

$$\min_{s.t.\ q \in S = \{q \in C^2([t_0,t_1],\mathbb{R}^2): q(t_0)=(-1,0), q(t_1)=(1,0)} J(q)$$

where $I(q) = \int_{t_0}^{t_1} \sqrt{x^{-2} + y^{-2}} dt = l > 2, J(q) = \int_{t_0}^{t_1} \frac{1}{2}(x\dot{y} - y\dot{x})dt$.

We have $F = \frac{1}{2}(x\dot{y} - y\dot{x}) - \lambda\sqrt{\dot{x}^2 + \dot{y}^2}$

Hence we get the EL equations

$$\frac{d}{dt}\left(\lambda\frac{\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2} - \frac{1}{2}y}\right) - \frac{1}{2}\dot{y} = 0$$
$$\frac{d}{dt}\left(\lambda\frac{\dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2} - \frac{1}{2}x}\right) - \frac{1}{2}\dot{x} = 0$$

Hence $c_0 = -\lambda\frac{y}{\dot{x}}\sqrt{\dot{x}^2 + \dot{y}^2} - y, c_1 = -\lambda\frac{\dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}} + x$.

This gives

$$(x - c_1)^2 + (y + c_0)^2 = \frac{\lambda^2\dot{x}^2}{\dot{x}^2 + \dot{y}^2} + \frac{\lambda^2\dot{y}^2}{\sqrt{\dot{x}^2 + \dot{y}^2}} = \lambda^2$$

This means the solution is a circular arc.

For non-rigidity, extremals for I fulfill

$$\frac{d}{dt}\left(\frac{\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}}\right) = 0$$
$$\frac{d}{dt}\left(\frac{\dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}}\right) = 0$$

This shows the EL equations for those extremals become $y = c_0, x = c_1$.

Hence, $\dot{x} = \dot{y} = 0$ if $t_1 - t_2 < l \implies$ extremals for I don't fulfill the isoperimetric constraint, meaning that all extremals for J fulfill the non-rigidity condition.

4.6 Holonomic and non-holonomic problems

Setting: Let $X = C^2([t_0, t_1], \mathbb{R}^n)$ with $t_0 < t_1 \in \mathbb{R}$ and $q \in X$ given by $q = (q_1, \ldots, q_n)$.

Definition 4.90. *A holonomic constraint is a condition of the form*

$$g(t, q) = g(t, q(x)) = 0, t \in [t_0, t_1]$$

A non-holonomic constraint is a condition of the form

$$g(t, q, \dot{q}) = g(t, g(t), \dot{g}(t)) = 0, t \in [t_0, t_1]$$

Dealing with holonomic constraints is simpler than dealing with non-holonomic constraints.

For simplicity, we consider only $n = 2$.

This means our problem is

$$\min_{\text{s.t. } q \in S, g(t,q)=0} J(q)$$

where $J(q) = \int_{t_0}^{t_1} L(t,q,\dot{q})dt, S = \{q \in X : q(t_0) = \overline{q_0}, q(t_1) = \overline{q_1}\}$ where $\overline{q_0}, \overline{q_1} \in \mathbb{R}^2$ are fixed end-point constraints.

For consistency we require $g(t_0, \overline{q_0}) = g(t_1, \overline{q_1}) = 0$.

We also assume that L, g are smooth. We assume that J has an extremum at $q \in S$ subject to $g(t,q) = 0$. We also assume that

$$\nabla g|_{(t,q)} = \left(\frac{\partial g}{\partial q_1}, \frac{\partial g}{\partial q_2}\right)\bigg|_{(t,q)} \neq 0, t \in [t_0, t_1]$$

Remark 4.91. *Because we assume $\nabla g|_{(t,q)} \neq 0$ in principle, using the implicity function theorem, we could directly solve*

$$g(t, q_1, q_2) = 0$$

for q_1 or q_2 i.e. $q_2(t) = h(t, g_1(t))$ s.t. $g(t, q_1(t), h(t, q_1(t))) = 0$ for $t \in [t_0, t_1]$.

And the nplug $q_2 = h(t, q_1)$ into J to make the problem an unconstrainted minimisation problem in one dependent variable q_1.

This approach has 2 issues:

1. *h might not be smooth*
2. *In terms of a physical/geometrical interpretation of the holonomic problem, it is not clear which dependent variable to choose, as for example "in space all directions are equal".*

Let J have an extremum at q subject to the holonomic constraints and perturb by $\eta \in C^2([t_0,t_1], \mathbb{R}^2)$ so we get $\hat{q} = q + \epsilon\eta$.

Definition 4.92. *A function $\hat{q} = (\hat{q}_1, \hat{q}_2)$ is called an allowable variation for q if $\hat{q}_k \in C^2([t_0, t_1]), \hat{q}(t_0) = \overline{q_0}, \hat{q}(t_1) = \overline{q_1}$ and if $g(t, \hat{q}) = 0, t \in [t_0, t_1]$*

Do allowable variations exist?

Let $\eta = (\eta_1, \eta_2)$, we need $\eta_1, \eta_2 \in C^2([t_0, t_1]), \eta(t_0) = \eta(t_1) = (0,0), g(t, q(t) + \epsilon\eta(t)) = 0, t \in [t_0, t_1]$.

Recall that q is fixed, $\nabla g|_{(t,q)} \neq 0$, suppose for simplicity that

$$\frac{\partial g}{\partial q_2}\bigg|_{(t,q)} \neq 0$$

Implicit function theorem assets that $g(t, q + \epsilon\eta) = 0$ can be solved for η_2 in terms of η_1 and ϵ_1 if $|\epsilon|$ is sufficiently small.

We have g smooth $\implies \eta_2$ is smooth.

Take $\eta_1 \in C^2([t_0, t_1])$ arbitrary with $\eta_1(t_0) = \eta_1(t_1) = 0$ and then we get η_2 as solution to equation $g(t, q + \epsilon\eta) = 0$.

At $t = t_0$ we have

$$0 = g(t_0, q + \epsilon\eta) = g(t_0 \overline{q_0} + \epsilon(0, \eta_2(t_0))) \implies \eta_2(t_0) = 0$$

Similarly $\eta_2(t_1) = 0$.

Hence, as long as $\nabla g|_{(t,q)} \neq 0$ we have non-trivial allowable variations.

Now, assume J has an extremum at y. Hence $J(\hat{q}) - J(q) = \mathcal{O}(\epsilon^2) \implies 0 = \delta J(\eta, q) = \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial q_1} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_1}\right)\right)\eta_1 + \left(\frac{\partial L}{\partial q_2} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_2}\right)\right)\eta_2 dt$

But η_2 is not independent of η_1. If η_1 is arbitrary but fixed we can regard η_2 as function of ϵ.

If $|\epsilon|$ is small then η_2 is smooth.

Now

$$
\begin{aligned}
0 = g(t,\hat{q}) \implies 0 &= \frac{d}{d\epsilon} g(t,\hat{q})|_{\epsilon=0} \\
&= \frac{\partial g}{\partial q_1}\eta_1 + \frac{\partial g}{\partial q_2}\left(\eta_2 + \epsilon\frac{\partial \eta_1}{\partial \epsilon}\right)\Big|_{\epsilon=0} \\
&= \frac{\partial g}{\partial q_1}\eta_1 + \frac{\partial g}{\partial q_2}\eta_2
\end{aligned}
$$

as $\frac{\partial g}{\partial q_2} \neq 0 \implies \eta_2 = -\frac{\frac{\partial g}{\partial q_1}}{\frac{\partial g}{\partial q_2}}\eta_1$.

Since L is smooth, $E_2(L) = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_2}\right) - \frac{\partial L}{\partial q_2}$ is continuous in t. Also $\frac{\partial g}{\partial q_2}$ is continuous in t. This gives

$$
\exists \lambda(t) \text{ s.t. } E_2(L) = \lambda(t)\frac{\partial g}{\partial q_2}
$$

So we have $\eta_2 = -\frac{\frac{\partial g}{\partial q_1}}{\frac{\partial g}{\partial q_2}}\eta_1$. Note that $\frac{\partial g}{\partial q_2} \neq 0$.

We get

$$
E_2(L) = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_2}\right) - \frac{\partial L}{\partial q_2}
$$

is continuous in t, and also $\frac{\partial g}{\partial q_2}$ is continuous in t.

Hence there exists a function λ depending on t s.t.

$$
E_2(L) = \lambda(t)\frac{\partial g}{\partial q_2}
$$

especially we can define $\lambda(t) = \frac{E_2(L)}{\frac{\partial g}{\partial q_2}}$

Now, recall

$$
\begin{aligned}
0 = \delta J(\eta, q) &= \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial q_1} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_1}\right)\right)\eta_1 + \left(\frac{\partial L}{\partial q_2} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_2}\right)\right)\eta_2 dt \\
&= \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial q_1} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_1}\right)\right)\eta_1 + \lambda(t)\frac{\partial g}{\partial q_2}\eta_2 dt \\
&= \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial q_1} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_1}\right) + \lambda(t)\frac{\partial g}{\partial q_1}\right)\eta_1 dt
\end{aligned}
$$

Since η_1 is arbitrary, we get

$$
\frac{\partial L}{\partial q_1} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_1}\right) + \lambda(t)\frac{\partial g}{\partial q_1} = 0
$$

Hence

$$
\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{q}_k}\right) - \frac{\partial F}{\partial q_k} = 0 \qquad k = 1, 2
$$

where $F = f - \lambda g$.

Theorem 4.93. *Suppose that J has a nextremum at $q = (q_1, q_2)$ subject to $q \in S, g(t,q) = 0$ and such that $\nabla g|_{(t,q)} \neq 0$ for all $t \in [t_0, t_1]$. Then there exists a function λ of t such that q solves $\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{q}_k}\right) - \frac{\partial F}{\partial q_k} = 0$ for $k = 1, 2$.*

Example 4.94. *Take* $J(q) = \int_0^{\frac{\pi}{2}} \sqrt{|\dot{q}|^2 + 1dt}$ *with* $0 = g(t,q) = |q|^2 - 1$ *where*

$$\overline{q}_0 == \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \overline{q}_1 == \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and $F = \sqrt{|\dot{q}|^2 + 1} - \lambda(t)(|q|^2 - 1)$.

This gives the EL-equation

$$\begin{aligned} 0 &= \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{q}_1}\right) - \frac{\partial F}{\partial q_1} = \frac{d}{dt}\left(\frac{\dot{q}_1}{\sqrt{|\dot{q}|^2+1}}\right) - \lambda(t)2q_1 \\ 0 &= \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{q}_2}\right) - \frac{\partial F}{\partial q_2} = \frac{d}{dt}\left(\frac{\dot{q}_2}{\sqrt{|\dot{q}|^2+2}}\right) - \lambda(t)2q_2 \end{aligned}$$

Since $|q|^2 = 1$, *we use substitutions*

$$\begin{aligned} q_1(t) = \cos(\phi(t)) &\implies \dot{q}_1(t) = -\sin(\phi(t))\dot{\phi}(t) \\ q_2(t) = \sin(\phi(t)) &\implies \dot{q}_2(t) = \cos(\phi(t))\dot{\phi}(t) \end{aligned}$$

This gives $|\dot{q}|^2 = \dot{\phi}^2$.

This gives the EL equations

$$\begin{aligned} 0 &= \frac{d}{dt}\left(\frac{\dot{\phi}\sin(\phi)}{\sqrt{\dot{\phi}^2+1}}\right) + 2\lambda(t)\cos(\phi) \\ 0 &= \frac{d}{dt}\left(\frac{\dot{\phi}\cos(\phi)}{\sqrt{\dot{\phi}^2-1}}\right) + 2\lambda(t)\sin(\phi) \end{aligned}$$

This gives

$$\begin{aligned} 0 &= \sin(\phi)\frac{d}{dt}\left(\frac{\dot{\phi}\sin(\phi)}{\sqrt{\dot{\phi}^2+1}}\right) + \cos(\phi)\frac{d}{dt}\left(\frac{\dot{\phi}\cos(\phi)}{\sqrt{\dot{\phi}^2+1}}\right) \\ &= \sin(\phi)\left(\frac{d}{dt}\left(\frac{\dot{\phi}}{\sqrt{\dot{\phi}^2+1}}\right)\sin(\phi) + \frac{\dot{\phi}}{\sqrt{\dot{\phi}^2+1}}\cos(\phi)\dot{\phi}\right) + \cos(\phi)\left(\frac{d}{dt}\left(\frac{\dot{\phi}}{\sqrt{\dot{\phi}^2+1}}\right)\cos(\phi) + \frac{\dot{\phi}}{\sqrt{\dot{\phi}^2+1}}(-\sin(\phi)\dot{\phi})\right) \\ &= \frac{d}{dt}\left(\frac{\dot{\phi}}{\sqrt{\dot{}^{2}\dot{}+\phi 1}}\right) \end{aligned}$$

This gives

$$\frac{\dot{(\phi)}}{\sqrt{\dot{\phi}^2+1}} = c \implies \dot{\phi} = a \implies \phi(t) = at + b$$

This gives

$$\begin{aligned} q_1(t) &= \cos(at+b) \\ q_2 2(t) &= \sin(at+b) \end{aligned}$$

Hence

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = q(0) = \begin{pmatrix} \cos(b) \\ \sin(b) \end{pmatrix} \implies b = 2n\pi, n \in \mathbb{Z}$$

We also have

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = q\left((\frac{\pi}{2}\right) = \begin{pmatrix} \cos(\frac{\pi}{2}a + 2n\pi) \\ \sin(\frac{\pi}{2}a + 2n\pi) \end{pmatrix} \implies a = 4m + 1, m \in \mathbb{Z}$$

If we pick $0 = n = m$*, we have*

$$q(t) = \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix}$$

And we can compute

$$Dg|_{(t,q)} = 2\begin{pmatrix} q_1 \\ q_2 \end{pmatrix}^\top \neq 0, \forall t \in [0, \frac{\pi}{2}]$$

Example 4.95 (Pendulum)**.** *Consider a pendulum of mass* m *and length* l*. It's position at time* t *is given by*

$$q(t) = \begin{pmatrix} q_1(t) \\ q_2(t) \end{pmatrix}$$

where q_2 *is the vertical component of the position motion of the pendulum between time* t_0 *and time* t_1 *minimisers the total energy*

$$J(q) = \int_{t_0}^{t_1} \frac{m}{2}|\dot{q}|^2 + gq_2 dt$$

subject to $0 = h(t, q) = q_1^2 + (q_2 - l)^2 - l^2$*, where* $F = \frac{m}{2}|\dot{q}|^2 + gq_2 + \lambda(t)(q_1^2 + (q_2 - l)^2 - l^2)$

This gives the EL equation

$$0 = \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{q}_1}\right) - \frac{\partial F}{\partial q_1} = \frac{d}{dt}(m\dot{q}_1) - (\lambda(t)2q_1) = m\ddot{q}_1 - 2\lambda(t)q_1$$

$$0 = \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{q}_2}\right) - \frac{\partial F}{\partial q_2} = \frac{d}{dt}(m\dot{q}_2) - (g + \lambda(t)2(q_2 - l)) = m\ddot{q}_2 - g - 2\lambda(t)(q_2 - l)$$

Those are the equations of motion of the pendulum.

Example 4.96 (Geodesics on a surface)**.** *Let* g *be a smooth function on* $\mathbb{R}^3$ *in variables* x, y, z *then the equation* $g(x, t, z) = 0$ *defines a surface (implicitly) as long as* $\nabla g \neq 0$*.*

For example

$$g(x, y, z) = x^2 + y^2 + z^2 - 1 = 0$$

describes a sphere of radius 1 centered at the origin. We can describe a curve $\gamma : [t_0, t_1] \to \mathbb{R}^3$ *by* $r(t) = (x(t), y(t), z(t))$ *for* $t \in [t_0, t_1]$*.*

The arclength of γ *is given by*

$$J(\gamma) = J(x, y, z) = \int_{t_0}^{t_1} |r'(t)|dt = \int_{t_0}^{t_1} \sqrt{(x')^2 + (y')^2 + (z')^2} dt$$

Let $\Sigma = \{(x, y, z) \in \mathbb{R}^3 : g(x, y, z) = 0\}$ *the surface described by* g*. Let* $p_0, p_1 \in \Sigma$ *be two fixed points. Then a geodesic on* Σ *is a curve* γ *in* Σ *(i.e.* $\gamma(t) \in \Sigma, \forall t$*) s.t.*

- $r(t_0) = p_0$

- $r(t_1) = p_1$
- $g(x(t), y(t), z(t)) = 0, \forall t \in [t_0, t_1]$
- γ *minimisers the arc length*

This means γ is a solution to a holonomic problem. Hence solves the EL equations

$$F = \sqrt{(x')^2 + (y')^2 + (z')^2} - \lambda(t) g(x, y, z)$$

The EL equation is

$$0 = \frac{d}{dt}\left(\frac{\partial F}{\partial x'}\right) - \frac{\partial F}{\partial x} = \frac{d}{dt}\left(\frac{x'}{\sqrt{(x')^2 + (y')^2 + (z')^2}}\right) - \left(-\lambda(t)\frac{\partial g}{\partial x}\right)$$
$$0 = \frac{d}{dt}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = \frac{d}{dt}\left(\frac{y'}{\sqrt{(x')^2 + (y')^2 + (z')^2}}\right) - \left(-\lambda(t)\frac{\partial g}{\partial y}\right)$$
$$0 = \frac{d}{dt}\left(\frac{\partial F}{\partial z'}\right) - \frac{\partial F}{\partial z} = \frac{d}{dt}\left(\frac{z'}{\sqrt{(x')^2 + (y')^2 + (z')^2}}\right) - \left(-\lambda(t)\frac{\partial g}{\partial z}\right)$$

Settings: Now, we consider the nonholonomic problem, which is

$$\min_{\text{s.t. } q \in S, g(t,q,q')=0} J(q)$$

where $S = \{q \in X : q(t_0) = \overline{q_0}, q(t_1) = \overline{q_1}\}$ for $\overline{q_0}, \overline{q_1} \in \mathbb{R}^n$ fixed. And $J(q) = \int_{t_0}^{t_1} L(t, q, \dot{q})dt$

Definition 4.97. *The above nonholonomic problem is called Lagrange problem*

Remark 4.98. *The Lagrange problem includes all holonomic problems as special case.*

Can we re-write the condition $g(t, q, \dot{q}) = 0$ as some $\hat{g}(t, q) = 0$. The general answer is no.
For example, if $n = 3$ and $0 = g(t, q, \dot{q}) = P(q)\dot{q}_1 + Q(q)\dot{q}_2 + R(q)\dot{q}_3$ then g is integrable if and only if

$$P\left(\frac{\partial Q}{\partial q_3} - \frac{\partial R}{\partial q_2}\right) + Q\left(\frac{\partial R}{\partial q_1} - \frac{\partial P}{\partial q_3}\right) + R\left(\frac{\partial P}{\partial q_2} - \frac{\partial Q}{\partial q_1}\right)$$

If $n = 2$ and $0 = g(t, q, \dot{q}) = P(q)\dot{q}_1 + Q(q)\dot{q}_2$ with $Q(q) \neq 0$. Then $\frac{dq_2}{dq_1} = -\frac{P(q)}{Q(q)}$. Therefore, Picard' Theorem gives a solution $q_2(q_1)$.

Remark 4.99. *We can convert isoperimetric problems into Lagrange problems if $I(q) = \int_{t_0}^{t_1} g(t, q, \dot{q})dt = L$ with $q = (q_1, \ldots, q_n)$.*

Define new dependent variable q_{n+1} by

$$\dot{q}_{n+1} = g(t, q, \dot{q})$$

so we get the condition

$$\dot{q}_{n+1} - g(t, q, \dot{q}) = 0 \qquad [\textit{Nonholonomic condition}]$$

hence we can impose boundary conditions on $q_{n+1}(t_0)$ and $q_{n+1}(t_1)$ such that

$$q_{n+1}(t_1) - q_{n+1}(t_0) = \int_{t_0}^{t_1} \dot{q}_{n+1}(t)dt = \int_{t_0}^{t_1} g(t, q, \dot{q})dt = I(q) = L$$

Remark 4.100. *Problems containing higher derivatives in the integrand f of J can also be recast as Lagrange problems: If we consider*

$$\min_{s.t.\ y\in S} J(y)$$

where $J(y)=\int_{x_0}^{x_1} f(x,y,y',y'')dx$, $S=\{y\in C^2([x_0,x_1]) : y(x_0)=y_0, y(x_1)=y_1\}$ *Denote* $t_0=x_0, t_1=x_1, q_1=y_1, q_2=y'$.

Then

$$J(q)=\int_{t_0}^{t_1} f(t,x,q_1,q_2,\dot{q}_2)dt$$

with constraint $0=g(t,q,\dot{q})=\dot{q}_1-q_2$

Example 4.101 (Rigid extremals)**.** *We want to find extremals for* $J(q)=\int_{t_0}^{t_1} q_1\sqrt{1+\dot{q}_2^2}dt$ *subject to* $0=g(t,q,\dot{q})=\dot{q}_1^2+\dot{q}_2^2$ *and such that* $q\in S$

Because of the nonholonomic condition $0=\dot{q}_1^2+\dot{q}_2^2$ *we get* $\dot{q}_1=\dot{q}_2=0$.

Hence, q_1, q_2 *is constant.*

If $\overline{q_0}\neq\overline{q_1}$*, there are no solutions to the problem.*

If $\overline{q_0}=\overline{q_1}$ *then* $q(t)=\overline{q_0}, \forall t\in[t_0,t_1]$ *then* $J(q)=\int_{t_0}^{t_1} q_1(0)\sqrt{1}dt=q_1(0)(t_1-t_0)$ *is constant.*

Hence, for this problem, no variations for q are available.

Theorem 4.102. *Let* $J(q)=\int_{t_0}^{t_1} L(t,q,\dot{q})dt$ *with L smooth. Suppose J has an extremum at* $q\in X=C^2([t_0,t_1],\mathbb{R}^n)$ *s.t.* $q\in S$ *subject to*

$$g(t,q,\dot{q})=0, \forall t\in[t_0,t_1]$$

with g smooth and such that

$$\frac{\partial g}{\partial \dot{q}_j}\neq 0 \text{ for some } 1\leq j\leq n$$

Then there exists a constant λ_0 and a function $\lambda_1(t)$ that are not both zero s.t. q solves

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}_k}\right)-\frac{\partial K}{\partial q_k}=0 \qquad k=1,\dots,n$$

where $K(t,q,\dot{q})=\lambda_0 L(t,q,\dot{q})-\lambda_1(t)g(t,q,\dot{q})$

Remark 4.103. *The above theorem is similar to a general Lagrange multiplier rule for finite dimensional optimisation problems*

$$\min_{s.t.\ h(x,y)=0} f(x,y)$$

where $(x,y)\in\Omega\subseteq\mathbb{R}^2$*, where the multiplier λ_0 is introduced to deal with the situation that* $\nabla h(x,y)=0$ *(i.e. (x,y) is not a regular point in the terminology in LY)*

More information about general Lagrange multiplier rules can be found in vB 4.1.3. (p 79)

If $\lambda_0=0$ then λ_1 is no the zero function on $[t_0,t_1]$ and we get from the Theorem that

$$0=\frac{d}{dt}\left(\lambda_1(t)\frac{\partial q}{\partial \dot{q}_k}\right)-\lambda_1(t)\frac{\partial g}{\partial q_k} \qquad k=1,\dots,n$$

Hence, λ_1 solves this ODE and must be non-trivial

Definition 4.104. *Let q be a smooth extremal for the Lagrange problem.*

We call q abnormal if there exists a non-trivial solution $\lambda_1(t)$ to the ODE

$$0 = \frac{d}{dt}\left(\lambda_1(t)\frac{\partial g}{\partial \dot{q}_k}\right) - \lambda_1(t)\frac{\partial g}{\partial g_k} \qquad k = 1, \ldots, n$$

Otherwise, we call q normal

Theorem 4.105. *Let J, L, g, q be as in Theorem 4.102 and assume that q is a normal extremal. Then there exists a function $\lambda_1(t)$ such that q solves*

$$\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{q}_k}\right) - \frac{\partial F}{\partial q_k} = 0 \qquad k = 1, \ldots, n$$

where $F(t, q, \dot{q}) = L(t, q, \dot{q}) - \lambda_1(t)g(t, q, \dot{q})$

Moreover, λ_1 is uniquely determined by q

Remark 4.106. *The system*

$$\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{q}_k}\right) - \frac{\partial F}{\partial q_k} = 0 \qquad k = 1, \ldots, n$$

contains the term $\dot{\lambda}_1$ (unlike other optimisation problems we have seen so far).

Example 4.107. *Let*

$$J(q) = \int_{t_0}^{t_1} q_1^2 + q_2^2 dt$$

and $g(t, q, \dot{q}) = \dot{q}_1 + q_1 + q_2$ we have

$$0 = \frac{d}{dt}\left(\lambda_1(t)\frac{\partial g}{\partial \dot{q}_2}\right) - \lambda_1(t)\frac{\partial g}{\partial q_2} = \frac{d}{dt}\left(\lambda_1(t) \cdot 0\right) - \lambda_1(t) = -\lambda_1(t)$$

Hence $\lambda_1(t) = 0$ is the only solution to this ODE.

Hence, every extremal is normal.

We have $F(t, q, \dot{q}) = q_1^2 + q_2^2 - \lambda_1(t)(\dot{q}_1 + q_1 + q_2)$. Hence, we get the system of equations

$$0 = \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{q}_1}\right) - \frac{\partial F}{\partial q_1} = \frac{d}{dt}(-\lambda_1(t)) - (2q_1 - \lambda_1(t)) = -\dot{\lambda}_1 - 2q_1 + \lambda_1$$

$$0 = \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{q}_2}\right) - \frac{\partial F}{\partial q_2} = \frac{d}{dt}(0) - (2q_2 - \lambda_1(t)) = -\lambda_1 - 2q_2$$

We solve the ODE system

$$\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = c_1 \begin{pmatrix} 1 - \sqrt{2} \\ 1 \end{pmatrix} e^{\sqrt{2}t} + c_2 \begin{pmatrix} 1 + \sqrt{2} \\ 1 \end{pmatrix} e^{-\sqrt{2}t}$$

Example 4.108 (Catenary)**.** *Assuming the distance between two points is less than l. The potential energy given by*

$$J(y) = \int_0^l y ds$$

where ds is the integration wrt arc length

To ensure that s is arclength, we assume that $\dot{x}^2 + \dot{y}^2 = 1$.

Define $q_1 = x, q_2 = y, t = s$. Hence, we want to solve

$$\min_{s.t.\ 0=g(\dot{q})=\dot{q}_1^2+\dot{q}_2^2-1} J(q) = \int_0^l q_2 dt$$

with $q(0) = (x_0, y_0)$ and $q(l) = (x_1, y_1)$

We have the ODEs

$$0 = \frac{d}{dt}\left(\lambda_1 \frac{\partial g}{\partial \dot{q}_1}\right) - \lambda_1 \frac{\partial g}{\partial q_1} = \frac{d}{dt}\left(\lambda_2 2\dot{q}_1\right)$$
$$0 = \frac{d}{dt}\left(\lambda_1 \frac{\partial g}{\partial \dot{q}_2}\right) - \lambda_1 \frac{\partial g}{\partial q_2} = \frac{d}{dt}(\lambda_1 2\dot{q}_2)$$

for q fixed extremal.

Suppose the ODEs have a non-trivial solution λ_1 then

$$c_1 = \lambda_1 \dot{q}_1$$
$$c_2 = \lambda_1 \dot{q}_2$$

Hence, we get

$$0 = g(\dot{q}) = \left(\frac{c_1}{\lambda_1}\right)^2 + \left(\frac{c_2}{\lambda_1}\right)^2 - 1 \implies \lambda_1^2 = c_1^2 + c_2^2$$

showing that λ_1 is a constant, hence $\dot{q}_1, \dot{q}_2$ is a constant as well. q_1, q_2 are linear in t.

Hence,

$$q_1(t) = \frac{x_1 - x_0}{l} t + x_0$$
$$q_2(t) = \frac{y_1 - y_0}{l} t + y_0$$

Hence,

$$\dot{q}_1^2 + \dot{q}_2^2 = \left(\frac{x_1 - x_0}{l}\right)^2 + \left(\frac{y_1 - y_0}{l}\right)^2 = \frac{1}{e^2}((x_1 - x_0)^2 + (y_1 - y_0)^2) < 1$$

Hence, the assumption that htere is a non-trivial λ_1 was false. Hence, all extremals are normal.

Hence, $F = q_2 - \lambda_1(t)(\dot{q}_1^2 + \dot{q}_2^2 - 1)$. This gives the EL equation

$$0 = \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{q}_1}\right) - \frac{\partial F}{\partial q_1} = \frac{d}{dt}(-2\lambda_1(t)\dot{q}_1)$$
$$0 = \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{q}_2}\right) - \frac{\partial F}{\partial q_2} = \frac{d}{dt}(-2\lambda_1(t)\dot{q}_2) - 1$$

This gives

$$c_1 = 2\lambda_1(t)\dot{q}_2 \implies 2\lambda_1(t)\dot{q}_2 = t + c_2 \implies 4\lambda_1^2(t)(\dot{q}_1^2 + \dot{q}_2^@) = c_1^2 + (t + c_2)^2 \implies \lambda_1(t) = \pm\frac{1}{2}\sqrt{c_1^2 + (t + c_2)^2}$$

Hence,

$$q_1(t) = \sinh^{-1}\left(\frac{t + c_2}{c_1}\right) + c_3$$
$$q_2(t) = \sqrt{c_1^2 + (t + c_2)^2} + c_4$$