MAT 354 LECTURE NOTES

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1 Introduction

Let's first focus our interest in solving a cubic equation given the form

$$x^3 + cx = d$$

for some $c, d \in \mathbb{R}$. Cardano gives the solution to this cubic equation in around 1535 to be

$$x = \sqrt[3]{\frac{d}{2} + \sqrt{\frac{d^2}{4} + \frac{c^3}{27}}} + \sqrt[3]{\frac{d}{2} - \sqrt{\frac{d^2}{4} + \frac{c^3}{27}}}$$

Now to solve a general cubic equation given the form

$$ax^3 + bx^2 + cx + d = 0$$

for some $a, b, c, d \in \mathbb{R}$. Tartaglia in around 1539 solves this by letting $x = t - \frac{b}{3a}$. This will therefore yields a cubic equation in t in the form of

$$At^3 + Bt = C$$

for some $A, B, C \in \mathbb{R}$

Therefore, we can then use the Cardano's formula to solve t and hence x.

However, this means any cubic equation has only one real solution, which is obviously not true. So is Cardano's formula wrong? No, the formula itself is correct. Consider $x^3 - 3x = 0$, its real roots are $x = 0, \pm\sqrt{3}$. Using the Cardano's formula with $c = -3, d = 0$, we have

$$\begin{aligned} x &= \sqrt[3]{0 + \sqrt{0 + \frac{-27}{27}}} + \sqrt[3]{0 - \sqrt{0 + \frac{-27}{27}}} \\ &= \sqrt[3]{\sqrt{-1}} + \sqrt[3]{-\sqrt{-1}} \\ &= \sqrt[3]{\sqrt{-1}} - \sqrt[3]{\sqrt{-1}} \\ &= \sqrt[3]{i} - \sqrt[3]{i} \end{aligned}$$

Note that i has 3 cubic roots, and all the combinations of these roots would gives us exactly 3 different solutions.

Remember that, every polynomial of degree n has n complex roots considering the multiplicity (the Fundamental Theorem of Calculus).

Despite the complex analysis requires complex numbers, it is not the analysis of complex numbers. Complex Analysis is also not a fancy generalization of Real Analysis. The following example can tell why

Example 1.1. Consider $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \{x^2 \text{ if } x \geq 0 - x^2 \text{ if } x < 0$$

We have

$$f'(x) = \{2x \text{ if } x \geq 0 - 2x \text{ if } x < 0$$

It is clear that f is differentiable at $x = 0$ but f' is not differentiable at $x = 0$.

It is obvious that in Real Analysis, a function is differentiable does not have to be second differentiable.

Soon, we will see that for any $f : \mathbb{C} \to \mathbb{C}$ is complex differentiable then f' is also complex differentiable. In fact, by induction, this means f is infinitely differentiable. This is because complex differentiability imposes a strict condition.

There are, however, some similarities. In Real Analysis, $f : \mathbb{R} \to \mathbb{R}$ is differentiable at x_0 if there is some $L \in \mathbb{R}$ such that for all $\varepsilon > 0$, there exists some $\delta > 0$ so that

$$\forall x, |x - x_0| < \delta \Longrightarrow |\frac{f(x) - f(x_0)}{x - x_0} - L| < \varepsilon$$

In Complex Analysis, $f : \mathbb{C} \to \mathbb{C}$ is differentiable at x_0 if there is some $L \in \mathbb{C}$ such that for all $\varepsilon > 0$, there exists some $\delta > 0$ so that

$$\forall x, |x - x_0| < \delta \implies |\frac{f(x) - f(x_0)}{x - x_0} - L| < \varepsilon$$

where you will see that the $|\cdot|$ operation is called the modulus.

Another difference between real and complex analysis can be shown as the following example.

Example 1.2. Consider $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \Big\{ e^{-\frac{1}{x^2}} \text{ if } x \neq 00 \text{ if } x = 0$$

We can show that for any $n, f^{(n)}(0) = 0$ showing that f is infinitely differentiable. The Taylor Series at $x = 0$ gives

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(0)}{n!} x^n = \sum_{n=0}^{\infty} 0 = 0$$

Showing that the Taylor series of f does not approach f at a neighbourhood of 0.

If a real function is infinitely differentiable, still their Taylor series might not approaches to the function. In this case, we call the function is not analytic.

However, in complex analysis, differentiable $\implies$ infinitely differentiable $\implies$ analytic.

Example 1.3. Here are some applications of Complex Analysis

- Solving PDE, ODE
- Solving some integrals, e.g. $\int_0^\infty \cos(x^2)\, \mathrm{d}x$
- Analytic Number Theory, e.g. the Prime Number Theorem:

$$\lim_{n \to \infty} \frac{\pi(n)}{\frac{n}{\log(n)}} = 1$$

, where one proof uses the Riemann Zeta Function.

2 The Algebra of Complex Numbers

2.1 Arithmetic of Complex Numbers.

Definition 2.1. A complex number z is written as a form of $a + bi$ where $a, b \in \mathbb{R}$.

We also write $a = \mathrm{Re}(z), b = \mathrm{Im}(z)$ where a is the real part, b is the imaginary part.

Remark 2.2. We can also think about complex numbers as $\mathbb{R}^2 = \left\{ \binom{a}{b} : a, b \in \mathbb{R} \right\}$, where $i \to \binom{0}{1}, 1 \to \binom{1}{0}$.

Definition 2.3. (Operations of Complex Numbers)

- Addition: $(a + bi) + (c + di) = (a + c) + (b + d)i$
- Multiplication: $(a + bi) \cdot (c + di) = ac + adi + bci + bdi^2 = (ac - bd) + (ad + bc)i$

Proposition 2.4. *$\mathbb{C}$ is a field with $+$ and $\cdot$.*

Proof. (by William He) For any $x = x_1 + x_2 i, y = y_1 + y_2 i, z = z_1 + z_2 i \in \mathbb{C}$,

- $+$ is commutative: We have

$$\begin{aligned} x + y &= (x_1 + x_2 i) + (y_1 + y_2 i) \\ &= (x_1 + y_1) + (x_2 + y_2)i \\ &= (y_1 + x_1) + (y_2 + x_2)i \quad [+ \text{ is comutative in } \mathbb{R}] \\ &= (y_1 + y_2 i) + (x_1 + x_2 i) \\ &= y + x \end{aligned}$$

- $\cdot$ is commutative: We have

$$\begin{aligned}
x \cdot y &= (x_1 + x_2 i) \cdot (y_1 + y_2 i) \\
&= (x_1 \cdot y_1 - x_2 \cdot y_2) + (x_1 \cdot y_2 + x_2 \cdot y_1)i \\
&= (y_1 \cdot x_1 - y_2 \cdot x_2) + (y_2 \cdot x_1 + y_1 \cdot x_2)i \quad [\cdot \text{ is comutative in } \mathbb{R}] \\
&= (y_1 + y_2 i) \cdot (x_1 + x_2 i) \\
&= y \cdot x
\end{aligned}$$

- $+$ is associative: We have

$$\begin{aligned}
x + (y + z) &= (x_1 + x_2 i) + ((y_1 + y_2 i) + (z_1 + z_2 i)) \\
&= (x_1 + x_2 i) + ((y_1 + z_1) + (y_2 + z_2)i) \\
&= (x_1 + (y_1 + z_1)) + (x_2 + (y_2 + z_2))i \\
&= ((x_1 + y_1) + z_1) + ((x_2 + y_2) + z_2)i \quad [+ \text{ is commutative in } \mathbb{R}] \\
&= ((x_1 + y_1) + (x_2 + y_2)i) + (z_1 + z_2 i) \\
&= ((x_1 + x_2 i) + (y_1 + y_2 i)) + (z_1 + z_2 i) \\
&= (x + y) + z
\end{aligned}$$

- $\cdot$ is associative: We have

$$\begin{aligned}
&x \cdot (y \cdot z) \\
&= (x_1 + x_2 i) \cdot ((y_1 + y_2 i) \cdot (z_1 + z_2 i)) \\
&= (x_1 + x_2 i) \cdot ((y_1 \cdot z_1 - y_2 \cdot z_2) + (y_1 \cdot z_2 + y_2 \cdot z_1)i) \\
&= (x_1 \cdot (y_1 \cdot z_1 - y_2 \cdot z_2) + (-(x_2 \cdot (y_1 \cdot z_2 + y_2 \cdot z_1)))) \\
&\qquad + (x_1 \cdot (y_1 \cdot z_2 + y_2 \cdot z_1) + x_2 \cdot (y_1 \cdot z_1 - y_2 \cdot z_2))i \\
&= (x_1 \cdot (y_1 \cdot z_1) + (-(x_1 \cdot (y_2 \cdot z_2))) + (-(x_2 \cdot y_1 \cdot z_2) + (-(x_2 \cdot y_2 \cdot z_1)))) \\
&\qquad + (x_1 \cdot y_1 \cdot z_2 + x_1 \cdot y_2 \cdot z_1 + x_2 \cdot y_1 \cdot z_1 + (-(x_2 \cdot y_2 \cdot z_2)))i \quad [\cdot \text{ is commuta/associa/distributive in } \mathbb{R}] \\
&= (x_1 \cdot (y_1 \cdot z_1) + (-(x_2 \cdot y_2 \cdot z_1)) + (-(x_1 \cdot (y_2 \cdot z_2))) + (-(x_2 \cdot y_1 \cdot z_2))) \\
&\qquad + (x_1 \cdot y_1 \cdot z_2 + (-(x_2 \cdot y_2 \cdot z_2)) + x_1 \cdot y_2 \cdot z_1 + x_2 \cdot y_1 \cdot z_1)i \quad [+ \text{ is commuta/associative in } \mathbb{R}] \\
&= ((x_1 \cdot y_1) \cdot z_1 + (-(x_2 \cdot y_2)) \cdot z_1 + (-(x_1 \cdot y_2 \cdot z_2)) + (-(x_2 \cdot y_1 \cdot z_2))) \\
&\qquad + (x_1 \cdot y_1 \cdot z_2 + (-(x_2 \cdot y_2) \cdot z_2) + x_1 \cdot y_2 \cdot z_1 + x_2 \cdot y_1 \cdot z_1)i \quad [\cdot \text{ is associa/distributive in } \mathbb{R}] \\
&= ((x_1 \cdot y_1) \cdot z_1 + (-(x_2 \cdot y_2)) \cdot z_1 + (-(x_1 \cdot y_2 \cdot z_2 + x_2 \cdot y_1 \cdot z_2))) \\
&\qquad + (x_1 \cdot y_1 \cdot z_2 + (-(x_2 \cdot y_2) \cdot z_2) + x_1 \cdot y_2 \cdot z_1 + x_2 \cdot y_1 \cdot z_1)i \quad [\cdot \text{ is distributive in } \mathbb{R}] \\
&= (x_1 \cdot y_1 + (-(x_2 \cdot y_2)) \cdot z_1 + (-((x_1 \cdot y_2 + x_2 \cdot y_1) \cdot z_2))) \\
&\qquad + ((x_1 \cdot y_1 + (-(x_2 \cdot y_2)) \cdot z_2 + (x_1 \cdot y_2 + x_2 \cdot y_1) \cdot z_1)i \quad [\cdot \text{ is distributive in } \mathbb{R}] \\
&= ((x_1 \cdot y_1 + (-(x_2 \cdot y_2))) + (x_1 \cdot y_2 + x_2 \cdot y_1)i) \cdot (z_1 + z_2 i) \\
&= ((x_1 + x_2 i) \cdot (y_1 + y_2 i)) \cdot (z_1 + z_2 i) \\
&= (x \cdot y) \cdot z
\end{aligned}$$

- distributive of $\cdot$ under $+$: We have

$$
\begin{aligned}
& x\cdot(y+z)\\
&=(x_1+x_2i)\cdot((y_1+y_2i)+(z_1+z_2i))\\
&=(x_1+x_2i)\cdot((y_1+z_1)+(y_2+z_2)i)\\
&=(x_1\cdot(y_1+z_1)+(-(x_2\cdot(y_2+z_2))))+(x_2\cdot(y_1+z_1)+x_1\cdot(y_2+z_2))i\\
&=(x_1\cdot y_1+x_1\cdot z_1+(-(x_2\cdot y_2+x_2\cdot z_2)))\\
&\qquad +(x_2\cdot y_1+x_2\cdot z_1+x_1\cdot y_2+x_1\cdot z_2)i \quad [\cdot \text{ is distributive in } \mathbb{R}]\\
&=(x_1\cdot y_1+x_1\cdot z_1+(-(x_2\cdot y_2))+(-(x_2\cdot z_2)))\\
&\qquad +(x_2\cdot y_1+x_2\cdot z_1+x_1\cdot y_2+x_1\cdot z_2)i \quad [\cdot \text{ is distributive in } \mathbb{R}]\\
&=(x_1\cdot y_1+(-(x_2\cdot y_2))+x_1\cdot z_1+(-(x_2\cdot z_2)))\\
&\qquad +(x_2\cdot y_1+x_1\cdot y_2+x_2\cdot z_1+x_1\cdot z_2)i \quad [+ \text{ is commutative in } \mathbb{R}]\\
&=((x_1\cdot y_1+(-(x_2\cdot y_2)))+(x_1\cdot z_1+(-(x_2\cdot z_2))))\\
&\qquad +((x_2\cdot y_1+x_1\cdot y_2)+(x_2\cdot z_1+x_1\cdot z_2))i \quad [+ \text{ is associative in } \mathbb{R}]\\
&=((x_1\cdot y_1+(-(x_2\cdot y_2)))+(x_2\cdot y_1+x_1\cdot y_2)i)\\
&\qquad +((x_1\cdot z_1+(-(x_2\cdot z_2)))+(x_2\cdot z_1+x_1\cdot z_2)i)\\
&=(x_1+x_2i)\cdot(y_1+y_2i)+(x_1+x_2i)\cdot(z_1+z_2i)\\
&=x\cdot y+x\cdot z
\end{aligned}
$$

- additive identity: Take $\mathbf{0}=0+0i\in\mathbb{C}$, we have

$$
\begin{aligned}
\mathbf{0}+x&=(0+0i)+(x_1+x_2i)\\
&=(0+x_1)+(0+x_2)i\\
&=x_1+x_2i \quad [\text{Additive Identity in } \mathbb{R}]\\
&=x
\end{aligned}
$$

- multiplicative identity: Take $\mathbf{1}=1+0i\in\mathbb{C}$, we have

$$
\begin{aligned}
\mathbf{1}\cdot x&=(1+0i)\cdot(x_1+x_2i)\\
&=(1\cdot x_1+0\cdot x_2\cdot(-1))+(0\cdot x_1+1\cdot x_2)i\\
&=(x_1+0)+(0+x_2)i \quad [0\in\mathbb{R} \text{ satisfies } \forall x\in\mathbb{R}, 0\cdot x=x]\\
&=x_1+x_2i \quad [\text{Additive Identity in } \mathbb{R}]\\
&=x
\end{aligned}
$$

- additive inverse: Take $-x=-x_1+(-x_2)i\in\mathbb{C}$ where $-x_1,-x_2\in\mathbb{R}$ satisfies $x_1+(-x_1)=0, x_2+(-x_2)=0$ by the definition of additive inverses in $\mathbb{R}$, we have

$$
\begin{aligned}
x+(-x)&=(x_1+x_2i)+(-x_1+(-x_2)i)\\
&=(x_1+(-x_1))+(x_2+(-x_2))i\\
&=0+0i \quad [\text{Additive inverse in } \mathbb{R}]\\
&=\mathbf{0}
\end{aligned}
$$

- multiplicative inverse: Suppose $x=x_1+x_2i\neq 0$, then we must have $|x|^2=x_1^2+x_2^2>0$ by the definition of a norm. Hence, $(x_1^2+x_2^2)^{-1}$ exists by the definition of a multiplicative inverse in $\mathbb{R}$. This also means $x_1\cdot(x_1^2+x_2^2)^{-1}, x_2\cdot(x_1^2+x_2^2)^{-1}\in\mathbb{R}\implies -\Big(x_2\cdot(x_1^2+x_2^2)^{-1}\Big)\in\mathbb{R}$ by the definition of a field. Now, take $x^{-1}=x_1\cdot(x_1^2+x_2^2)^{-1}+\Big(-\Big(x_2\cdot(x_1^2+x_2^2)^{-1}\Big)\Big)i$ where $x_1\cdot(x_1^2+x_2^2)^{-1}, -\Big(x_2\cdot(x_1^2+x_2^2)^{-1}\Big)\in\mathbb{R}$, we have

$$\begin{aligned}
&x \cdot x^{-1} \\
&= (x_1 + x_2 i) \cdot \Big(x_1 \cdot (x_1^2 + x_2^2)^{-1} + \Big(-\Big(x_2 \cdot (x_1^2 + x_2^2)^{-1}\Big)\Big) i\Big) \\
&= \Big(x_1^2 \cdot (x_1^2 + x_2^2)^{-1} + \Big(-\Big(-\Big(x_2^2 \cdot (x_1^2 + x_2^2)^{-1}\Big)\Big)\Big)\Big) \\
&\qquad + \Big(x_1 \cdot \Big(-\Big(x_2 \cdot (x_1^2 + x_2^2)^{-1}\Big)\Big) + x_2 \cdot x_1 \cdot (x_1^2 + x_2^2)^{-1}\Big) i \\
&= \Big(x_1^2 \cdot (x_1^2 + x_2^2)^{-1} + x_2^2 \cdot (x_1^2 + x_2^2)^{-1}\Big) \\
&\qquad + \Big(x_1 \cdot \Big(-\Big(x_2 \cdot (x_1^2 + x_2^2)^{-1}\Big)\Big) + x_1 \cdot x_2 \cdot (x_1^2 + x_2^2)^{-1}\Big) i \quad [\cdot \text{ is commutative in } \mathbb{R}] \\
&= \Big((x_1^2 + x_2^2) \cdot (x_1^2 + x_2^2)^{-1}\Big) \\
&\qquad + \Big(x_1 \cdot \Big(-\Big(x_2 \cdot (x_1^2 + x_2^2)^{-1}\Big) + x_2 \cdot (x_1^2 + x_2^2)^{-1}\Big)\Big) i \quad [\cdot \text{ is distributive in } \mathbb{R}] \\
&= 1 + (x_1 \cdot 0) i \quad [\text{Additive and Multiplicative inverse in } \mathbb{R}] \\
&= 1 + 0i \\
&= \mathbf{1}
\end{aligned}$$

□

2.2 Complex Conjugate.

Definition 2.5. Given $z = a + bi \in \mathbb{C}$, we say $\overline{z} = a - bi$ is the conjugate of z.

Example 2.6.
- $\overline{5 + 2i} = 5 - 2i$
- $\overline{5} = 5$
- $\overline{-2i} = 2i$

Remark 2.7. Given $z = a + bi \in \mathbb{C}$, we also have
- $a = \text{Re}(z) = \frac{z+\overline{z}}{2}$
- $b = \text{Im}(z) = \frac{z-\overline{z}}{2i}$

We can therefore write the equation of a line $ax + by = c$ (where obviously $a^2 + b^2 > 0$) in terms of z and $\overline{z}$. We get

$$a\Big(\frac{z + \overline{z}}{2}\Big) + b\Big(\frac{z - \overline{z}}{2i}\Big) = c \Longrightarrow \Big(\frac{a}{2} + \frac{b}{2i}\Big) z + \Big(\frac{a}{2} - \frac{b}{2i}\Big)\overline{z} = c$$

Remark 2.8. We can also show that for any complex numbers $a + bi \neq 0$, then $\frac{1}{a+bi}$ exists. As

$$\frac{1}{a + bi} = \frac{1}{a + bi} \cdot \frac{a - bi}{a - bi} = \frac{a - bi}{a^2 + b^2} = \frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2} i$$

where $a^2 + b^2 > 0$.

Proposition 2.9. (Properties of Complex Numbers) *Given* $a, b \in \mathbb{C}$
1. $\overline{a + b} = \overline{a} + \overline{b}$
2. $\overline{ab} = \overline{a}\,\overline{b}$

Proof. (by William He) Since $a, b \in \mathbb{C}$ we can write $a = a_1 + a_2 i, b = b_1 + b_2 i$ for some $a_1, a_2, b_1, b_2 \in \mathbb{R}$.

1. We have

$$\begin{aligned}\overline{a+b} &= \overline{(a_1+b_1)+(a_2+b_2)i}\\ &= (a_1+b_1)-(a_2+b_2)i\\ &= (a_1-a_2i)+(b_1-b_2i)\\ &= \overline{a}+\overline{b}\end{aligned}$$

2. We have

$$\begin{aligned}\overline{ab} &= \overline{(a_1b_1-a_2b_2)+(a_1b_2+a_2b_1)i}\\ &= (a_1b_1-a_2b_2)-(a_1b_2+a_2b_1)i\\ &= (a_1-a_2i)(b_1-b_2i)\\ &= \overline{a}\overline{b}\end{aligned}$$

□

2.3 Modulus/Absolute Value.

Definition 2.10. Suppose $a = x+iy \in \mathbb{C}$, we define $|a|^2 = a\overline{a} = (x+iy)(x-iy) = x^2+y^2 \geq 0$
We also define the modulus/absolute value of a to be $|a| = \sqrt{x^2+y^2} \geq 0$

Proposition 2.11. (Properties of Modulus) *For all* $z \in \mathbb{C}$

1. *If* $z \neq 0, \frac{1}{z}\cdot\frac{\overline{z}}{\overline{z}} = \frac{\overline{z}}{|z|^2}$
2. $|z| = |\overline{z}|$
3. $-|z| \leq \mathrm{Re}(z) \leq |z|$

Proof. (by William He)

1. By definition
2. We can write $z = x+iy$ for some $x, y \in \mathbb{R}$, we have

$$|z| = |x+iy| = \sqrt{x^2+y^2} = |x+i(-y)| = |x-iy| = |\overline{z}|$$

3. We can write $z = x+iy$ for some $x, y \in \mathbb{R}$, we have

$$-|z| = -\sqrt{x^2+y^2} \leq -\sqrt{x^2} \leq x \leq \sqrt{x^2} \leq \sqrt{x^2+y^2} = |z|$$

□

Proposition 2.12. (More Properties of Modulus) *For all* $a, b \in \mathbb{C}$

1. $|ab| = |a||b|$
2. $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}, b \neq 0$
3. $|a+b| \leq |a|+|b|$

For all $a_1, ..., a_n \in \mathbb{C}$

4. $|a_1...a_n| = |a_1|\ ...\ |a_n|$

Proof.

1. We have

$$|ab|^2 = (ab)\big(\overline{ab}\big) = ab\overline{a}\overline{b} = ab\overline{a}\overline{b} = a\overline{a}b\overline{b} = |a|^2|b|^2$$

Since all modulus are non-negative, we get $|ab| = |a||b|$

2. Since $b \neq 0, |b| > 0$, we have

$$\left|\frac{a}{b}\right| = \left|\frac{a}{b}\cdot 1\right| = \left|\frac{a}{b}\cdot\frac{\bar{b}}{\bar{b}}\right| = \left|\frac{a\bar{b}}{|b|^2}\right| = \frac{1}{|b|^2}\ |a\bar{b}| = \frac{1}{|b|^2}\ |a||\bar{b}| = \frac{1}{|b|^2}\ |a||b| = \frac{|a|}{|b|}$$

where we can pull the constants out as for all $z = u + iv \in \mathbb{C}$, for any $c \neq 0$, we have $\left|\frac{z}{c}\right| = \left|\frac{u}{c} + i\frac{v}{c}\right| = \sqrt{\frac{u^2}{c^2} + \frac{v^2}{c^2}} = \sqrt{\frac{u^2+v^2}{c^2}} = \left|\frac{1}{c}\right|\sqrt{u^2+v^2} = \left|\frac{1}{c}\right||z|$. In our case, $z = a\bar{b}, c = |b|$.

3. We have

$$\begin{aligned}
|a+b|^2 &= (a+b)\overline{a+b} \\
&= (a+b)\big(\bar{a}+\bar{b}\big) \\
&= a\bar{a} + b\bar{a} + a\bar{b} + b\bar{b} \\
&= |a|^2 + 2\operatorname{Re}\big(a\bar{b}\big) + |b|^2 \quad \Big[2\operatorname{Re}\big(a\bar{b}\big) = a\bar{b} + \overline{a\bar{b}} = a\bar{b} + \bar{a}\bar{\bar{b}} = a\bar{b} + b\bar{a}\Big] \\
&\leq |a|^2 + 2\ |a\bar{b}| + |b|^2 \quad [\texttt{Proposition 2.11}] \\
&= |a|^2 + 2|a||\bar{b}| + |b|^2 \quad [\text{By Part 1}] \\
&= |a|^2 + 2|a||b| + |b|^2 \\
&= (|a| + |b|)^2
\end{aligned}$$

Since all modulus are non-negative, we get $|a+b| = |a| + |b|$

4. We will prove by induction. The case where $n = 1$ is trivial.

Suppose $n = k$ is true, we show $n = k+1$ is also true, take arbitrary $a_1, ..., a_{k+1} \in \mathbb{C}$, we have

$$\begin{aligned}
|a_1...a_{k+1}| &= |(a_1...a_k)\cdot a_{k+1}| \\
&= |a_1...a_k| \cdot |a_{k+1}| \quad [\text{By Part 1}] \\
&= |a_1|\ ...\ |a_k| \cdot |a_{k+1}| \quad [\text{By Inductive Hypothesis}] \\
&= |a_1|\ ...\ |a_{k+1}|
\end{aligned}$$

□

2.4 **Exercises.**

Exercise 2.13. Find the values of

$$(1+2i)^3, \frac{5}{-3+4i}, \left(\frac{2+i}{3-2i}\right)^2, (1+i)^n + (1-i)^n$$

Exercise 2.14. If $z = x + iy$ (x and y real), find the real and imaginary parts of

$$z^4, \frac{1}{z}, \frac{z-1}{z+1}, \frac{1}{z^2}$$

Exercise 2.15. Compute $\sqrt[4]{i}$ and $\sqrt[4]{-i}$

Exercise 2.16. Solve the quadratic equation

$$z^2 + (\alpha + i\beta)z + \gamma + i\delta = 0$$

Exercise 2.17. Show that the system of all matrices of the special form

$$\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$$

combined by matrix addition and matrix multiplication, is isomorphic to the field of complex numbers.

Exercise 2.18. Verify by calculation that the values of

$$\frac{z}{z^2+1}$$

for $z = x + iy$ and $\overline{z} = x - iy$ are conjugate.

Exercise 2.19. Find the absolute values of

$$-2i(3+i)(2+4i)(1+i) \quad \text{and} \quad \frac{(3+4i)(-1+2i)}{(-1-i)(3-i)}$$

Exercise 2.20. Prove that

$$\left|\frac{a-b}{1-\overline{a}b}\right| = 1$$

if either $|a| = 1$ or $|b| = 1$. What exception must be made if $|a| = |b| = 1$?

Exercise 2.21. Prove that

$$\left|\frac{a-b}{1-\overline{a}b}\right| < 1$$

if $|a| < 1$ and $|b| < 1$.

Exercise 2.22. Prove Cauchy's inequality by induction.

Exercise 2.23. If $|a_i| < 1, \lambda_i \geq 0$ for $i = 1, ..., n$ and $\lambda_1 + ... + \lambda_n = 1$, show that

$$|\lambda_1 a_1 + ... + \lambda_n a_n| \leq 1$$

3 The Geometric Representation of Complex Numbers

3.1 Geometry of the Complex Numbers.

Geometrically (as shown in Figure 1), we can write a complex number $z = x + iy$ as

$$z = r(\cos(\theta) + i\sin(\theta))$$

where $r = \sqrt{x^2+y^2} \in \mathbb{R}, \theta = \arg(z) \in \mathbb{R}$

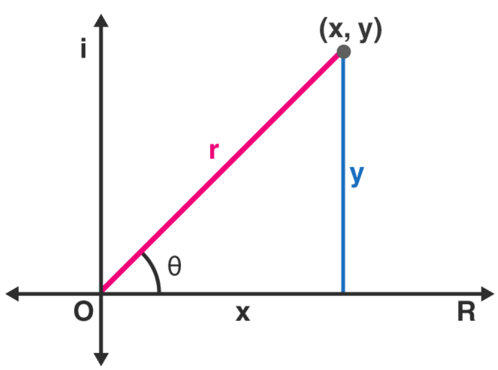

Figure 1. Complex Numbers Polar Form

Theorem 3.1. *For all $z, w \in \mathbb{C}$*

1. $\arg(zw) = \arg(z) + \arg(w)$
2. $\arg(\frac{z}{w}) = \arg(z) - \arg(w)$ *if* $w \neq 0$

The above are correct up to an integer multiple of 2π

Proof. Given $z = r_1(\cos(\theta_1) + i\sin(\theta_1)), w = r_2(\cos(\theta_2) + i\sin(\theta_2))$

1. We have

$$\begin{aligned}
&\arg(zw)\\
&= \arg(r_1(\cos(\theta_1) + i\sin(\theta_1))r_2(\cos(\theta_2) + i\sin(\theta_2)))\\
&= \arg(r_1r_2(\cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2) + i(\sin(\theta_1)\cos(\theta_2) + \cos(\theta_1)\sin(\theta_2))))\\
&= \arg(r_1r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)))\\
&= \theta_1 + \theta_2\\
&= \arg(z) + \arg(w)
\end{aligned}$$

2. If $w \neq 0$, we have

$$\begin{aligned}
&\arg\left(\frac{z}{w}\right)\\
&= \arg\left(\frac{r_1(\cos(\theta_1) + i\sin(\theta_1))}{r_2(\cos(\theta_2) + i\sin(\theta_2))}\right)\\
&= \arg\left(\frac{r_1}{r_2}\frac{\cos(\theta_1)\cos(\theta_2) + \sin(\theta_1)\sin(\theta_2) + i(\sin(\theta_1)\cos(\theta_2) - \cos(\theta_1)\sin(\theta_2))}{\cos^2(\theta_2) + \sin^2(\theta_2)}\right)\\
&= \arg\left(\frac{r_1}{r_2}(\cos(\theta_1)\cos(-\theta_2) - \sin(\theta_1)\sin(-\theta_2) + i(\sin(\theta_1)\cos(-\theta_2) + \cos(\theta_1)\sin(-\theta_2)))\right)\\
&= \arg\left(\frac{r_1}{r_2}(\cos(\theta_1 - \theta_2) + i(\sin(\theta_1 - \theta_2)))\right)\\
&= \theta_1 - \theta_2\\
&= \arg(z) - \arg(w)
\end{aligned}$$

□

Remark 3.2. Take $z = w = -1$, we have $\arg(z) = \arg(w) = \pi$. Hence

$$\arg(z) + \arg(w) = 2\pi = 0 \bmod 2\pi = \arg(1) = \arg(zw)$$

Remark 3.3. By `Theorem 3.1`, we also see that when we are computing zw, we are essentially multiplying the length of both z and w and rotate z by the angle of w.

3.2 De Moivre Formula.

Theorem 3.4. (De Moivre Formula) *For any* $z = r(\cos(\theta) + i\sin(\theta)), z^n = r^n(\cos(n\theta) + i\sin(n\theta))$ *for all* $n \in \mathbb{N}$

Proof. We prove by induction.

Base case ($n = 1$): This is trivially true.

Inductive step: Suppose this is true for some $n = k$, we show this is also true for $n = k + 1$, we have

$$\begin{aligned}
z^{k+1} &= z^k \cdot z\\
&= r^k(\cos(k\theta) + i\sin(k\theta)) \cdot r(\cos(\theta) + i\sin(\theta))\\
&= r^{k+1}(\cos(k\theta)\cos(\theta) - \sin(k\theta)\sin(\theta) + i(\cos(k\theta)\sin(\theta) + \sin(k\theta)\cos(\theta)))\\
&= r^{k+1}(\cos((k + 1)\theta) + i\sin((k + 1)\theta))
\end{aligned}$$

□

Remark 3.5. This gives a way to find the n-root of a complex number.

Suppose we want to solve all the possible z such that

$$z^n = a$$

for some $n \in \mathbb{N}, a \in \mathbb{C}$

Note that z can be written as some $z = r(\cos(\theta) + i\sin(\theta))$ by De Moivre Formula, $z^n = r^n(\cos(n\theta) + i\sin(n\theta))$. Similarly, we can write $a = \rho(\cos(\varphi) + i\sin(\varphi))$. This gives us

$$r^n(\cos(n\theta) + i\sin(n\theta)) = \rho(\cos(\varphi) + i\sin(\varphi))$$

$$\Rightarrow \begin{cases} r^n = \rho \\ n\theta = \varphi + 2k\pi \quad k \in \mathbb{N} \end{cases}$$

$$\Rightarrow \begin{cases} r = \sqrt[n]{\rho} \\ \theta = \frac{\varphi+2k\pi}{n} \quad k = 0, ..., n-1 \end{cases}$$

This gives the solutions $z = \sqrt[n]{\rho}\Big(\cos\Big(\frac{\varphi+2k\pi}{n}\Big) + i\sin\Big(\frac{\varphi+2k\pi}{n}\Big)\Big)$ for all $k = 0, ..., n-1$

Example 3.6. Find all z so that $z^3 = i$ (equivalently, find $\sqrt[3]{i}$)

Proof. Since z is a solution, we can denote $z = r(\cos(\theta) + i\sin(\theta))$ where

$$z^3 = r^3(\cos(3\theta) + i\sin(3\theta)) = i = 1 \cdot \left(\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right)$$

This gives

$$\begin{cases} r^3 = 1 \\ 3\theta = \frac{\pi}{2} + 2k\pi \quad k \in \mathbb{N} \end{cases} \Rightarrow \begin{cases} r = 1 \\ \theta = \frac{\frac{\pi}{2}+2k\pi}{3} \quad k = 0, 1, 2 \end{cases} \Rightarrow \begin{cases} r = 1 \\ \theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2} \end{cases}$$

Hence, $\sqrt[3]{i} = \cos\big(\frac{\pi}{6}\big) + i\sin\big(\frac{\pi}{6}\big), \cos\big(\frac{5\pi}{6}\big) + i\sin\big(\frac{5\pi}{6}\big), \cos\big(\frac{3\pi}{2}\big) + i\sin\big(\frac{3\pi}{2}\big)$

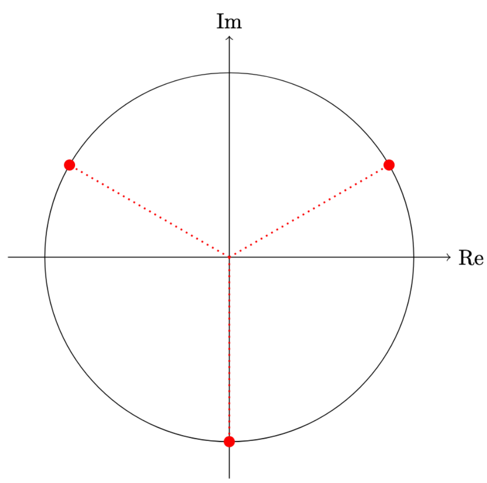

FIGURE 2. Cube roots of i

□

Remark 3.7. The n-roots of $z^n = r$ for some $r \in \mathbb{C}$ are always equally spaced on a circle with radius $\sqrt[n]{|r|}$.

Remark 3.8. Given some $n \in \mathbb{N}$ we have all the solutions z satisfying $z^n = 1$ can be written as

$$z = \cos\left(\frac{2k\pi}{n}\right) + i\sin\left(\frac{2k\pi}{n}\right) \quad k = 0, ..., n-1$$

Thus, all the answers can be written as the form $1, w, w^2, ..., w^{n-1}$ where

$$w = \cos\left(\frac{2\pi}{n}\right) + i\sin\left(\frac{2\pi}{n}\right)$$

3.3 The Spherical Representation.

Note first that before preceding complex analysis, ∞ is not yet defined under the current definition of the complex numbers. In the plane there is no room for a point corresponding to ∞, but we can of course introduce such an "ideal" point.

Definition 3.9. We denote

$$S = \{(x_1, x_2, x_3) : x_1^2 + x_2^2 + x_3^2 = 1, x_1, x_2, x_3 \in \mathbb{R}\}$$

as the unit sphere. We also denote $N = (0, 0, 1)$ as the north pole. Similarly, $S = (0, 0, -1)$ as the south pole.

We now aim to associate every point on such sphere, except for the point $(0, 0, 1)$, a complex number.

Proposition 3.10. *The function $f : S \setminus \{(0, 0, 1)\} \to \mathbb{C}$ by*

$$f(Z) = z = \frac{x_1 + ix_2}{1 - x_3}$$

would be a bijection that associates all complex numbers to the unit sphere excluding the north pole.

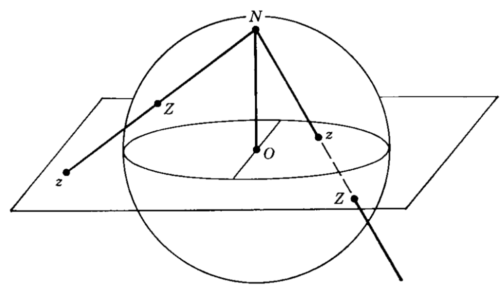

FIGURE 3. Cube roots of i

Proof. We will first find an inverse of f, denoting it by g. If z exists, we have

$$|z|^2 = \frac{x_1^2 + x_2^2}{(1 - x_3)^2} = \frac{1 + x_3}{1 - x_3} \implies |z|^2 - |z|^2\ x_3 = 1 + x_3 \implies x_3 = \frac{|z|^2 - 1}{|z|^2 + 1}$$

We also have

$$z + \overline{z} = \frac{2x_1}{1 - x_3} \implies x_1 = \frac{z + \overline{z}}{1 + |z|^2}$$

$$z - \overline{z} = \frac{2ix_2}{1 - x_3} \implies x_2 = \frac{z - \overline{z}}{i(1 + |z|^2)}$$

This gives the association that for all $z \in \mathbb{C}$, which can be expressed as a function $g : \mathbb{C} \to S \setminus \{(0, 0, 1)\}$ by

$$g(z) = Z = \left(\frac{z + \overline{z}}{1 + |z|^2}, \frac{z - \overline{z}}{i(1 + |z|^2)}, \frac{|z|^2 - 1}{|z|^2 + 1}\right)$$

which is indeed in $S \setminus \{(0, 0, 1)\}$ as

$$\left(\frac{z + \overline{z}}{1 + |z|^2}\right)^2 + \left(\frac{z - \overline{z}}{i(1 + |z|^2)}\right)^2 + \left(\frac{|z|^2 - 1}{|z|^2 + 1}\right)^2 = \frac{(|z|^2 - 1)^2 + 4z\overline{z}}{(|z|^2 + 1)^2} = 1$$

Now, we can show this association is indeed bijective by showing $f \circ g = g \circ f = \text{id}$. We have for all $z \in \mathbb{C}$,

$$\begin{aligned}
f(g(z)) &= f\left(\frac{z+\overline{z}}{1+|z|^2}, \frac{z-\overline{z}}{i(1+|z|^2)}, \frac{|z|^2-1}{|z|^2+1}\right) \\
&= \frac{\frac{z+\overline{z}}{1+|z|^2} + i\frac{z-\overline{z}}{i(1+|z|^2)}}{1-\frac{|z|^2-1}{|z|^2+1}} \\
&= \frac{z+\overline{z}+z-\overline{z}}{2} \\
&= z
\end{aligned}$$

Similarly, for any $Z = (x_1, x_2, x_3) \in S \setminus \{(0,0,1)\}$, we have

$$\begin{aligned}
g(f(Z)) &= g\left(\frac{x_1+ix_2}{1-x_3}\right) \\
&= \left(\frac{\frac{x_1+ix_2}{1-x_3} + \overline{\frac{x_1+ix_2}{1-x_3}}}{1+\left|\frac{x_1+ix_2}{1-x_3}\right|^2}, \frac{\frac{x_1+ix_2}{1-x_3} - \overline{\frac{x_1+ix_2}{1-x_3}}}{i\left(1+\left|\frac{x_1+ix_2}{1-x_3}\right|^2\right)}, \frac{\left|\frac{x_1+ix_2}{1-x_3}\right|^2-1}{\left|\frac{x_1+ix_2}{1-x_3}\right|^2+1}\right) \\
&= \frac{1}{\left(1+\left|\frac{x_1+ix_2}{1-x_3}\right|^2\right)}\left(\frac{2x_1}{1-x_3}, \frac{2x_2}{1-x_3}, \frac{x_1^2+x_2^2}{(1-x_3)^2}-1\right) \\
&= \frac{1}{\left(1+\frac{x_1^2+x_2^2}{(1-x_3)^2}\right)}\left(\frac{2x_1}{1-x_3}, \frac{2x_2}{1-x_3}, \frac{x_1^2+x_2^2}{(1-x_3)^2}-1\right) \\
&= \frac{1}{\left(1+\frac{1-x_3^2}{(1-x_3)^2}\right)}\left(\frac{2x_1}{1-x_3}, \frac{2x_2}{1-x_3}, \frac{1-x_3^2}{(1-x_3)^2}-1\right) \\
&= \frac{1}{\left(1+\frac{1+x_3}{1-x_3}\right)}\left(\frac{2x_1}{1-x_3}, \frac{2x_2}{1-x_3}, \frac{1+x_3}{1-x_3}-1\right) \\
&= \frac{1-x_3}{2}\left(\frac{2x_1}{1-x_3}, \frac{2x_2}{1-x_3}, \frac{2x_3}{1-x_3}\right) \\
&= (x_1, x_2, x_3) \\
&= Z
\end{aligned}$$

□

Remark 3.11. Note that under this association, if $z \in \mathbb{C}$ and $|z| < 1$ then $x_3 < 0$, and $x_3 > 0$ if $|z| > 1$.

Theorem 3.12. *The above transformation g from the complex plane to the unit sphere transforms every line in the complex plane into a circle on S which passes through the point $N = (0,0,1)$ and the converse is true as well.*

Proof. Consider a circle on S, this circle must lies on a plane $\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 = \alpha_0$. Substituting $(0,0,1)$ into the equation we get $\alpha_3 = \alpha_0$. Note by scaling, we can also safely assume that $\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1$ and $\alpha_3 \geq 0$. Furthermore, $\alpha_3 \neq 1$ as otherwise the plane only intersects the sphere at one point. Hence, we have $0 \leq \alpha_0 = \alpha_3 < 1$.

Write it in terms of z and $\overline{z}$ where $z = x + iy$ we get

$$\begin{aligned}
&\alpha_1 \frac{2x}{|z|^2+1} + \alpha_2 \frac{2y}{|z|^2+1} + \alpha_3 \frac{|z|^2-1}{|z|^2+1} = \alpha_0 \\
\implies& 2x \cdot \alpha_1 + 2y \cdot \alpha_2 + \alpha_3(|z|^2-1) = \alpha_0(|z|^2+1) \\
\implies& \alpha_1(z+\overline{z}) - \alpha_2 i(z-\overline{z}) + \alpha_3(|z|^2-1) = \alpha_0(|z|^2+1)
\end{aligned}$$

Rearrange this we would get

$$(\alpha_0 - \alpha_3)(x^2 + y^2) - 2\alpha_1 x - 2\alpha_2 y + \alpha_0 + \alpha_3 = 0$$

Since $\alpha_0 = \alpha_3$ then this is an equation of a line in the complex plane.

Conversely, any line in the complex plane can be written as the form

$$-2\alpha_1 x - 2\alpha_2 y + \alpha_0 + \alpha_3 = 0$$

such that $0 \leq \alpha_0 = \alpha_3 < 1$ and $\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1$. Working backwards we will see that this is a circle on the sphere that lines on the plane $\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 = \alpha_0$. □

Remark 3.13. Now, given $z, z' \in \mathbb{C}$, we can define a distance between the two using their stereographic projections. Suppose those points are denoted by $(x_1, x_2, x_3), ({x_1}', {x_2}', {x_3}')$ respectively. Then we have

$$\begin{aligned}
d(z, z') &= \sqrt{(x_1 - {x_1}')^2 + (x_2 - {x_2}')^2 + (x_3 - {x_3}')^2} \\
&= \sqrt{2 - 2(x_1 {x_1}' + x_2 {x_2}' + x_3 {x_3}')} \\
&= \sqrt{2 - 2\frac{(z + \overline{z})(z' + \overline{z'}) - (z - \overline{z})(z' - \overline{z'}) + (|z|^2 - 1)(|z'|^2 - 1)}{(1 + |z|^2)(1 + |z'|^2)}} \\
&= \sqrt{2 - 2\frac{(1 + |z|^2)(1 + |z'|^2) - 2|z - z'|^2}{(1 + |z|^2)(1 + |z'|^2)}} \\
&= \sqrt{\frac{4|z - z'|^2}{(1 + |z|^2)(1 + |z'|^2)}} \\
&= \frac{2|z - z'|}{\sqrt{(1 + |z|^2)(1 + |z'|^2)}}
\end{aligned}$$

If $z' = \infty$, we have

$$d(z, z') = \frac{2}{\sqrt{1 + |z|^2}}$$

This gives us a metric defined in $\mathbb{C} \cup \{\infty\}$, we can define an ε-neighbourhood of ∞ as follows

Definition 3.14. Given $\varepsilon > 0$, we have

$$\begin{aligned}
B(\infty, \varepsilon) &= B_\varepsilon(\infty) \\
&= \{z \in \mathbb{C} : d(z, \infty) < \varepsilon\} \\
&= \left\{ z \in \mathbb{C} : \frac{2}{\sqrt{1 + |z|^2}} < \varepsilon \right\} \\
&= \left\{ z \in \mathbb{C} : |z| > \sqrt{\frac{4}{\varepsilon^2} - 1} \right\}
\end{aligned}$$

3.4 **Exercises.**

Exercise 3.15. Find the symmetric points of a with respect to the lines which bisect the angles between the coordinate axes.

Exercise 3.16. Prove that the points a_1, a_2, a_3 are vertices of an equilateral triangle if and only if $a_1^2 + a_2^2 + a_3^2 = a_1 a_2 + a_2 a_3 + a_3 a_1$

Exercise 3.17. Suppose that a and b are two vertices of a square. Find the two other vertices in all possible cases.

Exercise 3.18. Find the center and the radius of the circle which circumscribes the triangle with vertices a_1, a_2, a_3. Express the result in symmetric form.

Exercise 3.19. Express the fifth and tenth roots of unity in algebraic form.

Exercise 3.20. When does $az + b\overline{z} + c = 0$ represent a line?

Exercise 3.21. Show that z and z' correspond to diametrically opposite points on the Riemann sphere if and only if $z\overline{z'} = -1$.

Exercise 3.22. Let Z, Z' denote the stereographic projections of z, z', and let N be the north pole. Show that the triangles NZZ' and Nzz' are similar, and use this to derive (28).

Exercise 3.23. Find the radius of the spherical image of the circle in the plane whose center is a and radius R.

4 Complex Functions

4.1 Limits and continuity.

Definition 4.1. Take $a \in \mathbb{C}, \delta > 0$, we define

$$B(a, \delta) = B_\delta(a) = \{z \in \mathbb{C} : |z - a| < \delta\}$$

to be the δ-neighbourhood of a

In limit context, we say ε-neighbourhood of ∞ is $B(\infty, \varepsilon) = \left\{z \in \mathbb{C} : |z| > \frac{1}{\varepsilon}\right\}$

Definition 4.2. The function $f(x)$ is said to have the limit L as x approaches a denoting $\lim_{x\to a} f(x) = A$, if and only if

$$\forall\varepsilon > 0, \exists\delta > 0, \text{ s.t. } 0 < |x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon$$

Remark 4.3. Note again in the real analysis, we might have $x \to \pm\infty$ or $f(x) \to \pm\infty$. However, for complex functions, as there is only one infinity, we can only have $z \to \infty$ or $f(z) \to \infty$.

Theorem 4.4. *Let $f : \mathbb{C} \to \mathbb{C}$ be a complex function and $\lim_{z\to a} f(z) = L$ (where a or/and L can be ∞), then:*

1. $\lim_{z\to a} \overline{f(z)} = \overline{L}$
2. $\lim_{z\to a} |f(z)| = |L|$
3. $\lim_{z\to a} \operatorname{Re}(f(z)) = \operatorname{Re}(L)$ *and* $\lim_{z\to a} \operatorname{Im}(f(z)) = \operatorname{Im}(L)$
4. $\lim_{z\to a} cf(z) = cL$ *for any* $c \in \mathbb{C}$

If $g : \mathbb{C} \to \mathbb{C}$ and $\lim_{z\to a} g(z) = M$

5. $\lim_{z\to a} f(z) \pm \lim_{z\to a} g(z) = L \pm M$
6. $\lim_{z\to a} f(z)g(z) = L \cdot M$ *(the case where $0 \cdot \infty$ is undefined)*
7. $\lim_{z\to a} \frac{f(z)}{g(z)} = \frac{L}{M}, M \neq 0$

Proof.

1. Since $\lim_{z\to a} f(z) = L$, we have

$$\begin{aligned} &\forall\varepsilon > 0, \exists\delta > 0, \text{ s.t. } 0 < |x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon \\ \Longrightarrow\ &\forall\varepsilon > 0, \exists\delta > 0, \text{ s.t. } 0 < |x - a| < \delta \Rightarrow |\overline{f(x) - L}| < \varepsilon \\ \Longrightarrow\ &\forall\varepsilon > 0, \exists\delta > 0, \text{ s.t. } 0 < |x - a| < \delta \Rightarrow |\overline{f(x)} - \overline{L}| < \varepsilon \end{aligned}$$

which is the definition of $\lim_{z\to a} \overline{f(z)} = L$.

2. Since $\lim_{z\to a} f(z) = L$, we have

$$\begin{aligned}
&\forall \varepsilon > 0, \exists \delta > 0, \text{ s.t. } 0 < |x-a| < \delta \Rightarrow |f(x) - L| < \varepsilon \\
\Longrightarrow\ &\forall \varepsilon > 0, \exists \delta > 0, \text{ s.t. } 0 < |x-a| < \delta \Rightarrow ||f(x)| - |L|| \leq |f(x) - L| < \varepsilon \\
\Longrightarrow\ &\forall \varepsilon > 0, \exists \delta > 0, \text{ s.t. } 0 < |x-a| < \delta \Rightarrow ||f(x)| - |L|| < \varepsilon
\end{aligned}$$

which is the definition of $\lim_{z\to a}|f(z)| = |L|$.

3. We have $\text{Re}(f(z)) = \frac{1}{2}(z + \overline{z})$ and $\text{Im}(f(z)) = \frac{1}{2i}(z - \overline{z})$, the limit property is satisfied by Part 1, 4, 5.
4. Given any $c \in \mathbb{C}$, and $\varepsilon > 0$, take δ such that for all $0 < |x-a| < \delta, |f(x) - L| < \frac{\varepsilon}{|c|}$. We have for all $0 < |x - a| < \delta$,

$$|cf(x) - cL| = |c(f(x) - L)| = |c|\ |f(x) - L| < \varepsilon$$

5. Given any $\varepsilon > 0$, take δ_1 such that for all $0 < |x-a| < \delta_1, |f(x) - L| < 0.5\varepsilon$. Take δ_2 such that for all $0 < |x-a| < \delta_2, |g(x) - M| < 0.5\varepsilon$. Finally, take $\delta = \min(\delta_1, \delta_2)$ we have for all $0 < |x-a| < \delta$,

$$|(f(x) \pm g(x)) - (L \pm M)| \leq |f(x) - L| + |g(x) - M| < \varepsilon$$

□

Definition 4.5. The function $f(x)$ is said to be continuous at a if and only if $\lim_{x\to a} f(z) = f(a)$, equivalently, this is

$$\forall \varepsilon > 0, \exists \delta > 0, \text{ s.t. } 0 \leq |x-a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$$

A function is said to be continuous if it is continuous at every point in its domain.

Example 4.6. Prove that $f : \mathbb{C} \to \mathbb{C}$ defined by $f(z) = \overline{z}$ is a continuous function.

Proof. We want to prove that for any $z_0 \in \mathbb{C}, \lim_{z\to z_0} f(z) = f(z_0)$.

Given $\varepsilon > 0$, take $\delta = \varepsilon$, we have for all $|z - z_0| < \delta = \varepsilon$

$$|f(z) - f(z_0)| = |\overline{z} - \overline{z_0}| = |\overline{z - z_0}| = |z - z_0| < \varepsilon$$

showing that f is continuous at z_0. Since z_0 is arbitrary, we just showed that f is continuous. □

Example 4.7. Prove that $f : \mathbb{C} \to \mathbb{C}$ is a continuous function, then $g : \mathbb{C} \to \mathbb{R}$ defined by $g(z) = \text{Re}(f(z))$ and $h : \mathbb{C} \to \mathbb{R}$ defined by $h(z) = \text{Im}(f(z))$ are all continuous.

Proof. The proof is essentially similar to the limit case in `Theorem 4.4`,. □

Remark 4.8. `Theorem 4.4` also works in the continuous case.

4.2 **Derivatives.**

Definition 4.9. The function $f(x)$ is said to be differentiable at a if and only if there exists some $f'(a) \in \mathbb{C}$ such that

$$f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a}$$

A function is said to be differentiable/analytic/holomorphic when it is differentiable on the entire domain.

Example 4.10.

1. If $f : \mathbb{R} \to \mathbb{R}$, applying the definition this is essentially the definition in the real analysis case.
2. If $f : \mathbb{R} \to \mathbb{C}$, we can write $f(t) = u(t) + iv(t)$. Hence, we have $f'(t) = u'(t) + iv'(t)$.
3. If $f : \mathbb{C} \to \mathbb{R}$, if the derivative exists at $a \in \mathbb{C}$ then $f'(a) = 0$ for all $a \in \mathbb{C}$. Indeed, by definition we have

$$f'(a) = \lim_{z \to a} \frac{f(z) - f(a)}{z - a} = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h} \in \mathbb{R}$$

We also have

$$f'(a) = \lim_{z \to a} \frac{f(z) - f(a)}{z - a} = \lim_{h \to 0} \frac{f(z+ih) - f(z)}{ih}$$

which is purely imaginary.

Hence, this means we must have $f'(a) = 0$.

4. If $f : \mathbb{C} \to \mathbb{C}$, we will only study this function in this course. $f : \mathbb{C} \to \mathbb{C}$ is differentiable at $a \in \mathbb{C}$ if and only if

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{z \to a} \frac{f(z) - f(a)}{z - a}$$

exists

Example 4.11. $f : \mathbb{C} \to \mathbb{C}$ defined by $f(z) = z^2$ is differentiable everywhere and $f'(z) = 2z$.

Proof. Indeed, for any $z \in \mathbb{C}$, we have

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h} = \lim_{h \to 0} \frac{(z+h)^2 - z^2}{h} = \lim_{h \to 0} \frac{2zh + h^2}{h} = \lim_{h \to 0} 2z + h = 2z$$

□

Example 4.12. $f : \mathbb{C} \to \mathbb{C}$ defined by $f(z) = \overline{z}$ is differentiable nowhere

Proof. For any $z \in \mathbb{C}$, we have

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h} = \lim_{h \to 0} \frac{\overline{z+h} - \overline{z}}{h} = \lim_{h \to 0} \frac{\overline{h}}{h}$$

Note that if we take $h \in \mathbb{R}$ (approach along the real axis), we have

$$\lim_{h \to 0} \frac{\overline{h}}{h} = \frac{h}{h} = 1$$

If we take $h = it$ (approach along the imaginary axis), we have

$$\lim_{h \to 0} \frac{\overline{h}}{h} = \frac{-h}{h} = -1$$

showing $f'(z)$ does not exist. □

Proposition 4.13. *Suppose $f : G \to \mathbb{C}$ is differentiable at a point $a \in G$. Then, f is continuous at a.*

Proof. (by Ahmad) Erm, consider

$$\lim_{z \to a} |f(z) - f(a)| = \left[\lim_{z \to a} \frac{|f(z) - f(a)|}{|z - a|} \right] \cdot \left[\lim_{z \to a} |z - a| \right] = f'(a) \cdot a = 0.$$

From here it is just a simple epsilon-delta argument. □

While not given in lecture (as far as i can remember -_-), the usual laws for differentiating sums, products, and quotients stay valid. In addition, the chain rule also holds (i refuse to prove it).

Proposition 4.14. *If $f : \mathbb{C} \to \mathbb{C}$ is differentiable at z_0, then*

$$\mathrm{Re}(f'(z_0)) = \lim_{\Delta z \to 0} \mathrm{Re}\left(\frac{f(z_0+\Delta z)-f(z_0)}{\Delta z}\right)$$
$$\mathrm{Im}(f'(z_0)) = \lim_{\Delta z \to 0} \mathrm{Im}\left(\frac{f(z_0+\Delta z)-f(z_0)}{\Delta z}\right)$$

Proof. This is straightforward from `Theorem 4.4`. □

Remark 4.15. Observe that we can write $\Delta z = \Delta x + i\Delta y$

Theorem 4.16. (Cauchy-Riemann Equations) *Suppose that $f(z) = u(x,y) + iv(x,y)$, and $f'(z_0)$ exists at a point $z_0 = x_0 + iy_0$, then*

$$\frac{\partial u}{\partial x}(x_0,y_0) = \frac{\partial v}{\partial y}(x_0,y_0)$$

and

$$\frac{\partial u}{\partial y}(x_0,y_0) = -\frac{\partial v}{\partial x}(x_0,y_0)$$

Proof. We have

$$\begin{aligned}\lim_{\Delta z\to 0}\frac{f(z_0+\Delta z)-f(z_0)}{\Delta z} &= \lim_{\Delta z\to 0}\frac{u(x_0+\Delta x, y_0+i\Delta y)-u(x_0,y_0)+iv(x_0+\Delta_x, y_0+i\Delta y)-iv(x_0,y_0)}{\Delta z}\\ &= \lim_{\Delta z\to 0}\frac{u(x_0+\Delta x, y_0+i\Delta y)-u(x_0,y_0)}{\Delta z}+i\frac{v(x_0+\Delta x, y_0+i\Delta y)-v(x_0,y_0)}{\Delta z}\end{aligned}$$

Now, let $\Delta z = \Delta x + 0i$, we have the above expression becomes

$$\begin{aligned}&\lim_{\Delta z\to 0}\frac{u(x_0+\Delta x, y_0)-u(x_0,y_0)}{\Delta x}+i\frac{v(x_0+\Delta x, y_0)-v(x_0,y_0)}{\Delta x}\\ &= \frac{\partial u}{\partial x}(x_0,y_0)+i\frac{\partial v}{\partial x}(x_0,y_0)\end{aligned}$$

Let $\Delta z = 0 + i\Delta y$, we have the above expression becomes

$$\begin{aligned}&\lim_{\Delta z\to 0}\frac{u(x_0, y_0+i\Delta y)-u(x_0,y_0)}{i\Delta y}+i\frac{v(x_0, y_0+i\Delta y)-v(x_0,y_0)}{i\Delta y}\\ &= \lim_{\Delta z\to 0}-i\frac{u(x_0, y_0+i\Delta y)-u(x_0,y_0)}{\Delta y}+\frac{v(x_0, y_0+i\Delta y)-v(x_0,y_0)}{\Delta y}\\ &= -i\frac{\partial u}{\partial y}(x_0,y_0)+\frac{\partial v}{\partial y}(x_0,y_0)\end{aligned}$$

Since $f'(z_0)$ exists, we must have

$$f'(z) = \frac{\partial u}{\partial x}(x_0,y_0)+i\frac{\partial v}{\partial x}(x_0,y_0) = -i\frac{\partial u}{\partial y}(x_0,y_0)+\frac{\partial v}{\partial y}(x_0,y_0)$$

Hence,

$$\frac{\partial u}{\partial x}(x_0,y_0) = \frac{\partial v}{\partial y}(x_0,y_0)$$

and

$$\frac{\partial u}{\partial y}(x_0,y_0) = -\frac{\partial v}{\partial x}(x_0,y_0)$$

□

Lemma 4.17. *If $u : \mathbb{R}^2 \to \mathbb{R}$ is C^1 then we have*

$$u(x+\Delta x, y+\Delta y) - u(x,y) = u_x(x,y)\Delta x + u_y(x,y)\Delta_y + E$$

where $\lim_{\Delta z \to 0} \frac{|E|}{|\Delta z|} = 0$

Proof. Define $P : [0,1] \to \mathbb{R}$ by $P(t) = u(x + t\Delta x, y + t\Delta y)$. We have $P(1) = u(x+\Delta x, y+\Delta y), P(0) = u(x,y)$.

By MVT, there exists some $c \in (0,1)$ such that

$$u(x+\Delta x, y+\Delta y) - u(x,y) = P(1) - P(0) = P'(c) = u_x(x + c\Delta x, y + c\Delta y)\Delta x + u_y(x + c\Delta x, y + c\Delta y)\Delta_y$$

We therefore have

$$E = u_x(x + c\Delta x, y + c\Delta y)\Delta x - u_x(x,y)\Delta x + u_y(x + c\Delta x, y + c\Delta y)\Delta y - u_y(x,y)\Delta_y$$

we will prove that $\lim_{\Delta z \to 0} \frac{|E|}{|\Delta z|} = 0$

Take any $\varepsilon > 0$ since u_x, u_y is continuous as u is C^1, there exists $\delta > 0$ such that

$$|u_x(x + c\Delta x, y + c\Delta y) - u_x(x,y)| < \varepsilon$$
$$\left|u_y(x + c\Delta x, y + c\Delta y) - u_y(x,y)\right| < \varepsilon$$

This gives

$$|E| \le |u_x(x + c\Delta x, y + c\Delta y)\Delta x - u_x(x,y)\Delta x| + |u_y(x + c\Delta x, y + c\Delta y)\Delta y - u_y(x,y)\Delta_y| < \varepsilon(\Delta x + \Delta y) \le 2\varepsilon\Delta z$$

Hence,

$$\lim_{\Delta z \to 0} \frac{|E|}{|\Delta z|} = 0$$

holds. $\square$

Theorem 4.18. *If $u(x,y), v(x,y)$ have continuous first order partial derivatives which satisfy the Cauchy-Riemann equations then $f(z) = u(z) + iv(z)$ is an analytic function.*

Proof. Since u, v have continuous partial derivatives we have

$$u(x+\Delta x, y+\Delta y) - u(x,y) = \frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y + E_1$$
$$v(x+\Delta x, y+\Delta y) - v(x,y) = \frac{\partial v}{\partial x}\Delta x + \frac{\partial v}{\partial y}\Delta y + E_2$$

where $\lim_{\Delta z \to 0} \frac{E_1}{|\Delta z|} = \lim_{\Delta z \to 0} \frac{E_2}{|\Delta z|} = 0$.

We therefore have

$$\begin{aligned} f(z+\Delta z) - f(z) &= u(x+\Delta x, y+\Delta y) - u(x,y) + i(v(x+\Delta x, y+\Delta y) - v(x,y)) \\ &= \frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y + E_1 + i\left(\frac{\partial v}{\partial x}\Delta x + \frac{\partial v}{\partial y}\Delta y + E_2\right) \\ &= \frac{\partial u}{\partial x}\Delta x - \frac{\partial v}{\partial x}\Delta y + E_1 + i\left(\frac{\partial v}{\partial x}\Delta x + \frac{\partial u}{\partial x}\Delta y + E_2\right) \quad \text{[By Cauchy-Riemann Equations]} \\ &= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)(\Delta x + i\Delta y) + E_1 + iE_2 \end{aligned}$$

Hence, this gives

$$\lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \lim_{\Delta z \to 0} \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} + \frac{E_1 + iE_2}{\Delta z} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

which is analytic as u, v have continuous first order derivatives. $\square$

Proposition 4.19. (Properties of the derivatives)

1. *If $f : \mathbb{C} \to \mathbb{C}$ defined by $f(z) = c$ for all $z \in \mathbb{C}$ then $f'(z) = 0$ for all $z \in \mathbb{C}$*

If $f, g : \mathbb{C} \to \mathbb{C}$ are analytic, then for all $z \in \mathbb{C}$

2. $(f(z) \pm g(z))' = f'(z) \pm g'(z)$
3. $(f(z)g(z))' = f'(z)g(z) + g'(z)f(z)$
4. $\left(\frac{f(z)}{g(z)}\right)' = \frac{f'(z)g(z) - g'(z)f(z)}{g(z)^2}$ *where* $g(z) \neq 0$
5. $f(g(z))' = f'(g(z))g'(z)$

If $f : \mathbb{C} \to \mathbb{C}$ defined by $f(z) = z^n$ for some $n \in Z_+$, for all $z \in \mathbb{C}$ then we have for all $z \in \mathbb{C}$

6. $f'(z) = nz^{n-1}$

Proof.

1. For any $z \in \mathbb{C}$, we have

$$f'(z) = \lim_{x \to z} \frac{f(x) - f(z)}{x - z} = \lim_{x \to z} \frac{0}{x - z} = 0$$

2. For any $z \in \mathbb{C}$, we have

$$\begin{aligned}(f(z) + g(z))' &= \lim_{x \to z} \frac{(f(x) + g(x)) - (f(z) + g(z))}{x - z} \\ &= \lim_{x \to z} \frac{f(x) - f(z)}{x - z} + \frac{g(x) - g(z)}{x - z} \\ &= \lim_{x \to z} \frac{f(x) - f(z)}{x - z} + \lim_{x \to z} \frac{g(x) - g(z)}{x - z} \quad \text{[Properties of a limit]} \\ &= f'(z) + g'(z)\end{aligned}$$

3. For any $z \in \mathbb{C}$, we have

$$\begin{aligned}(f(z)g(z))' &= \lim_{x \to z} \frac{f(x)g(x) - f(z)g(z)}{x - z} \\ &= \lim_{x \to z} \frac{f(x)g(x) - f(x)g(z) + f(x)g(z) - f(z)g(z)}{x - z} \\ &= \lim_{x \to z} \frac{f(x)(g(x) - g(z)) + g(z)(f(x) - f(z))}{x - z} \\ &= \lim_{x \to z} \frac{f(x)(g(x) - g(z))}{x - z} + \lim_{x \to z} \frac{g(z)(f(x) - f(z))}{x - z} \\ &= \lim_{x \to z} f(x) \lim_{x \to z} \frac{g(x) - g(z)}{x - z} + \lim_{x \to z} g(z) \lim_{x \to z} \frac{f(x) - f(z)}{x - z} \quad \text{[Properties of a limit]} \\ &= f(z)g'(z) + g(z)f'(z)\end{aligned}$$

4. Since $g(z) \neq 0$, define $h(z) = \frac{1}{g(z)} \implies h(z)g(z) = 1$. By part 3, we get

$$\begin{aligned}&h'(z)g(z) + h(z)g'(z) = 0 \\ &\implies h'(z) = -\frac{h(z)g'(z)}{g(z)} \\ &\implies h'(z) = -\frac{g'(z)}{g^2(z)}\end{aligned}$$

Now, again by part 3 we have

$$\left(\frac{f(z)}{g(z)}\right)' = (f(z)h(z))' = f(z)h'(z) + f'(z)h(z) = \frac{f'(z)g(z) - g'(z)f(z)}{g^2(z)}$$

5. Consider the functions $F : A \to \mathbb{C}$ and $G : B \to \mathbb{C}$ defined by

$$F(z) = \begin{cases} \frac{f(z)-f(z_0)}{z-z_0} & \text{if } \ z \neq z_0 \\ f'(z_0) & \text{if } \ z = z_0. \end{cases}$$

and

$$G(w) = \begin{cases} \frac{g(w)-g(w_0)}{g-g_0} & \text{if } \ w \neq w_0 \\ g'(w_0) & \text{if } w = w_0. \end{cases}$$

Because

$$\lim_{z\to z_0} F(z) = \lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0} = f'(z_0) = F(z_0),$$

it follows that F is continuous at z_0. Similarly, G is continuous at w_0 which implies that the composition $G \circ f$ is continuous at z_0 (f, being differentiable at z_0, is certainly continuous there). Therefore, for all $z \in A$ such that $z \neq z_0$, we have

6. For any $z \in \mathbb{C}$, we have

$$\begin{aligned} f'(z) &= \lim_{x\to z} \frac{f(x)-f(z)}{x-z} \\ &= \lim_{x\to z} \frac{x^n - z^n}{x-z} \\ &= \lim_{x\to z} \sum_{i=0}^{n-1} x^{n-1-i} z^i \\ &= \sum_{i=0}^{n-1} \lim_{x\to z} x^{n-1-i} z^i \quad \text{[Properties of a limit]} \\ &= \sum_{i=0}^{n-1} z^{n-1} \\ &= nz^{n-1} \end{aligned}$$

□

Corollary 4.20. *If $f : \mathbb{C} \to \mathbb{C}$ defined by $f(z) = z$ for all $z \in \mathbb{C}$ then we have for all $z \in \mathbb{C}, f'(z) = 1$*

Proof. Immediate from above. □

Definition 4.21. Suppose $S \subseteq \mathbb{C}$ is open and connected, then we call S is a domain.

Proposition 4.22. *If $f'(z) = 0$ on a domain S then $f(z)$ is constant on that domain*

Proof. Let $f(z) = f(x+iy) = u(x,y) + iv(x,y)$. Since $f'(z) = 0$ for all $z \in S$, then f is differentiable on S and so satisfies the Cauchy-Riemann equations.

$$0 = f'(z) = f'(x+iy) = \frac{\partial u}{\partial x}(x,y) + i\frac{\partial v}{\partial x}(x,y) \implies \frac{\partial u}{\partial x}(x,y) = \frac{\partial v}{\partial x}(x,y) = 0$$

By Cauchy-Riemann, we have

$$0 = f'(z) = f'(x+iy) = \frac{\partial u}{\partial x}(x,y) + i\frac{\partial v}{\partial x}(x,y) = \frac{\partial v}{\partial y}(x,y) - i\frac{\partial u}{\partial y}(x,y) \implies \frac{\partial u}{\partial y}(x,y) = \frac{\partial v}{\partial y}(x,y) = 0$$

Since we just showed all partial derivatives are continuous and are 0 in u, v, by multivariable calculus u, v are constant in S. Hence f is constant in S. □

Remark 4.23. Recall that using Cauchy-Riemann Equations we get

$$-i\frac{\partial f}{\partial y} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial f}{\partial x}$$

Note we also have that

$$x = \frac{1}{2}(z + \overline{z}), y = \frac{1}{2i}(z - \overline{z}) = -\frac{i}{2}(z - \overline{z})$$

This can be seen as a composition of functions. By chain rule we have that

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)$$

$$\frac{\partial f}{\partial \overline{z}} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \overline{z}} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)$$

Proposition 4.24. *f is analytic $\iff \frac{\partial f}{\partial \overline{z}} = 0$ for all z where f is defined.*

Proof. Given z where f is defined, by `Remark 4.23`, and the Cauchy-Riemann Equations

$$\frac{\partial f}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) = \frac{1}{2}\left(\frac{\partial f}{\partial x} - \frac{\partial f}{\partial x}\right) = \frac{1}{2} \cdot 0 = 0$$

□

Definition 4.25. $f : \mathbb{R}^2 \to \mathbb{R}$ is called harmonic if it satisfies the Laplace's equation

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

Proposition 4.26. *If $f = u + iv$ is an analytic function then u and v are harmonic functions.*

Proof. Since f is analytic, f satisfies the Cauchy-Riemann Equations

$$\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \end{cases}$$

By Cauchy-Riemann Equations, we have

$$\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \implies \frac{\partial^2 u}{\partial x^2} = \frac{\partial v}{\partial y \partial x} \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \implies \frac{\partial^2 u}{\partial y^2} = -\frac{\partial v}{\partial x \partial y} \end{cases}$$

$$\implies \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial v}{\partial y \partial x} - \frac{\partial v}{\partial x \partial y} = 0$$

Hence, u is harmonic.

Similarly, we have

$$\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \implies \frac{\partial u}{\partial x \partial y} = \frac{\partial^2 v}{\partial y^2} \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \implies \frac{\partial u}{\partial y \partial x} = -\frac{\partial^2 v}{\partial x^2} \end{cases}$$

$$\implies \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = -\frac{\partial u}{\partial y \partial x} + \frac{\partial v}{\partial x \partial y} = 0$$

Hence, v is harmonic. □

4.3 Polynomials.

Definition 4.27. A polynomial $P(z)$ of degree n in complex numbers is written as $P(z) = a_0 + a_1 z + ... + a_n z^n$, where $a_0, ..., a_n \in \mathbb{C}$.

If we see $P : \mathbb{C} \to \mathbb{C}$ as a function we have $P'(z) = a_1 + ... + n a_n z^{n-1}$.

Theorem 4.28. (Fundamental Theorem of Algebra) *Every polynomial of degree n has n roots (counting the multiplicity)*

Remark 4.29.

If α_1 is a root (that is, $P(\alpha_1) = 0$) then we can write

$$\begin{aligned} P(z) &= P(z) - 0 \\ &= P(z) - P(\alpha_1) \\ &= a_n z^n + ... + a_0 - (a_n \alpha_1^n + ... + a_0) \\ &= a_n(z^n - \alpha_1^n) + ... + a_1(z - \alpha_1) \\ &= (z - \alpha_1)(a_n(z^{n-1} + z^{n-2}\alpha_1 + ... + \alpha_1^n) + ... + \alpha_1(1)) \end{aligned}$$

This shows that $P(z) = (z - \alpha_1)P_1(z)$ for some polynomial $P_1(z)$ with degree $n - 1$.

Doing this inductively, we have $P(z) = a_n(z - \alpha_1)...(z - \alpha_n)$

Take the log we have

$$\log P(z) = a_n + \log(z - \alpha_1) + ... + \log(z - \alpha_n)$$

Take the derivative with respect to z, we get

$$\frac{P'(z)}{P(z)} = \frac{1}{z - \alpha_1} + ... + \frac{1}{z - \alpha_n}$$

This expression is useful when proving `Theorem 4.34`

Definition 4.30. A set S is convex if for all $x, y \in S, \alpha \in [0, 1], \alpha x + (1 - \alpha)y \in S$.

Definition 4.31. Given a set of points S, a convex hull S' is the smallest convex set such that $S \subseteq S'$.

Remark 4.32. When we have a finite points, a convex polygon is equivalent to the convex hull of its vertices (where all interior angles in such polygon must be at most π).

Definition 4.33. A point is in a convex n-polygon where $a_1, ...a_n$ are the vertices if we can write that point as some $m_1 a_1 + ... + m_n a_n$ such that $\sum_{i=1}^{n} m_i = 1$

Theorem 4.34. (Gauss-Lucas Theorem) *Let $P(z)$ be a complex polynomial with roots $\alpha_1, ..., \alpha_n \in \mathbb{C}$. Then the roots of $P'(z)$ lie in the convex hull of $\alpha_1, ..., \alpha_n$*

Proof. Let r be a root of $P'(z)$, if r is a root of $P(z)$ we are done. Suppose now that r is not a root of $P(z)$. We have

$$\begin{aligned}
\frac{P'(z)}{P(z)} &= \frac{1}{z-\alpha_1} + ... + \frac{1}{z-\alpha_n} \\
\implies \frac{P'(r)}{P(r)} &= \frac{1}{r-\alpha_1} + ... + \frac{1}{r-\alpha_n} \quad \text{[Plug in } r\text{]} \\
\implies 0 &= \frac{1}{r-\alpha_1} + ... + \frac{1}{r-\alpha_n} \\
\implies 0 &= \left(\frac{\overline{r-\alpha_1}}{|r-\alpha_1|^2} + ... + \frac{\overline{r-\alpha_n}}{|r-\alpha_n|^2} \right) \\
\implies 0 &= \frac{\overline{r}-\overline{\alpha_1}}{|r-\alpha_1|^2} + ... + \frac{\overline{r}-\overline{\alpha_n}}{|r-\alpha_n|^2} \\
\implies \overline{r}\left(\frac{1}{|r-\alpha_1|^2} + ... + \frac{1}{|r-\alpha_n|^2} \right) &= \frac{\overline{\alpha_1}}{|r-\alpha_1|^2} + ... + \frac{\overline{\alpha_n}}{|r-\alpha_n|^2} \\
\implies r\left(\frac{1}{|r-\alpha_1|^2} + ... + \frac{1}{|r-\alpha_n|^2} \right) &= \frac{\alpha_1}{|r-\alpha_1|^2} + ... + \frac{\alpha_n}{|r-\alpha_n|^2} \quad \text{[Take conjugate of both sides]} \\
\implies r &= \frac{\frac{\alpha_1}{|r-\alpha_1|^2} + ... + \frac{\alpha_n}{|r-\alpha_n|^2}}{A} \quad \left[A = \frac{1}{|r-\alpha_1|^2} + ... + \frac{1}{|r-\alpha_n|^2} \right] \\
\implies r &= \frac{1}{A}\frac{1}{|r-\alpha_1|^2}\alpha_1 + ... + \frac{1}{A}\frac{1}{|r-\alpha_n|^2}\alpha_n
\end{aligned}$$

we therefore have $m_i = \frac{1}{A}\frac{1}{|r-\alpha_i|^2}$ for all $i = 1, ..., n$ and we also get

$$\sum_{i=1}^{n} m_i = \frac{1}{A}\left(\frac{1}{|r-\alpha_1|^2} + ... + \frac{1}{|r-\alpha_n|^2} \right) = 1$$

□

4.4 **Rational Functions.**

Definition 4.35. Given two polynomials $P(z), Q(z)$ without any common factor, we call $R(z) = \frac{P(z)}{Q(z)}$ a rational function.

Remark 4.36. If we consider the extended complex plane $(\mathbb{C} \cup \{\infty\})$, where $R : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$, if $Q(z) = 0 \to R(z) = \infty$. Similarly, $P(z) = 0 \to R(z) = 0$

Definition 4.37. A pole of $R(z)$ is a point where $R(z) = \infty$ (equivalently, when $Q(z) = 0$, note that since P, Q does not have any common factor $P(z) \neq 0$)

Definition 4.38. The order of a pole is the order of corresponding zero of $Q(z)$.

Example 4.39. If α is a root of $Q(z)$ so that $Q(z) = (z-\alpha)^n Q_1(z)$ and α is not a root of $Q_1(z)$ then n is the order of the pole α, where we get $R(z) = \frac{P(z)}{(z-\alpha)^n Q_1(z)}$.

For cases where $Q(z) \neq 0, z \neq \alpha$, we have

$$\begin{aligned}
R'(z) &= \frac{P'(z)(z-\alpha)^n Q_1(z) - n(z-\alpha)^{n-1} Q_1(z) P(z) - (z-\alpha)^n Q_1'(z) P(z)}{(z-\alpha)^{2n} \left(Q_1(z)\right)^2} \\
&= \frac{P'(z)(z-\alpha) Q_1(z) - n Q_1(z) P(z) - (z-\alpha) Q_1'(z) P(z)}{(z-\alpha)^{n+1} \left(Q_1(z)\right)^2}
\end{aligned}$$

Since $P(z), Q_1(z)$ does not have a factor of $(z-\alpha)$, so the numerator does not have a factor of $(z-\alpha)$. Hence, the order at $R'(z)$ at pole α is $n+1$.

This concludes that $R'(z)$ has the same poles as $R(z)$, but the order of each pole being increased by one.

We may investigate under the extended complex plane, we can define $R(\infty)$ as the limit of $R(z)$ as $z \to \infty$, but this definition would not determine the order of a zero or pole at ∞. To do so, we introduce a new polynomial $R_1(z) = R(\frac{1}{z})$, where the following holds

- $R_1(\infty) = R(0)$
- $R_1(0) = R(\infty)$

We can write $P(z) = a_0 + ... + a_n z^n, Q(z) = b_0 + ... + b_m z^m$. This gives

$$R(z) = \frac{a_0 + ... + a_n z^n}{b_0 + ... + b_m z^m}$$
$$\implies R_1(z) = \frac{a_0 + ... + a_n z^{-n}}{b_0 + ... + b_m z^{-m}}$$
$$\implies R_1(z) = z^m \frac{a_0 + ... + a_n z^{-n}}{b_0 z^m + ... + b_m}$$
$$\implies R_1(z) = z^{m-n} \frac{a_0 z^n + ... + a_n}{b_0 z^m + ... + b_m}$$
$$\implies R(z) = z^{n-m} \frac{a_0 z^{-n} + ... + a_n}{b_0 z^{-m} + ... + b_m}$$

Some key observation including

1. If $m > n, R(z)$ has a zero of order $m - n$ at ∞
2. If $m < n, R(z)$ has a pole of order $n - m$ at ∞
3. If $m = n, R(\infty) = \frac{a_n}{b_m} \neq 0, \infty$

Remark 4.40. This gives some extra observations

1. If $m > n$, there are n zeros at finite numbers and a zero of order $m - n$ at ∞.
2. If $m < n$, there are n zeros at finite numbers and no zero at ∞.
3. If $m = n$, there are n zeros

To conclude, the number of zeros of $R(z)$ (including those of ∞) is the greater of n and m, this common number is called the order of the rational function $R(z)$.

4.5 **Exercises.**

Exercise 4.41. If $g(w)$ and $f(z)$ are analytic functions, show that $g(f(z))$ is also analytic.

Exercise 4.42. Verify Cauchy-Riemann's equations for the functions z^2 and z^3

Exercise 4.43. Use the method of the text to develop

$$\frac{z^4}{z^3 - 1} \quad \text{and} \quad \frac{1}{z(z+1)^2(z+2)^3}$$

in partial fractions.

Exercise 4.44. If Q is a polynomial with distinct roots $a_1, ..., a_n$, and if P is a polynomial of degree $< n$, show that

$$\frac{P(z)}{Q(z)} = \sum_{k=1}^{n} \frac{P(\alpha_k)}{Q'(\alpha_k)(z - \alpha_k)}$$

5 Power Series

5.1 **Sequences.**

Definition 5.1. The sequence $\{a_n\}_1^{\{\infty\}}$ has the limit A if for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ st. $|a_n - A| < \varepsilon$ for $n \geq n_0$

Definition 5.2. A sequence with a finite limit is said to be convergent; and any sequence which does not converge is divergent. If $\lim_{n\to\infty} a_n = \infty$, the sequence is divergent to ∞

Definition 5.3. A sequence is called fundamental or a Cauchy sequence, if it satisfies

$$\forall \varepsilon > 0, \exists n_0 \in \mathbb{N} \text{ st } |a_n - a_m| < \varepsilon \text{ whenever } n, m \geq n_0$$

Theorem 5.4. *A sequence is convergent iff it is a Cauchy sequence.*

Proof. The proof of necessity is easy: If $\lim_{n\to\infty} a_n = A$ then we can find n_0 st $|a_n - A| < \frac{\varepsilon}{2}$ for all $n \geq n_0$. If $m, n \geq n_0$ then $|a_m - a_n| \leq |a_m - A| + |a_n - A| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

To prove the sufficiency, note that the real and imaginary parts of a Cauchy sequence are again Cauchy sequences. Hence, the real and imaginary parts of a sequence are convergent, so is the original sequence. The argument of exchanging complex numbers with real/imaginary parts and vice versa are similar to a vector converges iff its components converge. $\square$

Definition 5.5. Consider a real sequence $\{a_n\}_1^\infty$, we define

$$\limsup a_n = \lim_{n\to\infty}\left(\sup_{m\geq n} a_m\right) = \inf_{n\geq 1}\left(\sup_{m\geq n} a_m\right)$$
$$\liminf a_n = \lim_{n\to\infty}\left(\inf_{m\geq n} a_m\right) = \sup_{n\geq 1}\left(\inf_{m\geq n} a_m\right)$$

These can be $\pm\infty$.

Remark 5.6. Try to see why the limit of these exists and why the last two of each equation are equivalent

Theorem 5.7. $\liminf \alpha_n = \limsup \alpha_n$ *iff the sequence converges to a finite number or diverges to* $\pm\infty$.

Proof. $(\Longrightarrow)$: Suppose $\liminf a_n = \limsup \alpha_n = L$

Case 1: If $L \in \mathbb{R}$, we show that $a_n \to L$. Given $\varepsilon > 0$, take n_0 such that for all $n \geq n_0$

$$\sup_{m\geq n} a_m < L + \varepsilon$$
$$\inf_{m\geq n} a_m > L - \varepsilon$$

Now, for all $n \geq n_0$, we have

$$\sup_{m\geq n} a_m < L + \varepsilon \Longrightarrow a_n < L + \varepsilon$$
$$\inf_{m\geq n} a_m > L - \varepsilon \Longrightarrow a_n > L + \varepsilon$$

This gives $|a_n - L| < \varepsilon$. Showing that $a_n \to L$.

Case 1: If $L = \infty$, we show $a_n \to \infty$. Given $M > 0$, since $\liminf a_n = \lim_{n\to\infty}(\inf_{m\geq n} a_m) = \infty$ we can take n_0 such that for all $n \geq n_0$

$$\inf_{m\geq n} a_m > M$$

Hence, for all $n \geq n_0$, we get

$$a_n \geq \inf_{m\geq n} a_m > M$$

Showing that $a_n \to \infty$. For the case where $L = -\infty$ is similar.

$(\Longleftarrow)$: Suppose the sequence $\{a_n\}_1^{\{\infty\}}$ converges.

Case 1: If $a_n \to L$ where $L \in \mathbb{R}$, we show that

$$\liminf a_n = \limsup a_n = L$$

Given $\varepsilon > 0$, we can find a n_0 such that for all $n \geq n_0$, $|a_n - L| < \varepsilon \Longrightarrow L - \varepsilon < a_n < L + \varepsilon$. Hence, for all $n \geq n_0$, we get

$$L - \varepsilon \leq \sup_{m \geq n} a_n \leq L + \varepsilon \Longrightarrow |\sup_{m \geq n} a_n - L| \leq \varepsilon$$
$$L - \varepsilon \leq \inf_{m \geq n} a_n \leq L + \varepsilon \Longrightarrow |\inf_{m \geq n} a_n - L| \leq \varepsilon$$

showing that $\liminf a_n = \limsup a_n = L$

Case 2: If $a_n \to \infty$, we show that

$$\liminf a_n = \limsup a_n = \infty$$

Given any $M > 0$, there exists some n_0 such that for all $n \geq n_0, a_n > M$. Hence, we have

$$\inf_{m \geq n_0} a_n \geq M \Longrightarrow \liminf a_n = \sup_{n \geq 1} \inf_{m \geq n} a_n \geq \inf_{m \geq n_0} a_n \geq M$$

showing that $\liminf a_n \to \infty$.

Finally, since $\inf_{m \geq n} a_n \leq \sup_{m \geq n} a_n$ for all $n \in \mathbb{N}$, we have

$$\infty = \liminf a_n = \sup_{n \geq 1} \inf_{m \geq n} a_n \leq \inf_{n \geq 1} \sup_{m \geq n} a_n = \limsup a_n \Longrightarrow \limsup a_n = \infty$$

The case for $a_n \to -\infty$ is similar. □

Theorem 5.8. *If a sequence $\{\alpha_n\}_1^\infty$ is Cauchy, then it converges*

Proof. Note that since $\{\alpha_n\}_1^\infty$ is Cauchy, given any $\varepsilon > 0$, we can take n_0 such that for all $n, m \geq n_0, |a_n - a_m| < \varepsilon$. It is also the case that we can fix $m = n_0$ and we get $|a_n - a_{n_0}| < \varepsilon$.

Now, take $A = \limsup \alpha_n, a = \liminf \alpha_n$, note that A, a are all finite (take $\varepsilon = 1$ and we get $1 \geq |\lim_{n \to \infty}(\sup_{m \geq n} a_m) - \alpha_{n_0}| = |A - \alpha_{n_0}|, 1 \geq |\lim_{n \to \infty}(\inf_{m \geq n} a_m) - \alpha_{n_0}| = |a - \alpha_{n_0}| \leq 1$). Suppose for a contradiction that $A \neq a$, take $\varepsilon = \frac{A-a}{3}$ and n_0 such that for all $n, m \geq n_0, |a_n - a_m| < \varepsilon$.

There exists some $n, m \geq n_0$ such that $a_n < a + \varepsilon, a_m > A - \varepsilon$ (if $a_n \geq a + \varepsilon$ for all n, then $\liminf a_n = \lim_{n \to \infty}(\inf_{m \geq n} a_m) \geq a + \varepsilon$ as $\inf_{m \geq n} a_m \geq a + \varepsilon$ for all n, similar for the case if $a_m \leq A - \varepsilon$).

Then, we get

$$A - a = (A - a_m) + (a_m - a_n) + (a_n - a) < 3\varepsilon = A - a$$

a contradiction. Hence, $A = a$, showing that the sequence converges. □

5.2 Series.

Definition 5.9. Consider a sequence $\{a_n\}_1^{\{\infty\}}$, we define the infinite series as a sequence of partial sums $\{s_n\}_1^{\{\infty\}}$ where

$$s_n = \sum_{i=1}^{n} a_i$$

we call each s_n as a partial sum.

Series can be seen as sequences where convergence is defined in the same way. If the series is convergent to some L, then we denote $L = \sum_{i=1}^\infty a_i$.

Remark 5.10. Note that the infinite series $\{s_n\}_1^{\{\infty\}}$ converges if and only if it is Cauchy. This is if and only if for any $\varepsilon > 0$, there exists some n_0 such that for all $p \geq 0$ (it is easy to check)

$$|a_n + ... + a_{n+p}| < \varepsilon$$

Definition 5.11. If the series $\sum_{n=1}^\infty |a_n|$ converges, then $\sum_{n=1}^\infty a_n$ converges absolutely.

Our first theorem is an easy consequence of the definition, and also a familiar result from singe-variable calculus.

Proposition 5.12. *If $\sum a_n$ converges absolutely, then it converges.*

Proof. (by Ahmad) Suppose $\sum a_n$ converges absolutely. Denote by (s_n) the partial sums of $\sum|a_n|$. For each natural n, define the sequence of partial sums $z_n = \sum_{i=0}^{n} a_i$. Our goal is to show that the sequence (z_n) converges. Since every Cauchy sequence is convergent, it suffices to prove that (z_n) is Cauchy. Fix $\varepsilon > 0$. By the convergence of (s_n), there exists $N \in \mathbb{N}$ such that for all $m \geq n \geq N$,

$$|s_m - s_n| = \sum_{i=n+1}^{m} |a_i| < \varepsilon$$

Taking the same N as above, fix $m, n \in \mathbb{N}$ such that $m \geq n \geq N$. Then

$$|z_m - z_n| = \left|\sum_{i=n+1}^{m} a_i\right| \leq \sum_{i=n+1}^{m} |a_i| < \varepsilon$$

Thus, the sequence (z_n) is Cauchy. It must also converge. □

5.3 **Uniform Convergence.**

Definition 5.13. Consider a sequence of functions $f_n(x)$ defined on the domain of E, we say that this sequence converges pointwise if $\forall \varepsilon > 0, \forall x \in E, \exists n_0$ such that $\forall n \geq n_0$

$$|f_n(x) - f(x)| < \varepsilon$$

Example 5.14.

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right) x = x$$

Indeed, given $\varepsilon > 0, x \in \mathbb{C}$, take $n_0 = \left\lceil \frac{|x|}{\varepsilon} \right\rceil$, we have for all $n \geq n_0$

$$|f_n(x) - f(x)| = \left|\frac{x}{n}\right| = \frac{|x|}{n} < \varepsilon$$

Definition 5.15. Consider a sequence of functions $f_{n(x)}$ defined on the domain of E, we say that this sequence converges uniformly to $f(x)$ if $\forall \varepsilon > 0, \exists n_0$ such that $\forall x \in E, \forall n \geq n_0$

$$|f_n(x) - f(x)| < \varepsilon$$

Example 5.16. $f_n(x) = \frac{\sin(nx+n)}{n}$ converges uniformly to $f(x) = 0$.

Indeed, given $\varepsilon > 0$, take $n_0 = \left\lceil \frac{1}{\varepsilon} \right\rceil$, we have for all $x \in \mathbb{R}, n \geq n_0$

$$|f_{n(x)} - f(x)| = \left|\frac{\sin(nx+n)}{n}\right| = \left|\frac{\sin(nx+n)}{n}\right| < \frac{1}{n} < \frac{1}{n_0} < \varepsilon$$

Example 5.17. $f_n(x) = \frac{x^2}{n}$ converges uniformly to $f(x) = 0$ on an interval $[a, b]$.

Indeed, given $\varepsilon > 0$, take $n_0 > \left\lceil \frac{\varepsilon}{\max(a^2,b^2)} \right\rceil$, we have for all $x \in \mathbb{R}, n \geq n_0$

$$|f_n(x) - f(x)| = \left|\frac{x^2}{n}\right| = \frac{|x^2|}{n} < \frac{\max(a^2, b^2)}{n} < \frac{\max(a^2, b^2)}{n_0} < \varepsilon$$

Theorem 5.18. *The limit function of a uniformly convergent sequence of continuous functions is itself continuous.*

Proof. Given $\varepsilon > 0, x_0 \in \mathbb{C}$, we show f is continuous at x_0. Take n_0 such that for all $n \geq n_0, x \in E, |f_n(x) - f(x)| < \frac{\varepsilon}{3}$. Similarly, take δ such that for all $|x - x_0| < \delta \Longrightarrow |f_{n_0}(x) - f_{n_0}(x_0)| < \frac{\varepsilon}{3}$. Hence, we now have

$$|f(x) - f(x_0)| \leq |f(x) - f_{n_0}(x)| + |f_{n_0}(x) - f_{n_0}(x_0)| + |f_{n_0}(x_0) - f(x_0)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

□

Theorem 5.19. *The sequence $f_n(x)$ converges uniformly on E iff for every $\varepsilon > 0$ there exists n_0 such that $|f_m(x) - f_n(x)| < \varepsilon$ for all $m, n \geq n_0, x \in E$*

Proof. $(\Longrightarrow)$: Given $\varepsilon > 0$, take n_0 such that for all $n \geq n_0, x \in E$

$$|f_n(x) - f(x)| < \frac{\varepsilon}{2}$$

Therefore, for all $m, n \geq n_0$,

$$|f_m(x) - f_n(x)| \leq |f_m(x) - f(x)| + |f(x) - f_n(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

$(\Longleftarrow)$: Define $f(x) = \lim_{n\to\infty} f_n(x)$, this exists as $f_n(x)$ is Cauchy for any x. We now show $f_n(x) \to f(x)$, indeed, for arbitrary $\varepsilon > 0$, there is some n_0 such that for all $m, n \geq n_0, x \in E$

$$|f_m(x) - f_n(x)| < \varepsilon$$

Take $m \to \infty$, we get for all $n \geq n_0, x \in E$

$$|f(x) - f_n(x)| < \varepsilon$$

showing $f_n(x)$ converges to $f(x)$ uniformly. $\square$

Theorem 5.20. *If $f_n(x)$ is a contraction of a convergence sequence a_n (for all $x \in E, |f_n(x) - f_m(x)| < |a_n - a_m|$), then $f_n(x)$ is uniformly convergent.*

Proof. Given $\varepsilon > 0$, since a_n is convergent, it is Cauchy, there exists some n_0 such that for all $n, m \geq n_0, |a_n - a_m| < \varepsilon$. We therefore get for all $x \in E, |f_n(x) - f_m(x)| < |a_n - a_m| < \varepsilon$. $\square$

Theorem 5.21. (Weierstrass M test) *If for some constant M, and sufficiently large $n, |f_n(x)| < Ma_n$ for some sequence a_n and if $\sum a_n$ is convergent then $\sum f_n(x)$ is uniformly convergent.*

Proof. It is enough to show that for all $\varepsilon > 0$, there exists n_0 such that for all $m_1, m_2 \geq n_0$

$$\left|\sum_{n=1}^{m_1} f_n(x) - \sum_{n=1}^{m_2} f_n(x)\right| < \varepsilon$$

Given ε, since $\sum a_n$ is convergent, there is some n_0 such that for all $m_1, m_2 \geq n_0$

$$\left|\sum_{n=1}^{m_1} a_n - \sum_{n=1}^{m_2} a_n\right| < \frac{\varepsilon}{M}$$

Take any $m_1, m_2 \geq n_0, x \in E$ (WLOG, assume $m_1 < m_2$), we get

$$\left|\sum_{n=1}^{m_1} f_n(x) - \sum_{n=1}^{m_2} f_n(x)\right| = \left|\sum_{n=m_1}^{m_2} f_n(x)\right| \leq \sum_{n=m_1}^{m_2} |f_n(x)| < M \sum_{n=m_1}^{m_2} a_n \leq M\left|\sum_{n=1}^{m_1} a_n - \sum_{n=1}^{m_2} a_n\right| < \varepsilon$$

Showing $\sum f_n(x)$ is uniformly convergent. $\square$

Definition 5.22. A power series is of the form $a_0 + a_1 z + ... + a_n z^n + ...$ where the coefficients a_n and the variable z are complex. We can consider $\sum_{n=1}^{\infty} a_n(z-a)^n$ which is the power series centered at a.

Example 5.23. Consider $1 + z + z^2 + ...$, the partial sums are

$$1 + z + ... + z^{n-1} = \frac{1 - z^n}{1 - z}$$

- If $|z| < 1, \lim_{n\to\infty} |z|^n = 0 \Longrightarrow \lim_{n\to\infty} z^n = 0$
- If $|z| \geq 1, \lim_{n\to\infty} |z|^n = 1$ or ∞

Hence, only when $|z| < 1$, we get

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$$

Theorem 5.24. (Abel's Theorem) *For every power series, there exists a number R, where $0 \leq R \leq \infty$, called radius of convergence, with the following properties:*

1. *The series converges absolutely for every z with $|z| < R$. If $0 < \rho < R$, the convergence is uniform for $|z| \leq \rho$.*
2. *If $|z| > R$ the terms of the series are unbounded and the series is divergent.*
3. *If $|z| < R$ the sum of the series is an analytic function. The derivative can be derived by term-wise differentiation, and the derived series has the same radius of convergence.*

The circle $|z| = R$ is called the circle of convergence; nothing is claimed about convergence on the circle.

Proof. I am not going to provide the proof here, it is way too long so probably not important to know how it is proven. If you are interested see Theorem 2 pp 38.

Important: The radius of convergence is $R = \frac{1}{\lim_{n\to\infty} \sup \sqrt[n]{a_n}}$, as given by the proof. This is known as the Hadamard's formula. Note that this formula does not show convergence on the circle or not (which you will need to check this based on the question settings). □

Theorem 5.25. *If $\sum_{n=0}^{\infty} a_n$ converges, then $f(z) = \sum_{n=0}^{\infty} a_n z^n$ tends to $f(1)$ as $z \to 1$ in such a way that $\frac{|1-z|}{1-|z|}$ remains bounded.*

Remark 5.26. In this Theorem we are assuming that $R = 1$, and the convergence is at the point $z = 1$, but this theorem is correct for any radius of convergence R and any point on the circle of convergence

Proof. Since $\frac{|1-z|}{1-|z|}$ is bounded, assume $\frac{|1-z|}{1-|z|} \leq k \implies |1-z| \leq k(1-|z|)$

Assume that $\sum_{n=0}^{\infty} a_n = 0$ (we can add a constant to a_0 to achieve this). Define $s_n = a_0 + ... + a_n$ for all $n \in \mathbb{N}$. Define

$$\begin{aligned} s_n(z) &= a_0 + a_1 z + ... + a_n z^n \\ &= s_0 + (s_1 - s_0)z + ... + (s_n - s_{n-1})z^n \\ &= s_0(1-z) + s_1(z - z^2) + ... + s_{n-1}(z^{n-1} - z^n) + s^n z^n \\ &= (1-z)(s_0 + s_1 z + ... + s_{n-1} z^{n-1}) + s_n z^n \end{aligned}$$

Note that as $n \to \infty, s_n \to 0, |z| < 1$, we have

$$s_n z^n \to 0$$

Define

$$f(z) = (1-z)\sum_{n=0}^{\infty} s_n z^n$$

Now, choose m large enough so that for all $n \geq m, |s_n| < \varepsilon$ then

$$\sum_{n=m}^{\infty} |s_n z^n| = \sum_{n=m}^{\infty} |s_n| \ |z|^n < \sum_{n=m}^{\infty} \varepsilon \ |z|^n = \varepsilon \frac{|z|^m}{1-|z|} < \frac{\varepsilon}{1-|z|}$$

Hence, it follows that

$$\begin{aligned}|f(z)| &\leq |1-z| \sum_{n=0}^{\infty} |s_n z^n| \\ &\leq |1-z| \left(\sum_{n=0}^{m-1} |s_n z^n| + \sum_{n=m}^{\infty} |s_n z^n| \right) \\ &< |1-z| \sum_{n=0}^{m-1} |s_n z^n| + |1-z| \frac{\varepsilon}{1-|z|} \\ &< |1-z| \sum_{n=0}^{m-1} |s_n z^n| + k\varepsilon\end{aligned}$$

Take z close enough to 1 □

5.4 **Exercises.**

Exercise 5.27. Prove that a convergent sequence is bounded.

Exercise 5.28. If $\lim_{n\to\infty} z_n = A$, prove that

$$\lim_{n\to\infty} (z_1 + z_2 + ... + z_n) = A$$

Exercise 5.29. Show that the sum of an absolutely convergent series does not change if the terms are rearranged.

Exercise 5.30. Discuss completely the convergence and uniform convergence of the sequence $\{nz_n\}_1^{\infty}$.

Exercise 5.31. Discuss the uniform convergence of the series

$$\sum_{n=1}^{\infty} \frac{x}{n(1+nx^2)}$$

for real values of x.

Exercise 5.32. If $U = u_1 + u_2 + ..., V = v_1 + v_2 + ...$ are convergent series, prove that $UV = u_1v_1 + (u_1v_2 + u_2v_1) + (u_1v_3 + u_2v_2 + u_3v_1) + ...$ provided that at least one of the series is absolutely convergent. (It is easy if both series are absolutely convergent. Try to arrange the proof so economically that the absolute convergence of the second series is not needed.)

Exercise 5.33. Expand $(1-z)^{-m}$, m a positive integer, in powers of z.

Exercise 5.34. Expand $\frac{2z+3}{z+1}$ powers of $z-1$. What is the radius of convergence?

Exercise 5.35. Find the radius of convergence of the following power series:

$$\sum n^p z^n \quad \sum \frac{z^n}{n!} \quad \sum n! z^n \quad \sum q^{n^2} z^n (|q| < 1) \quad \sum q^{n!}$$

Exercise 5.36. If $\sum a_n z^n$ has radius of convergence R, what is the radius of convergence of $\sum a_n z^{2n}$? of $\sum a_n^2 z^n$?

Exercise 5.37. If $f(z) = \sum a_n z_n$, what is $\sum n^3 a_n z^n$

Exercise 5.38. If $\sum a_n z^n$ and $\sum b_n z^n$ have radii of convergence R_1 and R_2, show that the radius of convergence of $\sum a_n b_n z^n$ is at least $R_1 R_2$

6 Exponential and Trigonometric Functions

6.1 **Exponential Function.**

Definition 6.1. $f(z)$ is an exponential function if it is the solution of the differential equation $f'(z) = f(z)$ and $f(0) = 1$.

Remark 6.2. By setting

$$f(z) = a_0 + a_1 z + ... + a_n z^n + ...$$
$$f'(z) = a_1 + a_2 z + ... + n a_n z^{n-1} + ...$$

Hence, $na_n = a_{n-1}$ for all $n = 1, 2, ...$

We also get $f(0) = 1 \Longrightarrow a_0 = 1$. Hence, $a_n = \frac{1}{n!}$ for all n.

Finally, we need to show $f(z)$ is defined.

Theorem 6.3. *For all $a, b \in \mathbb{C}$ $e^{a+b} = e^a e^b$*

Proof. We get

$$(e^z e^{c-z})' = e^z e^{c-z} + (-1)e^{c-z} e^z = 0 \Longrightarrow e^z e^{c-z} \text{ is a constant}$$

The value of the constant can be found by setting $z = 0$, we get

$$e^z e^{c-z} = e^c$$

Let $z = a, c = a + b$, we get

$$e^a e^b = e^{a+b}$$

□

Remark 6.4. If $z = x + 0i$ then we have real exponential function. If $x > 0$, then $e^x > 1$ and if $x < 0$, then $0 < e^x < 1$.

Remark 6.5. $e^{\overline{z}} = \overline{e^x}$

The proof is simple

$$e^{\overline{z}} = \sum_{k=0}^{\infty} \frac{\overline{z}^k}{k!} = \sum_{k=0}^{\infty} \frac{\overline{z^k}}{k!} = \overline{\sum_{k=0}^{\infty} \frac{z^k}{k!}} = \overline{e^z}$$

Remark 6.6. If $z = 0 + iy$, then $|e^{iy}|^2 = e^{iy}\overline{e^{iy}} = e^{iy}e^{\overline{iy}} = e^{iy}e^{-iy} = 1$. Therefore

$$\left|e^{x+iy}\right|^2 = e^{x+iy} \cdot \overline{e^{x+iy}} = e^{x+iy} \cdot e^{\overline{x+iy}} = e^{x+iy} e^{x-iy} = e^{2x} = (e^x)^2$$

6.2 **Trigonometric Functions.**

Definition 6.7. We define

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2}$$
$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i}$$

for all $z \in \mathbb{C}$

Remark 6.8. We can also define

$$\tan(z) = \frac{\sin(z)}{\cos(z)} = -i\frac{e^{iz} - e^{-iz}}{e^{iz} + e^{-iz}}$$

All trig functions are rational functions of e^{iz}

Remark 6.9.

$$\cos(z) = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - ...$$
$$\sin(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} - ...$$

The proof uses series of e^x, we get

$$\cos(z) = \frac{e^{iz}+e^{-iz}}{2} = \frac{\sum_{k=0}^{\infty}\frac{(iz)^k}{k!} + \sum_{k=0}^{\infty}\frac{(-iz)^k}{k!}}{2} = \frac{\sum_{k=0}^{\infty}\frac{(i^k+(-i)^k)z^k}{k!}}{2} = \frac{\sum_{k=0}^{\infty}\frac{2i^{2k}z^{2k}}{(2k)!}}{2} = \sum_{k=0}^{\infty} i^{2k}\frac{z^{2k}}{(2k)!} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - ...$$

$$\sin(z) = \frac{e^{iz}-e^{-iz}}{2i} = \frac{\sum_{k=0}^{\infty}\frac{(iz)^k}{k!} - \sum_{k=0}^{\infty}\frac{(-iz)^k}{k!}}{2i} = \frac{\sum_{k=0}^{\infty}\frac{(i^k-(-i)^k)z^k}{k!}}{2i} = \frac{\sum_{k=0}^{\infty}\frac{2i^{2k+1}z^{2k+1}}{(2k+1)!}}{2i} = \sum_{k=0}^{\infty} i^{2k}\frac{z^{2k+1}}{(2k+1)!} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - ...$$

Remark 6.10. By definition, we can obtain the following
- $e^{iz} = \cos(z) + i\sin(z)$.
- $\cos^2(z) + \sin^2(z) = 1$
- $(\sin(z))' = \cos(z)$
- $(\cos(z))' = -\sin(z)$

Remark 6.11. Using definition and properties of e^z, we can obtain
- $\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \sin(\beta)\cos(\alpha)$
- $\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\beta)\sin(\alpha)$

The proof is easy again

$$\begin{aligned}
\sin(\alpha)\cos(\beta) + \sin(\beta)\cos(\alpha) &= \frac{e^{i\alpha}-e^{-i\alpha}}{2i}\frac{e^{i\beta}+e^{-i\beta}}{2} + \frac{e^{i\beta}-e^{-i\beta}}{2i}\frac{e^{i\alpha}+e^{-i\alpha}}{2} \\
&= \frac{2e^{i(\alpha+\beta)} - 2e^{-i(\alpha+\beta)}}{4i} \\
&= \frac{e^{i(\alpha+\beta)} - e^{-i(\alpha+\beta)}}{2i} \\
&= \sin(\alpha+\beta) \\
\cos(\alpha)\cos(\beta) + \sin(\beta)\sin(\alpha) &= \frac{e^{i\alpha}+e^{-i\alpha}}{2}\frac{e^{i\beta}+e^{-i\beta}}{2} + \frac{e^{i\beta}-e^{-i\beta}}{2i}\frac{e^{i\alpha}-e^{-i\alpha}}{2i} \\
&= \frac{2e^{i(\alpha+\beta)} + 2e^{-i(\alpha+\beta)}}{4} \\
&= \frac{e^{i(\alpha+\beta)} + e^{-i(\alpha+\beta)}}{2} \\
&= \cos(\alpha+\beta)
\end{aligned}$$

6.3 **Periodicty.**

Definition 6.12. We say that $f(z)$ has the period c if $f(z+c) = f(z)$ for all z.

Example 6.13. Consider a function $f : \mathbb{C} \to \mathbb{C}$ defined by $f(z) = e^z$. We have

$$\begin{aligned}
f(z) &= e^z \\
&= e^{x+iy} \\
&= e^x e^{iy} \\
&= e^x \sum_{i=0}^{\infty} \frac{(iy)^n}{n!} \\
&= e^x(\cos(y) + i\sin(y))
\end{aligned}$$

Note that $y = \arg(e^z)$.

Furthermore, take $y = \pi$, we know that $e^{iy} = -1 \Longrightarrow e^{2iy} = 1$. Hence, $e^{z+2\pi i} = e^z$, showing that the complex exponential is $2\pi i$ periodic.

6.4 **Logarithm.**

Together with the exponential function we must also study its inverse function, the logarithm. By definition, $x + iy = z = \log(w)$ is a root of the equation $e^z = w$. First of all, $e^z \neq 0$ (as $e^x \neq 0$), it is impossible to solve for $e^z = w$ where $w = 0$.

If $e^z = e^{x+iy} = w \neq 0$, we get

$$e^x = |w| \Longrightarrow x = \log(|w|)$$

$$e^{iy} = \frac{w}{|w|} \Longrightarrow \cos(y) + i\sin(y) = \frac{w}{|w|} \Longrightarrow y = \arg(w)$$

This gives us

$$\log(w) = \log(|w|) + i\arg(w)$$

Note that any other y that differ from the argument of w by a multiple of 2π also satisfies. Hence, every complex number other than 0 has infinitely many logarithms which differ from each other by $2\pi n, n \in \mathbb{Z}$.

Remark 6.14. Note that for all $z \in \mathbb{C}$

$$e^{\log z} = e^{\log(|z|)+i\arg(z)} = |z|\ e^{i\arg(z)} = |z|\ (\cos(\arg(z)) + i\sin(\arg(z))) = z$$

However,

$$\log(e^z) = \{z + 2\pi n : n \in \mathbb{Z}\}$$

Definition 6.15. We call the principal branch of complex logarithm of z to be

$$\text{Log } z = \log|z| + i\arg z \text{ where } -\pi < \arg z \leq \pi$$

Proposition 6.16. *For two complex numbers z_1 and z_2: $\log(z_1 z_2) = \log(z_1) + \log(z_2)$ which is correct up to integer multiples of $2\pi i$.*

Proof. Consider $z_1, z_2 \in \mathbb{C}$, we get

$$\begin{aligned}
\log(z_1 z_2) &= \log(|z_1 z_2|) + i\arg(z_1 z_2) \\
&= \log(|z_1|\ |z_2|) + i(\arg(z_1) + \arg(z_2)) \\
&= (\log(|z_1|) + i\arg(z_1)) + (\log(|z_2|) + i\arg(z_2)) \\
&= \log(z_1) + \log(z_2)
\end{aligned}$$

□

6.5 **Exercises.**

Exercise 6.17. Find the values of $\sin(i), \cos(i), \tan(1+i)$.

Exercise 6.18. Use the addition formulas to separate $\cos(x+iy)$, $\sin(x+iy)$ in real and imaginary parts.

Exercise 6.19. Find the value of e^z for $z = -\frac{\pi i}{2}, \frac{3}{4}\pi i, \frac{2}{3}\pi i$

Exercise 6.20. For what values of z is e^z equal to $2, -1, i, -\frac{i}{2}, -1-i, 1+2i$?

Exercise 6.21. Find the real and imaginary parts of $\exp(e^z)$.

Exercise 6.22. Determine all values of $2^i, i^i, (-1)^{2i}$.

Exercise 6.23. Determine the real and imaginary parts of z^z.

7 Point Set Topology

7.1 Metric spaces.

Definition 7.1. A ball (disc) with center x and radius δ is:

$$B(x, \delta) = \{y \in S : d(x, y) < \varepsilon\}$$

Definition 7.2. A set $N \subset S$ is called a neighborhood of $x \in S$ if it contains a ball $B(x, \delta)$.

Definition 7.3. A set is open if it is a neighborhood of each of its elements

Remark 7.4. Empty set is an open set.

Proposition 7.5. *Every ball $B(x, \delta)$ is an open set.*

Proof. Given $y \in B(x, \delta)$, take $\delta' = \delta - d(x, y)$, we show $B(y, \delta') \subseteq B(x, \delta)$. For all $z \in B(y, \delta')$, we have

$$d(x, z) \leq d(x, y) + d(y, z) < d(x, y) + \delta - d(x, y) = \delta$$

showing $z \in B(x, \delta)$. Since y is arbitrary, we showed that $B(x, \delta)$ is open. □

Definition 7.6. A closed set is a complement of an open set.

Theorem 7.7. *The intersection of a finite number of open sets is open.*

Proof. Suppose $U_1, ..., U_n$ are open. Take $x \in U_1 \cap ... \cap U_n$, by definition, for all i there exists some δ_i such that $x \in B(x, \delta_i) \subseteq U_i$. Now, take $\delta = \min\{\delta_1, ..., \delta_n\}$, we have $B(x, \delta) \subseteq U_1 \cap ... \cap U_n$ showing $U_1 \cap ... \cap U_n$ is open. □

Theorem 7.8. *The union of any collection of open sets is open*

Proof. Suppose $\{U_i\}_{i \in I}$ are open. Take $x \in \bigcup_{i \in I} U_i$, by definition, $x \in U_i$ for some $i \in I$. Since U_i is open, there exists some δ such that $x \in B(x, \delta) \subseteq U_i$, showing $\bigcup_{i \in I} U_i$ is open. □

Theorem 7.9. *The union of finite number of closed sets is closed.*

Proof. Take $U_1, ..., U_n$ are closed so that $U_1^c, ..., U_n^c$ are open. We know that $U_1^c \cap ... \cap U_n^c$ is open. Hence, $(U_1)^c \cap ... \cap U_n^c)^c = U_1 \cup ... \cup U_n$ is closed. □

Theorem 7.10. *The intersection of any collection of closed sets is closed.*

Proof. Take $\{U_i\}_{i \in I}$ are closed so that $\{U_i^c\}_{i \in I}$ are open. We know that $\bigcup_{i \in I} U_i^c$ is open. Hence, $\left(\bigcup_{i \in I} U_i^c\right)^c = \bigcap_{i \in I} U_i$ is closed. □

Definition 7.11.

- The interior of X is the largest open set in X. (Union of all open sets inside X). Notation: $\text{int}(X)$ or X°.
- The closure of X is the smallest closed set which contains X. (Intersection of all closed sets that contain X. Notation: $\text{cl}(X)$ or $\overline{X}$.
- The boundary of X is the closure of X minus the interior of X. A point is on the boundary of X iff all of its neighborhoods intersects with both X and X^c. Notation: $\text{Bd}(X)$ or δX.
- The exterior of X is the interior of X^c. Notation: $(X^c)^\circ$

Remark 7.12.
- $X^\circ \subseteq X \subseteq \overline{X}$
- X is open $\iff X^\circ = X$
- X is closed $\iff \overline{X} = X$

Definition 7.13. We say $x \in X$ is an isolated point of X if x has a neighborhood whose intersection with X is only x.

Definition 7.14. We say $x \in \overline{X}$ is an accumulation point if any neighborhood of x contains infinitely many points from X. (It is also called limit point)

Proposition 7.15. *The accumulation points of any set form a closed set.*

Proof. Suppose A contains all accumulation points of a set $\overline{X}$, we show that $A^c = \overline{X}^c \cup \{x \in \overline{X} : x \text{ is not an accumulation point of } X\}$ is open.

Since $\overline{X}^c$ is open, it is enough to show $\{x \in \overline{X} : x \text{ is not an accumulation point of } X\}$ is also open.

For all $x \in \overline{X}$ such that x is not an accumulation point of $\overline{X}$. This means, there exists some neighbourhood U of x such that $U \cap \overline{X}$ is finite, we can write this as $U \cap X = \{x, a_1, ..., a_n\}$ for some $n \in \mathbb{N}, a_i \neq x$ for all $i \in [n]$. Now, take $\delta = \min(d(x, a_i) : i \in [n])$ and take $B(x, \delta)$, we have $B(x, \delta) \cap X = \{x\}$ □

7.2 Connectedness.

Definition 7.16. A subset S is called connected if whenever U_1, U_2 satisfies that

- $S = U_1 \cup U_2$
- U_1 and U_2 are open
- $U_1 \cap U_2 = \emptyset$

We must have $S \subseteq U_1$ or $S \subseteq U_2$

Theorem 7.17. *If $(S_\alpha)_{\alpha \in I}$ is a family of connected sets so that $\bigcap_{\alpha \in I} S_\alpha \neq \emptyset$ then $\bigcup_{\alpha \in I} S_\alpha$ is connected*

Proof. Assume there exists open A, B such that $\bigcup_{\alpha \in I} S_\alpha = A \cup B$ where $A \cap B = \emptyset$. Note first that $A \neq \emptyset, B \neq \emptyset$. Now, take any S_α where $\alpha \in I$, note that we must either have $S_\alpha \subseteq A$ or $S_\alpha \subseteq B$ by connectedness as otherwise, the set $S_\alpha \cap A$ and $S_\alpha \cap B$ are both open in S_α and it would contradict that S_α is connected. WLOG, we assume $S_\alpha \subseteq A$

Hence, for all other S_β where $\beta \neq \alpha$, since $S_\beta \cap S_\alpha \neq \emptyset$, this means $S_\beta \cap A \neq \emptyset$. Using a similar argument as the case for S_α, it must also be the case that $S_\beta \subseteq A$. Since β is arbitrary, we have $\bigcup_{\alpha \in I} S_\alpha \subseteq A$.

Hence, showing connectedness. □

Theorem 7.18. *If $S \subseteq \mathbb{C}$ then it can be uniquely decomposed by connected sets, that means $S = \bigcup U_\alpha$ where each U_α is connected.*

Proof. Given $s \in S$ and denote $C(s)$ to be the union of all the connected subsets of S that contains s. By `Theorem 7.17`, $C(s)$ is connected. Note that $C(s)$ would also be the largest connected subset that contains s. And also note that this set is unique.

Now, we prove that any two largest connected subsets are either disjoint or identical. Suppose $C(a), C(b)$ such that $x \in C(a) \cap C(b)$, we show these two subsets are the same. Indeed, we must have $C(a) \subseteq C(x) \implies a \in C(x) \implies C(x) \subseteq C(a)$. This shows that $C(a) = C(x)$. In a similar way we can find $C(b) = C(x)$. Hence, this shows that any two largest connected subsets are either disjoint or identical.

Hence, this obtains a unique partitions of S into largest connected subsets. □

Theorem 7.19. (Intermediate Value Theorem) *Let S be connected and f is a continuous function defined on S, then $f(S)$ is connected.*

Proof. Suppose for a contradiction that $f(S)$ is not connected. That is, we can write $f(S) = A \cup B$ for some disjoint A, B, where $f(S) \nsubseteq A$ and $f(S) \nsubseteq B$.

Then, we have

$$f^{-1}(f(S)) = f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$$

where by `Lemma 7.20`, $f^{-1}(A), f^{-1}(B)$ are open. Moreover, $f^{-1}(A) \cap f^{-1}(B)$ are disjoint since A, B are disjoint.

Finally, $S \not\subseteq f^{-1}(A)$ and $S \not\subseteq f^{-1}(B)$, showing that S is not connected, a contradiction. □

Lemma 7.20. *If $f : X \to Y$ is continuous where $B \subseteq Y$ is open, then $f^{-1}(B)$ is also open.*

Proof. Take $a \in f^{-1}(B)$ such that $f(a) \in B$. Since B is open, we can find an $\varepsilon > 0$ such that $B(f(a), \varepsilon) \subseteq B$.

Since f is continuous at a, given this $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $x \in X$

$$d(x, a) < \delta \implies d(f(x), f(a)) < \varepsilon \implies f(x) \in B$$

Hence, this shows that

$$B(a, \delta) \in f^{-1}(B)$$

□

Remark 7.21. Path-connectedness is not covered in Ahlfors, just take those results as granted.

Definition 7.22. A continuous $\gamma : [a, b] \to \mathbb{C}$ is called an arc or a path from $\gamma(a)$ to $\gamma(b)$.

Definition 7.23. $S \subset \mathbb{C}$ is called path-connected if $z_1, z_2 \in S$ then there exists a path $\gamma : [a, b] \to S$ so that $\gamma(t)$ is in S for any $t \in [a, b]$ such that $\gamma(a) = z_1, \gamma(b) = z_2$.

Theorem 7.24. *$B(z, \delta)$ and $\overline{B}(z, \delta)$ are path connected.*

Theorem 7.25. *Let $U \subset \mathbb{C}$ be open, then U is connected iff it is path connected*

Theorem 7.26. *A path-connected subset of $\mathbb{C}$ is always connected. But not all connected subsets of C are path-connected.*

Theorem 7.27. *If $U \subset \mathbb{C}$ is open then each of its connected components is open.*

Not from Ahlfors ends.

Definition 7.28. A region is an open connected subset of $\mathbb{C}$.

We will denote all analytic functions in a region D as $A(D)$.

Theorem 7.29. *If $f \in A(D)$ and $f'(z) = 0$ for any $z \in D$ then f is constant in D.*

Proof. See `Proposition 4.22`. □

7.3 Compactness.

Definition 7.30. K is compact if $K \subset \bigcup_{\alpha \in I} U_\alpha$, all U_α's are open, then there is a finite number of, $U_1, ..., U_n$ such that $K \subset U_1 \cup U_2 \cup ... \cup U_n$.

Theorem 7.31. *$K \subset \mathbb{C}$ is compact if and only if K is a closed and bounded in $\mathbb{C}$*

Theorem 7.32. *$K \subset \mathbb{C}$ is compact if and only if for every sequence $\{z_n\} \subset K$ there exists a subsequence z_{n_k} st. $\lim_{k\to\infty} z_{n_k} = z \in K$*

Theorem 7.33. *If K is compact, If $f : K \to \mathbb{R}/\mathbb{C}$ is continuous, then $f(K)$ is closed and bounded. And f is uniformly continuous.*

Theorem 7.34. *If for every n, $K_n \neq \emptyset$ is compact and $K_n \supset K_{n+1}$ then $\bigcap_1^\infty K_n \neq \emptyset$*

8 Conformality

8.1 Analytic Functions in Regions.

Definition 8.1. f is analytic on $S \subset C$, if there exist an open set U, such that $S \subset U$, and there exists $F : U \to \mathbb{C} \in A(U)$. Then we call $F_{|S} = f$

Definition 8.2. A complex-valued function $f(z)$, defined on a region Ω is said to be analytic in Ω if it has a derivative at each point of Ω.

Definition 8.3. A function $f(z)$ is analytic in an arbitrary set of points A if it is the restriction to A of a function which is analytic in some open neighborhood containing A.

When we say "let $f(z)$ be analytic at z_0", it means a function $f(z)$ is defined and has a derivative in some unspecified open neighborhood of z_0.

Remark 8.4. The above definition requires all analytic functions to be single-valued.

For multi-valued functions, it is possible to restrict their domain to a region where they are single valued.

Example 8.5. Let $f(z) = \sqrt{z}$. Recall that if $z = r(\cos(\theta) + i\sin(\theta))$, then $\sqrt{z}$ can be

$$\sqrt{r}\left(\cos\left(\frac{\theta}{2}\right) + i\sin\left(\frac{\theta}{2}\right)\right)$$

$$\sqrt{r}\left(\cos\left(\frac{\theta}{2}+\pi\right) + i\sin\left(\frac{\theta}{2}+\pi\right)\right) = -\sqrt{r}\left(\cos\left(\frac{\theta}{2}\right) + i\sin\left(\frac{\theta}{2}\right)\right)$$

To make f single-valued, restrict the domain to $\mathbb{C} \setminus (-\infty, 0]$, and define $\sqrt{z}$ with the root with a positive real part.

If we left the negative real axis in the domain: for example approaching $z = -1$. Approaching from above the axis we will have $\operatorname{Arg}(z) \to \pi$ so $\frac{\theta}{2} \to \frac{\pi}{2}$. Approaching from below the axis $\operatorname{Arg}(a) \to -\pi$ so $\frac{\theta}{2} \to -\frac{\pi}{2}$. This will make the square root to have a jump discontinuity.

Definition 8.6. A branch of f is a single valued function $F : U \to \mathbb{C}$ so that u is open and connected, $F(z)$ has only a single value and $F(z)$ is analytic in U.

8.2 Conformal Mapping.

We will see that how a conformal map is defined so that it preserves angles between curves that intersect at the same point.

Definition 8.7. An angle between curves is the angle between their tangent lines, at the point they are intersecting.

Example 8.8. Let $\gamma_1(t) = (\sin(t), \cos(t))$ for $0 \le t \le \pi$ and $\gamma_2(t) = (t, t_2)$ for $0 \le t \le 3$.

Let $f(z) = z^2$ be analytic. Then

$$f(\gamma_1(t)) = (\sin(t) + i\cos(t))^2 = (\sin^2(t) - \cos^2(t), 2\sin(t)\cos(t))$$

$$f(\gamma_2(t)) = \left(t + it^2\right)^2 = \left(t^2 - t^4, 2t^3\right)$$

If we graph those curves as shown in Figure 4, note that the angle between two curves does not change, which is what we want if we look at the conformal mapping.

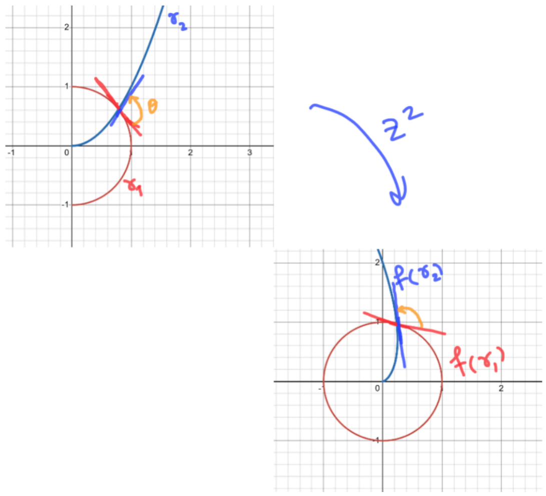

FIGURE 4. Conformal Map

Let $g(z) = \overline{z}$. Note that g is not analytic, then

$$g(\gamma_1(t)) = \overline{\sin(t) + i\cos(t)} = \sin(t) - i\cos(t) = (\sin(t), -\cos(t))$$
$$g(\gamma_2(t)) = \overline{t + it^2} = t - it^2 = (t, -t^2)$$

If we graph those curves as shown in Figure 5, note that the angle between two curves does change (as the sign has switched), in this case, g is not a conformal mapping.

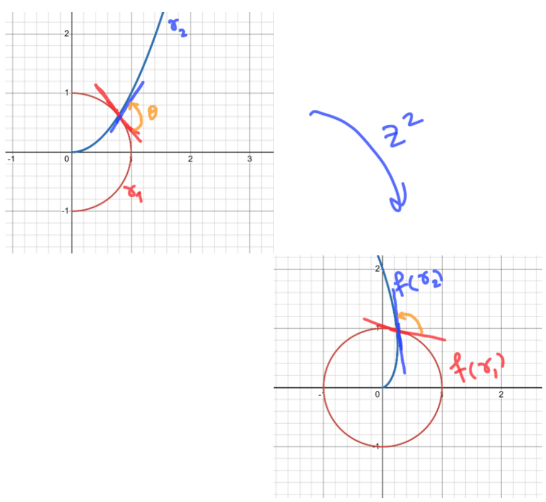

FIGURE 5. Conformal Map

Remark 8.9. To get some more ideas of conformal mapping. Let $\gamma(t)$ be an arc (or a curve) with $\alpha \le z \le \beta$ which is contained in a region Ω.

Let $f(z)$ be defined and is continuous in Ω. Then define $w(t) = f(\gamma(t))$.

If $f(z)$ is analytic in Ω and $\gamma(t)$ is piecewise smooth, then we have

$$w'(t) = \gamma'(t)f'(\gamma(t))$$

Assume $f'(\gamma(t_0)) \neq 0$ for some $t_0 \in [\alpha, \beta]$. Then:

$$w'(t_0) = \gamma'(t_0)f'(\gamma(t_0)) \Longrightarrow \operatorname{Arg}(w'(t_0)) = \operatorname{Arg}(\gamma'(t_0)) + \operatorname{Arg}(f'(\gamma(t_0)))$$

So the tangent line is only scaled and rotated (in the same angle. If we have another curve the same rotation will be applied to the image.

Definition 8.10. A function is conformal if it preserves angles between curves. More precisely, a smooth complex-valued function f (continuous with continuous partial derivatives) is conformal at z_0, if whenever the two curves γ_1 and γ_2 intersect at z_0 with non-zero tangents at z_0, $f \circ \gamma_1$ and $f \circ \gamma_2$ have non-zero tangents at $f(z_0)$ that intersect at same angle

Theorem 8.11. *If $f : U \to \mathbb{C}$ is analytic and if $z_0 \in U$ st. $f'(z_0) \neq 0$ then f is conformal at z_0.*

Definition 8.12. A function f is conformal on a region U if it is conformal at each point of U. It must also satisfy that f is analytic at all $z \in U$ and $f'(z) \neq 0$. Moreover, f is one-to-one and onto $f(U)$

Example 8.13. Let $f : \{z \in \mathbb{C} : \mathrm{Re}(z) > 0\} \to \mathbb{C} \setminus (-\infty, 0]$ defined by $f(z) = z^2$, then the function is conformal on $\{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$

Example 8.14. $f(z) = e^z$ is conformal at each point in each point of $\mathbb{C}$ and $f'(z) \neq 0$ in $\mathbb{C}$. So, f is conformal at each point of $\mathbb{C}$

However, since f is not one to one in $\mathbb{C}$ ($e^z = e^{z+2\pi i}$), it is not a conformal mapping from $\mathbb{C}$ to $\mathbb{C} \setminus \{0\}$.

Note also that f is locally conformal.

Remark 8.15. The function $f(z) = \overline{z}$ is one-to-one and onto $\mathbb{C} \to \mathbb{C}$ but reverses angles, while the size of the angle preserved, it is called indirectly conformal.

Example 8.16. Show directly (by computing the tangents and angles) that $f(z) = z^3$ maintains the angles between the curves $\gamma_1(t) = t + t^2 i$ and $\gamma_2(t) = t^2 + ti$, at the intersection point $z = 1 + i$.

Proof. Note that at $z = 1 + i$, we have $\gamma_1(1) = 1 + i = \gamma_2(1)$.

We also have

$$\gamma_1{}'(t) = 1 + 2ti \Longrightarrow \gamma_1{}'(1) = 1 + 2i$$
$$\gamma_2{}'(t) = 2t + i \Longrightarrow \gamma_1{}'(1) = 2 + i$$

This gives the angle to be

$$\arg\left(\frac{\gamma_2{}'(t)}{\gamma_1{}'(t)}\right) = \arg\left(\frac{2+i}{1+2i}\right) = \arg\left(\frac{(2+i)(1-2i)}{5}\right) = \arg\left(\frac{4-3i}{5}\right) = -\arctan\left(\frac{3}{4}\right)$$

Now, consider

$$f(\gamma_1(1))' = f'(\gamma_1(1))\gamma_1{}'(1) = 3\gamma_1(1)^2(1+2i) = 3(1+i)^2(1+2i) = 6i(1+2i)$$
$$f(\gamma_2(1))' = f'(\gamma_2(1))\gamma_2{}'(1) = 3\gamma_2(1)^2(2+i) = 3(1+i)^2(2+i) = 6i(2+i)$$

This gives the angle to be

$$\arg\left(\frac{f(\gamma_2{}'(t))}{f(\gamma_1{}'(t))}\right) = \arg\left(\frac{6i(2+i)}{6i(1+2i)}\right) = \arg\left(\frac{(2+i)}{1+2i}\right) = \arg\left(\frac{(2+i)(1-2i)}{5}\right) = \arg\left(\frac{4-3i}{5}\right) = -\arctan\left(\frac{3}{4}\right)$$

showing that $f(z)$ maintains the angle between the curves γ_1, γ_2 at the point $1 + i$. ☐

Theorem 8.17. *If f is angle preserving at z_0, and f has continuous partial derivatives at z_0, then f is analytic at z_0.*

Proof. Let $w(t) = f(z(t))$, where $z = z(t)$ is an arc defined for $\alpha \leq t \leq \beta$, and $z(t_0) = z_0$

Then, by partial derivative chain rule

$$\begin{aligned} w'(t_0) &= \frac{\partial f}{\partial x}x'(t_0) + \frac{\partial f}{\partial y}y'(t_0) \\ &= \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)z'(t_0) + \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)\overline{z'(t_0)} \\ &= \frac{\partial f}{\partial z}z'(t_0) + \frac{\partial f}{\partial \overline{z}}\overline{z'(t_0)} \end{aligned}$$

so

$$\frac{w'(t_0)}{z'(t_0)} = \frac{\partial f}{\partial z} + \frac{\partial f}{\partial \overline{z}}\frac{\overline{z'(t_0)}}{z'(t_0)}$$

$\mathrm{Arg}\left(\frac{w'(t_0)}{z'(t_0)}\right)$ is independent from $\mathrm{Arg}(z'(t_0))$ (as angles are preserved), but $\mathrm{Arg}\left(\frac{\overline{z'(t_0)}}{z'(t_0)}\right)$ is dependent to $\mathrm{Arg}(z'(t_0))$, so $\frac{\partial f}{\partial \overline{z}} = 0$. Note that

$$\frac{\partial f}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)$$

which means

$$\frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y}$$

If $f = u + iv$

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right) \implies \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \wedge \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

Since f has continuous partial derivatives, this means f is analytic. □

Now, recall that a function $h : \mathbb{R}^2 \to \mathbb{R}$ satisfies the Laplace's Equation if $h_{xx} + h_{yy} = 0$. Now, consider the following result, that says Conformal Mapping preserves Laplace's Equation.

Theorem 8.18. *Take $f : D \to S$ is analytic and $f'(z) \neq 0$, let $f = u + iv$, and h satisfies $h_{xx} + h_{yy} = 0$, then $h \circ f(x, y) = h(u(x, y), v(x, y))$ satisfies the Laplace's Equation (i.e. $h_{uu} + h_{vv} = 0$).*

Proof. We get

$$\begin{aligned} h_{xx} &= (h_x)_x \\ &= (h_u u_x + h_v v_x)_x \\ &= (h_u u_x)_x + (h_v v_x)_x \\ &= (h_u)_x u_x + h_u u_{xx} + (h_v)_x v_x + h_v v_{xx} \\ &= \big((h_u)_u u_x + (h_u)_v v_x\big)u_x + h_u u_{xx} + \big((h_v)_u u_x + (h_v)_v v_x\big)v_x + h_v v_{xx} \quad [h_u, h_v \text{ is a function of } u, v] \\ &= h_u u_{xx} + h_{uu}u_x^2 + h_{uv}u_x v_x + h_v v_{xx} + h_{vu}u_x v_x + h_{vv}v_x^2 \\ h_{yy} &= \big(h_y\big)_y \\ &= \big(h_u u_y + h_v v_y\big)_y \\ &= \big(h_u u_y\big)_y + \big(h_v v_y\big)_y \\ &= (h_u)_y u_y + h_u u_{yy} + (h_v)_y v_y + h_v v_{yy} \\ &= \big((h_u)_u u_y + (h_u)_v v_y\big)u_y + h_u u_{yy} + \big((h_v)_u u_y + (h_v)_v v_y\big)v_y + h_v v_{yy} \quad [h_u, h_v \text{ is a function of } u, v] \\ &= h_u u_{yy} + h_{uu}u_y^2 + h_{uv}u_y v_y + h_v v_{yy} + h_{vu}u_y v_y + h_{vv}v_y^2 \end{aligned}$$

Since f is analytic then u, v must satisfy the Laplace's Equation $u_{xx} + u_{yy} = 0 = v_{xx} + v_{yy}$.

Then, adding together we get

$$h_{xx}+h_{yy}=h_{uu}(u_x^2+u_y^2)+(h_{uv}+h_{vu})(u_xv_x+u_yv_y)+h_{vv}(v_x^2+v_y^2)$$

Since f is analytic u,v must satisfy Cauchy-Riemann Equation and this gives us

$$\begin{aligned} h_{xx}+h_{yy} &= (h_{uu}+h_{vv})(u_x^2+u_y^2) \\ &= (h_{uu}+h_{vv})|f'(z)|^2 \quad [\texttt{Theorem 4.18}] \end{aligned}$$

Since $f'(z)\neq 0$, we must have $h_{uu}+h_{vv}=0$. □

8.3 **Exercises.**

Exercise 8.19. Give a precise definition of a single-valued branch of $\sqrt{1+z}+\sqrt{1-z}$ in a suitable region, and prove that it is analytic.

Exercise 8.20. Same problem for $\log(\log(z))$

9 Linear Transformations

9.1 **Linear Map.**

Definition 9.1. $S(z):\mathbb{C}\to\mathbb{C}$ is called a linear transformations or Mobius transformations if it is of the form

$$S(z)=\frac{az+b}{cz+d}\implies S'(z)=\frac{ad-bc}{(cz+d)^2}$$

with $ad-bc\neq 0$. This is a special form of rational functions of order 1.

Remark 9.2. Note that

- $S^{-1}(w)=\frac{dw-b}{-cw+a}$
- $S(\infty)=\frac{a}{c}$
- $S\left(-\frac{d}{c}\right)=\infty$

Remark 9.3. Consider $M=\left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix} : ad-bc\neq 0\right\}$

This is a group under matrix multiplication.

Remark 9.4.

- $z+a$ is called a parallel translation
- $\frac{1}{z}$ is called an inversion

Theorem 9.5. *Any Mobius transformation is a composition of a rotation, dilation, translations and an inversion*

Proof. If $c=0$, then

$$\frac{az+b}{d}=\frac{a}{d}z+\frac{b}{d}$$

If $c\neq 0$ then

$$\frac{az+b}{cz+d}=\frac{a}{c}+\frac{bc-ad}{c^2}\left(\frac{1}{z+\frac{d}{c}}\right)$$

□

Theorem 9.6. *Any Mobius transformation maps circles and lines to circles and lines.*

Proof. This claim is obvious when we have translations or rotation and dilations, so we only need to prove it for inversions.

Consider a circle $|z-a| = r \implies |z-a|^2 = r^2$. For all $z \in \mathbb{C}$, we can write $z = \frac{1}{w}$ where $w \in \mathbb{C}$. We get

$$\left|\frac{1}{w} - a\right|^2 = r^2 \implies |1-aw|^2 = r^2\ |w|^2 \implies (1-aw)^2 = r^2\ |w|^2$$

If we write $w = u+iv$, we get

$$(a^2 - r^2)(u^2+v^2) + Au + Bv + 1 = 0$$

for some constants A, B

If $|a| = r$ then such equation becomes $Au + Bv + 1 = 0$ which is a line.

If $|a| \neq r$ then such equation represents a circle.

Now, consider a line $Ax + By = C$, we can rewrite this as $A\operatorname{Re}(z) + B\operatorname{Im}(z) = C$.

Consider $z = \frac{1}{w}$ where $w = u+iv$ (hence $z = \frac{u-iv}{u^2+v^2}$), we get

$$A(\operatorname{Re}\left(\frac{1}{w}\right) + B\left(\operatorname{Im}\left(\frac{1}{w}\right)\right) = C \implies A\left(\frac{u}{u^2+v^2}\right) + B\left(\frac{-v}{u^2+v^2}\right) = C \implies Au + B(-v) = C(u^2+v^2)$$

If $C \neq 0$, we get a circle. If $C = 0$, we get a line. □

9.2 **Cross Ratio.**

Theorem 9.7. *Given three distinct points z_2, z_3, z_4 in the extended plane, there exist linear transformations (a cross ratio) S, which carry them to $1, 0, \infty$ in this order.*

Moreover, such transformation is unique.

Proof. We first prove existence.

Case 1: If none of the z_2, z_3, z_4 are ∞, then set

$$S = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)}$$

Case 2: If $z_2 = \infty$, then set

$$S = \frac{z_1 - z_3}{z_1 - z_4}$$

Case 3: If $z_3 = \infty$, then set

$$S = \frac{z_2 - z_4}{z_1 - z_4}$$

Case 4: If $z_4 = \infty$, then set

$$S = \frac{z_1 - z_3}{z_2 - z_3}$$

Now, we prove uniqueness. Take T be another transformation such that $T(z_2) = 1, T(z_3) = 0, T(z_4) = \infty$. Note that mobius transformation is closed by composition and inversion (check).

We can assume $ST^{-1}(z_1) = \frac{az_1+b}{cz_1+d}$

Then,

$$ST^{-1}(0) = 0 \implies \frac{0+b}{0+d} = 0 \implies b = 0$$
$$ST^{-1}(\infty) = \infty \implies \frac{a}{c} = \infty \implies a \neq 0, c = 0$$
$$ST^{-1}(1) = 1 \implies \frac{a}{d} = 1 \implies b = a = d$$

Hence, this shows that for all $z_1 \in \mathbb{C}, ST^{-1}(z_1) = z_1 \implies S = T$ □

Example 9.8. Find the Mobius transformation T, so that $T(1) = i, T(2) = 1+i, T(3) = \infty$.

Proof. Let $Tz = \frac{az+b}{cz+d}$. We have $T(3) = \infty \implies 3c + d = 0$. WLOG, we can assume up to a scalar that $c = 1, d = -3$. Hence, $Tz = (az + b)(z - 3)$.

Now, since $T(1) = i, T(2) = 1 + i$, we have

$$T(1) = \frac{a+b}{-2} = i \implies a + b = -2i$$
$$T(2) = \frac{2a+b}{-1} = 1 + i \implies 2a + b = -1 - i$$

Hence, this gives $a = -1 + i, b = 1 - 3i$.

Therefore,

$$Tz = \frac{(-1+i)z + (1-3i)}{z-3}$$

□

Definition 9.9. The cross ratio (z_1, z_2, z_3, z_4) is the image of z_1 under the map which takes distinct z_2, z_3, z_4 to $1, 0, \infty$, respectively.

Theorem 9.10. *If z_1, z_2, z_3, z_4 are distinct points in the extended plane and T any linear transformation, then $(Tz_1, Tz_2, Tz_3, Tz_4) = (z_1, z_2, z_3, z_4)$.*

Proof. Consider $Sz_1 = (z_1, z_2, z_3, z_4)$, take any linear transformation T. Then consider ST^{-1}, we have

$$ST^{-1}(T(z_2)) = S(z_2) = 1$$
$$ST^{-1}(T(z_3)) = S(z_3) = 0$$
$$ST^{-1}(T(z_4)) = S(z_4) = \infty$$

Hence, we get $ST^{-1}(z_1) = (z_1, Tz_2, Tz_3, Tz_4)$, therefore

$$(Tz_1, Tz_2, Tz_3, Tz_4) = ST^{-1}(T(z_1)) = S(z_1) = (z_1, z_2, z_3, z_4)$$

□

Theorem 9.11. *The cross-ratio (z_1, z_2, z_3, z_4) is real if and only if the four points lie on a circle or a line*

Proof. Consider $Sz_1 = (z_1, z_2, z_3, z_4)$, we have (z_1, z_2, z_3, z_4) is real $\iff (Sz_1, Sz_2, Sz_3, Sz_4)$ is real $\iff$ $(Sz_1, 1, 0, \infty)$ is real $\iff Sz_1$ is on the same line generated by $1, 0, \infty$ ($1, 0$, together with ∞ lies on the real axis, where Sz_1 is real on the real axis as well) $\iff z_1$ is on the same circle or line as z_2, z_3, z_4. □

9.3 **Symmetry.**

Definition 9.12. The points z and z^* are said to be symmetric with respect to the circle C through z_1, z_2, z_3 iff

$$(z^*, z_1, z_2, z_3) = \overline{(z, z_1, z_2, z_3)}$$

or $\left(Sz^* = \overline{Sz}\right)$

Theorem 9.13. *If a linear transformation carries a circle C_1 to a circle C_2, then it transforms any pair of symmetric points with respect to C_1 into a pair of symmetric points with respect to C_2.*

9.4 **Exercises.**

Exercise 9.14. Show that the reflection $z \to \overline{z}$ is not a linear transformation.

Exercise 9.15. If

$$T_1 z = \frac{z+2}{z+3}, \quad T_2 z = \frac{z}{z+1}$$

find $T_1 T_2 z, T_2 T_1 z$ and $T_1^{-1} T_2 z$

Exercise 9.16. Find the linear transformation which carries $0, i, -i$ into $1, -1, 0$.

Exercise 9.17. Express the cross ratios corresponding to the 24 permutations of four points in terms of $\lambda = (z_1, z_2, z_3, z_4)$.

Exercise 9.18. If the consecutive vertices $z_1, z_2, z_3 z_4$ of a quadrilateral lie on a circle, prove that

$$|z_1 - z_3| \cdot |z_2 - z_4| = |z_1 - z_2| \cdot |z_3 - z_4| + |z_2 \cdot z_3| \cdot |z_1 - z_4|$$

and interpret the result geometrically.

Exercise 9.19. Prove that every reflection carries circles into circles.

Exercise 9.20. Reflect the imaginary axis, the line $x = y$, and the circle $|z| = 1$ in the circle $|z - 2| = 1$.

Exercise 9.21. Find the linear transformation which carries the circle $|z| = 2$ into $|z + 1| = 1$, the point -2 into the origin, and the origin into i.

Example 9.22. Show that any linear transformation which transforms the real axis into itself, can be written with real coefficients.

10 Fundamental Theorems

10.1 Line Integral.

In real calculus, given a continuous function $f : [a, b] \to \mathbb{R}$, the integral $\int_a^b f(x)\,\mathrm{d}x$ has a clear meaning.

If $a < b$ there is only one path to go from a to b, but if a, b are complex and considering the complex plane, there are infinite paths between them.

Let's first consider a complex function f defined over a real interval, the integral is defined as follows

Definition 10.1. Take f defined on $[a, b]$ where $f(t) = u(t) + iv(t)$. We define

$$\int_a^b f(t)\,\mathrm{d}t = \int_a^b u(t)\,\mathrm{d}t + i \int_a^b v(t)\,\mathrm{d}t$$

Theorem 10.2. *Let f, g be complex functions defined on the interval $[a, b]$ by $f(t) = u(t) + iv(t), g(t) = u_1(t) + iv_1(t)$ and $c = \alpha + i\beta$ is a complex number, then:*

1. $\int_a^b cf(t)\,\mathrm{d}t = c\int_a^b f(t)\,\mathrm{d}t$
2. $\int_a^b \overline{f(t)}\,\mathrm{d}t = \overline{\int_a^b f(t)\,\mathrm{d}t}$
3. $\int_a^b f(t) \pm g(t)\,\mathrm{d}t = \int_a^b f(t)\,\mathrm{d}t \pm \int_a^b g(t)\,\mathrm{d}t$
4. *If* $a \le b$, $\left|\int_a^b f(t)\,\mathrm{d}t\right| \le \int_a^b |f(t)|\,\mathrm{d}t$

Proof. We will prove 4 only as others are trivial from real functions. Consider $\int_a^b f(t)\,\mathrm{d}t = c = re^{i\theta}$, where we have

$$\mathrm{Re}\big(ce^{-i\arg(c)}\big) = \mathrm{Re}\big(re^{i\theta}e^{-i\theta}\big) = \mathrm{Re}(r) = r = |c|$$

Therefore, we have

$$\begin{aligned}
\left|\int_a^b f(t)\,\mathrm{d}t\right| &= |c| \\
&= \operatorname{Re}\left(ce^{-i\arg(c)}\right) \\
&= \operatorname{Re}\left(e^{-i\theta}\int_a^b f(t)\,\mathrm{d}t\right) \\
&= \operatorname{Re}\left(\int_a^b e^{-i\theta} f(t)\,\mathrm{d}t\right) \\
&= \int_a^b \operatorname{Re}\left(e^{-i\theta} f(t)\right)\mathrm{d}t \\
&\leq \int_a^b |e^{-i\theta}|\ |f(t)|\,\mathrm{d}t \\
&= \int_a^b |f(t)|\,\mathrm{d}t
\end{aligned}$$

□

Now, we look at complex functions defined on complex domain.

Definition 10.3. Let γ be a differentiable arc with the equation $z(t), a \leq t \leq b$. If $f(z)$ is defined and continuous on the arc γ, then

$$\int_\gamma f(z)\,\mathrm{d}z = \int_a^b f(z(t))z'(t)\,\mathrm{d}t$$

If γ is peicewise differentiable over $[a, b]$, then we can consider the pieces to be $\gamma_1, ..., \gamma_n$ with $\gamma_i = z_{i(t)}$ for $t_i \leq t \leq t_{i+1}$ where $t_1 = a, t_{n+1} = b$ then

$$\int_\gamma f(z)\,\mathrm{d}z = \sum_{i=1}^n \int_{t_i}^{t_{i+1}} f\left(z_{i(t)}\right){z_i}'(t)\,\mathrm{d}t = \int_{\gamma_1+\ldots+\gamma_n} f(z)\,\mathrm{d}z$$

Remark 10.4. (Invariance under the change of parameters) Let $l(\tau)$ maps intervals from $\alpha \leq \tau \leq \beta$ to $a \leq l \leq b$ and assume l be piecewise differentiable, then

$$\int_a^b f'(z(t))z'(t)\,\mathrm{d}t = \int_\alpha^\beta f(z(l(\tau)))z'(l(\tau))l'(\tau)\,\mathrm{d}\tau$$

where $z'(l(\tau))l'(\tau) = z(l(\tau))'$

Definition 10.5. If γ is an arc defined by $z(t)$ for $a \leq t \leq b$, then $-\gamma$ is defined by $z(-t)$ for $-b \leq t \leq -a$ is called the opposite of γ.

Remark 10.6. Note by such definition, we have

$$
\begin{aligned}
\int_{-\gamma} f(z)\,\mathrm{d}z &= \int_{-b}^{-a} f(z(-t))(-z'(-t))\,\mathrm{d}t \\
&= -\int_{-b}^{-a} f(z(-t))z'(-t)\,\mathrm{d}t \\
&= \int_{b}^{a} f(z(u))z'(u)\,\mathrm{d}u \\
&= -\int_{a}^{b} f(z(u))z'(u)\,\mathrm{d}u \\
&= -\int_{\gamma} f(z)\,\mathrm{d}z
\end{aligned}
$$

Definition 10.7. A closed curve is a curve γ represented by $z(t)$ for $a \le t \le b$ and $z(a) = z(b)$.

Remark 10.8. Integral over a closed curve is also invariant under change of parameter. We can divide it into γ_1 and γ_2 so that $\gamma = \gamma_1 + \gamma_2$, and all properties will be the same.

Definition 10.9. We define

$$
\int_{\gamma} f(z)\overline{\mathrm{d}z} = \int_{a}^{b} f(z(t))\overline{z'(t)}\,\mathrm{d}z
$$

as the integral with respect to $\overline{z}$

Remark 10.10. We get

$$
\begin{aligned}
\int_{\gamma} f(z)\overline{\mathrm{d}z} &= \int_{a}^{b} f(z(t))\overline{z'(t)}\,\mathrm{d}t \\
&= \int_{a}^{b} \overline{\overline{f(z(t))}z'(t)}\,\mathrm{d}t \\
&= \overline{\int_{a}^{b} \overline{f(z(t))}z'(t)\,\mathrm{d}t} \\
&= \overline{\int_{\gamma} \overline{f}\,\mathrm{d}z}
\end{aligned}
$$

Remark 10.11. Recall that $x = \frac{z+\overline{z}}{2}, y = \frac{z-\overline{z}}{2i}$, we get

$$
\begin{aligned}
\int_{\gamma} f\,\mathrm{d}x &= \frac{1}{2}\left(\int_{\gamma} f\,\mathrm{d}z + \int_{\gamma} f\overline{\mathrm{d}z}\right) \\
\int_{\gamma} f\,\mathrm{d}y &= \frac{1}{2i}\left(\int_{\gamma} f\,\mathrm{d}z - \int_{\gamma} f\overline{\mathrm{d}z}\right)
\end{aligned}
$$

Definition 10.12. We define the integral with respect to arc length to be

$$
\int_{\gamma} f\,\mathrm{d}s = \int_{\gamma} f\ |\mathrm{d}z| = \int_{a}^{b} f(z(t))\ |z'(t)|\,\mathrm{d}t
$$

Remark 10.13. This integral again is independent of the choice of parameter. We also have

$$\begin{aligned}\int_{-\gamma} f \ |\mathrm{d}z| &= \int_{-b}^{-a} f(z(-t)) \ |-z'(-t)| \,\mathrm{d}t \\ &= -\int_b^a f(z(u)) \ |-z'(u)| \,\mathrm{d}u \\ &= \int_a^b f(z(u)) \ |z'(u)| \,\mathrm{d}u \\ &= \int_\gamma f \ |\mathrm{d}z|\end{aligned}$$

Remark 10.14. Note that if $f = 1$, then $\int_\gamma f \ |\mathrm{d}z|$ is the length of γ.

Example 10.15. Let $z(t)$ be a circle and $z(t) = a + re^{it}$ for $0 \leq t \leq 2\pi$. Then

$$\int_\gamma |\mathrm{d}z| = \int_0^{2\pi} |z'(t)| \,\mathrm{d}t = \int_0^{2\pi} |ire^{it}| \,\mathrm{d}t = \int_0^{2\pi} r \,\mathrm{d}t = 2\pi r$$

10.2 **Line Integrals as Functions of Arcs.**

The general line integrals are of the form

$$\int_\gamma p \,\mathrm{d}x + q \,\mathrm{d}y$$

where $p(x, y), q(x, y)$ are continuous functions on the region Ω, and γ is some path that varies in Ω

Example 10.16. Consider $z(t) = (\cos(t), \sin(t))$ with $0 \leq t \leq \pi$ (the upper half circle). We have

$$\mathrm{d}x = x'(t) \,\mathrm{d}t = -\sin(t) \,\mathrm{d}t$$
$$\mathrm{d}y = y'(t) \,\mathrm{d}t = \cos(t) \,\mathrm{d}t$$

Moreover,

$$\begin{aligned}\int_\gamma y \,\mathrm{d}x + x \,\mathrm{d}y &= \int_0^\pi (-\sin^2(t) + \cos^2(t)) \,\mathrm{d}t \\ &= \int_0^\pi \cos(2t) \,\mathrm{d}t \\ &= \left[\frac{1}{2}\sin(2t)\right]_0^\pi \\ &= 0\end{aligned}$$

Theorem 10.17. *The line integral $\int_\gamma p \,\mathrm{d}x + q \,\mathrm{d}y$ defined in a region Ω, depends only on the end points of γ if and only if there exists a function $U(x, y)$ in Ω with partial derivatives $\frac{\partial U}{\partial x} = p$ and $\frac{\partial U}{\partial y} = q$.*

Note: *If such function U exists then U is unique up to an additive constant.*

Theorem 10.18. *The integral $\int_\gamma f \,\mathrm{d}z$, with continuous f depends only on the end points of γ if and only if f is the derivative of an analytic function in Ω*

Proof. If $f = F'$ for some analytic F. Take any path γ where $z : [a, b] \to \Omega$ we get

$$\int_\gamma f \,\mathrm{d}z = \int_a^b f(z(t))z'(t) \,\mathrm{d}t = \int_a^b \frac{\mathrm{d}}{\mathrm{d}t}F(z(t)) \,\mathrm{d}t = F(z(b)) - F(z(a))$$

showing that this depends only on the end points of γ.

If $\int_\gamma f \,\mathrm{d}z$ depends only on the end points, let's write

$$\int_\gamma f\,\mathrm{d}z = \int_\gamma f(\mathrm{d}x + i\,\mathrm{d}y) = \int_\gamma f\,\mathrm{d}x + if\,\mathrm{d}y$$

By `Theorem 10.17`, there exists an F such that $\frac{\partial F}{\partial x} = f$, $\frac{\partial F}{\partial y} = if$. If we write $F(x,y) = u(x,y) + iv(x,y)$, $f(x,y) = p(x,y) + iq(x,y)$ we get

$$\frac{\partial F}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = p + iq$$
$$\frac{\partial F}{\partial y} = \frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y} = -q + ip$$

This gives

$$\frac{\partial u}{\partial x} = p = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = q = -(-q) = -\frac{\partial u}{\partial y}$$

which is the Cauchy-Riemann Equation for F. Since f is by assumption continuous, showing that F has continuous partial derivatives, then F is analytic, and its derivative is $F'(z) = \frac{\partial F}{\partial x} = f(z)$. □

Example 10.19.

1. If $n \geq 0$ then $\int_\gamma (z-a)^n\,\mathrm{d}z = 0$ for all closed curves γ.
2. If $n < 0$ and $n \neq -1$, we still have $\int_\gamma (z-a)^n\,\mathrm{d}z = 0$ for all closed curves γ which do not passes through a.
3. If $n = -1$, and γ is a circle with center a and radius r then

$$\int_\gamma (z-a)^n\,\mathrm{d}z = 2\pi i$$

Proof.

1. By `Theorem 10.18`, $(z-a)^n$ has an anti-derivative. Hence $\int_\gamma (z-a)^n = 0$.
2. Same as part 1.
3. We have $z = a + re^{i\theta}$ where $0 \leq \theta \leq 2\pi$. Hence $\mathrm{d}z = ire^{i\theta}d\theta$. Hence,

$$\begin{aligned}\int_\gamma (z-a)^n\,\mathrm{d}z &= \int_0^{2\pi} \frac{ire^{i\theta}}{re^{i\theta}}\,\mathrm{d}\theta \\ &= \int_0^{2\pi} i\,\mathrm{d}\theta \\ &= 2\pi i\end{aligned}$$

□

10.3 **Cauchy's Theorem for a Rectangle.**

Theorem 10.20. *Consider a function $f : \mathbb{C} \to \mathbb{C}$ if $f(z)$ is analytic on R, then*

$$\int_{\delta R} f(z)\,\mathrm{d}z = 0$$

where $R = \{x + iy : a \leq x \leq b, c \leq y \leq d\}$

Remark 10.21. Note that R is closed, hence, f is analytic in R means that it is analytic on an open set that contains R.

The boundary of R can be considered as a simple closed curve consisting of four lines segments.

We choose the direction to be anti-clockwise

Proof. Given $f : \mathbb{C} \to \mathbb{C}$, we will define

$$\eta(R) = \int_{\delta R} f(z)\,\mathrm{d}z$$

We bisect the rectangle like the following

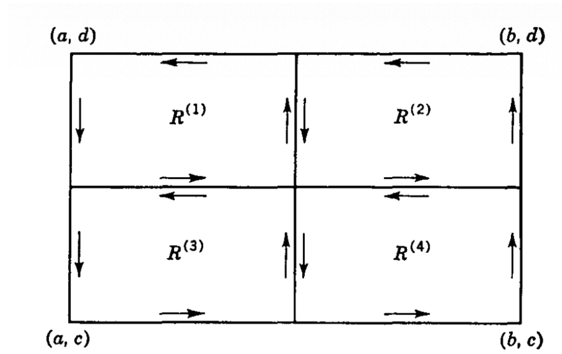

FIGURE 6. Bisection of a rectangle

Note that the common sides cancel each other because of the opposite directions. Hence we have

$$\eta(R) = \eta(R_1) + \eta(R_2) + \eta(R_3) + \eta(R_4)$$

Note that there must be some $k \in [4]$ that satisfies

$$|\eta(R_k)| \geq \frac{1}{4}|\eta(R)|$$

WLOG, let's call this R_1, continuing this process by bisecting R_1 and we can obtain a sequence

$$R \supset R_1 \supset R_2 \supset \cdots$$

such that for all $n \in \mathbb{N}$

$$|\eta(R_n)| \geq 4^{-n}|\eta(R)|$$

Hence, $\bigcap_{n=1}^{\infty} R_n = \{z^*\}$ for some $z^* \in R$. Pick some small δ_1 such that $f(z)$ is analytic on all $|z - z^*| < \delta_1$.

Given some $\varepsilon > 0$, choose some δ_2 such that

$$\left|\frac{f(z) - f(z^*)}{z - z^*} - f'(z^*)\right| < \varepsilon \implies |f(z) - f(z^*) - (z - z^*)f'(z^*)| < \varepsilon|z - z^*|$$

for all $|z - z^*| < \delta_2$. Pick $\delta = \min(\delta_1, \delta_2)$, pick n large enough such that for all $z \in R_n, |z - z^*| < \delta$, we get

$$\eta(R_n) = \int_{\delta R_n} f(z)\,\mathrm{d}z = \int_{\delta R_n} f(z) - f(z^*) - (z - z^*)f'(z^*)\,\mathrm{d}z$$

This is because

$$\int_{\delta R_n} \mathrm{d}z = 0 = \int_{\delta R_n} z\,\mathrm{d}z$$

Hence,

$$\begin{aligned}|\eta(R_n)| &= \left|\int_{\delta R_n} f(z) - f(z^*) - (z - z^*) f'(z^*) \, \mathrm{d}z\right| \\ &\leq \int_{\delta R_n} |f(z) - f(z^*) - (z - z^*) f'(z^*)||\mathrm{d}z| \quad \text{[By Ahlfors, to formally prove, parameterise } \delta R_n\text{]} \\ &\leq \int_{\delta R_n} \varepsilon |z - z^*||\mathrm{d}z|\end{aligned}$$

If we write d_n to be the length of diagonal of R_n, L_n to be the length of perimeter of R_n we get

$$|\eta(R_n)| \leq \varepsilon d_n L_n = 4^{-n} \varepsilon d L \Longrightarrow |\eta(R)| \leq \varepsilon d L$$

since ε is arbitrary, $|\eta(R)| = 0$. □

Theorem 10.22. *Let $f(z)$ be analytic on the set R' obtained from a rectangle R by omitting a finite number of interior points ξ_i · If it is true that*

$$\lim_{z \to \xi_j} (z - \xi_j) f(z) = 0$$

for all j, then

$$\int_{\delta R} f(z) \, \mathrm{d}z = 0$$

Proof. It is sufficient to consider the case of a single exceptional point ξ, for evidently R can be divided into smaller rectangles which contain at most one ξ_i.

Given $\varepsilon > 0$, we divide R into nine rectangles, where R_0 is the middle rectangle that contains ξ, as shown below

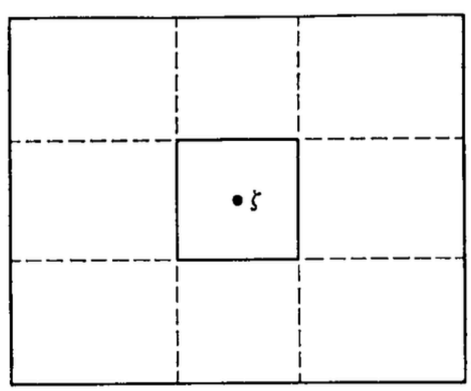

Figure 7. Trisection of a rectangle

where we get $\int_{\delta R'} f \, \mathrm{d}z = 0$ for all other $R' \neq R_0$. Hence,

$$\int_{\delta R} f \, \mathrm{d}z = \int_{\delta R_0} f \, \mathrm{d}z$$

We can continue to do this until R_0 is small and ξ is the centre of R_0 that

$$|f(z)| \leq \frac{\varepsilon}{|z - \xi|}$$

on δR_0. This gives us

$$\begin{aligned}\left|\int_{\delta R_0} f \,\mathrm{d}z\right| &\leq \varepsilon \int_{\delta R_0} \frac{1}{|z-\xi|}\,|\,\mathrm{d}z| \\ &< 8\varepsilon \quad [\text{Assuming } \xi \text{ is the centre of } R_0]\end{aligned}$$

□

Example 10.23. Evaluate $\int_C \overline{z}\,\mathrm{d}z$, first let C be the upper half circle from -1 to 1, then let C be the lower half circle from -1 to 1.

Proof. We can parameterise the half circle as $z(t) = e^{it}$, we have

$$\int_C \overline{z}\,\mathrm{d}z = \int_\pi^0 e^{-it}(ie^{it})\,\mathrm{d}t = \int_\pi^0 i\,\mathrm{d}t = -i\pi$$

$$\int_C \overline{z}\,\mathrm{d}z = \int_\pi^{2\pi} e^{-it}(ie^{it})\,\mathrm{d}t = \int_\pi^{2\pi} i\,\mathrm{d}t = i\pi$$

□

Example 10.24. Evaluate $\int_C z^{0.5}\,\mathrm{d}z$, where C is any path above the x-axis from 3 to -3.

Proof. $z^{0.5}$ is continuous at $\{z : \mathrm{Im}(z) \geq 0\}$. Hence, the integral is independent of the path from 3 to -3 above the x-axis.

We can parameterise the half circle from 3 to -3 as $z(t) = 3e^{it}$, we have

$$\int_C z^{0.5}\,\mathrm{d}z = \int_0^\pi (3e^{it})^{0.5} 3ie^{it}\,\mathrm{d}t = 3\sqrt{3}i\int_0^\pi e^{i1.5t}\,\mathrm{d}t = 3\sqrt{3}i\frac{1}{1.5i}\left[e^{i1.5t}\right]_0^\pi = 2\sqrt{3}(-i-1)$$

□

Example 10.25. Evaluate $\int_C z^m\overline{z}^n\,\mathrm{d}z$, where $m, n \in \mathbb{Z}$ and C is the circle $|z| = 1$ taken counterclockwise.

10.4 **Cauchy's Theorem in a Disk.**

We will assume for simplicity that an open disk $\Delta = \{z : |z - z_0| < \rho\}$ for some $z_0 \in \mathbb{C}, \rho > 0$.

Theorem 10.26. *If $f(z)$ is analytic in an open disc Δ, then $\int_\gamma f(z)\,\mathrm{d}z = 0$ for every closed curve γ in Δ*

Proof. Define $F(z) = \int_\sigma f(z)\,\mathrm{d}z$ where σ is a path from $(x_0, y_0) \to (x, y)$

We have

$$F(z) = \int_\sigma f(z)\,\mathrm{d}z = \int_\sigma f(z)\,\mathrm{d}x + \int_\sigma if(z)\,\mathrm{d}y$$

When σ is a grid like path from $(x_0, y_0) \to (x, y_0) \to (x, y)$, this gives $\int_\sigma f(z)\,\mathrm{d}x$ to be constant. Hence,

$$\frac{\partial F}{\partial y} = if(z)$$

When σ is a grid like path from $(x_0, y_0) \to (x_0, y) \to (x, y)$, this gives $\int_\sigma if(z)\,\mathrm{d}y$ to be constant. Hence,

$$\frac{\partial F}{\partial x} = f(z)$$

which satisfies the Cauchy Riemann equations. Showing that $\int f(z)\,\mathrm{d}z$ is independent from the paths. And $\int_\gamma f(z)\,\mathrm{d}z = 0$. □

Theorem 10.27. *Let $f(z)$ be analytic in the region Δ' obtained by omitting a finite number of points ξ_j from an open disc Δ. If $f(z)$ satisfies the condition $\lim_{z\to\xi_j}(z - \xi_j)f(z) = 0$ for j, then $\int_\gamma f(z)\,\mathrm{d}z = 0$ for any closed curve in Δ'.*

Proof. If ξ not on the original path, we are done. If ξ on the original path, just redirect the path.

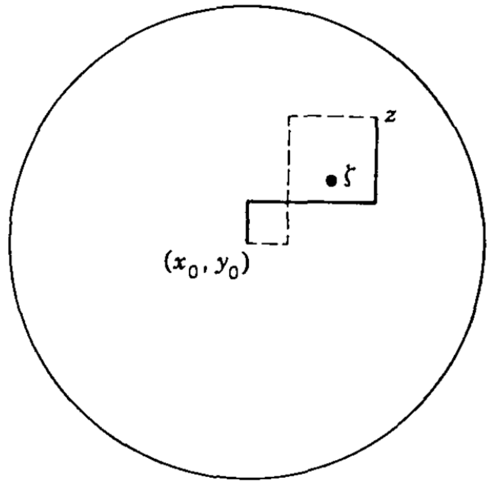

FIGURE 8. Redirect the path

□

10.5 **Exercises.**

Exercise 10.28. Compute

$$\int_\gamma x \, \mathrm{d}z$$

where γ is the directed line segment from 0 to $1+i$.

Exercise 10.29. Compute

$$\int_{|z|=r} x \, \mathrm{d}z$$

for the positive sense of the circle, in two ways: first, by use of a parameter, and second, by observing that

$$x = \frac{1}{2}(z+\overline{z}) = \frac{1}{2}\left(z + \frac{r^2}{z}\right)$$

on the circle.

Exercise 10.30. Compute

$$\int_{|z|=2} \frac{\mathrm{d}z}{z^2-1}$$

for the positive sense of the circle.

Exercise 10.31. Compute

$$\int_{|z|=1} |z-1| \, |\mathrm{d}z|$$

11 Cauchy's Integral Formula

11.1 **The Index.**

Lemma 11.1. *If the piecewise differentiable closed curve γ does not pass through the point a, then the value of the integral*

$$\int_\gamma \frac{\mathrm{d}z}{z-a}$$

is a multiple of $2\pi i$

Proof. If the equation of γ is $z = z(t)$, where $\alpha \leq t \leq \beta$. Consider the function

$$h(t) = \int_\alpha^t \frac{z'(t)}{z(t)-a}\,\mathrm{d}t$$

is defined on all $\alpha \leq t \leq \beta$. We have for all $\alpha \leq t \leq \beta$, whenever $z'(t)$ is continuous.

$$h'(t) = \frac{z'(t)}{z(t)-a}$$

Consider

$$g(t) = e^{-h(t)}(z(t)-a)$$

we have

$$\begin{aligned} g'(t) &= -h'(t)e^{-h(t)}(z(t)-a) + e^{-h(t)}z'(t) \\ &= -\frac{z'(t)}{z(t)-a}e^{-h(t)}(z(t)-a) + e^{-h(t)}z'(t) \\ &= 0 \end{aligned}$$

showing that g is a constant.

We have

$$h(t) = \int_\alpha^t \frac{z'(t)}{z(t)-a}\,\mathrm{d}t = (\log(|z(t)-a|))_\alpha^t = \log\left(\left|\frac{z(t)-a}{z(\alpha)-a}\right|\right) \implies e^{h(t)} = \frac{z(t)-a}{z(\alpha)-a}$$

Since $z(\alpha) = z(\beta)$

$$e^{h(\beta)} = 1 \implies h(\beta) = n2\pi i$$

Showing that

$$n2\pi i = h(\beta) = \int_\alpha^\beta \frac{z'(t)}{z(t)-a}\,\mathrm{d}t = \int_\gamma \frac{\mathrm{d}z}{z-a}$$

□

Definition 11.2. Index of a point a with respect to the curve γ is

$$n(\gamma, \alpha) = \frac{1}{2\pi i}\int_\gamma \frac{\mathrm{d}z}{z-a}$$

which is also called winding number of γ with respect to a.

Proposition 11.3.

1. *$-n(\gamma, a) = n(-\gamma, a)$*
2. *If γ lies in a circle, then $n(\gamma, a) = 0$ for all points outside of the same circle.*
3. *As a function of a, the index $n(\gamma, a)$ is constant in each of the regions determined by γ, and 0 in the unbounded region*

11.2 **The Cauchy's Integral Formula.**

Theorem 11.4. *Suppose that $f(z)$ is analytic in an open disk Δ, and let γ be a closed curve in Δ. For any point a not on γ*

$$n(\gamma, a) f(a) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-a} \, \mathrm{d}z$$

where $n(\gamma, a)$ is the index of a with respect to γ

Remark 11.5. The most common application is when $n(\gamma, a) = 1$ then

$$f(a) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-a} \, \mathrm{d}z$$

Also we will show later that

$$f^{(n)}(a) = \frac{n!}{2\pi i} \int_\gamma \frac{f(z)}{(z-a)^{n+1}} \, \mathrm{d}z$$

Example 11.6.

$$\int_{|z|=1} \sec(z) \, \mathrm{d}z = 0$$

as $\sec(z) = \frac{1}{\cos(z)}$ is analytic at $|z| \leq 1$, we can use the Cauchy's Theorem.

11.3 **Higher Derivatives.**

Theorem 11.7. *Let $f(z, \xi)$ be a function. Suppose that γ is a continuous curve, for fixed ξ, $f(z, \xi)$ is continuous on γ, $f(z, \xi)$ is uniform continuous on z, then:*

$$\lim_{z \to z_0} \int_\gamma f(z, \xi) \, \mathrm{d}\xi = \int_\gamma \lim_{z \to z_0} f(z, \xi) \, \mathrm{d}\xi$$

Proof.

□

Lemma 11.8. *Suppose $\phi(\xi)$ is continuous on the arc γ, then the function*

$$F_n(z) = \int_\gamma \frac{\phi(\xi)}{(\xi - z)^n} \, \mathrm{d}\xi$$

analytic in each of the regions determined by γ, and its derivative is $F_n{}'(z) = nF_{n+1}(z)$

Corollary 11.9. *Suppose $f(a)$ and γ is defined similarly as `Theorem 11.4`, where*

$$f(a) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-a} \, \mathrm{d}z$$

then

$$f^{(n)}(a) = \frac{n!}{2\pi i} \int_\gamma \frac{f(z)}{(z-a)^{n+1}} \, \mathrm{d}z$$

Corollary 11.10. *An analytic function f has derivative of any order that can be represented as above.*

Theorem 11.11. (Morera's Theorem) *If $f(z)$ is defined and continuous in a region Ω, and if $\int_\gamma f(z) \, \mathrm{d}z = 0$ for all closed curves γ in Ω, then $f(z)$ is analytic in Ω.*

Proof. For any path γ_1 from p to q, take any path γ_2 from q to p, we have

$$\int_{\gamma_1} f(z)\,\mathrm{d}z + \int_{\gamma_2} f(z)\,\mathrm{d}z = 0 \implies \int_{\gamma_1} f(z)\,\mathrm{d}z = -\int_{\gamma_2} f(z)\,\mathrm{d}z$$

Since γ_1 is arbitrary, this shows that the integral is independent on the path taken.

Hence, by `Theorem 10.18`, f is a derivative of some function F in Ω. Since F is analytic, then f also is. □

Theorem 11.12. (Liouville's Theorem) *A function which is analytic and bounded in the whole plane must reduce to a constant.*

Proof. Suppose γ to be a closed circle around some $a \in \mathbb{C}$ with radius r and assume that $|f(\xi)| \leq M$ for all $\xi \in \gamma$. Then

$$f^{(n)}(a) = \frac{n!}{2\pi i} \int_\gamma \frac{f(\xi)}{(\xi - a)^{n+1}}\,\mathrm{d}\xi$$

$$\implies \left|f^{(n)}(a)\right| \leq \frac{n!}{2\pi} \int_\gamma \frac{|f(\xi)|}{|(\xi - a|)^{n+1}}\,|\mathrm{d}\xi| \leq \frac{n!}{2\pi} \frac{M}{r^{n+1}} \int_\gamma |d\xi| = \frac{Mn!}{r^n}$$

For the case where $n = 1$,

$$|f'(a)| \leq \frac{M}{r}$$

when $r \to \infty$, we get $f'(a) = 0$. Since a is arbitrary, we have $f'(z) = 0$ for all $z \in \mathbb{C}$. Hence, the function is constant. □

Theorem 11.13. (Fundamental Theorem of Algebra) *Every non-constant polynomial $p(z)$, with complex coefficients, has at least one complex root. (Indeed, there are n roots for a polynomial of degree n).*

Proof. Suppose that $p(z) \neq 0$ for any z. The function $\frac{1}{p(z)}$ would be analytic in the whole plane. Since $p(z) \to \infty$ as $z \to \infty$, we have $\frac{1}{p(z)} \to 0$.

Consider $f(z) = \left|\frac{1}{p(z)}\right| = \frac{1}{|p(z)|}$ where $f(z) \to 0$ as $z \to \infty$. We have f is continuous and thus bounded (clearly bounded for $|z| > M$ for some large enough M. By EVT, $|p(z)|$ is also closed for $\{|z| \leq M\}$. Hence, does not approaches to 0).

By Liouville's Theorem $\frac{1}{p(z)}$ would be constant, similarly for $p(z)$. Since this is not so, the equation $p(z) = 0$ must have a root. □

11.4 **Exercises.**

Exercise 11.14. Compute

$$\int_{|z|=1} \frac{e^z}{z}\,\mathrm{d}z$$

Exercise 11.15. Compute

$$\int_{|z|=2} \frac{\mathrm{d}z}{z^2+1}$$

by decomposition of the integrand in partial fractions.

Proof. We have

$$\begin{aligned}\int_{|z|=2} \frac{\mathrm{d}z}{z^2+1} &= \int_{|z|=2} \frac{1}{(z-i)(z+i)}\,\mathrm{d}z \\ &= \frac{1}{2i}\int_{|z|=2} \frac{1}{z-i} - \frac{1}{z+i}\,\mathrm{d}z \\ &= \frac{1}{2i}\left(\int_{|z|=2}\frac{\mathrm{d}z}{z-i} - \int_{|z|=2}\frac{\mathrm{d}z}{z+i}\right) \\ &= \frac{1}{2i}2\pi i(n(\gamma,i)-n(\gamma,-i)) \\ &= 0\end{aligned}$$

□

Exercise 11.16. Compute

$$\int_{|z|=\rho} \frac{|\mathrm{d}z|}{|z-a|^2}$$

under the condition $|a| \neq \rho$. Hint: make use of the equations $z\overline{z} = \rho^2$ and

$$|\mathrm{d}z| = -i\rho\frac{\mathrm{d}z}{z}$$

Exercise 11.17. Compute

$$\int_{|z|=1} e^z z^{-n}\,\mathrm{d}z, \quad \int_{|z|=2} z^n(1-z)^m\,\mathrm{d}z, \quad \int_{|z|=\rho} |z-a|^{-4}\,|\mathrm{d}z|$$

where $|a| \neq \rho$.

Exercise 11.18. Prove that a function which is analytic in the whole plane and satisfies the inequality $|f(z)| < |z|^n$ for some n and all sufficiently large $|z|$ reduces to a polynomial.

Exercise 11.19. If $f(z)$ is analytic and $|f(z)| \leq M$ for $|z| \leq R$, find an upper bound for $|f^n(z)|$ in the disk $|z| \leq \rho < R$.

Exercise 11.20. If $f(z)$ is analytic for $|z| < 1$ and $|f(z)| \leq \frac{1}{1-|z|}$, find the best estimate of $|f^n(0)|$ that Cauchy's inequality will yield.

Exercise 11.21. Show that the successive derivatives of an analytic function at a point can never satisfy $|f^n(z)| > n!n^n$. Formulate a sharper theorem of the same kind.

12 Local Properties of Analytic Functions

Theorem 12.1. (Taylor's Theorem) *Suppose a function f is analytic throughout a disk $|z-z_0| < R_0$, centered at z_0 and with radius R_0. Then $f(z)$ has a power series representation*

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n \quad , \quad |z-z_0| < R_0$$

where $a_n = \frac{f^n(z_0)}{n!}$.

The series is convergent to $f(z)$ if z is in the stated open disk.

Proof. We prove it for the case $z_0 = 0$, for the general case of z_0, consider $g(z) = f(z+z_0)$, so $g(0) = f(z_0)$.

Assume f is analytic for $|z| \leq R_0$. Consider r so that $|z| = r$. Let C_0 denotes the circle with radius r_0, positively oriented, r_0 is selected so that $r < r_0 < R_0$. By Cauchy integral formula we have

$$f(z) = \frac{1}{2\pi i}\int_{C_0} \frac{f(\xi)}{\xi - z}\,\mathrm{d}\xi$$

Since for all $z \neq 1$

$$\frac{1}{1-z} = \sum_{n=0}^{N-1} z^n + \frac{z^N}{1-z}$$

by writing $\frac{1}{\xi - z} = \frac{1}{\xi}\frac{1}{1-\frac{z}{\xi}}$ we have

$$\frac{1}{\xi - z} = \sum_{n=0}^{N-1}\frac{1}{\xi^{n+1}}z^n + z^N\frac{1}{(\xi - z)\xi^N} \implies \frac{1}{2\pi i}\int_{C_0}\frac{f(\xi)}{\xi - z}\,\mathrm{d}\xi = \frac{1}{2\pi i}\sum_{n=0}^{N-1}\int_{C_0}\frac{f(\xi)}{\xi^{n+1}}z^n\,\mathrm{d}\xi + \frac{1}{2\pi i}\int_{C_0} z^N\frac{f(\xi)}{(\xi - z)\xi^N}\,\mathrm{d}\xi$$

where by `Corollary 11.9`

$$\frac{1}{2\pi i}\int_{C_0}\frac{f(\xi)}{\xi^{n+1}}z^n\,\mathrm{d}\xi = \frac{f^{(n)}(0)}{n!}$$

Let

$$p_N(z) = \frac{1}{2\pi i}\int_{C_0} z^N\frac{f(\xi)}{(\xi - z)\xi^N}\,\mathrm{d}\xi$$

This gives

$$f(z) = \sum_{n=0}^{N-1}\frac{f^{(n)}(0)}{n!}z^n + p_N(z)$$

We will now show that $\lim_{N\to\infty} p_{N(z)} \to 0$

Since $r_0 > r$, where ξ is a point on the circle C_0 with radius r_0 and z is a point inside C_0. We have

$$|\xi - z| \geq ||\xi| - |z|| = |r_0 - r| = r_0 - r$$

Moreover, $f(C_0)$ contains a maximum M as C_0 is closed and f is continuous.

We have

$$\begin{aligned}
&\lim_{N\to\infty} p_N(z) = \lim_{N\to\infty}\frac{1}{2\pi i}\int_{C_0} z^N\frac{f(\xi)}{(\xi - z)\xi^N}\,\mathrm{d}\xi \\
\implies &\lim_{N\to\infty}|p_N(z)| \leq \lim_{N\to\infty}\frac{1}{2\pi}\int_{C_0}|z|^N\,\frac{|f(\xi)|}{|(\xi - z)|\;|\xi|^N}\;|\mathrm{d}\xi| \\
\implies &\lim_{N\to\infty}|p_N(z)| \leq \lim_{N\to\infty}\frac{1}{2\pi}\int_{C_0} r^N\frac{M}{(r_0 - r)r_0^N}\;|\mathrm{d}\xi| \\
\implies &\lim_{N\to\infty}|p_N(z)| \leq \lim_{N\to\infty}\frac{1}{2\pi}r^N\frac{M}{(r_0 - r)r_0^N}(2\pi r_0) \\
\implies &\lim_{N\to\infty}|p_{N(z)}| = \lim_{N\to\infty}\frac{Mr_0}{r_0 - r}\left(\frac{r}{r_0}\right)^N \to 0 \\
\implies &\lim_{N\to\infty} p_{N(z)} \to 0
\end{aligned}$$

□

Example 12.2. Write the Taylor series for $\frac{1}{1-z}$ centered at i. What is the radius of convergence?

Proof.

$$\begin{aligned}\frac{1}{1-z} &= \frac{1}{1-i+i-z} \\ &= \frac{1}{(1-i)-(z-i)} \\ &= \frac{1}{1-i}\frac{1}{\frac{(1-i)-(z-i)}{1-i}} \\ &= \frac{1}{1-i}\frac{1}{1-\frac{z-i}{1-i}} \\ &= \frac{1}{1-i}\sum_{n=0}^{\infty}\frac{(z-i)^n}{(1-i)^n} \\ &= \sum_{n=0}^{\infty}\frac{(z-i)^n}{(1-i)^{n+1}}\end{aligned}$$

This converge if and only if $\left|\frac{z-i}{1-i}\right| < 1 \iff |z-i| < |1-i| = \sqrt{2}$ □

Example 12.3. Write the Maclaurin series for $g(z) = z^3e^{2z}$

Proof. We have

$$z^3e^{2z} = z^3\sum_{n=0}^{\infty}\frac{(2z)^n}{n!} = \sum_{n=0}^{\infty}\frac{2^n z^{n+3}}{n!}$$

□

Theorem 12.4. (Removable singularities) *If f is analytic in $\Omega \setminus \{a\}$, there exists an analytic function in Ω which coincides $f(z)$ in $\Omega \setminus \{a\}$* $\iff \lim_{z\to a} f(z)(z-a) = 0$

Proof. $\Longrightarrow$: If f has an extension $\tilde{f}$ on Ω where $f = \tilde{f}$ on $\Omega \setminus \{a\}$ and $\tilde{f}$ is analytic in Ω. Hence, $\tilde{f}$ is continuous at a. Therefore

$$\lim_{z\to a} f(z)(z-a) = \lim_{z\to a}\tilde{f}(z)(z-a) = \tilde{f}(z)(a-a) = 0$$

$\Longleftarrow$: Assume $\lim_{z\to a} f(z)(z-a) = 0$, define

$$h(z) = \begin{cases}(z-a)^2 f(z) & \text{if } z \neq a \\ 0 & \text{if } z = a\end{cases}$$

Clearly, h is analytic at $\Omega \setminus \{a\}$, we also have

$$h'(a) = \lim_{z\to a}\frac{h(z)-h(a)}{z-a} = \lim_{z\to a}\frac{(z-a)^2 f(z)}{z-a} = \lim_{z\to a}(z-a)f(z) = 0$$

hence, h is analytic at all Ω.

By `Theorem 12.4`,

$$h(z) = \sum_{n=0}^{\infty} c_n(z-a)^n = \sum_{n=2}^{\infty} c_n(z-a)^n$$

as $h(a) = h'(a) = 0$

For $z \neq a$, we have

$$f(z) = \frac{h(z)}{(z-a)^2} = \sum_{n=0}^{\infty} c_{n+2}(z-a)^n = c_2 + c_3(z-a) + c_4(z-a)^2 + \cdots$$

Denote

$$\tilde{f}(z) = \begin{cases} f(z) & \text{if } z \neq a \\ c_2 & \text{if } z = a\end{cases}$$

this is an analytic extension of f, as

$$\tilde{f}'(a) = \lim_{z \to a} \frac{\tilde{f}(z) - \tilde{f}(a)}{z-a} = \lim_{z \to a} \frac{c_2 + c_3(z-a) + c_4(z-a)^2 + \cdots - c_2}{z-a} = \lim_{z \to a} c_3 + c_4(z-a) + c_5(z-a)^2 + \cdots = c_3$$

□

Exercise 12.5. $\lim_{z\to a} f(z)(z-a) = 0 \iff f$ is bounded in a neighbourhood of a.

12.1 Zeros and Poles.

Proposition 12.6. *If f is analytic and $f^{(k)}(a) = 0$ for all $k = 0, 1, \ldots$, then $f \equiv 0$*

Proof.

□

Corollary 12.7. *If f is analytic and not identically zero, if $f(a) = 0$, there exists a smallest h such that $f^{(h)}(a) \neq 0$.*

Moreover, we can write

$$f(z) = (z-a)^h f_h(z)$$

for some neighbourhood of a, where $f_h(z)$ is analytic and $f_h(a) \neq 0$

Proof. The first part is trivial.

Since f is analytic, we have

$$\begin{aligned} f(z) &= f(a) + \frac{f'(a)}{1!}(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \ldots \\ &= \frac{f^{(h)}(a)}{h!}(z-a)^h + \frac{f^{(h+1)}(a)}{(h+1)!}(z-a)^{h+1} + \ldots \\ &= (z-a)^h \left(\frac{f^{(h)}(a)}{h!} + \frac{f^{(h+1)}(a)}{(h+1)!}(z-a) + \ldots \right) \\ &= (z-a)^h f_h(z) \end{aligned}$$

Obviously $f_h(z)$ is analytic and $f_h(a) \neq 0$. □

Definition 12.8. In the above case, we call such h the order of f at a.

Proposition 12.9. *If f is analytic that is not identically 0, then zeros of f are isolated points.*

Proof.

□

Definition 12.10. If $f(z)$ is analytic in a neighborhood of a except for a, then a is called an (isolated) singularity of f.

An isolated singularity can be:

- a removable singularity if for some $h \in \mathbb{N}$

$$\lim_{z \to a} (z-a) f(z) = 0$$

- an essential singularity if for all $h \in \mathbb{N}$

$$\lim_{z \to a} (z-a)^h f(z) \neq 0$$

- a pole if

$$\lim_{z \to a} f(z) = \infty$$

Example 12.11. The function $g(z) = \frac{1}{f(z)}$ is defined and analytic in a neighborhood of the point a. a is a removable singularity in this case.

If we define $g(a) = 0$, then g is analytic at a

Example 12.12.

- $e^{\frac{1}{z}}$ has isolated singularity at 0
- $\frac{1}{\sin(\frac{1}{z})}$ has non-isolated singularity at 0

Example 12.13.

- $f(z) = \frac{z^2+4}{(z^2-9)(z^2+16)} = \frac{z^2+4}{(z-3)(z+3)(z-4i)(z+4i)}$ has a pole at $z = 3$
- $g(z) = \frac{\sin(z)}{z}$ has a removable singularity at $z = 0$.
- $h(z) = e^{\frac{1}{z}}$ has an essential singularity at $z = 0$. It is not a pole as the limit is 0 when coming from the negative real axis. It is not a removable singularity as the limit grows exponentially when coming from the positive real axis.

Since $g(z) \not\equiv 0$, we can write $g(z) = (z-a)^h g_h(z)$ for some $h \in \mathbb{N}, g_h(z)$ analytic such that $g_h(a) \neq 0$. In this case, $f(z) = \frac{(z-a)^{-h}}{g_h(z)} = (z-a)^{-h} f_h(z)$, we can define the pole of $f(z)$ at a to be h. This gives us the following definition

Definition 12.14. If $\lim_{z \to a} f(z) = \infty$, we say the order of the pole of $f(z)$ at a is h if the order of zeros of $g(z) = \frac{1}{f(z)}$ at a is h

Definition 12.15. A function is called meromorphic if it is analytic except for poles.

Remark 12.16. Poles of a meromorphic functions are isolated by definition

Theorem 12.17. (Cassorati-Weierstrass) *An analytic function comes arbitrarily close to any complex value in every neighborhood of an essential singularity.*

I.e. If f is analytic such that f has an essential singularity at a. Then for arbitrary open neighbourhood V of a satisfy that $f(V/\{a\})$ is dense in $\mathbb{C}$.

12.2 **The Local Mapping Theorem.**

We will look at a general formula which enables us to determine the number of zeros of an analytic function.

Theorem 12.18. *Let z_i be the zeros of a function $f(z)$ which is analytic in a disk Δ, and is not identically the zero function. Each zero being counted as many times as its order indicates. Then, for every closed curve γ in Δ which does not pass through a zero*

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} \,\mathrm{d}z = \sum_i n(\gamma, z_i)$$

Proof. For simplicity we suppose that $f(z)$ has finite number of zeros $z_1, ..., z_n$ which might be repeated. We can write

$$f(z) = (z - z_1)(z - z_2) \cdots (z - z_n) g(z)$$

where $g(z)$ is analytic and $g(z) \neq 0$ in Δ. Using log differentiation for all $z \neq z_i$

$$\frac{f'(z)}{f(z)} = \frac{1}{z - z_1} + \frac{1}{z - z_2} + ... + \frac{1}{z - z_n} + \frac{g'(z)}{g(z)}$$

Since $g(z) \neq 0, g(z)$ is analytic which implies $g'(z)$ is analytic, $\frac{g'(z)}{g(z)}$ is also analytic. By Cauchy Theorem $\int_\gamma \frac{g'(z)}{g(z)} = 0$

Hence,

$$n(\gamma, z_1) + ... + n(\gamma, z_n) = \frac{1}{2\pi i}\int_\gamma \frac{1}{z-z_1} + ... + \frac{1}{z-z_n}\,\mathrm{d}z = \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,\mathrm{d}z$$

If $f(z)$ has infinitely many zeros in Δ. γ is contained in a concentric disk Δ' smaller than Δ. If there were infinitely many zeros inside Δ', there would be an accumulation point in the closure of Δ'. and this is impossible. We can now use the same previous equation on Δ', where z_i satisfies $f(z_i) = 0$ outside of Δ' satisfy $n(\gamma, z_i) = 0$ and hence do not contribute to the previous equation. □

Remark 12.19. The function $w = f(z)$ maps γ to some path Γ in the w-plane. So by chain rule, we get

$$\int_\Gamma \frac{\mathrm{d}w}{w} = \int_\gamma \frac{f'(z)}{f(z)}\,\mathrm{d}z$$

So

$$n(\Gamma, 0) = \sum_i n(\gamma, z_i)$$

Corollary 12.20. *The total number of zeros enclosed by γ is*

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,\mathrm{d}z$$

Proof. Note that $n(\gamma, z_i)$ must be either 0 or 1, then we can count the number of zeros of $f(z)$ enclosed by the curve γ. This yields the formula from `Theorem 12.18` □

Remark 12.21. Let a be an arbitrary complex value, and apply `Theorem 12.18` to $f(z) - a$. The zeros of $f(z) - a$ are the roots of the equation $f(z) = a$. Denote them by $z_i(a)$. We obtain the formula

$$\sum_i n(\gamma, z_i(a)) = \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)-a}\,\mathrm{d}z$$

If $f(z)$ maps γ onto Γ where $f(z) \neq a$ for all $z \in \gamma$ then we obtain

$$n(\Gamma, a) = \sum_i n(\gamma, z_i(a))$$

Remark 12.22. If $n(\Gamma, a) = n(\Gamma, b)$ then we have

$$\sum_i n(\gamma, z_i(a)) = \sum_i n(\gamma, z_i(b))$$

If γ is a circle, it follows that $f(z)$ takes the values a and b equally many times inside of γ ·

Theorem 12.23. *Suppose that $f(z)$ is analytic at z_0, $f(z_0) = w_0$ and that $f(z) - w_0$ has a zero of order n at z_0. If $\varepsilon > 0$ is sufficiently small, there exists a corresponding $\delta > 0$ such that for all a with $|a - w_0| < \delta$ the equation $f(z) - a$ has exactly n roots for the disk $|z - z_0| < \varepsilon$.*

Proof. We can choose $\varepsilon > 0$ so that $f(z)$ is defined and analytic for $|z - z_0| \leq \varepsilon$ so that z_0 is the only zero of $f(z) - w_0$ in the disk. Let γ be the circle $|z - z_0| = \varepsilon$ and Γ is its image under the mapping $w = f(z)$. Since $w_0 = f(z_0) \notin \Gamma$ there exist a neighborhood $|w - w_0| < \delta$ which does not intersect Γ.

It follows for any a such that $|a - w_0| < \delta$ in this neighborhood will take the same number of times inside Γ. We therefore get

$$\sum_i n(\gamma, z_i(a)) = n(\Gamma, a) = n(\Gamma, w_0) = \sum_j n\big(\gamma, z_j(z_0)\big)$$

hence every value a is taken n times. □

Remark 12.24. Multiple roots are counted according to their multiplicity, but if ε is sufficiently small we can assume that all roots of the equation $f(z) = a$ are simple for $a \neq w_0$.

Hence, it is sufficient to choose ε so that $f'(z)$ does not vanish for $0 < |z - z_0| < \varepsilon$.

Corollary 12.25. (Local Mapping Theorem) *A nonconstant analytic functions maps open sets onto open sets.*

Another way to say this is that the image of every sufficiently small disk $|z - z_0| < \varepsilon$ contains a neighborhood $|w - w_0| < \delta$.

Remark 12.26. In $n = 1$ there is a one-to-one correspondence between the disk $|w - w_0| < \delta$ and $|z - z_0| < \varepsilon$.

Since open sets in the z-plane correspond to open sets in the w-plane the inverse function of $f(z)$ is continuous, and the mapping is topological. The mapping can be restricted to a neighborhood of z_0 contained in Δ, we have

Corollary 12.27. *If $f(z)$ is analytic at z_0 with $f'(z_0) \neq 0$, it maps a neighborhood of z_0 conformally and topologically onto a region.*

Proof. By assignment 1, the continuity of the inverse function implies that the inverse function is analytic, and is conformal also by $f'(z_0) \neq 0$. $\square$

Example 12.28. Let $f(z) = z^4 - 2z^2, f(z) = 0$ has four zeros at 0 with multiplicity of 2, $\sqrt{2}, -\sqrt{2}$. So there is some neighbourhood U around 0 that f have 4 solutions.

If we let $U = \{z : |z| < 2\}, a = -1$, then $z^4 - 2z^2 = -1 \implies \left(z^2 - 1\right)^2 = 0$ so $z = \pm 1 \in U$, so there is a neighbourhood around -1 that maps conformally and topologically onto a region.

But this does not apply to a neighborhood around 0 as $f'(0) = 0$.

12.3 **The Maximum Principle.**

Theorem 12.29. (The Maximum Principle) *If $f(z)$ is analytic and non-constant in a region Ω, then $|f(z)|$ has no maximum in Ω.*

Proof. Suppose for a contradiction that $a \in \Omega$ such that $|f(a)|$ is the maximum of $|f(z)|$ (note that $|f(a)| \neq 0$ otherwise f is constant). Take U be a neighbourhood of a in Ω. Since f is analytic and non-constant, $f(U)$ is open.

Now, let ε small enough such that $B(f(a), \varepsilon) \subseteq f(U)$.

Let $v = \frac{f(a)}{|f(a)|}$ and denote $w = f(a) + tv$ for some $0 < t < \varepsilon$. Note that $w \in f(U) \implies w = f(b)$ for some $b \in U$, then

$$\begin{aligned}
|f(b)|^2 &= (f(a) + tv)\overline{f(a) + tv} \\
&= |f(a)|^2 + tv\overline{f(a)} + t\overline{v}f(a) + t^2\ |v|^2 \\
&= |f(a)|^2 + 2t\ |f(a)| + t^2\ |v|^2 \\
&> |f(a)|^2
\end{aligned}$$

showing that $|f(b)| > |f(a)|$. $\square$

Theorem 12.30. *If $f(z)$ is defined and continuous on a closed bounded set E and analytic on the interior of E, then the maximum of $|f(z)|$ on E is on the boundary of E.*

Proof. By the Maximum Principle, the maximum of $f(z)$ is not on E° hence the maximum is on the boundary of E. $\square$

Theorem 12.31. (Schwarz Lemma) *If $f(z)$ is analytic for $|z| < 1$ and satisfies $f(0) = 0$ and for all $z, |f(z)| \leq 1$, then $|f(z)| \leq |z|$ and $|f'(0)| \leq 1$.*

If $|f(z)| = |z|$ for some $z \neq 0, |z| < 1$, or if $|f'(0)| = 1$, then $f(z) = cz$ with $c \in \mathbb{C}, |c| = 1$.

Proof. Let

$$f_1(z) = \begin{cases} \frac{f(z)}{z} & \text{if } z \neq 0 \\ f'(0) & \text{if } z = 0 \end{cases}$$

If $f_1(z)$ is constant then for all $|z| < 1, f_1(z) = f'(0) \implies f(z) = f'(0)z$. Given that $|f(z)| = |f'(0)z| \leq 1 \implies |f'(0)| \leq 1$. Hence, $|f(z)| = |f'(0)z| \leq |z|$.

If $f_1(z)$ is not constant. Consider some $0 < r < 1$, for all $|z| < r$, consider $\overline{B}(0, r)$, by Maximum Principle, the maximum is on the boundary. Hence, we have

$$|f_1(z)| = \left|\frac{f(z)}{z}\right| \leq \frac{|f(z)|}{r} \leq \frac{1}{r}$$

Let $r \to 1$ we get $|f_1(z)| \leq 1 \implies |f(z)| \leq |z|$.

For the second part, if $|f(z)| = |z|$ for some $z \neq 0, |z| < 1$ or $|f'(0)| = 1$ then $|f_1(z)| = 1$ for some $|z| < 1$. Then f_1 attains its maximum in an open set. So $f_1(z) = c$ where c is a constant and $|c| = 1$. So $f(z) = cz$. □

12.4 **Exercises.**

Exercise 12.32. If $f(z)$ and $g(z)$ have the algebraic orders h and k at $z = a$, show that fg has the order $h + k$, f/g the order $h - k$, and $f + g$ an order which does not exceed $\max(h, k)$.

Exercise 12.33. Show that the functions $e^z, \sin(z), \text{and} \cos(z)$ have essential singularities at ∞.

Exercise 12.34. Show by use of , or directly, that $|f(z)| \leq 1$ for $|z| \leq 1$ implies

$$\frac{|f'(z)|}{1 - |f(z)|^2} \leq \frac{1}{1 - |z|^2}$$

Exercise 12.35. If $f(z)$ is analytic and $\operatorname{Im} f(z) \geq 0$ for $\operatorname{Im} z > 0$, show that

$$\frac{|f(z) - f(z_0)|}{\left|f(z) - \overline{f(z_0)}\right|} \leq \frac{|z - z_0|}{|z - \overline{z_0}|}$$

and

$$\frac{|f'(z)|}{\operatorname{Im} f(z)} \leq \frac{1}{y} \quad z = x + iy$$

13 The General Form of Cauchy's Theorem

13.1 **Simply Connected Regions.**

Definition 13.1. A region (an open, connected set in $\mathbb{C}$) is simply connected if its complement with respect to the extended plane is connected.

Remark 13.2. This is used as a formal definition of a region that does not have holes.

We can apply this definition on the regions on the Riemann Sphere, if the complement of the region is connected on the Riemann Sphere, then the region is simply connected.

We will see that Cauchy's theorem is universally valid on simply connected regions.

Example 13.3. A disk, a half plane, and a parallel strip are simply connected, where the last one is because taking the complement with respect to the extended plane.

Theorem 13.4. *A region Ω is simply connected if and only if $n(\gamma, a) = 0$ for all cycles γ in Ω and all points $a \notin \Omega$.*

Proof. See Ahlfors. □

Definition 13.5. A cycle γ in an open set Ω is said to be homologous to zero with respect to Ω if $n(\gamma, a) = 0$ for all $a \notin \Omega$.

13.2 The General Statement of Cauchy's Theorem.

Theorem 13.6. (The General Statement of Cauchy's Theorem) *If $f(z)$ is analytic in Ω, then*

$$\int_\gamma f(z)\,\mathrm{d}z = 0$$

for every cycle γ which is homologous to zero in Ω.

Proof. See Ahlfors. □

Corollary 13.7. *If $f(z)$ is analytic in a simply connected region Ω, then for all cycles γ in Ω*

$$\int_\gamma f(z)\,\mathrm{d}z = 0$$

13.3 Multiply Connected Regions.

Definition 13.8. A region is multiple connected if it is not simply connected.

Ω is said to have the finite connectivity n if the complement of Ω has exactly n components and infinite connectivity if the complement has infinitely many components.

Theorem 13.9. *Suppose that*

1. *c is a simple closed curve with positive direction (counter-clockwise)*
2. *For $k = 1, ..., n, c_k$ are simple closed curves inside c with opposite directions. And c, c_k's are disjoint.*

If f is analytic on all curves, and the region inside c and outside c_k's, then

$$\int_c f(z)\,\mathrm{d}z + \sum_{k=1}^{n} \int_{c_k} f(z)\,\mathrm{d}z = 0$$

Corollary 13.10. *Let c_1, c_2 be positively oriented simple closed curves, where c_1 is inside c_2. If f is a function that is analytic on both curves as well as the region between the curves then*

$$\int_{c_1} f(z)\,\mathrm{d}z = \int_{c_2} f(z)\,\mathrm{d}z$$

14 Laurent Series

14.1 The Basics.

Theorem 14.1. *Suppose f is analytic in a region $R_1 < |z - z_0| < R_2$ centered at z_0, and c be any positive oriented simple closed curve around z_0 in the region. Then at each point z in the region, $f(z)$ has the representation*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$$

where

$$a_n = \frac{1}{2\pi i} \int_c \frac{f(z)}{(z - z_0)^{n+1}}\,\mathrm{d}z$$

$$b_n = \frac{1}{2\pi i} \int_c \frac{f(z)}{(z - z_0)^{-n+1}}\,\mathrm{d}z$$

Example 14.2. The function $f(z) = \frac{1}{(z-i)^2}$ has singularities at $z = i$, and is already in the form of Laurent Series for $|z - i| > 0$.

Example 14.3. The function $f(z) = \frac{1}{z(z^2+1)}$ has singularities at $z = 0, \pm i$. The Laurent Series at z such that $0 < |z| < 1$ is

$$\frac{1}{1+z^2} = \sum_{n=0}^{\infty} (-1)^n z^{2n} \implies \frac{1}{z(1+z^2)} = \sum_{n=0}^{\infty} (-1)^n z^{2n-1}$$

Example 14.4. The function $f(z) = z^2 \sin\left(\frac{1}{z}\right)$ has singularities at $z = 0$. The Laurent Series at z such that $0 < |z|$ is

$$z^2 \sin\left(\frac{1}{z}\right) = z^2 \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{1}{z}\right)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)! z^{2n-1}}$$

Theorem 14.5. *If a power series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ converges at $z = z_1$ where $z_1 \neq z_0$ then it is absolutely and uniformly convergent at each z in the open disk $|z - z_0| < |z_1 - z_0|$.*

14.2 Uniqueness of Laurent Series.

Theorem 14.6. *If a power series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ converges to $f(z)$ at all points in some disk centered at z_0, then it is the Taylor Series expansion of f within the disk.*

The same applies to Laurent series.

15 The Calculus of Residues

15.1 The Residue Theorem.

Remark 15.1. Recall from `Theorem 14.1` that if z_0 is an isolated singularity for $f(z)$, there is $R > 0$ such that f is analytic at all $0 < |z - z_0| < R$. $f(z)$ has a Laurent Series of

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$$

where $b_n = \frac{1}{2\pi i} \int_c \frac{f(z)}{(z-z_0)^{-n+1}} \, \mathrm{d}z$, for c to be any positive oriented simple closed curve in the region. When $n = 1$, we have

$$b_1 = \frac{1}{2\pi i} \int_c f(z) \, \mathrm{d}z$$

Definition 15.2. If $f(z)$ has a Laurent Series centered at z_0 of

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$$

then b_1 the coefficient of $\frac{1}{z-z_0}$ is called the residue of f at z_0, denoted as

$$b_1 = \mathrm{Res}_{z=z_0} f(z)$$

Example 15.3. Evaluate the integral $\int_c \frac{e^z-1}{z^4} \, \mathrm{d}z$ where c is the positively oriented circle $|z| = 1$.

Proof. The function $f(z) = \frac{e^z-1}{z^4}$ at $|z| > 0$ has a Laurent Series of

$$\frac{\sum_{n=0}^{\infty} \frac{z^n}{n!} - 1}{z^4} = \sum_{n=1}^{\infty} \frac{z^{n-4}}{n!} = \frac{1}{z^3} + \frac{1}{2}\frac{1}{z^2} + \frac{1}{6}\frac{1}{z} + \frac{1}{24} + ...$$

Since c is a positive oriented simple closed curve in the region, the residue of f at $z = 0$ is $\frac{1}{6} = \frac{1}{2\pi i} \int_{|z|=1} \frac{e^z-1}{z^4} \, \mathrm{d}z \implies \int_c \frac{e^z-1}{z^4} \, \mathrm{d}z = \frac{\pi i}{3}$ □

Example 15.4. Evaluate the integral $\int_c \frac{1}{z(z-2)^5}\,\mathrm{d}z$ where c is the positively oriented circle $|z-2|=1$.

Proof. The function $f(z)=\frac{1}{z(z-2)^5}$ at $0<|z-2|<2$ has a Laurent Series of

$$\begin{aligned}\frac{1}{(z-2)^5}\frac{1}{z-2+2} &= \frac{1}{(z-2)^5}\frac{1}{2}\frac{1}{\frac{z-2}{2}+1}\\ &= \frac{1}{(z-2)^5}\frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z-2}{2}\right)^n\\ &= \sum_{n=0}^{\infty}(-1)^n\frac{(z-2)^{n-5}}{2^{n+1}}\end{aligned}$$

Since c is a positive oriented simple closed curve in the region, the residue of f at $z=2$ is $\frac{1}{32}=\frac{1}{2\pi i}\int_{|z-2|=1}\frac{1}{z(z-2)^5}\,\mathrm{d}z \Longrightarrow \int_{|z-2|=1}\frac{1}{z(z-2)^5}\,\mathrm{d}z=\frac{\pi i}{16}$ □

Theorem 15.5. (Cauchy Residue Theorem) *Let c be a simple closed curve, positively oriented. If a function f is analytic inside and on c except for finite isolated points $z_k, k=1,...,n$ in c then*

$$\int_c f(z)\,\mathrm{d}z = 2\pi i\sum_{k=1}^{n}\mathrm{Res}_{z=z_k}f(z)$$

Proof. Take c_k be the positive-oriented circles with center z_k where the radius is small enough such that each c_k is in the interior of c and disjoint with other c_k's.

Note that f is analytic in the region inside c and outside c_k's. By `Theorem 13.9`

$$\int_c f(z)\,\mathrm{d}z-\sum_{k=1}^{n}\int_{c_k}f(z)\,\mathrm{d}z=0 \Longrightarrow \int_c f(z)\,\mathrm{d}z=\sum_{k=1}^{n}\int_{c_k}f(z)\,\mathrm{d}z$$

By definition of a residue this further equivalent to $2\pi i\sum_{k=1}^{n}\mathrm{Res}_{z=z_k}f(z)$ □

Example 15.6. Evaluate $\int_c \frac{4z-5}{z(z-1)}\,\mathrm{d}z$ where c is the circle $|z|=2$. Note that f is analytic inside c except at $z=0,1$. Hence,

$$\int_c f(z)\,\mathrm{d}z = 2\pi i(\mathrm{Res}_{z=0}f(z)+\mathrm{Res}_{z=1}f(z))$$

We have

$$f(z)=\frac{4z-5}{z}\frac{1}{z-1}=\left(4-\frac{5}{z}\right)\left(-\sum_{n=0}^{\infty}z^n\right)=-4\sum_{n=0}^{\infty}z^n+\sum_{n=0}^{\infty}5z^{n-1}\Longrightarrow \mathrm{Res}_{z=0}f(z)=5$$

$$f(z)=\frac{4(z-1)-1}{z-1}\frac{1}{1+(z-1)}=\left(4-\frac{1}{z-1}\right)\sum_{n=0}^{\infty}(-1)^n(z-1)^n=4\sum_{n=0}^{\infty}(-1)^n(z-1)^n-\sum_{n=0}^{\infty}(-1)^n(z-1)^{n-1}\Longrightarrow \mathrm{Res}_{z=1}f(z)=-1$$

Hence,

$$\int_c f(z)\,\mathrm{d}z = 8\pi i$$

Definition 15.7. If f is analytic in a neighbourhood of z_0, and $f(z)$ has a pole of order m at z_0 if its Laurent Series is in the form

$$f(z)=\frac{b_m}{(z-z_0)^m}+...\frac{b_1}{z-z_0}+\sum_{n=0}^{\infty}a_n(z-z_0)^n$$

Equivalently, $\lim_{z\to z_0}(z-z_0)^{m+1}f(z)=0$ and $m+1$ is the smallest number that makes it 0.

Theorem 15.8. (Cauchy Residue Theorem at Poles) *Let z_0 be an isolated singular point of $f(z)$. The following statements are equivalent*

1. *z_0 is a pole of order m for some $m \in \mathbb{Z}^+$ of f*
2. *f can be written in the form $f(z) = \frac{\varphi(z)}{(z-z_0)^m}$ where $\varphi(z)$ is analytic and non-zero at $z = z_0$, for some $m \in \mathbb{Z}^+$.*

If the above holds, then

$$\mathrm{Res}_{z=z_0} f(z) = \begin{cases} \varphi(z_0) & \text{if } m = 1 \\ \frac{\varphi^{m-1}(z_0)}{(m-1)!} & \text{if } m \geq 2 \end{cases}$$

Example 15.9. Let $f(z) = \frac{z+4}{z^2+1} = \frac{z+4}{(z+i)(z-i)}$. f has poles at $\pm i$.

For $z = i, f(z) = \frac{\varphi(z)}{z-i}$ where $\varphi(z) = \frac{z+4}{z+i}$. Since $m = 1$ in this case, $\mathrm{Res}_{z=i} f(z) = \varphi(i) = \frac{i+4}{2i}$

For $z = -i, f(z) = \frac{\varphi(z)}{z+i}$ where $\varphi(z) = \frac{z+4}{z-i}$. Since $m = 1$ in this case, $\mathrm{Res}_{z=-i} f(z) = \varphi(-i) = \frac{i-4}{2i}$

Considering positively oriented circle only, by Cauchy Residue Theorem, we have

$$\int_{|z|=2} f(z)\,\mathrm{d}z = 2\pi i\left(\frac{i+4}{2i} + \frac{i-4}{2i}\right) = 2\pi i$$

Example 15.10. Let $f(z) = \frac{z^3+2z}{(z-i)^3}$, it has a pole at $z = i$ with order 3. We can write $f(z) = \frac{\varphi(z)}{(z-i)^3}$ where $\varphi(z) = z^3 + 2z$. This gives

$$\mathrm{Res}_{z=i} f(z) = \frac{\varphi^2(i)}{2!} = \frac{6i}{2} = 3i$$

Let c be the positively oriented circle $|z| = 2$, by Cauchy Residue Theorem, we get

$$\int_c f(z)\,\mathrm{d}z = 2\pi i(3i) = -6\pi$$

Example 15.11. Considering positively oriented circle only, we have

$$\int_{|z-2|=2} \frac{3z^2+2}{(z-1)(z^2+9)}\,\mathrm{d}z = \pi i$$

$$\int_{|z|=4} \frac{3z^2+2}{(z-1)(z^2+9)}\,\mathrm{d}z = 6\pi i$$

15.2 **The Argument Principle.**

Assume now that we have a function f, a positively oriented closed simple curve $\gamma : [a, b] \to \mathbb{C}$. Moreover, f is meromorphic inside γ, analytic and non-zero on γ.

We also call Γ is the image $f(\gamma)$, so Γ does not pass the origin.

Consider some $z_0 \in \gamma$ and define $w_0 = f(z_0), \Phi_0 = \arg(w_0)$, also define some $z \in \gamma, w = f(z), \Phi = \arg(w)$. Define γ_1 to be the subpath of γ that goes from z_0 to z. We will denote the change of argument as

$$\Delta_{\gamma_1} \arg(f(z)) = \Phi - \Phi_0$$

Note that if we have $w_0 = w$, then we must have for some $n \in \mathbb{Z}$

$$\Delta_{\gamma_1} \arg(f(z)) = n2\pi$$

Since γ is closed, we must have $\gamma(a) = \gamma(b) \implies f(\gamma(a)) = f(\gamma(b))$ hence for some $n \in \mathbb{Z}$

$$\Delta_{\gamma} \arg(f(z)) = 2\pi n(\Gamma, 0) = \frac{1}{i}\int_{\Gamma} \frac{1}{w}\,\mathrm{d}w$$

This gives us the following theorem

Theorem 15.12. (Argument Principle) *Suppose γ is a positively oriented closed simple curve, f is a function that is*

- *meromorphic inside γ*

- *analytic and non-zero on* γ
- *Counting multiplicities* Z *is the number of zeros and* P *is the number of poles of* $f(z)$ *inside* γ

Then

$$n(\Gamma, 0) = \frac{1}{2\pi}\Delta_\gamma \arg(f(z)) = Z - P$$

Example 15.13. Consider analytic $f(z) = z^2$ and γ to be the positive oriented simple closed curve $|z| = 1$ denoted by $\gamma(\theta) = e^{i\theta}, 0 \le \theta \le 2\pi$. For all $0 \le \theta \le 2\pi, f(\gamma(\theta)) = e^{2i\theta}$

Since $f(0) = f(2\pi) = 1$, we have

$$\Delta_\gamma \arg(f(z)) = 2\pi n(\Gamma, 0) = 4\pi \implies \frac{1}{2\pi}\Delta_\gamma \arg(f(z)) = Z - P = 2 - 0 = 2$$

Example 15.14. Let $f(z) = \frac{z^3+2}{z}$. And consider γ be the positive oriented simple closed curve of $|z| = 1$. Certainly, f is meromorphic in γ and analytic and non-zero on γ. There is one pole and no zeros in γ. Hence,

$$\Delta_\gamma \arg(f(z)) = 2\pi(Z - P) = -2\pi$$

Showing that

$$n(\Gamma, 0) = -1$$

which the image of γ loops around the origin once in the opposite (counter-clockwise) direction.

Theorem 15.15. (Rouche's Theorem) *Let* γ *be a closed simple curve and* f, g *analytic in and on* γ *so that* $|f(z)| > |g(z)|$ *at all points on* γ

Then, $f(z), f(z) + g(z)$ *have the same number of zeros counting multiplicities in* γ.

Example 15.16. Consider the polynomial $z^4 + 3z^3 + 6 = 0, \gamma$ be the closed simple curve $|z| = 2$.

Consider $f(z) = 3z^3, g(z) = z^4 + 6$, then for all $|z| = 2$

$$|f(z)| = 3|z|^3 = 24 > 22 = 6 + |z|^4 \ge |g(z)|$$

Thus, there are three zeros in the closed simple curve.

15.3 Evaluation of Definite Integrals.

Case 1: We will first look at integrals of the form

$$\int_0^{2\pi} F(\sin(\theta), \cos(\theta))\, \mathrm{d}\theta$$

where F is rational can be computed by residues. By substituting $z = e^{i\theta}$ we get

$$\frac{\mathrm{d}z}{\mathrm{d}\theta} = iz \wedge \sin(\theta) = \frac{1}{2i}\left(z - \frac{1}{z}\right), \cos(\theta) = \frac{1}{2}\left(z + \frac{1}{z}\right)$$

Then,

$$\int_0^{2\pi} F(\sin(\theta), \cos(\theta))\, \mathrm{d}\theta = -i\int_{|z|=1} \frac{F\left(\frac{1}{2i}\left(z - \frac{1}{z}\right), \frac{1}{2}\left(z + \frac{1}{z}\right)\right)}{iz}\, \mathrm{d}z$$

Example 15.17. To compute

$$\int_0^{2\pi} \frac{1}{5 + 4\sin(\theta)}\, \mathrm{d}\theta$$

We substitute $z = e^{i\theta}$ and get

$$\begin{aligned}
\int_0^{2\pi} \frac{1}{5+4\sin(\theta)}\,\mathrm{d}\theta &= \int_{|z|\,=1} \frac{1}{iz\big(5+4\big(\frac{1}{2i}\big)\big(z-\frac{1}{z}\big)\big)}\,\mathrm{d}z \\
&= \int_{|z|\,=1} \frac{1}{5iz+2z^2-2}\,\mathrm{d}z \\
&= \int_{|z|\,=1} \frac{1}{2\big(z+\frac{i}{2}\big)(z+2i)}\,\mathrm{d}z \\
&= 2\pi i \mathrm{Res}_{z=-\frac{i}{2}} f(z) \\
&= 2\pi i \mathrm{Res}_{z=-\frac{i}{2}} \frac{\frac{1}{2(z+2i)}}{z+\frac{i}{2}} \\
&= 2\pi i \frac{1}{3i} \\
&= \frac{2}{3}\pi
\end{aligned}$$

Example 15.18. We show the integral

$$\int_0^{\pi} \frac{\cos(2\theta)}{1-2a\cos(\theta)+a^2}\,\mathrm{d}\theta = \frac{a^2\pi}{1-a^2} \quad -1<a<1$$

If $a=0$, we have the trivial case

$$\int_0^{\pi} \frac{\cos(2\theta)}{1}\,\mathrm{d}\theta = \left[\frac{\sin(2\theta)}{2}\right]_0^{\pi} = 0$$

Now, assume $a \neq 0$, we have

$$\begin{aligned}
\int_0^{2\pi} \frac{\cos(2\theta)}{1-2a\cos(\theta)+a^2}\,\mathrm{d}\theta &= \int_0^{2\pi} \frac{\frac{z^2+\frac{1}{z^2}}{2}}{1-2a\frac{z+\frac{1}{z}}{2}+a^2}\frac{1}{iz}\,\mathrm{d}z \\
&= \frac{1}{2i}\int_0^{2\pi} \frac{z^2+\frac{1}{z^2}}{1-az-\frac{a}{z}+a^2}\frac{1}{z}\,\mathrm{d}z \\
&= \frac{1}{2i}\int_0^{2\pi} \frac{z^2+\frac{1}{z^2}}{z-az^2-a+a^2z}\,\mathrm{d}z \\
&= \frac{1}{2i}\int_0^{2\pi} \frac{z^4+1}{z^3-az^4-az^2+a^2z^3}\,\mathrm{d}z \\
&= \frac{1}{2i}\int_0^{2\pi} \frac{z^4+1}{z^2(z-az^2-a+a^2z)}\,\mathrm{d}z \\
&= \frac{1}{2i}\int_0^{2\pi} \frac{z^4+1}{z^2(z-a)(1-az)}\,\mathrm{d}z \\
&= \pi(\mathrm{Res}_{z=a} f(z) + \mathrm{Res}_{z=0} f(z)) \\
&= \pi\left(\frac{a^4+1}{a^2(1-a^2)} + \left(-\frac{a^2+1}{a^2}\right)\right) \\
&= \frac{2\pi a^2}{1-a^2}
\end{aligned}$$

Now, for the original integral, we have

$$\int_0^{2\pi} \frac{\cos(2\theta)}{1-2a\cos(\theta)+a^2}\,\mathrm{d}\theta = 2\int_0^{\pi} \frac{\cos(2\theta)}{1-2a\cos(\theta)+a^2}\,\mathrm{d}\theta$$

where

$$\int_\pi^{2\pi} \frac{\cos(2\theta)}{1-2a\cos(\theta)+a^2}\,\mathrm{d}\theta = \int_0^{\pi} \frac{\cos(2\theta)}{1-2a\cos(\theta)+a^2}\,\mathrm{d}\theta$$

as for each θ,

$$\frac{\cos(2(2\pi-\theta))}{1-2a\cos(2\pi-\theta)+a^2} = \frac{\cos(2\theta)}{1-2a\cos(\theta)+a^2}$$

Case 2: We will now look at the integral of the form

$$\int_{-\infty}^{\infty} R(x)\,\mathrm{d}x$$

where $R(x)$ is rational. We will need to assume that $R(x)$ does not have any pole on the real line and at least one pole above the real axis.

We will utilise the following theorem for simple pole in the next example

Theorem 15.19. *If p, q are analytic at z_0 and $p(z_0) \neq 0, q(z_0) = 0, q'(z_0) \neq 0$ then*

$$\mathrm{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}$$

Example 15.20. Consider

$$\int_{-\infty}^{\infty} \frac{1}{1+x^6}\,\mathrm{d}x$$

Take C as the path of semi-circle that has corners $-R, R$ and the arc goes above the real axis. If $R \geq 1$, we have three poles in C, which are

$$e^{i\frac{\pi}{6}}, e^{i\frac{3\pi}{6}}, e^{i\frac{5\pi}{6}}$$

This gives

$$\begin{aligned}
\int_C \frac{1}{1+x^6}\,\mathrm{d}x &= 2\pi i\Big(\mathrm{Res}_{z=e^{i\frac{\pi}{6}}} f(z) + \mathrm{Res}_{z=e^{i\frac{3\pi}{6}}} f(z) + \mathrm{Res}_{z=e^{i\frac{5\pi}{6}}} f(z)\Big) \\
&= 2\pi i\left(\frac{1}{6\left(e^{\left(i\frac{\pi}{6}\right)^5}\right)} + \frac{1}{6\left(e^{\left(i\frac{3\pi}{6}\right)^5}\right)} + \frac{1}{6\left(e^{\left(i\frac{5\pi}{6}\right)^5}\right)}\right) \\
&= 2\pi i\left(-\frac{2}{6}i\right) \\
&= \frac{2}{3}\pi
\end{aligned}$$

Now, split C into the upper arc C_R and the path from $-R$ to R and let $R \to \infty$, we have

$$\left|\int_{C_R} \frac{1}{z^6+1}\,\mathrm{d}z\right| \leq \int_{C_R} \frac{1}{|(z^6+1)|}|\mathrm{d}z| \leq \frac{\pi R}{R^6-1} \to 0$$

Hence,

$$\int_{-\infty}^{\infty} \frac{1}{1+x^6}\,\mathrm{d}x = \lim_{R\to\infty}\left(\int_C \frac{1}{1+x^6}\,\mathrm{d}x - \int_{C_R} \frac{1}{1+x^6}\,\mathrm{d}x\right) = \frac{2\pi}{3}$$

This gives us

$$\int_{-\infty}^{\infty} R(x)\,\mathrm{d}x = 2\pi i \sum_{y>0} \mathrm{Res}_z(f(z))$$

Example 15.21.

$$\int_{-\infty}^{\infty} \frac{x}{(x^2+1)(x^2+2x+2)}\,\mathrm{d}x$$

i has a pole at $i, -i, -1+i, -1-i$. Thus is analytic at the real-axis. Moreover, the integral must converge by the p-test.

This gives us

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{x}{(x^2+1)(x^2+2x+2)}\,\mathrm{d}x = 2\pi i(\mathrm{Res}_{z=i} f(z) + \mathrm{Res}_{z=-1+i} f(z)) &= 2\pi i\left(\frac{i}{2i\cdot(1+2i)\cdot 1} + \frac{-1+i}{2i\cdot(-1+2i)\cdot(-1)}\right)\\ &= \pi\left(\frac{i}{1+2i} + \frac{-1+i}{2i-1}\right)\\ &= \pi\left(\frac{i+2}{5} + \frac{-1-2-i}{5}\right)\\ &= -\frac{\pi}{5}\end{aligned}$$

Case 3: Now, we look at the integral with a general form of

$$\int_{-\infty}^{\infty} R(x)e^{ix}\,\mathrm{d}x$$

using the same idea as the previous case, we have that the integral over the semicircle tends to 0, as $\left|e^{iz}\right| = e^{-r}$ tends to 0 as r increases. Hence, we get

$$\int_{-\infty}^{\infty} R(x)e^{ix}\,\mathrm{d}x = 2\pi i \sum_{y>0} \mathrm{Res}_z(f(z))e^{iz}$$

Remark 15.22. Note that

$$\mathrm{Re}\left(\int_{-\infty}^{\infty} R(x)e^{ix}\,\mathrm{d}x\right) = \int_{-\infty}^{\infty} R(x)\cos(x)\,\mathrm{d}x$$

$$\mathrm{Im}\left(\int_{-\infty}^{\infty} R(x)e^{ix}\,\mathrm{d}x\right) = \int_{-\infty}^{\infty} R(x)\sin(x)\,\mathrm{d}x$$

15.4 **Exercises.**

Exercise 15.23. Find the poles and residues of the following functions:

1. $\frac{1}{z^2+5z+6}$
2. $\frac{1}{(z^2-1)^2}$
3. $\frac{1}{\sin(z)}$
4. $\cot(z)$
5. $\frac{1}{\sin^2}(z)$
6. $\frac{1}{z^m(1-z)^n}$

Exercise 15.24. Evaluate the following integrals by the method of residues:

1. $\int_0^{\frac{\pi}{2}} \frac{1}{a+\sin^2(x)}\,\mathrm{d}x, |a| > 1$
2. $\int_0^{\infty} \frac{x^2}{x^4+5x^2+6}\,\mathrm{d}x$
3. $\int_{-\infty}^{\infty} \frac{x^2-x+2}{x^4+10x^2+9}\,\mathrm{d}x$

4. $\int_0^\infty \frac{x^2}{(x^2+a^2)^3}\,\mathrm{d}x$, a is real
5. $\int_0^\infty \frac{\cos(x)}{x^2+a^2}\,\mathrm{d}x$, a is real
6. $\int_0^\infty \frac{x\sin(x)}{x^2+a^2}\,\mathrm{d}x$, a is real