2.1 Strings

Definition 2.1.1

Let X be a set and let $n \in \mathbb{Z}_{\geq 0}$ be a whole number

An X-String of length n is a function $f: \{1, \dots, n\} \to X$

It is denoted as $\overline{\underset{f(1)}{\underset{\shortparallel}{x_1}}\, \underset{f(2)}{\underset{\shortparallel}{x_2}} \cdots \underset{f(n)}{\underset{\shortparallel}{x_n}}}$

Notation 2.1.2

We write $[n] = \{1, 2, \dots, n\}$

Example 2.1.3

(1) $X = \{a, b, \dots, z\}$ be the set of English alphabet, cbabbac is an X-string of length 7. $(f: \{1, \dots, 7\} \to X)$

(2) X is the set of polygons, ⬡ △ □ ⬠ is an X-string of length 4. $(f: \{1, \dots, 4\} \to X)$

(3) $X = \{a, b, \dots, z\}$, $abc \neq acb \neq acc$

Theorem 2.1.4

The number of X-strings of length n is $|X|^n$.

Proof

Let $x = \overline{x_1 x_2 \cdots x_n}$ be an X-string of length n.

For x_1 we have $|X|$ choices

Same for $x_2, \dots, x_n$.

So, in total, we have $|X|^n$ choices. All of these choices are distinct.

More formal way: Induction.

$\square$

Example 2.1.5

(1) A binary string is a $\{0,1\}$ string

2^n binary strings of length n

(2) 10^7 decimal numbers of length 7

(3) An Ontario number plate is of the form ABCD 123, there are $26^4 \times 10^3$ choices.

2.2 Permutations (No Repetitions)

Definition 2.2.1

An X-string x of length n is called a permutation if $x = \overline{x_1 x_2 \cdots x_n}$ such that $x_i \neq x_j$ for all $i \neq j$.

Example 2.2.2

Let $X = \{a, b, \dots, z\}$ be the set of English alphabet

abc, adb, ab, cdba are permutations

aab, abca, ccba, ababc are not permutations

Example 2.2.3

There are $26 \times 25 \times 24 \times 23 \times 10 \times 9 \times 8$ distinct permutations for an Ontario number plate

Notation 2.2.4

We write $p(m,n) = \frac{m!}{(m-n)!}$ where $n! = n \times \cdots \times 1$

Proposition 2.2.5

Let X be a set where $|X| = m$. The number of X-permutation of length n is $\begin{cases} 0 & \text{if } n > m \\ \frac{m!}{(m-n)!} & \text{otherwise} \end{cases}$

Proof

Let $x = \overline{x_1 \cdots x_n}$ be an X-permutation of length n

There are m choices for x_1, $m-1$ for $x_2, \dots, m-(n-1)$ for x_n

So in total, $m(m-1)(m-n+1) = \frac{m!}{(m-n)!}$

More formally, use pigeonhole for $n>m$, induction for others. $\square$

Example 2.2.6

How many ways can we let 4 different people sit at a round table with 7 chairs?

Rotation is not counted.

Solution

$$\frac{7!}{(7-4)! \times 7}$$

Repetition

Example 2.2.7

How many strings of length 16 we can have for

COMBINATORICSISM

How many strings of length 7 we can have for

TORONTO

Solution

$$\frac{16!}{2! \times 2! \times 2! \times 3!}$$

Repetitions

$$\frac{7!}{2! \times 3!}$$

2.3 Combinations

Definition 2.3.1

An X-combination of size n is a subset $E \subseteq X$ of size n

Proposition 2.3.2

Let $n := |X|$. The number of X-combination of size m is $\binom{n}{m} := \frac{n!}{m!(n-m)!}$

Proof

Let $E = \{e_1, \dots, e_m\}$

The number of ways to choose e_1 is n.

For e_2 is $n-1, \dots,$ for e_m is $n-(m-1) = n-m+1$

And we need to divide the answer by $m!$ to account for overcounting.

Answer is $\frac{n \cdot (n-1) \cdot (n-m+1)}{m!} = \frac{n!}{m!(n-m)!}$

$\square$

2.4 Combinatorial Proof

Proposition 2.4.2

$$\binom{n}{m} = \binom{n-1}{m} + \binom{n-1}{m-1}$$

Proof

Solution 1: Algebraic way

Solution 2: Fix a person A.

Case 1: If A is chosen in m people then we have n-1 choose m-1 people.

Case 2: If not, we have n-1 choose m people.

Hence $\binom{n}{m} = \binom{n-1}{m} + \binom{n-1}{m-1}$ $\square$

Proposition 2.4.3

$$\binom{n}{k} = \binom{n}{n-k}$$

Proof

$\binom{n}{k}$ is the number of ways to choose k of them.

$\binom{n}{n-k}$ is the number of ways to choose not to have n-k, so $\binom{n}{k} = \binom{n}{n-k}$. $\square$

Proposition 2.4.4

$$\binom{n}{k+1} = \binom{n-1}{k} + \binom{n-2}{k} + \cdots + \binom{k}{k}$$

Proof

Solution 1:
$$\begin{aligned}
\binom{n}{k+1} &= \binom{n-1}{k} + \binom{n-1}{k+1} \\
&= \binom{n-1}{k} + \left[\binom{n-2}{k} + \binom{n-2}{k+1}\right] \\
&\vdots \\
&= \binom{n-1}{k} + \binom{n-2}{k} + \cdots + \binom{k+1}{k} + \binom{k+1}{k+1} \\
&= \binom{n-1}{k} + \binom{n-2}{k} + \cdots + \binom{k+1}{k} + \binom{k}{k}
\end{aligned}$$

Solution 2: Let $S \subseteq \{1, \dots, n\} = [n]$ where $|S| = k+1$

Let r be the highest number in S, where $k+1 \leq r \leq n$

Then, we have to choose $S' \subseteq \{1, \dots, r-1\} = [r-1]$

We have $\binom{r-1}{k}$ ways of choosing.

Hence, $\binom{n}{k+1} = \sum_{r=1}^{n} \binom{r-1}{k} = \sum_{r=1}^{n-1} \binom{r}{k}$ $\square$

Proposition 2.4.5

$$\binom{2n}{n} = \binom{n}{0}^2 + \cdots + \binom{n}{n}^2$$

Proof

We will choose $S \subseteq [2n]$ such that $|S| = n$.

There are $\binom{2n}{n}$ ways of choosing.

Let $[2n] = X_1 \cup X_2$ where $X_1 = [n]$, $X_2 = [2n] \setminus [n]$

Let $S_i := X_i \cap S$ for $i = 1, 2$ and let $|S_1| = k$.

Therefore, $|S_2| = n-k$.

There are $\binom{n}{k}$ ways of choosing S_1, $\binom{n}{n-k}$ ways for S_2.

In total, there are $\binom{n}{k}\binom{n}{n-k}$ choices.

Since k can be $0, 1, \dots, n$, in total there are $\sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k} = \sum_{k=0}^{n} \binom{n}{k}^2$ $\square$

2.6 Binomial Theorem

Theorem 2.6.1

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Proof

Let $n_k = [x^k y^{n-k}]$ denotes the coefficient of $x^k y^{n-k}$ in the expression of $(x+y)^n$

And $n_k = \binom{n}{k}$, number of ways to choose k xs out of n $(x+y)$.

Hence, $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$

Corollary 2.6.2

$$2^n = \sum_{i=0}^{n} \binom{n}{i}$$

Proof

$$2^n = (1+1)^n = \sum_{i=0}^{n} \binom{n}{i} 1^i 1^{n-i} = \sum_{i=0}^{n} \binom{n}{i} \qquad \square$$

2.5 The Ubiquitous nature of Binomial Coefficient

Example 2.5.1

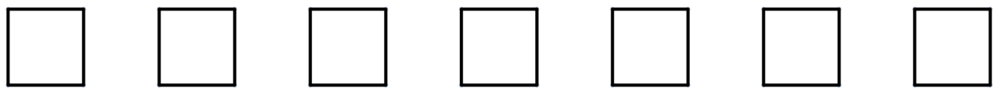
1、How many ways can we distribute 7 M&Ms to 3 people? Assume you can give 0 M&Ms to some people.

2、In general, how many ways can we distribute n M&Ms to k people? Assume you can give 0 M&Ms to some people.

3、What if you don't have to distribute all n M&Ms?

Solution

1、We can model this question as a box and stick diagram



We need to put two sticks to divide three parts

In total there is $\binom{9}{2} = 36$ ways

2、Similarly, $\binom{n+k-1}{k-1}$

3、This means there could be the $k+1$ person who gets the remaining. Hence $\binom{n+k-1}{k}$

Definition 2.5.2

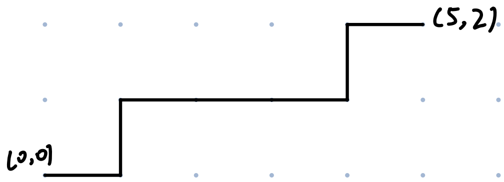
A lattice path is a sequence of pairs of integers $(m_1, n_1), (m_2, n_2), \ldots, (m_t, n_t)$ such that for all $i = 1, \ldots, t-1$, either

(1) $m_{i+1} = m_i + 1$ and $n_{i+1} = n_i$

(2) $m_{i+1} = m_i$ and $n_{i+1} = n_i + 1$

Example 2.5.3

This is a lattice path.



Example 2.5.4

How many lattice paths are there shooting from $(0,0)$ to (m,n) where $n,m>0$

Solution

Define R to be the movement to the right, i.e. $(i,j)\to(i+1,j)$

Define U to be the movement upwards, i.e. $(i,j)\to(i,j+1)$

The path from $(0,0)$ to (m,n) always have m Rs, n Us. Hence there are $\frac{(m+n)!}{m!\,n!}=\binom{m+n}{n}$ ways.

Example 2.5.5

How many lattice paths are there shooting from $(0,0)$ to (n,n) where $n,m>0$ and never go above the upper diagonal.

Solution

Let P be all the paths starting from $(0,0)$ and ending at (n,n). $|P|=\binom{2n}{n}$

G be all the paths starting from $(0,0)$ and ending at (n,n), and never go above the upper diagonal.

B be all the paths starting from $(0,0)$ and ending at (n,n), and crosses the upper diagonal

We have $P=G\sqcup B$ $\quad$ $\sqcup$ disjoint union

$$\Rightarrow \binom{2n}{n}=|G|+|B|$$

We will define a bijection $B\cong P(n-1,n+1)$

Let $p=(s_1,\dots,s_{2n})\in B$, let s_i be the first point where p crosses the diagonal.

Construct path $\tilde{p}=s_1\dots s_i\tilde{s}_{i+1}\dots\tilde{s}_{2n}$ where $\tilde{s}_k=\begin{cases}R & \text{if } s_k=U\\ U & \text{if } s_k=R\end{cases}$

$$\begin{aligned}
\#L\text{s in } \tilde{p} &= \#L\text{s in } \overline{s_1\dots s_i} + \#L\text{s in } \overline{\tilde{s}_{i+1}\dots\tilde{s}_{2n}}\\
&= \#L\text{s in } \overline{s_1\dots s_i} + \#U\text{s in } \overline{s_{i+1}\dots s_{2n}}\\
&= \#L\text{s in } \overline{s_1\dots s_i} + n - \#U\text{s in } \overline{s_1\dots s_i}\\
&= j+n-(j+1)\\
&= n-1
\end{aligned}$$

[Note that in $\overline{s_1\dots s_i}$, $\#L+1=\#U$ since i just crosses the diagonal]

$$\begin{aligned}
\#U\text{s in } \tilde{p} &= 2n-\#L\text{s in } \tilde{p}\\
&= n+1
\end{aligned}$$

Hence, $\tilde{p}\in P(n-1,n+1)$

Conversely, take any $p=(s_1,\dots,s_{2n})\in P(n-1,n+1)$, let s_i be the first point where p crosses the diagonal.

Construct path $\tilde{p}=s_1\dots s_i\tilde{s}_{i+1}\dots\tilde{s}_{2n}$ where $\tilde{s}_k=\begin{cases}R & \text{if } s_k=U\\ U & \text{if } s_k=R\end{cases}$

$$\begin{aligned}\#L\text{s in }\tilde{p} &= \#L\text{s in }\overline{s_1\dots s_i}+\#L\text{s in }\overline{\tilde{s}_{i+1}\dots\tilde{s}_{2n}}\\ &= \#L\text{s in }\overline{s_1\dots s_i}+\#U\text{s in }\overline{s_{i+1}\dots s_{2n}}\\ &= \#L\text{s in }\overline{s_1\dots s_i}+(n+1)-\#U\text{s in }\overline{s_1\dots s_i}\\ &= j+(n+1)-(j+1)\\ &= n\end{aligned}$$

$$\begin{aligned}\#U\text{s in }\tilde{p} &= 2n-\#L\text{s in }\tilde{p}\\ &= n\end{aligned}$$

Since we also have s_i is above the diagonal $p\in B$.

And we have $B\cong P(n-1,n+1)\Rightarrow |B|=|P(n-1,n+1)|=\binom{2n}{n-1}$

Hence $|G|=\binom{2n}{n}-\binom{2n}{n-1}=\frac{1}{n+1}\binom{2n}{n}$ $\square$

2.7 Multinomial Coefficient

Theorem 2.7.1

Let $n\in\mathbb{N}$ and $k_1,\dots,k_r\in\mathbb{Z}_{\geq 0}$ such that $n=k_1+\dots+k_r$

The number of ways of partitioning the set $[n]$ into an ordered list $[n]=A_1\sqcup A_2\dots\sqcup A_r$ such that $A_s\subseteq[n]$ of size k_s is $\binom{n}{k_1,k_2,\dots,k_r}=\frac{n!}{k_1!k_2!\dots k_r!}$

Remark 2.7.2

$n!$ is the number of permutation for $[n]$. Each permutation $s=x_1\cdots x_n$ gives

$A_1=\{x_1,\dots,x_{k_1}\}$

$A_2=\{x_{k_1+1},\dots,x_{k_1+k_2}\}$

$\vdots$

$A_r=\{x_{k_1+\dots+k_{r-1}+1},\dots,x_n\}$

Each $A_1,\dots,A_r$ comes from $k_1!,\dots,k_n!$ permutations. Hence, $n!=k_1!\cdots k_n!\cdot$(# of such lists of subsets).

Theorem 2.7.2 (Multinomial Theorem)

$$(x_1+\dots+x_r)^n=\sum_{l_1=0,\dots,l_r=0}^{k_1,\dots,k_r}\binom{n}{l_1,\dots,l_r}x_1^{l_1}\dots x_r^{l_r}\quad\text{where } l_1+\dots+l_r=n$$

3.1 Induction & Recursion

Example 3.1.1

Consider a box of size $2 \times n$

In how many ways can you tile 2×1 cookies?

Solution

The possible distribution of cookies in the end is either a vertical bar (the number of ways to tile $2 \times (n-1)$)

or two horizontal bars (the number of ways to tile $2 \times (n-2)$)

Hence, the number of solutions for n is $F_n = F_{n-1} + F_{n-2}$ where $F_1 = 1, F_2 = 2$ $\square$

Example 3.1.1 (Catalan Number)

In Example 2.5.5, we found the number of lattice paths are there shooting from $(0,0)$ to (n,n) where $n,m>0$

and never go above the upper diagonal.

We will now calculate this using recursion.

Let $k \in \{1, \dots, n\}$ be the smallest number such that the path s goes through (k,k)

Then we have $$D_n = \sum_{k=1}^{n} \left| \left\{ \begin{matrix} \text{paths from } (0,0) \\ \text{to } (k,k) \text{ which does} \\ \text{not touch the diagonal} \end{matrix} \right\} \right| \left| \left\{ \begin{matrix} \text{paths from } (k,k) \\ \text{to } (n,n) \text{ without} \\ \text{passing across the diagonal} \end{matrix} \right\} \right|$$

Note that $\left| \left\{ \begin{matrix} \text{paths from } (k,k) \\ \text{to } (n,n) \text{ without} \\ \text{passing across the diagonal} \end{matrix} \right\} \right| = D_{n-k}$

For any paths $s \in \left\{ \begin{matrix} \text{paths from } (0,0) \\ \text{to } (k,k) \text{ which does} \\ \text{not touch the diagonal} \end{matrix} \right\}$, it must starts with moving right (R) and ends with moving up (U)

We can write $s = R s_2 s_3 \dots s_{2k-2} U$

Note that $s_2 s_3 \dots s_{2k-2} \in$ path from $(1,0)$, $(k, k-1)$, it's also a path that should not pass across it own diagonal from $(1,0)$ to $(k,k-1)$

as otherwise the path s would touch the diagonal. Hence

$$\left| \left\{ \begin{matrix} \text{paths from } (0,0) \\ \text{to } (k,k) \text{ which does} \\ \text{not touch the diagonal} \end{matrix} \right\} \right| = D_{k-1}$$

And so $D_n = \sum_{k=1}^{n} D_{k-1} \cdot D_{n-k}$ where $D_0 = 1, D_1 = 1$

Example 3.1.2

Given $n \in \mathbb{N}$, show that $1+3+\cdots+(2n-1) = \sum_{k=1}^{n}(2k-1) = n^2$

Proof

Induction:

$P(1)$ is true, $1=1$

Inductive hypothesis: Suppose $P(i)$. That is, $1+3+\cdots+(2i-1) = \sum_{k=1}^{i}(2k-1) = i^2$ for some $i \in \mathbb{N}$

$P(i+1)$: $\sum_{k=1}^{i+1}(2k-1) = \sum_{k=1}^{i}(2k-1) + 2(i+1)-1 = i^2+2i+1 = (i+1)^2$

Hence, $P(i+1)$ is true.

Since $P(1)$ is true and $P(i+1)$ is true whenever $P(i)$ is true, $P(n)$ is true for all $n \in \mathbb{N}$. $\square$

Example 3.1.3

Prove that $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

Proof

Induction. $\square$

Example 3.1.4

Define a sequence $a_1, a_2, \ldots$ recursively by

$a_1 = 3$
$a_2 = 5$
$a_n = 2a_{n-1} - a_{n-2}$ for $n \geq 3$

What is a_{2024}

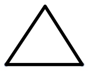
Solution

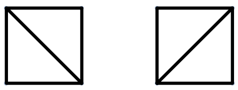
$a_n = 2n+1$ (Proof by Induction). Hence $a_{2024} = 4049$. $\square$

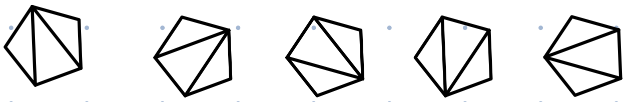
Example 3.1.5

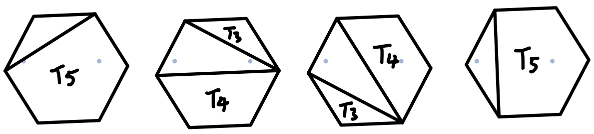
In how many ways can we triangulate a regular polygon of n sides?

Solution

$n=3$  1 way

$n=4$  2 ways

$n=5$  5 ways

$n=6$ 

$T_6 = T_2 \times T_5 + T_3 \times T_4 + T_3 \times T_4 + T_5 \times T_2$

Proof

The edge $\overline{1n}$ has to be the edge of some triangle. Let $k=2,\dots,n-1$ be s.t. $\overline{1n}$ belongs triangle $1kn$.

This splits the n-gon to 1 k-gon, 1 $n-k+1$-gon, and the triangle.

k-gon can be done in T_k ways, T_{n-k+1} ways for $n-k+1$ gon. Each triangulation is distinct. Thus, it takes $T_k \times T_{n-k+1}$ ways.

For each k, cases are mutually exclusive. Hence $T_n = \sum_{k=2}^{n-1} T_{n-k+1} T_k$. $\square$

4.1 Pigeonhole Principle

If we place $mn+1$ beans in n bedrooms, there will be at least one bedroom with at least $m+1$ beans.

Example 4.1.1

There are at least 2 students, out of 410 students registered in MAT344, who share the same birthday.

Solution

There are 366 days in 2024, we have 366 bedroom, ≥ 367 beans. Thus there are at least 2 students who share the same birthday

Example 4.1.2

There are at least 40 residents of the GTA who have the same number of hair on their head.

Solution

A human have 1-150000 hair, there are 6 million residents of the GTA

By PHP, 150000 bedrooms, $\geq 39 \times 150000 + 1$ beans. Hence, there are at least 40 residents of the GTA who have the same number of hair on their head.

Example 4.1.3

In a randomly chosen group of n people, there are at least 2 people who know the same number of people in that group.

Proof

Any person can know $0, 1, \dots, n-1$ people.

There are n beans for n bedrooms. But it is impossible for a person knows n people (all people) and the other knows 0 people. So, there are only maximumally $n-1$ bedrooms that can be occupied. And by PHP, we are done.

Example 4.1.3

In a group of six people, there are at least 3 people who all know each other or they all don't know each other

Proof

Given a person s_1, he has relationship with other five people. Since five people can either be known or unknown, there are at least known or three unknown to that person. WLOG, say three known person, s_2, s_3, s_4. Now, if a pair of people out of s_2, s_3, s_4 know each other then this pair together with s_1 would be the 3 people who all know each other. Otherwise, there is no pair out of s_2, s_3, s_4 who know each other. Hence, s_2, s_3, s_4 would be the 3 people who don't know each other. $\square$

Example 4.1.4

Let $m, n \in \mathbb{N}$. Any sequence of $mn+1$ distinct real numbers either has an increasing sequence of length $m+1$ or a decreasing sequence of length $n+1$.

Proof

Let a_i = longest increasing subsequence starting with term i

b_i = longest decreasing subsequence starting with term i

If some $a_i \geq m+1$, or $b_i \geq n+1$ we are done.

We take pairs of integers $(a_i, b_i)_{i=1,\dots,mn+1}$

Suppose not, we assume that $a_i \leq m$ and $b_i \leq n$ for all i.

Then, we take pairs of integers (a_i, b_i) for $i = 1, \dots, mn+1$

Since $1 \leq a_i \leq m$ and $1 \leq b_i \leq n$, there can be at most mn-distinct pairs of integers $\{(a_i, b_i)\}$

But there are $mn+1$ many pairs of (a_i, b_i).

Hence, by PHP, there exists some i, j such that $(a_i, b_i) = (a_j, b_j)$ where $i \neq j$. WLOG, let $i < j$.

So we have $a_i = a_j$, $b_i = b_j$.

However, if the ith term is greater than jth term, then $b_i > b_j$, since a decreasing sequence starts at ith term can have the sequence that starts at jth term. Similarly, if the ith term is less than jth term, then $a_i > a_j$. Contradiction. Hence, there must be some $a_i \geq m+1$, or $b_i \geq n+1$. $\square$

5.1 Graph Theory

Definition 5.1.1

A graph G is a pair (V, E) where $E \subseteq V \times V$ where

V is called the set of verticies

E is called the set of edges.

If edges are not directed, then $(a,b) \in E$ is an edge connecting a and b.

Example 5.1.2

Let $G = (V, E)$ be a graph of train stations in Canada

$V = \{\text{Toronto, London, Windsor}\}$

$E = \{(\text{Toronto, London}), (\text{London, Windsor})\}$

To go from Toronto to Windsor, we cannot go directly, we need to go to London and then to Windsor

Definition 5.1.2

For $a, b \in V$, we say that a and b are neighbours / adjacent if they are connected by a single path $(a,b) \in E$.

Definition 5.1.3

A complete graph of $G = (V, E)$ is a graph where for all $v_1, v_2 \in V$, $(v_1, v_2) \in E$. There are edges between any two verticies.

Theorem 5.1.4

For any graph $G = (V, E)$, $\sum_{v \in V} \deg_G(v) = 2|E|$

Proof

Let $E_1 = \{(i,j) \mid i,j \in V \text{ and } (i,j) \in E\}$

$E_2 = \{(i,j) \mid i,j \in V \text{ and } (j,i) \in E\}$

Take $E_1 \cup E_2$. we want to show $|E_1 \cup E_2| = |\bigcup_{i \in V} Adj_G(i)|$ by giving a bijection.

Hence, we have $|E_1 \cup E_2| = |\bigcup_{i \in V} Adj_G(i)|$

$$\Rightarrow 2|E| = \sum_{i \in G} deg_G(i)$$

$\square$

Corollary 5.1.5

G has even number of odd degree vertices

Definition 5.1.6

Let $G=(V,E)$ be a graph. A walk is a sequence $(v_1, \dots, v_n)$ of vertices in V such that $(v_i, v_{i+1}) \in E$ $\forall i = 1, \dots, n-1$.

A path $(v_1, \dots, v_n)$ is a walk if $v_i \neq v_j$ $\forall i,j$.

The length of this path is then $n-1$.

A cycle $(v_1, \dots, v_n)$ is a path if $(v_n, v_1) \in E$. It has a length of cycle of n.

We say that two vertices $v \neq w \in V$ are connected if $\exists$ a path P starting with v and ending with w.

A graph G is called connected if vertices v, w are connected $\forall v, w \in V$.

A graph is disconnected if it is not connected.

A graph is acyclic if it has no cycles.

Definition 5.1.7

An isomorphism $f: G \to G'$ where $G=(V,E)$ and $G'=(V',E')$ is a bijection $f: V \to V'$ such that $\forall v, w \in V$, we have

$(f(v), f(w)) \in E' \Leftrightarrow (v,w) \in E$

Definition 5.1.8

A subgraph of a graph $G=(V,E)$ is a graph $G'=(V',E')$ such that $V' \subseteq V$, and $E' \subseteq E$

We call that the subgraph is spanning if $V=V'$. A graph is induced if $\forall x, y \in V'$, $(x,y) \in E \Rightarrow (x,y) \in E'$.

Definition 5.1.9

A **forest** is an acyclic graph

A **tree** is a connected forest

A **leaf** in a forest is a vertex of degree

Definition 5.1.10

An **Eulerian walk** is a walk $(v_0, \dots, v_n)$ such that

(1) Each edge (v_i, v_{i+1}) is distinct from each other

(2) All the edges appears exactly once in (v_i, v_{i+1})

An **Eulerian circuit** is an Eulerian walk $(v_0, \dots, v_n)$ such that $v_0 = v_n$.

A graph is called **Eulerian** if it has an Eulerian circuit and connected.

Theorem 5.1.11

A connected graph has an Eulerian walk starting at v and ending at w, where $v \neq w \in V \iff \deg_G(u)$ is even $\forall u \neq v, w \in V$

A graph is Eulerian $\iff \deg_G(u)$ is even $\forall u \in V$.

Proof

$\Rightarrow$ If a graph has a Eulerian walk then except the starting and ending have even degrees. They must have same numbers of edges going in and going out, where each edges are distinct as they must appear only once.

$\Leftarrow$ This would be an algorithm to find an Eulerian circuit. Using a partial Eulerian circuit (PEC), which is a walk $(x_0, \dots, x_t)$ such that $x_t = x_0$, $x_i x_{i+1}$ are distinct paths.

(1) Start with (x_0) for any $x_0 \in V$. This is a PEC.

(2) If all edges have been traversed in this PEC, we are done. Else, go to (3).

(3) This means there is a minimal $i \in \{0, \dots, t-1\}$ such that x_i is in an edge that has not been traversed.

(4) Take any walk $(u_0, \dots u_s)$ starting at $u_0 = x_i = u_s$ as long as possible while not traversing an edge that has been traversed in $(x_0, \dots, x_t)$.

(5) Then, $(x_0, \dots, x_{i-1}, u_0, \dots, u_s, x_{i+1}, \dots, x_t)$ is a larger PEC. And go back to (2). $\square$

5.2 Hamiltonian Graph

Definition 5.2.1

A graph G is hamiltonian if there is a cycle $(x_0, \dots, x_n)$ that exists each vertex exactly once.

Theorem 5.2.2

Let G be a graph such that $\deg(v) \geq \frac{|V|}{2}$ $\forall v \in V$, then G is hamiltonian.

Proof

Let $n = |V|$, $P = (x_1, \dots, x_t)$ be the longest path, we have $\text{length}(P) = t-1$

We must have all neighbours of x_1 appear in P, as otherwise, we can put the neighbour in the beginning and obtain a longer path. Hence, we have $t-1 \geq \deg(x_1) \geq \frac{n}{2} \Rightarrow t > \frac{n}{2}$

Similarly, all neighbours in x_t is in P.

Now, if P is a cycle, if $t=n$, we are done.

If $t<n$, then there is a $w \in V$ and $w \notin P$.

However, $\deg_G(w) \geq \frac{n}{2} > n-t =$ number of elements outside P.

By PHP, there is an $x_i \in P$ such that $(w, x_i) \in E$

But we have a path longer than P, namely $P = (w, x_i, x_{i+1}, \dots, x_t, x_1, \dots, x_{i-1})$, contradiction.

Suppose P is not a cycle, let $X = \{(x_1, x_{i+1}) : 1 \leq i \leq t-1 \text{ s.t. } (x_1, x_{i+1}) \in E\} \cup \{(x_i, x_t) : 1 \leq i \leq t-1 \text{ s.t. } (x_i, x_t) \in E\}$

Define a function $f : X \to \{1, \dots, t-1\}$ where $f(x_1, x_{i+1}) = i$, $f(x_i, x_t) = i$

Note that $|X| = |\deg(x_1)| + |\deg(x_t)| \geq n > t-1$

By PHP, f is not injective. There is an i such that $(x_1, x_{i+1}) \in E$, $(x_i, x_t) \in E$

We write $P' = \{x_1, \dots, x_i, x_t, x_{t-1}, \dots, x_{i+1}\}$, we know that P' is a cycle. And $\text{length}(P') = \text{length}(P)$.

And by previous, P' is hamiltonian.

$\square$

5.3 Planar Graphs

Definition 5.3.1

A graph is planar if it can be drawn in $\mathbb{R}^2$ without edge crossing.

A face of a planar embedding of G is a region bounded by edges and vertices.

Theorem 5.3.2 (Euler's formula)

Let G be a planar graph. Any planar drawing of G satisfies $\chi(G) = |V| - |E| + |F| = 2$

Proof

We induct on $|E|$

Base case: $|E| = 0$, since G is connected, $|V| = 1$, so $|F| = 1$. And so $|V| - |E| + |F| = 2$.

Suppose this is true for all $|E| = k$, take $G = (V, E)$ where $|E| = k+1$, and suppose $G' = (V', E')$ where $V = V'$, $E' = E \setminus \{e\}$ for some $e \in E$.

Case 1: If G' is connected, we must have $|F'| = |F| - 1$

By Induction Hypothesis $|V'| - |E'| + |F'| = 2$. Hence $|V| - |E| + |F| = |V| - |E'| - 1 + |F'| + 1 = 2$.

Case 2: If G' is not connected, it has two components G_1', G_2', by Induction Hypothesis we have $|V_1'| - |E_1'| + |F_1'| = 2 = |V_2'| - |E_2'| + |F_2'|$.

But $|F_1'| + |F_2'| = |F| + 1$.

Hence $|V| - |E| + |F| = |V_1'| + |V_2'| - |E_1'| - |E_2'| - 1 + |F_1'| + |F_2'| - 1 = 4 - 2 = 2$ $\square$

Corollary 5.3.3

Let $G = (V, E)$ be a planar graph

Then $|E| \leq 3|V| - 6$

Moreover, if G has no triangles.

Then $|E| \leq 2|V| - 4$

Proof

Let $P=\{(e,f)$ where $e\in E, f\in F$ and e is a boundary of $F\}$

Since each edge divides at most two faces (each edge can only be boundary of two faces), we must have $|P|\leq 2|E|$

Since each face at least 3 edges on its boundary, we have $|P|\geq 3|F|$

So $3|F|\leq|P|\leq 2|E| \Rightarrow |F|\leq\frac{2}{3}|E|$

By Euler's formula, $|V|-|E|+|F|=2 \Rightarrow 2-|V|+|E|=|F|$

$$\Rightarrow 2-|V|+|E|\leq\frac{2}{3}|E|$$

$$\Rightarrow \frac{1}{3}|E|\geq|V|-2$$

$$\Rightarrow |E|\geq 3|V|-6$$

For G having no triangle, the proof is essentially the same except each face at least 4 edges on its boundary, we have $|P|\geq 4|F|$.

Definition 5.3.4

A graph is K_n if it is a complete graph with n vertices.

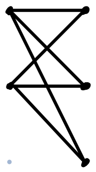
A bipartite graph is a graph $G=(V,E)$ such that $V=A\cup B$ where $A\cap B=\emptyset$ and the subgraphs induced by A and B are independent.

(i.e. there is no edge with vertices both in A or both in B).

A complete bipartite graph $K_{m,n}$ is the graph $G=(V,E)$ where $V=V_1\cup V_2$, $|V_1|=m$, $|V_2|=n$, $V_1\cap V_2=\emptyset$, and $xy\in E \Leftrightarrow x\in V_1, y\in V_2$.

Example 5.3.5

The following graph is for $K_{2,3}$.



Example 5.3.6

K_5 is not planar, as $|V|=5$, $|E|=10$, and $|E|=10\nleq 3\times5-6=3\times|V|-6$

Definition 5.3.1

A proper colouring of G is a set C of colours together with a function $\phi: V \to C$ called a colouring such that $\phi(x) \neq \phi(y)$ if $xy \in E$.

The smallest $|C|$ such that there is a proper colouring $\phi: G \to C$ is called the chromatic number $\chi(G)$ of G.

Theorem 5.3.2

The following are equivalent

(1) G is a 2-colourable, i.e. $\chi(G) \leq 2$.

(2) $|V| = |A| \sqcup |B|$ such that induced subgraphs A and B are independent.

Such a graph is called bipartite.

(3) G has no odd length cycle.

Proof

(1) $\Rightarrow$ (2):

Take $A = \{v \in V : v \text{ is in a colour}\}$, $B = \{v \in V : v \text{ is in another colour}\}$ and we are done.

(2) $\Rightarrow$ (3):

Suppose there is a cycle of odd length $\{v_1, \ldots, v_{2n+1}\}$.

If $v_1 \in A$, then we must have $v_2 \in B$. This leads to $v_3 \in A$, $v_4 \in B, \ldots$ Hence $v_i \in A$ for i being odd and $v_i \in B$ for i being even.

But this means $v_{2n+1} \in A$ but we also have $v_{2n+1}v_1 \in E$. Contradiction.

Hence, there is no cycles of odd length.

(3) $\Rightarrow$ (1):

Take $v, w \in V$, define $d(v,w)$ to be the length of the smallest path from v to w.

Let's fix a vertex $v \in V$.

$A = \{w \in V \mid d(w,v) \text{ is even}\}$

$B = \{w \in V \mid d(w,v) \text{ is odd}\}$

Suppose A and B is not independent, then we can find some w and $w' \in A$ such that $ww' \in E$.

By assumption there exists a path $P = (v, \ldots, w)$ of even length. There also exist a path $(v, \ldots, w')$ of even length.

But then we can obtain a cycle of odd length $(v, \ldots, w, w', \ldots, v)$. Contradiction. Similar for B.

□

5.4 Extremal Graph Theory

Definition 5.4.1

A clique in a graph $G=(V,E)$ is a subset $K \subseteq V$ such that the subgraph induced by K is complete.

The clique number of G, denoted by $w(G)$ is the largest size of a clique.

We also write $\Delta(G) = \max_{v \in V} \deg_G(v)$.

Example 5.4.2

We have $w(K_n) = n$

$w(P_n) = 2$, a path

$w(C_n) = \begin{cases} 3 & \text{if } n=3 \\ 2 & \text{otw} \end{cases}$, a cycle

Theorem 5.4.3

$w(G) \leq \chi(G)$. And (without proof) this is a tight lower bound on $\chi(G)$.

Proof

Let $d = w(G)$

Note that $\chi(K_d) = d$. Also $K_d \subseteq G \Rightarrow \chi(K_d) \leq \chi(G)$

$\Rightarrow d \leq \chi(G)$.

$\square$

Theorem 5.4.4

$\chi(G) \leq \Delta(G) + 1$. And (without proof) this is a tight upper bound on $\chi(G)$.

Proof

We use the Greedy Algorithm

We number the vertices $v = \{1, \dots, n\}$ and take $C = \{1, \dots, \infty\}$ be the set of colours.

Let v_k be the kth vertex where $k = 1, \dots, n$

Step 1: Colour vertex v_1 with colour c_1.

Step k for all $k \geq 2$: Look at neighbours of v_k inside $\{v_1, \dots, v_{k-1}\}$ and choose the smallest number r such that any $v_j \in \{v_1, \dots, v_{k-1}\}$ is not coloured by c_r.

This ensures that at most $\Delta(G)+1$ is used. $\square$

Corollary 5.4.5

$\chi(K_n) \leq n$

Proof

$\chi(K_n) \leq \Delta(K_n)+1 = (n-1)+1 = n$ $\square$

Theorem 5.4.6

$\binom{\chi(G)}{2} \leq E(G)$

Proof

Take any two colours, there must be at least one edge that has those two colours.

If not, we can repaint one colour with another and the graph would be a k-1 proper colouring. Hence $\binom{\chi(G)}{2} \leq E(G)$. $\square$

7.2 Inclusion & Exclusion

Example 7.2.1

How many numbers in [100] is divisible by 2 or 5?

Solution

$\frac{100}{2} + \frac{100}{5} - \frac{100}{10} = 60$

Example 7.2.2

How many numbers in [100] is divisible by 2 or 5?

Solution

$\frac{100}{2} + \lfloor \frac{100}{3} \rfloor + \frac{100}{5} - \lfloor \frac{100}{2\times 3} \rfloor - \lfloor \frac{100}{3\times 5} \rfloor - \frac{100}{2\times 5} + \lfloor \frac{100}{2\times 3\times 5} \rfloor$

Theorem 7.2.3

Let X be a finite set. $A_1, A_2, A_3 \subseteq X$. Then

$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cup A_2| - |A_1 \cup A_3| - |A_2 \cup A_3| + |A_1 \cup A_2 \cup A_3|$

Let X be a finite set. $A_1, A_2, A_3, A_4 \subseteq X$. Then

$|\cup A_i| = \sum |A_i| - \sum_{i<j} |A_i \cap A_j| + \sum_{i<j<k} |A_i \cap A_j \cap A_k| - \sum_{i<j<k<l} |A_i \cap A_j \cap A_k \cap A_l|$

Example 7.2.4

$P_{\text{black jack}} = \frac{\#\text{ black jack}}{\#\text{ total}} = \frac{4\times 16}{\binom{52}{2}}$

$P_{\text{no black jack}} = 1 - P_{\text{at least one black jack}}$

$= 1 - 2P_{\text{black jack}} + P_{\text{both}}$

$= 1 - 2P_{\text{black jack}} + \frac{4\times 16\times 3\times 15}{\binom{52}{2}\binom{50}{0}}$

$= 0.905$

Theorem 7.2.5 (Inclusion-Exclusion Formula)

X be a finite set of cardinality m. $A_1, \dots, A_m \subseteq X$

Then $\left| X \setminus \bigcup A_i \right| = \sum_{S \subseteq [m]} (-1)^{|S|} \left| \bigcap_{i \in S} A_i \right|$

If $S = \emptyset \subseteq [m]$, we have $\bigcap_{i \in S} A_i = X$.

Theorem 7.2.6

Let $m, n \in \mathbb{N}$. The number of surjections $f : [n] \to [m]$ is $\sum_{k=0}^{n} (-1)^k \binom{m}{k} (m-k)^n$

Proof

$P_i = \{ f : [n] \to [m] \mid \forall j \in [n],\ f(j) \neq i \}$, for $i = 1, \dots, n$

Then $\bigcup_{i=1}^{m} P_i = \{\text{non-surjective functions}\} = \{\text{all functions}\} - \{\text{surjective}\}$

We also have $|P_i| = (m-1)^n$

Take $S \subseteq [m]$, define $P_S = \{ f : [n] \to [m] \mid \forall j \in [n],\ P(j) \neq i\ \forall i \in S \}$

in other words, S is not the image

Then, $|P_S| = (m - |S|)^n$

Hence, the number of surjection is $\left| X \setminus \bigcup P_i \right| = \sum_{S \subseteq [n]} (-1)^{|S|} \left| \bigcap_{i \in S} P_i \right|$ [by Theorem 7.2.5]

$$= \sum_{k=0}^{m} \sum_{\substack{S \subseteq [n] \\ |S| = k}} (-1)^k \left| \bigcap_{i \in S} P_i \right|$$

$$= \sum_{k=0}^{m} (-1)^k (m-k)^n \sum_{\substack{S \subseteq [n] \\ |S| = k}} 1 \quad \left[\left| \bigcap_{i \in S} P_i \right| = \left| \{ f : [n] \to [m] \mid \forall j \in [n],\ f(j) \notin S \} \right| \right]$$

$$= \sum_{k=0}^{m} (-1)^k (m-k)^n \binom{m}{k}$$

$\square$

7.4 Derangements

Definition 7.4.1

A permutation is a bijection $\sigma:[n]\to[n]$

A derangement is a permutation p such that $p(i)\neq i \ \forall i\in[n]$ (there is no fixed points)

Theorem 7.4.2

The number of permutations $\sigma:[n]\to[n]$ such that i is a fixed point is $(n-1)!$

The number of permutations $\sigma:[n]\to[n]$ such that S is a fixed point where $S\subseteq[n]$ is $(n-|S|)!$

Theorem 7.4.3

The number of derangements of $[n]$ is $\sum_{k=0}^{n}(-1)^k\binom{n}{k}(n-k)!$

Proof

Let X be the set of permutations of $[n]$.

Let $A_i=\{\sigma:\sigma(i)=i\}$

Then we have $|X\setminus\bigcup A_i| = \sum_{S\subseteq[n]}(-1)^{|S|}\left|\bigcap_{i\in S}A_i\right|$

$$= \sum_{k=0}^{n}\sum_{\substack{S\subseteq[n]\\|S|=k}}(-1)^{|S|}\left|\bigcap_{i\in S}A_i\right|$$

$$= \sum_{k=0}^{n}\sum_{\substack{S\subseteq[n]\\|S|=k}}(-1)^{k}(n-k)! \qquad \left[\left|\bigcap_{i\in S}A_i\right| = \left|\{f:[n]\to[m] \mid \forall i\in S, A[i]=i\}\right|\right]$$

$$= \sum_{k=0}^{n}(-1)^k(n-k)!\sum_{\substack{S\subseteq[n]\\|S|=k}}1$$

$$= \sum_{k=0}^{n}(-1)^k(n-k)!\binom{n}{k}$$

$\square$

7.5 Euler phi-function

Definition 7.5.1

Let $n \in \mathbb{N}$. define $\varphi(n) = \{m \in [n] \mid \gcd(m,n) = 1\}$

Example 7.5.2

$\varphi(p) = p-1$ when p is prime

$\varphi(p^2) = p^2 - p$

$\varphi(4) = 2$

Theorem 7.4.2

Let $n \in \mathbb{N}$ such that $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ be its prime factorisation then $\varphi(n) = n\left(\frac{p_1-1}{p_1}\right) \cdots \left(\frac{p_k-1}{p_k}\right)$

Proof

Assume $k=3$.

Let $X = [n]$, $P_r = \{m \in [n] \mid p_r \text{ divides } m\}$

$$\varphi(n) = |X \setminus \{\cdots\}| = |X \setminus (P_1 \cup P_2 \cup P_3)| = |X| - (|P_1| + |P_2| + |P_3|) + (|P_1 \cap P_2| + |P_1 \cap P_3| + |P_2 \cap P_3|) - |P_1 \cap P_2 \cap P_3|$$

$$= n - \frac{n}{p_1} - \frac{n}{p_2} - \frac{n}{p_3} + \frac{n}{p_1 p_2} + \frac{n}{p_1 p_3} + \frac{n}{p_2 p_3} - \frac{n}{p_1 p_2 p_3}$$

$$= n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\left(1 - \frac{1}{p_3}\right)$$

$\square$

8.1 Generating Functions

Definition 8.1.1

A generating function of a sequence $(a_0, a_1, \dots)$ is the power series $\sum_{n=0}^{\infty} a_n x^n$.

Proposition 8.1.2

$$F(t) = 1 + t + \cdots = \frac{1}{1-t}$$

Proof

$$(1-t)(1+t+\cdots+t^n) = 1 - t^{n+1}$$

Hence, $\lim_{n\to\infty} (1-t)(1+t+\cdots+t^n) = 1$ for $|t|<1$

Therefore, $F(t) = \lim_{n\to\infty} 1+t+\cdots+t^n = \frac{1}{1-t}$ $\square$

Example 8.1.3

Let $f_n = f_{n-1} + f_{n-2}$ $\forall n \geq 3$, and $f(0)=1, f(1)=1$.

We then have $f_2 t^2 = f_1 t^2 + f_0 t^2$

$$f_3 t^3 = f_2 t^3 + f_1 t^3$$
$$\vdots$$

Let $F(t) = f_0 + f_1 t + f_2 t^2 + \cdots$

We have $f_2 t^2 + f_3 t^3 + \cdots = (f_1 t^2 + f_2 t^3 + \cdots) + (f_0 t^2 + f_1 t^3 + \cdots)$

And so $F(t) - f_0 - f_1 t = f_2 t^2 + f_3 t^3 + \cdots$

$$= (f_1 t^2 + f_2 t^3 + \cdots) + (f_0 t^2 + f_1 t^3 + \cdots)$$
$$= t(F(t) - f_0) + t^2 F(t)$$

$$\Rightarrow F(t) - tF(t) - t^2 F(t) = f_0 + f_1 t - f_0 t$$

$$\Rightarrow F(t)(1-t-t^2) = f_0 + (f_1 - f_0)t$$
$$= 1 + 0t$$
$$= 1$$

$$\Rightarrow F(t) = \frac{1}{1-t-t^2}$$

$$= \frac{1}{\sqrt{5}}\left(\frac{1}{1-\frac{1+\sqrt{5}}{2}t} - \frac{1}{1-\frac{1-\sqrt{5}}{2}t}\right)^2$$

$$= \frac{1}{\sqrt{5}}\left[\left(1+\frac{1+\sqrt{5}}{2}t+\left(\frac{1+\sqrt{5}}{2}t\right)^2+\cdots\right)-\left(1+\frac{1-\sqrt{5}}{2}t+\left(\frac{1-\sqrt{5}}{2}t\right)^2+\cdots\right)\right]$$

$$= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}-\frac{1-\sqrt{5}}{2}\right)t+\left(\left(\frac{1+\sqrt{5}}{2}\right)^2-\left(\frac{1-\sqrt{5}}{2}\right)^2\right)t^2+\cdots\right]$$

Hence, $f_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n-\left(\frac{1-\sqrt{5}}{2}\right)^n\right]$ $\square$

Proposition 8.1.4

Let $A(x)=\sum_{n=0}^{\infty}a_n x^n$, $B(x)=\sum_{n=0}^{\infty}b_n x^n$ be generating functions.

Then $C(x)=A(x)B(x)=\sum_{n=0}^{\infty}c_n x^n$ where $c_n=\sum_{k=0}^{n}a_k b_{n-k}$

Example 8.1.5

$\frac{x}{1-x}=1+x+\cdots=\sum_{n=1}^{\infty}x^n$ is the generating function for the number of ways to distribute n apples to 1 person.

where $a_n=1$ represents the way of distributing n apples to 1 person.

$\left(\frac{x}{1-x}\right)^k$ is the generating function for the number of ways to distribute n apples to k people.

Take $k=5$ as an example

$$\left(\frac{x}{1-x}\right)^5=\frac{x^5}{(1-x)^5}$$

$$=\frac{x^5}{4!}\frac{d^4}{dx^4}\frac{1}{1-x}$$

$$=\frac{x^5}{4!}\frac{d^4}{dx^4}\sum_{n=0}^{\infty}x^n$$

$$=\frac{x^5}{4!}\sum_{n=0}^{\infty}n(n-1)(n-2)(n-3)x^{n-4}$$

$$=\sum_{n=0}^{\infty}\frac{n(n-1)(n-2)(n-3)}{4}x^{n+1}$$

$$=\sum_{n=0}^{\infty}\binom{n}{4}x^{n+1}$$

$$=\sum_{n=1}^{\infty}\binom{n-1}{4}x^n$$ $\square$

which is exactly what we want

Example 8.1.6

How many ways can we distribute 20 fruits such that there is at least one apple, no more than 3 pears, and number of oranges is a multiple of 4.

Solution

Generating function for apple is $\frac{x}{1-x}$, only positive powers

Generating function for pears is $1+x+x^2+x^3$

Generating function for oranges $= \frac{1}{1-x^4}$

In total, we have $\frac{x}{1-x}(1+x+x^2+x^3)\frac{1}{1-x^4} = \frac{x}{(1-x)^3} = \frac{x}{2}\frac{d^2}{dx^2}\frac{1}{1-x} = \frac{x}{2}\sum_{n=0}^{\infty} n(n-1)x^{n-2} = \sum_{n=0}^{\infty}\binom{n+1}{2}x^n$

There are $C(n+1,2) = C(21,2) = 210$ ways.

Example 8.1.7

Example 8.6. Find the number of integer solutions to the equation

$$x_1 + x_2 + x_3 = n$$

($n \geq 0$ an integer) with $x_1 \geq 0$ even, $x_2 \geq 0$, and $0 \leq x_3 \leq 2$.

Solution

x_1: $\frac{1}{1-x^2}$

x_2: $\frac{1}{1-x}$

x_3: $1+x+x^2$

Hence $\frac{1+x+x^2}{(1-x)(1-x^2)}$

$$\frac{1+x+x^2}{(1+x)(1-x)^2} = \frac{A}{1+x} + \frac{B}{1-x} + \frac{C}{(1-x)^2}$$

for appropriate constants, A, B, and C. To find the constants, we clear the denominators, giving

$$1+x+x^2 = A(1-x)^2 + B(1-x^2) + C(1+x).$$

Equating coefficients on terms of equal degree, we have:

$$\begin{aligned} 1 &= A+B+C \\ 1 &= -2A+C \\ 1 &= A-B \end{aligned}$$

Solving the system, we find $A = 1/4$, $B = -3/4$, and $C = 3/2$. Therefore, our generating function is

$$\begin{aligned} &\frac{1}{4}\frac{1}{1+x} - \frac{3}{4}\frac{1}{1-x} + \frac{3}{2}\frac{1}{(1-x)^2} \\ =&\frac{1}{4}\sum_{n=0}^{\infty}(-1)^n x^n - \frac{3}{4}\sum_{n=0}^{\infty} x^n + \frac{3}{2}\sum_{n=0}^{\infty} n x^{n-1}. \end{aligned}$$

The solution to our question is thus the coefficient on x^n in the above generating function, which is

$$\frac{(-1)^n}{4} - \frac{3}{4} + \frac{3(n+1)}{2},$$

8.3 Newton's Binomial Theorem

Definition 8.3.1

Let $P: (\mathbb{R} \times \mathbb{Z}_{\geq 0}) \to \mathbb{R}$ is defined recursively as

1. $P(p,0) = 1$, for all $p \in \mathbb{R}$
2. $P(p,k) = pP(p-1,k-1)$

$\Rightarrow P(n,k) = \frac{n!}{(n-k)!}$, $\binom{n}{k} = \frac{P(n,k)}{k!}$

Theorem 8.3.2 (Newton's Binomial Theorem)

$$(1+x)^{\alpha} = \sum_{n \geq 0} \binom{\alpha}{n} x^n$$

$$(1+x)^{-1} = \frac{1}{1+x} = 1 - x + x^2 - x^3 \cdots$$

Theorem 8.3.3

The function $f(x) = \frac{1}{\sqrt{1-4x}}$ is the generating function of the number of lattice path from $(0,0)$ to (n,n)

Proof

We want to show that $\frac{1}{\sqrt{1-4x}} = \sum_{n=0}^{\infty} \binom{2n}{n} x^n$

By Theorem 8.3.2, $\frac{1}{\sqrt{1-4x}} = \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} (-4x)^n = \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} (-1)^n 2^{2n} x^n$

$$= \sum_{n=0}^{\infty} 2^{2n} (-1)^n \frac{(-\frac{1}{2})(-\frac{3}{2}) \cdots (-\frac{1}{2}-n+1)}{n!} x^n$$

$$= \sum_{n=0}^{\infty} 2^n \times (-1)^n \frac{(-1)\times(-3)\times\cdots(-2n+1)}{n!} x^n$$

$$= \sum_{n=0}^{\infty} 2^n \times \frac{1\times3\times\cdots\times(2n-1)}{n!} x^n$$

$$= \sum_{n=0}^{\infty} 2^n \times \frac{2n!}{n!\times2^n\times n!} x^n$$

$$= \sum_{n=0}^{\infty} \binom{2n}{n} x^n$$

$\square$

8.5 Integer Partitions

Definition 8.5.1

A partition of integers $n \geq 0$ is an unordered collection of not necessarily distinct natural numbers adding to n.

$P_n = \#$ partition of n

Example 8.5.2

$P(1)=1$: 1

$P(2)=2$: $2=1+1$

$P(3)=3$: $3=1+2=1+1+1$

$P(4)=5$: $4=1+3=1+1+2=2+2=1+1+1+1$

Theorem 8.5.3

The generating function for P_n is $\prod_{i=1}^{\infty} \frac{1}{1-x^k}$

Theorem 8.5.4

For each $n \geq 1$, the number of partitions of n into distinct parts is equal to the number of partitions of n into odd parts.

Proof

Let $D(x)$ be the generating functions for distinct parts. $O(x)$ for odd parts. We have

$$O(x) = \prod_{n=1}^{\infty} \frac{1}{1-x^{2n-1}}$$

$$D(x) = \sum_{n=0}^{\infty} \left(\sum_{\substack{m_1, \ldots \in \{0,1\} \\ m_1+2m_2+\cdots=n}} 1 \right) x^n$$

$$= \left(\sum_{m_1=0}^{1} x^{m_1} \right) \left(\sum_{m_2=0}^{1} x^{2m_2} \right) \cdots$$

$$= (1+x)(1+x^2)\cdots$$

$$= \prod_{n=1}^{\infty} (1+x^n)$$

To see $D(x)=O(x)$, we have

$$D(x) = \prod_{n=1}^{\infty} (1+x^n) = \prod_{n=1}^{\infty} \frac{1-x^{2n}}{1-x^n} = \frac{\prod_{n=1}^{\infty} (1-x^{2n})}{\prod_{n=1}^{\infty} (1-x^{2n-1}) \prod_{n=1}^{\infty} (1-x^{2n})} = \prod_{n=1}^{\infty} \frac{1}{(1-x^{2n-1})} = O(x)$$

8.6 Exponential Generating Functions

Definition 8.6.1

An exponential generating function of a sequence (a_n) is $\sum_{n=0}^{\infty} \frac{a_n x^n}{n!}$

Example 8.6.2

Given sequence $(1,1,1,\dots)$, the exponential generating function is $\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$

Given sequence $(1,2,4,\dots)$, the exponential generating function is $\sum_{n=0}^{\infty} \frac{2^n x^n}{n!} = e^{2x}$

Given two exponential generating functions $\sum_{n=0}^{\infty} \frac{2^n x^n}{n!}$, $\sum_{n=0}^{\infty} \frac{3^n x^n}{n!}$.

We have $$e^{5x} = e^{2x} \times e^{3x} = \left(\sum_{n=0}^{\infty} \frac{2^n x^n}{n!}\right)\left(\sum_{n=0}^{\infty} \frac{3^n x^n}{n!}\right)$$

Proposition 8.6.3

$A(x) = \sum_{n=0}^{\infty} \frac{a_n}{n!} x^n$ and $B(x) = \sum_{n=0}^{\infty} \frac{b_n}{n!} x^n$.

Then
$$\begin{aligned} A(x)B(x) &= \left(\sum_{n=0}^{\infty} \frac{a_n}{n!} x^n\right)\left(\sum_{n=0}^{\infty} \frac{b_n}{n!} x^n\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{m+k=n} \frac{a_k}{k!}\frac{b_m}{m!}\right) x^n \\ &= \sum_{n=0}^{\infty}\left(\sum_{m+k=n} \frac{a_k}{k!}\frac{b_m}{m!}\right) \frac{n!}{n!} x^n \\ &= \sum_{n=0}^{\infty}\left(\sum_{m+k=n} \frac{n!}{k!\,m!} a_k b_m\right) \frac{1}{n!} x^n \\ &= \sum_{n=0}^{\infty} \frac{\sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}}{n!} x^n \end{aligned}$$

which is the exponential generating function of $c_n = \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}$

Example 8.6.4

We have
$$\begin{aligned} e^{mx} \cdot e^{lx} &= \sum_{n=0}^{\infty} \frac{(mx)^n}{n!} \sum_{n=0}^{\infty} \frac{(lx)^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} \binom{n}{k} m^k l^{n-k}\right)\frac{x^n}{n!} \quad \text{[by Proposition 8.6.3]} \\ &= \sum_{n=0}^{\infty} \frac{(m+l)^n}{n!} x^n \end{aligned}$$

Example 8.6.5

The number of ternary strings of length n with even number of "0"s.

Let $F_0(x)$ be the EGF to "0"s

We have $F_0(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$

Note that $\frac{e^x + e^{-x}}{2} = \frac{1}{2}\left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots\right) + \frac{1}{2}\left(1 - \frac{x}{1!} + \frac{x^2}{2!} + \cdots\right)$

$= 1 + \frac{x^2}{2!} + \frac{x^4}{4!}$

$= F_0(x)$

We also have $F_1(x) = F_2(x) = e^x$

And $F(x) = \left(\frac{e^x + e^{-x}}{2}\right) e^x \cdot e^x = \frac{1}{2}(e^{3x} + e^x)$

$= \sum_{n=0}^{\infty} \left(\frac{3^n + 1}{2}\right) \frac{x^n}{n!}$

9.1 Recurrence Relations

Definition 9.1.1

A linear recurrence equation is a recurrence relation of the form:

$$c_0 a_{n+k} + c_1 a_{n+k-1} + \cdots + c_k a_n = g(n)$$

where c_i are constants, $c_0 \neq 0$

A homogenous linear recurrence equation if $g(n)=0$.

Definition 9.1.2

Let V be the space of functions $f: \mathbb{N} \to \mathbb{C}$

The advancement operator $A: V \to V$ such that for all n, $Af(n) = f(n+1)$

Equivalently, $A: \mathbb{C}^{\mathbb{N}} \to \mathbb{C}^{\mathbb{N}}$.

Note that you can repeatedly apply an operator

Example 9.1.3

Let f be the fibonacci sequence $f(0)=0, f(1)=1, f(2)=1, \ldots$

Then $Af = \{1, 1, 2, 3, \ldots\}$

$A^2 f = \{1, 2, 3, 5, \ldots\}$

Example 9.1.4

Find all solution to the linear recurrence equation

$a_n = 3a_{n-1}, n \geq 1$

Solution

Note that

$\{a_1, a_2, a_3, \dots\} - 3\{a_0, a_1, a_2, \dots\} = 0$

$\Rightarrow Af - 3f = 0$

$\Rightarrow (A-3)f = 0$

In general, $(c \cdot 3^n)_{n \geq 0}$ is a solution for any c.

Note, if given an initial condition, the solution is unique.

Remark 9.1.5

In general, for a homogenous linear recurrence equation

$a_n + c_{n-1}a_{n-1} + \dots + c_{n-k}a_{n-k} = 0$ for $n \geq k$

We find an advancement operator equation

$\{a_n, a_{n+1}, \dots\} + c_{n-1}\{a_{n-1}, a_n, \dots\} + c_{n-k}\{a_{n-k}, a_{n-k+1}, \dots\} = 0$

$(A^k + c_{n-1}A^{k-1} + \dots c_{n-k})\{a_{n-k}, a_{n-k+1}, \dots\} = 0$

Where we can factor $(A^k + c_{n-1}A^{k-1} + \dots c_{n-k})$ into linear group where the general solution will be solution to each of these linear group.

Example 9.1.6

For Fibonacci sequence $f_n = f_{n-1} + f_{n-2}$, $f(0) = 0$, $f(1) = 1$

$\{f_{n+2}\} - \{f_{n+1}\} - \{f_n\} = 0$

$(A^2 - A - 1)\{f_n\} = 0$

$(A - \varphi)(A - \psi)\{f_n\} = 0$

where $\varphi = \frac{1+\sqrt{5}}{2}$, $\psi = \frac{1-\sqrt{5}}{2}$

Set $\{g_n\} = (A - \varphi)\{f_n\}$, we have that $\{g_n\}$ solves $(A - \psi)\{g_n\} = 0$

So $\{g_n\} = \{c\psi^n\}$

This gives $(A - \varphi)\{f_n\} = \{c\psi^n\}$

$\Rightarrow \{f_n\} = \{c_1\psi^n + c_2\varphi^n\}$

By initial condition $c_1 = \frac{-1}{\sqrt{5}}$, $c_2 = \frac{1}{\sqrt{5}}$

So $f_n = -\frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n + \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n$

Example 9.1.7

Let f_n be the ternary string of length n which doesn't contain $\overline{02}$ as a substring

$f_1=3, f_2=8, f_n=3f_{n-1}-f_{n-2}$

$$\Rightarrow A^2f=3Af-f$$

$$\Rightarrow (A^2-3A+I)f=0$$

$$\Rightarrow f_n=c_1\left(\frac{3+\sqrt{5}}{2}\right)^n+c_2\left(\frac{3-\sqrt{5}}{2}\right)^n$$

$$\Rightarrow c_1=\frac{7\sqrt{5}}{10}+\frac{3}{2},\ c_2=-\frac{7\sqrt{5}}{10}+\frac{3}{2}$$

Example 9.1.8

Suppose $f_{n-4}=-2f_{n-3}+12f_{n-2}-14f_{n-1}+5f_n$

$$A^4f=-2A^3f+12A^2f-14Af+5f$$

$$\Rightarrow (A^4+2A^3-12A^2+14A-5)f=0$$

$$(A-1)^3(A-5)f=0$$

Then $f_n=c_1+c_2n+c_3n^2+c_4(-5)^n$

Theorem 9.1.9

The general solution to $(A-r)^kf=0$ for $r\neq 0, k\in\mathbb{N}$, is

$$f(n)=c_1r^n+c_2nr^n+\cdots+c_kn^{k-1}r^n$$

Example 9.1.10

Suppose $(A+2)(A-6)f=3^n$

The solution of such non-homogenous equation has the form $f=f_H+f_P$

where f_H is the solution to homogenous equation, f_P is a solution to non-homogenous equation

For $(A+2)(A-6)f_H=0$

$$\Rightarrow f_H=c_1(-2)^n+c_2 6^n$$

Let $f_p=c3^n$, let's find c

$$(A+2)(A-6)f_p=-5c3^{n+1}=3^n$$

$$\Rightarrow c=-\frac{1}{15}$$

Hence $f=c_1(-2)^n+c_2 6^n-\frac{1}{15}3^n$

Now, let $f_p=c6^n$, let's find c

$(A+2)(A-6)f_p=0\neq 3^n$

So there is no such c works

Theorem 9.1.11

For non-homogenous equation $P(A)f = (c_0A^k + c_1A^{k-1} \cdots + c_k)f = g$ with $c_0, c_k \neq 0$

Let $W \subseteq V$ be the subspace of all solutions to the equation $P(A)f = 0$

If f_p is a solution to $P(A)f = g$, then every solution to $P(A)f = g$ is of the form of $f = f_H + f_p$ where $f_H \in W$

Example 9.1.12

Let f_n be the number of faces being split by n lines. Note $f_1 = 2$.

Then $f_n - f_{n-1} = n+1$

$\Rightarrow (A-1)f = n+1$

Solving the homogenous equation we have $f_H(n) = c_1$

Guess $f_p(n) = c_2 + c_3 n$

Then $(A-1)f_p = c_3 n - c_3(n-1) = c_3$ X

Guess $f_p(n) = c_3 n + c_4 n^2$

Then $(A-1)f_p = c_3 + c_4 2n + c_4$

$\Rightarrow c_3 + c_4 = 1$

$\Rightarrow c_4 = \frac{1}{2}$

Hence $f_p(n) = \frac{1}{2}n + \frac{1}{2}n^2$

Thus, all solutions are written as $\frac{1}{2}n^2 + \frac{1}{2}n + c_1$

Since $f_1 = 2$, $c_1 = 0$. The solution is $\frac{1}{2}n^2 + \frac{1}{2}n$

Example 9.1.13

Suppose $(A-2)^2 f = 3^n + 2n$

We have $f_H(n) = c_1 2^n + n c_2 2^n$

Try $f_p(n) = c_3 3^n + c_4 n + c_5 n^2 + c_6$

Then $(A-2)^2 f = 3^n + 2n$

We have $c_5 = 0, c_4 = 2, c_3 = 1, c_6 = 4$

So $f_p(n) = 3^n + 2n + 4$

13.1 Network Flows

Definition 13.1

An orientated graph $G=(V,E)$ is a graph with vertices V and edges $E \subseteq V \times V$ of induced faces of vertices $x,y \in V$ s.t.

(1) If $(x,y) \in E$ then $(y,x) \notin E$

(2) $(x,x) \notin E$ for all $x \in V$.

A network is an oriented graph $G=(V,E)$ together with

(1) A choice of a source $S \in V$ and a sink $T \in V$ s.t. all edges incident to S are oriented away from S.
all edges incident to T are oriented towards T.

(2) A function of capacity $C: E \to \mathbb{Z}_{\geq 0}$

A flow on the network is a function $\phi: E \to \mathbb{Z}_{\geq 0}$ s.t. $\phi(e) \leq c(e)$ $\forall e \in E$ and satisfying the laws of conservation

(1) $\displaystyle\sum_{\substack{x \in V \\ \text{s.t. } (x,v) \in E}} \phi(x,v) = \sum_{\substack{x \in V \\ \text{s.t. } (v,x) \in E}} \phi(v,x) \quad \forall x \in V \setminus \{S,T\}$

(2) $\displaystyle\sum_{\substack{x \in V \\ \text{s.t. } (s,x) \in E}} \phi(s,x) = \sum_{\substack{x \in V \\ \text{s.t. } (x,t) \in E}} \phi(x,t) = \text{value}(\phi)$

Let $(G=(V,E), S, T, c)$ be a network

A cut is a partition $V = L \sqcup R$ s.t. $S \in L$ & $T \in R$

The capacity of a cut (L,R) is $\displaystyle C(L,R) = \sum_{\substack{x \in L \\ y \in R \\ (x,y) \in E}} c(x,y)$

Theorem 13.2

$\text{value}(\phi) \leq C(L,R) \quad \forall \text{cut}$

$\max(\text{value}(\phi)) = \min(C(L,R))$

Definition 13.2

Let $(G=(V,E), S, T, c)$ be a network and ϕ be a flow.

An edge $e \in E$ is used if $\phi(e) > 0$

full if $\phi(e) = c(e)$

has space capacity if $\phi(e) < c(e)$

empty if $\phi(e) = 0$

Definition 13.3

An augmenting path is a sequence of distinct vertices $P=(x_0,\dots,x_m)$ s.t. $x_0=s, x_m=t$

And for all $i=1,\dots,m$. either $(x_{i-1},x_i)\in E$ and it has space capacity. In this case it is called a forward edge.

or $(x_i,x_{i-1})\in E$ and it is used

Definition 13.4

Let $P=(x_0,\dots,x_m)$ be an augmenting path

Let $\delta_1=\min\{C(x_{i-1},x_i)-\phi(x_{i-1},x_i):(x_{i-1},x_i) \text{ is forward}\}>0$

Let $\delta_2=\min\{\phi(x_i,x_{i-1}):(x_i,x_{i-1}) \text{ is backward}\}>0$

And now, we set $\delta=\min\{\delta_1,\delta_2\}>0$

Proposition 13.5

Let $(G=(V,E),S,T,C)$ be a network, ϕ is a flow, and $P=(x_0,\dots,x_m)$ an augmenting path.

We modify ϕ by

$+\delta$ on each forward edge of P

$-\delta$ on each backward edge of P

Then, the result is a new flow $\hat{\phi}$ with $\text{value}(\hat{\phi})=\text{value}(\phi)+\delta>\text{value}(\phi)$