Step 1: Express the Variance of the Linear Combination

Let $Z = \alpha X + (1-\alpha)Y$. Then, the variance of Z is

$$\mathrm{Var}(Z) = \mathrm{Var}(\alpha X + (1-\alpha)Y).$$

Using the properties of variance, we expand this expression as follows:

$$\mathrm{Var}(Z) = \alpha^2 \mathrm{Var}(X) + (1-\alpha)^2 \mathrm{Var}(Y) + 2\alpha(1-\alpha)\mathrm{Cov}(X,Y).$$

Now, substituting $\mathrm{Var}(X) = \sigma_X^2$, $\mathrm{Var}(Y) = \sigma_Y^2$, and $\mathrm{Cov}(X,Y) = \sigma_{XY}$, we get:

$$\mathrm{Var}(Z) = \alpha^2 \sigma_X^2 + (1-\alpha)^2 \sigma_Y^2 + 2\alpha(1-\alpha)\sigma_{XY}.$$

Step 2: Expand the Terms

Let's expand each term:

1. **Expand $(1-\alpha)^2\sigma_Y^2$:**

$$(1-\alpha)^2\sigma_Y^2 = (\sigma_Y^2)(1 - 2\alpha + \alpha^2) = \sigma_Y^2 - 2\alpha\sigma_Y^2 + \alpha^2\sigma_Y^2.$$

2. **Expand $2\alpha(1-\alpha)\sigma_{XY}$:**

$$2\alpha(1-\alpha)\sigma_{XY} = 2\alpha\sigma_{XY} - 2\alpha^2\sigma_{XY}.$$

Substituting these expanded terms back into $\mathrm{Var}(Z)$, we get:

$$\mathrm{Var}(Z) = \alpha^2\sigma_X^2 + \sigma_Y^2 - 2\alpha\sigma_Y^2 + \alpha^2\sigma_Y^2 + 2\alpha\sigma_{XY} - 2\alpha^2\sigma_{XY}.$$

Now, we can combine the α^2 terms and the terms with α:

$$\mathrm{Var}(Z) = \alpha^2(\sigma_X^2 + \sigma_Y^2 - 2\sigma_{XY}) + \alpha(-2\sigma_Y^2 + 2\sigma_{XY}) + \sigma_Y^2.$$

Step 3: Differentiate with Respect to α and Set to Zero

To minimize $\mathrm{Var}(Z)$, we differentiate with respect to α and set the result to zero:

$$\frac{d}{d\alpha}\mathrm{Var}(Z) = 2\alpha(\sigma_X^2 + \sigma_Y^2 - 2\sigma_{XY}) + (-2\sigma_Y^2 + 2\sigma_{XY}) = 0.$$

Dividing by 2:

$$\alpha(\sigma_X^2 + \sigma_Y^2 - 2\sigma_{XY}) = \sigma_Y^2 - \sigma_{XY}.$$

Solving for α, we find:

$$\alpha = \frac{\sigma_Y^2 - \sigma_{XY}}{\sigma_X^2 + \sigma_Y^2 - 2\sigma_{XY}}.$$

Conclusion

Thus, we've shown that the value of α that minimizes $\mathrm{Var}(\alpha X + (1-\alpha)Y)$ is indeed

$$\alpha = \frac{\sigma_Y^2 - \sigma_{XY}}{\sigma_X^2 + \sigma_Y^2 - 2\sigma_{XY}}.$$