Step 1: Expanding $p_k(x)$

Since we need to maximize $p_k(x)$ to determine the class, we can focus on maximizing $\log(p_k(x))$, which is equivalent because the logarithm is a monotonically increasing function.

Taking the logarithm of $p_k(x)$:

$$\log(p_k(x)) = \log\left(\frac{\pi_k}{\sqrt{2\pi\sigma^2}}\right) - \frac{(x-\mu_k)^2}{2\sigma^2}$$

Expanding $\log(p_k(x))$:

$$\log(p_k(x)) = \log(\pi_k) - \frac{1}{2}\log(2\pi\sigma^2) - \frac{(x-\mu_k)^2}{2\sigma^2}$$

Since $\frac{1}{2}\log(2\pi\sigma^2)$ is a constant and does not depend on k, we can ignore it for comparison purposes. Thus, we aim to maximize:

$$\log(\pi_k) - \frac{(x-\mu_k)^2}{2\sigma^2}$$

Step 2: Expanding $(x-\mu_k)^2$

Now expand the term $\frac{(x-\mu_k)^2}{2\sigma^2}$:

$$\frac{(x-\mu_k)^2}{2\sigma^2} = \frac{x^2 - 2x\mu_k + \mu_k^2}{2\sigma^2} = \frac{x^2}{2\sigma^2} - \frac{x\mu_k}{\sigma^2} + \frac{\mu_k^2}{2\sigma^2}$$

Substituting this into $\log(p_k(x))$:

$$\log(p_k(x)) = \log(\pi_k) - \frac{x^2}{2\sigma^2} + \frac{x\mu_k}{\sigma^2} - \frac{\mu_k^2}{2\sigma^2}$$

Step 3: Simplifying to the Discriminant Function $\delta_k(x)$

Again, the term $-\frac{x^2}{2\sigma^2}$ does not depend on k, so we can ignore it for comparison purposes. Thus, we are left with:

$$\delta_k(x) = \frac{x\mu_k}{\sigma^2} - \frac{\mu_k^2}{2\sigma^2} + \log(\pi_k)$$

This expression $\delta_k(x) = \frac{x\mu_k}{\sigma^2} - \frac{\mu_k^2}{2\sigma^2} + \log(\pi_k)$ is known as the **discriminant function**. The Bayes classifier will assign x to the class k for which $\delta_k(x)$ is largest, as required.

Conclusion

We have shown that maximizing $p_k(x)$ is equivalent to maximizing the discriminant function:

$$\delta_k(x) = \frac{x\mu_k}{\sigma^2} - \frac{\mu_k^2}{2\sigma^2} + \log(\pi_k)$$