1. Using basic statistical properties of the variance, as well as single-variable calculus, derive (5.6). In other words, prove that $\alpha$ given by (5.6) does indeed minimize $\mathrm{Var}(\alpha X + (1-\alpha)Y)$.

$$\alpha = \frac{\sigma_Y^2 - \sigma_{XY}}{\sigma_X^2 + \sigma_Y^2 - 2\sigma_{XY}}, \qquad (5.6)$$

- 假設 Z=αX+(1−α)Y，希望最小化 Z 的變異數，也就是 Var(Z)

又 Var(Z)=Var(αX+(1−α)Y)

- 利用 Variance 特性展開上述式子得：

  Var(Z)=α2Var(X)+(1−α)2Var(Y)+2α(1−α)Cov(X,Y)

令 $\sigma_X^2 = \mathrm{Var}(X)$、$\sigma_Y^2 = \mathrm{Var}(Y)$、以及 $\sigma_{XY} = \mathrm{Cov}(X, Y)$

- 帶入 Var(Z)：

$$\mathrm{Var}(Z) = \alpha^2\sigma_X^2 + (1-\alpha)^2\sigma_Y^2 + 2\alpha(1-\alpha)\sigma_{XY}$$

- 展開式子：

$$\mathrm{Var}(Z) = \alpha^2\sigma_X^2 + (\alpha^2 - 2\alpha + 1)\sigma_Y^2 + 2\alpha(1-\alpha)\sigma_{XY}$$

- 簡化式子：

$$\mathrm{Var}(Z) = \alpha^2(\sigma_X^2 + \sigma_Y^2 - 2\sigma_{XY}) + 2\alpha(\sigma_{XY} - \sigma_Y^2) + \sigma_Y^2$$

- 為了找到使 Var(Z)\text{Var}(Z)Var(Z) 最小的 α 值，我們對 α 求導數並令其為零：

$$\frac{d}{d\alpha}\mathrm{Var}(Z) = 2\alpha(\sigma_X^2 + \sigma_Y^2 - 2\sigma_{XY}) + 2(\sigma_{XY} - \sigma_Y^2) = 0$$

- 得出：

$$\alpha = \frac{\sigma_Y^2 - \sigma_{XY}}{\sigma_X^2 + \sigma_Y^2 - 2\sigma_{XY}}$$