1. Using basic statistical properties of the variance, as well as single-variable calculus, derive (5.6). In other words, prove that α given by (5.6) does indeed minimize Var(αX + (1 − α)Y).

$$\alpha = \frac{\sigma_Y^2 - \sigma_{XY}}{\sigma_X^2 + \sigma_Y^2 - 2\sigma_{XY}}, \tag{5.6}$$

Let $Z = \alpha X + (1-\alpha) Y$

$$\Rightarrow \operatorname{Var}(Z) = \operatorname{Var}[\alpha X + (1-\alpha) Y]$$

$$= \alpha^2 \operatorname{Var}(X) + (1-\alpha)^2 \operatorname{Var}(Y) + 2\alpha(1-\alpha) \operatorname{Cov}(X,Y)$$

$$= \alpha^2 \sigma_X^2 + (1-2\alpha+\alpha^2)\sigma_Y^2 + 2\alpha(1-\alpha)\sigma_{XY}$$

$$= (\sigma_X^2 + \sigma_Y^2 - 2\sigma_{XY})\alpha^2 + (-2\sigma_Y^2 + 2\sigma_{XY})\alpha + \sigma_Y^2$$

To minimize Var(Z), take $\frac{d}{d\alpha} \operatorname{Var}(Z) = 0$

$$\frac{d}{d\alpha} \operatorname{Var}(Z) = 2\alpha(\sigma_X^2 + \sigma_Y^2 - 2\sigma_{XY}) + (-2\sigma_Y^2 + 2\sigma_{XY}) = 0$$

$$\Rightarrow \alpha(\sigma_X^2 + \sigma_Y^2 - 2\sigma_{XY}) = \sigma_Y^2 - \sigma_{XY}$$

$$\Rightarrow \alpha = \frac{\sigma_Y^2 - \sigma_{XY}}{\sigma_X^2 + \sigma_Y^2 - 2\sigma_{XY}}$$