HW 1112

Q1

$$f_{u_1}(u_1) = \begin{cases} 1, & 0 \le u_1 \le 1 \\ 0, & \text{otherwise} \end{cases}$$

$$f_{u_2}(u_2) = \begin{cases} 1, & 0 \le u_2 \le 1 \\ 0, & \text{otherwise} \end{cases}$$

$$f_{u_1,u_2}(u_1,u_2) = f_{u_1}(u_1) f_{u_2}(u_2) = 1$$

for $0 \le u_1 < 1$, $0 \le u_2 \le 1$

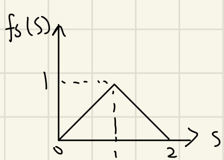
$$f_S(s) = \int_{-\infty}^{\infty} f_{u_1}(u_1) f_{u_2}(s-u_1)\, du_1$$

$$f_S(s) = \int_0^s 1\, du_1 = s \quad \text{for } 0 \le s < 1$$

$$f_S(s) = \int_{s-1}^1 1\, du_1 = 1-(s-1) = 2-s \quad \text{for } 1 \le s \le 2$$

$$f_S(s) = 0 \quad \text{for } s<0 \;\; s>2$$

Ans



Q2

$$P(N=k) = \frac{\lambda^k e^{-\lambda}}{k!}$$

$$N = N_1 + N_2$$

$$M_X(t) = E[e^{tX}]$$

$$M_N(t) = E[e^{tN}] = e^{\lambda(e^t-1)}$$

MGF of N_1: $e^{\lambda_1(e^t-1)}$

MGF of N_2: $e^{\lambda_2(e^t-1)}$

MGF of $N = N_1 + N_2$: $e^{\lambda_1(e^t-1)} e^{\lambda_2(e^t-1)}$

$$= e^{(\lambda_1+\lambda_2) e^t - 1)}$$

Since $N = N_1 + N_2$, $N \sim \text{Poisson}(\lambda_1+\lambda_2)$

Q 3

$$p(Z=z)=\begin{cases}\frac{1}{9} & z=0\\ \frac{2}{9} & z=1\\ \frac{1}{3} & z=2\\ \frac{2}{9} & z=3\\ \frac{1}{9} & z=4\end{cases}$$

Q 4

$$Z = XY \Rightarrow X = \frac{Z}{Y}$$

Jacobian determinant : $\left|\frac{\partial(X,Y)}{\partial(Z,Y)}\right| = \frac{1}{|Y|}$

pdf of Z : $f_Z(z) = \int_{-\infty}^{\infty} f\left(y, \frac{z}{y}\right) \frac{1}{|y|}\, dy$

Q 5

$f_{X,Y}(x,y) = f_X(x)\, f_Y(y) = 1$ for $0 \le x \le 1$, $0 \le y \le 1$

CDF of Z

$$F_Z(z) = P\left(\frac{X}{Y} \le z\right) = P(X \le zY)$$

$$= \int_0^1 \int_0^{\min(zy,1)} 1 \, dx\, dy$$

$$F_Z(z) = \begin{cases} \frac{z}{2} & \text{for } 0 \le z \le 1 \\ 1 - \frac{1}{2z^2} & \text{for } z > 1 \end{cases}$$

$$f_Z(z) = \frac{d}{dz} F_Z(z)$$

$$\Rightarrow f_Z(z) = \begin{cases} \frac{1}{2} & \text{for } 0 \le z \le 1 \\ \frac{1}{2z^2} & \text{for } z > 1 \end{cases}$$

Q 6

$$f_{X,Y,Z}(x,y,z) = \frac{1}{(2\pi\sigma^2)^{\frac{3}{2}}} e^{-\frac{x^2+y^2+z^2}{2\sigma^2}}$$

$$dx\,dy\,dz = r^2 \sin\phi \, dr\, d\phi\, d\theta$$

$$f_{R,\Phi,\Theta}(r,\phi,\theta) = \frac{1}{(2\pi\sigma^2)^{\frac{3}{2}}} e^{\frac{-r^2}{2\sigma^2}} r^2 \sin\phi$$

$$f_R(r) = \int_0^{2\pi}\int_0^{\pi} f_{R,\Phi,\Theta}(r,\phi,\theta) \sin\phi \, d\phi\, d\theta$$

$$= \frac{2r^2}{(2\pi\sigma^2)^{\frac{3}{2}}} e^{-\frac{r^2}{2\sigma^2}}$$

$$f_\Phi(\phi) = \int_0^{\infty}\int_0^{2\pi} f_{R,\Phi,\Theta}(r,\phi,\theta)\, d\theta\, dr$$

$$= \frac{1}{(2\pi\sigma^2)^{\frac{3}{2}}} 2\pi \sin\phi \frac{\sqrt{\pi}}{2} \sigma^3 = \frac{1}{2}\sin\phi$$

$$f_R(r) = \frac{2r^2}{(2\pi\sigma^2)^{\frac{3}{2}}} e^{-\frac{r^2}{2\sigma^2}}$$

$$f_\Theta(\theta) = \frac{1}{2\pi} \quad 0 \le \theta \le 2\pi$$

$$f_\Phi(\phi) = \frac{1}{2}\sin\phi \quad 0 \le \phi \le \pi$$