Hw 1126 Q1

Mgf :

$p(x=1)=p \quad p(X=0)=1-p$

$\hookrightarrow e^{tx}=e^{t} \quad \hookrightarrow e^{t\cdot 0}=1$

$\hookrightarrow E[e^{tx}]=pe^{t}+(1-p)$

$$M_x(t)=E[e^{tx}]=p\cdot e^{t\cdot 1}+(1-p)e^{t\cdot 0}=1-p+pe^{t}$$

Mean of Bernoullis Mgf :

$$M_x'(t)=pe^{t}$$

$$E[X]=M_x'(0)=p$$

$$Var(X)=E[X^2]-E[X]^2$$

because $X^2=X$ $(X=0,1)$ so $E[X^2]=E[X]=p$

$$Var(X)=p-p^2=p(1-p)$$

HW 1126 Q2

$$f_x(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

$$M_x(t) = E[e^{tx}] = \int_0^\infty e^{tx} \cdot \lambda e^{-\lambda x} dx = \frac{\lambda}{\lambda - t}, \quad t < \lambda$$

$$E[X] = M'(t) = \frac{1}{\lambda}$$

$$Var[X] = M''(0) = \frac{1}{\lambda^2}$$

$$M_x(t) = \int_0^\infty e^{tx} \lambda e^{-\lambda x} dx = \lambda \int_0^\infty e^{x(t-\lambda)} dx$$

令 $p = x(t-\lambda)$

$dp = (t-\lambda)dx$

$$\int \frac{e^p}{t-\lambda} dx$$

$$\lambda \left[\frac{e^{x(t-\lambda)}}{t-\lambda} \right]_0^\infty \Rightarrow t < \lambda$$

HW 1126 Q3

$$M_x(t) = E[e^{tx}] = \int_0^\infty e^{tx} \cdot \frac{\lambda^k x^{k-1} e^{-\lambda x}}{\Gamma(k)} dx$$

$$= \frac{\lambda^k}{\Gamma(k)} \int_0^\infty x^{k-1} e^{-(\lambda - t)x} dx = \left(\frac{\lambda}{\lambda - t}\right)^k, \quad t < \lambda$$

$$M'_x(t) = k\left(\frac{\lambda}{\lambda - t}\right)^{k-1} \cdot \frac{\lambda}{(\lambda - t)^2} = \frac{k}{\lambda} = E[X]$$

$$Var(X) = M''_x(0) = k(k-1)\frac{\lambda^2}{\lambda^3} + k\frac{2\lambda}{\lambda^3} = \frac{k}{\lambda^2}$$

$$\int_0^\infty e^{tx} \cdot \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} dx$$

$$= \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^\infty e^{x(t-\lambda)} \cdot x^{\alpha-1} dx = \text{''} \int_0^\infty e^p \left(\frac{p}{t-\lambda}\right)^{\alpha-1} \frac{1}{t-\lambda} dp$$

$$\rightarrow \text{''} \int_0^\infty e^p p^{\alpha-1} \left(\frac{1}{t-\lambda}\right)^\alpha$$

$p = x(t-\lambda)$ $\quad x = \frac{p}{\lambda}$

$dp = (t-\lambda)dx$

$$\boxed{\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t}}$$

HW1126 Q4

$$MGF = \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} = e^{\mu t + \frac{\sigma^2 t^2}{2}}$$

$$E(X) = M'_X(t) = \mu e^{\mu t + \frac{\sigma^2 t^2}{2}} + \sigma^2 t e^{\mu t + \frac{\sigma^2 t^2}{2}} = \mu$$

$$Var(X) = M''_X(0) - (M'_X(0))^2 = (\mu^2 + \sigma^2) - \mu^2 = \sigma^2$$

HW1126 Q5

$f_Y: \int_0^y 2 \, dy$

$f_X: \int_0^1 2 \, dx$

a.

$$Cov(X,Y) = E[XY] - E[X]E[Y]$$

$$E[X] = \int_0^1 [x^2]_0^y \, 2 \, dy = \frac{1}{3}$$

$$E[Y] = \int_0^1 \int_0^y x \cdot y \cdot f(x,y) \, dx \, dy = \int_0^1 2y \left[\frac{x^2}{2}\right]_0^y dy = \frac{2}{3}$$

$$Cov(X,Y) = E[XY] - E[X]E[Y] = \frac{1}{36}$$

$$Var(X) = E[X^2] - E[X]^2 = \frac{1}{18}$$

$$Var(Y) = E[Y^2] - E[Y]^2 = \frac{1}{18}$$

$$Cor(X,Y) = \frac{1}{2}$$

b.

$$f(x|y) = \frac{1}{y}$$

$$E(X|Y) = \int_0^y \frac{x}{y} \, dx = \frac{y}{2}$$

$$f(Y|X) = \frac{f_{XY}(x,y)}{f_X(x)} = \frac{1}{1-X}$$

$$E(Y|X) = \int_X^1 y \frac{1}{1-X} \, dy = \frac{1+X}{2}$$

c.

$$E[X|Y=y] = \frac{y}{2}$$

$$f_{E[X|Y]}(z) = 2 \quad 0 \le z \le 1$$

$$E[Y|X=x] = \frac{1-x^2}{2(1-x)}$$

d.

$$\hat{Y} = a + bX$$

$$b = \frac{Cov(X,Y)}{Var(X)} = r\frac{\sigma_Y}{\sigma_X}$$

$$\frac{Cov(x,y)}{\sqrt{Var(x)Var(y)}} \cdot \frac{\sqrt{Var(y)}}{\sqrt{Var(x)}}$$

$$\hat{Y} = \frac{11}{18} + \frac{X}{6}$$

e.

$$MSE = Var(Y) - Var[Y|X]$$

$$= \frac{11}{72}$$

Hw 1126 Q7

a.

$\mathrm{Cov}(X,Y) = E[XY] - E[X]E[Y]$

$f_X(x) = \int_x^\infty e^{-y}\,dy = e^{-x}$ for $0 \le x \le \infty$

$f_Y(y) = \int_0^y e^{-y}\,dx = y e^{-y}$ for $0 \le y \le \infty$

$E[X] = \int_0^\infty x e^{-x}\,dx = 1$ $\quad \int x e^{-x}$

$u\ x \quad dv\ e^{-x}$

$du\ 1 \quad v\ -e^{-x}$

$-xe^{-x} + \int e^{-x}$

$= [-xe^{-x} - e^{-x}]_0^\infty$

$E[X^2] = \int_0^\infty x^2 e^{-x}\,dx = 2$

$E[Y] = \int_0^\infty y^2 e^{-y}\,dy = 2$

$E[Y^2] = \int_0^\infty y^3 e^{-y}\,dy = 6$

$E[XY] = \int_0^\infty \int_0^y xy e^{-y}\,dx\,dy = 3$

$\mathrm{Cov}(X,Y) = 1 \qquad \mathrm{Corr}(X,Y) = \frac{\sqrt{2}}{2}$

b.

$f(x|y) = \frac{e^{-y}}{ye^{-y}} = \frac{1}{y}$

$E(X|Y=y) = \frac{y}{2} = \int_0^y x\frac{1}{y}\,dx = \frac{y}{2}$

$E(Y|X=x) = 1+X = \int_0^\infty$

C.

$f_{E[X|Y]} = 4ye^{-2y} \quad 0 \le y \le \infty$

$f_{E[Y|X]} = e^{1+x} \quad 0 \le x \le \infty$

$f_y(y) = \int_0^y e^{-y}\,dx$

$f_X(x) = \int_x^\infty e^{-y}\,dy$

HW1126 Q8

$f(x) = 2x$

$M_X(t) = E[e^{tx}] = \int_0^1 2x e^{tx} = 2\left(\frac{e^t}{t} - \frac{e^t - 1}{t^2}\right)$

① $E(X) = M_X'(0) = \frac{d}{dt} 2\left(\frac{e^t}{t} - \frac{e^t - 1}{t^2}\right) = \frac{2}{3}$

$\downarrow$

$\int_0^1 x\, 2x = \frac{2}{3}$

② $E(X^2) = \int_0^1 2x^3 = \frac{1}{2}$

$M''(t) = \lim_{t \to 0} 2 \frac{3e^t t^2 + e^t t^3 - 3e^t t^2}{4t^3} = \frac{1}{2}$

HW1126 Q9

$X_i \sim N(\mu_i, \sigma_i^2)$

$M_X(t) = E[e^{tX}] = e^{\mu_i t + \frac{1}{2}\sigma_i^2 t^2}$

$M_Y(t) = E\left[e^{t \sum_{i=1}^{n} \alpha_i X_i}\right] = \prod_{i=1}^{n} M_X(\alpha_i t)$

$= \prod_{i=1}^{n} e^{t \alpha_i \mu_i + \frac{t^2 \alpha_i^2 \sigma^2}{2}}$

$Y \sim N\left(\sum_{i=1}^{n} \alpha_i \mu_i, \sum_{i=1}^{n} \alpha_i^2 \sigma_i^2\right)$

$E[Y] = E\left[\sum_{i=1}^{n} \alpha_i X_i\right] = \sum_{i=1}^{n} \alpha_i \mu_i$

$Var(Y) = \sum_{i=1}^{n} \alpha_i^2 \sigma_i^2$