HW 1210 Q1

$X \sim F_{n,m}$, $X^{-1} \sim F_{m,n}$

$X = \frac{Y_1/n}{Y_2/m} \Rightarrow Y_1 \sim \chi_n^2$, $Y_2 \sim \chi_m^2$

$X^{-1} = \frac{Y_2/m}{Y_1/n} = \frac{Y_2}{Y_1} \times \frac{m}{n} \sim F_{n,m}$

$f_X(x) = \frac{x^{\alpha-1} e^{-\frac{x}{\beta}}}{\beta^{\alpha}\Gamma(\alpha)} \rightarrow \int_0^\infty e^{-t} t^{\alpha-1} dt$

$z^2 = \frac{1}{2\pi} e^{-}$

$$\boxed{f_X(x) = \frac{x^{\frac{n}{2}-1} e^{-\frac{x}{2}}}{2^{\frac{n}{2}} \Gamma(\frac{n}{2})} \quad x \ge 0}$$

$Z^2 \sim \chi_n^2$

$f_X(x) = \frac{x^{-1} e^{-\frac{x}{2}}}{2\Gamma(1)}$

HW1210 Q2

$Z \sim N(0,1)$

$T = \frac{Z}{\sqrt{V/n}}$ $V \sim \chi_n^2$

$F_{1,n} = \frac{U/1}{V/n}$ for $U \sim \chi_1^2$, $V \sim \chi_n^2$

$T^2 = \frac{Z^2}{V/n} \Rightarrow U = Z^2 \sim \chi_1^2$, $V \sim \chi_n^2 = F_{1,n}$

HW 1210 Q3

t pdf : $p(x) = \frac{(1+x^2/r)^{-\frac{r+1}{2}}}{B(r/2, 1/2)}$

$\int_0^\infty t^{z-1} e^{-t} dt = \Gamma(z)$

Cauchy pdf : $p(x) = \frac{1}{\pi(1+x^2)}$

$$f(t) = \frac{\Gamma[(n+1)/2]}{\sqrt{n\pi}\,\Gamma(n/2)}\left(1+\frac{t^2}{n}\right)^{-(n+1)/2}.$$

$= \frac{1}{\sqrt{v}\, B(\frac{1}{2}, \frac{v}{2})} \left(1 + \frac{t^2}{v}\right)^{-\frac{(v+1)}{2}}$

if $r=1$, t-distribution : $\frac{1}{B(1/2, 1/2)(1+x^2)} = \frac{1}{\pi(1+x^2)}$

$$f(x) = \frac{1}{2^{k/2}\Gamma(k/2)} x^{k/2-1} e^{-x/2}$$

$F \Rightarrow \frac{W/n_1}{V/n_2}$

$k=2$ $f(x) = \frac{e^{-\frac{x}{2}}}{2\Gamma(1)} = \frac{e^{-\frac{x}{2}}}{2}$

$X \Rightarrow e^{-x}$

HW 1210 Q4

$f_X(x) = e^{-x}$ x $f_Y(y) - e^{-xy}$ $Z = \frac{X}{Y}$

$f_Z(z) = \int_0^\infty |y| f_X(zy) f_Y(y) dy = \int_0^\infty y e^{-zy} e^{-y}$

$= \int_0^\infty y e^{-y(1+z)}$

$\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} dt$

$\Gamma(2) = \int_0^\infty e^{-t} t$

$e^{-x} \sim \chi_2^2$

HW 12 10 Q5

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$$

$$\frac{(n-1)s^2}{\sigma^2} \sim \chi^2_{n-1}$$

$$E\left(\frac{(n-1)s^2}{\sigma^2}\right) = E(\chi^2_{n-1}) = n-1$$

$$\frac{n-1}{\sigma^2}E(s^2) = n-1$$

$$E(s^2) = \sigma^2$$

$$\frac{(n-1)s^2}{\sigma^2} \sim \chi^2_{n-1}$$

$$\Rightarrow \mathrm{var}\left(\frac{(n-1)s^2}{\sigma^2}\right) = \mathrm{var}(\chi^2_{n-1}) = 2(n-1)$$

$$\frac{(n-1)^2}{\sigma^4}\mathrm{var}(s^2) = 2(n-1)$$

$$\mathrm{var}(s^2) = \frac{2\sigma^4}{(n-1)}$$

HW 12 10 Q6

a.

$$\frac{1}{3}\theta \qquad -\frac{4}{3}\theta \qquad -\theta$$

$$E(X) = 0\times \cdot\cdot + 1\times\frac{1}{3}\theta + 2\times\frac{2}{3}(1-\theta) + 3\times\frac{1}{3}(1-\theta)$$

$$= \frac{1}{3}\theta + \frac{4}{3} - \frac{4}{3}\theta + 1 - \theta = \frac{7}{3} - 2\theta = \frac{3}{2}$$

$$= \bar{X} = 1.5 = \frac{3}{2}$$

$$\theta = \frac{5}{12}$$

b.

$$SE = \frac{\sigma}{\sqrt{n}}$$

$X=0 \Rightarrow \frac{10}{36}$

$X=1 \Rightarrow \frac{5}{36}$

$X=2 \Rightarrow \frac{14}{36}$

$X=3 \Rightarrow \frac{7}{36}$

$E(X) = 1.5$

$SE = 0.3456$

(c)

$$L(\theta) = \left(\frac{2}{3}\theta\right)^2 \times \left(\frac{1}{3}\theta\right)^3 \times \left(\frac{2}{3}(1-\theta)\right)^3 \times \left(\frac{1}{3}(1-\theta)\right)^2 = \theta^5(1-\theta)^5\frac{2^5}{3^{10}}$$

$$\log L(\theta) = 5\log(\theta) + 5\log(1-\theta) + C$$

$$\frac{\partial}{\partial\theta}\log L(\theta) = \frac{5}{\theta} + \frac{5}{1-\theta} = 0$$

$$\theta = \frac{1}{2}$$

(d)

$$SE = \frac{\sigma}{\sqrt{n}} \doteq 0.158$$

$0 \Rightarrow \frac{1}{3}$

$1 \Rightarrow \frac{1}{6}$

$2 \Rightarrow \frac{1}{3}$

$3 \Rightarrow \frac{1}{6}$

$$\frac{1}{6} + \frac{4}{3} + \frac{9}{6} - \left(\frac{8}{6}\right)^2$$

(e)

$$\hat{\theta}_{MAP} = 0.5$$

HW 1210 Q 7

$p(X=1)=\theta \quad p(X=2)=1-\theta$

$X_1=1 \quad X_2=2 \quad X_3=2$

a.

$1\times\theta + 2(1-\theta) = \frac{5}{3}$

$\theta=\frac{1}{3}$

b.

$L(\theta) = \theta \times (1-\theta)^2$

c.

$\log l(\theta) = \log(\theta) + 2\log(1-\theta)$

$\frac{\partial}{\partial\theta}\log l(\theta) = \frac{1}{\theta} - \frac{2}{1-\theta} = 0$

$1-\theta=2\theta \ni \theta=\frac{1}{3}$

d.

Beta $(2,3)$

HW 1210 Q 8

a. $S^2 = \frac{2}{9(3\alpha+1)}$

$\alpha = \frac{1}{3}\left(\frac{2}{9S^2}-1\right)$

b.

log likelihood : $\sum_{i=1}^{n} \log(f(x_i|\alpha))$

$\sum_{i=1}^{n}\{\psi(2\alpha) - 2\psi(\alpha) + \log(x_i) + \log(1-x_i)\}=0$

$\downarrow$

digamma function

$\Gamma(\alpha) = \int_0^\infty e^{-t}\, t^{\alpha-1}$

likelihood f : $\prod_{i=1}^{n} \frac{\Gamma(3\alpha)}{\Gamma(\alpha)\Gamma(2\alpha)} x_i^{\alpha-1}(1-x_i)^{2\alpha-1}$

$\log l$: $n\left[\ln(\Gamma(3\alpha)) - \ln(\Gamma(\alpha)) - \ln(\Gamma(2\alpha)) + (\alpha-1)\log(x_i) + (2\alpha-1)\log(1-x_i)\right]' = 0$