4.29 Consider a model for household expenditure as a function of household income using the 2013 data from the Consumer Expenditure Survey, *cex5_small*. The data file *cex5* contains more observations. Our attention is restricted to three-person households, consisting of a husband, a wife, plus one other. In this exercise, we examine expenditures on a staple item, food. In this extended example, you are asked to compare the linear, log-log, and linear-log specifications.

a. Calculate summary statistics for the variables: *FOOD* and *INCOME*. Report for each the sample mean, median, minimum, maximum, and standard deviation. Construct histograms for both variables. Locate the variable mean and median on each histogram. Are the histograms symmetrical and “bell-shaped” curves? Is the sample mean larger than the median, or vice versa? Carry out the Jarque–Bera test for the normality of each variable.

b. Estimate the linear relationship $FOOD = \beta_1 + \beta_2 INCOME + e$. Create a scatter plot *FOOD* versus *INCOME* and include the fitted least squares line. Construct a 95% interval estimate for β_2. Have we estimated the effect of changing income on average *FOOD* relatively precisely, or not?

c. Obtain the least squares residuals from the regression in (b) and plot them against *INCOME*. Do you observe any patterns? Construct a residual histogram and carry out the Jarque–Bera test for normality. Is it more important for the variables *FOOD* and *INCOME* to be normally distributed, or that the random error *e* be normally distributed? Explain your reasoning.

d. Calculate both a point estimate and a 95% interval estimate of the elasticity of food expenditure with respect to income at $INCOME = 19, 65$, and 160, and the corresponding points on the fitted line, which you may treat as not random. Are the estimated elasticities similar or dissimilar? Do the interval estimates overlap or not? As *INCOME* increases should the income elasticity for food increase or decrease, based on Economics principles?

e. For expenditures on food, estimate the log-log relationship $\ln(FOOD) = \gamma_1 + \gamma_2 \ln(INCOME) + e$. Create a scatter plot for ln(*FOOD*) versus ln(*INCOME*) and include the fitted least squares line. Compare this to the plot in (b). Is the relationship more or less well-defined for the log-log model relative to the linear specification? Calculate the generalized R^2 for the log-log model and compare it to the R^2 from the linear model. Which of the models seems to fit the data better?

f. Construct a point and 95% interval estimate of the elasticity for the log-log model. Is the elasticity of food expenditure from the log-log model similar to that in part (d), or dissimilar? Provide statistical evidence for your claim.

g. Obtain the least squares residuals from the log-log model and plot them against ln(*INCOME*). Do you observe any patterns? Construct a residual histogram and carry out the Jarque–Bera test for normality. What do you conclude about the normality of the regression errors in this model?

h. For expenditures on food, estimate the linear-log relationship $FOOD = \alpha_1 + \alpha_2 \ln(INCOME) + e$. Create a scatter plot for *FOOD* versus ln(*INCOME*) and include the fitted least squares line. Compare this to the plots in (b) and (e). Is this relationship more well-defined compared to the others? Compare the R^2 values. Which of the models seems to fit the data better?

i. Construct a point and 95% interval estimate of the elasticity for the linear-log model at $INCOME = 19, 65$, and 160, and the corresponding points on the fitted line, which you may treat as not random. Is the elasticity of food expenditure similar to those from the other models, or dissimilar? Provide statistical evidence for your claim.

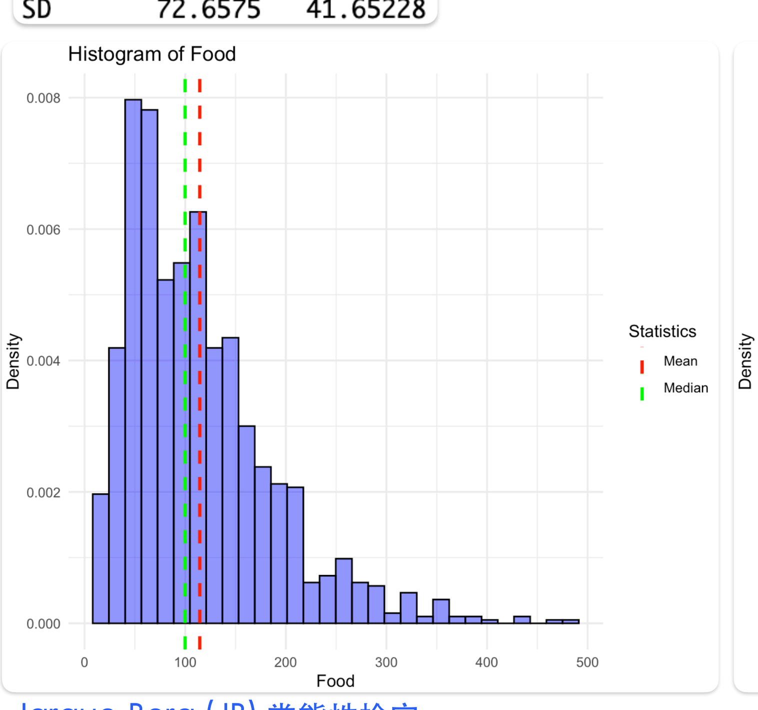
j. Obtain the least squares residuals from the linear-log model and plot them against ln(*INCOME*). Do you observe any patterns? Construct a residual histogram and carry out the Jarque–Bera test for normality. What do you conclude about the normality of the regression errors in this model?

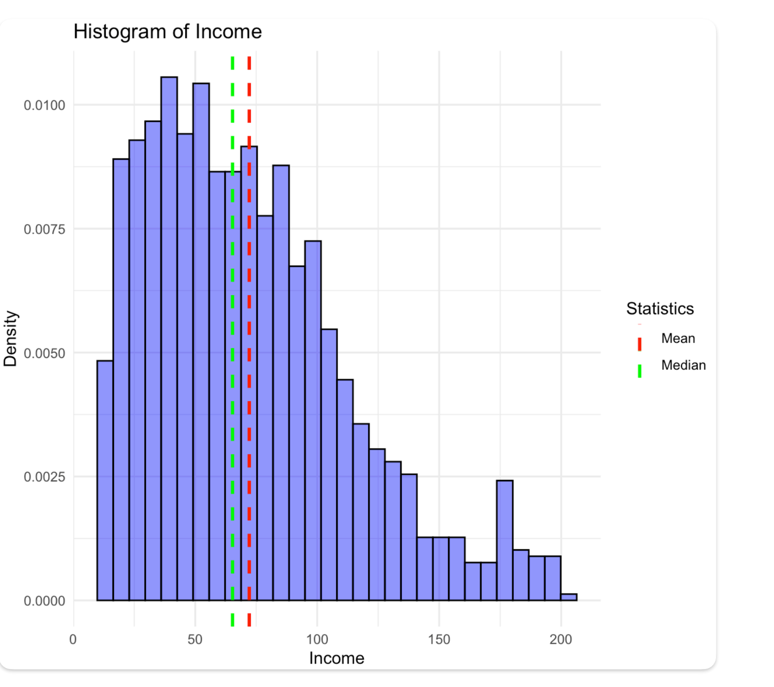
k. Based on this exercise, do you prefer the linear relationship model, or the log-log model or the linear-log model? Explain your reasoning.

a.

```
> print(stats_table)
            Food      Income
N      1200.0000 1200.00000
Mean    114.4431   72.14264
Median   99.8000   65.29000
Min       9.6300   10.00000
Max     476.6700  200.00000
SD       72.6575   41.65228
```

Food 和 Income 的平均數皆大於中位數，顯示 正偏態 (right-skewed)。
Food 和 Income 的分布都顯示 右偏 (右尾較長)，不是 bell-shaped 或對稱。





Jarque-Bera (JB) 常態性檢定
H0 ：資料符合常態分布　　H1 ：資料不符合常態分布
臨界值 (5% 顯著水準)：$\chi^2(2)=5.99$
結論：因為 JB 統計量 (food: 648. 65, income: 148.21) 都遠大於 5.99 ，且 P 值 = 0.00 (小於 0.05)，拒絕 H0 ，表示 food 和 income 都不符合常態分布。

```
> print(jb_table)
                Food   Income
JB_Statistic 648.6476 148.2112
P_Value        0.0000   0.0000
```

b.

```
> summary(lm_model)

Call:
lm(formula = food ~ income, data = data)

Residuals:
    Min      1Q  Median      3Q     Max 
-145.37  -51.48  -13.52   35.50  349.81 

Coefficients:
            Estimate Std. Error t value Pr(>|t|)    
(Intercept) 88.56650    4.10819  21.559  < 2e-16 ***
income       0.35869    0.04932   7.272 6.36e-13 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 71.13 on 1198 degrees of freedom
Multiple R-squared:  0.04228,	Adjusted R-squared:  0.04148 
F-statistic: 52.89 on 1 and 1198 DF,  p-value: 6.357e-13
```

Linear Model: FOOD = ß1 + ß2*INCOME + e

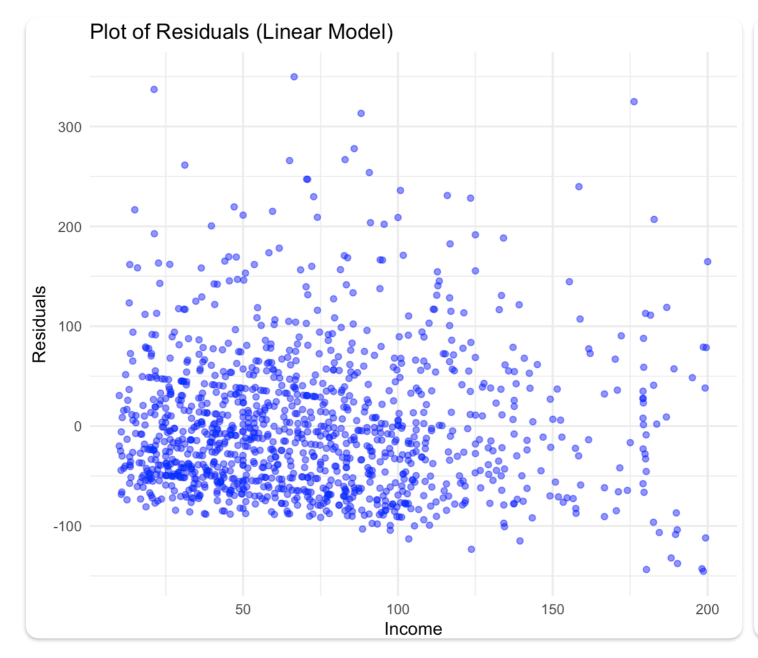
	估計值	標準誤
截距 (Intercept)	88.5665	4.1082
收入 (Income)	0.3587	0.0493

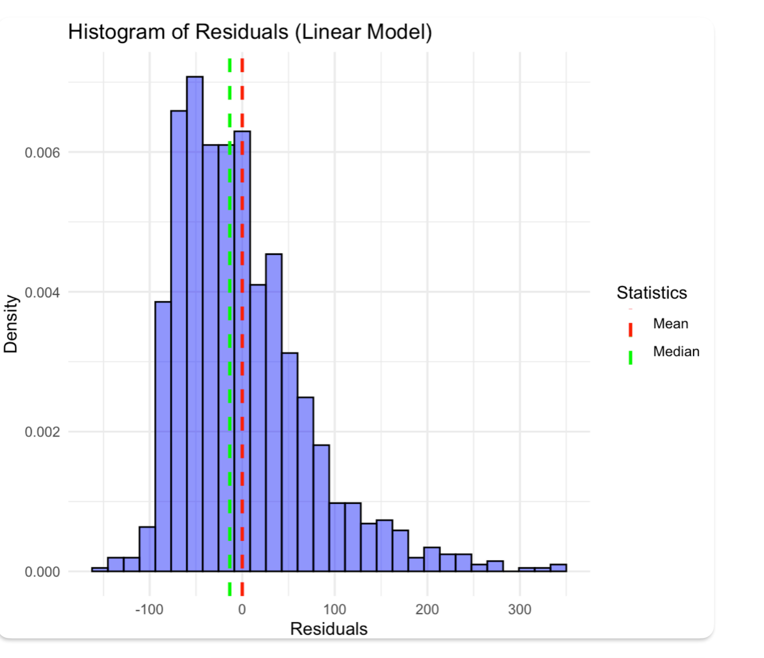
N = 1200, R^2 值 = 0.0423
解釋力偏低，表示收入並不是食品支出的主要決定因素。
ß2 95% 信賴區間為 [0.2619, 0.4555]

```
> conf_interval <- confint(lm_model, level = 0.95)
> print(conf_interval)
                 2.5 %    97.5 %
(Intercept) 80.5064570 96.626543
income       0.2619215  0.455452
```

從標準誤與信賴區間來看，Linear Model 對收入變動對食物支出的影響估計相對精確；但由於 R^2 很低，表示收入對食物支出的解釋力有限，可能還需要考慮其他影響因素，因此模型的整體預測能力仍然較弱。

c.





由殘差散布圖可知，Linear Model 的殘差沒有明顯的模式，未呈現隨機分佈。
由殘差直方圖可知，Linear Model 的殘差呈現正偏態。
Skewness = 1.30 > 0 : 右偏 (Right-skewed)
Kurtosis = 4.40 > 3 : 高峰分佈 (Leptokurtic)

```
> cat("Skewness:", skewness_value_lm, "\n")
Skewness: 1.29554
> cat("Kurtosis:", kurtosis_value_lm, "\n")
Kurtosis: 5.402088
```

Jarque-Bera (JB) 常態性檢定
H0 ：資料符合常態分布　　H1 ：資料不符合常態分布
臨界值 (5% 顯著水準)：$\chi^2(2)=5.99$
結論：因為 JB 統計量 (624.19) 遠大於 5.99，且 P 值 = 0.00 (小於 0.05)，拒絕 H0 ，表示 Linear Model 的殘差不符合常態分布。

```
> print(jb_table_residuals_linear)
             Residuals
JB_Statistic   624.186
P_Value          0.000
```

隨機誤差項 e的常態性比 FOOD 和 INCOME 的常態性更重要，因為它影響 OLS 估計的推論與檢定的準確性。 如果 e 服從常態分布，則 OLS 估計的係數 (如 ß1, ß2) 會是 最優線性不偏估計量 (BLUE)。即使 FOOD 和 INCOME 偏離常態，也可以通過 變數轉換 (如對數轉換) 來改善模型擬合。

d.

```
> print(elasticity_table)
  INCOME  FOOD_hat          ε       se_ε ε_lower_bound ε_upper_bound
1     19  95.38155 0.07145038 0.00982475    0.05219423    0.09070654
2     65 111.88114 0.20838756 0.02865423    0.15222630    0.26454882
3    160 145.95638 0.39319883 0.05406661    0.28723022    0.49916745
```

- Linear Model 彈性估計比較：彈性 (ε) 隨著 INCOME 的增加而上升， 食品支出的收入彈性在不同收入水準下顯著不同。
- 彈性的信賴區間沒有重疊，表示不同收入群體的食品支出行為在統計上具有顯著差異。
- 從 經濟學預期 來看，食物為必需品，食物的 所得彈性 (income elasticity) 通常應該 隨收入上升而下降或趨於穩定，但這次的結果顯示 彈性隨收入上升而增加，這與經濟學的一般預測不完全一致。

e.

```
> summary(log_log_model)  # 顯示回歸結果

Call:
lm(formula = ln_food ~ ln_income, data = data_log)

Residuals:
     Min       1Q   Median       3Q      Max 
-2.48175 -0.45497  0.06151  0.46063  1.72315 

Coefficients:
            Estimate Std. Error t value Pr(>|t|)    
(Intercept)  3.77893    0.12035  31.400   <2e-16 ***
ln_income    0.18631    0.02903   6.417    2e-10 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 0.6418 on 1198 degrees of freedom
Multiple R-squared:  0.03323,	Adjusted R-squared:  0.03242 
F-statistic: 41.18 on 1 and 1198 DF,  p-value: 1.999e-10
```

Log-Log Model: ln(FOOD) = γ1+γ2*ln(INCOME) + e

	估計值	標準誤
截距 (Intercept)	3.7789	0.1204
收入 (ln_Income)	0.1863	0.0290

N = 1200, R^2 值 = 0.0332
解釋力偏低，表示收入並不是食品支出的主要決定因素。

比較 Linear Model 與 Log-Log Model 散佈圖，Log-Log Model 曲線較線性模型更符合數據分佈。

```
> print(r2_comparison_e)
          Model         R2 Generalized_R2
1  Linear Model 0.04228120             NA
2 Log-Log Model 0.03322915     0.03965161
```

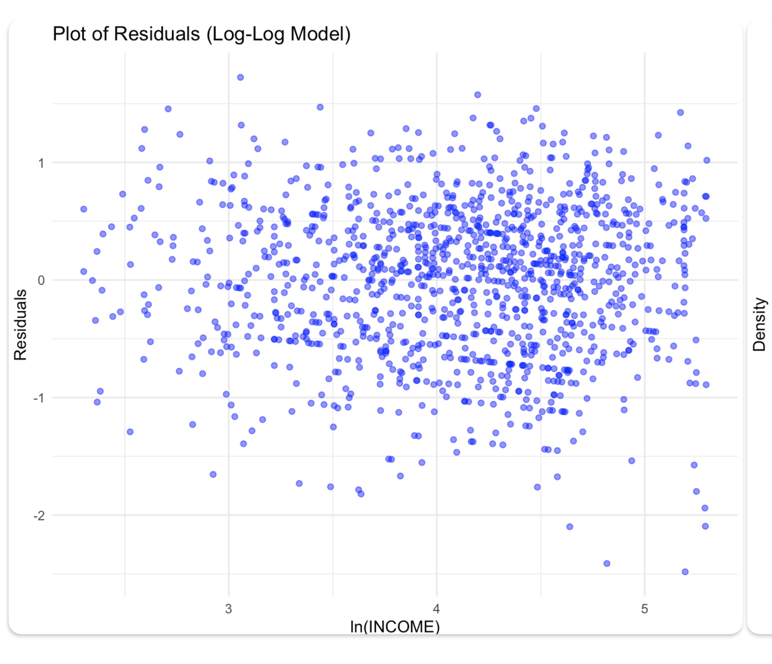
Log-Log Model 的 Generalized R^2（0.0397）接近但仍低於 Linear Model 的 R^2（0.0423）。
這表示 Log-Log Model 的預測力與 Linear Model 差異不大，但仍略低。
在考慮 R^2 下，Linear Model 的解釋能力最好。

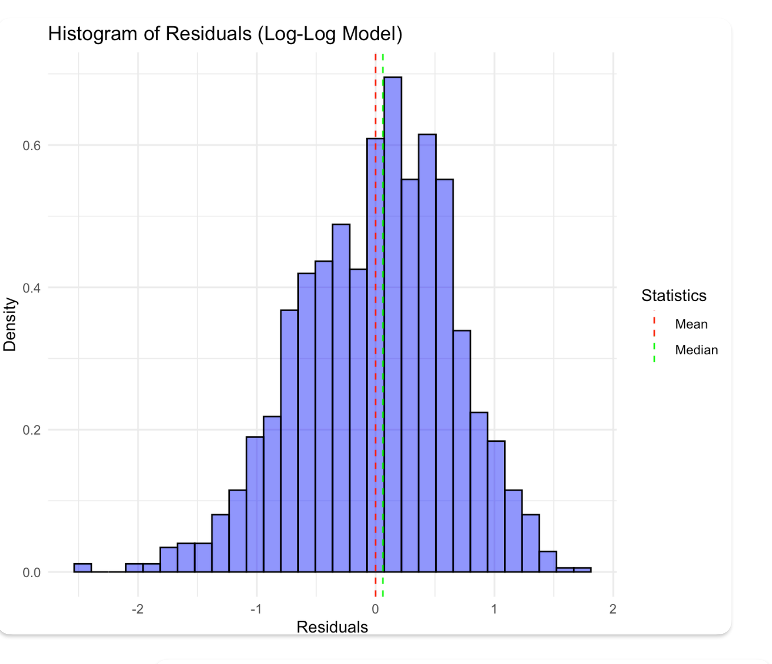
f.

```
> print(elasticity_table_loglog)
          Elasticity ε_Lower_Bound ε_Upper_Bound
ln_income     0.1863        0.1293        0.2433
```

- Log-Log Model 的彈性固定為 0.1863，其 95% 信賴區間為 [0.1293, 0.2433]。
- Log-Log Model 的彈性固定，但 Linear Model 的彈性會隨收入增加而上升。
- Log-Log Model 的彈性與在中等收入時最相似，但在低收入與高收入時存在顯著差異。

g.





由殘差散布圖可知，Log-Log Model 的殘差整體分佈較為均勻。
由殘差直方圖可知，Log-Log Model 的殘差呈現些微的負偏態。
Skewness = -0.36 < 0 : 左偏 (Left-skewed)
Kurtosis = 3.07≈ 3 : 接近常態分佈

```
> cat("Skewness:", skewness_value_loglog, "\n")
Skewness: -0.3577097
> cat("Kurtosis:", kurtosis_value_loglog, "\n")
Kurtosis: 3.071935
```

Jarque-Bera (JB) 常態性檢定
H0 ：資料符合常態分布　　H1 ：資料不符合常態分布
臨界值 (5% 顯著水準)：$\chi^2(2)=5.99$
結論：因為 JB 統計量 (25.85) 遠大於 5.99，且 P 值 = 2.436e-06 (小於 0.05)，拒絕 H0 ，表示 Log-Log Model 的殘差不符合常態分布。

```
	Jarque Bera Test

data:  data_log$residuals_loglog
X-squared = 25.85, df = 2, p-value = 2.436e-06
```

Log-Log Model 的殘差比 Linear Model 更接近常態分布。

h.

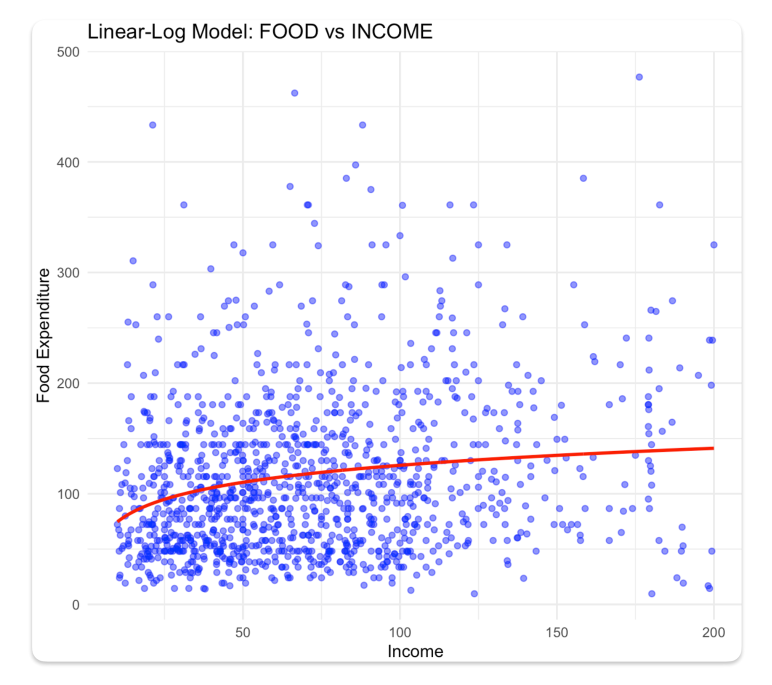
```
> summary(linear_log_model)

Call:
lm(formula = food ~ ln_income, data = data_linear_log)

Residuals:
    Min      1Q  Median      3Q     Max 
-129.18  -51.47  -13.98   35.05  345.54 

Coefficients:
            Estimate Std. Error t value Pr(>|t|)    
(Intercept)   23.568     13.370   1.763   0.0782 .  
ln_income     22.187      3.225   6.879 9.68e-12 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 71.29 on 1198 degrees of freedom
Multiple R-squared:  0.038,	Adjusted R-squared:  0.0372 
F-statistic: 47.32 on 1 and 1198 DF,  p-value: 9.681e-12
```



Linear-Log Model: FOOD = a1+a2*ln(INCOME) + e

	估計值	標準誤
截距 (Intercept)	23.568	13.370
收入 (ln_Income)	22.187	3.225

N = 1200, R^2 值 = 0.038
解釋力偏低，表示收入並不是食品支出的主要決定因素。

比較三模型散佈圖， Linear-Log Model 的回歸線更接近於 Linear Model 的回歸線。
Linear Model：擬合線較為筆直。Log-Log Model：擬合線最彎。Linear-Log Model ：擬合線稍微平滑。

與 Log-Log Model 的 Generalized R^2（0.0397）、 Linear Model 的 R^2（0.0423）相比，Linear-Log Model 的 R^2（0.038）是三者中最小的。這表示在考慮模型擬合度的情況下， Linear-Log Model 的解釋能力是三者中最差的， Linear Model 的解釋能力是三者中最好的。

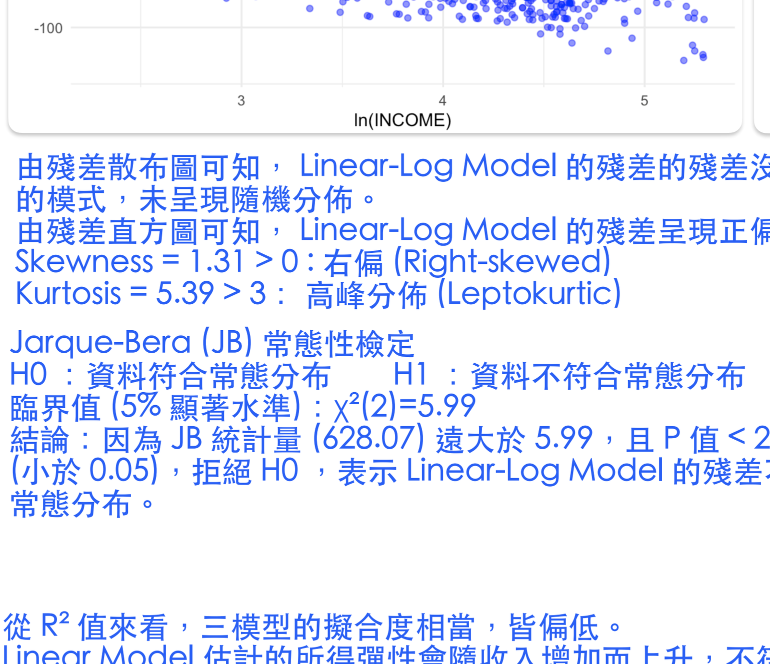
```
> print(r2_comparison_h)
                             Model         R2
1                     Linear Model 0.04228120
2 Log-Log Model(Generalized R²) 0.03965161
3                Linear-Log Model 0.03799984
```

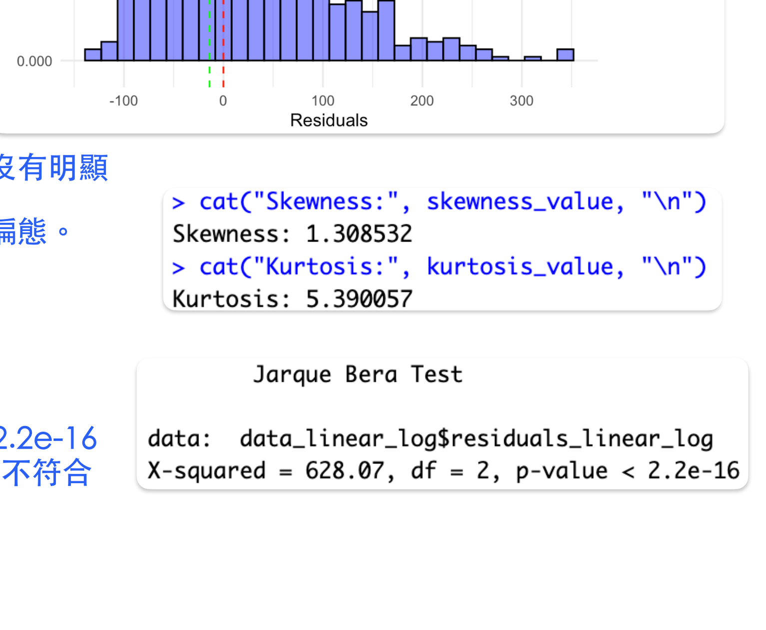
i.

```
> print(elasticity_table_i)
  INCOME  FOOD_hat         ε       se_ε ε_lower_bound ε_upper_bound
1     19  88.89788 0.2495828 0.03628131     0.1784728     0.3206929
2     65 116.18722 0.1909624 0.02775978     0.1365542     0.2453705
3    160 136.17332 0.1629349 0.02368549     0.1165122     0.2093576
```

- Linear-Log Model 彈性估計比較：彈性 (ε) 隨著 INCOME 的增加而下降。
- 從 經濟學預期 來看，食物為必需品，食物的 所得彈性 (income elasticity) 通常應該 隨收入上升而下降或趨於穩定，但這次的結果顯示 彈性隨收入上升而下降，這與經濟學的一般預測一致。
- Linear Model 的彈性的彈性則隨收入增加而上升， Log-Log Model 的彈性固定，而 Linear-Log Model 的彈性隨收入增加而減小。

j.





由殘差散布圖可知， Linear-Log Model 的殘差的殘差沒有明顯的模式，未呈現隨機分佈。
由殘差直方圖可知， Linear-Log Model 的殘差呈現正偏態。
Skewness = 1.31 > 0 : 右偏 (Right-skewed)
Kurtosis = 5.39 > 3 : 高峰分佈 (Leptokurtic)

```
> cat("Skewness:", skewness_value, "\n")
Skewness: 1.308532
> cat("Kurtosis:", kurtosis_value, "\n")
Kurtosis: 5.390057
```

Jarque-Bera (JB) 常態性檢定
H0 ：資料符合常態分布　　H1 ：資料不符合常態分布
臨界值 (5% 顯著水準)：$\chi^2(2)=5.99$
結論：因為 JB 統計量 (628.07) 遠大於 5.99，且 P 值 < 2.2e-16 (小於 0.05)，拒絕 H0 ，表示 Linear-Log Model 的殘差不符合常態分布。

```
	Jarque Bera Test

data:  data_linear_log$residuals_linear_log
X-squared = 628.07, df = 2, p-value < 2.2e-16
```

k. 從 R^2 值來看，三模型的擬合度相當，皆偏低。
Linear Model 估計的所得彈性會隨收入增加而上升，不符合經濟學預期。
Linear-Log Model 雖然符合經濟學理論，但殘差的分佈模式並非理想的隨機散佈。
Log-Log Model 假設所得彈性在所有收入水準下皆為固定值，可能限制了靈活性。
然而， Log-Log Mode l的殘差分佈最接近隨機分佈，且根據偏態（skewness）與峰度（kurtosis），其殘差的非常態性最小。
基於這些理由，對 Log-Log Model 似乎是較好的選擇。.