

$$y_1 = \alpha_1 y_2 + e_1$$

$$y_2 = \alpha_2 y_1 + \beta_1 x_1 + \beta_2 x_2 + e_2$$

We assume that x_1 and x_2 are exogenous and uncorrelated with the error terms e_1 and e_2 .

a. Solve the two structural equations for the reduced-form equation for y_2 , that is, $y_2 = \pi_0 + \pi_1 x_1 + \pi_2 x_2 + v_2$. Express the reduced-form parameters in terms of the **structural parameters** and the reduced-form error in terms of the structural parameters and e_1 and e_2 . Show that y_2 is correlated with e_1 .

b. Which equation parameters are consistently estimated using OLS? Explain.

c. Which parameters are "identified," in the simultaneous equations sense? Explain your reasoning.

$$a. y_1 = \alpha_1 y_2 + e_1$$

$$y_2 = \alpha_2 y_1 + \beta_1 x_1 + \beta_2 x_2 + e_2$$

$$y_2 = \alpha_2 y_1 + \beta_1 x_1 + \beta_2 x_2 + e_2$$

$$= \alpha_2 (\alpha_1 y_2 + e_1) + \beta_1 x_1 + \beta_2 x_2 + e_2$$

$$= \alpha_1 \alpha_2 y_2 + \alpha_2 e_1 + \beta_1 x_1 + \beta_2 x_2 + e_2$$

$$y_2 (1 - \alpha_1 \alpha_2) = \alpha_2 e_1 + \beta_1 x_1 + \beta_2 x_2 + e_2$$

$$y_2 = \frac{\beta_1}{1 - \alpha_1 \alpha_2} x_1 + \frac{\beta_2}{1 - \alpha_1 \alpha_2} x_2 + \frac{\alpha_2 e_1 + e_2}{1 - \alpha_1 \alpha_2}$$

$$= \pi_1 x_1 + \pi_2 x_2 + v_2$$

$$\text{cov}(y_2, e_1 | x)$$

$$= E(y_2 e_1 | x)$$

$$= E \left[\left(\frac{\beta_1}{1 - \alpha_1 \alpha_2} x_1 + \frac{\beta_2}{1 - \alpha_1 \alpha_2} x_2 + \frac{\alpha_2 e_1 + e_2}{1 - \alpha_1 \alpha_2} \right) e_1 | x \right]$$

$$= E \left[\frac{\beta_1}{1 - \alpha_1 \alpha_2} x_1 e_1 | x \right] + E \left[\frac{\beta_2}{1 - \alpha_1 \alpha_2} x_2 e_1 | x \right] + E \left[\left(\frac{\alpha_2 e_1 + e_2}{1 - \alpha_1 \alpha_2} \right) e_1 | x \right]$$

$$= 0 + 0 + E \left[\left(\frac{\alpha_2 e_1^2 + e_1 e_2}{1 - \alpha_1 \alpha_2} \right) | x \right]$$

$$\text{cov}(y_2, e_1 | x) = E(y_2 e_1 | x)$$

$$= \frac{E(e_2 e_1 | x) + \alpha_2 E(e_1^2 | x)}{(1 - \alpha_1 \alpha_2)}$$

$$= \frac{\alpha_2}{(1 - \alpha_1 \alpha_2)} \sigma_1^2 \neq 0$$

b. \rightarrow 個都 inconsistently,

因 2 個 equation 都有内生變數

OLS 會 bias & inconsistent.

reduced form 則 consistently (OLS)

因其式無内生變數

c. 2 個 simultaneous equation $M=2$.

因此至少要有 $(2-1)$ 個變數

absent 才能 identified

equation 1: x_1, x_2 absent

\rightarrow identified.

equation 2: 沒有變數 absent

\rightarrow unidentified.

- d. To estimate the parameters of the reduced-form equation for y_2 using the method of moments (MOM), which was introduced in Section 10.3, the two moment equations are

$$N^{-1} \sum x_{i1}(y_2 - \pi_1 x_{i1} - \pi_2 x_{i2}) = 0$$

$$N^{-1} \sum x_{i2}(y_2 - \pi_1 x_{i1} - \pi_2 x_{i2}) = 0$$

Explain why these two moment conditions are a valid basis for obtaining consistent estimators of the reduced-form parameters.

- e. Are the MOM estimators in part (d) the same as the OLS estimators? Form the sum of squared errors function for $y_2 = \pi_1 x_1 + \pi_2 x_2 + v_2$ and find the first derivatives. Set these to zero and show that they are equivalent to the two equations in part (d).
- f. Using $\sum x_{i1} = 1$, $\sum x_{i2} = 1$, $\sum x_{i1}x_{i2} = 0$, $\sum x_{i1}y_{i1} = 2$, $\sum x_{i1}y_{i2} = 3$, $\sum x_{i2}y_{i1} = 3$, $\sum x_{i2}y_{i2} = 4$, and the two moment conditions in part (d) show that the MOM/OLS estimates of π_1 and π_2 are $\hat{\pi}_1 = 3$ and $\hat{\pi}_2 = 4$.
- g. The fitted value $\hat{y}_2 = \hat{\pi}_1 x_1 + \hat{\pi}_2 x_2$. Explain why we can use the moment condition $\sum \hat{y}_{i2}(y_{i1} - \alpha_1 y_{i2}) = 0$ as a valid basis for consistently estimating α_1 . Obtain the IV estimate of α_1 .
- h. Find the 2SLS estimate of α_1 by applying OLS to $y_1 = \alpha_1 \hat{y}_2 + e_1^*$. Compare your answer to that in part (g).

d. These moment conditions arise from the assumptions that the x_i 's are exogenous.

$$\Rightarrow E(X_{i1}v_{i2}|X) = E(X_{i2}v_{i2}|X) = 0$$

$$y_2 = \frac{\beta_1}{1-\alpha_1\alpha_2} x_1 + \frac{\beta_2}{1-\alpha_1\alpha_2} + \frac{\alpha_2 e_1 + e_2}{1-\alpha_1\alpha_2} = \pi_1 x_1 + \pi_2 x_2 + v_2$$

reduced form error v_2 uncorrelated with x

$$\Rightarrow E\left[X_{i1} \left[\frac{\alpha_2 e_1 + e_2}{1-\alpha_1\alpha_2} \right] | X\right] = E\left[\frac{1}{(1-\alpha_1\alpha_2)} X_{i1} e_2 | X\right] + E\left[\frac{\alpha_2}{(1-\alpha_1\alpha_2)} X_{i1} e_1 | X\right] = 0 + 0$$

$$e. \frac{\partial S(\pi_1, \pi_2 | y, x)}{\partial \pi_1} = 2 \sum (y_2 - \pi_1 x_1 - \pi_2 x_2) x_1 = 0$$

$$\frac{\partial S(\pi_1, \pi_2 | y, x)}{\partial \pi_2} = 2 \sum (y_2 - \pi_1 x_1 - \pi_2 x_2) x_2 = 0$$

f. moment conditions:

$$N^{-1} \sum x_{i1} (y_2 - \pi_1 x_{i1} - \pi_2 x_{i2}) = 0$$

$$N^{-1} \sum x_{i2} (y_2 - \pi_1 x_{i1} - \pi_2 x_{i2}) = 0$$

$$\Rightarrow \sum x_{i1} y_{i2} - \pi_1 \sum x_{i1}^2 - \pi_2 \sum x_{i1} x_{i2} = 0$$

$$\sum x_{i2} y_{i2} - \pi_1 \sum x_{i1} x_{i2} - \pi_2 \sum x_{i2}^2 = 0$$

$$3 - \pi_1 = 0 \Rightarrow \pi_1 = 3$$

$$4 - \pi_2 = 0 \Rightarrow \pi_2 = 4$$

g.

$$y_1 = \alpha_1 y_2 + e_1$$

$$E[(\pi_1 x_1 + \pi_2 x_2) e_1 | x] = E[(\pi_1 x_1 + \pi_2 x_2)(y_1 - \alpha_1 y_2) | x] = 0$$

$$N^{-1} \sum (\pi_{1i} x_{1i} + \pi_{2i} x_{2i})(y_{1i} - \alpha_1 y_{2i}) = 0$$

In large samples the consistent estimators converge to true parameter values

$$\text{plim } \hat{\alpha}_1 = \alpha_1, \quad \text{plim } \hat{\alpha}_2 = \alpha_2$$

$$\Rightarrow \sum (\hat{\pi}_{1i} x_{1i} + \hat{\pi}_{2i} x_{2i})(y_{1i} - \alpha_1 y_{2i}) = \sum \hat{y}_{12}(y_{1i} - \alpha_1 y_{2i}) = 0$$

$$\sum \hat{y}_{12} y_{1i} - \alpha_1 \sum \hat{y}_{12} y_{2i} = 0 \Rightarrow \hat{\alpha}_1, \text{IV} = \frac{\sum \hat{y}_{12} y_{1i}}{\sum \hat{y}_{12} y_{2i}}$$

$$\hat{\alpha}_1, \text{IV} = \frac{\sum (\hat{\pi}_{1i} x_{1i} + \hat{\pi}_{2i} x_{2i}) y_{1i}}{\sum (\hat{\pi}_{1i} x_{1i} + \hat{\pi}_{2i} x_{2i}) y_{2i}} = \frac{\hat{\pi}_{1i} \sum x_{1i} y_{1i} + \hat{\pi}_{2i} \sum x_{2i} y_{1i}}{\hat{\pi}_{1i} \sum x_{1i} y_{2i} + \hat{\pi}_{2i} \sum x_{2i} y_{2i}} = \frac{18}{25}$$

h

$$\hat{\alpha}_{1, 2SLS} = \frac{\sum \hat{y}_{12} y_{1i}}{\sum \hat{y}_{12}^2}, \quad \hat{v}_2 = y_2 - \hat{y}_2 \Rightarrow \hat{y}_2 = y_2 - \hat{v}_2$$

$$\sum \hat{y}_{12}^2 = \sum \hat{y}_{12} (y_2 - \hat{v}_2) = \sum \hat{y}_{12} y_2 - \sum \hat{y}_{12} \hat{v}_2 = \sum \hat{y}_{12} y_2$$

$$\sum \hat{y}_{12} \hat{v}_2 = \sum (\hat{\pi}_{1i} x_{1i} + \hat{\pi}_{2i} x_{2i}) \hat{v}_2 = \hat{\pi}_{1i} \sum x_{1i} \hat{v}_2 + \hat{\pi}_{2i} \sum x_{2i} \hat{v}_2 = 0$$

$$\Rightarrow \sum x_{1i} \hat{v}_2 = 0, \quad \sum x_{2i} \hat{v}_2 = 0$$

11.16 Consider the following supply and demand model

Demand: $Q_i = \alpha_1 + \alpha_2 P_i + e_{di}$, Supply: $Q_i = \beta_1 + \beta_2 P_i + \beta_3 W_i + e_{si}$

where Q is the quantity, P is the price, and W is the wage rate, which is assumed exogenous. Data on these variables are in Table 11.7.

TABLE 11.7 Data for Exercise 11.16		
Q	P	W
4	2	2
6	4	3
9	3	1
3	5	1
8	8	3

- Derive the algebraic form of the reduced-form equations, $Q = \theta_1 + \theta_2 W + v_2$ and $P = \pi_1 + \pi_2 W + v_1$, expressing the reduced-form parameters in terms of the structural parameters.
- Which structural parameters can you solve for from the results in part (a)? Which equation is "identified"?
- The estimated reduced-form equations are $\hat{Q} = 5 + 0.5W$ and $\hat{P} = 2.4 + 1W$. Solve for the identified structural parameters. This is the method of **indirect least squares**.
- Obtain the fitted values from the reduced-form equation for P , and apply 2SLS to obtain estimates of the demand equation.

$Q = \alpha_1 + \alpha_2 P$

a.
$$\begin{cases} Q_i = \alpha_1 + \alpha_2 P_i + e_{di} \\ Q_i = \beta_1 + \beta_2 P_i + \beta_3 W_i + e_{si} \end{cases}$$

$$0 = (\alpha_1 - \beta_1) + (\alpha_2 - \beta_2) P_i - \beta_3 W_i + e_{di} - e_{si}$$

$$P_i = \frac{-(\alpha_1 - \beta_1) + \beta_3 W_i - (e_{di} - e_{si})}{\alpha_2 - \beta_2}$$

$$= \frac{\beta_1 - \alpha_1}{\alpha_2 - \beta_2} + \frac{\beta_3}{\alpha_2 - \beta_2} W_i + \frac{e_{si} - e_{di}}{\alpha_2 - \beta_2}$$

$$Q_i = \alpha_1 + \alpha_2 P_i + e_{di}$$

$$\begin{aligned} \Rightarrow Q_i &= \alpha_1 + \alpha_2 \left(\frac{\beta_1 - \alpha_1}{\alpha_2 - \beta_2} + \frac{\beta_3}{\alpha_2 - \beta_2} W_i + \frac{e_{si} - e_{di}}{\alpha_2 - \beta_2} \right) + e_{di} \\ &= \alpha_1 + \frac{\alpha_2 \beta_1 - \alpha_2 \alpha_1}{\alpha_2 - \beta_2} + \frac{\alpha_2 \beta_3}{\alpha_2 - \beta_2} W_i + \frac{\alpha_2 (e_{si} - e_{di})}{\alpha_2 - \beta_2} + e_{di} \end{aligned}$$

b. $M=2$, 至少需要 absent $(2-1)$ 个 variable.

因此 demand 可以 identified,
supply unidentified.

可由 (a) 解: α_1, α_2 ,
 β_1 需另解

$$c. \quad \hat{Q} = \overset{\theta_1}{5} + \overset{\theta_2}{0.5} W$$

$$\hat{P} = \overset{\pi_1}{2.4} + \overset{\pi_2}{1} W$$

$$\theta_1 = 5$$

$$\theta_2 = 0.5$$

$$\pi_1 = 2.4$$

$$\pi_2 = 1$$

$$Q = \alpha_1 + \alpha_2 (\pi_1 + \pi_2 W + v_2) + v_2$$

$$\alpha_1 + \alpha_2 \pi_1 = \theta_1$$

$$\alpha_2 \pi_2 = \theta_2, \quad \alpha_2 \cdot 1 = 0.5, \quad \alpha_2 = 0.5$$

$$\alpha_1 + 0.5 \cdot 2.4 = 5 \quad \alpha_1 + 1.2 = 5 \quad \alpha_1 = 3.8$$

∴ Demand:

$$Q = 3.8 + 0.5P$$

supply: unidentified, 無法求唯一解

11.16 Consider the following supply and demand model

$$\text{Demand: } Q_i = \alpha_1 + \alpha_2 P_i + e_{di}, \quad \text{Supply: } Q_i = \beta_1 + \beta_2 P_i + \beta_3 W_i + e_{si}$$

where Q is the quantity, P is the price, and W is the wage rate, which is assumed exogenous. Data on these variables are in Table 11.7.

TABLE 11.7		Data for Exercise 11.16
Q	P	W
4	2	2
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- Derive the algebraic form of the reduced-form equations, $Q = \theta_1 + \theta_2 W + v_2$ and $P = \pi_1 + \pi_2 W + v_1$, expressing the reduced-form parameters in terms of the structural parameters.
- Which structural parameters can you solve for from the results in part (a)? Which equation is "identified"?
- The estimated reduced-form equations are $\hat{Q} = 5 + 0.5W$ and $\hat{P} = 2.4 + 1W$. Solve for the identified structural parameters. This is the method of **indirect least squares**.
- Obtain the fitted values from the reduced-form equation for P , and apply 2SLS to obtain estimates of the demand equation.

d. first-stage:

$$\hat{p} = 2.4 + 1w$$

w_i	\hat{p}_i
2	4.4
3	5.4
1	3.4
1	3.4
3	5.4

Second stage:

Q_i	\hat{p}_i
4	4.4
6	5.4
9	3.4
3	3.4
8	5.4
$\bar{Q} = 6$	$\bar{\hat{p}} = 4.4$

$$\alpha_2 = \frac{\sum (\hat{p}_i - \bar{\hat{p}})(Q_i - \bar{Q})}{\sum (\hat{p}_i - \bar{\hat{p}})^2}$$

$$= 0.5$$

$$\alpha_1 = \bar{Q} - \alpha_2 \bar{\hat{p}} = 6 - 0.5 \times 4.4 = 3.8$$

$$2SLS: Q = 3.8 + 0.5 \hat{p}$$

11.17 Example 11.3 introduces Klein's Model I.

- Do we have an adequate number of IVs to estimate each equation? Check the necessary condition for the identification of each equation. The necessary condition for identification is that in a system of M equations at least $M - 1$ variables must be omitted from each equation.
- An equivalent identification condition is that the number of excluded exogenous variables from equation must be at least as large as the number of included right-hand side endogenous variables. Check that this condition is satisfied for each equation.
- Write down in econometric notation the first-stage equation, the reduced form, for W_{1t} , w_t , workers earned in the private sector. Call the parameters π_1, π_2, \dots
- Describe the two regression steps of 2SLS estimation of the consumption function. This is question about a computer software command.
- Does following the steps in part (d) produce regression results that are identical to the 2 estimates provided by software specifically designed for 2SLS estimation? In particular, will t -values be the same?

$$CN_t = \alpha_1 + \alpha_2(W_{1t} + W_{2t}) + \alpha_3 P_t + \alpha_4 P_{t-1} + e_{1t}$$

$$I_t = \beta_1 + \beta_2 P_t + \beta_3 P_{t-1} + \beta_4 K_{t-1} + e_{2t}$$

$$W_{1t} = \gamma_1 + \gamma_2 E_t + \gamma_3 E_{t-1} + \gamma_4 TIME_t + e_{3t}$$

a. $M=3$, 每個至少要有2個外生變數 absent

外生: $G, W_2, Tx, Time, P-1, K-1, E-1$ 等物

内生: CN, I_t, W_1, P, E 87個 (有3個是從model來的)

3個 equation 都至少有2個外生變數 absent

b. identified $\#$
 CN : 内生2個... 排除5個外生

I_t : 内生1個 ... =

W_{1t} : 内生1個 ... =

排除的外生 = 内生 ... satisfied $\#$

(c) $W_{it} = \pi_1 + \pi_2 G + \pi_3 W_{it} + \pi_4 TX_t + \pi_5 Time_t + \pi_6 P_{t-1} + \pi_7 K_{t-1} + \pi_8 E_{t-1} + V$

(d) obtain fitted values \hat{w}_{it} from the estimated reduced form equation in part (c) and similarly obtain \hat{p}_t . Create $W_{it}^* = \hat{w}_{it} + W_{it}$.

Regress CN_t on W_{it}^* , \hat{p}_t and P_{t-1} plus a constant by OLS,

(e) The coefficient estimates will be the same. The t-values will not be because the standard errors in part (d) are not correct OLS standard errors.