11.1 Our aim is to estimate the parameters of the simultaneous equations model

$$y_1 = \alpha_1 y_2 + e_1$$

$$y_2 = \alpha_2 y_1 + \beta_1 x_1 + \beta_2 x_2 + e_2$$

We assume that x_1 and x_2 are exogenous and uncorrelated with the error terms e_1 and e_2.

a. Solve the two structural equations for the reduced-form equation for y_2, that is, $y_2 = \pi_1 x_1 + \pi_2 x_2 + v_2$. Express the reduced-form parameters in terms of the **structural parameters** and the reduced-form error in terms of the structural parameters and e_1 and e_2. Show that y_2 is correlated with e_1.

b. Which equation parameters are consistently estimated using OLS? Explain.

c. Which parameters are "identified," in the simultaneous equations sense? Explain your reasoning.

a. $y_2 = a_2(a_1 y_2 + e_1) + \beta_1 x_1 + \beta_2 x_2 + e_2$

$$\Rightarrow y_2 = \frac{\beta_1}{1-a_2 \cdot a_1} x_1 + \frac{\beta_2}{1-a_2 \cdot a_1} x_2 + \frac{a_2 \cdot e_1 + e_2}{1-a_2 \cdot a_1}$$

$$= \pi_1 x_1 + \pi_2 x_2 + v_2 \text{, where } \pi_1 = \frac{\beta_1}{1-a_2 \cdot a_1},\ \pi_2 = \frac{\beta_2}{1-a_2 \cdot a_1},\ v_2 = \frac{a_2 e_1 + e_2}{1-a_2 \cdot a_1}$$

$$Corr(y_2, e_1) = \sigma_{y_2} \cdot \sigma_{e_1} \cdot Cov(y_2, e_1) = \sigma_{y_2} \cdot \sigma_{e_1} \cdot \frac{a_2}{1-a_2 a_1} \cdot \sigma_{e_1}^2 \neq 0$$

b.

Since both equation contain endogenous variable (y_1 & y_2), The OLS estimator won't be consistent.

c.

There are 2 equations, so there must be $2-1=1$ variable be omitted to make the equation identified, in equation (1), there are 2 variables absent $\Rightarrow$ identified, while equation have no variable absent

Therefore, only $y_1 = a_1 y_2 + e_i$ is identified.

d. To estimate the parameters of the reduced-form equation for y_2 using the method of moments (MOM), which was introduced in Section 10.3, the two moment equations are

$$N^{-1}\sum x_{i1}(y_2 - \pi_1 x_{i1} - \pi_2 x_{i2}) = 0$$

$$N^{-1}\sum x_{i2}(y_2 - \pi_1 x_{i1} - \pi_2 x_{i2}) = 0$$

Explain why these two moment conditions are a valid basis for obtaining consistent estimators of the reduced-form parameters.

e. Are the MOM estimators in part (d) the same as the OLS estimators? Form the sum of squared errors function for $y_2 = \pi_1 x_1 + \pi_2 x_2 + v_2$ and find the first derivatives. Set these to zero and show that they are equivalent to the two equations in part (d).

f. Using $\sum x_{i1}^2 = 1$, $\sum x_{i2}^2 = 1$, $\sum x_{i1}x_{i2} = 0$, $\sum x_{i1}y_{1i} = 2$, $\sum x_{i1}y_{2i} = 3$, $\sum x_{i2}y_{1i} = 3$, $\sum x_{i2}y_{2i} = 4$, and the two moment conditions in part (d) show that the MOM/OLS estimates of π_1 and π_2 are $\hat{\pi}_1 = 3$ and $\hat{\pi}_2 = 4$.

d. Combine two equations we have $E(X_{i1}\cdot V_2 \mid X_1, X_2) = E(X_{i2}\cdot V_2 \mid X_1, X_2) = 0$

replace V_2 by $\frac{a_2 - e_1 + e_2}{1 - a_2 \cdot a_1}$ we have

$$\begin{cases} \frac{a_2}{1-a_2\cdot a_1}\cdot E(X_{i1}\cdot e_1 \mid X_1, X_2) + \frac{1}{1-a_2\cdot a_1} E(X_{i1}\cdot e_1 \mid X_1, X_2) = 0 \\ \frac{a_2}{1-a_2\cdot a_1} - E(X_{i2}\cdot e_1 \mid X_1, X_2) + \frac{1}{1-a_2\cdot a_1}\cdot E(X_{i2}\cdot e_1 \mid X_1, X_2) = 0 \end{cases}$$

while $E(X_{ik}\cdot e_1 \mid X_1 \cdots X_k) = 0 = E(X_{ik}\cdot e_2 \mid X_1 \cdots X_k)$

makes X_1, X_2 consistent

e.

OLS: $\min \sum (y_2 - \pi_1 X_1 - \pi_2 X_2)^2$ by F.O.C. $\begin{cases} -2\cdot\sum X_1 (y_2 - \pi_1 X_1 - \pi_2 X_2) = 0 \\ -2\cdot\sum X_2 (y_2 - \pi_1 X_1 - \pi_2 X_2) = 0 \end{cases}$

are equivalent to 2 equations in part (d)

f.

equations in part d. $\begin{cases} \sum X_1 y_2 - \pi_1 \cdot \sum X_1^2 - \pi_2 \cdot \sum X_1 X_2 = 0 \\ \sum X_2 y_2 - \pi_1 \cdot \sum X_2 X_1 - \pi_2 \cdot \sum X_2^2 = 0 \end{cases} \Rightarrow \begin{cases} \pi_1 + 0\pi_2 = 3 \\ \Rightarrow \pi_1 = 3 \\ 0\pi_1 + \pi_2 = 4 \\ \Rightarrow \pi_2 = 4 \end{cases}$

$x_1 = 3$ and $x_2 = 4$.

g. The fitted value $\hat{y}_2 = \hat{\pi}_1 x_1 + \hat{\pi}_2 x_2$. Explain why we can use the moment condition $\sum \hat{y}_{i2}(y_{i1} - \alpha_1 y_{i2}) = 0$ as a valid basis for consistently estimating α_1. Obtain the IV estimate of α_1.

h. Find the 2SLS estimate of α_1 by applying OLS to $y_1 = \alpha_1 \hat{y}_2 + e_1^*$. Compare your answer to that in part (g).

g.

$\sum \hat{y}_2 (y_1 - a_1 y_2) = 0 \Rightarrow \alpha_1 = \dfrac{\sum \hat{y}_2 \cdot y_1}{\sum \hat{y}_2 \cdot y_2}$, plug $\hat{y}_2 = \hat{\pi}_1 x_1 + \hat{\pi}_2 x_2$ into it

$$\Rightarrow \hat{\alpha}_1 = \frac{\sum(\hat{\pi}_1 \cdot x_1 + \hat{\pi}_2 \cdot x_2) y_1}{\sum(\hat{\pi}_1 \cdot x_1 + \hat{\pi}_2 \cdot x_2) y_2} = \frac{3 \cdot \sum(x_1 y_1) + 4\sum(x_2 \cdot y_1)}{3 \cdot \sum(x_1 \cdot y_2) + 4 \cdot \sum(x_2 \cdot y_2)} = \frac{3\times2+4\times3}{3\times3+4\times4} = \frac{18}{25}$$

This is consistent because we have $y_1 = \alpha_1 y_2 + e_1$, and the moment condition of (y_2, e_1) make $\hat{\alpha}_1$ consistent

h.

$\hat{v}_2 = y_2 - \hat{y}_2 \Rightarrow \hat{y}_2 = y_2 - \hat{v}_2$, $\hat{\alpha}_{1,2SLS} = \dfrac{\sum \hat{y}_2 \cdot y_1}{\sum \hat{y}_2^2}$, so we need to prove

$$\sum \hat{y}_2^2 = \sum \hat{y}_2 \cdot y_2 \Rightarrow \sum \hat{y}_2 (y_2 - \hat{v}_2) = \sum \hat{y}_2 \cdot y_2$$

$$\Rightarrow \sum \hat{y}_2 y_2 - \sum \hat{y}_2 \cdot \hat{v}_2 = \sum \hat{y}_2 \cdot y_2$$, note that $\sum \hat{y}_2 \cdot \hat{v}_2 = 0$ if $cov(\hat{y}_2, \hat{v}_2) = 0$

$$\Rightarrow \sum \hat{y}_2 y_2 - 0 = \sum \hat{y}_2 \cdot y_2 \ \#$$

11.16 Consider the following supply and demand model

$$\text{Demand: } Q_i = \alpha_1 + \alpha_2 P_i + e_{di}, \quad \text{Supply: } Q_i = \beta_1 + \beta_2 P_i + \beta_3 W_i + e_{si}$$

where Q is the quantity, P is the price, and W is the wage rate, which is assumed exogenous. Data on these variables are in Table 11.7.

TABLE 11.7 Data for Exercise 11.16

Q	P	W
4	2	2
6	4	3
9	3	1
3	5	1
8	8	3

a. Derive the algebraic form of the reduced-form equations, $Q = \theta_1 + \theta_2 W + v_2$ and $P = \pi_1 + \pi_2 W + v_1$, expressing the reduced-form parameters in terms of the structural parameters.

b. Which structural parameters can you solve for from the results in part (a)? Which equation is "identified"?

c. The estimated reduced-form equations are $\hat{Q} = 5 + 0.5W$ and $\hat{P} = 2.4 + 1W$. Solve for the identified structural parameters. This is the method of **indirect least squares**.

d. Obtain the fitted values from the reduced-form equation for P, and apply 2SLS to obtain estimates of the demand equation.

a.

$$Q_i = \alpha_1 + \alpha_2 P_i + e_{di} = \beta_1 + \beta_2 P_i + \beta_3 W_i + e_{si}$$

$$\Rightarrow P_i = \frac{\beta_1 - \alpha_1}{\alpha_2 - \beta_2} + \frac{\beta_3}{\alpha_2 - \beta_2} W_i + \frac{e_{di} + e_{si}}{\alpha_2 - \beta_2} = \pi_1 + \pi_2 W + v_1$$

$$Q_1 = \alpha_1 + \alpha_2(\pi_1 + \pi_2 W + v_1) + e_{di}$$

$$= \left(\alpha_1 + \frac{\alpha_2(\beta_1 - \alpha_1)}{\alpha_2 - \beta_2}\right) + \frac{\alpha_2 \cdot \beta_3}{\alpha_2 - \beta_2} W_i + \left(e_{di} + \frac{\alpha_2(e_{di} + e_{si})}{\alpha_2 - \beta_2}\right) = \theta_1 + \theta_2 W_i + v_2$$

b.

Only Demand Equation is "identified" because $M=2$, and there is zero variable being omitted in Supply equation, which require at least $2-1=1$ variable being omitted to make equation "identified" $\Rightarrow$ α_1, α_2 can be solved c.

$$5+0.5W = \alpha_1+\alpha_2(2.4+W) \Rightarrow \begin{cases} 2.4\alpha_2+\alpha_1 = 5 \\ \alpha_2 = 0.5 \end{cases} \Rightarrow \alpha_1 = 3.8,\ \alpha_2 = 0.5$$

d.

$\hat{p} = 2.4+W$, $\bar{\hat{p}} = 4.4$, $\bar{Q} = 6$

	W	$\hat{p}$	$\hat{p}-\bar{\hat{p}}$	$Q-\bar{Q}$	$(\hat{p}-\bar{\hat{p}})^2$	$(\hat{p}-\bar{\hat{p}})\cdot(Q-\bar{Q})$
	2	4.4	0	-2	0	0
	3	5.4	1	0	1	0
	1	3.4	-1	3	1	-3
	1	3.4	-1	-3	1	3
	3	5.4	1	2	1	2
sum	10	22	0	0	4	2

$Q_i = \alpha_1 - \alpha_2\hat{p} + e_i$

$$\alpha_2 = \frac{\Sigma(\hat{p}_i-\bar{\hat{p}}_i)(Q_i-\bar{Q})}{\Sigma(\hat{p}_i-\bar{\hat{p}})^2} = \frac{2}{4} = \frac{1}{2}$$

$$\Rightarrow \hat{\alpha}_1 = \bar{Q} - \alpha_2\bar{\hat{p}} = 6-\frac{1}{2}\cdot 4.4 = 3.8$$

$$\Rightarrow \hat{Q} = 3.8+0.5p$$

11.17 Example 11.3 introduces Klein's Model I.

a. Do we have an adequate number of IVs to estimate each equation? Check the necessary condition for the identification of each equation. The necessary condition for identification is that in a system of M equations at least $M - 1$ variables must be omitted from each equation.

b. An equivalent identification condition is that the number of excluded exogenous variables from the equation must be at least as large as the number of included right-hand side endogenous variables. Check that this condition is satisfied for each equation.

c. Write down in econometric notation the first-stage equation, the reduced form, for W_{1t}, wages of workers earned in the private sector. Call the parameters $\pi_1, \pi_2, \ldots$

d. Describe the two regression steps of 2SLS estimation of the consumption function. This is not a question about a computer software command.

e. Does following the steps in part (d) produce regression results that are identical to the 2SLS estimates provided by software specifically designed for 2SLS estimation? In particular, will the t-values be the same?

$$CN_t = \alpha_1 + \alpha_2 (W_{1t} + W_{2t}) + \alpha_3 P_t + \alpha_4 P_{t-1} + e_{1t} \quad (11.17)$$

$$I_t = \beta_1 + \beta_2 P_t + \beta_3 P_{t-1} + \beta_4 K_{t-1} + e_{2t} \quad (11.18)$$

$$W_{1t} = \gamma_1 + \gamma_2 E_t + \gamma_3 E_{t-1} + \gamma_4 TIME_t + e_{3t} \quad (11.19)$$

a. M=8, Endogenous = 8, Exogenous = 8, at least 8-1=7 variable should be omitted to make equation identified.

Consumption: 5 variable included, 11 omitted ⇒ identified

Investment: 4 " 12 " ⇒ "

Wage : 4 " 11 " ⇒ "

b.

Consumption: 2 endogenous variables included and exclude 5 exogenous

Investment: 1 " " 5 "

Wage : 1 " " 5 "

⇒ all satisfied

c. $W_{1t} = \pi_1 + \pi_2 G_t + \pi_3 W_{2t} + \pi_4 TX_t + \pi_5 TIME_t + \pi_6 P_{t-1} + \pi_7 K_{t-1} + \pi_8 E_{t-1}$

d. from (c), we get $\hat{w}_{1t}$, and apply same method to obtain $\hat{p}_t$,

then regress CN_t by OLS with $\hat{w}_{1t}$ and $\hat{p}_t$

e.

Coefficient will be the same, but t-values won't.

Q28

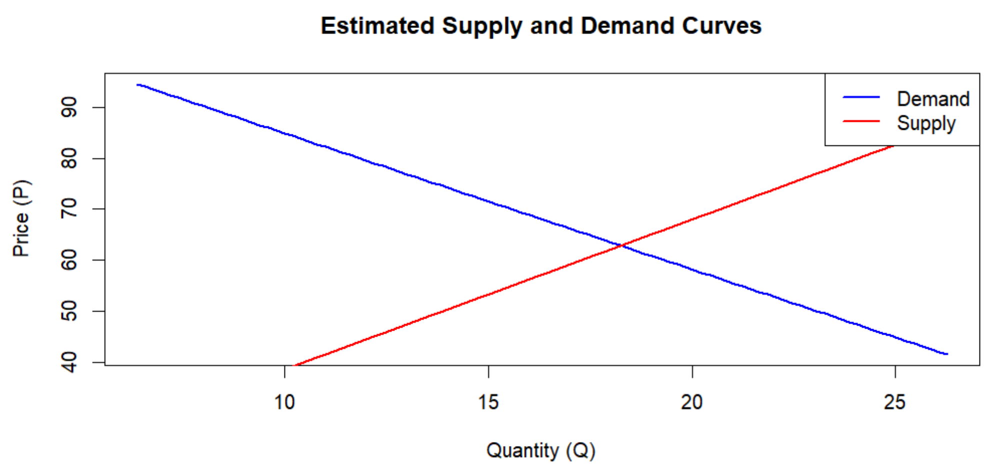
(b)需求與供給模型係數皆符合預期(上圖為需求，下圖為供給)

```
Call:
ivreg(formula = p ~ q + ps + di | ps + di + pf, data = truffles)

Residuals:
    Min      1Q  Median      3Q     Max 
-39.661  -6.781   2.410   8.320  20.251 

Coefficients:
            Estimate Std. Error t value Pr(>|t|)    
(Intercept)  -11.428     13.592  -0.841  0.40810    
q             -2.671      1.175  -2.273  0.03154 *  
ps             3.461      1.116   3.103  0.00458 ** 
di            13.390      2.747   4.875 4.68e-05 ***
Call:
ivreg(formula = p ~ q + pf | pf + ps + di, data = truffles)

Residuals:
    Min      1Q  Median      3Q     Max 
-9.7983 -2.3440 -0.6281  2.4350 11.1600 

Coefficients:
            Estimate Std. Error t value Pr(>|t|)    
(Intercept) -58.7982     5.8592  -10.04 1.32e-10 ***
q             2.9367     0.2158   13.61 1.32e-13 ***
pf            2.9585     0.1560   18.97  < 2e-16 ***
```

(c)平均數的點彈性為-1.272

```
> gamma_2 <- coef(demand_2sls)["q"]
> dq_dp <- 1 / gamma_2
> elasticity <- dq_dp * (mean_p / mean_q)
> cat("需求的價格彈性(at the means)為：", round(elasticity, 3), "\n")
需求的價格彈性(at the means)為： -1.272
```

(d)給定外生變數的需求與供給曲線如下圖



(e) 需求與供給的均衡為Q=18.25;P=62.843

```
> cat("結構模型下的均衡數量 Q* =", round(q_eq, 3), "\n")
結構模型下的均衡數量 Q* = 18.25
> cat("結構模型下的均衡價格 P* =", round(p_eq, 3), "\n")
結構模型下的均衡價格 P* = 62.843
```

(f)使用OLS模型中需求等式的q的係數不顯著且與預期不符，供給方程式則相對有一致性。

```
Call:
lm(formula = p ~ q + ps + di, data = truffles)

Residuals:
     Min       1Q   Median       3Q      Max 
-25.0753  -2.7742  -0.4097   4.7079  17.4979 

Coefficients:
            Estimate Std. Error t value Pr(>|t|)    
(Intercept) -13.6195     9.0872  -1.499   0.1460    
q             0.1512     0.4988   0.303   0.7642    
ps            1.3607     0.5940   2.291   0.0303 *  
di           12.3582     1.8254   6.770 3.48e-07 ***
```

```
Call:
lm(formula = p ~ q + pf, data = truffles)

Residuals:
    Min      1Q  Median      3Q     Max 
-8.4721 -3.3287  0.1861  2.0785 10.7513 

Coefficients:
            Estimate Std. Error t value Pr(>|t|)    
(Intercept) -52.8763     5.0238  -10.53 4.68e-11 ***
q             2.6613     0.1712   15.54 5.42e-15 ***
pf            2.9217     0.1482   19.71  < 2e-16 ***
```

Q30

(a)11.18之OLS結果如下圖

```
Call:
lm(formula = i ~ p + plag + klag, data = klein)

Residuals:
     Min       1Q   Median       3Q      Max 
-2.56562 -0.63169  0.03687  0.41542  1.49226 

Coefficients:
            Estimate Std. Error t value Pr(>|t|)    
(Intercept) 10.12579    5.46555   1.853 0.081374 .  
p            0.47964    0.09711   4.939 0.000125 ***
plag         0.33304    0.10086   3.302 0.004212 ** 
klag        -0.11179    0.02673  -4.183 0.000624 ***
---
```

(b)將殘差及phat存入原資料

```
> klein$vt <- residuals(profit_model)
> klein$phat <- fitted(profit_model)
```

(c)加入殘差進行Hausmen test，vt之t-test的p-value為0.973875，代表在5%的信心水準下無充足證據說明vt的係數不為0，及investment與殘差無關。

```
Call:
lm(formula = i ~ p + plag + klag + vt, data = klein)

Residuals:
     Min       1Q   Median       3Q      Max 
-2.56520 -0.63656  0.03554  0.40691  1.48042 

Coefficients:
              Estimate Std. Error t value Pr(>|t|)    
(Intercept)  1.016e+01  5.719e+00   1.776 0.094700 .  
p            4.792e-01  1.010e-01   4.746 0.000219 ***
plag         3.335e-01  1.050e-01   3.176 0.005862 ** 
klag        -1.120e-01  2.800e-02  -3.999 0.001035 ** 
vt           1.458e+12  4.382e+13   0.033 0.973875    
```

(d) 2SLS結果如下，與OLS結果相差無幾。

```
Call:
ivreg(formula = i ~ p + plag + klag | w1 + w2 + g + tx + elag + 
    e + plag + klag, data = klein)

Residuals:
     Min       1Q   Median       3Q      Max 
-2.56562 -0.63169  0.03687  0.41542  1.49226 

Coefficients:
            Estimate Std. Error t value Pr(>|t|)    
(Intercept) 10.12579    5.46555   1.853 0.081374 .  
p            0.47964    0.09711   4.939 0.000125 ***
plag         0.33304    0.10086   3.302 0.004212 ** 
klag        -0.11179    0.02673  -4.183 0.000624 ***
```

(e)手動2SLS與ivreg結果無異

```
Call:
lm(formula = i ~ p_hat + plag + klag, data = klein)

Residuals:
     Min       1Q   Median       3Q      Max 
-2.56562 -0.63169  0.03687  0.41542  1.49226 

Coefficients:
            Estimate Std. Error t value Pr(>|t|)    
(Intercept) 10.12579    5.46555   1.853 0.081374 .  
p_hat        0.47964    0.09711   4.939 0.000125 ***
plag         0.33304    0.10086   3.302 0.004212 ** 
klag        -0.11179    0.02673  -4.183 0.000624 ***
```

(f)

```
> if (sargan_stat < crit_val) {
+   cat("✅ 結論：無法拒絕 H0，工具變數有效。\n")
+ } else {
+   cat("❌ 結論：拒絕 H0，工具變數可能無效。\n")
+ }
❌ 結論：拒絕 H0，工具變數可能無效。
```