11.1 Our aim is to estimate the parameters of the simultaneous equations model

$$y_1 = \alpha_1 y_2 + e_1 \quad \text{—(1)}$$
$$y_2 = \alpha_2 y_1 + \beta_1 x_1 + \beta_2 x_2 + e_2 \quad \text{—(2)}$$

We assume that x_1 and x_2 are exogenous and uncorrelated with the error terms e_1 and e_2.

a. Solve the two structural equations for the reduced-form equation for y_2, that is, $y_2 = \pi_1 x_1 + \pi_2 x_2 + v_2$. Express the reduced-form parameters in terms of the **structural parameters** and the reduced-form error in terms of the structural parameters and e_1 and e_2. Show that y_2 is correlated with e_1.

將①代入②式

$$y_2 = \alpha_2(\alpha_1 y_2 + e_1) + \beta_1 x_1 + \beta_2 x_2 + e_2$$
$$= \alpha_1\alpha_2 y_2 + \alpha_2 e_1 + \beta_1 x_1 + \beta_2 x_2 + e_2$$
$$y_2 - \alpha_1\alpha_2 y_2 = \alpha_2 e_1 + \beta_1 x_1 + \beta_2 x_2 + e_2$$
$$y_2(1-\alpha_1\alpha_2) = \beta_1 x_1 + \beta_2 x_2 + e_2 + \alpha_2 e_1$$
$$y_2 = \frac{\beta_1}{(1-\alpha_1\alpha_2)} x_1 + \frac{\beta_2}{(1-\alpha_1\alpha_2)} x_2 + \frac{e_2+\alpha_2 e_1}{(1-\alpha_1\alpha_2)} = \pi_1 x_1 + \pi_2 x_2 + v_2 \; \#$$

To show the correlation:

$$\mathrm{cov}(y_2, e_1|X) = E[(y_2 - E(y_2|X))(e_1 - E(e_1|X))|X] \quad (因\ E(e_1|X)=0)$$
$$= E(y_2 e_1|X) = E\left[\left(\frac{\beta_1}{(1-\alpha_1\alpha_2)} x_1 + \frac{\beta_2}{(1-\alpha_1\alpha_2)} x_2 + \frac{e_2+\alpha_2 e_1}{(1-\alpha_1\alpha_2)}\right) e_1 \Big| X\right]$$
$$= E\left[\frac{\beta_1}{(1-\alpha_1\alpha_2)} x_1 e_1 \Big| X\right] + E\left[\frac{\beta_2}{(1-\alpha_1\alpha_2)} x_2 e_1 \Big| X\right] + E\left[\left(\frac{e_2+\alpha_2 e_1}{(1-\alpha_1\alpha_2)}\right) e_1 \Big| X\right]$$
$$= 0 + 0 + E\left[\left(\frac{e_2 e_1 + \alpha_2 e_1^2}{(1-\alpha_1\alpha_2)}\right)\Big| X\right]$$

因 X是外生變數與 random errors 無關

$$= \frac{E(e_2 e_1|X) + \alpha_2 E(e_1^2|X)}{(1-\alpha_1\alpha_2)} = \frac{\alpha_2}{(1-\alpha_1\alpha_2)} \sigma_1^2 \neq 0$$

(因假設兩方程式的誤差不相關) (除非 $\alpha_2=0$, 即沒同時性)

b. Which equation parameters are consistently estimated using OLS? Explain.

兩個結構方程式都不是 reduced-form。兩方程式右邊都有內生變數，因此用 OLS 都是不一致且有偏誤的。

c. Which parameters are "identified," in the simultaneous equations sense? Explain your reasoning.

$M=2$，要缺少 $2-1=1$ 個外生變數才是 identified

$$y_1 = \alpha_1 y_2 + e_1 \rightarrow \text{缺 2 個} \rightarrow \text{identified}$$

$$y_2 = \alpha_2 y_1 + \beta_1 x_1 + \beta_2 x_2 + e_2 \rightarrow \text{缺 0 個} \rightarrow \text{非 identified}$$

It is possible to estimate α_1 consistently.

d. To estimate the parameters of the reduced-form equation for y_2 using the method of moments (MOM), which was introduced in Section 10.3, the two moment equations are

$$\underbrace{N^{-1}}_{\frac{1}{N}} \sum x_{i1}(y_2 - \pi_1 x_{i1} - \pi_2 x_{i2}) = 0$$

$$N^{-1} \sum x_{i2}(y_2 - \pi_1 x_{i1} - \pi_2 x_{i2}) = 0$$

Explain why these two moment conditions are a valid basis for obtaining consistent estimators of the reduced-form parameters.

two moment equations mean that V_2 is uncorrelated with X_1, X_2.

$$N^{-1} \sum X_{i1} V_{i1} = 0 \leftarrow E(X_{i1} V_{i1} | X) = 0$$

$$N^{-1} \sum X_{i2} V_{i2} = 0 \leftarrow E(X_{i2} V_{i2} | X) = 0 \Rightarrow \text{consistent}$$

↑ 來自

因 X_1 和 X_2 與 e_1、e_2 不相關，故推導可得 X_1、X_2 和簡化型誤差項 V_2 也不相關

reduced-form: $y_2 = \pi_1 X_1 + \pi_2 X_2 + V_2$

$$= \frac{\beta_1}{1-\alpha_1\alpha_2} x_1 + \frac{\beta_2}{1-\alpha_1\alpha_2} x_2 + \frac{e_2 + \alpha_2 e_1}{1-\alpha_1\alpha_2} \quad \text{From (A)}$$

$$E\left[X_{ik}\left(\frac{e_2+\alpha_2 e_1}{1-\alpha_1\alpha_2}\right) \Big| X\right] = E\left[\frac{1}{1-\alpha_1\alpha_2} \underbrace{X_{ik} e_2}_{\text{題意不相關}} \Big| X\right] + E\left[\frac{\alpha_2}{1-\alpha_1\alpha_2} \underbrace{X_{ik} e_1}_{\text{題意不相關}} \Big| X\right] = 0 + 0 = 0$$

e. Are the MOM estimators in part (d) the same as the OLS estimators? Form the sum of squared errors function for $y_2 = \pi_1 x_1 + \pi_2 x_2 + v_2$ and find the first derivatives. Set these to zero and show that they are equivalent to the two equations in part (d).

OLS-SSE $\rightarrow S(\pi_1,\pi_2 | y, x) = \Sigma (y_2 - \pi_1 x_1 - \pi_2 x_2)^2$

The first derivatives are

$$\frac{\partial S(\pi_1,\pi_2 | y, x)}{\partial \pi_1} = 2\Sigma (y_2 - \pi_1 x_1 - \pi_2 x_2) x_1 = 0$$

$$\frac{\partial S(\pi_1,\pi_2 | y, x)}{\partial \pi_2} = 2\Sigma (y_2 - \pi_1 x_1 - \pi_2 x_2) x_2 = 0$$

MOM

$N^{-1} \Sigma x_{i1} v_{i1} = 0$

$N^{-1} \Sigma x_{i2} v_{i2} = 0$

$\downarrow$

$\hookrightarrow$ Divide these equations by 2 $=$ Multiply the moment equations by N

f. Using $\sum x_{i1}^2 = 1$, $\sum x_{i2}^2 = 1$, $\sum x_{i1}x_{i2} = 0$, $\sum x_{i1}y_{1i} = 2$, $\sum x_{i1}y_{2i} = 3$, $\sum x_{i2}y_{1i} = 3$, $\sum x_{i2}y_{2i} = 4$, and the two moment conditions in part (d) show that the MOM/OLS estimates of π_1 and π_2 are $\hat{\pi}_1 = 3$ and $\hat{\pi}_2 = 4$.

Two moment conditions are

$N^{-1} \Sigma x_{i1} (y_2 - \pi_1 x_{i1} - \pi_2 x_{i2}) = 0$

$N^{-1} \Sigma x_{i2} (y_2 - \pi_1 x_{i1} - \pi_2 x_{i2}) = 0$

$\Sigma x_{i1} y_{2i} = \pi_1 \Sigma x_{i1}^2 + \pi_2 \Sigma x_{i1} x_{i2} \Rightarrow 3 = \hat{\pi}_1 \times 1 + \hat{\pi}_2 \times 0 \Rightarrow \hat{\pi}_1 = 3$ #

$\Sigma x_{i2} y_{2i} = \pi_1 \Sigma x_{i1} x_{i2} + \pi_2 \Sigma x_{i2}^2 \Rightarrow 4 = \hat{\pi}_1 \times 0 + \hat{\pi}_2 \times 1 \Rightarrow \hat{\pi}_2 = 4$ #

g. The fitted value $\hat{y}_2 = \hat{\pi}_1 x_1 + \hat{\pi}_2 x_2$. Explain why we can use the moment condition $\sum \hat{y}_{i2}(y_{i1} - \alpha_1 y_{i2}) = 0$ as a valid basis for consistently estimating α_1. Obtain the IV estimate of α_1.

The first structural equation is $y_1 = \alpha_1 y_2 + e_1$，因 $\hat{y}_{2i}$ 是用外生變數預測出來的 y_2 擬合值，故它和 e_i 不相關，因此 $E[(\pi_1 x_1 + \pi_2 x_2) e_1 | x] = E[(\pi_1 x_1 + \pi_2 x_2)(y_1 - \alpha_1 y_2) | x] = 0$

$\Rightarrow$ 對應式為 $N^{-1} \Sigma (\pi_1 x_{i1} + \pi_2 x_{i2})(y_{i1} - \alpha_1 y_{i2}) = 0$ $\quad \text{plim}\, \hat{\pi}_1 = \pi_1$ and $\hat{\pi}_2 = \pi_2$

$\Sigma (\hat{\pi}_1 x_{i1} + \hat{\pi}_2 x_{i2})(y_{i1} - \alpha_1 y_{i2}) = \Sigma \hat{y}_{i2}(y_{i1} - \alpha_1 y_{i2}) = 0$ #

$\Sigma \hat{y}_{i2} y_{i1} - \alpha_1 \Sigma \hat{y}_{i2} y_{i2} = 0 \quad \hat{\alpha}_1 = \dfrac{\Sigma \hat{y}_{i2} y_{i1}}{\Sigma \hat{y}_{i2} y_{i2}}$

$$\hat{\alpha}_1 = \frac{\Sigma \hat{y}_{i2} y_{i1}}{\Sigma \hat{y}_{i2} y_{i2}} = \frac{\Sigma (\hat{\pi}_1 x_{i1} + \hat{\pi}_2 x_{i2}) y_{i1}}{\Sigma (\hat{\pi}_1 x_{i1} + \hat{\pi}_2 x_{i2}) y_{i2}} = \frac{\hat{\pi}_1 \Sigma x_{i1} y_{i1} + \hat{\pi}_2 \Sigma x_{i2} y_{i1}}{\hat{\pi}_1 \Sigma x_{i1} y_{i2} + \hat{\pi}_2 \Sigma x_{i2} y_{i2}} = \frac{3 \times 2 + 4 \times 3}{3 \times 3 + 4 \times 4}$$

$$= \frac{18}{25}$$ #

h. Find the 2SLS estimate of α_1 by applying OLS to $y_1 = \alpha_1 \hat{y}_2 + e_1^*$. Compare your answer to that in part (g).

$$\hat{\alpha}_{1,2SLS} = \frac{\sum \hat{y}_{i2} y_{i1}}{\sum \hat{y}_{i2}^2}$$

To prove $\hat{\alpha}_{1,2SLS} = \hat{\alpha}$ (by moment condition) #

we need to prove $\sum \hat{y}_2^2 = \sum \hat{y}_2 y_2$ $\quad \hat{V}_2 = y_2 - \hat{y}_2 \Rightarrow \hat{y}_2 = y_2 - \hat{V}_2$

$$\sum \hat{y}_2^2 = \sum \hat{y}_2 (y_2 - \hat{V}_2) = \sum \hat{y}_2 y_2 - \underbrace{\cancel{\sum \hat{y}_2 \hat{V}_2}}_{=0} = \sum \hat{y}_2 y_2 \text{ (∵解釋變數和誤差無關)}$$

11.16 Consider the following supply and demand model

$$\text{Demand: } Q_i = \alpha_1 + \alpha_2 P_i + e_{di}, \quad \text{Supply: } Q_i = \beta_1 + \beta_2 P_i + \beta_3 W_i + e_{si}$$

where Q is the quantity, P is the price, and W is the wage rate, which is assumed exogenous. Data on these variables are in Table 11.7.

TABLE 11.7 Data for Exercise 11.16

Q	P	W
4	2	2
6	4	3
9	3	1
3	5	1
8	8	3

a. Derive the algebraic form of the reduced-form equations, $Q = \theta_1 + \theta_2 W + v_2$ and $P = \pi_1 + \pi_2 W + v_1$, expressing the reduced-form parameters in terms of the structural parameters.

$$\alpha_1 + \alpha_2 P_i + e_{di} = \beta_1 + \beta_2 P_i + \beta_3 W_i + e_{si}$$

$$(\alpha_2 - \beta_2) P_i = \beta_1 - \alpha_1 + \beta_3 W_i + e_{si} - e_{di}$$

$$P_i = \frac{\beta_1 - \alpha_1}{\alpha_2 - \beta_2} + \frac{\beta_3}{\alpha_2 - \beta_2} W_i + \frac{e_{si} - e_{di}}{\alpha_2 - \beta_2} = \pi_1 + \pi_2 W + V_1$$

$$\pi_1 = \frac{\beta_1 - \alpha_1}{\alpha_2 - \beta_2} \quad \pi_2 = \frac{\beta_3}{\alpha_2 - \beta_2} \quad V_1 = \frac{e_{si} - e_{di}}{\alpha_2 - \beta}$$

$$Q_i = \alpha_1 + \alpha_2(\pi_1 + \pi_2 W + V_1) + e_{di}$$

$$= (\alpha_1 + \alpha_2 \pi_1) + \alpha_2 \pi_2 W + (\alpha_2 V_1 + e_{di})$$

$$= \left(\alpha_1 + \alpha_2 \times \frac{\beta_1 - \alpha_1}{\alpha_2 - \beta_2}\right) + \left(\alpha_2 \times \frac{\beta_3}{\alpha_2 - \beta_2}\right) W + \left(\alpha_2 \times \frac{e_{si} - e_{di}}{\alpha_2 - \beta_2} + e_{di}\right)$$

$$= \frac{\alpha_2 \beta_1 - \alpha_1 \beta_2}{\alpha_2 - \beta_2} + \frac{\alpha_2 \beta_3}{\alpha_2 - \beta_2} W + \frac{\alpha_2 e_{si} - \beta_2 e_{di}}{\alpha_2 - \beta_2}$$

$$= \theta_1 + \theta_2 W + V_2$$

$$\theta_1 = \frac{\alpha_2 \beta_1 - \alpha_1 \beta_2}{\alpha_2 - \beta_2}, \quad \theta_2 = \frac{\alpha_2 \beta_3}{\alpha_2 - \beta_2}, \quad V_2 = \frac{\alpha_2 e_{si} - \beta_2 e_{di}}{\alpha_2 - \beta_2}$$

b. Which structural parameters can you solve for from the results in part (a)? Which equation is "identified"?

(1) $\alpha_1 . \alpha_2$

(2) $M = 2$ $\quad 2-1=1$ 要少至少1個 variable

Demand → identified (少1個)

Supply → not identified (少0個)

c. The estimated reduced-form equations are $\hat{Q} = 5 + 0.5W$ and $\hat{P} = 2.4 + 1W$. Solve for the identified structural parameters. This is the method of **indirect least squares**.

$\hat{P} = 2.4 + 1W = \pi_1 + \pi_2 W \qquad \hat{\pi}_1 = 2.4 \quad \hat{\pi}_2 = 1$

$\hat{Q} = 5 + 0.5W = \theta_1 + \theta_2 W \qquad \hat{\theta}_1 = 5 \quad \hat{\theta}_2 = 0.5$

$5 + 0.5W = \alpha_1 + \alpha_2(2.4 + 1W) = \alpha_1 + 2.4\alpha_2 + \alpha_2 W$

$\alpha_2 = 0.5$ # $\qquad \alpha_1 = 3.8$ #

d. Obtain the fitted values from the reduced-form equation for P, and apply 2SLS to obtain estimates of the demand equation.

$\hat{P}_i = 2.4 + W_i$

$Q_i = \alpha_1 + \alpha_2 \hat{P}_i + e_i$

$\alpha_2 = \dfrac{\Sigma(\hat{P}_i - \bar{\hat{P}})(Q_i - \bar{Q})}{\Sigma(\hat{P}_i - \bar{\hat{P}})^2} = \dfrac{2}{4}$

$= 0.5$

$\bar{Q}_i = \alpha_1 + 0.5\bar{\hat{P}}_i$

$\alpha_1 = 6 - 0.5 \times 4.4 = 3.8$

$\Rightarrow \hat{Q} = 3.8 + 0.5\hat{P}$ #

TABLE 11.7 Data for Exercise 11.16

Q	P	W	$\hat{P}_i$	$\hat{P}_i - \bar{\hat{P}}_i$	$Q_i - \bar{Q}$
4	2	2	4.4	0	-2
6	4	3	5.4	1	0
9	3	1	3.4	-1	3
3	5	1	3.4	-1	-3
8	8	3	5.4	1	2

$\bar{\hat{P}}_i = 4.4$

11.17 Example 11.3 introduces Klein's Model I.

a. Do we have an adequate number of IVs to estimate each equation? Check the necessary condition for the identification of each equation. The necessary condition for identification is that in a system of M equations at least $M-1$ variables must be omitted from each equation.

b. An equivalent identification condition is that the number of excluded exogenous variables from the equation must be at least as large as the number of included right-hand side endogenous variables. Check that this condition is satisfied for each equation.

c. Write down in econometric notation the first-stage equation, the reduced form, for W_{1t}, wages of workers earned in the private sector. Call the parameters $\pi_1, \pi_2, \ldots$

d. Describe the two regression steps of 2SLS estimation of the consumption function. This is not a question about a computer software command.

e. Does following the steps in part (d) produce regression results that are identical to the 2SLS estimates provided by software specifically designed for 2SLS estimation? In particular, will the t-values be the same?

a. M=8 至少要少 8-1=7個變數在每个方程式

$CN_t = \alpha_1 + \alpha_2(W_{1t} + W_{2t}) + \alpha_3 P_t + \alpha_4 P_{t-1} + e_{1t}$ ⇒ 包含6個變數,省略10個 >7 → identification

$I_t = \beta_1 + \beta_2 P_t + \beta_3 P_{t-1} + \beta_4 K_{t-1} + e_{2t}$ ⇒ 包含5個變數,省略11個 >7 → identification

$W_{1t} = \gamma_1 + \gamma_2 E_t + \gamma_3 E_{t-1} + \gamma_4 TIME_t + e_{3t}$ ⇒ 包含5個變數,省略11個 >7 → identification

b. as IVs. The exogenous variables are government spending, G_t, public sector wages, W_{2t}, taxes, TX_t, and the time trend variable, $TIME_t$. Another exogenous variable is the constant term, the "intercept" variable in each equation, $X_{1t} \equiv 1$. The predetermined variables are lagged profits, P_{t-1}, the lagged capital stock, K_{t-1}, and the lagged total national product minus public sector wages, E_{t-1}.

每个方程式中被排除的外生變數的數量,必須至少和該方程式中包含的右边内生數量一樣多 → 另一等價識別

$CN_t = \alpha_1 + \alpha_2(W_{1t} + W_{2t}) + \alpha_3 P_t + \alpha_4 P_{t-1} + e_{1t}$ ⇒ 外生共8個,這有3個,排除5個外生 >2個内生 ⇒ identification

$I_t = \beta_1 + \beta_2 P_t + \beta_3 P_{t-1} + \beta_4 K_{t-1} + e_{2t}$ ⇒ 外生共8個,這有3個,排除5個外生 > 1個内生 ⇒ identification

$W_{1t} = \gamma_1 + \gamma_2 E_t + \gamma_3 E_{t-1} + \gamma_4 TIME_t + e_{3t}$ ⇒ 外生共8個,這有3個,排除5個外生 > 1個内生 ⇒ identification

c. $W_{1t} = \pi_1 + \pi_2 G_t + \pi_3 W_{2t} + \pi_4 TX_t + \pi_5 TIME_t + \pi_6 P_{t-1} + \pi_7 K_{t-1} + \pi_8 E_{t-1} + v$

d. Obtain fitted values $\hat{W}_{1t}$ from the estimated reduced form equation in part (c) and similarly obtain $\hat{P}_t$. Create $W_t^* = \hat{W}_{1t} + W_{2t}$. Regress CN_t on W_t^*, $\hat{P}_t$ and P_{t-1} plus a constant by OLS.

e. The coefficient estimates will be the same. The t-values will not be because the standard errors in part (d) are not correct 2SLS standard errors.