**11.1** Our aim is to estimate the parameters of the simultaneous equations model

$$y_1 = \alpha_1 y_2 + e_1 \quad \cdots\cdots (1)$$
$$y_2 = \alpha_2 y_1 + \beta_1 x_1 + \beta_2 x_2 + e_2 \quad \cdots\cdots (2)$$

We assume that $x_1$ and $x_2$ are exogenous and uncorrelated with the error terms $e_1$ and $e_2$.

**a.** Solve the two structural equations for the reduced-form equation for $y_2$, that is, $y_2 = \pi_1 x_1 + \pi_2 x_2 + v_2$. Express the reduced-form parameters in terms of the **structural parameters** and the reduced-form error in terms of the structural parameters and $e_1$ and $e_2$. Show that $y_2$ is correlated with $e_1$.

目標：將 $y_2$ 完全用外生變數 $x_1, x_2$ 表示

(1) 代入 (2)

$$y_2 = \alpha_2(\alpha_1 y_2 + e_1) + \beta_1 x_1 + \beta_2 x_2 + e_2$$

$$\Rightarrow (1-\alpha_1\alpha_2) y_2 = \beta_1 x_1 + \beta_2 x_2 + \alpha_2 e_1 + e_2$$

$$\Rightarrow y_2 = \underbrace{\frac{\beta_1}{1-\alpha_1\alpha_2}}_{\pi_1} x_1 + \underbrace{\frac{\beta_2}{1-\alpha_1\alpha_2}}_{\pi_2} x_2 + \underbrace{\frac{\alpha_2 e_1 + e_2}{1-\alpha_1\alpha_2}}_{v_2}$$

!! $v_2 = \dfrac{\alpha_2 e_1 + e_2}{1-\alpha_1\alpha_2}$ 包含 $e_1$，∴ $y_2$ 包含 $e_1$ 的部分，$y_2$ is correlated with $e_1$

**b.** Which equation parameters are consistently estimated using OLS? Explain.

$\alpha_1$ 不能用 OLS 因為 $y_2$ 與 $e_1$ 相關

$\alpha_2$ 不能用 OLS 因為 $y_1$ 由 $y_2$ 決定，可能與 $e_2$ 相關

Reduce form parameters $(\pi_1, \pi_2)$ 可以用 OLS

**c.** Which parameters are "identified," in the simultaneous equations sense? Explain your reasoning.

內生變數 $M=2$ $(y_1, y_2)$

外生變數：$x_1, x_2$

fcn (1)：$y_1 = \alpha_1 y_2 + e_1$

使用的內生變數：$y_2$

使用的外生變數：0

階數條件：要識別一條包含 $M=2$ 個內生變數的方程，至少需要排除 $M-1=1$ 個外生變數

∵ 排除了兩個 ∴ 可識別，$\alpha_1$ is identified

fcn (2)：$y_2 = \alpha_2 y_1 + \beta_1 x_1 + \beta_2 x_2 + e_2$

使用的內生變數：$y_1$

使用的外生變數：$x_1, x_2$

∵ 排除了 0 個 ∴ 不可識別，$\alpha_2, \beta_1, \beta_2$ are unidentified

**d.** To estimate the parameters of the reduced-form equation for $y_2$ using the method of moments (MOM), which was introduced in Section 10.3, the two moment equations are

$$N^{-1}\sum x_{i1}\left(y_2 - \pi_1 x_{i1} - \pi_2 x_{i2}\right) = 0$$

$$N^{-1}\sum x_{i2}\left(y_2 - \pi_1 x_{i1} - \pi_2 x_{i2}\right) = 0$$

Explain why these two moment conditions are a valid basis for obtaining consistent estimators of the reduced-form parameters.

valid：① $x_1, x_2$ are exogenous

② $E[X_1 V_2]=0$ , $E[X_2V_2]=0$ ($V_2$與$X_1$,$X_2$無關)

consistent: $\hat{\theta}_N \xrightarrow{P} \theta_0$, $N\to\infty$,

$N\to\infty$

$$\frac{1}{N}\sum_{i=1}^{N} x_{ij}(y_{2i}-\pi_1 x_{i1}-\pi_2 x_{i2}) \xrightarrow{P} E[X_j(Y_2-\pi_1X_1-\pi_2X_2)]=0$$

$j=1,2$

$\therefore (\hat{\pi}_1, \hat{\pi}_2) \xrightarrow{P} (\pi_1,\pi_2)$ when $N\to\infty$

$\therefore$ it is consistent

**e.** Are the MOM estimators in part (d) the same as the OLS estimators? Form the sum of squared errors function for $y_2 = \pi_1 x_1 + \pi_2 x_2 + v_2$ and find the first derivatives. Set these to zero and show that they are equivalent to the two equations in part (d).

By OLS, $\min S(\pi_1,\pi_2)=\sum_{i=1}^{N}(y_{2i}-\pi_1 x_{i1}-\pi_2 x_{i2})^2$

一階偏微

$$\frac{\partial S}{\partial \pi_1} = -\cancel{2}\sum_{i=1}^{N} x_{i1}(y_{2i}-\pi_1 x_{i1}-\pi_2 x_{i2})=0$$

$$\frac{\partial S}{\partial \pi_2} = -\cancel{2}\sum_{i=1}^{N} x_{i2}(y_{2i}-\pi_1 x_{i1}-\pi_2 x_{i2})=0$$

同除N，結果與MOM估計量相同

**f.** Using $\sum x_{i1}^2 = 1$, $\sum x_{i2}^2 = 1$, $\sum x_{i1}x_{i2} = 0$, $\sum x_{i1}y_{1i} = 2$, $\sum x_{i1}y_{2i} = 3$, $\sum x_{i2}y_{1i} = 3$, $\sum x_{i2}y_{2i} = 4$, and the two moment conditions in part (d) show that the MOM/OLS estimates of $\pi_1$ and $\pi_2$ are $\hat{\pi}_1 = 3$ and $\hat{\pi}_2 = 4$.

原式 $\Rightarrow \sum x_{i1} y_{2i} = \pi_1 \sum x_{i1}^2 + \pi_2 \sum x_{i1} x_{i2}$

$\sum x_{i2} y_{2i} = \pi_1 \sum x_{i1} x_{i2} + \pi_2 \sum x_{i2}^2$

代入資料

$\Rightarrow \quad 3 = \hat{\pi}_1 + \hat{\pi}_2 \cdot 0 \quad \Rightarrow \hat{\pi}_1 = 3$

$4 = \hat{\pi}_1 \cdot 0 + \hat{\pi}_2 \quad\quad \hat{\pi}_2 = 4$

**g.** The fitted value $\hat{y}_2 = \hat{\pi}_1 x_1 + \hat{\pi}_2 x_2$. Explain why we can use the moment condition $\sum \hat{y}_{i2}(y_{i1} - \alpha_1 y_{i2}) = 0$ as a valid basis for consistently estimating $\alpha_1$. Obtain the IV estimate of $\alpha_1$.

$\hat{y}_{2i} = 3x_{i1} + 4x_{i2}$

$\because$ ① $x_1, x_2$ are exogeneous

② $\hat{y}_{2i}$ is uncorrelated with $e_{1i}$

$\therefore$ $\hat{y}_{2i}$ 可以當作 IV

$\because \sum \hat{y}_{2i} y_{1i} = \alpha_1 \sum \hat{y}_{2i} y_{2i}$

$$\Rightarrow \hat{\alpha}_1 = \frac{\sum \hat{y}_{2i} y_{1i}}{\sum \hat{y}_{2i} y_{2i}} = \frac{\sum (3x_{i1} + 4x_{i2}) y_{1i}}{\sum (3x_{i1} + 4x_{i2}) y_{2i}}$$

$$= \frac{3\sum x_{i1} y_{1i} + 4\sum x_{i2} y_{1i}}{3\sum x_{i1} y_{2i} + 4\sum x_{i2} y_{2i}}$$

$$= \frac{3 \cdot 2 + 4 \cdot 3}{3 \cdot 3 + 4 \cdot 4} = \frac{18}{25}$$

**h.** Find the 2SLS estimate of $\alpha_1$ by applying OLS to $y_1 = \alpha_1 \hat{y}_2 + e_1^*$. Compare your answer to that in part (g).

By OLS, $\min S(\alpha_1) = \sum (y_{1i} - \alpha_1 \hat{y}_{2i})^2$

$$\frac{dS}{d\alpha_1} = \sum_{i=1}^{N} 2(y_{1i} - \alpha_1 \hat{y}_{2i})(-\hat{y}_{2i}) = 0$$

$$\Rightarrow \sum_{i=1}^{N} \hat{y}_{2i}(y_{1i} - \alpha_1 \hat{y}_{2i}) = 0$$

展開 $\Rightarrow \sum_{i=1}^{N} \hat{y}_{2i} y_{1i} - \alpha_1 \sum_{i=1}^{N} \hat{y}_{2i}^2 = 0$

$$\Rightarrow \alpha_1 = \frac{\sum \hat{y}_{2i} y_{1i}}{\sum \hat{y}_{2i}^2} = \frac{18}{\sum (3x_{i1} + 4x_{i2})^2}$$

$$= \frac{18}{\sum [9x_{i1}^2 + 24x_{i1}x_{i2} + 16x_{i2}^2]}$$

$$= \frac{18}{9 \cdot (1) + 24(0) + 16 \cdot (1)}$$

$$= \frac{18}{25}$$

$$\therefore \hat{\alpha}_1^{2LS} = \hat{\alpha}_1^{IV} = \frac{18}{25}$$

**11.16** Consider the following supply and demand model

$$\text{Demand: } Q_i = \alpha_1 + \alpha_2 P_i + e_{di}, \ (1) \qquad \text{Supply: } Q_i = \beta_1 + \beta_2 P_i + \beta_3 W_i + e_{si} \ \cdots(2)$$

where $Q$ is the quantity, $P$ is the price, and $W$ is the wage rate, which is assumed exogenous. Data on these variables are in Table 11.7.

**TABLE 11.7** Data for Exercise 11.16

| $Q$ | $P$ | $W$ |
|---|---|---|
| 4 | 2 | 2 |
| 6 | 4 | 3 |
| 9 | 3 | 1 |
| 3 | 5 | 1 |
| 8 | 8 | 3 |

**a.** Derive the algebraic form of the reduced-form equations, $Q = \theta_1 + \theta_2 W + v_2$ and $P = \pi_1 + \pi_2 W + v_1$, expressing the reduced-form parameters in terms of the structural parameters.

combine (1) and (2)

$$\alpha_1 + \alpha_2 P_i + e_{di} = \beta_1 + \beta_2 P_i + \beta_3 W_i + e_{si}$$

$$\Rightarrow (\alpha_2 - \beta_2) P_i = \beta_1 - \alpha_1 + \beta_3 W_i + e_{si} - e_{di}$$

If $\alpha_2 \neq \beta_2$

$$\Rightarrow P_i = \underbrace{\frac{\beta_1 - \alpha_1}{\alpha_2 - \beta_2}}_{\pi_1} + \underbrace{\frac{\beta_3}{\alpha_2 - \beta_2}}_{\pi_2} W_i + \underbrace{\frac{e_{si} - e_{di}}{\alpha_2 - \beta_2}}_{v_{1i}} \cdots (3)$$

(3) 代回 (2)

$$Q_i = \beta_1 + \beta_2\left(\frac{\beta_1 - \alpha_1}{\alpha_2 - \beta_2} + \frac{\beta_3}{\alpha_2 - \beta_2} W_i + \frac{e_{si} - e_{di}}{\alpha_2 - \beta_2}\right) + \beta_3 W_i + e_{si}$$

$$= \underbrace{\left(\beta_1 + \beta_2 \frac{\beta_1 - \alpha_1}{\alpha_2 - \beta_2}\right)}_{\theta_1} + \underbrace{\left(\beta_3 + \frac{\beta_2 \beta_3}{\alpha_2 - \beta_2}\right)}_{\theta_2} W_i + \underbrace{\left(\frac{\beta_2(e_{si} - e_{di})}{\alpha_2 - \beta_2} + e_{si}\right)}_{v_{2i}}$$

**b.** Which structural parameters can you solve for from the results in part (a)? Which equation is "identified"?

M=2，至少需排除 2−1 個外生變數

Demand fcn 排除 $W_i$ ∴ identified

Supply fcn 沒有排除 ∴ not identified

**c.** The estimated reduced-form equations are $\hat{Q} = \overset{\theta_1}{5} + \overset{\theta_2}{0.5}W$ and $\hat{P} = \overset{\pi_1}{2.4} + \overset{\pi_2}{1}W$. Solve for the iden~~tified structural parameters.~~ This is the method of **indirect least squares**.

(demand fcn ⇒ $\alpha_1, \alpha_2$)

$$Q_i = \theta_1 + \theta_2 W_i + V_{2i} = \alpha_1 + \alpha_2 P_i + e_{di}$$

$P_i = \pi_1 + \pi_2 W_i + V_{1i}$ 代入

$$Q_i = (\alpha_1 + \alpha_2\pi_1) + \alpha_2\pi_2 W_i + \varepsilon$$

$$\therefore \begin{matrix} \theta_1 = \alpha_1 + \alpha_2\pi_1 \\ \theta_2 = \alpha_2\pi_2 \end{matrix} \Rightarrow \begin{cases} 5 = \alpha_1 + \alpha_2 \cdot (2.4) \\ 0.5 = \alpha_2 \cdot (1) \end{cases} \Rightarrow \begin{cases} \alpha_1 = 3.8 \\ \alpha_2 = 0.5 \end{cases}$$

**d.** Obtain the fitted values from the reduced-form equation for *P*, and apply 2SLS to obtain estimates of the demand equation.

first-stage

$\hat{P} = 2.4 + 1W$

| $W_i$ | $\hat{P}_i$ |
|---|---|
| 2 | 4.4 |
| 3 | 5.4 |
| 1 | 3.4 |
| 1 | 3.4 |
| 3 | 5.4 |

second-stage

| $Q_i$ | $\hat{P}_i$ | $\hat{P}_i - \bar{\hat{P}}$ | $Q_i - \bar{Q}$ |
|---|---|---|---|
| 4 | 4.4 | 0 | −2 |
| 6 | 5.4 | 1 | 0 |
| 9 | 3.4 | −1 | 3 |
| 3 | 3.4 | −1 | −3 |
| 8 | 5.4 | 1 | 2 |

$\bar{Q} = 6$ $\quad \bar{\hat{P}} = 4.4$

$$\hat{\alpha}_2 = \frac{\Sigma(\hat{P}_i - \bar{\hat{P}})(Q_i - \bar{Q})}{\Sigma(\hat{P}_i - \bar{\hat{P}})^2} = \frac{2}{4} = 0.5$$

$$\hat{\alpha}_1 = \bar{Q} - \alpha_2\bar{\hat{P}} = 6 - 0.5(4.4) = 3.8$$

By 2LS: $Q = 3.8 + 0.5P$

**11.17** Example 11.3 introduces Klein's Model I.

**a.** Do we have an adequate number of IVs to estimate each equation? Check the necessary condition for the identification of each equation. The necessary condition for identification is that in a system of $M$ equations at least $M - 1$ variables must be omitted from each equation.

Klein's Model I

consumption fcn: $CN_t = \alpha_1 + \alpha_2(W_{1t} + W_{2t}) + \alpha_3 P_t + \alpha_4 P_{t-1} + e_{1t}$

investment equation: $I_t = \beta_1 + \beta_2 P_t + \beta_3 P_{t-1} + \beta_4 K_{t-1} + e_{2t}$

wage equation: $W_{1t} = \gamma_1 + \gamma_2 E_t + \gamma_3 E_{t-1} + \gamma_4 TIME_t + e_{3t}$

$W_t = W_{1t} + W_{2t}$. $E_t = CN_t + I_t + (G_t - W_{2t})$. $TIME_t = YEAR_t - 1931$

$M = 8$（三條估計式和五條定義式）

至少需排除7個外生變數

共16個 variables

8 endogenous variabes: $CN_t, I_t, W_{1t}, W_t, P_t, E_t, Y_t, K_t$

8 exogenous variables: $G_t, W_{2t}, TX_t, TIME_t, X_{1t}, P_{t-1}, K_{t-1}, E_{t-1}$

consumption fcn: use 6 variable and omit 10

investment fcn: use 5 variable and omit 11

wage equation: use 5 variable and omit 11

All three fcn are identified.

**b.** An equivalent identification condition is that the number of excluded exogenous variables from the equation must be at least as large as the number of included right-hand side endogenous variables. Check that this condition is satisfied for each equation.

方程排除的外生變數 ≥ 方程右側包含的內生變數

| | 排除的外生 | 包含的內生 |
|---|---|---|
| consumption fcn: | $G_t, TX_t, TIME_t, K_{t-1}, E_{t-1}$ | $W_{1t}, P_t$ |
| investment fcn: | $G_t, TX_t, TIME_t, K_{t-1}, E_{t-1}$ | $P_t$ |
| wage equation: | $G_t, W_{2t}, TX_t, P_{t-1}, K_{t-1}$ | $E_t$ |

All three fcn satisfied

**c.** Write down in econometric notation the first-stage equation, the reduced form, for $W_{1t}$, wages of workers earned in the private sector. Call the parameters $\pi_1, \pi_2, \ldots$

不使用內生變數表示

$$W_{1t} = \pi_1 + \pi_2 G_t + \pi_3 W_{2t} + \pi_4 TIME_t + \pi_5 TX_t + \pi_6 P_{t-1} + \pi_7 K_{t-1} + \pi_8 E_{t-1} + v_t$$

**d.** Describe the two regression steps of 2SLS estimation of the consumption function. This is not a question about a computer software command.

對內生變數做IV回歸

$$\hat{W}_{1t} = \pi_{11} + \pi_{12} G_t + \pi_{13} W_{2t} + \pi_{14} TIME_t + \pi_{15} TX_t + \pi_{16} P_{t-1} + \pi_{17} K_{t-1} + \pi_{18} E_{t-1}$$

$$\hat{P}_t = \pi_{21} + \pi_{22} G_t + \pi_{23} W_{2t} + \pi_{24} TIME_t + \pi_{25} TX_t + \pi_{26} P_{t-1} + \pi_{27} K_{t-1} + \pi_{28} E_{t-1}$$

用預測值做OLS

$$CN_t = \alpha_1 + \alpha_2 (\hat{W}_{1t} + W_{2t}) + \alpha_3 \hat{P}_t + \alpha_4 P_{t-1} + u_t$$

**e.** Does following the steps in part (d) produce regression results that are identical to the 2SLS estimates provided by software specifically designed for 2SLS estimation? In particular, will the *t*-values be the same?

估計值相同

t值不同 (part d 低估 SE)