11.1 Our aim is to estimate the parameters of the simultaneous equations model

$$y_1 = \alpha_1 y_2 + e_1$$
$$y_2 = \alpha_2 y_1 + \beta_1 x_1 + \beta_2 x_2 + e_2$$

We assume that x_1 and x_2 are exogenous and uncorrelated with the error terms e_1 and e_2.

a. Solve the two structural equations for the reduced-form equation for y_2, that is, $y_2 = \pi_1 x_1 + \pi_2 x_2 + v_2$. Express the reduced-form parameters in terms of the **structural parameters** and the reduced-form error in terms of the structural parameters and e_1 and e_2. Show that y_2 is correlated with e_1.

b. Which equation parameters are consistently estimated using OLS? Explain.

c. Which parameters are "identified," in the simultaneous equations sense? Explain your reasoning.

d. To estimate the parameters of the reduced-form equation for y_2 using the method of moments (MOM), which was introduced in Section 10.3, the two moment equations are

$$N^{-1}\sum x_{i1}(y_2 - \pi_1 x_{i1} - \pi_2 x_{i2}) = 0$$
$$N^{-1}\sum x_{i2}(y_2 - \pi_1 x_{i1} - \pi_2 x_{i2}) = 0$$

Explain why these two moment conditions are a valid basis for obtaining consistent estimators of the reduced-form parameters.

e. Are the MOM estimators in part (d) the same as the OLS estimators? Form the sum of squared errors function for $y_2 = \pi_1 x_1 + \pi_2 x_2 + v_2$ and find the first derivatives. Set these to zero and show that they are equivalent to the two equations in part (d).

f. Using $\sum x_{i1}^2 = 1$, $\sum x_{i2}^2 = 1$, $\sum x_{i1}x_{i2} = 0$, $\sum x_{i1}y_{1i} = 2$, $\sum x_{i1}y_{2i} = 3$, $\sum x_{i2}y_{1i} = 3$, $\sum x_{i2}y_{2i} = 4$, and the two moment conditions in part (d) show that the MOM/OLS estimates of π_1 and π_2 are $\hat{\pi}_1 = 3$ and $\hat{\pi}_2 = 4$.

g. The fitted value $\hat{y}_2 = \hat{\pi}_1 x_1 + \hat{\pi}_2 x_2$. Explain why we can use the moment condition $\sum \hat{y}_{i2}(y_{i1} - \alpha_1 y_{i2}) = 0$ as a valid basis for consistently estimating α_1. Obtain the IV estimate of α_1.

h. Find the 2SLS estimate of α_1 by applying OLS to $y_1 = \alpha_1 \hat{y}_2 + e_1^*$. Compare your answer to that in part (g).

(a) $y_2 = \alpha_2(\alpha_1 y_2 + e_1) + \beta_1 x_1 + \beta_2 x_2 + e_2$

$\Rightarrow y_2(1-\alpha_1\alpha_2) = \beta_1 x_1 + \beta_2 x_2 + (\alpha_2 e_1 + e_2)$

$\Rightarrow y_2 = \left(\dfrac{\beta_1}{1-\alpha_1\alpha_2}\right)x_1 + \left(\dfrac{\beta_2}{1-\alpha_1\alpha_2}\right)x_2 + \left(\dfrac{\alpha_2 e_1 + e_2}{1-\alpha_1\alpha_2}\right)$

$= \pi_1 x_1 + \pi_2 x_2 + v_2$

$\operatorname{cov}(y_2, e_1 | x) = E(y_2 \cdot e_1 | x) = E\left[\left(\dfrac{\beta_1}{1-\alpha_1\alpha_2}x_1 + \dfrac{\beta_2}{1-\alpha_1\alpha_2}x_2 + \dfrac{\alpha_2 e_1 + e_2}{1-\alpha_1\alpha_2}\right)e_1 \,\middle|\, x\right]$

$= E\left[\left(\dfrac{\beta_1}{1-\alpha_1\alpha_2}x_1 e_1\right)\middle| x\right] + E\left[\left(\dfrac{\beta_1}{1-\alpha_1\alpha_2}x_2 e_1\right)\middle| x\right] + E\left[\left(\dfrac{\alpha_2 e_1 + e_2}{1-\alpha_1\alpha_2}e_1\right)\middle| x\right]$

$= 0 + 0 + E\left[\left(\dfrac{\alpha_2 e_1 + e_2}{1-\alpha_1\alpha_2}e_1\right)\middle| x\right]$

$\to \operatorname{cov}(y_2, e_1 | x) = E(y_2 \cdot e_1 | x) = \dfrac{E(e_1 e_2 | x) + \alpha_2 E(e_1^2 | x)}{1-\alpha_1\alpha_2} = \dfrac{\alpha_2 \cdot \sigma_1^2}{1-\alpha_1\alpha_2}$

by consider the error of two equation are uncorrelated

$\operatorname{cov}(y_2, e_1 | x)$ isn't zero unless $\alpha_2 = 0$.

(b) Since equation (1), (2) have an endogenous variable on the right-hand side, the OLS is biased and inconsistent

(c) $M=2$ At least $M-1=1$ variable needs to be omitted from equation

Equation (1) omitted two exogenous variables larger than one variable $\to$ "identified"

Equation (2) omitted no variables $\to$ "not indentified"

(d) $E(x_{i1} v_{i1} | x) = E(x_{i2} v_{i2} | x) = 0$

$\Rightarrow E(x_{ik} (\frac{\alpha_1 e_1 + e_2}{1-\alpha_1\alpha_2}) | x]$

$= E[\frac{\alpha_1}{1-\alpha_1\alpha_2} e_1 x_{ik} | x] + E[\frac{1}{1-\alpha_1\alpha_2} e_2 x_{ik} | x] = 0$

(e) $\min \sum (y_2 - \pi_1 x_1 - \pi_2 x_2)^2$

对 π_1 偏微 : $2\sum(y_2 - \pi_1 x_1 - \pi_2 x_2) \times (-x_1) = 0$

对 π_2 偏微 : $2\sum(y_2 - \pi_1 x_1 - \pi_2 x_2) \times (-x_2) = 0$

$\hookrightarrow \div 2, \times N^{-1}$

so they are equivalente to the two equations.

(f.) $N^{-1}\sum x_{i1}(y_2 - \pi_1 x_{i1} - \pi_2 x_{i2}) = 0 \Rightarrow \sum x_{i1} y_2 - \pi_1 \sum x_{i1}^2 - \pi_2 \sum x_{i1} x_{i2} = 0$

$N^{-1}\sum x_{i2}(y_2 - \pi_1 x_{i1} - \pi_2 x_{i2}) = 0 \Rightarrow \sum x_{i2} y_2 - \pi_1 \sum x_{i1} x_{i2} - \pi_2 \sum x_{i2}^2 = 0$

$\Rightarrow \begin{cases} 3 - \hat{\pi}_1 \times 1 - \pi_2 \times 0 = 0 \\ 4 - \hat{\pi}_1 \times 0 - \pi_2 \times 1 = 0 \end{cases} \Rightarrow \begin{cases} \hat{\pi}_1 = 3 \\ \hat{\pi}_2 = 4 \end{cases}$

(g) $\because y_1 = \alpha_1 y_2 + e_1$ $\quad \hat{y}_2 = \hat{\pi}_1 x_1 + \hat{\pi}_2 x_2$

moment condition $(y_2, e_1) \Rightarrow \sum \hat{y}_2 (y_1 - \alpha_1 y_2) = 0 \Rightarrow \sum \hat{y}_2 y_1 - \alpha_1 \sum \hat{y}_2 y_2 = 0 \Rightarrow \alpha_1 = \frac{\sum \hat{y}_2 y_1}{\sum \hat{y}_2 y_2}$

$\Rightarrow \hat{\alpha}_1 = \frac{\sum(\hat{\pi}_1 x_1 + \hat{\pi}_2 x_2) y_1}{\sum(\hat{\pi}_1 x_1 + \hat{\pi}_2 x_2) y_2} = \frac{3\sum x_1 y_1 + 4\sum x_2 y_1}{3\sum x_1 y_2 + 4\sum x_2 y_2} = \frac{3\times 2 + 4\times 3}{3\times 3 + 4\times 4} = \frac{18}{25}$

(h.) 原 $\hat{\alpha}_{1,2SLS} = \frac{\sum \hat{y}_2 y_1}{\sum \hat{y}_2^2}$

To prove $\hat{\alpha}_{1,2SLS} = \hat{\alpha}_1$ (by moment condition)

we need to prove $\sum \hat{y}_2^2 = \sum \hat{y}_2 y_2$

$\sum \hat{y}_2^2 = \sum \hat{y}_2 (y_2 - \hat{v}_2) = \sum \hat{y}_2 y_2 - \underbrace{\cancel{\sum \hat{y}_2 \hat{v}_2}}_{0} = \sum \hat{y}_2 y_2$ ($\because$ 解釋變數和誤差無關)

11.16 Consider the following supply and demand model

$$\text{Demand: } Q_i = \alpha_1 + \alpha_2 P_i + e_{di}, \quad \text{Supply: } Q_i = \beta_1 + \beta_2 P_i + \beta_3 W_i + e_{si}$$

where Q is the quantity, P is the price, and W is the wage rate, which is assumed exogenous. Data on these variables are in Table 11.7.

TABLE 11.7 **Data for Exercise 11.16**

Q	P	W
4	2	2
6	4	3
9	3	1
3	5	1
8	8	3

a. Derive the algebraic form of the reduced-form equations, $Q = \theta_1 + \theta_2 W + v_2$ and $P = \pi_1 + \pi_2 W + v_1$, expressing the reduced-form parameters in terms of the structural parameters.

b. Which structural parameters can you solve for from the results in part (a)? Which equation is "identified"?

c. The estimated reduced-form equations are $\hat{Q} = 5 + 0.5W$ and $\hat{P} = 2.4 + 1W$. Solve for the identified structural parameters. This is the method of **indirect least squares**.

d. Obtain the fitted values from the reduced-form equation for P, and apply 2SLS to obtain estimates of the demand equation.

a, reduced-form

Demand = Supply

$\Rightarrow \alpha_1 + \alpha_2 P_i + e_{di} = \beta_1 + \beta_2 P_i + \beta_3 W_i + e_{si}$

$(\alpha_2 - \beta_2) P_i = (\beta_1 - \alpha_1) + \beta_3 W_i + (e_{si} - e_{di})$

$\Rightarrow P_i = \dfrac{\beta_1 - \alpha_1}{\alpha_2 - \beta_2} + \dfrac{\beta_3}{\alpha_2 - \beta_2} W_i + \dfrac{e_{si} - e_{di}}{\alpha_2 - \beta_2}$

$Q_i = \alpha_1 + \alpha_2 \left(\dfrac{\beta_1 - \alpha_1}{\alpha_2 - \beta_2} + \dfrac{\beta_3}{\alpha_2 - \beta_2} W_i + \dfrac{e_{si} - e_{di}}{\alpha_2 - \beta_2} \right) + e_{di}$

$\Rightarrow Q_i = \alpha_1 + \dfrac{\beta_1 - \alpha_1}{\alpha_2 - \beta_2} \alpha_2 + \dfrac{\beta_3}{\alpha_2 - \beta_2} W_i \alpha_2 + \dfrac{e_{si} - e_{di}}{\alpha_2 - \beta_2} \alpha_2 + e_{di}$

b, M=2, omitted at least one variable

equation (1) → one → "identified" → 可推出 α_1, α_2

equation (2) → zero → "not indentified" ⇒ 不可推出 α_1, α_2

c, $\hat{Q}_1 = 5 + 0.5W$, $\hat{P} = 2.4 + 1W$

$5 + 0.5W = \alpha_1 + \alpha_2 (2.4 + 1W)$

$= (\alpha_1 + \alpha_2 \times 2.4) + \alpha_2 W$

$\Rightarrow \alpha_2 = 0.5$ $\quad 5 = \alpha_1 + 0.5 \times 2.4$

$\alpha_1 = 3.8$

d, $\hat{P} = 2.4 + 1W$

		$\hat{P} - \bar{\hat{P}}$	$Q - \bar{Q}$
$W=2$	$\hat{P} = 4.4$	0	-2
$W=3$	$\hat{P} = 5.4$	1	0
$W=1$	$\hat{P} = 3.4$	-1	3
$W=1$	$\hat{P} = 3.4$	-1	-3
$W=3$	$\hat{P} = 5.4$	1	2

$\bar{\hat{P}} = 4.4$

$\bar{Q} = 6$

$Q = \alpha_1 + \alpha_2 \hat{P} + e_i$

$\hat{\alpha}_2 = \dfrac{\sum (\hat{P}_i - \bar{\hat{P}})(Q_i - \bar{Q})}{\sum (\hat{P}_i - \bar{\hat{P}})^2} = \dfrac{-3+3+2}{4} = \dfrac{1}{2}$

$\hat{\alpha}_1 = \bar{Q} - \alpha_2 \bar{P} = 6 - 0.5 \times 4.4 = 3.8$

$\Rightarrow \hat{Q} = 3.8 + 0.5P$

11.17 Example 11.3 introduces Klein's Model I.

- **a.** Do we have an adequate number of IVs to estimate each equation? Check the necessary condition for the identification of each equation. The necessary condition for identification is that in a system of M equations at least $M-1$ variables must be omitted from each equation.
- **b.** An equivalent identification condition is that the number of excluded exogenous variables from the equation must be at least as large as the number of included right-hand side endogenous variables. Check that this condition is satisfied for each equation.
- **c.** Write down in econometric notation the first-stage equation, the reduced form, for W_{1t}, wages of workers earned in the private sector. Call the parameters $\pi_1, \pi_2, \ldots$
- **d.** Describe the two regression steps of 2SLS estimation of the consumption function. This is not a question about a computer software command.
- **e.** Does following the steps in part (d) produce regression results that are identical to the 2SLS estimates provided by software specifically designed for 2SLS estimation? In particular, will the t-values be the same?

P_{t-1}. Thus, the consumption function is

$$CN_t = \alpha_1 + \alpha_2(W_{1t} + W_{2t}) + \alpha_3 P_t + \alpha_4 P_{t-1} + e_{1t} \quad (11.17)$$

$$I_t = \beta_1 + \beta_2 P_t + \beta_3 P_{t-1} + \beta_4 K_{t-1} + e_{2t} \quad (11.18)$$

$$W_{1t} = \gamma_1 + \gamma_2 E_t + \gamma_3 E_{t-1} + \gamma_4 TIME_t + e_{3t} \quad (11.19)$$

a. $M=8$, requiring 7 omitted variables, total 16 variable

The consumption function	includes 6 variables	and omits 10 variables	
The investment function	5	11	⇒ all satisfied necessary condition for identification
The wage function	5	11	

b. consumption function : 2 RHS endogenous variables and excludes 5 exogenous variables

Investment function : 1 RHS — 5

wage function : 1 RHS — 5

⇒ all satisfied

c. $W_{1t} = \pi_1 + \pi_2 G_t + \pi_3 W_{2t} + \pi_4 Tx_t + \pi_5 TIME_t + \pi_6 P_{t-1} + \pi_7 K_{t-1} + \pi_8 E_{t-1} + v$

d. Get fitted value $\hat{W}_{1t}$ from (c), and using the same method $\hat{P}_t$, create $W_t^* = \hat{W}_{1t} + W_{2t}$

Regress CN_t by OLS.