11.1 Our aim is to estimate the parameters of the simultaneous equations model

$$y_1 = \alpha_1 y_2 + e_1$$

$$y_2 = \alpha_2 y_1 + \beta_1 x_1 + \beta_2 x_2 + e_2$$

We assume that x_1 and x_2 are exogenous and uncorrelated with the error terms e_1 and e_2.

a. Solve the two structural equations for the reduced-form equation for y_2, that is, $y_2 = \pi_1 x_1 + \pi_2 x_2 + v_2$. Express the reduced-form parameters in terms of the **structural parameters** and the reduced-form error in terms of the structural parameters and e_1 and e_2. Show that y_2 is correlated with e_1.

$$y_2 = \alpha_2 y_1 + \beta_1 x_1 + \beta_2 x_2 + e_2$$

$$= \alpha_2(\alpha_1 y_2 + e_1) + \beta_1 x_1 + \beta_2 x_2 + e_2$$

$$= \alpha_1\alpha_2 y_2 + \alpha_2 e_1 + \beta_1 x_1 + \beta_2 x_2 + e_2$$

$$\Rightarrow (1-\alpha_1\alpha_2) y_2 = \beta_1 x_1 + \beta_2 x_2 + (\alpha_2 e_1 + e_2)$$

$$\Rightarrow y_2 = \frac{\beta_1}{1-\alpha_1\alpha_2} x_1 + \frac{\beta_2}{1-\alpha_1\alpha_2} x_2 + \frac{\alpha_2 e_1 + e_2}{1-\alpha_1\alpha_2}$$

$$= \pi_1 x_1 + \pi_2 x_2 + v_2$$

$$\pi_1 = \frac{\beta_1}{1-\alpha_1\alpha_2}, \quad \pi_2 = \frac{\beta_2}{1-\alpha_1\alpha_2}, \quad v_2 = \frac{\alpha_2 e_1 + e_2}{1-\alpha_1\alpha_2}$$

$$\text{cov}(y_2, e_1) = \text{cov}\left(\frac{\beta_1}{1-\alpha_1\alpha_2} x_1 + \frac{\beta_2}{1-\alpha_1\alpha_2} x_2 + \frac{\alpha_2 e_1 + e_2}{1-\alpha_1\alpha_2}, e_1\right)$$

$$= \frac{\beta_1}{1-\alpha_1\alpha_2}\text{cov}(x_1, e_1) + \frac{\beta_2}{1-\alpha_1\alpha_2}\text{cov}(x_2, e_1) + \text{cov}\left(\frac{\alpha_2 e_1}{1-\alpha_1\alpha_2}, e_1\right) + \text{cov}\left(\frac{e_2}{1-\alpha_1\alpha_2}, e_1\right)$$

$$= 0 + 0 + \frac{\alpha_2}{1-\alpha_1\alpha_2}\sigma_{e_1}^2 + 0$$

$$= \frac{\alpha_2}{1-\alpha_1\alpha_2}\sigma_{e_1}^2 \neq 0 \quad \text{if } \alpha_2 \neq 0$$

b. Which equation parameters are consistently estimated using OLS? Explain.

Both $y_1 = \alpha_1 y_2 + e_1$ and $y_2 = \alpha_2 y_1 + \beta_1 x_1 + \beta_2 x_2 + e_2$ are inconsistent.
The reduced form equation (a) parameters are consistenly estimated using OLS.

c. Which parameters are "identified," in the simultaneous equations sense? Explain your reasoning.

$M=2$ at least absent $M-1=1$ variable

$\therefore$ $y_1 = \alpha_1 y_2 + e_1$ (absent x_1, x_2 2 variables) is identified

$y_2 = \alpha_2 y_1 + \beta_1 x_1 + \beta_2 x_2 + e_2$ (absent 0 variables) is not identified

d. To estimate the parameters of the reduced-form equation for y_2 using the method of moments (MOM), which was introduced in Section 10.3, the two moment equations are

$$N^{-1}\sum x_{i1}(y_2 - \pi_1 x_{i1} - \pi_2 x_{i2}) = 0$$

$$N^{-1}\sum x_{i2}(y_2 - \pi_1 x_{i1} - \pi_2 x_{i2}) = 0$$

Explain why these two moment conditions are a valid basis for obtaining consistent estimators of the reduced-form parameters.

$E(X_{i1} V_{i2} \mid X) = E(X_{i2} V_{i2} \mid X) = 0$ $\quad$ (X_1, X_2 are exogenous)

$E\left(X_{ik}\left(\frac{\alpha_2 e_1 + e_2}{1-\alpha_1\alpha_2}\right) \mid X\right) = \frac{\alpha_2}{1-\alpha_1\alpha_2} E(X_{ik} e_1 \mid X) + \frac{1}{1-\alpha_1\alpha_2} E(X_{ik} e_2 \mid X)$ $\quad k = 1, 2$

$= 0 + 0 = 0$

e. Are the MOM estimators in part (d) the same as the OLS estimators? Form the sum of squared errors function for $y_2 = \pi_1 x_1 + \pi_2 x_2 + v_2$ and find the first derivatives. Set these to zero and show that they are equivalent to the two equations in part (d).

$S(\pi_1, \pi_2 \mid y, X) = \sum_{i=1}^{N} (y_{i2} - \pi_1 X_{i1} - \pi_2 X_{i2})^2$

$\frac{\partial S}{\partial \pi_1} = \sum 2(y_2 - \pi_1 x_1 - \pi_2 x_2)(-x_1) = 0 \quad \Rightarrow \frac{1}{N}\sum (y_2 - \pi_1 x_1 - \pi_2 x_2) x_1 = 0$

$\frac{\partial S}{\partial \pi_2} = \sum 2(y_2 - \pi_1 x_1 - \pi_2 x_2)(-x_2) = 0 \quad \Rightarrow \frac{1}{N}\sum (y_2 - \pi_1 x_1 - \pi_2 x_2) x_2 = 0$ #

f. Using $\sum x_{i1}^2 = 1$, $\sum x_{i2}^2 = 1$, $\sum x_{i1}x_{i2} = 0$, $\sum x_{i1}y_{1i} = 2$, $\sum x_{i1}y_{2i} = 3$, $\sum x_{i2}y_{1i} = 3$, $\sum x_{i2}y_{2i} = 4$, and the two moment conditions in part (d) show that the MOM/OLS estimates of π_1 and π_2 are $\hat{\pi}_1 = 3$ and $\hat{\pi}_2 = 4$.

$\sum (y_2 - \pi_1 x_1 - \pi_2 x_2) x_1 = 0 \Rightarrow \sum (x_1 y_2 - \pi_1 x_1^2 - \pi_2 x_1 x_2) = 0$

$= 3 - \hat{\pi}_1 \cdot 1 - \hat{\pi}_2 \cdot 0 = 0 \Rightarrow \hat{\pi}_1 = 3$

$\sum (y_2 - \pi_1 x_1 - \pi_2 x_2) x_2 = 0 \Rightarrow \sum (x_2 y_2 - \pi_1 x_1 x_2 - \pi_2 x_2^2) = 0$

$= 4 - \hat{\pi}_1 \cdot 0 - \hat{\pi}_2 \cdot 1 = 0 \Rightarrow \hat{\pi}_2 = 4$

g. The fitted value $\hat{y}_2 = \hat{\pi}_1 x_1 + \hat{\pi}_2 x_2$. Explain why we can use the moment condition $\sum \hat{y}_{i2}(y_{i1} - \alpha_1 y_{i2}) = 0$ as a valid basis for consistently estimating α_1. Obtain the IV estimate of α_1. $\quad y_1 = \alpha_1 y_2 + e_1$

$E(y_2 e_1 \mid X) = E[(\pi_1 X_1 + \pi_2 X_2) e_1 \mid X] = E[(\pi_1 X_1 + \pi_2 X_2)(y_1 - \alpha_1 y_2) \mid X] = 0$

$\hat{\pi}_1 \xrightarrow{p} \pi, \hat{\pi}_2 \xrightarrow{p} \pi \quad \sum[(\hat{\pi}_1 X_{i1} + \hat{\pi}_2 X_{i2})(y_{i1} - \alpha_1 y_{i2})] = \sum \hat{y}_{i2}(y_{i1} - \alpha_1 y_{i2}) = 0$

$\Rightarrow \hat{\alpha}_{1,IV} = \frac{\sum \hat{y}_{i2} y_{i1}}{\sum \hat{y}_{i2} y_{i2}} = \frac{\sum(\hat{\pi}_1 X_{i1} + \hat{\pi}_2 X_{i2}) y_{i1}}{\sum(\hat{\pi}_1 X_{i1} + \hat{\pi}_2 X_{i2}) y_{i2}} = \frac{3\times 2 + 4\times 3}{3\times 3 + 4\times 4} = \frac{18}{25}$

h. Find the 2SLS estimate of α_1 by applying OLS to $y_1 = \alpha_1 \hat{y}_2 + e_1^*$. Compare your answer to that in part (g).

$\hat{\alpha}_{1,2SLS} = \frac{\sum \hat{y}_{i2} y_{i1}}{\sum \hat{y}_{i2}^2} = \frac{\sum \hat{y}_{i2} y_{i1}}{\sum \hat{y}_{i2}(y_2 - \hat{v}_2)} = \frac{\sum \hat{y}_{i2} y_{i1}}{\sum \hat{y}_{i2} y_{i2} - \sum \hat{y}_{i2}\hat{v}_2} = \frac{\sum \hat{y}_{i2} y_{i1}}{\sum \hat{y}_{i2} y_{i2}} = (g)$

$\sum \hat{y}_{i2}\hat{v}_2 = \sum(\hat{\pi}_1 X_{i1} + \hat{\pi}_2 X_{i2})\hat{v}_2 = \hat{\pi}_1 \underbrace{\sum X_{i1}\hat{v}_2}_{0} + \hat{\pi}_2 \underbrace{\sum X_{i2}\hat{v}_2}_{0} = 0$

11.16 Consider the following supply and demand model

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$$\text{Demand: } Q_i = \alpha_1 + \alpha_2 P_i + e_{di}, \quad \text{Supply: } Q_i = \beta_1 + \beta_2 P_i + \beta_3 W_i + e_{si}$$

where Q is the quantity, P is the price, and W is the wage rate, which is assumed exogenous. Data on these variables are in Table 11.7.

TABLE 11.7 Data for Exercise 11.16

Q	P	W
4	2	2
6	4	3
9	3	1
3	5	1
8	8	3

a. Derive the algebraic form of the reduced-form equations, $Q = \theta_1 + \theta_2 W + v_2$ and $P = \pi_1 + \pi_2 W + v_1$, expressing the reduced-form parameters in terms of the structural parameters.

$$Q = \alpha_1 + \alpha_2 P + e_d = \beta_1 + \beta_2 P + \beta_3 W + e_s$$

$$(\alpha_2 - \beta_2)P = (\beta_1 - \alpha_1) + \beta_3 W + e_s - e_d$$

$$P = \frac{\beta_1 - \alpha_1}{\alpha_2 - \beta_2} + \frac{\beta_3}{\alpha_2 - \beta_2} W + \frac{e_s - e_d}{\alpha_2 - \beta_2}$$

$$= \pi_1 + \pi_2 W + v_1, \quad \pi_1 = \frac{\beta_1 - \alpha_1}{\alpha_2 - \beta_2}, \ \pi_2 = \frac{\beta_3}{\alpha_2 - \beta_2}, \ v_1 = \frac{e_s - e_d}{\alpha_2 - \beta_2}$$

$$Q = \alpha_1 + \alpha_2(\pi_1 + \pi_2 W + v_1) + e_d = (\alpha_1 + \alpha_2 \pi_1) + \alpha_2 \pi_2 W + (\alpha_2 v_1 + e_d)$$

$$= \left(\alpha_1 + \alpha_2 \cdot \frac{\beta_1 - \alpha_1}{\alpha_2 - \beta_2}\right) + \alpha_2 \cdot \frac{\beta_3}{\alpha_2 - \beta_2} W + \left(\alpha_2 \cdot \frac{e_s - e_d}{\alpha_2 - \beta_2} + e_d\right)$$

$$= \frac{\alpha_1 \beta_2 + \alpha_2 \beta_1}{\alpha_2 - \beta_2} + \frac{\alpha_2 \beta_3}{\alpha_2 - \beta_2} W + \frac{\alpha_2 e_s - \beta_2 e_d}{\alpha_2 - \beta_2}$$

$$= \theta_1 + \theta_2 W + v_2, \quad \theta_1 = \frac{\alpha_1 \beta_2 + \alpha_2 \beta_1}{\alpha_2 - \beta_2}, \ \theta_2 = \frac{\alpha_2 \beta_3}{\alpha_2 - \beta_2}, \ v_2 = \frac{\alpha_2 e_s - \beta_2 e_d}{\alpha_2 - \beta_2}$$

b. Which structural parameters can you solve for from the results in part (a)? Which equation is "identified"?

$$\theta_1 = \alpha_1 + \alpha_2 \pi_1 \quad \Rightarrow \text{can solve } \alpha_1, \alpha_2$$
$$\theta_2 = \alpha_2 \pi_2$$

M=2 absent at least 1 variable

① : $Q = \alpha_1 + \alpha_2 P + e_d$ (absent W, 1 variable) ⇒ identified

② $Q = \beta_1 + \beta_2 P + \beta_3 W + e_s$ (absent 0 variable) ⇒ not identified.

c. The estimated reduced-form equations are $\hat{Q} = 5 + 0.5W$ and $\hat{P} = 2.4 + 1W$. Solve for the identified structural parameters. This is the method of **indirect least squares**.

$\hat{\theta}_1 = 5 \quad \hat{\theta}_2 = 0{,}5$

$\hat{\pi}_1 = 2{,}4 \;,\; \hat{\pi}_2 = 1$

$\hat{\alpha}_1 + \hat{\alpha}_2 \cdot 2{,}4 = 5$

$\hat{\alpha}_2 \cdot 1 = 0{,}5$

$\Rightarrow \hat{\alpha}_2 = 0{,}5$

$\Rightarrow \hat{\alpha}_1 = 5 - 2{,}4 \times 0{,}5 = 3{,}8$

d. Obtain the fitted values from the reduced-form equation for P, and apply 2SLS to obtain estimates of the demand equation.

$\hat{P} = 2{,}4 + W$

$Q = \alpha_1 + \alpha_2 \hat{P}_i + e_i$

$$\alpha_2 = \frac{\Sigma(\hat{P}_i - \bar{\hat{P}})(Q_i - \bar{Q})}{\Sigma(\hat{P}_i - \bar{\hat{P}})^2} = \frac{2}{4} = 0{,}5$$

TABLE 11.7 Data for Exercise 11.16

Q	P	W	$\hat{P}_i$	$\hat{P}_i - \bar{\hat{P}}$	$Q_i - \bar{Q}$
4	2	2	4,4	0	−2
6	4	3	5,4	1	0
9	3	1	3,4	−1	3
3	5	1	3,4	−1	−3
8	8	3	5,4	1	2

$\bar{Q} = 6$ $\qquad \bar{\hat{P}} = 4{,}4$

$\alpha_1 = \bar{Q} - \alpha_2 \bar{P}$

$= 6 - 0{,}5 \times 4{,}4 = 3{,}8$

$\Rightarrow \hat{Q} = 3{,}8 + 0{,}5 \hat{P}$

11.17 Example 11.3 introduces Klein's Model I.

a. Do we have an adequate number of IVs to estimate each equation? Check the necessary condition for the identification of each equation. The necessary condition for identification is that in a system of M equations at least $M - 1$ variables must be omitted from each equation.

M=8 at least absent 8-1=7 variables.

# of variables = 8 exodogenous + 8 endogenous = 16

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内 $CN_t = \alpha_1 + \alpha_2(W_{1t} + W_{2t}) + \alpha_3 P_t + \alpha_4 P_{t-1} + e_{1t}$ (11.17) (absent 16-6=10)

外

内 $I_t = \beta_1 + \beta_2 P_t + \beta_3 P_{t-1} + \beta_4 K_{t-1} + e_{2t}$ (11.18) (absent 16-5=11)

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内 $W_{1t} = \gamma_1 + \gamma_2 E_t + \gamma_3 E_{t-1} + \gamma_4 TIME_t + e_{3t}$ (11.19) (absent 16-5 =11)

外: G, TX

Three of equations are identified.

b. An equivalent identification condition is that the number of excluded exogenous variables from the equation must be at least as large as the number of included right-hand side endogenous variables. Check that this condition is satisfied for each equation.

(11.17): 1 endo = 2 endogenous (W_1, P_t) + 3 exogenous excluded 8-3=5 > 2 satisfied

(11.18) 1 endo = 1 endogenous + 3 exogenous 8-3=5 > 1 satisfied

(11.19) 1 endo = 1 endogenous + 3 exogenous 8-3=5 > 1 satisfied

c. Write down in econometric notation the first-stage equation, the reduced form, for W_{1t}, wages of workers earned in the private sector. Call the parameters $\pi_1, \pi_2, \ldots$

$$W_{1t} = \pi_1 + \pi_2 G_t + \pi_3 W_{2t} + \pi_4 TX_t + \pi_5 P_{t-1} + \pi_6 K_{t-1} + \pi_7 E_{t-1} + \pi_8 TIME_t + v$$

d. Describe the two regression steps of 2SLS estimation of the consumption function. This is not a question about a computer software command.

Step 1: Obtain fitted value $\hat{W}_{1t}$, $\hat{P}_t$ (endogenous) from the estimated reduced form equation using all exogenous variables.
Create $W_t^* = \hat{W}_{1t} + W_{2t}$

2: Regress CN_t on W_t^*, $\hat{P}_t$, P_{t-1} and constant by OLS

e. Does following the steps in part (d) produce regression results that are identical to the 2SLS estimates provided by software specifically designed for 2SLS estimation? In particular, will the t-values be the same?

The coefficient estimates will be the same.
The t-values will not be the same, the standard error will be underestimated because manual 2SLS does not account for the uncertainty in the first-stage prediction.