Q1: Let $k=2$, in p.28 of slides in CH5

the multiple regression model is $Y = X\beta + e$

$Y = (y_1, \dots, y_n)'$

$b = (b_1, b_2)'$

$e = (e_1, \dots, e_n)'$ and

$$X = \begin{bmatrix} 1 & x_{1,2} \\ \vdots & \vdots \\ 1 & x_{n,2} \end{bmatrix}, \text{ so we know } X' = \begin{bmatrix} 1 & \cdots & 1 \\ x_{1,2} & \cdots & x_{n,2} \end{bmatrix}$$

and in p.29 of slides in CH5, we got the LSE for β is

$$b = (X'X)^{-1}(X'y)$$

$$X'X = \begin{bmatrix} 1 & \cdots & 1 \\ x_{1,2} & \cdots & x_{n,2} \end{bmatrix} \begin{bmatrix} 1 & x_{1,2} \\ \vdots & \vdots \\ 1 & x_{n,2} \end{bmatrix} = \begin{bmatrix} N & \Sigma x_i \\ \Sigma x_i & \Sigma x_i^2 \end{bmatrix}$$

$$(X'X)^{-1} = \frac{1}{n\Sigma x_i^2 - (\Sigma x_i)^2} \begin{bmatrix} \Sigma x_i^2 & -\Sigma x_i \\ -\Sigma x_i & n \end{bmatrix}$$

$$X'y = \begin{bmatrix} 1 & \cdots & 1 \\ x_1 & \cdots & x_{n,2} \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} \Sigma y_i \\ \Sigma x_i y_i \end{bmatrix}$$

展開

b_2

$$b = (X'X)^{-1}X'y = \frac{1}{n\Sigma x_i^2 - (\Sigma x_i)^2} \begin{bmatrix} \Sigma x_i^2 & -\Sigma x_i \\ -\Sigma x_i & n \end{bmatrix} \begin{bmatrix} \Sigma y_i \\ \Sigma x_i y_i \end{bmatrix}$$

b_1

而 b is the regression vector， $b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

其中 b_1 is the intercept，and b_2 is the slope

可得知 $b_2 = \dfrac{-\Sigma x_i \Sigma y_i + n\Sigma x_i y_i}{n\Sigma x_i^2 - (\Sigma x_i)^2}$

在 CH2 的 2.7 − 2.8 可知

$\bar{y} = \frac{1}{n}\Sigma y_i$ ， $\bar{x} = \frac{1}{n}\Sigma x_i$ 代入上述 b_2 可得

$$b_2 = \frac{n\Sigma x_i y_i - n^2\bar{x}\bar{y}}{n\Sigma x_i^2 - n^2\bar{x}} = \frac{\Sigma(x_i-\bar{x})(y_i-\bar{y})}{\Sigma(x_i-\bar{x})^2}$$ refer to (2.7)

$b_1 = \dfrac{\Sigma x_i^2 \Sigma y_i - \Sigma x_i \Sigma x_i y_i}{n\Sigma x_i^2 - (\Sigma x_i)^2}$ ，用 $\Sigma y_i = n\bar{y}$ ，$\Sigma x_i = n\bar{x}$ 整理

$$= \frac{\Sigma x_i^2(\cancel{n}\bar{y}) - (\cancel{n}\bar{x})\Sigma x_i y_i}{\cancel{n}\Sigma x_i^2 - n^{\cancel{2}}\bar{x}^2} = \frac{\Sigma x_i^2\bar{y} - \bar{x}\Sigma x_i y_i}{\Sigma x_i^2 - n\bar{x}^2}$$

相同

refer to $b_1 = \bar{y} - b\bar{x}$ (2.8)

$$\bar{y} - \frac{\Sigma x_i y_i - n\bar{x}\bar{y}}{\Sigma x_i^2 - n\bar{x}^2}\bar{x} = \bar{y} - \frac{\bar{x}\Sigma x_i y_i - n\bar{x}^2\bar{y}}{\Sigma x_i^2 - n\bar{x}^2}$$

$$= \frac{\bar{y}\Sigma x_i^2 - \cancel{\bar{y}n\bar{x}^2} - \bar{x}\Sigma x_i y_i + \cancel{n\bar{x}^2}y}{\Sigma x_i^2 - n\bar{x}^2} = \frac{\Sigma x_i^2\bar{y} - \bar{x}\Sigma x_i y_i}{\Sigma x_i^2 - n\bar{x}^2}$$

Q2 : from (2.14) to (2.16), we can know the below formula

$$var(b_1|x) = \sigma^2 \left[\frac{\Sigma x_i^2}{n\Sigma(x_i-\bar{x})^2} \right] \quad (2.14)$$

$$var(b_2|x) = \frac{\sigma^2}{\Sigma(x_i-\bar{x})^2} \quad (2.15)$$

$$cov(b_1, b_2|x) = \sigma^2 \left[\frac{-\bar{x}}{\Sigma(x_i-\bar{x})^2} \right] \quad (2.16)$$

and the variance in p 30 of slides in Ch 5,
we know $var(b) = \sigma^2 (x'x)^{-1}$
follow Q1 :

$$Var(b) = \sigma^2 \frac{1}{n\Sigma x_i^2 - (\Sigma x_i)^2} \begin{bmatrix} \Sigma x_i^2 & -\Sigma x_i \\ -\Sigma x_i & n \end{bmatrix}$$

$$= \begin{array}{cc} Var(b_1) & cov(b_1, b_2) \end{array}$$

$$= \begin{bmatrix} \frac{\sigma^2 \Sigma x_i^2}{n\Sigma x_i^2 - (\Sigma x_i)^2} & \frac{-\sigma^2 \Sigma x_i}{n\Sigma x_i^2 - (\Sigma x_i)^2} \\ \frac{-\sigma^2 \Sigma x_i}{n\Sigma x_i^2 - (\Sigma x_i)^2} & \frac{\sigma^2 n}{n\Sigma x_i^2 - (\Sigma x_i)^2} \end{bmatrix}$$

$$\begin{array}{cc} cov(b_1, b_2) & var(b_2) \end{array}$$