8.18

Consider the wage equation,

$$\ln(WAGE_i) = \beta_1 + \beta_2 EDUC_i + \beta_3 EXPER_i + \beta_4 EXPER_i^2 + \beta_5 FEMALE_i + \beta_6 BLACK_i + \beta_7 METRO_i + \beta_8 SOUTH_i + \beta_9 MIDWEST_i + \beta_{10} WEST_i + e_i$$

where $WAGE$ is measured in dollars per hour, education and experience are in years, and $METRO$= 1 if the person lives in a metropolitan area. Use the data file $cps5$ for the exercise.

a. We are curious whether holding education, experience, and $METRO$ equal, there is the same amount of random variation in wages for males and females. Suppose $\text{var}(e_i|\mathbf{x}_i, FEMALE = 0) = \sigma_M^2$ and $\text{var}(e_i|\mathbf{x}_i, FEMALE = 1) = \sigma_F^2$. We specifically wish to test the null hypothesis $\sigma_M^2 = \sigma_F^2$ against $\sigma_M^2 \neq \sigma_F^2$. Carry out a Goldfeld–Quandt test of the null hypothesis at the 5% level of significance.Clearly state the value of the test statistic and the rejection region, along with your conclusion.

Ans.

左圖為男性 OLS 模型的模型結果，右圖為女性 OLS 模型的模型結果

```
Call:
lm(formula = log(wage) ~ educ + exper + I(exper^2) + black +
    metro + south + midwest + west, data = data_male)

Residuals:
     Min       1Q   Median       3Q      Max
-2.24372 -0.30740  0.00184  0.31690  2.02197

Coefficients:
              Estimate Std. Error t value Pr(>|t|)
(Intercept)  1.227e+00  4.141e-02  29.636  < 2e-16 ***
educ         9.571e-02  2.285e-03  41.893  < 2e-16 ***
exper        3.501e-02  1.757e-03  19.919  < 2e-16 ***
I(exper^2)  -5.400e-04  3.488e-05 -15.483  < 2e-16 ***
black       -1.701e-01  2.486e-02  -6.844 8.55e-12 ***
metro        1.009e-01  1.658e-02   6.087 1.23e-09 ***
south       -3.312e-02  1.853e-02  -1.788   0.0739 .
midwest     -3.724e-02  1.931e-02  -1.929   0.0538 .
west         1.038e-03  1.934e-02   0.054   0.9572
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 0.4699 on 5415 degrees of freedom
Multiple R-squared:  0.317,	Adjusted R-squared:  0.316
F-statistic: 314.1 on 8 and 5415 DF,  p-value: < 2.2e-16
```

```
Call:
lm(formula = log(wage) ~ educ + exper + I(exper^2) + black +
    metro + south + midwest + west, data = data_female)

Residuals:
     Min       1Q   Median       3Q      Max
-2.32121 -0.29665 -0.01285  0.28314  2.97588

Coefficients:
              Estimate Std. Error t value Pr(>|t|)
(Intercept)  0.9681271  0.0507790  19.066  < 2e-16 ***
educ         0.1094453  0.0027551  39.724  < 2e-16 ***
exper        0.0233511  0.0019397  12.039  < 2e-16 ***
I(exper^2)  -0.0003278  0.0000404  -8.113 6.38e-16 ***
black       -0.0565417  0.0229841  -2.460  0.01393 *
metro        0.1425532  0.0182903   7.794 8.06e-15 ***
south       -0.0620508  0.0197754  -3.138  0.00171 **
midwest     -0.0960054  0.0205206  -4.678 2.98e-06 ***
west        -0.0132876  0.0214842  -0.618  0.53629
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 0.4584 on 4366 degrees of freedom
Multiple R-squared:  0.3079,	Adjusted R-squared:  0.3066
F-statistic: 242.7 on 8 and 4366 DF,  p-value: < 2.2e-16
```

$H_0 : \sigma_M^2 = \sigma_F^2 \qquad H_1 : \sigma_M^2 \neq \sigma_F^2$

$\alpha = 0.05$

$$F = \frac{\hat{\sigma}_M^2}{\hat{\sigma}_F^2} = 1.05076$$

```
> cat("F統計量值：", F_stat,"\n")
F統計量值： 1.05076
> cat("5%顯著水準的臨界值：", F_critical_lower,"和", F_critical_upper, "\n")
5%顯著水準的臨界值： 0.9452566 和 1.058097
```

由於計算得到的 F 統計量為 1.05076，沒有落在拒絕域（0.9452566 < 1.05076 < 1.058097），因此我們無法拒絕虛無假設 $H_0 : \sigma_M^2 = \sigma_F^2$。

結論：在 5% 顯著水平下，沒有足夠的證據表明男性和女性的薪資變異數存在顯著差異。

b. Estimate the model by OLS. Carry out the NR^2 test using the right-hand-side variables $METRO$, $FEMALE$, $BLACK$ as candidates related to the heteroskedasticity. What do we conclude about heteroskedasticity, at the 1% level? Do these results support your conclusions in (a)? Repeat the test using all model explanatory variables as candidates related to the heteroskedasticity.

Ans.

下圖為 OLS 模型的模型結果

```
Call:
lm(formula = log(wage) ~ educ + exper + I(exper^2) + female +
    black + metro + south + midwest + west, data = data)

Residuals:
     Min       1Q   Median       3Q      Max
-2.31711 -0.30038 -0.00584  0.30238  3.00061

Coefficients:
              Estimate Std. Error t value Pr(>|t|)
(Intercept)  1.201e+00  3.211e-02  37.409  < 2e-16 ***
educ         1.012e-01  1.758e-03  57.574  < 2e-16 ***
exper        2.962e-02  1.300e-03  22.780  < 2e-16 ***
I(exper^2)  -4.458e-04  2.635e-05 -16.915  < 2e-16 ***
female      -1.655e-01  9.529e-03 -17.368  < 2e-16 ***
black       -1.115e-01  1.694e-02  -6.583 4.86e-11 ***
metro        1.190e-01  1.231e-02   9.671  < 2e-16 ***
south       -4.576e-02  1.356e-02  -3.374 0.000744 ***
midwest     -6.394e-02  1.410e-02  -4.534 5.86e-06 ***
west        -6.589e-03  1.440e-02  -0.458 0.647321
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 0.466 on 9789 degrees of freedom
Multiple R-squared:  0.3173,    Adjusted R-squared:  0.3167
F-statistic: 505.6 on 9 and 9789 DF,  p-value: < 2.2e-16
```

$H_0 : \alpha_5 = \alpha_6 = \alpha_7 = 0$ (homoskedasticity) H_1 : not all the α_i in H_0 are zero (hetroskedasticity)

$\alpha = 0.01 \qquad S = 4$

$NR^2 = 23.55681 \qquad \geq \; \chi^2_{0.99,\,4-1} = 11.34487$

由於檢定統計量 23.55681 遠大於臨界值 11.34487，落在拒絕域，我們強烈拒絕虛無假設。

結論：在 1% 顯著性水平下，METRO、FEMALE、BLACK 與誤差變異數顯著相關，表示存在異質變異數現象。可能暗示誤差變異不僅與性別有關，而是與多個因素（如種族、居住區域）共同作用。

```
> cat("使用 METRO、FEMALE、BLACK: \n")
使用 METRO、FEMALE、BLACK:
> cat("NR^2:", NR2, "\n")
NR^2: 23.55681
> cat("1%顯著水準的χ²臨界值 =", critical_value, "\n")
1%顯著水準的χ²臨界值 = 11.34487
```

$H_0 : \alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = \alpha_6 = \alpha_7 = \alpha_8 = \alpha_9 = \alpha_{10} = 0$ (homoskedasticity)

H_1 : not all the α_i in H_0 are zero (hetroskedasticity)

$\alpha = 0.01 \qquad S = 10$

$NR^2 = 109.4243 \qquad \geq \; \chi^2_{0.99,\,10-1} = 21.66599$

由於檢定統計量 109.4243 遠大於臨界值 21.66599，落在拒絕域，我們強烈拒絕虛無假設。

結論：在 1% 顯著性水平下，當考慮所有解釋變數時，數據中存在更顯著的異質變異性。

```
> cat("使用所有解釋變數: \n")
使用所有解釋變數:
> cat("NR^2:", NR2_all, "\n")
NR^2: 109.4243
> cat("1%顯著水準的χ²臨界值 =", critical_value_all, "\n")
1%顯著水準的χ²臨界值 = 21.66599
```

c. **Carry out the White test for heteroskedasticity. What is the 5% critical value for the test? What do you conclude?**

```
        studentized Breusch-Pagan test

data:  model_ols
BP = 194.44, df = 44, p-value < 2.2e-16
```

```
> qchisq(1-0.05, white_test$parameter)
[1] 60.48089
```

Ans.

H_0 : homoskedasticity　　H_1 : hetroskedasticity

$\alpha = 0.05$　臨界值：$\chi^2_{0.95,\,44} = 60.48089$

$NR^2 = 194.4447 \quad \geq \chi^2_{0.95,\,44} = 60.48089$

由於檢定統計量 194.4447 遠大於臨界值 60.48089，落在拒絕域，我們強烈拒絕虛無假設。

結論：在 5% 顯著性水平下，數據中存在顯著的異質變異性。這表明回歸模型中存在異質變異，即誤差變異與解釋變量及其平方項、交叉項相關。

d. **Estimate the model by OLS with White heteroskedasticity robust standard errors. Compared to OLS with conventional standard errors, for which coefficients have interval estimates gotten narrower? For which coefficients have interval estimates gotten wider? Is there an inconsistency in the results?**

Ans.

下圖為 White Robust OLS 模型的模型結果（樣本數為 9799，屬於大樣本，使用 HC0）

```
t test of coefficients:

               Estimate  Std. Error  t value  Pr(>|t|)    
(Intercept)  1.2014e+00  3.2777e-02  36.6527 < 2.2e-16 ***
educ         1.0123e-01  1.9048e-03  53.1431 < 2.2e-16 ***
exper        2.9622e-02  1.3142e-03  22.5391 < 2.2e-16 ***
I(exper^2)  -4.4578e-04  2.7583e-05 -16.1615 < 2.2e-16 ***
female      -1.6550e-01  9.4834e-03 -17.4517 < 2.2e-16 ***
black       -1.1153e-01  1.6085e-02  -6.9333 4.371e-12 ***
metro        1.1902e-01  1.1576e-02  10.2814 < 2.2e-16 ***
south       -4.5755e-02  1.3895e-02  -3.2931 0.0009946 ***
midwest     -6.3943e-02  1.3717e-02  -4.6615 3.180e-06 ***
west        -6.5891e-03  1.4549e-02  -0.4529 0.6506470    
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1
```

SE、CI width 比較表 (OLS vs Robust OLS)

	OLS_SE	OLS_Robust_SE	SE_Change	OLS_Width	OLS_Robust_Width	CI_Change
(Intercept)	3.211489e-02	3.277743e-02	變大	0.1259036061	0.1285010626	變寬
educ	1.758260e-03	1.904848e-03	變大	0.0068931063	0.0074677917	變寬
exper	1.300342e-03	1.314237e-03	變大	0.0050978780	0.0051523513	變寬
I(exper^2)	2.635448e-05	2.758278e-05	變大	0.0001033205	0.0001081359	變寬
female	9.529136e-03	9.483417e-03	變小	0.0373581454	0.0371789103	變窄
black	1.694240e-02	1.608548e-02	變小	0.0664212100	0.0630617199	變窄
metro	1.230675e-02	1.157624e-02	變小	0.0482475482	0.0453836492	變窄
south	1.356134e-02	1.389454e-02	變大	0.0531660647	0.0544723330	變寬
midwest	1.410367e-02	1.371725e-02	變小	0.0552922035	0.0537772884	變窄
west	1.440237e-02	1.454941e-02	變大	0.0564632233	0.0570397063	變寬

截距和變數 educ, exper, $exper^2$, south, west：SE 變大、CI 變寬，表示原本可能低估不確定性， White Robust SE 更保守。

變數 female, black, metro, midwest：SE 變小、CI 變窄，表示原本可能高估不確定性， White Robust SE 更精確。

無不一致性。若存在異質變異（heteroskedasticity），則傳統 OLS 標準誤是錯誤的，其可能大於也可能小於穩健標準誤，而 Robust OLS 是一致估計量（Consistent Estimator）。

e. Obtain FGLS estimates using candidate variables $METRO$ and $EXPER$. How do the interval estimates compare to OLS with robust standard errors, from part (d)?

Ans.

輔助迴歸式 $\ln(\hat{e}_i^2) = -3.1560 + 0.2372METRO + 0.00499EXPER$

(se) (0.057883) (0.001739)

下圖為 FGLS 模型的模型結果

```
Call:
lm(formula = log(wage) ~ educ + exper + I(exper^2) + female +
    black + metro + south + midwest + west, data = data, weights = weights)

Weighted Residuals:
    Min      1Q  Median      3Q     Max
-4.7199 -0.6168 -0.0112  0.6182  6.1542

Coefficients:
              Estimate Std. Error t value Pr(>|t|)
(Intercept)  1.196e+00  3.184e-02  37.571  < 2e-16 ***
educ         1.015e-01  1.761e-03  57.604  < 2e-16 ***
exper        2.986e-02  1.299e-03  22.988  < 2e-16 ***
I(exper^2)  -4.510e-04  2.657e-05 -16.971  < 2e-16 ***
female      -1.658e-01  9.505e-03 -17.446  < 2e-16 ***
black       -1.112e-01  1.697e-02  -6.553 5.91e-11 ***
metro        1.184e-01  1.186e-02   9.979  < 2e-16 ***
south       -4.527e-02  1.354e-02  -3.343 0.000833 ***
midwest     -6.355e-02  1.405e-02  -4.524 6.13e-06 ***
west        -6.060e-03  1.439e-02  -0.421 0.673671
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 0.948 on 9789 degrees of freedom
Multiple R-squared:  0.3193,    Adjusted R-squared:  0.3187
F-statistic: 510.2 on 9 and 9789 DF,  p-value: < 2.2e-16
```

SE、CI width 比較表 (Robust OLS vs FGLS)

	OLS_Robust_SE	FGLS_SE	SE_Change	OLS_Robust_Width	FGLS_Width	CI_Change
(Intercept)	3.277743e-02	3.184437e-02	變小	0.1285010626	0.124843079	變窄
educ	1.904848e-03	1.761461e-03	變小	0.0074677917	0.006905656	變窄
exper	1.314237e-03	1.298873e-03	變小	0.0051523513	0.005092118	變窄
I(exper^2)	2.758278e-05	2.657195e-05	變小	0.0001081359	0.000104173	變窄
female	9.483417e-03	9.505454e-03	變大	0.0371789103	0.037265303	變寬
black	1.608548e-02	1.696582e-02	變大	0.0630617199	0.066513034	變寬
metro	1.157624e-02	1.186360e-02	變大	0.0453836492	0.046510222	變寬
south	1.389454e-02	1.354227e-02	變小	0.0544723330	0.053091297	變窄
midwest	1.371725e-02	1.404549e-02	變大	0.0537772884	0.055064111	變寬
west	1.454941e-02	1.438967e-02	變小	0.0570397063	0.056413445	變窄

截距和變數 educ, exper, $exper^2$, south, west：SE 變小、CI 變窄，表示 FGLS 在這些變數上更有效率，估計誤差變異結構後精準度提升。

變數 female, black, metro, midwest：SE 變大、CI 變寬，表示 FGLS 在這些變數上效率較低，信賴區間變寬，估計不確定性增加。

係數估計值本身變化不大。即使存在異質變異， Robust OLS 仍是一致估計量，FGLS 同樣也是一致估計量。

f. Obtain FGLS estimates with robust standard errors using candidate variables $METRO$ and $EXPER$. How do the interval estimates compare to those in part (e) and OLS with robust standard errors, from part (d)?

Ans. 下圖為 Robust FGLS 模型的模型結果（樣本數為 9799，屬於大樣本，使用 HC0）

```
t test of coefficients:

              Estimate  Std. Error  t value  Pr(>|t|)
(Intercept)  1.1964e+00  3.2509e-02  36.8028 < 2.2e-16 ***
educ         1.0147e-01  1.8953e-03  53.5352 < 2.2e-16 ***
exper        2.9858e-02  1.3071e-03  22.8441 < 2.2e-16 ***
I(exper^2)  -4.5096e-04  2.7444e-05 -16.4319 < 2.2e-16 ***
female      -1.6583e-01  9.4452e-03 -17.5569 < 2.2e-16 ***
black       -1.1118e-01  1.5959e-02  -6.9669 3.450e-12 ***
metro        1.1838e-01  1.1559e-02  10.2414 < 2.2e-16 ***
south       -4.5266e-02  1.3842e-02  -3.2703  0.001078 **
midwest     -6.3548e-02  1.3690e-02  -4.6419 3.497e-06 ***
west        -6.0599e-03  1.4507e-02  -0.4177  0.676152
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1
```

SE、CI width 比較表 (Robust OLS vs Robust FGLS)

Variable	OLS_Robust_SE	FGLS_Robust_SE	SE_Change	OLS_Robust_Width	FGLS_Robust_Width	CI_Change
(Intercept)	3.277743e-02	3.250910e-02	變小	0.1285010626	0.1274490902	變窄
educ	1.904848e-03	1.895323e-03	變小	0.0074677917	0.0074304492	變窄
exper	1.314237e-03	1.307055e-03	變小	0.0051523513	0.0051241957	變窄
I(exper^2)	2.758278e-05	2.744395e-05	變小	0.0001081359	0.0001075916	變窄
female	9.483417e-03	9.445177e-03	變小	0.0371789103	0.0370289927	變窄
black	1.608548e-02	1.595853e-02	變小	0.0630617199	0.0625640260	變窄
metro	1.157624e-02	1.155933e-02	變小	0.0453836492	0.0453173516	變窄
south	1.389454e-02	1.384176e-02	變小	0.0544723330	0.0542654167	變窄
midwest	1.371725e-02	1.369010e-02	變小	0.0537772884	0.0536708228	變窄
west	1.454941e-02	1.450663e-02	變小	0.0570397063	0.0568719873	變窄

在使用 Robust FGLS 後， 所有變數的 SE 變小、CI 變窄，這表明因為 FGLS 的權重讓估計更精確，即使用了 Robust FGLS。

SE、CI width 比較表 (FGLS vs Robust FGLS)

	FGLS_SE	FGLS_Robust_SE	SE_Change	FGLS_Width	FGLS_Robust_Width	CI_Change
(Intercept)	3.184437e-02	3.250910e-02	變大	0.124843079	0.1274490902	變寬
educ	1.761461e-03	1.895323e-03	變大	0.006905656	0.0074304492	變寬
exper	1.298873e-03	1.307055e-03	變大	0.005092118	0.0051241957	變寬
I(exper^2)	2.657195e-05	2.744395e-05	變大	0.000104173	0.0001075916	變寬
female	9.505454e-03	9.445177e-03	變小	0.037265303	0.0370289927	變窄
black	1.696582e-02	1.595853e-02	變小	0.066513034	0.0625640260	變窄
metro	1.186360e-02	1.155933e-02	變小	0.046510222	0.0453173516	變窄
south	1.354227e-02	1.384176e-02	變大	0.053091297	0.0542654167	變寬
midwest	1.404549e-02	1.369010e-02	變小	0.055064111	0.0536708228	變窄
west	1.438967e-02	1.450663e-02	變大	0.056413445	0.0568719873	變寬

截距和變數 educ, exper, $exper^2$, south, west：SE 變大、CI 變寬，表示 Robust FGLS 在這些變數上，估計更保守。

變數 female, black, metro, midwest：SE 變小、CI 變窄，表示 FGLS 標準誤可能高估變異數（權重模型不夠準確）， Robust FGLS 修正了過高估計。

g. **If reporting the results of this model in a research paper which one set of estimates would you present? Explain your choice.**

Ans.

模型版本	優點	缺點
OLS (a)	計算簡單，易於理解。 若無異質變異數，效率高。	不適合有異質變異數的數據。
Robust OLS (d)	簡單且穩健，無需指定變異數模型，適用於大多數異質變異數情況。 信賴區間可靠，推論正確。	效率可能低於 FGLS（若權重模型正確），因為未利用變異數結構資訊。 信賴區間普遍較寬。
FGLS (e)	若權重模型正確，效率高於 OLS 和 Robust OLS。 利用變異數結構，減少標準誤。	若權重模型錯誤，標準誤可能偏誤。 需假設變異數結構，增加模型風險。
Robust FGLS (f)	結合 FGLS 的加權效率（權重提高精確度）和穩健標準誤的可靠性（不假設權重模型正確）。	計算較複雜。 若權重模型完全錯誤，效率提升有限（但穩健標準誤確保可靠性）。

選擇 Robust FGLS 作為報告結果。

8.18c 確認數據存在異質變異數（$NR^2 = 194.4447$），傳統 OLS（8.18d）標準誤不可靠。

Robust FGLS 提供比 Robust OLS 略高的效率（所有變數信賴區間變窄），同時保持穩健性，在效率與可靠性間取得最佳平衡，可以提供精確且可信的結果。