11.1 Our aim is to estimate the parameters of the simultaneous equations model

$$y_1 = \alpha_1 y_2 + e_1$$

$$y_2 = \alpha_2 y_1 + \beta_1 x_1 + \beta_2 x_2 + e_2$$

We assume that x_1 and x_2 are exogenous and uncorrelated with the error terms e_1 and e_2.

a. Solve the two structural equations for the reduced-form equation for y_2, that is, $y_2 = \pi_1 x_1 + \pi_2 x_2 + v_2$. Express the reduced-form parameters in terms of the **structural parameters** and the reduced-form error in terms of the structural parameters and e_1 and e_2. Show that y_2 is correlated with e_1.

b. Which equation parameters are consistently estimated using OLS? Explain.

c. Which parameters are "identified," in the simultaneous equations sense? Explain your reasoning.

d. To estimate the parameters of the reduced-form equation for y_2 using the method of moments (MOM), which was introduced in Section 10.3, the two moment equations are

$$N^{-1}\sum x_{i1}\left(y_2 - \pi_1 x_{i1} - \pi_2 x_{i2}\right) = 0$$

$$N^{-1}\sum x_{i2}\left(y_2 - \pi_1 x_{i1} - \pi_2 x_{i2}\right) = 0$$

Explain why these two moment conditions are a valid basis for obtaining consistent estimators of the reduced-form parameters.

e. Are the MOM estimators in part (d) the same as the OLS estimators? Form the sum of squared errors function for $y_2 = \pi_1 x_1 + \pi_2 x_2 + v_2$ and find the first derivatives. Set these to zero and show that they are equivalent to the two equations in part (d).

f. Using $\sum x_{i1}^2 = 1$, $\sum x_{i2}^2 = 1$, $\sum x_{i1}x_{i2} = 0$, $\sum x_{i1}y_{1i} = 2$, $\sum x_{i1}y_{2i} = 3$, $\sum x_{i2}y_{1i} = 3$, $\sum x_{i2}y_{2i} = 4$, and the two moment conditions in part (d) show that the MOM/OLS estimates of π_1 and π_2 are $\hat{\pi}_1 = 3$ and $\hat{\pi}_2 = 4$.

g. The fitted value $\hat{y}_2 = \hat{\pi}_1 x_1 + \hat{\pi}_2 x_2$. Explain why we can use the moment condition $\sum \hat{y}_{i2}\left(y_{i1} - \alpha_1 y_{i2}\right) = 0$ as a valid basis for consistently estimating α_1. Obtain the IV estimate of α_1.

h. Find the 2SLS estimate of α_1 by applying OLS to $y_1 = \alpha_1 \hat{y}_2 + e_1^*$. Compare your answer to that in part (g).

<sol) a. $\begin{cases} y_1 = \alpha_1 y_2 + e_1 \\ y_2 = \alpha_2 y_1 + \beta_1 X_1 + \beta_2 X_2 + e_2 \end{cases}$

$$\Rightarrow y_2 = \alpha_2(\alpha_1 y_2 + e_1) + \beta_1 X_1 + \beta_2 X_2 + e_2$$

$$= \alpha_1\alpha_2 y_2 + \alpha_2 e_1 + \beta_1 X_1 + \beta_2 X_2 + e_2$$

$$\Rightarrow (1-\alpha_1\alpha_2) y_2 = \alpha_2 e_1 + \beta_1 X_1 + \beta_2 X_2 + e_2$$

$$\Rightarrow y_2 = \frac{\beta_1}{1-\alpha_1\alpha_2} X_1 + \frac{\beta_2}{1-\alpha_1\alpha_2} X_2 + \frac{\alpha_2 e_1 + e_2}{1-\alpha_1\alpha_2} = \pi_1 X_1 + \pi_2 X_2 + V_2$$

$$Cov(y, e_1|X) = E(y_2, e_1|X)$$

$$= E\left[\left(\frac{\beta_1}{1-\alpha_1\alpha_2} X_1 + \frac{\beta_2}{1-\alpha_1\alpha_2} X_2 + \frac{\alpha_2 e_1 + e_2}{1-\alpha_1\alpha_2}\right) e_1 | X\right]$$

$$= E\left[\frac{\beta_1}{1-\alpha_1\alpha_2} X_1 e_1 | X\right] + E\left[\frac{\beta_2}{1-\alpha_1\alpha_2} X_2 e_1 | X\right] + E\left[\left(\frac{\alpha_2 e_1 + e_2}{1-\alpha_1\alpha_2}\right) e_1 | X\right]$$

$$= 0 + 0 + E\left[\left(\frac{\alpha_2 e_1^2 + e_1 e_2}{1-\alpha_1\alpha_2}\right) | X\right]$$

$$Cov(y_2, e_1|X) = E(y_2, e_1|X) = \frac{E(e_2 e_1|X) + \alpha_2 E(e_1^2|X)}{1-\alpha_1\alpha_2} = \frac{\alpha_2}{1-\alpha_1\alpha_2}\sigma_1^2$$

b、2個都 inconsistenly，∵2條 equation都有內生變數

OLS會 bias and inconsistent、

reduced form 則 consistently (OLS)，∵其右式沒有內生變數

c、2條 simultaneous equation，$M=2$

因此至少要有 $(2-1)$ 個 variable，absent 才能 identified

equation 1：x_1, x_2 absent $\Rightarrow$ identified

equation 2：沒有變數 absent $\Rightarrow$ unidentified

d、These moment conditions arise from the assumptions that the x's are exougenous、

$\Rightarrow E(X_{i1}V_{i1}|X)=E(X_{i2}V_{i2}|X)=0$

$$y_2=\frac{\beta_1}{1-\alpha_1\alpha_2}x_1+\frac{\beta_2}{1-\alpha_1\alpha_2}x_2+\frac{\alpha_2e_1+e_2}{1-\alpha_1\alpha_2}=\pi_1X_1+\pi_2X_2+V_2$$

reduced form error is uncorrelated with x

$$\Rightarrow E\left[X_{ik}\left(\frac{\alpha_2e_1+e_2}{1-\alpha_1\alpha_2}\right)|x\right]=E\left[\frac{1}{1-\alpha_1\alpha_2}X_{ik}e_2|x\right]+E\left[\frac{\alpha_2}{1-\alpha_1\alpha_2}X_{ik}e_1|x\right]=0+0$$

e、$$\frac{\partial S(\pi_1,\pi_2|y,x)}{\partial\pi_1}=2\Sigma(y_2-\pi_1X_1-\pi_2X_2)X_1=0$$

$$\frac{\partial S(\pi_1,\pi_2|y,x)}{\partial\pi_2}=2\Sigma(y_2-\pi_1X_1-\pi_2X_2)X_2=0$$

f、moment conditions：$N^{-1}\Sigma X_{i1}(y_2-\pi_1X_{i1}-\pi_2X_{i2})=0$

$N^{-1}\Sigma X_{i2}(y_2-\pi_1X_{i1}-\pi_2X_{i2})=0$

$\Rightarrow \Sigma X_{i1}y_{i2}-\pi_1\Sigma X_{i1}^2-\pi_2\Sigma X_{i1}X_{i2}=0$

$\Sigma X_{i2}y_{i2}-\pi_1\Sigma X_{i1}X_{i2}-\pi_2\Sigma X_{i2}^2=0$

$\Rightarrow 3-\hat{\pi}_1=0 \Rightarrow \hat{\pi}_1=3$

$4-\hat{\pi}_2=0 \Rightarrow \hat{\pi}_2=4$

g、$y_1=\alpha_1y_2+e_1$

$E[(\pi_1X_1+\pi_2X_2)e_1|X]=E[(\pi_1X_1+\pi_2X_2)(y_1-\alpha_1y_2)|X]=0$

$N^{-1}\Sigma(\pi_1X_{i1}+\pi_2X_{i2})(y_{i1}-\alpha_1y_{i2})=0$

in large samples the consistent estimators converge to true parameter values，$\text{plim}\,\hat{\pi}_1=\pi_1$，$\text{plim}\,\hat{\pi}_2=\pi_2$

$\Rightarrow \Sigma(\hat{\pi}_1X_{i1}+\hat{\pi}_2X_{i2})(y_{i1}-\alpha_1y_{i2})=\Sigma\hat{y}_{i2}(y_{i1}-\alpha_1y_{i2})=0$

$$\Sigma\hat{y}_{i2}y_{i1}-\alpha_1\Sigma\hat{y}_{i2}y_{i2}=0 \Rightarrow \hat{\alpha}_{1,IV}=\frac{\Sigma\hat{y}_{i2}y_{i1}}{\Sigma\hat{y}_{i2}y_{i2}}$$

$$\hat{\alpha}_{1,IV}=\frac{\Sigma(\hat{\pi}_1X_{i1}+\hat{\pi}_2X_{i2})y_{i1}}{\Sigma(\hat{\pi}_1X_{i1}+\hat{\pi}_2X_{i2})y_{i2}}=\frac{\hat{\pi}_1\Sigma X_{i1}y_{i1}+\hat{\pi}_2\Sigma X_{i2}y_{i1}}{\hat{\pi}_1\Sigma X_{i1}y_{i2}+\hat{\pi}_2\Sigma X_{i2}y_{i2}}=\frac{18}{25}$$

h. $\hat{\alpha}_{1,2SLS} = \frac{\sum \hat{y}_{i2} y_{i1}}{\sum \hat{y}_{i2}^2}$, $\hat{v}_2 = y_2 - \hat{y}_2 \Rightarrow \hat{y}_2 = y_2 - \hat{v}_2$

$\sum \hat{y}_{i2}^2 = \sum \hat{y}_{i2} (y_2 - \hat{v}_2) = \sum \hat{y}_{i2} y_2 - \sum \hat{y}_{i2} \hat{v}_2 = \sum \hat{y}_{i2} y_2$

$\sum \hat{y}_{i2} \hat{v}_{i2} = \sum (\hat{\pi}_1 x_{i1} + \hat{\pi}_2 x_{i2}) \hat{v}_{i2} = \hat{\pi}_1 \sum x_{i1} \hat{v}_{i2} + \hat{\pi}_2 \sum x_{i2} \hat{v}_{i2} = 0$

$\Rightarrow \sum x_{i1} \hat{v}_{i2} = 0$, $\sum x_{i2} \hat{v}_{i2} = 0$

11.16 Consider the following supply and demand model

$$\text{Demand: } Q_i = \alpha_1 + \alpha_2 P_i + e_{di}, \quad \text{Supply: } Q_i = \beta_1 + \beta_2 P_i + \beta_3 W_i + e_{si}$$

where Q is the quantity, P is the price, and W is the wage rate, which is assumed exogenous. Data on these variables are in Table 11.7.

TABLE 11.7 **Data for Exercise 11.16**

Q	P	W
4	2	2
6	4	3
9	3	1
3	5	1
8	8	3

a. Derive the algebraic form of the reduced-form equations, $Q = \theta_1 + \theta_2 W + v_2$ and $P = \pi_1 + \pi_2 W + v_1$, expressing the reduced-form parameters in terms of the structural parameters.

b. Which structural parameters can you solve for from the results in part (a)? Which equation is "identified"?

c. The estimated reduced-form equations are $\hat{Q} = 5 + 0.5W$ and $\hat{P} = 2.4 + 1W$. Solve for the identified structural parameters. This is the method of **indirect least squares**.

d. Obtain the fitted values from the reduced-form equation for P, and apply 2SLS to obtain estimates of the demand equation.

(sol) a. $\alpha_1 + \alpha_2 P = \beta_1 + \beta_2 P + \beta_3 W$

$\Rightarrow P = \pi_1 + \pi_2 W + v_1$

$\pi_1 = \dfrac{\beta_1 - \alpha_1}{\alpha_2 - \beta_2}$, $\pi_2 = \dfrac{\beta_3}{\alpha_2 - \beta_2}$, $v_1 = \dfrac{e_s - e_d}{\alpha_2 - \beta_2}$

代回需求式：

$Q = \alpha_1 + \alpha_2(\pi_1 + \pi_2 W + v_1) + e_d = \theta_1 + \theta_2 W + v_2$

$\therefore \theta_1 = \alpha_1 + \alpha_2 \pi_1$, $\theta_2 = \alpha_2 \pi_2$, $v_2 = \alpha_2 v_1 + e_d$

b. $\pi_2 = \dfrac{\beta_3}{\alpha_2 - \beta_2}$, $\theta_2 = \alpha_2 \pi_2 \Rightarrow \alpha_2 = \dfrac{\theta_2}{\pi_2}$ …… identified

$\theta_1 = \alpha_1 + \alpha_2 \pi_1 \Rightarrow \alpha_1$ 亦 identified

但 $\beta_1, \beta_2, \beta_3$ 僅以 π_1, π_2, α 固定一條關係，無法唯一解出 $\Rightarrow$ not identified

$\therefore$ 需求方程式 (α_1, α_2) 被識別，供給方程式未識別

c. $\hat{\alpha}_2 = \dfrac{\hat{\theta}_2}{\hat{\pi}_2} = \dfrac{0.5}{1} = 0.5$

$\hat{\alpha}_1 = \hat{\theta}_1 - \hat{\alpha}_2 \hat{\pi}_1 = 5 - 0.5 \times 2.4 = 3.8$

d、第一階段：$\hat{P}_i = 2.4 + W_i$

第二階段：$\hat{Q} = \hat{\alpha}_1 + \hat{\alpha}_2 \hat{P}$

$\begin{pmatrix} \hat{\alpha}_1 \\ \hat{\alpha}_2 \end{pmatrix} = (X'X)^{-1} X'Q \Rightarrow \hat{\alpha}_1 = 3.8,\ \hat{\alpha}_2 = 0.5$

和(c)之結果相同，驗證識別良好且僅一內生變數時，

ILS之結果 = 2SLS之結果

11.17 Example 11.3 introduces Klein's Model I.

a. Do we have an adequate number of IVs to estimate each equation? Check the necessary condition for the identification of each equation. The necessary condition for identification is that in a system of M equations at least $M - 1$ variables must be omitted from each equation.

b. An equivalent identification condition is that the number of excluded exogenous variables from the equation must be at least as large as the number of included right-hand side endogenous variables. Check that this condition is satisfied for each equation.

c. Write down in econometric notation the first-stage equation, the reduced form, for W_{1t}, wages of workers earned in the private sector. Call the parameters $\pi_1, \pi_2, \ldots$

d. Describe the two regression steps of 2SLS estimation of the consumption function. This is not a question about a computer software command.

e. Does following the steps in part (d) produce regression results that are identical to the 2SLS estimates provided by software specifically designed for 2SLS estimation? In particular, will the t-values be the same?

(sol) a、系統共有M條，必要條件：每條方程式至少要省略M-1個外生變數，Klein I 皆滿足，故至少能識別

b、每條方程式被排除的外生變數數目 ≥ 該方程式右側內生變數數目

⇒ Kelin I 方程式皆可估

c、令所有外生變數為 z_t

$$W_t^p = \pi_0 + \pi_1 z_{1t} + \pi_2 z_{2t} + \cdots + v_t$$

其中 π 係數可用 OLS 拿到 fitted value $\hat{W}_t^p$，供第二階段使用

d、第一步：對所有右側外生變數(ex:所得，工資)以全部外生變數 z_t 為 IV 作 OLS，得到 $\hat{Y}_t$

第二步：以原方程的因變數(消費)對第一步 fitted values 和所有外生變數重新作 OLS，即得 2SLS 係數

e、係數估計值：相同

t值/標準誤：不一定 ⇒ if 手動套用 OLS，標準誤而未調整第一步估計誤差的不確定性，則標準誤會被低估，t-ratio 會偏大。專門的 2SLS 指令會給正確的漸近變異係數。