

Q1: Let $k=2$, in p.28 of slides in CH5

the multiple regression model is $y = x\beta + e$

$$Y = (y_1, \dots, y_n)'$$

$$b = (b_1, b_2)'$$

$$e = (e_1, \dots, e_n)' \text{ and}$$

$$X = \begin{bmatrix} 1 & x_{1,2} \\ \vdots & \vdots \\ 1 & x_{n,2} \end{bmatrix}, \text{ so we know } X' = \begin{bmatrix} 1 & \dots & 1 \\ x_{1,2} & \dots & x_{n,2} \end{bmatrix}$$

and in p.29 of slides in CH5, we got the LSE for β is

$$b = (X'X)^{-1} (X'Y)$$

$$X'X = \begin{bmatrix} 1 & \dots & 1 \\ x_{1,2} & \dots & x_{n,2} \end{bmatrix} \begin{bmatrix} 1 & x_{1,2} \\ \vdots & \vdots \\ 1 & x_{n,2} \end{bmatrix} = \begin{bmatrix} N & \sum x_i \\ \sum x_i & \sum x_i^2 \end{bmatrix}$$

$$(X'X)^{-1} = \frac{1}{n \sum x_i^2 - (\sum x_i)^2} \begin{bmatrix} \sum x_i^2 & -\sum x_i \\ -\sum x_i & n \end{bmatrix}$$

$$X'Y = \begin{bmatrix} 1 & \dots & 1 \\ x_1 & \dots & x_{n,2} \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} \sum y_i \\ \sum x_i y_i \end{bmatrix}$$

展開

$$b = (X'X)^{-1} X'Y = \frac{1}{n \sum x_i^2 - (\sum x_i)^2} \begin{bmatrix} \sum x_i^2 & -\sum x_i \\ -\sum x_i & n \end{bmatrix} \begin{bmatrix} \sum y_i \\ \sum x_i y_i \end{bmatrix}$$

b_2 (above the top-right element)
 b_1 (below the bottom-right element)

而 b is the regression vector, $b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

其中 b_1 is the intercept, and b_2 is the slope

可得知 $b_2 = \frac{-\sum x_i \sum y_i + n \sum x_i y_i}{n \sum x_i^2 - (\sum x_i)^2}$

在 CH2 的 2.7 - 2.8 節

$\bar{y} = \frac{1}{n} \sum y_i$, $\bar{x} = \frac{1}{n} \sum x_i$ 代入上述 b_2 節

$$b_2 = \frac{n \sum x_i y_i - n^2 \bar{x} \bar{y}}{n \sum x_i^2 - n^2 \bar{x}^2} = \frac{\sum (x_i - \bar{x})(y_i - \bar{y})}{\sum (x_i - \bar{x})^2} \quad \text{refer to (2.7)}$$

$b_1 = \frac{\sum x_i^2 \sum y_i - \sum x_i \sum x_i y_i}{n \sum x_i^2 - (\sum x_i)^2}$, 用 $\sum y_i = n \bar{y}$, $\sum x_i = n \bar{x}$ 整理

$$= \frac{\sum x_i^2 (n \bar{y}) - (n \bar{x}) \sum x_i y_i}{n \sum x_i^2 - n^2 \bar{x}^2} = \frac{\sum x_i^2 \bar{y} - \bar{x} \sum x_i y_i}{\sum x_i^2 - n \bar{x}^2}$$

refer to $b_1 = \bar{y} - b \bar{x}$ (2.8)

$$\bar{y} - \frac{\sum x_i y_i - n \bar{x} \bar{y}}{\sum x_i^2 - n \bar{x}^2} \bar{x} = \bar{y} - \frac{\bar{x} \sum x_i y_i - n \bar{x}^2 \bar{y}}{\sum x_i^2 - n \bar{x}^2}$$

$$= \frac{\bar{y} \sum x_i^2 - \cancel{\bar{y} n \bar{x}^2} - \bar{x} \sum x_i y_i + \cancel{n \bar{x}^2} \bar{y}}{\sum x_i^2 - n \bar{x}^2} = \frac{\sum x_i^2 \bar{y} - \bar{x} \sum x_i y_i}{\sum x_i^2 - n \bar{x}^2}$$

相同

Q2: from (2.14) to (2.16), we can know the below formula

$$\text{var}(b_1|x) = \sigma^2 \left[\frac{\sum x_i^2}{n \sum (x_i - \bar{x})^2} \right] \quad (2.14)$$

$$\text{var}(b_2|x) = \frac{\sigma^2}{\sum (x_i - \bar{x})^2} \quad (2.15)$$

$$\text{cov}(b_1, b_2|x) = \sigma^2 \left[\frac{-\bar{x}}{\sum (x_i - \bar{x})^2} \right] \quad (2.16)$$

and the variance in p 30 of slides in Ch 5,
we know $\text{var}(b) = \sigma^2 (x'x)^{-1}$
follow Q1:

$$\begin{aligned} \text{Var}(b) &= \sigma^2 \frac{1}{n \sum x_i^2 - (\sum x_i)^2} \begin{bmatrix} \sum x_i^2 & -\sum x_i \\ -\sum x_i & n \end{bmatrix} \\ &= \begin{bmatrix} \frac{\text{Var}(b_1)}{\sigma^2 \sum x_i^2} & \frac{\text{cov}(b_1, b_2)}{-\sigma^2 \sum x_i} \\ \frac{-\sigma^2 \sum x_i}{n \sum x_i^2 - (\sum x_i)^2} & \frac{\sigma^2 n}{n \sum x_i^2 - (\sum x_i)^2} \end{bmatrix} \\ &\quad \text{cov}(b_1, b_2) \quad \text{var}(b_2) \end{aligned}$$