11.1 Our aim is to estimate the parameters of the simultaneous equations model

$$y_1 = \alpha_1 y_2 + e_1$$
$$y_2 = \alpha_2 y_1 + \beta_1 x_1 + \beta_2 x_2 + e_2$$

We assume that x_1 and x_2 are exogenous and uncorrelated with the error terms e_1 and e_2.

a. Solve the two structural equations for the reduced-form equation for y_2, that is, $y_2 = \pi_1 x_1 + \pi_2 x_2 + v_2$. Express the reduced-form parameters in terms of the **structural parameters** and the reduced-form error in terms of the structural parameters and e_1 and e_2. Show that y_2 is correlated with e_1.

b. Which equation parameters are consistently estimated using OLS? Explain.

c. Which parameters are "identified," in the simultaneous equations sense? Explain your reasoning.

a. 要求 y_2 寫成 $y_2=\pi_1x_1+\pi_2x_2+v_2$，並用 structual parameters $(\alpha_1,\alpha_2,\beta_1,\beta_2)$ 表示 π_2, v_2

同時證 y_2 與 e_1 相关性

$y_1=\alpha_1y_2+e_1$

$y_2=\alpha_2y_1+\beta_1x_1+\beta_2x_2+e_2$

將 y_1 代入 $y_2 \Rightarrow \alpha_2(\alpha_1y_2+e_1)+\beta_1x_1+\beta_2x_2+e_2=\alpha_1\alpha_2y_2+\alpha_2e_1+\beta_1x_1+\beta_2x_2+e_2 \Rightarrow y_2=(1-\alpha_1\alpha_2)=\beta_1x_1+\beta_2x_2+e_2+\alpha_2e_1 \Rightarrow y_2=\frac{\beta_1}{(1-\alpha_1\alpha_2)}x_1+\frac{\beta_2}{(1-\alpha_1\alpha_2)}x_2+\frac{e_2+\alpha_2e_1}{(1-\alpha_1\alpha_2)}=\pi_1x_1+\pi_2x_2+v_2$

而 $Cov(y_2,e_1|x)=E(y_2,e_1|x)=E\left[\left(\frac{\beta_1}{1-\alpha_1\alpha_2}x_1+\frac{\beta_2}{1-\alpha_1\alpha_2}x_2+\frac{e_2+\alpha_2e_1}{1-\alpha_1\alpha_2}\right)\cdot e_1|x\right]=E\left[\frac{\beta_1}{1-\alpha_1\alpha_2}x_1e_1|x\right]+E\left[\frac{\beta_2}{1-\alpha_1\alpha_2}x_2e_1|x\right]+E\left[\frac{e_2+\alpha_2e_1^2}{1-\alpha_1\alpha_2}|x\right]$

因 X 與 e 不相关且 $E(e|x)=0$ 下具外生性 $Cov(y_2,e_1|x)=0+0+E\left[\frac{e_2+\alpha_2e_1^2}{1-\alpha_1\alpha_2}|x\right]$

在假設 2 者誤差不相关下 $cov(y_2,e_1|x)=E(y_2,e_1|x)=\frac{E(e_2e_1|x)+\alpha_2E(e_1^2|x)}{1-\alpha_1\alpha_2}=\frac{\alpha_2}{1-\alpha_1\alpha_2}\sigma^2$ 除非 $\alpha_2=0$，否則 $cov(y_2,e_1|x)$ 為非 0 值

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b. 這 2 個式子都不能用 OLS 得一致估計量，因右邊都有一個內生變數，OLS 有偏誤且不一致

另外，可用 OLS 一致估計簡化形式的方程式參數，發生在右邊只有外生變數

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c. 有 $M=2$ 的同時方程式中，一個結構方程要能被識別，要排除至少 $M-1$ 個外生變數，即 1

對 1 式：右邊只有 1 個內生變數 y_2，無任何外生變數，滿足識別條件，可用 IV 或 2SLS 估 α_1

對 2 式：無任何外生變數被排除，不符合識別條件

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d. x_1, x_2 為外生變數下的矩條件 $\Rightarrow E(x_{i1}v_{i1}|x)=E(x_{i2}v_{i2}|x)=0$ 從 a 可知 $y_2=\frac{\beta_1}{1-\alpha_1\alpha_2}x_1+\frac{\beta_2}{(1-\alpha_1\alpha_2)}x_2+\frac{e_2+\alpha_2e_1}{1-\alpha_1\alpha_2}=\pi_1x_1+\pi_2x_2+v_2$

而因 $E\left[x_{ik}\left(\frac{e_2+\alpha_2e_1}{1-\alpha_1\alpha_2}\right)|x\right]=E\left[\frac{1}{1-\alpha_1\alpha_2}x_{ik}e_2|x\right]+E\left[\frac{\alpha_2}{1-\alpha_1\alpha_2}x_{ik}e_1|x\right]=0+0$ 因此其 $y_2=\pi_1x_1+\pi_2x_2+v_2$ 的誤差項與 x 無相关性

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e. 关於 OLS 與 MOM 的等價性省略，的平方誤差函數為 $S(\pi_1,\pi_2|y,x)=\Sigma(y_2-\pi_1x_1-\pi_2x_2)^2$

一階微分 = $\frac{\partial S}{\partial \pi_1}=2\Sigma(y_2-\pi_1x_1-\pi_2x_2)x_1=0$、$\frac{\partial S}{\partial \pi_2}=2\Sigma(y_2-\pi_1x_1-\pi_2x_2)x_2=0$ 同除 2 下，再將矩條件乘上樣本數 N，就知 2 者等價

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f. 矩條件為：$\frac{1}{N}\Sigma x_1(y_2-\pi_1x_1-\pi_2x_2)=0$
$\frac{1}{N}\Sigma x_2(y_2-\pi_1x_1-\pi_2x_2)=0$

而得 $\Sigma x_1y_2-\pi_1\cdot\Sigma x_1^2-\pi_2\cdot\Sigma x_1x_2=0$
$\Sigma x_2y_2-\pi_1\Sigma x_1x_2-\pi_2\Sigma x_2^2=0$

代入已知值後：$3-\pi_1=0 \Rightarrow \pi_1=3$
$4-\pi_2=0 \Rightarrow \pi_2=4$

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g. 第一結構方程為 $y_1=\alpha_1y_2+e_1$，因此矩條件為：$E[(\pi_1x_1+\pi_2x_2)\cdot e_1|x]=E[(\pi_1x_1+\pi_2x_2)(y_1-\alpha_1y_2)|x]=0$ 可用 $\frac{1}{N}\Sigma x_1(y_1-\alpha_1y_2)=0$，$\frac{1}{N}\Sigma x_2(y_1-\alpha_1y_2)=0$

矩條件樣本為：$N^{-1}\Sigma(\pi_1x_1+\pi_2x_2)(y_1-\alpha_1y_2)=0$，若知 π_1,π_2 即可直接用矩條件解 α_1 的估計量

雖不知 π_1,π_2 真實值，但可從 reduced-form 得一致估計量，當樣本夠大，估計值趨近真值：$\hat{\pi}_1\to\pi_1$，$\hat{\pi}_2\longrightarrow\pi_2$

將矩條件未知數替為估計值 $\Sigma(\hat{\pi}_1x_1+\hat{\pi}_2x_2)(y_1-\alpha_1y_2)=\Sigma \hat{y}_2(y_1-\alpha_1y_2)=0$

因此 $\Sigma\hat{y}_2y_1-\alpha_1\Sigma\hat{y}_{i2}y_{i2}=0 \Rightarrow \hat{\alpha}_{1,IV}=\frac{\Sigma\hat{y}_2y_1}{\Sigma\hat{y}_{i2}y_2}$，即 IV 估計公式，帶值後得 $\hat{\alpha}_{1,IV}=\frac{18}{25}=0.72$

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h. 無截距的簡單線性回歸中，最小平方法解為 $\hat{\alpha}=\frac{\Sigma xy}{\Sigma x^2}$

將 $x\to\hat{y}_{i2}, y\to y_1$，則 $\hat{\alpha}_{1,2SLS}=\frac{\Sigma\hat{y}_2y_1}{\Sigma\hat{y}_2^2}$

為證公式相等，殘差定義為：$\hat{v}_2=y_2-\hat{y}_2 \Rightarrow y_2=\hat{y}_2+\hat{v}_2$

因此 $\hat{y}_2^2=\Sigma\hat{y}_2(y_2-\hat{v}_2)=\Sigma\hat{y}_2y_2-\Sigma\hat{y}_2\hat{v}_2=\Sigma\hat{y}_2y_2$

而 $\Sigma\hat{y}_2\hat{v}_2=\Sigma(\hat{\pi}_1x_1+\hat{\pi}_2x_2)\hat{v}_2=\hat{\pi}_1\Sigma x_1\hat{v}_2+\hat{\pi}_2\Sigma x_2\hat{v}_2=0$

因 x_1, x_2 在第一階段做 OLS 變數，其殘差 $\hat{v}_2$ 不相关下 $\Sigma x_1\hat{v}_2=0$，$\Sigma x_2\hat{v}_2=0$

即 OLS 一個基本性質

Consider the following supply and demand model

$$\text{Demand: } Q_i = \alpha_1 + \alpha_2 P_i + e_{di}, \quad \text{Supply: } Q_i = \beta_1 + \beta_2 P_i + \beta_3 W_i + e_{si}$$

where Q is the quantity, P is the price, and W is the wage rate, which is assumed exogenous. Data on these variables are in Table 11.7.

TABLE 11.7 Data for Exercise 11.16

Q	P	W
4	2	2
6	4	3
9	3	1
3	5	1
8	8	3

a. Derive the algebraic form of the reduced-form equations, $Q = \theta_1 + \theta_2 W + v_2$ and $P = \pi_1 + \pi_2 W + v_1$, expressing the reduced-form parameters in terms of the structural parameters.

b. Which structural parameters can you solve for from the results in part (a)? Which equation is "identified"?

c. The estimated reduced-form equations are $\hat{Q} = 5 + 0.5W$ and $\hat{P} = 2.4 + 1W$. Solve for the identified structural parameters. This is the method of **indirect least squares**.

d. Obtain the fitted values from the reduced-form equation for P, and apply 2SLS to obtain estimates of the demand equation.

a. Reduced-form 用 θ、π 表示

因市場均衡 $Q_d = Q_s = Q \Rightarrow \alpha_1 + \alpha_2 P + e_d = \beta_1 + \beta_2 P + \beta_3 W + e_s \Rightarrow (\alpha_2 - \beta_2)P = (\beta_1 - \alpha_1) + \beta_3 W + (e_s - e_d)$

則 $P = \frac{\beta_1 - \alpha_1}{\alpha_2 - \beta_2} + \frac{\beta_3}{\alpha_2 - \beta_2} W + \frac{e_s - e_d}{\alpha_2 - \beta_2}$，因此 reduced form $P = \pi_1 + \pi_2 W + v_1$

而 $Q = \alpha_1 + \alpha_2 P + e_d = \alpha_1 + \alpha_2(\pi_1 + \pi_2 W + v_1) + e_d = (\alpha_1 + \alpha_2 \pi_1) + \alpha_2 \pi_2 W + (\alpha_2 v_1 + e_d)$

則 reduced-form $Q = \theta_1 + \theta_2 W + v_2$

b. reduced-form 有 $\pi_1, \pi_2, \theta_1, \theta_2$ 4個估計值，但未知參數有 $\alpha_1, \alpha_2, \beta_1, \beta_2, \beta_3$ 5個，估計值 < 未知參數，則 Q_d 無法識別

而對 Q_s 來說，因 $\pi_2 = \frac{\beta_3}{\alpha_2 - \beta_3}$，而 β_3 可識別下，只要 $\beta_3 \neq 0$，Q_s 的參數都可識別

c. 給定 $\hat{Q} = 5 + 0.5W$，$\hat{P} = 2.4 + 1W$ 解 $\alpha_1, \alpha_2, \beta_1, \beta_2, \beta_3$

從 reduced-form：$\theta_1 = 5$，$\theta_2 = 0.5$，$\pi_1 = 2.4$，$\pi_2 = 1$

而 $\theta_1 = \alpha_1 + \alpha_2 \pi_1$，$\theta_2 = \alpha_2 \pi_2$ 下帶值 $=$ $\alpha_1 + \alpha_2(2.4) = 5$，$\alpha_2(1) = 0.5$，而 $\alpha_2 = 0.5$，$\alpha_1 = 3.8$

$\pi_1 = \frac{\beta_1 - \alpha_1}{\alpha_2 - \beta_2}$，$\pi_2 = \frac{\beta_3}{\alpha_2 - \beta_2} \Rightarrow \beta_3 = 1(0.5 - \beta_2) = 0.5 - \beta_2$，$\beta_1 = 5 - 2.4\beta_2$，2者都依賴 β_2

因此 Q_s 可識別但 Q_d 不能

d. 已知 $\hat{P}_i = 2.4 + 1W_i$

$\Rightarrow$

W	$\hat{P}$
2	$2.4 + 1 \cdot 2 = 4.4$
3	$2.4 + 1 \cdot 3 = 5.4$
1	$2.4 + 1.1 = 3.4$
1	$2.4 + 1.1 = 3.4$
3	$2.4 + 1.3 = 5.4$

$\Rightarrow \hat{P} = [4.4, 5.4, 3.4, 3.4, 5.4]$

用 $\hat{P}_i$ 當 P_i 的工具變數做 2SLS

11.17 Example 11.3 introduces Klein's Model I.

a. Do we have an adequate number of IVs to estimate each equation? Check the necessary condition for the identification of each equation. The necessary condition for identification is that in a system of M equations at least $M - 1$ variables must be omitted from each equation.

b. An equivalent identification condition is that the number of excluded exogenous variables from the equation must be at least as large as the number of included right-hand side endogenous variables. Check that this condition is satisfied for each equation.

c. Write down in econometric notation the first-stage equation, the reduced form, for W_{1t}, wages of workers earned in the private sector. Call the parameters $\pi_1, \pi_2, \ldots$

d. Describe the two regression steps of 2SLS estimation of the consumption function. This is not a question about a computer software command.

e. Does following the steps in part (d) produce regression results that are identical to the 2SLS estimates provided by software specifically designed for 2SLS estimation? In particular, will the t-values be the same?

a. 系統中共有 $M=8$ 個方程式，而每個方程要省略7個變數，而整個系統有16個變數

消費：6個變數 ⇒ 省略10個

投資：5個變數 ⇒ 省略11個

私部門工資：5個變數 ⇒ 省略11個

3者滿足識別條件

b. 消費方程含2個內生變數，排除5個外生變數

投資 & 私部門工資方程各含1個右邊的內生變數，排除5個外生變數

c. 私部門工資的 reduced-form 方程為：$W_{1t} = \pi_1 + \pi_2 G_t + \pi_3 W_{2t} + \pi_4 TX_t + \pi_5 TIME_t + \pi_6 P_{t-1} + \pi_7 K_{t-1} + \pi_8 E_{t-1} + v$

d. 從 c 的 reduced-form 方程取 $\hat{W}_{1t}$ & $\hat{P}_t$ 的預測值

建合成變數：$W_t^* = \hat{W}_{1t} + W_{2t}$

用 OLS 將 CN_t（消費）對 W_t^*、$\hat{P}_t$、P_{t-1}、一個截距項回歸

e. 回歸係數估計值會和正確的2SLS結果相同

但t值不會一樣，因在d中是用錯的2SLS標準誤