CH11

(a) $\begin{cases} y_1 = \alpha_1 y_2 + e_1 \\ y_2 = \alpha_2 y_1 + \beta_1 X_1 + \beta_2 X_2 + e_2 \end{cases}$

$$y_2 = \alpha_2(\alpha_1 y_2 + e_1) + \beta_1 X_1 + \beta_2 X_2 + e_2$$
$$y_2(1-\alpha_1\alpha_2) = \beta_1 X_1 + \beta_2 X_2 + (\alpha_2 e_1 + e_2)$$
$$y_2 = \frac{\beta_1}{1-\alpha_1\alpha_2} X_1 + \frac{\beta_2}{1-\alpha_1\alpha_2} X_2 + \frac{\alpha_2 e_1 + e_2}{1-\alpha_1\alpha_2}$$
$$\pi_1 = \frac{\beta_1}{1-\alpha_1\alpha_2}, \quad \pi_2 = \frac{\beta_2}{1-\alpha_1\alpha_2}, \quad v_2 = \frac{\alpha_2 e_1 + e_2}{1-\alpha_1\alpha_2}$$

$$\begin{aligned} \mathrm{cov}(y_2, e_1 \mid x) &= E(y_2, e_1 \mid x) \\ &= E\left[\left(\frac{\beta_1}{1-\alpha_1\alpha_2} X_1 + \frac{\beta_2}{1-\alpha_1\alpha_2} X_2 + \frac{\alpha_2 e_1 + e_2}{1-\alpha_1\alpha_2}\right) e_1 \mid x\right] \\ &= E\left[\left(\frac{\beta_1}{1-\alpha_1\alpha_2} X_1 e_1 \mid x\right)\right] + E\left[\left(\frac{\beta_2}{1-\alpha_1\alpha_2} X_2 e_1 \mid x\right)\right] + E\left[\left(\frac{\alpha_2 e_1 + e_2}{1-\alpha_1\alpha_2} e_1 \mid x\right)\right] \\ &= E\left[\left(\frac{\alpha_2 e_1 + e_2}{1-\alpha_1\alpha_2} e_1 \mid x\right)\right] \\ &= \frac{\alpha_2 E(e_1^2 \mid x) + E(e_1 e_2 \mid x)}{1-\alpha_1\alpha_2} = \frac{\alpha_2 \sigma_1^2}{1-\alpha_1\alpha_2} > 0 \text{ unless } \alpha_2 = 0. \end{aligned}$$

(b) Since both equations have endogeneous variables, the OLS is biased and consistent.

(c) Since $M=2$ and $2-1=1$, at least 1 variable needs to be omitted from equations.
(1) It omitted two exogeneous varibables. $\longrightarrow$ "identified"
(2) It omitted no variables. $\longrightarrow$ "not identified"

(d) $E(X_{i1} V_{i2} \mid x) = E(X_{i2} V_{i2} \mid x) = 0$
Therefore, $E\left[X_{ik}\left(\frac{\alpha_1 e_1 + e_2}{1-\alpha_1\alpha_2} \mid x\right)\right] = E\left[\frac{\alpha_1}{1-\alpha_1\alpha_2} e_1 X_{ik} \mid x\right] + E\left[\frac{1}{1-\alpha_1\alpha_2} e_2 X_{ik} \mid x\right] = 0.$

(e) $\begin{cases} \frac{d}{d\pi_1} \Sigma(y_2 - \pi_1 X_1 - \pi_2 X_2)^2 = 2\Sigma(y_2 - \pi_1 X_1 - \pi_2 X_2) \times (-X_1) = 0 \longrightarrow N^{-1}\Sigma X_{i1}(y_2 - \pi_1 X_{i1} - \pi_2 X_{i2}) = 0 \\ \frac{d}{d\pi_2} \Sigma(y_2 - \pi_1 X_1 - \pi_2 X_2)^2 = 2\Sigma(y_2 - \pi_1 X_1 - \pi_2 X_2) \times (-X_2) = 0 \longrightarrow N^{-1}\Sigma X_{i2}(y_2 - \pi_1 X_{i1} - \pi_2 X_{i2}) = 0 \end{cases}$

(f) $\begin{cases} \Sigma X_{i1}(y_2 - \pi_1 X_{i1} - \pi_2 X_{i2}) = 0 \longrightarrow \Sigma X_{i1} y_2 - \pi_1 \Sigma X_{i1}^2 - \pi_2 \Sigma X_{i1} X_{i2} = 0 \longrightarrow 3 - \pi_1 = 0, \ \pi_1 = 3 \\ \Sigma X_{i2}(y_2 - \pi_1 X_{i1} - \pi_2 X_{i2}) = 0 \longrightarrow \Sigma X_{i2} y_2 - \pi_1 \Sigma X_{i1} X_{i2} - \pi_2 \Sigma X_{i2}^2 = 0 \longrightarrow 4 - \pi_2 = 0, \ \pi_2 = 4 \end{cases}$

(g) $\because \hat{y}_2 = \hat{\pi}_1 X_1 + \hat{\pi}_2 X_2$
$\therefore \Sigma \hat{y}_2 (y_1 - \alpha_1 y_2) = 0, \ \Sigma \hat{y}_2 y_1 - \alpha_1 \Sigma \hat{y}_2 y_2 = 0, \ \alpha_1 = \frac{\Sigma \hat{y}_2 y_1}{\Sigma \hat{y}_2 y_2}$
$\rightarrow \hat{\alpha}_1 = \frac{\Sigma(\hat{\pi}_1 X_1 + \hat{\pi}_2 X_2) y_1}{\Sigma(\hat{\pi}_1 X_1 + \hat{\pi}_2 X_2) y_2} = \frac{3\Sigma X_1 y_1 + 4\Sigma X_2 y_1}{3\Sigma X_1 y_2 + 4\Sigma X_2 y_2} = \frac{3\times2 + 4\times3}{3\times3 + 4\times4} = \frac{18}{25}$

(h) To prove $\hat{\alpha}_{1,2SLS} = \frac{\Sigma \hat{y}_2 y_1}{\Sigma \hat{y}_2^2} = \hat{\alpha}_1$, we need to prove $\Sigma \hat{y}_2^2 = \Sigma \hat{y}_2 y_2$.
And, $\Sigma \hat{y}_2^2 = \Sigma \hat{y}_2 (y_2 - \hat{v}_2) = \Sigma \hat{y}_2 y_2 - \Sigma \hat{y}_2 \hat{v}_2 = \Sigma \hat{y}_2 y_2$.

16.

(a) $\alpha_1 + \alpha_2 P_i + e_{di} = \beta_1 + \beta_2 P_i + \beta_3 W_i + e_{si}$

$(\alpha_2 - \beta_2) P_i = (\beta_1 - \alpha_1) + \beta_3 W_i + (e_{si} - e_{di})$

$P_i = \frac{\beta_1 - \alpha_1}{\alpha_2 - \beta_2} + \frac{\beta_3}{\alpha_2 - \beta_2} W_i + \frac{e_{si} - e_{di}}{\alpha_2 - \beta_2}$

$Q_i = \alpha_1 + \alpha_2 \left(\frac{\beta_1 - \alpha_1}{\alpha_2 - \beta_2} + \frac{\beta_3}{\alpha_2 - \beta_2} W_i + \frac{e_{si} - e_{di}}{\alpha_2 - \beta_2} \right) + e_{di}$

$Q_i = \alpha_1 + \frac{\beta_1 - \alpha_1}{\alpha_2 - \beta_2} \alpha_2 + \frac{\beta_3}{\alpha_2 - \beta_2} W_i \alpha_2 + \frac{e_{si} - e_{di}}{\alpha_2 - \beta_2} \alpha_2 + e_{di}$

(b) Since $M = 2$ and $2 - 1 = 1$, at least 1 variable needs to be omitted from equations.

(1) It omitted two exogeneous varibables. $\longrightarrow$ "identified"

(2) It omitted no variables. $\longrightarrow$ "not identified"

(c) $\begin{cases} \hat{Q} = 5 + 0.5W & \longrightarrow \ 5 + 0.5W = \alpha_1 + \alpha_2(2.4 + W) = (\alpha_1 + 2.4\alpha_2) + \alpha_2 W \\ \hat{P} = 2.4 + W & \longrightarrow \ \alpha_1 = 3.8,\ \alpha_2 = 0.5 \end{cases}$

(d) $\hat{P} = 2.4 + W$

W	$\hat{P}$	$\hat{P} - \bar{\hat{P}}$	$Q - \bar{Q}$
2	4.4	0	-2
3	5.4	1	0
1	3.4	-1	3
1	3.4	-1	-3
3	5.4	1	2

$\longrightarrow \bar{\hat{P}} = 4.4,\ \bar{Q} = 6$

$Q = \alpha_1 + \alpha_2 \hat{P} + e_i$

$\hat{\alpha}_2 = \frac{\sum(\hat{P}_i - \bar{\hat{P}})(Q_i - \bar{Q})}{\sum(\hat{P}_i - \bar{\hat{P}})^2} = \frac{1}{2}$

$\hat{\alpha}_1 = \bar{Q} - \alpha_2 \bar{P} = 6 - 0.5 \times 4.4 = 3.8$

$\longrightarrow \hat{Q} = 3.8 + 0.5P$

17.

(a) Since $M=8$ and $8-1=7$, at least 7 variable needs to be omitted from equations.

Consumption — It omitted 10 exogeneous varibables.

Investment — It omitted 11 exogeneous varibables.

Wage ——— It omitted 11 exogeneous varibables.

$\longrightarrow$ all functions are "identified."

(b)

	endogeneous variables		exogeneous variables
Consumption —	2	<	5
Investment —	1	<	5
Wage ———	1	<	5

$\longrightarrow$ all functions are satisfied.

(c) $W_{1t} = \pi_1 + \pi_2 G_t + \pi_3 W_{2t} + \pi_4 Tx_t + \pi_5 TIME_t + \pi_6 P_{t-1} + \pi_7 K_{t-1} + \pi_8 E_{t-1} + V$

(d) 1. Get $\hat{W}_{1t}$ from (c)

2. Use the same method as $\hat{P}_t$.

3. Create $W_t^* = \hat{W}_{1t} + W_{2t}$

4. Regress CN_t by OLS.

(e) Two coefficient will be the same, but t-value will be different.

28.

(a) Demand Equation:

$\alpha_2 P_i = Q_i - (\alpha_1 + \alpha_3 PS_i + \alpha_4 DI_i + e_{di})$

$P_i = -\frac{\alpha_1}{\alpha_2} + \frac{1}{\alpha_2} Q_i - \frac{\alpha_3}{\alpha_2} PS_i - \frac{\alpha_4}{\alpha_2} DI_i + \frac{1}{\alpha_2} e_{di}$

$= \delta_1 + \delta_2 Q_i + \delta_3 PS_i + \delta_4 DI_i + u^d$

$\Rightarrow \delta_2 < 0$, law of demand

$\delta_3 > 0$, substitute goods

$\delta_4 > 0$, normal goods

Supply Equation:

$\beta_2 P_i = Q_i - (\beta_1 + \beta_3 PF_i + e_{si})$

$P_i = -\frac{\beta_1}{\beta_2} + \frac{1}{\beta_2} Q_i + \frac{\beta_3}{\beta_2} PF_i + \frac{1}{\beta_2} e_{si}$

$= \pi_1 + \pi_2 Q_i + \pi_3 PF_i + u^s$

$\Rightarrow \pi_2 > 0$, supply increases with price

$\pi_3 > 0$, cost of a factor of production

(b)

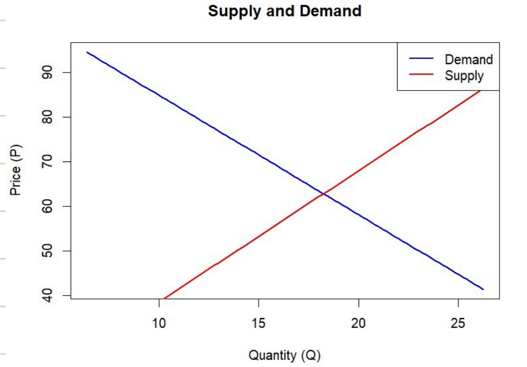
```
> summary(supply_2sls)

Call:
ivreg(formula = p ~ q + pf | ps + di + pf, data = truffles)

Residuals:
    Min      1Q  Median      3Q     Max 
-9.7983 -2.3440 -0.6281  2.4350 11.1600 

Coefficients:
            Estimate Std. Error t value Pr(>|t|)    
(Intercept) -58.7982     5.8592  -10.04 1.32e-10 ***
q             2.9367     0.2158   13.61 1.32e-13 ***
pf            2.9585     0.1560   18.97  < 2e-16 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 4.399 on 27 degrees of freedom
Multiple R-Squared: 0.9486,	Adjusted R-squared: 0.9448 
Wald test: 232.7 on 2 and 27 DF,  p-value: < 2.2e-16
```

```
> summary(demand_2sls)

Call:
ivreg(formula = p ~ q + ps + di | ps + di + pf, data = truffles)

Residuals:
    Min      1Q  Median      3Q     Max 
-39.661  -6.781   2.410   8.320  20.251 

Coefficients:
            Estimate Std. Error t value Pr(>|t|)    
(Intercept)  -11.428     13.592  -0.841  0.40810    
q             -2.671      1.175  -2.273  0.03154 *  
ps             3.461      1.116   3.103  0.00458 ** 
di            13.390      2.747   4.875 4.68e-05 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 13.17 on 26 degrees of freedom
Multiple R-Squared: 0.5567,	Adjusted R-squared: 0.5056 
Wald test: 17.37 on 3 and 26 DF,  p-value: 2.137e-06
```

$\longrightarrow$ All coefficients are statistically significant different from zero, corresponding with the theory.

(c) price elasticity = -1.2725

(d)



(e) $Q\_eq = 18.25022$, $P\_eq = 62.84226$
$\hat{Q} = 18.2604$, $\hat{P} = 62.8154$
→ The two approaches are in very good agreement.

(f)
```
> summary(supply_ols)

Call:
lm(formula = p ~ q + pf, data = truffles)

Residuals:
    Min      1Q  Median      3Q     Max 
-8.4721 -3.3287  0.1861  2.0785 10.7513 

Coefficients:
            Estimate Std. Error t value Pr(>|t|)    
(Intercept) -52.8763     5.0238  -10.53 4.68e-11 ***
q             2.6613     0.1712   15.54 5.42e-15 ***
pf            2.9217     0.1482   19.71  < 2e-16 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 4.202 on 27 degrees of freedom
Multiple R-squared:  0.9531,	Adjusted R-squared:  0.9496 
F-statistic: 274.4 on 2 and 27 DF,  p-value: < 2.2e-16
```

```
> summary(demand_ols)

Call:
lm(formula = p ~ q + ps + di, data = truffles)

Residuals:
     Min       1Q   Median       3Q      Max 
-25.0753  -2.7742  -0.4097   4.7079  17.4979 

Coefficients:
            Estimate Std. Error t value Pr(>|t|)    
(Intercept) -13.6195     9.0872  -1.499   0.1460    
q             0.1512     0.4988   0.303   0.7642    
ps            1.3607     0.5940   2.291   0.0303 *  
di           12.3582     1.8254   6.770 3.48e-07 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 8.814 on 26 degrees of freedom
Multiple R-squared:  0.8013,	Adjusted R-squared:  0.7784 
F-statistic: 34.95 on 3 and 26 DF,  p-value: 2.842e-09
```

→ Except for demand (q — 2SLS) , all the other are correct.
Except for demand (q — OLS) , all the other are statiscally significant different from zero.

30.
(a)
```
Call:
lm(formula = i ~ p + plag + klag, data = klein)

Residuals:
     Min       1Q   Median       3Q      Max 
-2.56562 -0.63169  0.03687  0.41542  1.49226 

Coefficients:
            Estimate Std. Error t value Pr(>|t|)    
(Intercept) 10.12579    5.46555   1.853 0.081374 .  
p            0.47964    0.09711   4.939 0.000125 ***
plag         0.33304    0.10086   3.302 0.004212 ** 
klag        -0.11179    0.02673  -4.183 0.000624 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 1.009 on 17 degrees of freedom
  (因為不存在，1 個觀察量被刪除了)
Multiple R-squared:  0.9313,	Adjusted R-squared:  0.9192 
F-statistic: 76.88 on 3 and 17 DF,  p-value: 4.299e-10
```

→
- Current Profits (p): The coefficient is positive and significant, suggesting that higher current profits lead to higher investment — consistent with the idea that profits provide internal funds for investment.
- Lagged Profits (plag): Also positive and significant, indicating that past profitability continues to influence current investment — possibly through expectations or retained earnings.
- Lagged Capital Stock (klag): Negative and significant, as expected: higher existing capital stock reduces the need for new investment (diminishing marginal returns or capital adjustment costs).

(b)

```
Call:
lm(formula = p ~ g + w2 + tx + time + plag + klag + elag, data = klein_b)

Residuals:
    Min      1Q  Median      3Q     Max 
-3.9067 -1.3050  0.3226  1.3613  2.8881 

Coefficients:
            Estimate Std. Error t value Pr(>|t|)  
(Intercept) 50.38442   31.63026   1.593   0.1352  
g            0.43902    0.39114   1.122   0.2820  
w2          -0.07961    2.53382  -0.031   0.9754  
tx          -0.92310    0.43376  -2.128   0.0530 .
time         0.31941    0.77813   0.410   0.6881  
plag         0.80250    0.51886   1.547   0.1459  
klag        -0.21610    0.11911  -1.814   0.0928 .
elag         0.02200    0.28216   0.078   0.9390  
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 2.183 on 13 degrees of freedom
Multiple R-squared:  0.8261,	Adjusted R-squared:  0.7324 
F-statistic: 8.821 on 7 and 13 DF,  p-value: 0.0004481

Analysis of Variance Table

Model 1: p ~ plag + klag
Model 2: p ~ g + w2 + tx + time + plag + klag + elag
  Res.Df    RSS Df Sum of Sq      F Pr(>F)
1     18 108.04                           
2     13  61.95  5    46.093 1.9345 0.1566
> F_crit <- qf(0.95, df1 = 5, df2 = 13)
> cat("Critical F(5,13;0.95) =", round(F_crit, 3), "\n")
Critical F(5,13;0.95) = 3.025
```

→ Since $F = 1.9345 < F_{crit} = 3.025$,
we fail to reject H_0 that all coefficients are zero.

(c)

```
Call:
lm(formula = i ~ p + plag + klag + vhat, data = klein_b)

Residuals:
     Min       1Q   Median       3Q      Max 
-1.04645 -0.56030  0.06189  0.25348  1.36700 

Coefficients:
            Estimate Std. Error t value Pr(>|t|)    
(Intercept) 20.27821    4.70179   4.313 0.000536 ***
p            0.15022    0.10798   1.391 0.183222    
plag         0.61594    0.10147   6.070 1.62e-05 ***
klag        -0.15779    0.02252  -7.007 2.96e-06 ***
vhat         0.57451    0.14261   4.029 0.000972 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 0.7331 on 16 degrees of freedom
Multiple R-squared:  0.9659,	Adjusted R-squared:  0.9574 
F-statistic: 113.4 on 4 and 16 DF,  p-value: 1.588e-11
```

Since $\hat{v}$ is significant at 1% level, P is endogeneous.
This is what we expected in simultaneous equation model.

(d)

```
Call:
lm(formula = i ~ p + plag + klag + vhat, data = klein_b)

Residuals:
     Min       1Q   Median       3Q      Max 
-1.04645 -0.56030  0.06189  0.25348  1.36700 

Coefficients:
            Estimate Std. Error t value Pr(>|t|)    
(Intercept) 20.27821    4.70179   4.313 0.000536 ***
p            0.15022    0.10798   1.391 0.183222    
plag         0.61594    0.10147   6.070 1.62e-05 ***
klag        -0.15779    0.02252  -7.007 2.96e-06 ***
vhat         0.57451    0.14261   4.029 0.000972 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 0.7331 on 16 degrees of freedom
Multiple R-squared:  0.9659,	Adjusted R-squared:  0.9574 
F-statistic: 113.4 on 4 and 16 DF,  p-value: 1.588e-11
```

→ The 2SLS results differ meaningfully from OLS, particularly in the coefficient on current profits (p). This supports the idea that p is endogenous in the investment equation and justifies the use of instrumental variables. The increase in the estimated coefficient on lagged profits (plag) and the continued significance of lagged capital (klag) reinforce the dynamic structure of Klein’s Model I. Overall, 2SLS provides more reliable estimates by correcting for endogeneity bias.

(e)

```
Call:
lm(formula = i ~ phat + plag + klag, data = klein_b)

Residuals:
    Min      1Q  Median      3Q     Max 
-3.8778 -1.0029  0.3058  0.7275  2.1831 

Coefficients:
            Estimate Std. Error t value Pr(>|t|)   
(Intercept) 20.27821    9.97663   2.033  0.05802 . 
phat         0.15022    0.22913   0.656  0.52084   
plag         0.61594    0.21531   2.861  0.01083 * 
klag        -0.15779    0.04778  -3.302  0.00421 **
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 1.556 on 17 degrees of freedom
Multiple R-squared:  0.837,	Adjusted R-squared:  0.8082 
F-statistic: 29.09 on 3 and 17 DF,  p-value: 6.393e-07
```

→ All coefficient estimates are the same. Only standard errors vary.

(f)

```
Call:
lm(formula = resid_2sls ~ plag + klag + g + w2 + tx + time + 
    elag, data = klein_b)

Residuals:
    Min      1Q  Median      3Q     Max 
-3.4087 -0.8799  0.2702  1.0011  2.4987 

Coefficients:
             Estimate Std. Error t value Pr(>|t|)
(Intercept)  7.671103  24.976416   0.307    0.764
plag         0.189896   0.409708   0.463    0.651
klag        -0.002262   0.094056  -0.024    0.981
g            0.034277   0.308861   0.111    0.913
w2          -0.704649   2.000800  -0.352    0.730
tx          -0.022846   0.342512  -0.067    0.948
time         0.283921   0.614439   0.462    0.652
elag        -0.116046   0.222807  -0.521    0.611

Residual standard error: 1.724 on 13 degrees of freedom
Multiple R-squared:  0.06102,	Adjusted R-squared:  -0.4446 
F-statistic: 0.1207 on 7 and 13 DF,  p-value: 0.9953

> cat("Sargan test statistic TR^2 =", sargan_stat, "\n")
Sargan test statistic TR^2 = 1.281519 
> cat("Critical value from Chi-square(4, 0.95) =", qchisq(0.95, df = 4), "\n")
Critical value from Chi-square(4, 0.95) = 9.487729 
```

→ Fail to reject H_0, surplus instruments are valid.