HW0505

11.1 Our aim is to estimate the parameters of the simultaneous equations model

$$y_1 = \alpha_1 y_2 + e_1$$
$$y_2 = \alpha_2 y_1 + \beta_1 x_1 + \beta_2 x_2 + e_2$$

We assume that x_1 and x_2 are exogenous and uncorrelated with the error terms e_1 and e_2.

a. Solve the two structural equations for the reduced-form equation for y_2, that is, $y_2 = \pi_1 x_1 + \pi_2 x_2 + v_2$. Express the reduced-form parameters in terms of the **structural parameters** and the reduced-form error in terms of the structural parameters and e_1 and e_2. Show that y_2 is correlated with e_1.

b. Which equation parameters are consistently estimated using OLS? Explain.

c. Which parameters are "identified," in the simultaneous equations sense? Explain your reasoning.

CHAPTER 11 Simultaneous Equations Models

d. To estimate the parameters of the reduced-form equation for y_2 using the method of moments (MOM), which was introduced in Section 10.3, the two moment equations are

$$N^{-1}\sum x_{i1}(y_2 - \pi_1 x_{i1} - \pi_2 x_{i2}) = 0$$
$$N^{-1}\sum x_{i2}(y_2 - \pi_1 x_{i1} - \pi_2 x_{i2}) = 0$$

Explain why these two moment conditions are a valid basis for obtaining consistent estimators of the reduced-form parameters.

e. Are the MOM estimators in part (d) the same as the OLS estimators? Form the sum of squared errors function for $y_2 = \pi_1 x_1 + \pi_2 x_2 + v_2$ and find the first derivatives. Set these to zero and show that they are equivalent to the two equations in part (d).

f. Using $\sum x_{i1}^2 = 1$, $\sum x_{i2}^2 = 1$, $\sum x_{i1}x_{i2} = 0$, $\sum x_{i1}y_{1i} = 2$, $\sum x_{i1}y_{2i} = 3$, $\sum x_{i2}y_{1i} = 3$, $\sum x_{i2}y_{2i} = 4$, and the two moment conditions in part (d) show that the MOM/OLS estimates of π_1 and π_2 are $\hat{\pi}_1 = 3$ and $\hat{\pi}_2 = 4$.

g. The fitted value $\hat{y}_2 = \hat{\pi}_1 x_1 + \hat{\pi}_2 x_2$. Explain why we can use the moment condition $\sum \hat{y}_{i2}(y_{i1} - \alpha_1 y_{i2}) = 0$ as a valid basis for consistently estimating α_1. Obtain the IV estimate of α_1.

h. Find the 2SLS estimate of α_1 by applying OLS to $y_1 = \alpha_1 \hat{y}_2 + e_1^*$. Compare your answer to that in part (g).

a.

對 (2) 式進行 reduced-form 轉換：

代入 (1) 得

y2=α2(α1y2+e1)+β1x1+β2x2+e2⇒y2(1−α1α2)=α2e1+β1x1+β2x2+e2⇒y2=β1 1−α1α2x1+β21−α1α2x2+α2e1+e21−α1α2y_2 = \alpha_2(\alpha_1 y_2 + e_1) + \beta_1 x_1 + \beta_2 x_2 + e_2 \Rightarrow y_2 (1 - \alpha_1 \alpha_2) = \alpha_2 e_1 + \beta_1 x_1 + \beta_2 x_2 + e_2 \Rightarrow y_2 = \frac{\beta_1}{1 - \alpha_1 \alpha_2} x_1 + \frac{\beta_2}{1 - \alpha_1 \alpha_2} x_2 + \frac{\alpha_2 e_1 + e_2}{1 - \alpha_1 \alpha_2}y2=α2(α1y2+e1)+β1x1+β2x2+e2⇒y2(1−α1α2)=α2e1+β1x1+β2x2+e2⇒y2=1−α1α2β1x1+1−α1α2β2x2+1−α1α2α2e1+e2

→ y2y_2y2 的 reduced-form 包含 x1,x2x_1, x_2x1,x2 與複合誤差項。

---

b.
若用 OLS 估計 (1)，因為 y2y_2y2 的 reduced-form 中含有 e1e_1e1，所以 y2y_2y2 與 e1e_1e1 相關聯，違反 OLS 的外生性假設，估計會有偏誤（inconsistent）。
→ OLS 不適用於 (1)；但 (2) 中右邊只含一個內生變數（y1y_1y1），其餘是外生變數（x1,x2x_1, x_2x1,x2），因此 (2) 可用 IV 或 2SLS 處理。

---

c.
識別條件：每個結構方程需排除至少一個外生變數，使之可識別。
(1) 中無任何外生變數，難以識別。
(2) 中含 x1,x2x_1, x_2x1,x2，但 (1) 無外生變數可作 instrument，故 **(1) 不可識別，(2) 可識別。**

---

d.
因為 y2y_2y2 與 e1e_1e1 相關，所以 y2y_2y2 是內生變數。必須使用 2SLS 或其他 IV 技術來估計 α1\alpha_1α1。

---

e.
如果要用 2SLS 估計 α1\alpha_1α1，第一階段可使用 (a) 中的 reduced-form：

y^2=π1x1+π2x2\hat{y}_2 = \pi_1 x_1 + \pi_2 x_2y^2=π1x1+π2x2

然後第二階段把 y^2\hat{y}_2y^2 放進原式 (1)：

y1=α1y^2+誤差 y_1 = \alpha_1 \hat{y}_2 + \text{誤差}y1=α1y^2+誤差

即可得到一致估計。

---

f.

此模型中 x1,x2x_1, x_2x1,x2 為外生變數，可作為 y2y_2y2 的工具變數。因為這些工具變數不在方程 (1) 中，且與 y2y_2y2 有關，符合有效工具變數條件。

---

g.

若只用 x1x_1x1 為工具變數，需檢查其相關性（relevance）與外生性（exogeneity）。若 x1x_1x1 無法強力解釋 y2y_2y2，則為弱工具變數（weak IV），將導致估計不準或偏誤。因此僅用 x1x_1x1 是風險較高的選擇。

---

h.

若 e1e_1e1 與 e2e_2e2 有關(即誤差項同時相關)，則使用 OLS 或 2SLS 須考慮同時性偏誤與誤差的相關性問題。這時可能需要使用 **三階段最小平方法（3SLS）**，才能達到有效率的估計。

11.16 Consider the following supply and demand model

$$\text{Demand: } Q_i = \alpha_1 + \alpha_2 P_i + e_{di}, \quad \text{Supply: } Q_i = \beta_1 + \beta_2 P_i + \beta_3 W_i + e_{si}$$

where Q is the quantity, P is the price, and W is the wage rate, which is assumed exogenous. Data on these variables are in Table 11.7.

TABLE 11.7 Data for Exercise 11.16

Q	P	W
4	2	2
6	4	3
9	3	1
3	5	1
8	8	3

a. Derive the algebraic form of the reduced-form equations, $Q = \theta_1 + \theta_2 W + v_2$ and $P = \pi_1 + \pi_2 W + v_1$, expressing the reduced-form parameters in terms of the structural parameters.

b. Which structural parameters can you solve for from the results in part (a)? Which equation is "identified"?

c. The estimated reduced-form equations are $\hat{Q} = 5 + 0.5W$ and $\hat{P} = 2.4 + 1W$. Solve for the identified structural parameters. This is the method of **indirect least squares**.

d. Obtain the fitted values from the reduced-form equation for P, and apply 2SLS to obtain estimates of the demand equation.

a.
聯立 demand 與 supply 式解出 reduced-form for PiP_iPi：
設 Demand = Supply：

α1+α2Pi+edi=β1+β2Pi+β3Wi+esi⇒Pi=π1+π2Wi+vi\alpha_1 + \alpha_2 P_i + e_{di} = \beta_1 + \beta_2 P_i + \beta_3 W_i + e_{si} \Rightarrow P_i = \pi_1 + \pi_2 W_i + v_iα1+α2Pi+edi=β1+β2Pi+β3Wi+esi⇒Pi=π1+π2Wi+vi

其中：

π1=β1−α1α2−β2,π2=β3α2−β2,vi=esi−ediα2−β2\pi_1 = \frac{\beta_1 - \alpha_1}{\alpha_2 - \beta_2}, \quad \pi_2 = \frac{\beta_3}{\alpha_2 - \beta_2}, \quad v_i = \frac{e_{si} - e_{di}}{\alpha_2 - \beta_2}π1=α2−β2β1−α1,π2=α2−β2β3,vi=α2−β2esi−edi

---

b.
從 reduced-form 代入 PiP_iPi 到 demand 式：

Qi=α1+α2(π1+π2Wi+vi)+edi⇒Qi=(α1+α2π1)+α2π2Wi+(α2vi+edi)Q_i = \alpha_1 + \alpha_2 (\pi_1 + \pi_2 W_i + v_i) + e_{di} \Rightarrow Q_i = (\alpha_1 + \alpha_2 \pi_1) + \alpha_2 \pi_2 W_i + (\alpha_2 v_i + e_{di})Qi=α1+α2(π1+π2Wi+vi)+edi⇒Qi=(α1+α2π1)+α2π2Wi+(α2vi+edi)

→ 若能觀察 Q 與 W，則可回推 α2\alpha_2α2，再推得 α1\alpha_1α1，因此 demand 方程為識別的。

---

c.
已知 reduced-form：

Q^=5+0.5W,P^=2.4+1.0W\hat{Q} = 5 + 0.5 W, \quad \hat{P} = 2.4 + 1.0 WQ^=5+0.5W,P^=2.4+1.0W

將 P^\hat{P}P^ 代入 demand 式：

Q^=α1+α2P^⇒5+0.5W=α1+α2(2.4+1.0W)\hat{Q} = \alpha_1 + \alpha_2 \hat{P} \Rightarrow 5 + 0.5W = \alpha_1 + \alpha_2 (2.4 + 1.0 W)Q^=α1+α2P^⇒5+0.5W=α1+α2(2.4+1.0W)

解聯立式得：

- α2=0.5\alpha_2 = 0.5α2=0.5
- α1=5−0.5×2.4=3.8\alpha_1 = 5 - 0.5 \times 2.4 = 3.8α1=5−0.5×2.4=3.8

---

d.
在 2SLS 中，第一階段用 WiW_iWi 對 PiP_iPi 回歸，得

P^i=2.4+1.0Wi\hat{P}_i = 2.4 + 1.0 W_iP^i=2.4+1.0Wi

第二階段用 P^i\hat{P}_iP^i 回歸 Q：

Qi=α1+α2P^i+誤差 Q_i = \alpha_1 + \alpha_2 \hat{P}_i + \text{誤差}Qi=α1+α2P^i+誤差

此估計量是一致的，即使 PiP_iPi 是內生的。

11.17 Example 11.3 introduces Klein's Model I.

a. Do we have an adequate number of IVs to estimate each equation? Check the necessary condition for the identification of each equation. The necessary condition for identification is that in a system of M equations at least $M-1$ variables must be omitted from each equation.

b. An equivalent identification condition is that the number of excluded exogenous variables from the equation must be at least as large as the number of included right-hand side endogenous variables. Check that this condition is satisfied for each equation.

c. Write down in econometric notation the first-stage equation, the reduced form, for W_{1t}, wages of workers earned in the private sector. Call the parameters $\pi_1, \pi_2, \ldots$

d. Describe the two regression steps of 2SLS estimation of the consumption function. This is not a question about a computer software command.

e. Does following the steps in part (d) produce regression results that are identical to the 2SLS estimates provided by software specifically designed for 2SLS estimation? In particular, will the t-values be the same?

a.
是否所有結構方程皆識別？
答：要滿足階數條件（order condition）與秩條件（rank condition）。Klein I 模型為經典模型，設計上通常每個方程都有排除的外生變數，因此每個結構方程皆可識別。

---

b.
利用階數條件檢查是否識別：

→ 計算每個方程中排除的外生變數數量是否 ≥ 其內生變數數量 - 1。
→ 若是，則符合階數條件，表示**至少局部可識別（locally identified）**。

---

c.
reduced-form 表示每個內生變數作為所有外生變數的函數，如：

W1=π1Z1+π2Z2+⋯+vW_1 = \pi_1 Z_1 + \pi_2 Z_2 + \cdots + vW1=π1Z1+π2Z2+⋯+v

其中 ZiZ_iZi 為所有外生變數。這表示用外生變數來預測內生變數，是 2SLS 的第一階段。

---

d.
2SLS 兩階段如下：

1. 第一階段：對內生變數使用所有外生變數進行回歸，取得 fitted values。
2. 第二階段：將 fitted values 代入原結構式，進行 OLS 估計。

---

e.
手動 2SLS 與直接使用 2SLS 套件指令，在理論上會產生**相同的參數估計值**。
但 t 值可能不同，因為標準誤差必須考慮第一階段誤差帶來的不確定性。
→ 套件會自動做誤差修正（如 robust 或 HAC），手動時需自行計算修正標準誤差。