$$\hat{b}_2 = \frac{\sum (x_i - \bar{x})(y_i - \bar{y})}{\sum (x_i - \bar{x})^2}$$

The model : $y_i = b_1 + b_2 x_i + \varepsilon_i$

$$\hat{b}_1 = \bar{y} - \hat{b}_2 \bar{x}$$

Matrix form : $Y = Xb + \varepsilon$

where : $Y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$; $\varepsilon = \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{pmatrix}$

$$X'X = \begin{pmatrix} n & \sum X_i \\ \sum X_i & \sum x_i^2 \end{pmatrix}$$

$b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ $\quad X = (\iota \ \sum X_i) = \begin{pmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{pmatrix}$

$$X'Y = \begin{pmatrix} \sum y_i \\ \sum X_i Y_i \end{pmatrix}$$

$$(X'X)^{-1}(X'Y) = \frac{1}{n\sum X_i^2 - (\sum X_i)^2} \begin{pmatrix} \sum x_i^2 & -\sum X_i \\ -\sum X_i & n \end{pmatrix} \begin{pmatrix} \sum Y_i \\ \sum X_i Y_i \end{pmatrix}$$

$$= \frac{1}{n\sum X_i^2 - (\sum X_i)^2} \begin{pmatrix} \sum x_i^2 \sum y_i - \sum X_i \sum X_i Y_i \\ -\sum X_i \sum Y_i + n \sum X_i \cdot Y_i \end{pmatrix}$$

we have $b_1 = \dfrac{\sum x_i^2 \sum y_i - \sum X_i \sum X_i Y_i}{n\sum X_i^2 - (\sum X_i)^2}$

$$b_2 = \frac{-\sum X_i \sum Y_i + n \sum X_i \cdot Y_i}{n\sum X_i^2 - (\sum X_i)^2} = \frac{\sum (X_i - \bar{X})(Y_i - \bar{Y})}{\sum (X_i - \bar{X})^2} = \hat{b}_2$$

$$b_1 = \frac{\sum x_i^2 \sum y_i - \sum x_i \sum x_i y_i}{n\sum x_i^2 - (\sum x_i)^2} = \frac{\sum x_i \left(\sum x_i - \sum x_i y_i\right)}{n\sum x_i^2 - (\sum x_i)^2}$$

$$b_1 = \bar{y} - b_2 \bar{x} = \frac{\sum y_i}{n} - \frac{\sum (x_i - \bar{x})(y_i - \bar{y})}{\sum (x_i - \bar{x})^2} \frac{\sum x_i}{n}$$

$$= \frac{\sum y_i \sum x_i^2 - n\bar{x}^2\bar{y} - \left[\sum x_i y_i - 2n\bar{x}\bar{y} + \bar{x}\bar{y}\right]\sum x_i}{n\sum (x_i - \bar{x})^2}$$

$$= \frac{\sum y_i \sum x_i^2 - n\cancel{\bar{x}^2}\bar{y} - \sum x_i y_i \sum x_i + n\cancel{\bar{x}^2}\bar{y}}{n\sum (x_i - \bar{x})^2}$$

$$= \frac{\sum y_i \sum x_i^2 - \sum x_i y_i \sum x_i}{n\sum x_i^2 - (\sum x_i)^2} = b_1$$

$$\mathrm{cov}(b_1, b_2) = \sigma^2 (X'X)^{-1}$$

$$= \sigma^2 \frac{1}{n\sum x_i^2 - (\sum x_i)^2} \begin{pmatrix} \sum x_i^2 & -\sum x_i \\ -\sum x_i & n \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\sigma^2 \sum x_i^2}{n\sum (x_i - \bar{x})^2} & \frac{-\sigma^2 \bar{x}^2}{\sum (x_i - \bar{x})^2} \\ \frac{-\sigma^2 \bar{x}^2}{\sum (x_i - \bar{x})^2} & \frac{\sigma^2}{\sum (x_i - \bar{x})^2} \end{pmatrix} = \begin{pmatrix} \mathrm{var}(b_1) & \mathrm{cov}(b_1, b_2) \\ \mathrm{cov}(b_1, b_2) & \mathrm{var}(b_2) \end{pmatrix}$$