Chap 2: The simple linear regression model

Financial Econometrics

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2.1 An economic model

Fig 2.6: Data for the food expenditure example.

$y = \beta_1 + \beta_2 x + e$

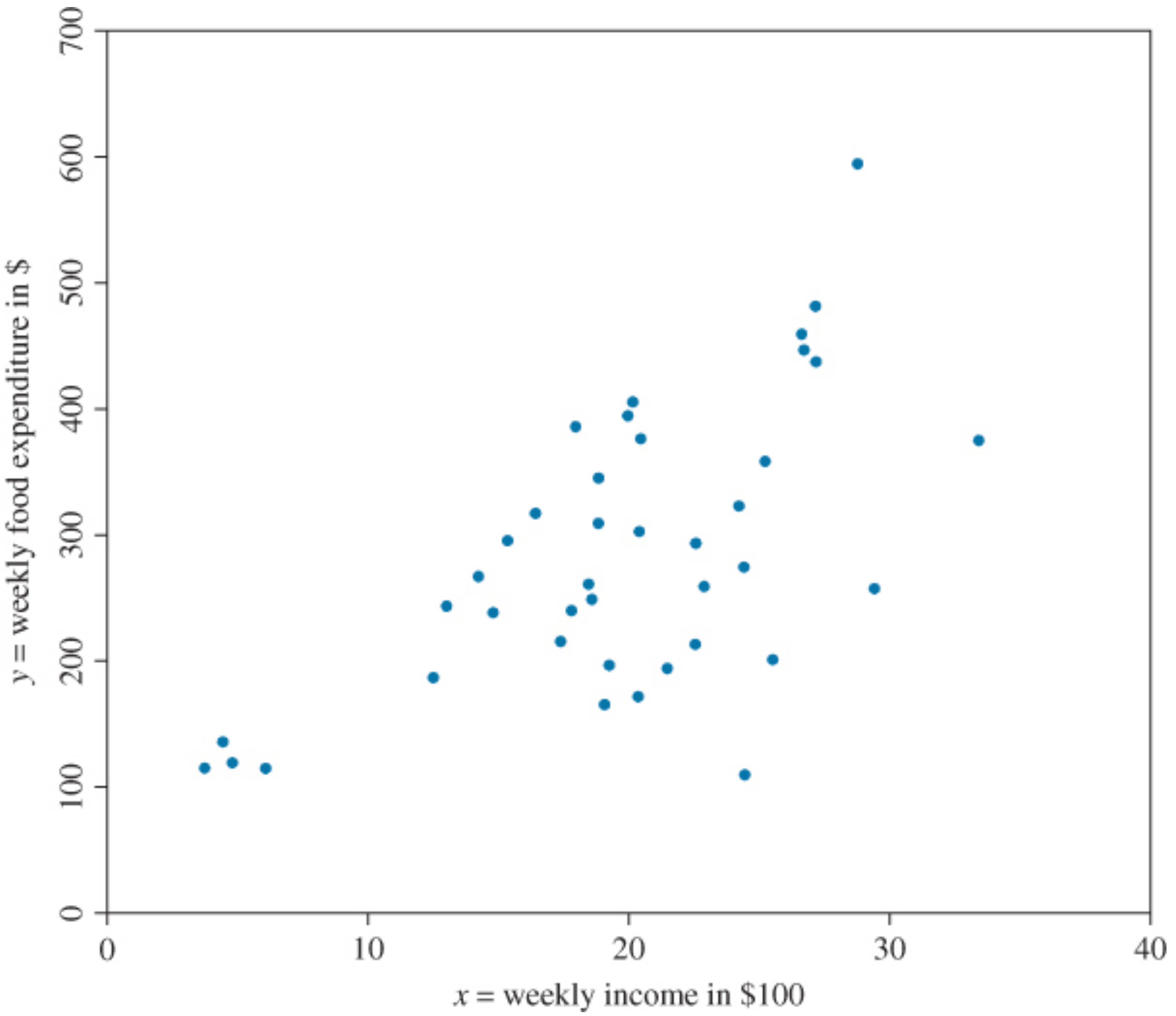
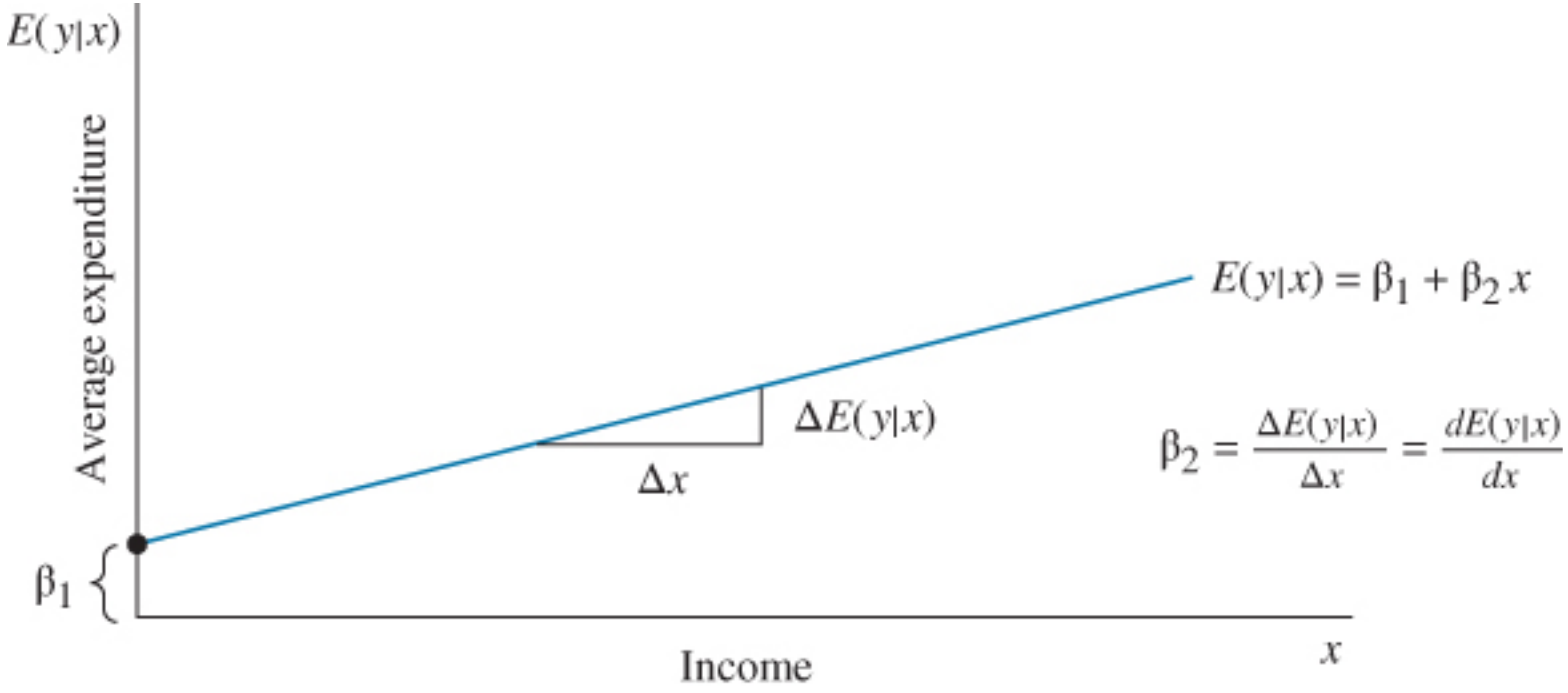


Fig 2.2. The economic model: a linear relationship between average per person food expenditure and income.



2.2 An econometric model

Fig 2.3 Conditional probability densities for *e* and *y*

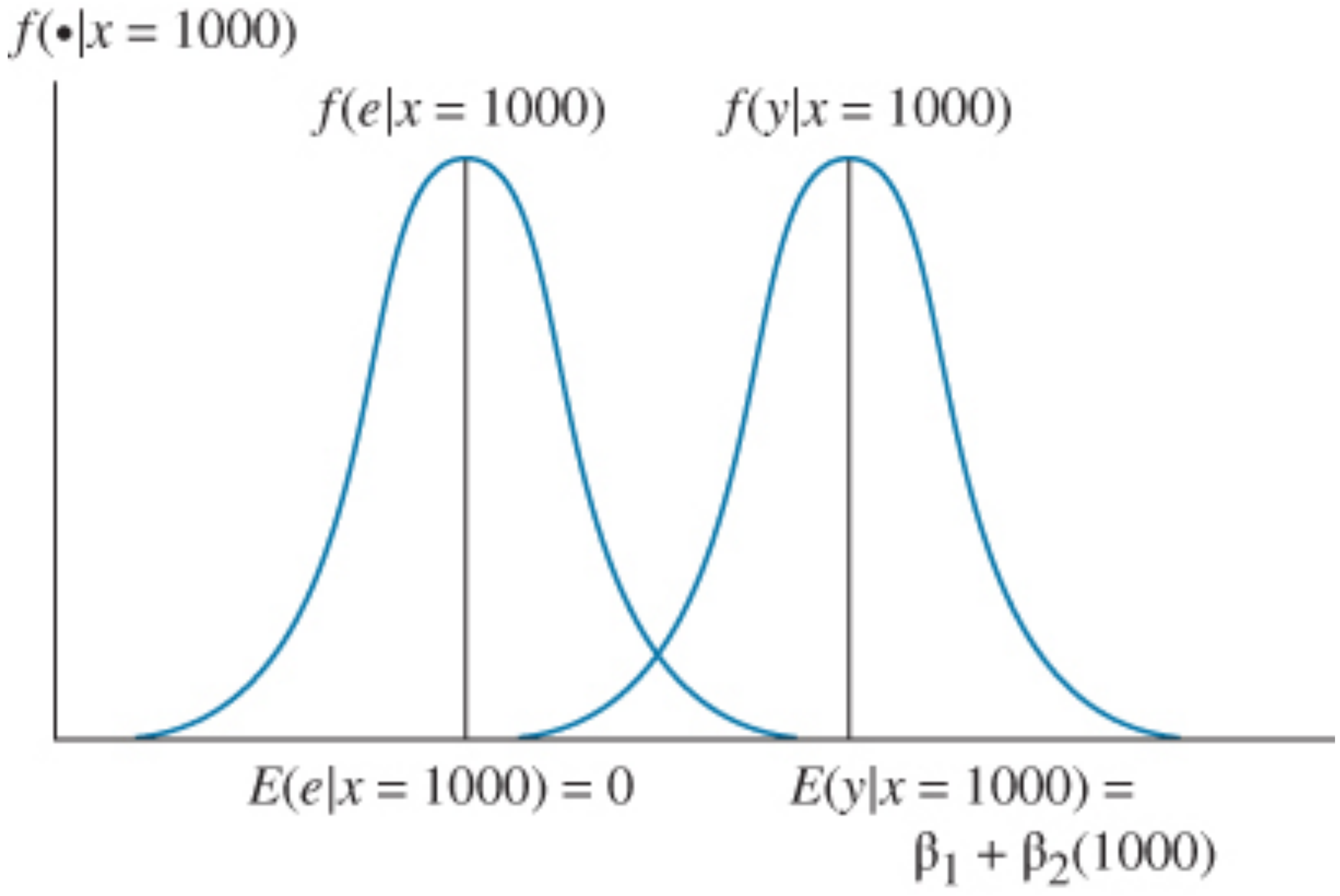


Fig 2.4. The random error

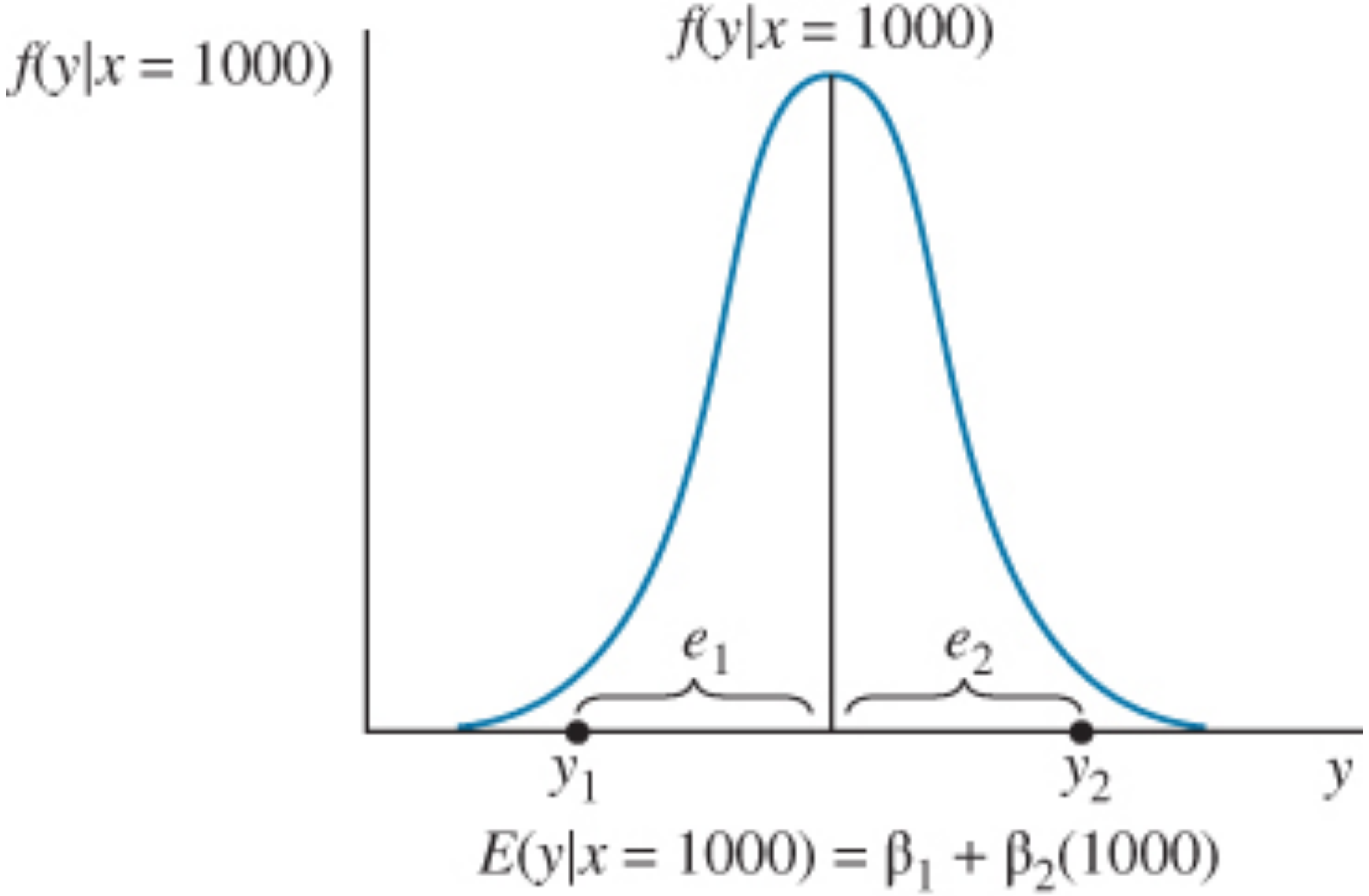
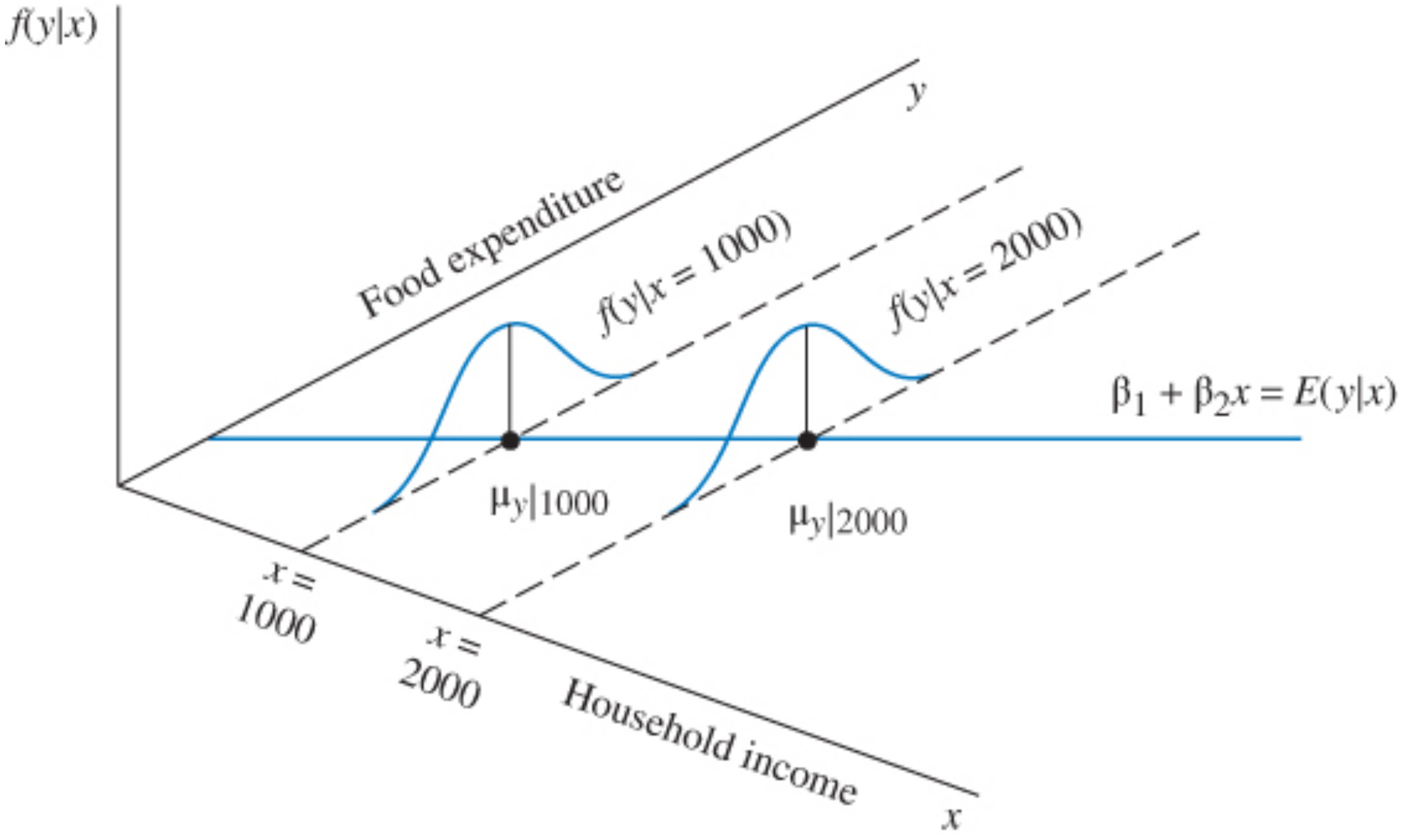


Fig 2.5 The conditional probability density function for *y*, food expenditure, at two levels of income.



2.3 Estimating the Regression Parameters

Review i

Economists are interested in studying relationship between variables

1. y: dependent variable (also known as target or response variable)
2. x: independent, explanatory variable. In a multiple regression, $x = (x_1, \ldots, x_p)$ indicate a vector of p dimensions.

Recipe in fitting a model i

1. An economic model is of the following form,

$$y = f(x, \beta)\,.$$

Suppose a functional form of $f(x,\beta)$ is decided based on domain knowledge or explanatory data analysis. β is called the parameter.

2. Econometric model links the data in practice to the theoretic economic model by an error term:

$$y = f(x, \beta) + e,$$

where e is the random error.

Recipe in fitting a model ii

3. Collect the data.
4. Estimate the parameter using one method from the following:

   least squares estimates (LSE), maximum likelihood estimate (MLE), Moment methods, others.

5. Perform model diagnostic to check model misspecification.

   5.1 Visualization tools: scatter plot (check weird pattern), density plot (check normality), qqplot (check normality).
   5.2 Goodness-of-fit test: Komogorove-Smirnov typed test, Others.

6. If Step (e) is passed, interpret the model parameters.

   Otherwise, it is meaningless to do so.

A simple regression model i

A simple regression model is proposed by

$$y = \beta_1 + \beta_2 x + e,$$

where β_1 is the intercept and β_2 is the slope. Further assumptions:

1. The variable x is not random and must take at least two different values.
2. e are i.i.d. $N(0, \sigma^2)$.

Assumptions of the Simple Linear Regression Model

SR 1: Econometric model

$$y_i = \beta_1 + \beta_2 x_i + \varepsilon_i, i = 1, \ldots, N.$$

SR 2: Strict Exogeneity. Let $\mathbf{x} = (x_1, \ldots, x_N)$, $E(e_i | \mathbf{x}) = 0$

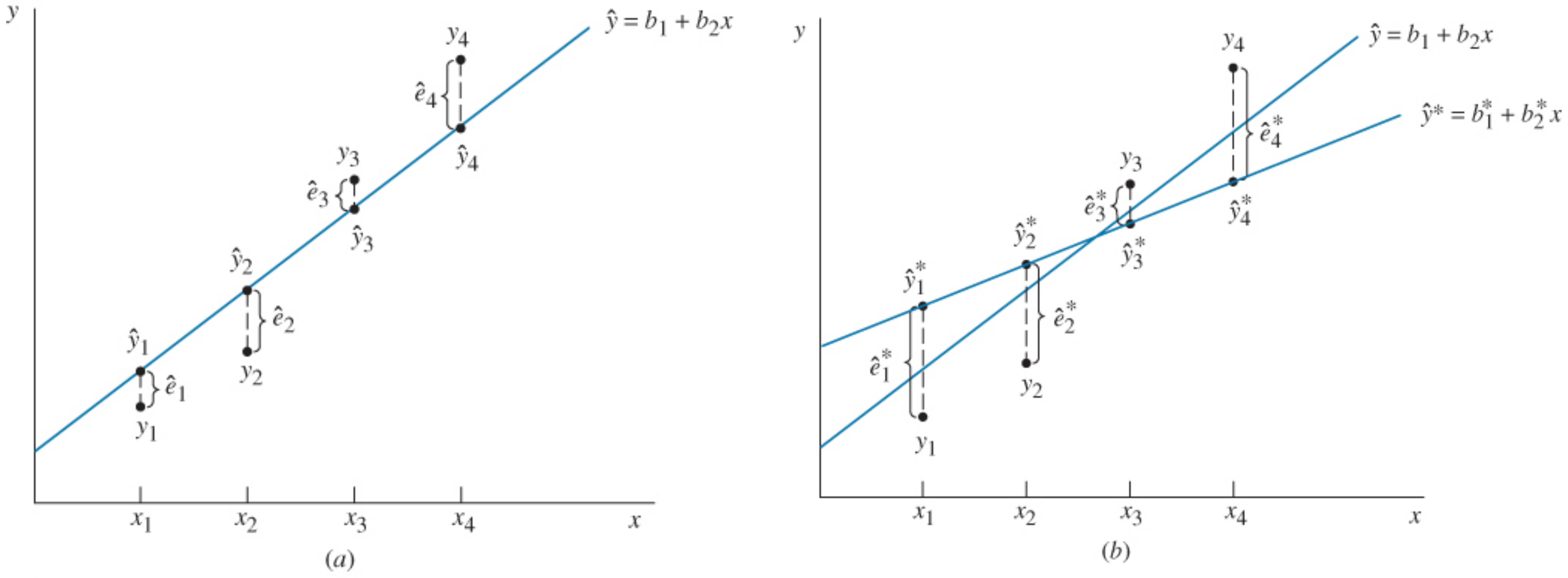
SR 3: Conditional Homoskedasticity $Var(e_i | \mathbf{x}) = \sigma^2$

SR 4: Conditional Uncorrelated Errors $Cov(e_i, e_j | \mathbf{x}) = 0$

SR 5: Explainatory Variable Must Vary: The variable x_i is not random and must take at least two different values.

SR 6: Error Normality (optional) $e_i | \mathbf{x} \sim N(0, \sigma^2)$

Fig 2.7 (a) The relationship among y, $\hat{e}$, and the fitted regression line, (b) The residuals from another fitted line.



Estimate parameters using LSE.

Define the sum of squares:

$$S(\beta_1, \beta_2) = \sum_{i=1}^{N} (y_i - \beta_1 - \beta_2 x_i)^2 \,.$$

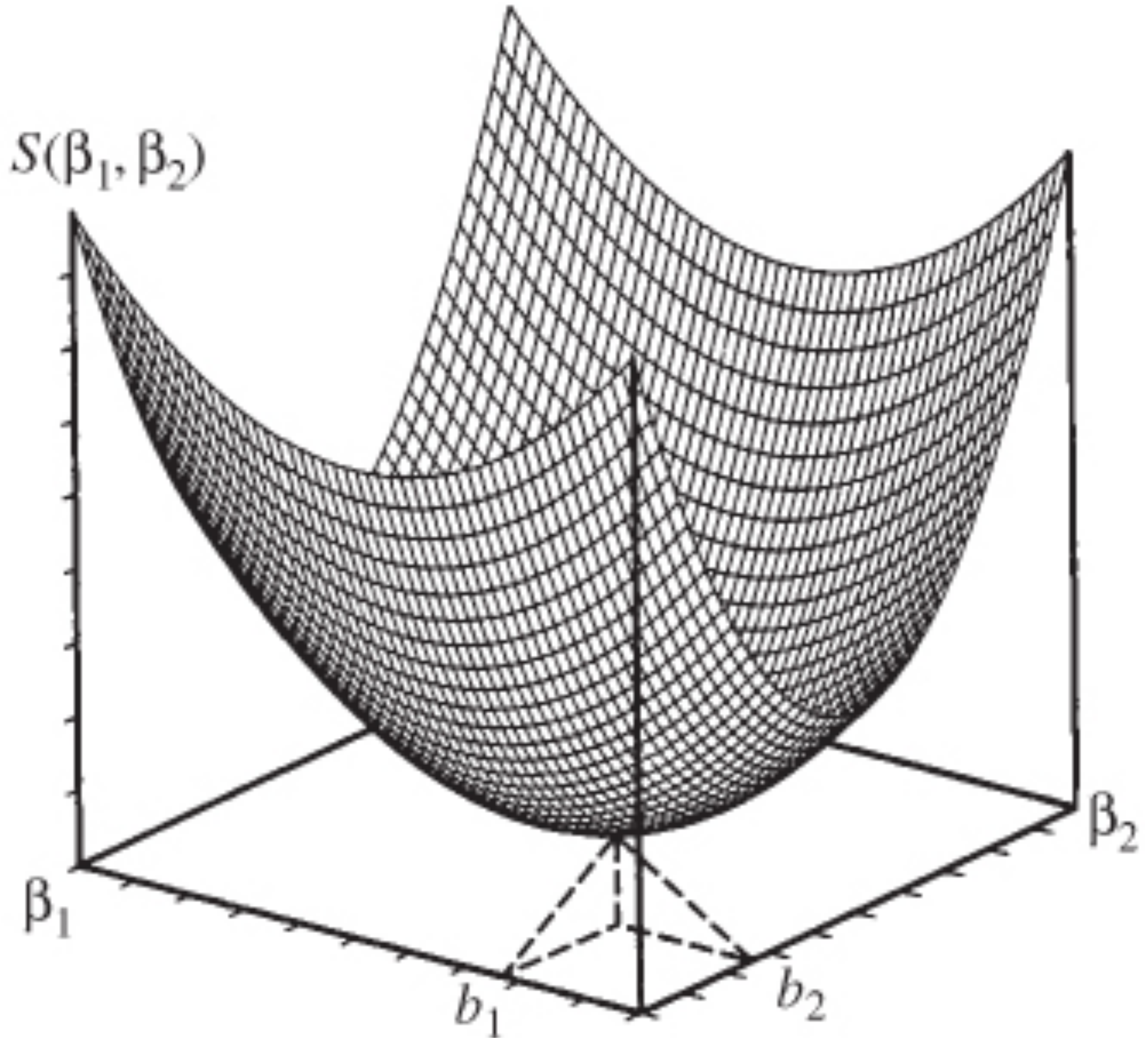
Least squares estimates b_1, b_2 satisfy

$$b_1, b_2 = \arg\min_{\beta_1, \beta_2} S(\beta_1, \beta_2) \,.$$

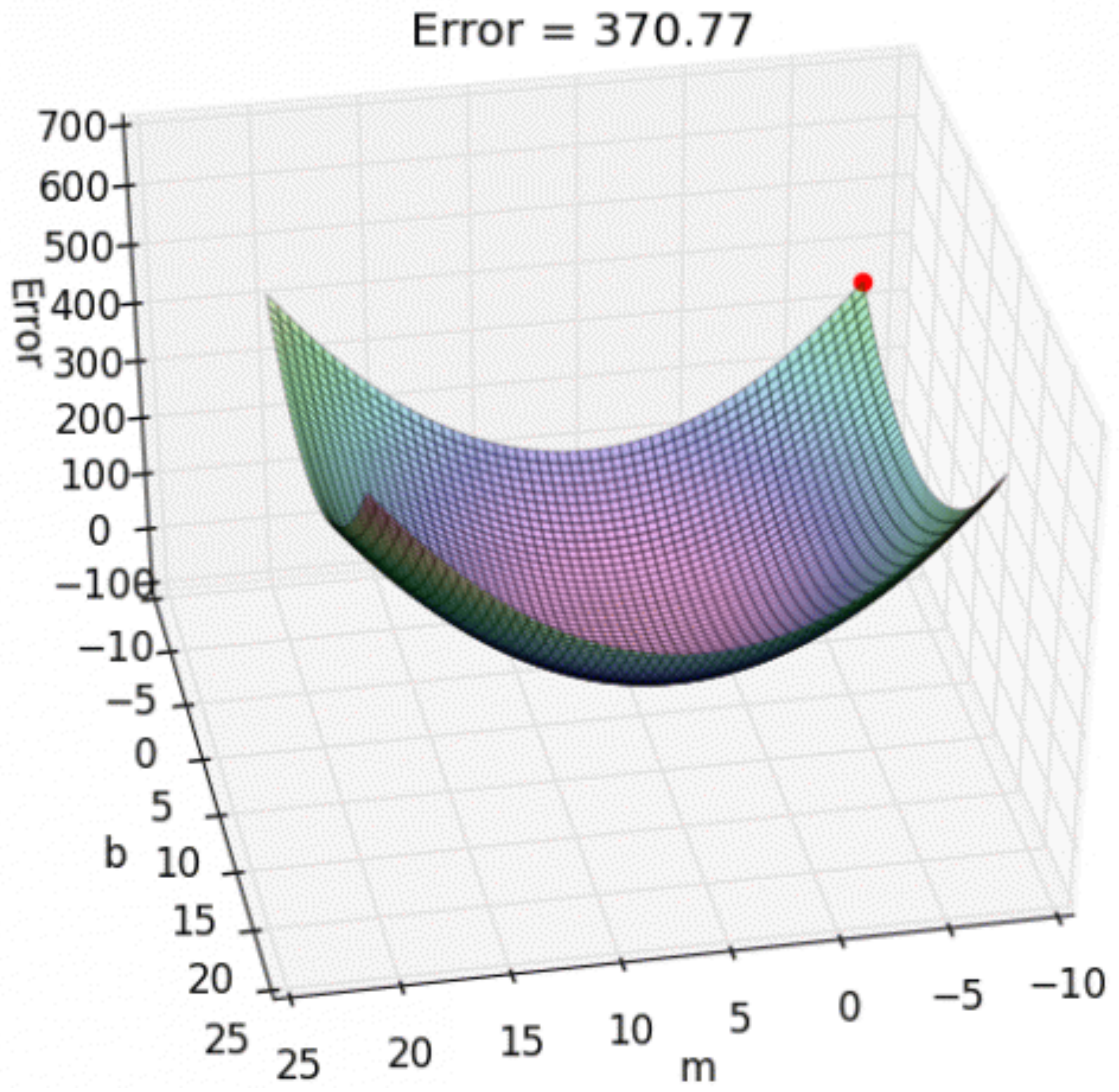
Or, least squares estimates b_1, b_2 solve

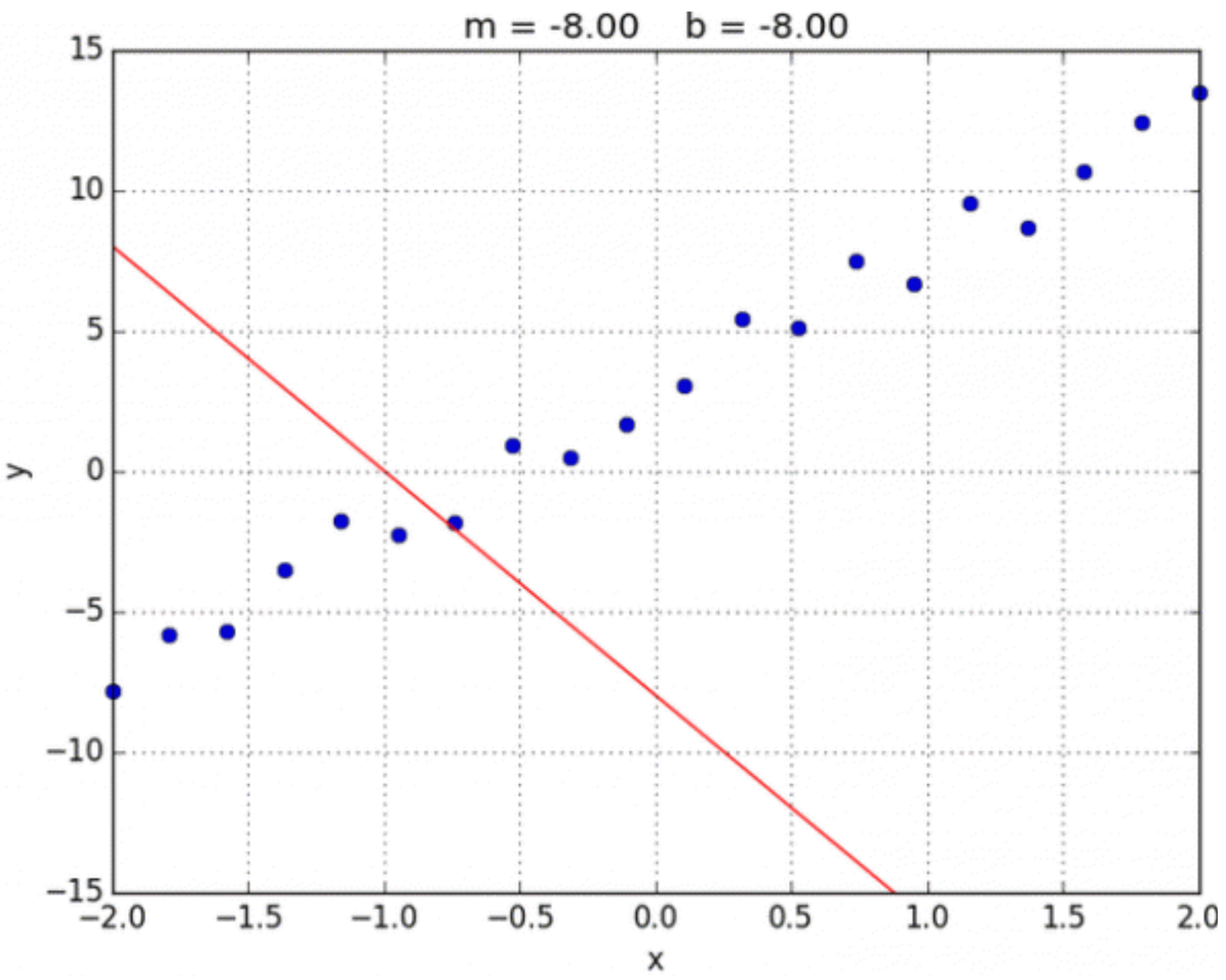
$$\min S(\beta_1, \beta_2) \,.$$

Fig. 2A.1: The sum of squares function and the minimizing values b_1 and b_2



- A demo





- https://medium.com/@savannahar68/getting-started-with-regression-a39aca03b75f

The resulting estimators are:

$$b_2 = \frac{\sum (x_i - \bar{x})(y_i - \bar{y})}{\sum (x_i - \bar{x})^2},$$

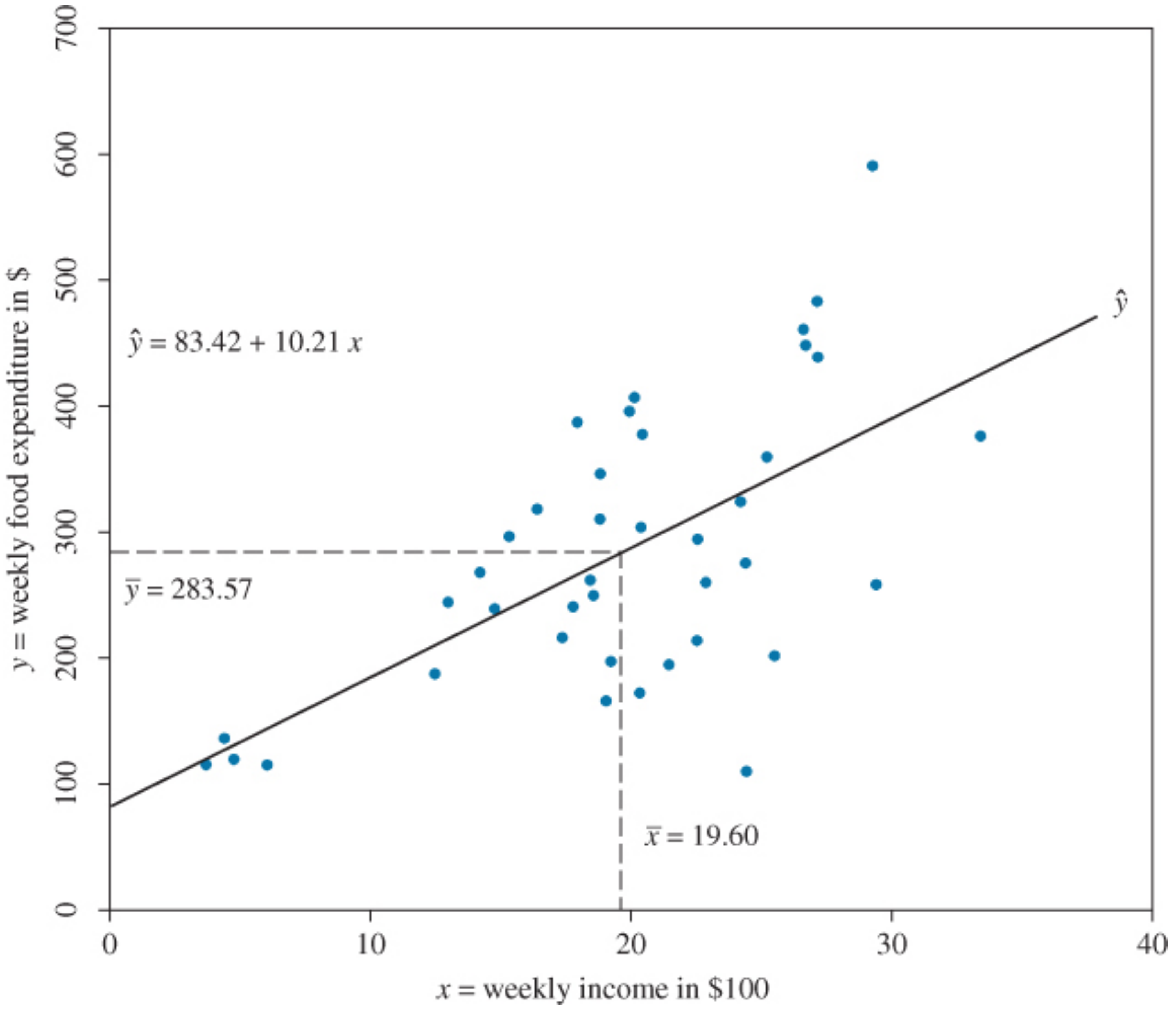
$$b_1 = \bar{y} - b_2\bar{x}\,.$$

Derivations to obtain b_1 and b_2 are deferred to the Appendix.

The estimated or fitted regression line is:

$$\hat{y}_i = b_1 + b_2 x_i\,.$$

Fig 2.8. The fitted regression



Example: Food expenditure model

We have

$$\hat{y}_i = 83.42 + 10.21x_i$$

Interpretations on the parameters:

1. The intercept estimate $b_1 = 83.42$ is an estimate of the weekly

food expenditure on food for a household with zero income.

2. The value $b_2 = 10.21$ is an estimate of β_2. We estimate that if the income goes up by \$100, expected weekly expenditure on food will increase approximately by \$10.21.

Elasticity

The elasticity of mean expenditure with respect to income is:

$$\varepsilon = \frac{\text{Percentage change in } y}{\text{percentage change in } x} = \frac{\triangle E(y)/E(y)}{\triangle x/x} = \frac{\triangle E(y)x}{\triangle xE(y)}$$

$$= \frac{\triangle E(y)}{\triangle x} \times \frac{x}{y} = \beta_2 \frac{x}{\beta_1 + \beta_2 x}$$

We estimate the elasticity by

$$\hat{\varepsilon} = b_2 \frac{\bar{x}}{\bar{y}} = 10.21 \times \frac{19.60}{283.57} = 0.71$$

Prediction

To predict weekly food expenditure for a household with a weekly income of \$2000, we plugging $x = 20$ into our estimated equation to obtained

$$\hat{y} = 83.42 + 10.21x_i = 83.42 + 10.21(20) = 287.61.$$

We *predict* that a household with a weekly income of \$2000 will spend \$287.61 per week on food.

R

```
# For details, see https://bookdown.org/ccolonescu/RPoE4/#
# plot the data
food = data(food);

plot(food$income, food$food_exp,
     ylim=c(0, max(food$food_exp)),
     xlim=c(0, max(food$income)),
     xlab="weekly income in $100",
     ylab="weekly food expenditure in $", type = "p")
```

```
# fit the model to the data: EXP = beta 1+ beta2 INCOME + e
mod1 <- lm(food_exp ~ income, data = food)
b1 <- coef(mod1)[[1]]
b2 <- coef(mod1)[[2]]
smod1 <- summary(mod1)
smod1
abline(b1,b2) # add the estimated (fitted) regression line



names(mod1)
names(smod1)
mod1$coefficients
smod1$coefficients
coef(mod1)
```

```
# retrieve the residuals and do a simple diagnostic test
r= resid(mod1)
plot(r) # scattor plot
hist(r) # histogram
plot(density(r)) # density plot
qqnorm(r) # qqplot
```

```r
# prediction
newx <- data.frame(income = c(20, 25, 27))
yhat <- predict(mod1, newx)
names(yhat) <- c("income=$2000", "$2500", "$2700")
yhat  # prints the result
```

Appendix. Derivations of the LSE for b_1 andb_2 i

First-order-Condition requires the partial derivatives of $S(\beta_1, \beta_2)$ to be zeros:

$$\frac{\partial S(\beta_1, \beta_2)}{\partial \beta_1} = -2\sum (y_i - \beta_1 - \beta_2 x_i) = 0,$$

$$\frac{\partial S(\beta_1, \beta_2)}{\partial \beta_2} = -2\sum x_i(y_i - \beta_1 - \beta_2 x_i) = 0.$$

Simple math gives

$$\sum y_i = N\beta_1 + (\sum x_i)\beta_2, \quad (1)$$

$$\sum x_i y_i = (\sum x_i)\beta_1 + (\sum x_i^2)\beta_2\,. \quad (2)$$

Appendix. Derivations of the LSE for b_1 andb_2 ii

Multiply (1) by $(\sum x_i)$ and multiply (2) by N, we have

$$(\sum x_i)(\sum y_i) = N(\sum x_i)\beta_1 + (\sum x_i)^2\beta_2, \tag{3}$$

$$N(\sum x_i y_i) = N(\sum x_i)\beta_1 + N(\sum x_i^2)\beta_2\,. \tag{4}$$

Let $\bar{x} = (\sum x_i)/N$ and $\bar{y} = (\sum y_i)/N$. We have

$$b_2 = \frac{N(\sum x_i y_i) - (\sum x_i \sum y_i)}{N(\sum x_i)^2 - (\sum x_i)^2} = \frac{\sum x_i y_i - N\bar{x}\bar{y}}{\sum x_i^2 - N\bar{x}^2} = \frac{\sum (x_i - \bar{x})(y_i - \bar{y})}{\sum (x_i - \bar{x})^2}\,. \qquad 5()$$

Plug b_2 in (5) into (1), we have

$$b_1 = \bar{y} - \bar{x}b_2\,.$$

Appendix. Derivations of the LSE for b_1 and b_2 iii

For hand calculations, we obtain the following identities,

$$\begin{aligned}
\sum (x_i - \bar{x})^2 &= \sum (x_i^2 - 2\bar{x}x_i + \bar{x}^2) \\
&= \sum x_i^2 - 2\bar{x}(N\bar{x}) + N\bar{x}^2 \\
&= \sum x_i^2 - N\bar{x}^2, \\
\sum (x_i - \bar{x})(y_i - \bar{y}) &= \sum (x_i y_i - \bar{x}y_i - \bar{y}x_i + \bar{x}\bar{y}) \\
&= \sum x_i y_i - \bar{x}(N\bar{y}) - \bar{y}N\bar{x} + N\bar{x}\bar{y} \\
&= \sum x_i y_i - N\bar{x}\bar{y}\,.
\end{aligned}$$

2.4 Assessing the Least Squares Estimators

Expected values of b_1 and b_2

$$E(b_1 \,|\, \mathbf{x}) = \beta_1, E(b_2 \,|\, \mathbf{x}) = \beta_2 \,.$$

TABLE 2.2 **Estimates from 10 Hypothetical Samples**

Sample	b_1	b_2
1	93.64	8.24
2	91.62	8.90
3	126.76	6.59
4	55.98	11.23
5	87.26	9.14
6	122.55	6.80
7	91.95	9.84
8	72.48	10.50
9	90.34	8.75
10	128.55	6.99

Figure 1: Table 2.2. Estimates from 10 Hypothetical Samples

Variance and covariances

$$Var(b_1) = \sigma^2 \frac{\sum x_i^2}{N \sum (x_i - \bar{x})^2}, \quad Var(b_2) = \frac{\sigma^2}{\sum (x_i - \bar{x})^2},$$

$$\sigma_{b_1} = \sqrt{Var(b_1)}, \quad \sigma_{b_2} = \sqrt{Var(b_2)},$$

$$Cov(b_1, b_2) = \sigma^2 \frac{-\bar{x}}{\sum (x_i - \bar{x})^2}.$$

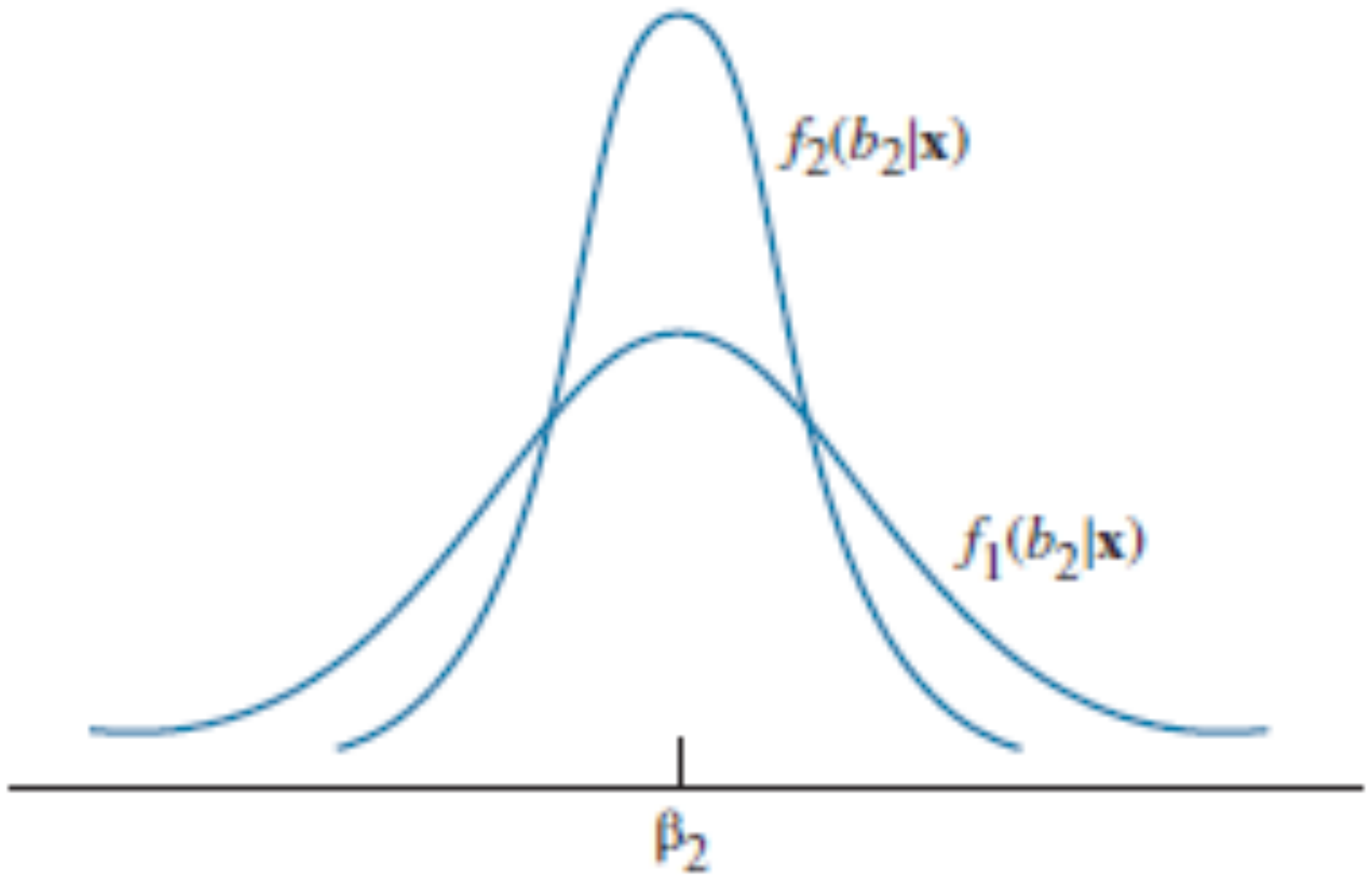


Figure 2: $Fig2.10$ Two possible probability density function for b_2

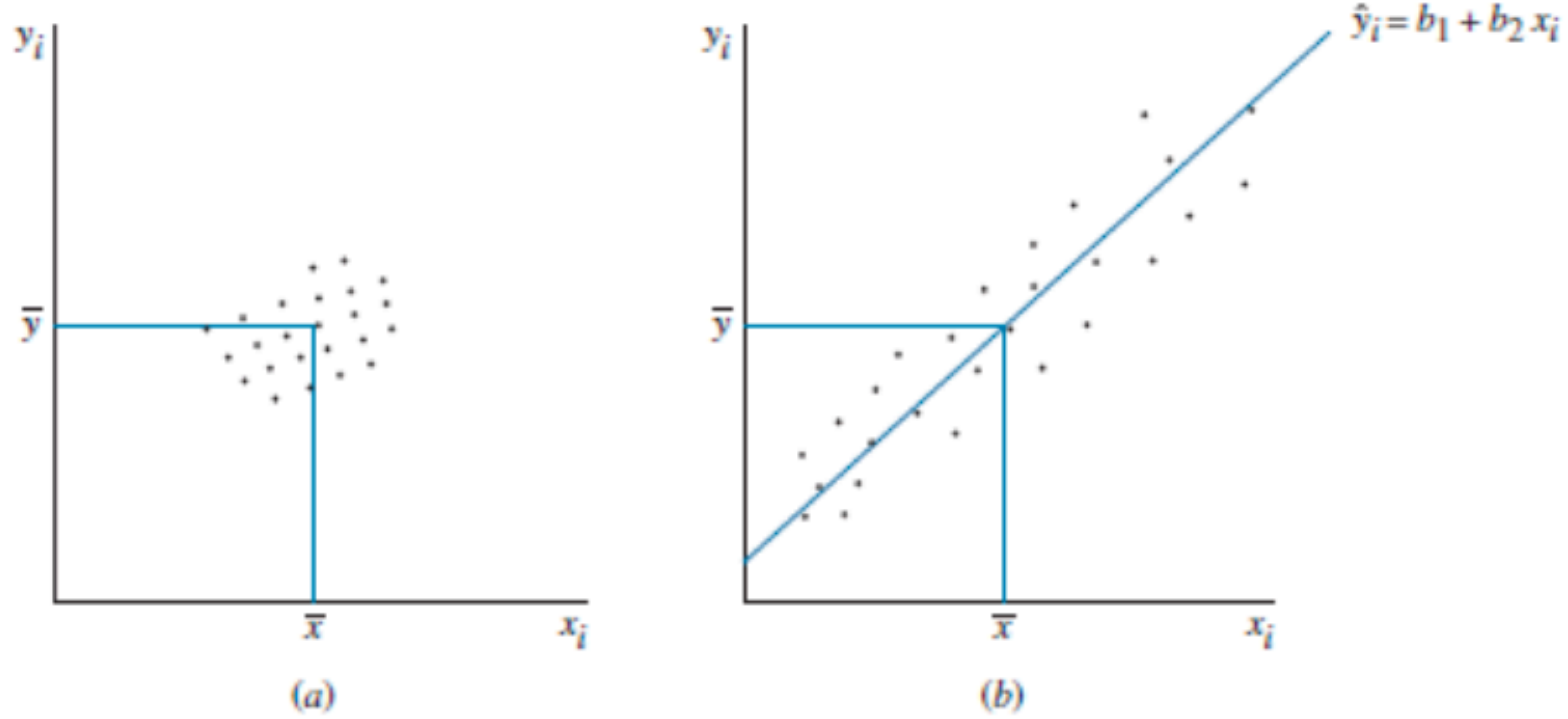


Figure 3: Fig 2.11 The influence of variation in the explanatory variable *x* on precision of estimation: (a) lower *x* variation, low precision: (b) high *x* variation, high precision.

2.5 Gauss-Markov Theorem

Assumptions in p. 58

Assumptions of the Simple Linear Regression Model

SR1: Econometric Model All data pairs (y_i, x_i) collected from a population satisfy the relationship

$$y_i = \beta_1 + \beta_2 x_i + e_i, \quad i = 1, \ldots, N$$

SR2: Strict Exogeneity The conditional expected value of the random error e_i is zero. If $\mathbf{x} = (x_1, x_2, \ldots, x_N)$, then

$$E(e_i | \mathbf{x}) = 0$$

If strict exogeneity holds, then the population regression function is

$$E(y_i | \mathbf{x}) = \beta_1 + \beta_2 x_i, \quad i = 1, \ldots, N$$

and

$$y_i = E(y_i | \mathbf{x}) + e_i, \quad i = 1, \ldots, N$$

SR3: Conditional Homoskedasticity The conditional variance of the random error is constant.

$$\operatorname{var}(e_i | \mathbf{x}) = \sigma^2$$

SR4: Conditionally Uncorrelated Errors The conditional covariance of random errors e_i and e_j is zero.

$$\operatorname{cov}(e_i, e_j | \mathbf{x}) = 0 \quad \text{for} \quad i \neq j$$

SR5: Explanatory Variable Must Vary In a sample of data, x_i must take at least two different values.

SR6: Error Normality (*optional*) The conditional distribution of the random errors is normal.

$$e_i | \mathbf{x} \sim N(0, \sigma^2)$$

Gauss-Markov Theorem

Given x under the assumptions SR1-SR5 of the linear regression model, the estimators b_1 and b_2 have the smallest variance of all linear and unbiased estimators of β_1 and β_2. They are the best linear unbiased estimators (BLUE) of β_1 and β_2.

2.6 The Probability Distributions of Least Squares Estimators

Sampling distribution

If SR6 holds, we have:

$$b_1 \sim N\left(\beta_1, \frac{\sigma^2 \sum x_i^2}{N \sum (x_i - \bar{x})^2}\right), \tag{1}$$

$$b_2 \sim N\left(\beta_2, \frac{\sigma^2}{\sum (x_i - \bar{x})^2}\right). \tag{2}$$

A Central Limit Theorem: If assumptions SR1-SR5 hold, and if the sample size N is sufficiently large, then the least square estimators have a distribution that approximate the normal distribution shown in (1) and (2).

Why we need (1) and (2)?

1. Confidence interval
2. Hypothesis test
    1.

A simulation study

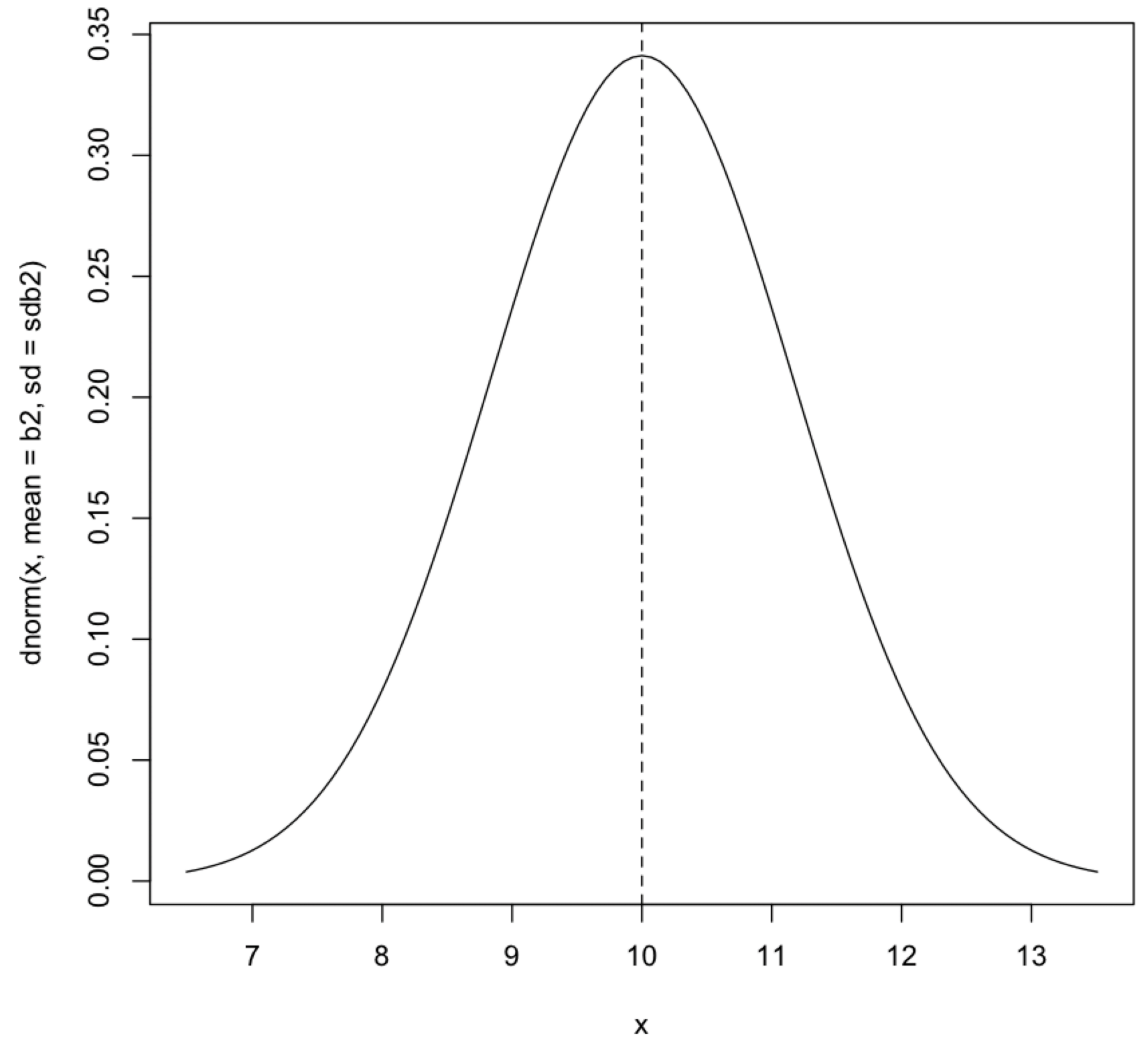
To show that b_2 has the distribution

$$N\left(\beta_2, \frac{\sigma^2}{\sum (x_i - \bar{x})^2}\right)$$

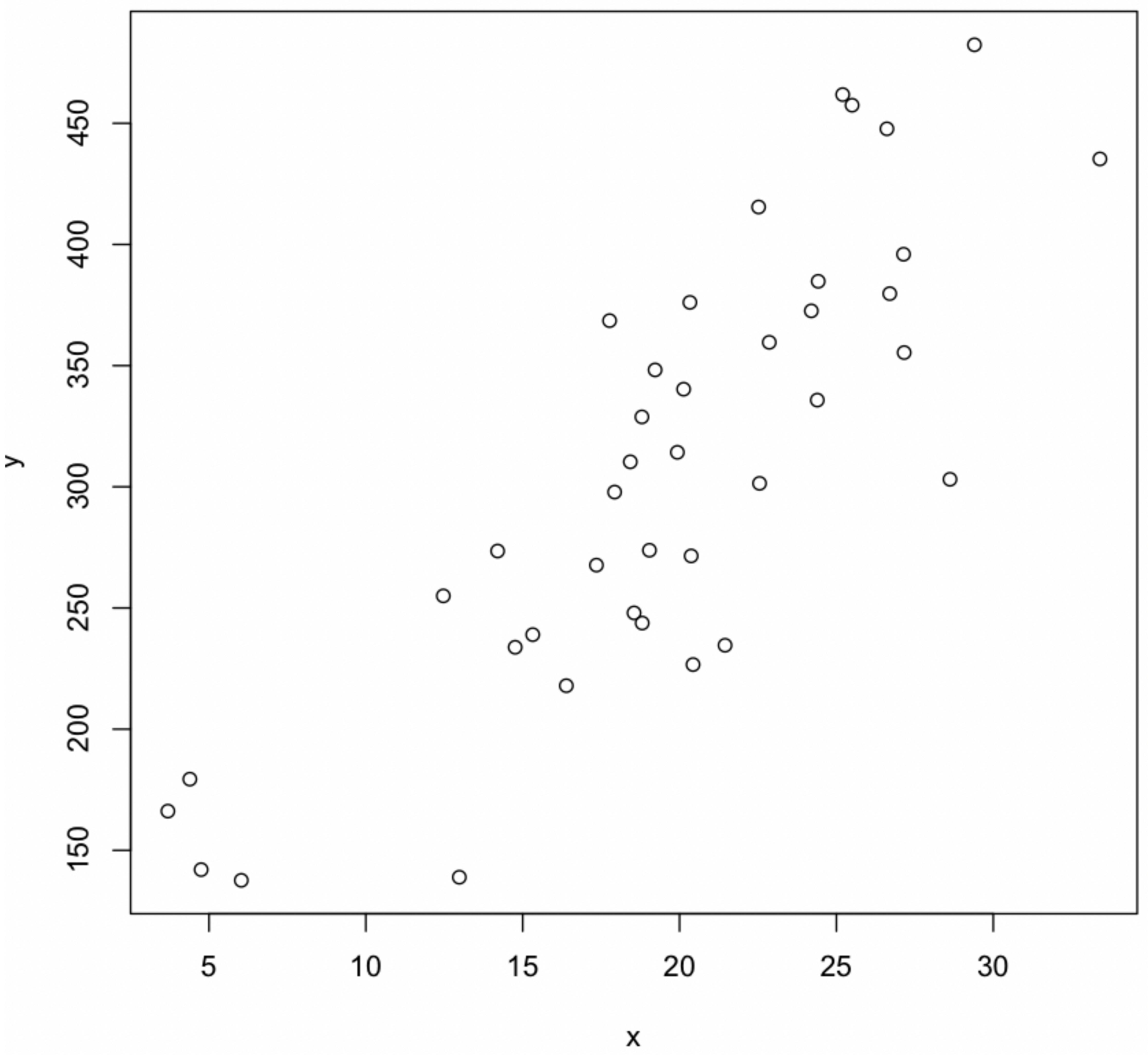
Model: $y = \beta_1 + \beta_2 x + \varepsilon, \varepsilon \sim N(0,\sigma^2)$

- x are from food data
- $\beta_1 = 100, \beta_2 = 10, \sigma^2 = 2500$
- $N = 40$

```
# Demonstrate this part in class.
# Start from here Teng: 20240308
# (A) Calculate the theoretical distribution of b_2
x <- food$income
x
xbar <- mean(x)
sumx2 <- sum((x-xbar)^2)
varb2 <- sig2e/sumx2
sdb2 <- sqrt(varb2)
leftlim <- b2-3*sdb2
rightlim <- b2+3*sdb2
curve(dnorm(x, mean=b2, sd=sdb2), leftlim, rightlim)
abline(v=b2, lty=2)
```

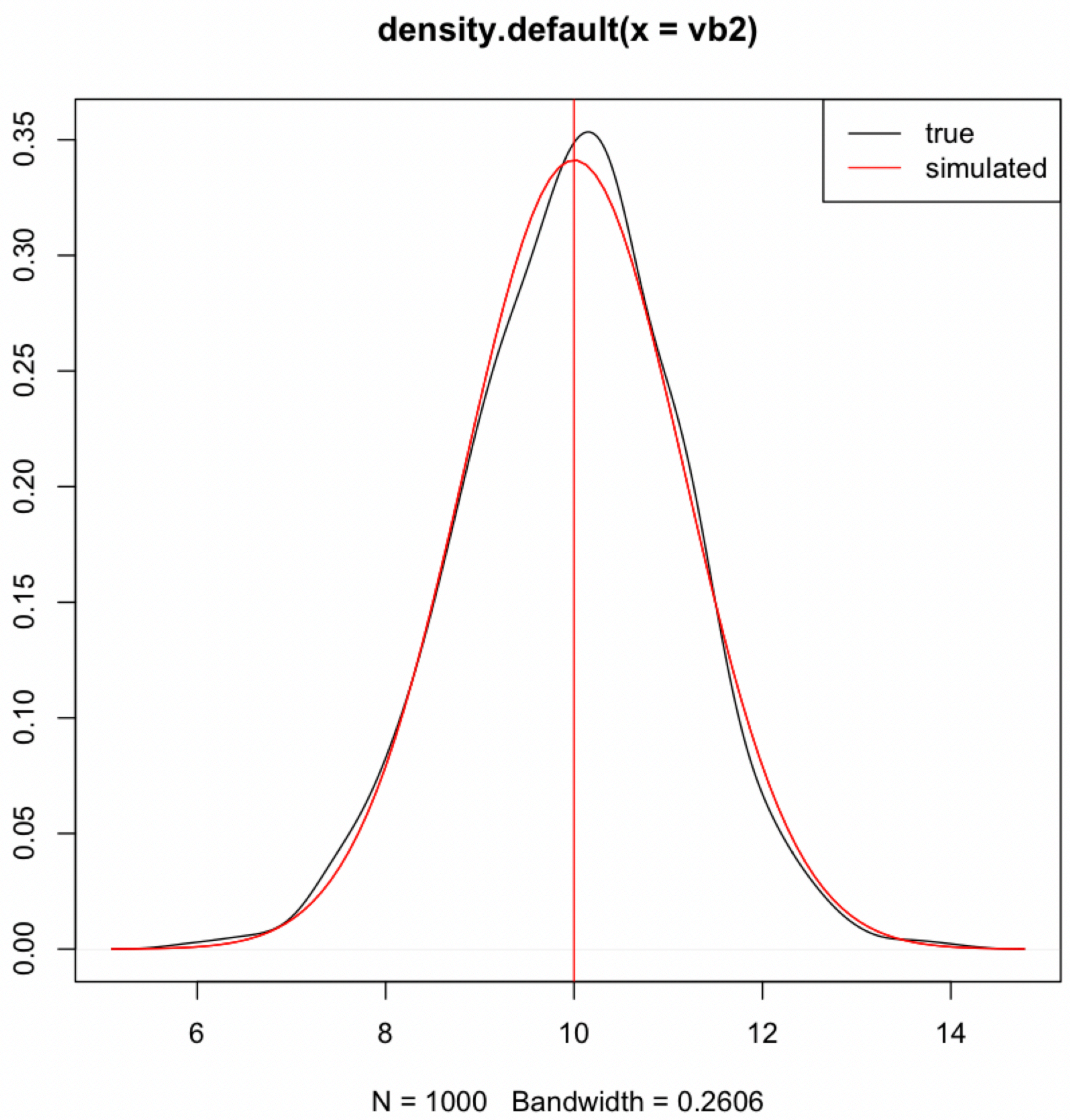


```r
# (B) Use x in the food dataset to generate price y.
# Check b1hat, b2hat, and seb2hat
# imagine we are the creator will produce data (x,y) using the model
# model: y = b1 + b2 *x + error
# Assume: b1, b2, x are known
set.seed(12345)
x <- food$income
y <- b1+b2*x+rnorm(N, mean=0, sd=sde)
plot(x, y) # simulated data
mod6 <- lm(y~x)
b1hat <- coef(mod6)[[1]]
b2hat <- coef(mod6)[[2]]
mod6summary <- summary(mod6) #the summary contains the standard errors
seb2hat <- coef(mod6summary)[2,2]
b1hat
b2hat
seb2hat
```



```r
# (C) Use x in the food dataset and implement a simulation study
# to understand the sampling properties of b1, b2, and the erros.
N <- 40
x <- food$income
nrsim <- 1000
sde <- 50
vb2 <- numeric(nrsim) #stores the estimates of b2
for (i in 1:nrsim){
  set.seed(12345+10*i)
  y <- b1+b2*x+rnorm(N, mean=0, sd=sde)
  mod7 <- lm(y~x)
  vb2[i] <- coef(mod7)[[2]]
}
mb2 <- mean(vb2)
seb2 <- sd(vb2)

plot(density(vb2))
curve(dnorm(x, b2, sdb2), col="red", add=TRUE)
abline(v = b2, col="red");
legend("topright", legend=c("true", "simulated"),
       lty=1, col=c("black", "red")) # modified by Teng on 2021/3/9
curve(dnorm(x, mean=b2, sd=sdb2), col="red", add=TRUE) #
```



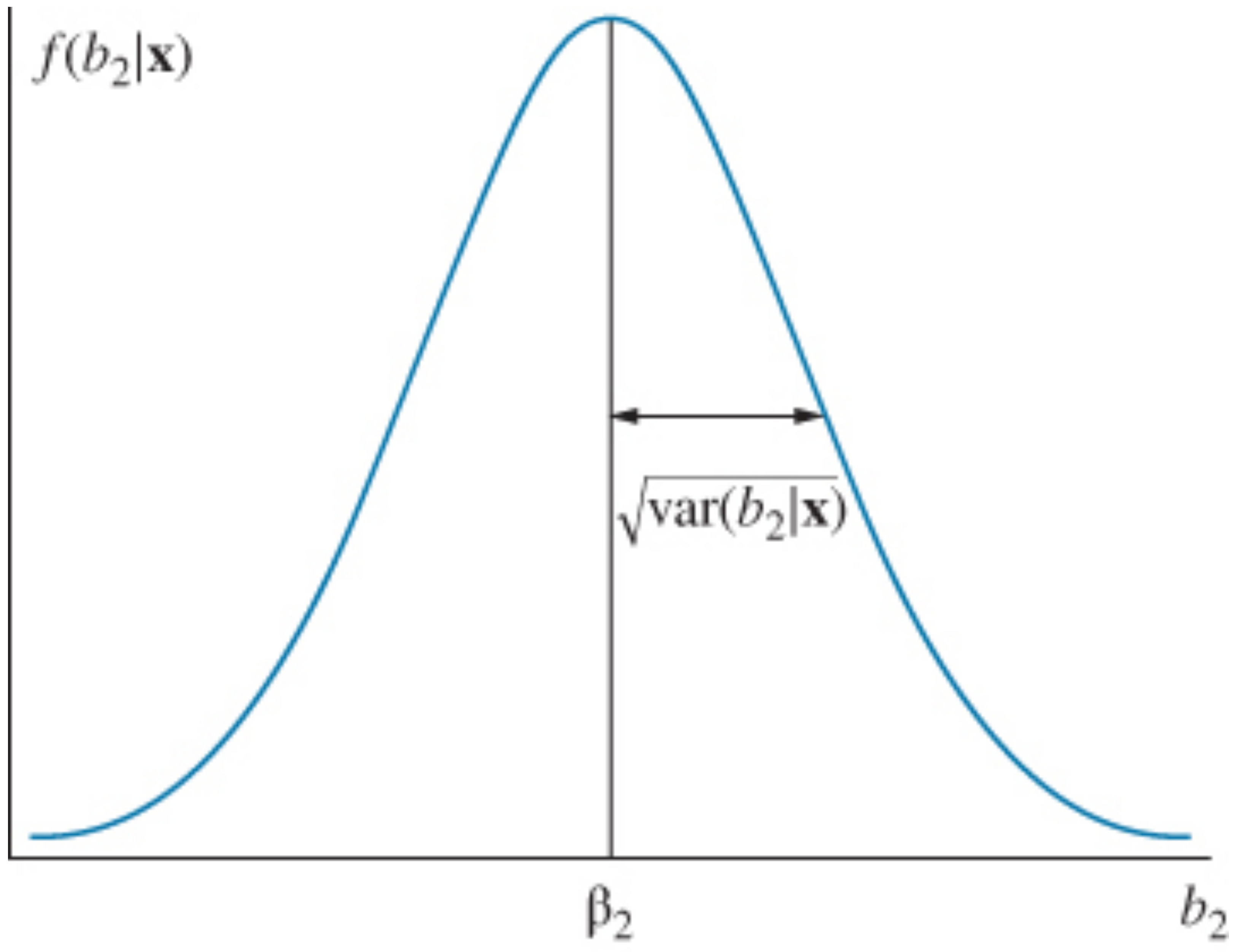
2.7 Estimating the Variance of the Error Term

How do we estimate $Var(b_2 | x)$ in (2)? Need to estimate σ^2!

Note $e_i \sim N(0,\sigma^2)$. Thus, we have

$$E(e_i^2) = \sigma^2 .$$

We use "a sample moment estimator" to estimate σ^2 .



The *residual* approximates the error:

$$\hat{e}_i = y_i - \hat{y}_i = y_i - b_1 - b_2 x_i \,.$$

An unbiased estimator for σ^2 is

$$\hat{\sigma}^2 = \frac{\sum (y_i - b_1 - b_2 x_i)^2}{N - 2},$$

We have $E(\hat{\sigma}^2) = \sigma^2$.

- Because $e \sim N(0, \sigma^2)$, $E[e^2] = \mu^2 + \sigma^2 = 0^2 + \sigma^2 = \sigma^2$.

We obtain estimates:

$$\hat{Var}(b_1) = \hat{\sigma}^2 \frac{\sum x_i^2}{N \sum (x_i - \bar{x})^2}, \quad \hat{Var}(b_2) = \frac{\hat{\sigma}^2}{\sum (x_i - \bar{x})^2},$$

$$\hat{Cov}(b_1, b_2) = \hat{\sigma}^2 \frac{-\bar{x}}{\sum (x_i - \bar{x})^2},$$

$$\hat{\sigma}_{b_1} = \sqrt{\hat{Var}(b_1)}, \quad \hat{\sigma}_{b_2} = \sqrt{\hat{Var}(b_2)}\,.$$

Summarize the estimated variances and covariance as:

$$\begin{pmatrix} \hat{Var}(b_1) & \hat{Cov}(b_1, b_2) \\ \hat{Cov}(b_1, b_2) & \hat{Var}(b_2) \end{pmatrix}.$$

R

```
# Many applications require estimates of the
# variances and covariances of the
# regression coefficients.
# R stores them in the a matrix vcov():
varb1 <- vcov(mod1)[1, 1]
varb1
varb2 <- vcov(mod1)[2, 2]
varb2
covb1b2 <- vcov(mod1)[1,2]
covb1b2
vcov(mod1)
```

2.8 Estimating Nonlinear Relationships

Overview

Recall the linear model of house price:

$$PRICE = \beta_1 + \beta_2 SQFT + e\,.$$

Many economic relationships are represented by curved lines, and are said to display *curvlinear* forms.

We may consider using $SQFT^2$ or ln(PRICE) as an alternative model.

For a general class of models, see chapter 4.1.

The quadratic model

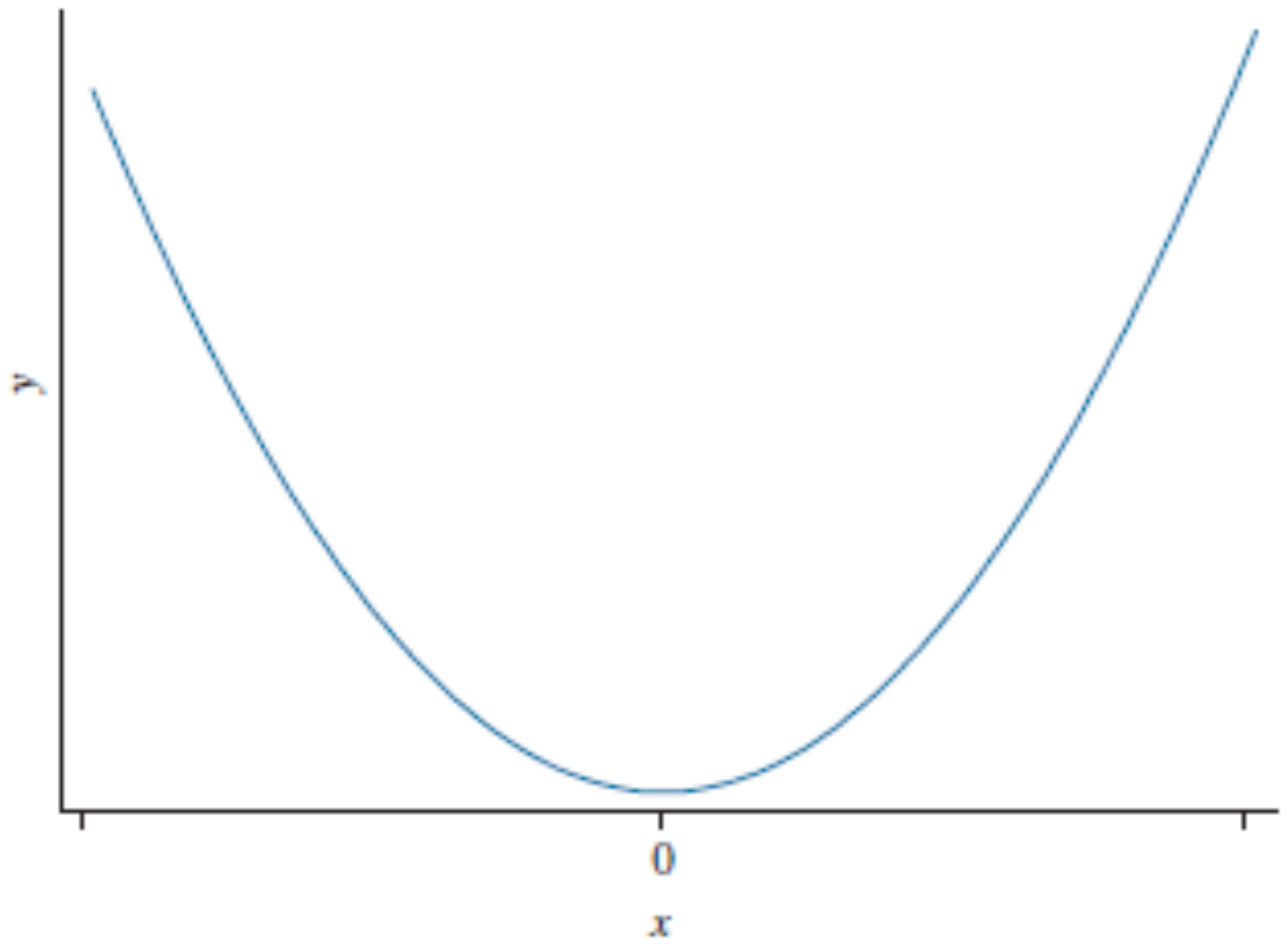


FIGURE 2.13 **A quadratic function, $y = a + bx^2$.**

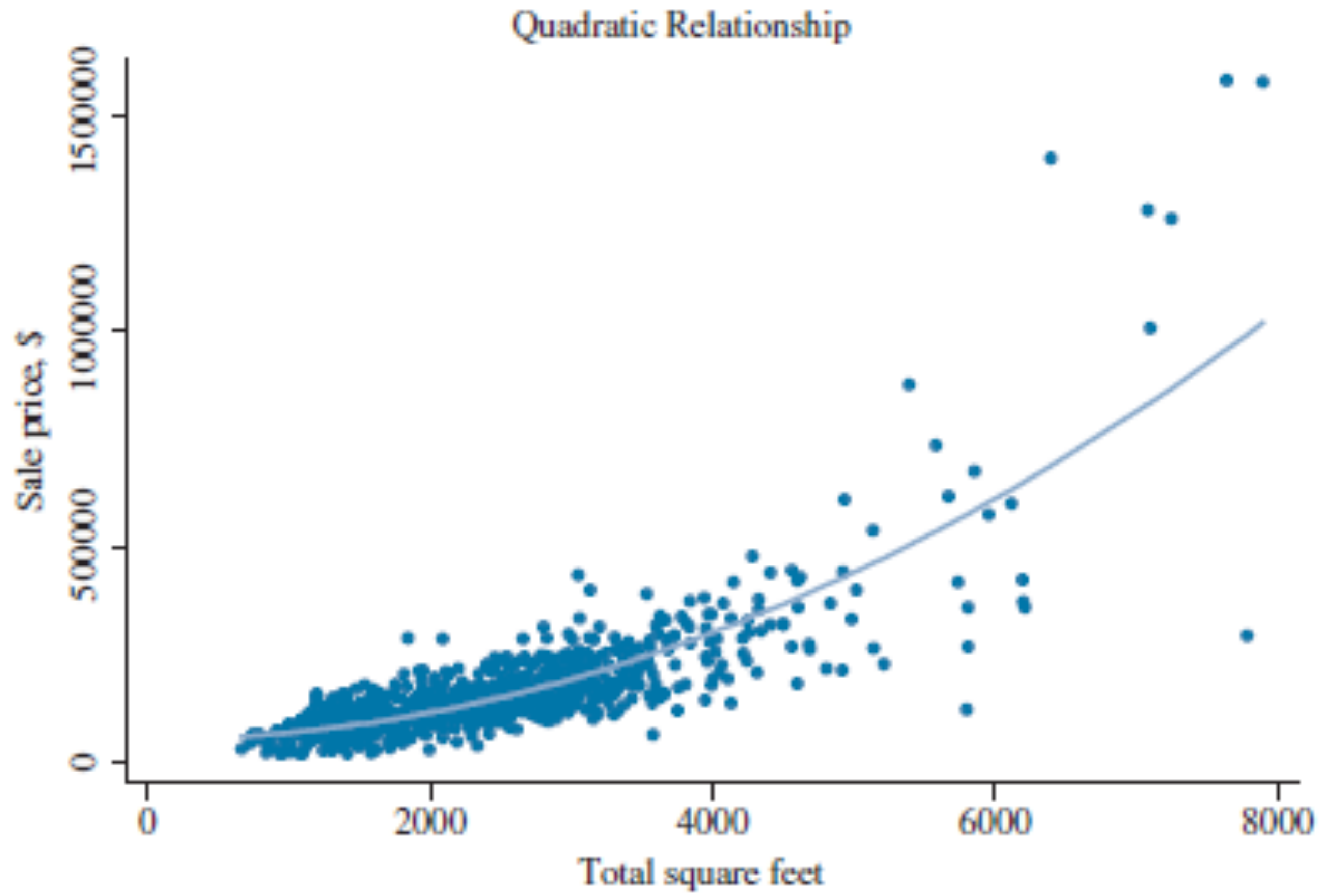


FIGURE 2.14 **A fitted quadratic relationship.**

The slope

Consider $SQFT^2$ as the explanatory variable:

$$\text{PRICE} = \beta_1 + \beta_2 SQFT^2 + e \,.$$

The slope is

$$m = \frac{dPRICE}{dSQFT} = 2\beta_2 SQFT \,.$$

We estimate the slope by

$$\hat{m} = 2b_2 SQFT \,.$$

If $b_2 > 0$, a larger house have large slop, and larger estimated price per additional square foot.

The elasticity

The elasticity is

$$\varepsilon = \frac{\triangle y/y}{\triangle x/x} = \frac{\triangle y}{\triangle x}\frac{x}{y} = m\frac{SQFT}{PRICE}$$
$$= (2\beta_2 SQFT)\frac{SQFT}{PRICE} = 2\beta_2\frac{SQFT^2}{PRICE} = 2\beta_2\frac{SQFT^2}{\beta_1 + \beta_2 SQFT^2}.$$

Hence, we estimate the elasticity by

$$\hat{\varepsilon} = 2b_2\frac{SQFT^2}{PRICE} = 2b_2\frac{SQFT^2}{b_1 + b_2}$$

R

```
# PRICE = beta1+ beta2*SQFT^2 + e
mod3 <- lm(price~I(sqft^2), data=br)
summary(mod3)
b1 <- coef(mod3)[[1]]
b2 <- coef(mod3)[[2]]
sqftx=c(2000, 4000, 6000)      #given values for pricex=b1+b2*sqftx^2 #prices corresponding to given sqft
DpriceDsqft <- 2*b2*sqftx.      # marginal effect of sqft on price
elasticity = DpriceDsqft*sqftx/pricex
curve(b1+b2*x^2, col="red", add=TRUE)     # add the quadratic curve to the scatter plot
```

Log transformation

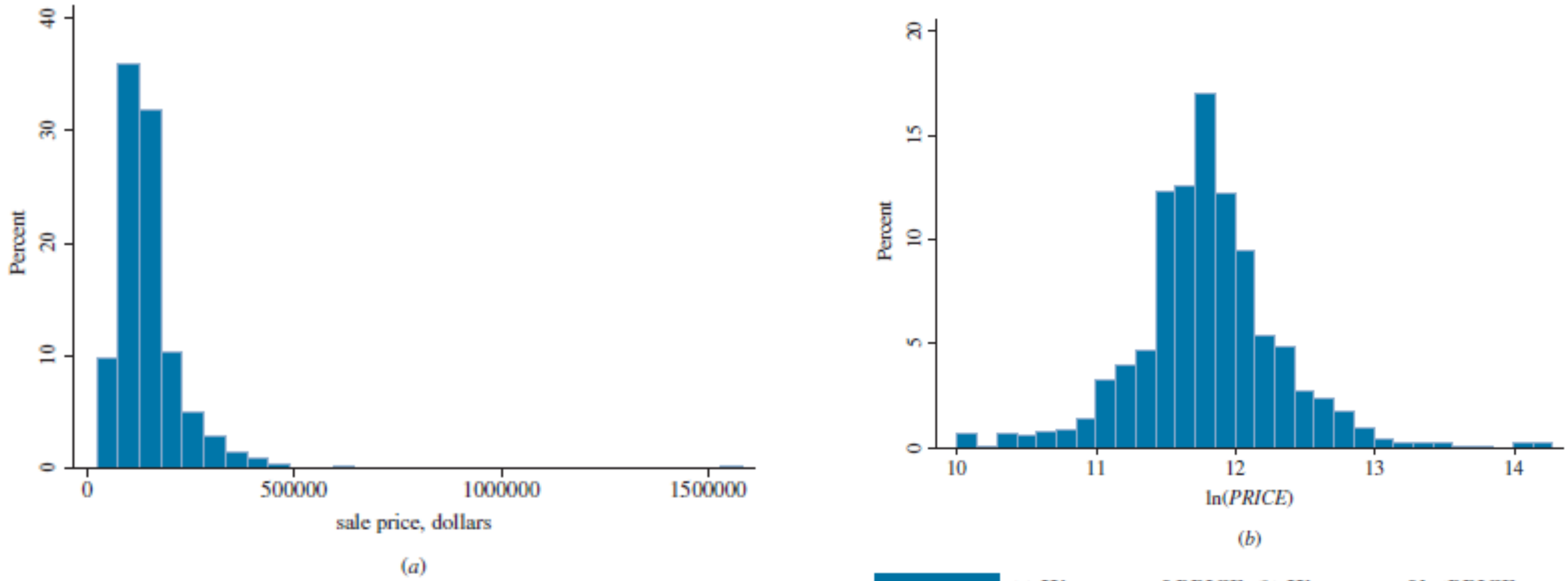


FIGURE 2.16 (*a*) Histogram of *PRICE*. (*b*) Histogram of ln(*PRICE*).

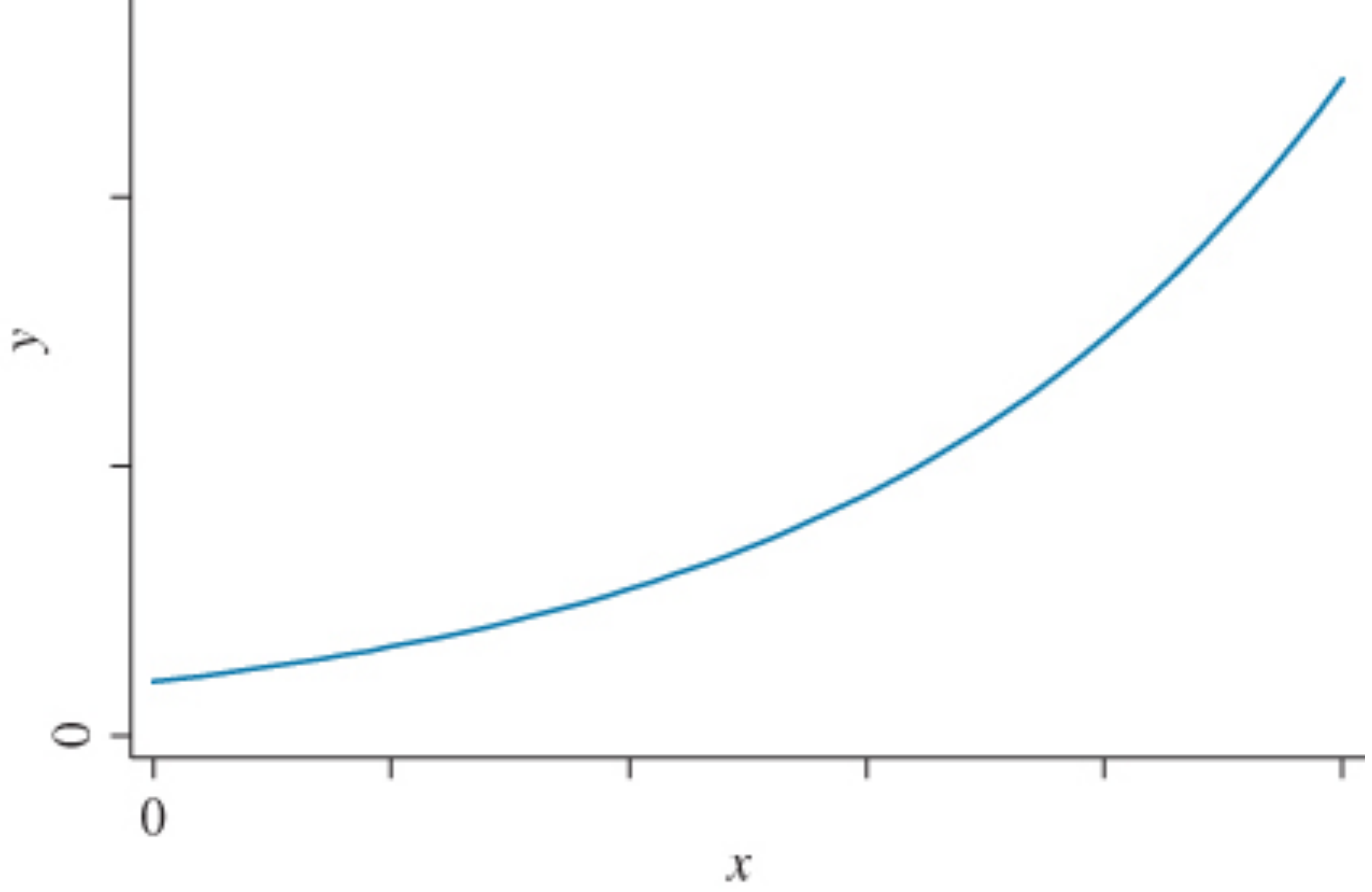
The log-linear model

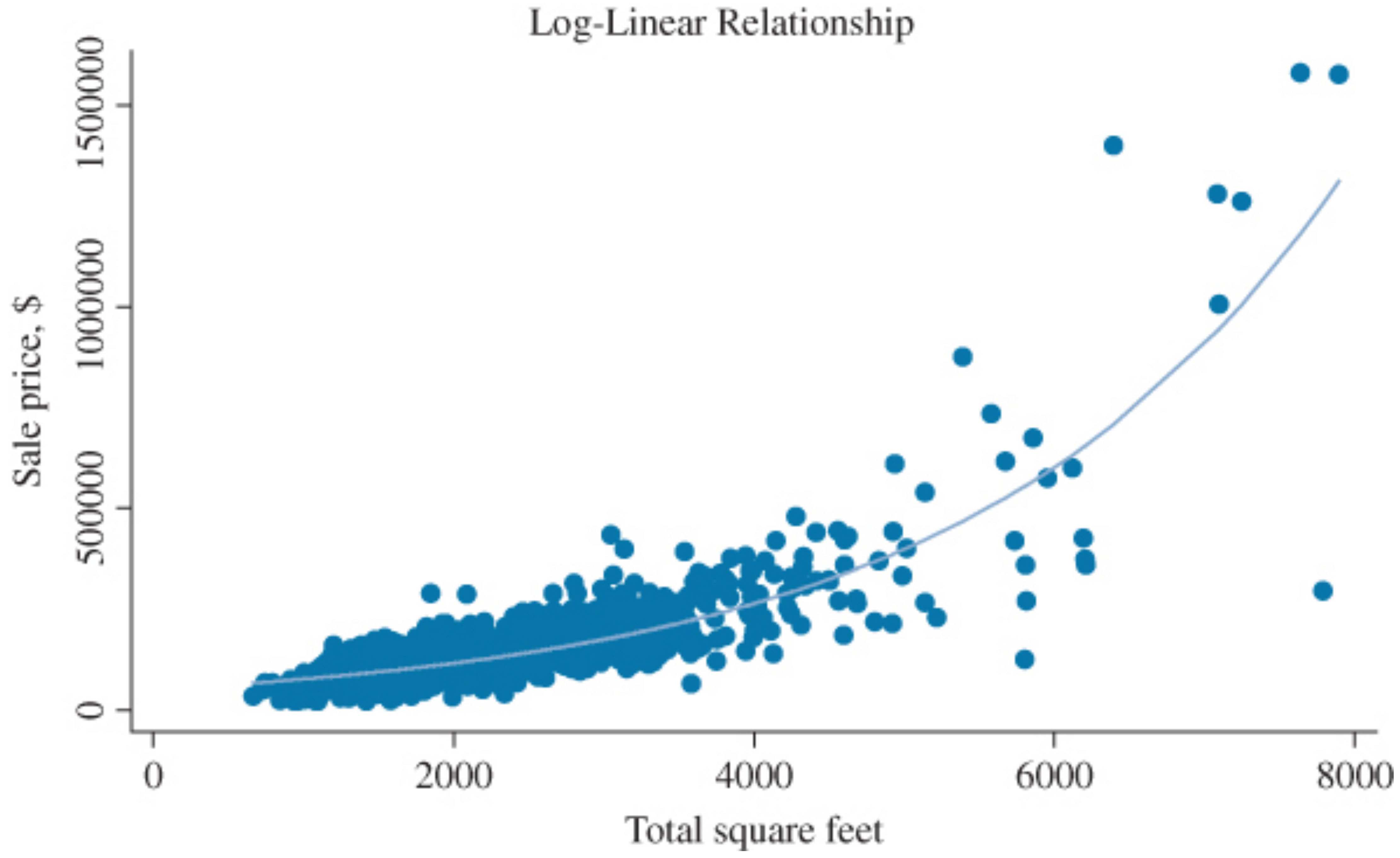
Log-Linear equation:

$$\log(y) = \beta_1 + \beta_2 x$$

$$\log(y) = \beta_1 + \beta_2 x + \varepsilon$$

$$y = \exp(\beta_1 + \beta_2 x)$$





The slope

The log-linear equation:

$$log(PRICE) = \beta 1 + \beta 2SQFT\,.$$

It is easy to see

$$PRICE = \exp(\beta_1 + \beta_2 SQFT)\,.$$

The slope is

$$m = \frac{dy}{dx} = \beta_2 \exp(\beta_1 + \beta_2 SQFT),$$

$$\hat{m} = b_2 \exp(b_1 + b_2 SQFT)\,.$$

Interpretations: When the size of the house is SQFT , the expected PRICE increases about $\beta_2 exp(\beta_1 + \beta_2 SQFT)$ unit with an additional square foot.

The elasticity

The elasticity is

$$\begin{aligned}\varepsilon &= \frac{\triangle y/y}{\triangle x/x} \\ &= m\frac{x}{y} \\ &= (\beta_2 \exp(\beta_1 + \beta_2 SQFT))\frac{SQFT}{PRICE} \\ &= \beta_2 SQFT,\end{aligned}$$

We estimate the elasticity by

$$\hat{\varepsilon} = b_2 SQFT\,.$$

Interpretations: While the size of the house is $SQFT$ and it increases one percent, the expected RRICE increases about $(b_2 SQFT) \times 100\,\%$.

R

```
# log(SQFT) = beta1 + beta2 SQFT + e_i
data(br)
hist(br$price, col='grey')
hist(log(br$price), col='grey')
mod4 <- lm(log(price)~sqft, data=br)
mod4
ordat <- br[order(br$sqft), ] #order the dataset
mod4 <- lm(log(price)~sqft, data=ordat)
mod4
plot(br$sqft, br$price, col="grey")
lines(exp(fitted(mod4))~ordat$sqft,
      col="blue", main="Log-linear Model")
```

Choosing a functional form

A challenging question:

- Sum squared of residuals (SSE), essentially $\hat{\sigma}^2$
- Informal way using visualization tools

2.9 Regression with Indicator Variables

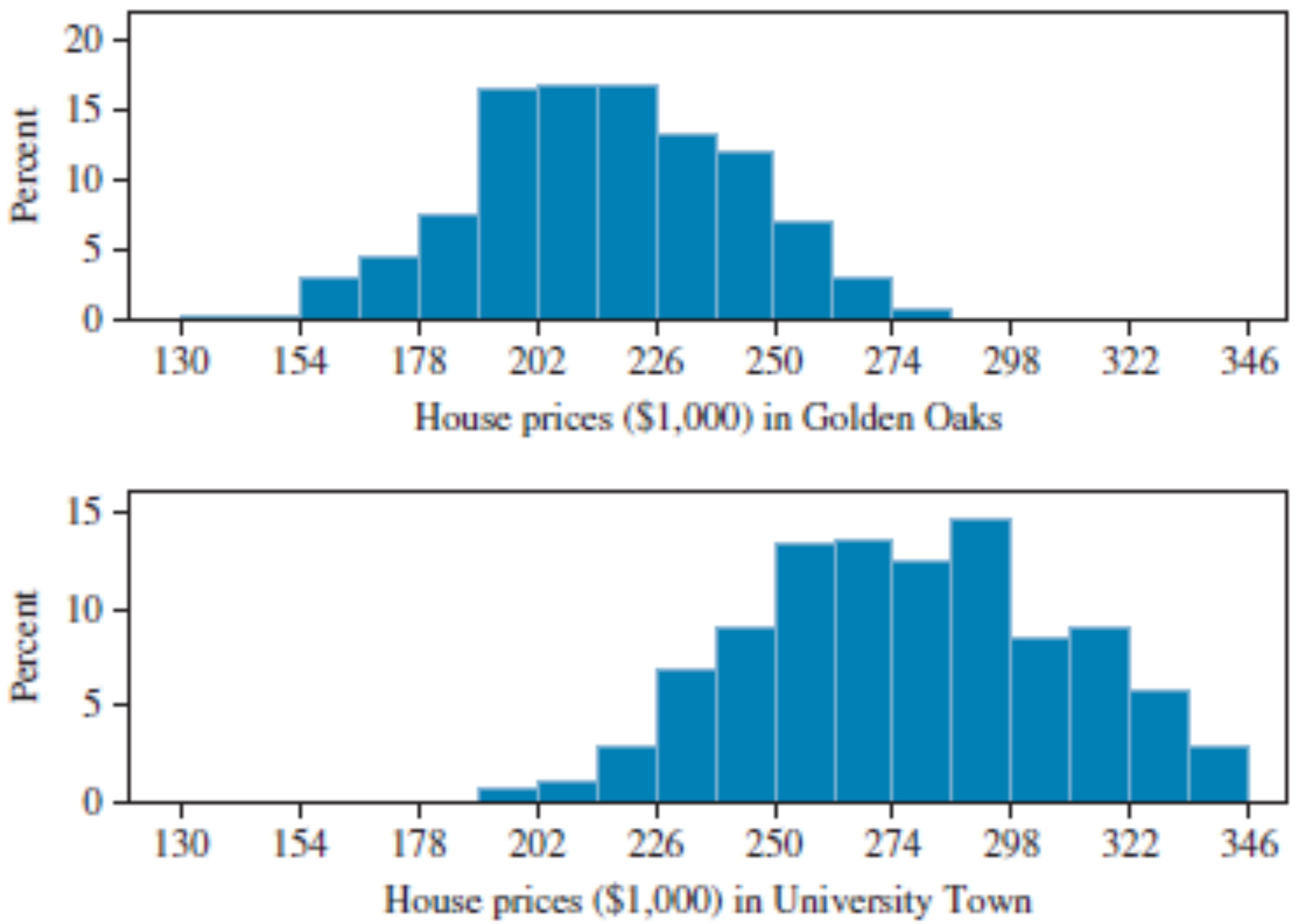


FIGURE 2.18 **Distributions of house prices.**

Dummy variable

Let

$$UTOWN = \begin{cases} 1, & \text{if a house is in University Town,} \\ 0, & \text{if a house is in G.} \end{cases}$$

Then, the model is

$$PRICE = \beta_1 + \beta_2 UTOWN + e\,.$$

- β_2 is the difference between the population means for house prices in the two neighborhoods.
- The expected price in University Town is $\beta_1 + \beta_2$.
- The expected price in Golden Oaks is β_1.

The estimated regression is

$$\widehat{PRICE} = b_1 + b_2 UTOWN = 215.7325 + 61.5091 UTOWN\,.$$

R

```
data(utown)
?utown
price0bar <- mean(utown$price[which(utown$utown==0)])
price1bar <- mean(utown$price[which(utown$utown==1)])
# See the difference
mod5 <- lm(price~utown, data=utown)
b1 <- coef(mod5)[[1]]
b2 <- coef(mod5)[[2]]
```

Appendix: Sampling distributions of b_1 and b_2

Setting

- Assume $x_1, \ldots, x_N$ are known
- Model: $y_i = \beta_1 + \beta_2 x_i + \varepsilon_i$, for $i = 1,\ldots,N$.
  - Parameters: β_1, β_2, (unknown) non-random numbers.
  - $\varepsilon_i \overset{i.i.d.}{\sim} N(0,\sigma^2)$
- Another presentation (Generalized linear model):

$$y_i \mid x_i \sim N(\beta_1 + \beta_2 x_i, \sigma^2)$$

The sampling distributions of LSE

$$b_1 \sim N\left(\beta_1, \frac{\sigma^2 \sum x_i^2}{N \sum (x_i - \bar{x})^2}\right),$$

$$b_2 \sim N\left(\beta_2, \frac{\sigma^2}{\sum (x_i - \bar{x})^2}\right)$$

Theorem A: Linear combinations of independent normal distributions remain a normal distribution. Specifically, if $X_i \sim N(\mu_i, \sigma^2)$ and X_i are independent, then

$$\sum a_i X_i \sim N\left(\sum a_i \mu_i, \sigma^2 (\sum a_i^2)\right).$$

Derivations for $b_2 \sim N(\beta_2, \frac{\sigma^2}{\sum (x_i - \bar{x})^2})$.

Because $\sum (x_i - \bar{x}) = 0$, we rewrite b_2 by

$$b_2 = \sum \frac{(x_i - \bar{x})(y_i - \bar{y})}{\sum (x_i - \bar{x})^2} = \sum \frac{(x_i - \bar{x})y_i}{\sum (x_i - \bar{x})^2}$$

$$= \sum \left(\frac{x_i - \bar{x}}{\sum (x_i - \bar{x})^2} \right) y_i = \sum w_i y_i,$$

where $w_i = \frac{x_i - \bar{x}}{\sum (x_i - \bar{x})^2}$.

b_2 is a linear combination of normal distribution, and hence a normal distribution.

To find the expectation and variance of b_2, note the following identities:

1. $\sum w_i = 0$
2. $\sum w_i x_i = \sum \frac{(x_i - \bar{x})x_i}{\sum (x_i - \bar{x})^2} = \sum \frac{(x_i - \bar{x})x_i - (x_i - \bar{x})\bar{x}}{\sum (x_i - \bar{x})^2} = \sum \frac{(x_i - \bar{x})(x_i - \bar{x})}{\sum (x_i - \bar{x})^2} = 1$
3. $\sum w_i^2 = \sum \left(\frac{(x_i - \bar{x})}{\sum (x_i - \bar{x})^2} \right)^2 = \frac{1}{\sum (x_i - \bar{x})^2}$

Rewrite

$$\begin{aligned} b_2 &= \sum w_i(\beta_1 + \beta_2 x_i + e_i) \\ &= \beta_1 \sum w_i + \beta_2 \sum w_i x_i + \sum w_i e_i \\ &= \beta_2 + \sum w_i e_i \,. \end{aligned}$$

Therefore,

$$E(b_2) = \beta_2$$

$$var(b_2) = \sum w_i^2 \sigma^2 = \sigma^2 \frac{1}{\sum (x_i - \bar{x})^2} \,.$$

Derivations for $b_1 \sim N(\beta_1, \sigma^2 \frac{\sum x_i^2}{N \sum (x_i - \bar{x})^2})$.

Now, we write

$$
\begin{aligned}
b_1 &= \bar{y} - b_2\bar{x} \\
&= \sum (\frac{1}{N} - \bar{x}w_i)y_i \\
&= \sum (\frac{1}{N} - \bar{x}w_i)(\beta_1 + \beta_2 x_i + e_i) \\
&= (\beta_1 - \beta_1\bar{x} \sum w_i) + \left(\beta_2\bar{x} - \bar{x}\beta_2 \sum x_i w_i\right) + \sum (\frac{1}{N} - \bar{x}w_i)e_i \\
&= \beta_1 + \sum (\frac{1}{N} - \bar{x}w_i)e_i\,.
\end{aligned}
$$

Hence, b_1 is a normal distribution.

To find the expectation and variance of b_1, it is easy that

$$E(b_1) = \beta_1 + \sum [(\frac{1}{N} - \bar{x}w_i)0] = \beta_1$$

$$\begin{aligned}
var(b_1) &= \sum (\frac{1}{N} - \bar{x}w_i)^2\sigma^2 \\
&= \sigma^2 \left(\sum \frac{1}{N^2} - 2\sum \frac{\bar{x}}{N}w_i + \bar{x}^2 \sum w_i^2 \right) \\
&= \sigma^2 \left(\frac{1}{N} + \bar{x}^2 \frac{1}{\sum (x_i - \bar{x})^2} \right) \\
&= \sigma^2 \frac{\sum (x_i - \bar{x})^2 + N\bar{x}^2}{N \sum (x_i - \bar{x})^2} \\
&= \sigma^2 \frac{(\sum x_i^2 - 2N\bar{x}^2 + N\bar{x}^2) + N\bar{x}^2}{N\sum (x_i - \bar{x})^2} = \sigma^2 \frac{\sum x_i^2}{N\sum (x_i - \bar{x})^2}.
\end{aligned}$$