11.1 Our aim is to estimate the parameters of the simultaneous equations model

$$y_1 = \alpha_1 y_2 + e_1$$
$$y_2 = \alpha_2 y_1 + \beta_1 x_1 + \beta_2 x_2 + e_2$$

We assume that x_1 and x_2 are exogenous and uncorrelated with the error terms e_1 and e_2.

a. Solve the two structural equations for the reduced-form equation for y_2, that is, $y_2 = \pi_1 x_1 + \pi_2 x_2 + v_2$. Express the reduced-form parameters in terms of the **structural parameters** and the reduced-form error in terms of the structural parameters and e_1 and e_2. Show that y_2 is correlated with e_1.

b. Which equation parameters are consistently estimated using OLS? Explain.

c. Which parameters are "identified," in the simultaneous equations sense? Explain your reasoning.

a. $y_2 = \alpha_2(\alpha_1 y_2 + e_1) + \beta_1 x_1 + \beta_2 x_2 + e_2$

$\Rightarrow y_2 = \frac{1}{1-\alpha_2\alpha_1}(\beta_1 x_1 + \beta_2 x_2 + e_2 + \alpha_2 e_1)$

$= \frac{\beta_1}{1-\alpha_2\alpha_1} x_1 + \frac{\beta_2}{1-\alpha_2\alpha_1} x_2 + \frac{e_2 + \alpha_2 e_1}{1-\alpha_2\alpha_1}$

Thus, $\pi_1 = \frac{\beta_1}{1-\alpha_2\alpha_1}$, $\pi_2 = \frac{\beta_2}{1-\alpha_2\alpha_1}$, $v_2 = \frac{e_2+\alpha_2 e_1}{1-\alpha_2\alpha_1}$

prove $cov(y_2, e_1) \neq 0$:

$cov(y_2, e_1) = E[(y_2 - E(y_2))(e_1 - E(e_1))]$

$= E[y_2 e_1]$

$= \frac{\beta_1}{1-\alpha_2\alpha_1} E[x_1 e_1] + \frac{\beta_2}{1-\alpha_2\alpha_1} E[x_2 e_1] + \frac{1}{1-\alpha_2\alpha_1}(E[e_2 e_1] + \alpha E[e_1^2])$

$= \frac{1}{1-\alpha_2\alpha_1} \alpha E[e_1^2]$

$\neq 0$ we finish the proof.

b.

By the proof above, we know that $cov(y_2, e_1) \neq 0 \Rightarrow$ OLS fail

By the rule of identifying, first equation is identified

$\Rightarrow$ can be consistently estimated by OLS.

C.

By the rule of identifing, since we have $M=2$ equations, at least $M-1=1$ variable is omitted from each equation to be identified for it.

Since $y_1 = \alpha_1 y_2 + e_1$

$y_2 = \alpha_2 y_1 + \beta_1 x_1 + \beta_2 x_2 + e_2$, x_1 and x_2 are obviously omitted

$\Rightarrow$ identified

d. To estimate the parameters of the reduced-form equation for y_2 using the method of moments (MOM), which was introduced in Section 10.3, the two moment equations are

$$N^{-1}\sum x_{i1}(y_2 - \pi_1 x_{i1} - \pi_2 x_{i2}) = 0$$
$$N^{-1}\sum x_{i2}(y_2 - \pi_1 x_{i1} - \pi_2 x_{i2}) = 0$$

Explain why these two moment conditions are a valid basis for obtaining consistent estimators of the reduced-form parameters.

e. Are the MOM estimators in part (d) the same as the OLS estimators? Form the sum of squared errors function for $y_2 = \pi_1 x_1 + \pi_2 x_2 + v_2$ and find the first derivatives. Set these to zero and show that they are equivalent to the two equations in part (d).

f. Using $\sum x_{i1}^2 = 1$, $\sum x_{i2}^2 = 1$, $\sum x_{i1}x_{i2} = 0$, $\sum x_{i1}y_{1i} = 2$, $\sum x_{i1}y_{2i} = 3$, $\sum x_{i2}y_{1i} = 3$, $\sum x_{i2}y_{2i} = 4$, and the two moment conditions in part (d) show that the MOM/OLS estimates of π_1 and π_2 are $\hat{\pi}_1 = 3$ and $\hat{\pi}_2 = 4$.

g. The fitted value $\hat{y}_2 = \hat{\pi}_1 x_1 + \hat{\pi}_2 x_2$. Explain why we can use the moment condition $\sum \hat{y}_{i2}(y_{i1} - \alpha_1 y_{i2}) = 0$ as a valid basis for consistently estimating α_1. Obtain the IV estimate of α_1.

h. Find the 2SLS estimate of α_1 by applying OLS to $y_1 = \alpha_1 \hat{y}_2 + e_1^*$. Compare your answer to that in part (g).

d.

① Exogeneity assumption: $E[v_2 x_1] = 0$, $E[v_2 x_2] = 0$

② moment condition $= 0$

③ consistency: These two equations are identified $\Rightarrow$ consistency

e.

$SSE = \sum(y_2 - \pi_1 x_1 - \pi_2 x_2)^2 \Rightarrow \frac{\partial SSE}{\partial \pi_1} = 2\sum(y_2 - \pi_1 x_1 - \pi_2 x_2)\cdot(-x_1) = 0$

$\frac{\partial SSE}{\partial \pi_2} = 2\sum(y_2 - \pi_1 x_1 - \pi_2 x_2)\cdot(-x_2) = 0$

, which are same as the first-order conditions that minimize OLS

f.

$3 - \hat{\pi}_1 - 0 = 0 \Rightarrow \hat{\pi}_1 = 3$

$4 - 0 - \hat{\pi}_2 = 0 \Rightarrow \hat{\pi}_2 = 4$

g. since x_1, x_2 are exogenous, they can be instrument for y_2.

Thus, $\hat{y}_2$ is uncorrelated with $e_1 \Rightarrow \alpha_1$ can be consistently estimated

$\sum \hat{y}_{i2}(y_{i1} - \alpha_1 \hat{y}_{i2}) = 0$

$\Rightarrow \sum \hat{y}_{i2} y_{i1} - \alpha_1 \sum \hat{y}_{i2}^2 = 0$

$\Rightarrow \hat{\alpha}_1 = \frac{\sum \hat{y}_{i2} y_{i1}}{\sum \hat{y}_{i2}^2} = \frac{18}{25} = 0.72$

h. $\hat{\alpha}_1 = \frac{\sum \hat{y}_2 y_1}{\sum \hat{y}_2^2}$ and $y_2 = \hat{y}_2 + \hat{v}_2$

$\Rightarrow \sum \hat{y}_2 y_2 = \sum \hat{y}_2(\hat{y}_2 + \hat{v}_2) = \sum \hat{y}_2^2 + \sum \hat{y}_2 \hat{v}_2$

$= \sum \hat{y}_2^2$ (Since $\sum \hat{y}_2 \hat{v}_2 = 0$)

$\Rightarrow$ OLS estimator is equal to IV estimator

11.16 Consider the following supply and demand model

$$\text{Demand: } Q_i = \alpha_1 + \alpha_2 P_i + e_{di}, \quad \text{Supply: } Q_i = \beta_1 + \beta_2 P_i + \beta_3 W_i + e_{si}$$

where Q is the quantity, P is the price, and W is the wage rate, which is assumed exogenous. Data on these variables are in Table 11.7.

TABLE 11.7 Data for Exercise 11.16

Q	P	W
4	2	2
6	4	3
9	3	1
3	5	1
8	8	3

a. Derive the algebraic form of the reduced-form equations, $Q = \theta_1 + \theta_2 W + v_2$ and $P = \pi_1 + \pi_2 W + v_1$, expressing the reduced-form parameters in terms of the structural parameters.

b. Which structural parameters can you solve for from the results in part (a)? Which equation is "identified"?

c. The estimated reduced-form equations are $\hat{Q} = 5 + 0.5W$ and $\hat{P} = 2.4 + 1W$. Solve for the identified structural parameters. This is the method of **indirect least squares**.

d. Obtain the fitted values from the reduced-form equation for P, and apply 2SLS to obtain estimates of the demand equation.

a. $\alpha_1 + \alpha_2 P + e_d = \beta_1 + \beta_2 P + \beta_3 W + e_s$

$\Rightarrow (\alpha_2 - \beta_2) P = (\beta_1 - \alpha_1) + \beta_3 W + e_s - e_d$

$\Rightarrow P = \frac{\beta_1 - \alpha_1}{\alpha_2 - \beta_2} + \frac{\beta_3}{\alpha_2 - \beta_2} W + \frac{e_s - e_d}{\alpha_2 - \beta_2}$

$\Rightarrow \pi_1 = \frac{\beta_1 - \alpha_1}{\alpha_2 - \beta_2}, \ \pi_2 = \frac{\beta_3}{\alpha_2 - \beta_2}, \ V_1 = \frac{e_s - e_d}{\alpha_2 - \beta_2}$

Since $P = \frac{(Q - e_d - \alpha_1)}{\alpha_2} = \frac{Q - e_s - \beta_1 - \beta_3 W}{\beta_2}$,

$\beta_2 Q - \beta_2 \alpha_1 - \beta_2 e_d = \alpha_2 Q - \alpha_2 e_s - \alpha_2 \beta_1 - \alpha_2 \beta_3 W$

$\Rightarrow (\beta_2 - \alpha_2) Q = (\alpha_1 \beta_2 - \alpha_2 \beta_1) - \alpha_2 \beta_3 W + (\beta_2 e_d - \alpha_2 e_s)$

$\Rightarrow Q = \frac{\alpha_1 \beta_2 - \alpha_2 \beta_1}{\beta_2 - \alpha_2} + \frac{-\alpha_2 \beta_3}{\beta_2 - \alpha_2} W + \frac{\beta_2 e_d - \alpha_2 e_s}{\beta_2 - \alpha_2}$

$\Rightarrow \theta_1 = \frac{\alpha_1 \beta_2 - \alpha_2 \beta_1}{\beta_2 - \alpha_2}, \ \theta_2 = \frac{-\alpha_2 \beta_3}{\beta_2 - \alpha_2}, \ V_2 = \frac{\beta_2 e_d - \alpha_2 e_s}{\beta_2 - \alpha_2}$

b. The Demand equation is identisied since it has at least one omitted variable W.

α_2 can be solve since Demand equation is identisied, its parameter can be consistently estimated by OLS.

c. $5 = \theta_1 = \frac{\alpha_1 \beta_2 - \beta_1 \alpha_2}{\beta_2 - \alpha_2}, \ 0.5 = \theta_2 = \frac{-\alpha_2 \beta_3}{\beta_2 - \alpha_2}$

$2.4 = \pi_1 = \frac{\beta_1 - \alpha_1}{\alpha_2 - \beta_2}, \ 1 = \pi_2 = \frac{\beta_3}{\alpha_2 - \beta_2}$ and $\alpha_1 = 6 - \alpha_2 \cdot 4.4$

$\Rightarrow \alpha_2 = \frac{1}{2}, \ \alpha_1 = 3.8$

$\beta_2 = 0, \ \beta_1 = 5, \ \beta_3 = \frac{1}{2}$

d. $\hat{P} = 2.4 + W$

$\Rightarrow Q = \alpha_1 + \alpha_2 \cdot 2.4 + \alpha_2 W$

$= 5 + 0.5W$

$\Rightarrow \alpha_2 = \frac{1}{2}, \ \alpha_1 = 3.8$

11.17 Example 11.3 introduces Klein's Model I.

a. Do we have an adequate number of IVs to estimate each equation? Check the necessary condition for the identification of each equation. The necessary condition for identification is that in a system of M equations at least $M - 1$ variables must be omitted from each equation.

b. An equivalent identification condition is that the number of excluded exogenous variables from the equation must be at least as large as the number of included right-hand side endogenous variables. Check that this condition is satisfied for each equation.

c. Write down in econometric notation the first-stage equation, the reduced form, for W_{1t}, wages of workers earned in the private sector. Call the parameters $\pi_1, \pi_2, \ldots$

d. Describe the two regression steps of 2SLS estimation of the consumption function. This is not a question about a computer software command.

e. Does following the steps in part (d) produce regression results that are identical to the 2SLS estimates provided by software specifically designed for 2SLS estimation? In particular, will the t-values be the same?

a. For first equation: $CN_t = \alpha_1 + \alpha_2 (W_{1t} + W_{2t}) + \alpha_3 P_t + \alpha_4 P_{t-1} + e_{1t}$

It has $2 = 3-1 = M-1$ exogeneous variables $\Rightarrow$ adequate

It has at least $2 = 3-1$ variables that are omitted from others

$\Rightarrow$ identified.

For second equation: $I_t = \beta_1 + \beta_2 P_t + \beta_3 P_{t-1} + \beta_4 K_{t-1} + e_{2t}$

It only has $1 < M-1$ exogeneous variable $\Rightarrow$ Not adequate

It has at least $2 = 3-1$ variables that are omitted from others $\Rightarrow$ identified.

For third equation: $W_{1t} = \gamma_1 + \gamma_2 E_t + \gamma_3 E_{t-1} + \gamma_4 TIME + e_{3t}$

It only has $1 < 2$ exogenous variable $\Rightarrow$ Not edequate

It has at least 2 variables that are omitted from others

$\Rightarrow$ identified

b. first equation: 2 excluded exogenous variables $\Rightarrow$ identisided

2 endogenous variables at RHS

second equation: 3 excluded exogenous variables $\Rightarrow$ identified

2 endogenous variables at RHS

third equation: 3 excluded exogenous variables $\Rightarrow$ identified

2 endogenous variables at RHS

c. $W_{1t} = \pi_1 + \pi_2 Z_2 + \pi_3 Z_3 + \ldots \pi_k Z_k$

d. first step: find the first stage equation for the endogenous variable on the RHS.

second step: find the fitted value for these endogenous variable

third step: Replace the RHS endogenous variable with these fitted value, then estimate the parameters

e. The estimation may be identical, however, standard error might be different (ex: Robust), then t-value would be different.