

11.1 Our aim is to estimate the parameters of the simultaneous equations model

$$y_1 = \alpha_1 y_2 + e_1$$

$$y_2 = \alpha_2 y_1 + \beta_1 x_1 + \beta_2 x_2 + e_2$$

¶ 2.

We assume that x_1 and x_2 are exogenous and uncorrelated with the error terms e_1 and e_2 .

- Solve the two structural equations for the reduced-form equation for y_2 , that is, $y_2 = \pi_1 x_1 + \pi_2 x_2 + v_2$. Express the reduced-form parameters in terms of the structural parameters and the reduced-form error in terms of the structural parameters and e_1 and e_2 . Show that y_2 is correlated with e_1 .
- Which equation parameters are consistently estimated using OLS? Explain.
- Which parameters are “identified,” in the simultaneous equations sense? Explain your reasoning.

$$(a) y_2 = \alpha_2(\alpha_1 y_2 + e_1) + \beta_1 x_1 + \beta_2 x_2 + e_2$$

$$\Rightarrow y_2(1 - \alpha_1 \alpha_2) = \beta_1 x_1 + \beta_2 x_2 + (\alpha_2 e_1 + e_2)$$

$$\Rightarrow y_2 = \left(\frac{\beta_1}{1 - \alpha_1 \alpha_2} \right) x_1 + \left(\frac{\beta_2}{1 - \alpha_1 \alpha_2} \right) x_2 + \left(\frac{\alpha_2 e_1 + e_2}{1 - \alpha_1 \alpha_2} \right) = \pi_1 x_1 + \pi_2 x_2 + v_2$$

$$\begin{aligned} \text{cov}(y_2, e_1 | x) &= E(y_2 e_1 | x) = E\left[\left(\frac{\beta_1}{1 - \alpha_1 \alpha_2} x_1 + \frac{\beta_2}{1 - \alpha_1 \alpha_2} x_2 + \frac{\alpha_2 e_1 + e_2}{1 - \alpha_1 \alpha_2}\right) e_1 | x\right] \\ &= E\left[\left(\frac{\beta_1}{1 - \alpha_1 \alpha_2} x_1 e_1\right) | x\right] + E\left[\left(\frac{\beta_2}{1 - \alpha_1 \alpha_2} x_2 e_1\right) | x\right] + E\left[\left(\frac{\alpha_2 e_1 + e_2}{1 - \alpha_1 \alpha_2} e_1\right) | x\right] \\ &= 0 + 0 + E\left[\left(\frac{\alpha_2 e_1 + e_2}{1 - \alpha_1 \alpha_2} e_1\right) | x\right] \end{aligned}$$

$$\Rightarrow \text{cov}(y_2, e_1 | x) = E(y_2 e_1 | x) = \frac{E(e_1 e_2 | x) + \alpha_2 E(e_1^2 | x)}{1 - \alpha_1 \alpha_2} = \frac{\alpha_2 \cdot \sigma_1^2}{1 - \alpha_1 \alpha_2}$$

by consider the error of two equation are uncorrelated.

$\text{cov}(y_2, e_1 | x)$ isn't zero unless $\alpha_2 = 0$.

(b) Since equation (1) and (2) have an endogenous variable on the right-hand side, the OLS is biased and inconsistent.

(c) $M=2$ at least $M-1=1$ variable needs to be omitted from equation.

Equation (1) omitted two exogenous variables larger than one variable \rightarrow "identified"

Equation (2) omitted no variables. \rightarrow "not identified"

- d. To estimate the parameters of the reduced-form equation for y_2 using the method of moments (MOM), which was introduced in Section 10.3, the two moment equations are

$$N^{-1} \sum x_{i1}(y_2 - \pi_1 x_{i1} - \pi_2 x_{i2}) = 0$$

$$N^{-1} \sum x_{i2}(y_2 - \pi_1 x_{i1} - \pi_2 x_{i2}) = 0$$

Explain why these two moment conditions are a valid basis for obtaining consistent estimators of the reduced-form parameters.

- e. Are the MOM estimators in part (d) the same as the OLS estimators? Form the sum of squared errors function for $y_2 = \pi_1 x_1 + \pi_2 x_2 + v_2$ and find the first derivatives. Set these to zero and show that they are equivalent to the two equations in part (d).

- f. Using $\sum x_{i1}^2 = 1$, $\sum x_{i2}^2 = 1$, $\sum x_{i1}x_{i2} = 0$, $\sum x_{i1}y_{i1} = 2$, $\sum x_{i2}y_{i1} = 3$, $\sum x_{i2}y_{i2} = 4$, and the two moment conditions in part (d) show that the MOM/OLS estimates of π_1 and π_2 are $\hat{\pi}_1 = 3$ and $\hat{\pi}_2 = 4$.

- g. The fitted value $\hat{y}_2 = \hat{\pi}_1 x_1 + \hat{\pi}_2 x_2$. Explain why we can use the moment condition $\sum \hat{y}_{i2}(y_{i1} - \alpha_1 y_{i2}) = 0$ as a valid basis for consistently estimating α_1 . Obtain the IV estimate of α_1 .

- h. Find the 2SLS estimate of α_1 by applying OLS to $y_1 = \alpha_1 \hat{y}_2 + e_1^*$. Compare your answer to that in part (g).

$$f. N^{-1} \sum x_{i1}(y_2 - \pi_1 x_{i1} - \pi_2 x_{i2}) = 0 \Rightarrow \sum x_{i1}y_2 - \pi_1 \sum x_{i1}^2 - \pi_2 \sum x_{i1}x_{i2} = 0$$

$$N^{-1} \sum x_{i2}(y_2 - \pi_1 x_{i1} - \pi_2 x_{i2}) = 0 \Rightarrow \sum x_{i2}y_2 - \pi_1 \sum x_{i1}x_{i2} - \pi_2 \sum x_{i2}^2 = 0$$

$$\begin{aligned} & 3 - \hat{\pi}_1 x_1 - \pi_2 x_0 = 0 \\ & 4 - \hat{\pi}_1 x_0 - \pi_2 x_1 = 0 \end{aligned} \Rightarrow \begin{cases} \hat{\pi}_1 = 3 \\ \hat{\pi}_2 = 4 \end{cases}$$

$$(d) E[X_{i1}V_{i1}|x] = E[X_{i2}V_{i2}|x] = 0$$

$$\rightarrow E[X_{ik} \left(\frac{\alpha_1 e_1 + e_2}{1 - \alpha_1 \alpha_2} \right) | x]$$

$$= E\left[\frac{\alpha_1}{1 - \alpha_1 \alpha_2} e_1 | x_{ik} | x\right] + E\left[\frac{1}{1 - \alpha_1 \alpha_2} e_2 | x_{ik} | x\right] = 0.$$

$$e. \min \sum (y_2 - \pi_1 x_1 - \pi_2 x_2)^2$$

$$\text{對 } \pi_1 \text{ 的偏微分: } 2 \sum (y_2 - \pi_1 x_1 - \pi_2 x_2) \times (-x_1) = 0$$

$$\text{對 } \pi_2 \text{ 的偏微分: } 2 \sum (y_2 - \pi_1 x_1 - \pi_2 x_2) \times (-x_2) = 0$$

$$\hookrightarrow \div 2, \times N^{-1}$$

So they are equivalent to the two equations.

$$g. \because y_1 = \alpha_1 y_2 + \epsilon_1 \quad \hat{y}_2 = \hat{\alpha}_1 x_1 + \hat{\alpha}_2 x_2$$

moment condition $(y_2, \epsilon_1) \Rightarrow \sum \hat{y}_2 (y_1 - \alpha_1 y_2) = 0 \Rightarrow \sum \hat{y}_2 y_1 - \alpha_1 \sum \hat{y}_2 y_2 = 0 \Rightarrow \alpha_1 = \frac{\sum \hat{y}_2 y_1}{\sum \hat{y}_2 y_2}$

$$\Rightarrow \hat{\alpha}_1 = \frac{\sum (\hat{\alpha}_1 x_1 + \hat{\alpha}_2 x_2) y_1}{\sum (\hat{\alpha}_1 x_1 + \hat{\alpha}_2 x_2) y_2} = \frac{3 \sum x_1 y_1 + 4 \sum x_2 y_1}{3 \sum x_1 y_2 + 4 \sum x_2 y_2} = \frac{3 \times 2 + 4 \times 3}{3 \times 3 + 4 \times 4} = \frac{18}{25}$$

$$h. \text{ 原 } \hat{\alpha}_1, \text{ vs } \hat{\alpha}_1 = \frac{\sum \hat{y}_2 y_1}{\sum \hat{y}_2^2}$$

To prove $\hat{\alpha}_1, \text{ vs } \hat{\alpha}_1 = \hat{\alpha}_1$ (by moment condition)

We need to prove $\sum \hat{y}_2^2 = \sum \hat{y}_2 y_2$

$$\sum \hat{y}_2^2 = \sum \hat{y}_2 (y_2 - \hat{v}_2) = \sum \hat{y}_2 y_2 - \underbrace{\sum \hat{y}_2 \hat{v}_2}_0 = \sum \hat{y}_2 y_2 \quad (\because \text{解釋變數和誤差無關})$$

11.16 Consider the following supply and demand model

$$\text{Demand: } Q_i = \alpha_1 + \alpha_2 P_i + e_{di}, \quad \text{Supply: } Q_i = \beta_1 + \beta_2 P_i + \beta_3 W_i + e_{si}$$

where Q is the quantity, P is the price, and W is the wage rate, which is assumed exogenous. Data on these variables are in Table 11.7.

TABLE 11.7 Data for Exercise 11.16		
Q	P	W
4	2	2
6	4	3
9	3	1
3	5	1
8	8	3

- a. Derive the algebraic form of the reduced-form equations, $Q = \theta_1 + \theta_2 W + v_2$ and $P = \pi_1 + \pi_2 W + v_1$, expressing the reduced-form parameters in terms of the structural parameters.
- b. Which structural parameters can you solve for from the results in part (a)? Which equation is "identified"?
- c. The estimated reduced-form equations are $\hat{Q} = 5 + 0.5W$ and $\hat{P} = 2.4 + 1W$. Solve for the identified structural parameters. This is the method of **indirect least squares**.
- d. Obtain the fitted values from the reduced-form equation for P , and apply 2SLS to obtain estimates of the demand equation.

b. M=2, omitted at least one variable

equation (1) \rightarrow one \rightarrow "identified" \rightarrow 可推出 α_1, α_2

equation (v) \rightarrow zero. \rightarrow "not identified" \rightarrow 可推出 $\beta_1, \beta_2, \beta_3$.

$$c. \hat{Q} = 5 + 0.5W, \hat{P} = 2.4 + 1W$$

$$5 + 0.5W = \alpha_1 + \alpha_2(2.4 + W)$$

$$= (\alpha_1 + \alpha_2 \times 2.4) + \alpha_2 W$$

$$\hat{\alpha}_2 = 0.5 \quad 5 = \alpha_1 + 0.5 \times 2.4$$

$$\alpha_1 = 3.8$$

a. reduced-form

Demand = Supply

$$\Rightarrow \alpha_1 + \alpha_2 P_i + e_{di} = \beta_1 + \beta_2 P_i + \beta_3 W_i + e_{si}$$

$$(\alpha_2 - \beta_2) P_i = (\beta_1 - \alpha_1) + \beta_3 W_i + (e_{si} - e_{di})$$

$$\Rightarrow P_i = \frac{\beta_1 - \alpha_1}{\alpha_2 - \beta_2} + \frac{\beta_3}{\alpha_2 - \beta_2} W_i + \frac{e_{si} - e_{di}}{\alpha_2 - \beta_2}$$

$$Q_i = \alpha_1 + \alpha_2 \left(\frac{\beta_1 - \alpha_1}{\alpha_2 - \beta_2} + \frac{\beta_3}{\alpha_2 - \beta_2} W_i + \frac{e_{si} - e_{di}}{\alpha_2 - \beta_2} \right) + e_{di}$$

$$\Rightarrow Q_i = \alpha_1 + \frac{\beta_1 - \alpha_1}{\alpha_2 - \beta_2} \bar{P} + \frac{\beta_3}{\alpha_2 - \beta_2} \bar{W} + \frac{e_{si} - e_{di}}{\alpha_2 - \beta_2} \bar{d}_2 + e_{di}$$

$$d. \hat{P} = 2.4 + W$$

$\bar{P} - \hat{P}$	$\bar{Q} - \hat{Q}$
0	-2
1	0
-1	3
1	-3
1	2

$$\bar{P} = 4.4$$

$$\bar{Q} = 6$$

$$\bar{Q} = \alpha_1 + \alpha_2 \hat{P} + e_i$$

$$\hat{\alpha}_2 = \frac{\sum (\hat{P}_i - \bar{P})(\bar{Q}_i - \bar{Q})}{\sum (\hat{P}_i - \bar{P})^2} = \frac{-3 + 3 + 2}{4} = \frac{1}{2}$$

$$\hat{\alpha}_1 = \bar{Q} - \hat{\alpha}_2 \bar{P} = 6 - 0.5 \times 4.4 = 3.8$$

$$\Rightarrow \hat{Q} = 3.8 + 0.5 \hat{P}$$

11.17 Example 11.3 introduces Klein's Model I.

- Do we have an adequate number of IVs to estimate each equation? Check the necessary condition for the identification of each equation. The necessary condition for identification is that in a system of M equations at least $M - 1$ variables must be omitted from each equation.
- An equivalent identification condition is that the number of excluded exogenous variables from the equation must be at least as large as the number of included right-hand side endogenous variables. Check that this condition is satisfied for each equation.
- Write down in econometric notation the first-stage equation, the reduced form, for W_{1t} , wages of workers earned in the private sector. Call the parameters π_1, π_2, \dots
- Describe the two regression steps of 2SLS estimation of the consumption function. This is not a question about a computer software command.
- Does following the steps in part (d) produce regression results that are identical to the 2SLS estimates provided by software specifically designed for 2SLS estimation? In particular, will the t -values be the same?

a. $M=8$ (requiring 7 omitted variables), total 16 variables.

The consumption function includes 6 variables and omits 10 variables

The investment function 5 11

The wage function 5 11

→ all satisfied necessary condition for identification.

b. Consumption function: 2 R/H endogenous variables and excludes 5 exogenous variables

Investment function: 1 R/H

Wage function: 1 R/H

→ all satisfied.

$$C_t = \alpha_1 + \alpha_2 Y_t + \alpha_3 W_{2t} + \alpha_4 T_{xt} + \alpha_5 TIME_t + \alpha_6 P_{1t} + \alpha_7 K_{t-1} + \alpha_8 E_{t-1} + v$$

d. Get fitted value \hat{W}_{1t} from (c), and using the same method \hat{P}_t , create $\hat{W}_t^* = \hat{W}_{1t} + W_{2t}$

Regress CN_t by \hat{W}_t^* .

P_{t-1} . Thus, the consumption function is

$$CN_t = \alpha_1 + \alpha_2 (W_{1t} + W_{2t}) + \alpha_3 P_t + \alpha_4 P_{t-1} + e_{1t} \quad (11.17)$$

investment equation is

$$I_t = \beta_1 + \beta_2 P_t + \beta_3 P_{t-1} + \beta_4 K_{t-1} + e_{2t} \quad (11.18)$$

wage equation is

$$W_{1t} = \gamma_1 + \gamma_2 E_t + \gamma_3 E_{t-1} + \gamma_4 TIME_t + e_{3t} \quad (11.19)$$

d. The coefficient will be the same, but t-values will not