1. $y_i = b_1 + b_2 x_i + e_i$

$Y = X\beta + e$

其中 $Y = (y_1, y_2 \cdots y_N)^{-1}$ $\quad \beta = (\beta_1, \beta_2)^{-1}$ $\quad e = (e_1, e_2 \cdots e_N)^{-1}$ $\quad X = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_N \end{bmatrix}$

$$b = (X'X)^{-1}(X'Y)$$

$$X'X = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_N \end{bmatrix} \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_N \end{bmatrix} = \begin{bmatrix} N & \sum_{i=1}^{N} x_i \\ \sum_{i=1}^{N} x_i & \sum_{i=1}^{N} (x_i)^2 \end{bmatrix}$$

$$X'Y = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_N \end{bmatrix} \times \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{N} y_i \\ \sum_{i=1}^{N} x_i y_i \end{bmatrix}$$

$$b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \boxed{\frac{1}{N \cdot \sum_{i=1}^{N} (x_i)^2 - \left(\sum_{i=1}^{N} x_i\right)^2} \begin{bmatrix} \sum_{i=1}^{N} (x_i)^2 & -\sum_{i=1}^{N} x_i \\ -\sum_{i=1}^{N} x_i & N \end{bmatrix}}^{\,(X'X)^{-1}} \begin{bmatrix} \sum_{i=1}^{N} y_i \\ \sum_{i=1}^{N} x_i y_i \end{bmatrix}$$

$$= \frac{1}{N \cdot \sum_{i=1}^{N} (x_i)^2 - \left(\sum_{i=1}^{N} x_i\right)^2} \begin{bmatrix} \sum_{i=1}^{N} (x_i)^2 \sum_{i=1}^{N} y_i - \sum_{i=1}^{N} x_i \cdot \sum_{i=1}^{N} x_i y_i \\ -\sum_{i=1}^{N} x_i \sum_{i=1}^{N} y_i + N \sum_{i=1}^{N} x_i y_i \end{bmatrix}$$

$$\Rightarrow b_2 = \frac{-\sum_{i=1}^{N} x_i \sum_{i=1}^{N} y_i + N \sum_{i=1}^{N} x_i y_i}{N \cdot \sum_{i=1}^{N} (x_i)^2 - \left(\sum_{i=1}^{N} x_i\right)^2} \xrightarrow{\times \frac{1}{n}} = \frac{\sum_{i=1}^{N} x_i y_i - \frac{1}{N}\left(\sum_{i=1}^{N} x_i \sum_{i=1}^{N} y_i\right)}{\sum_{i=1}^{N} (x_i)^2 - \frac{1}{n}\left(\sum_{i=1}^{N} x_i\right)^2} = \frac{\sum_{i=1}^{N} (x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{N} (x_i - \bar{x})^2}$$

formula 2.7

$$\sum (x_i - \bar{x})(y_i - \bar{y})$$
$$= \sum (x_i y_i - x_i \bar{y} - \bar{x} y_i + \bar{x}\bar{y})$$
$$= \sum x_i y_i - \bar{y} \sum x_i - \bar{x} \sum y_i + N\bar{x}\bar{y}$$
$$= \sum x_i y_i - \bar{y} \cdot N\bar{x} - \bar{x} N \bar{y} + N\bar{x}\bar{y}$$
$$= \sum x_i y_i - N\bar{x}\bar{y}$$
$$= \sum x_i y_i - N\left(\frac{1}{n}\sum x_i\right)\left(\frac{1}{N}\sum y_i\right)$$
$$= \sum x_i y_i - \frac{1}{N} \sum x_i y_i$$

$$\sum (x_i - \bar{x})^2 = \sum (x_i^2 - 2x_i\bar{x} + \bar{x}^2)$$
$$= \sum x_i^2 - 2\bar{x}\sum x_i + N\bar{x}^2$$
$$= \sum x_i^2 - 2\bar{x} \cdot N\bar{x} + N\bar{x}^2$$
$$= \sum x_i^2 - N\bar{x}^2$$
$$= \sum x_i^2 - N\left(\frac{1}{N}\sum x_i\right)^2$$
$$= \sum x_i^2 - \frac{1}{N}\sum x_i^2$$

$$\Rightarrow b_1 = \frac{\sum_{i=1}^{N}(x_i)^2 \sum_{i=1}^{N} y_i - \sum_{i=1}^{N} x_i \cdot \sum_{i=1}^{N} x_i y_i}{N \sum_{i=1}^{N}(x_i)^2 - (\sum_{i=1}^{N} x_i)^2} = \frac{\sum_{i=1}^{N}(x_i)^2 \cdot \cancel{N}\bar{y} - \cancel{N}\bar{x} \cdot \sum_{i=1}^{N} x_i y_i}{\cancel{N} \sum_{i=1}^{N}(x_i)^2 - (N\bar{x})^2} = \frac{\bar{y}\sum_{i=1}^{N}(x_i)^2 - \bar{x}(\sum_{i=1}^{N} x_i y_i)}{\sum_{i=1}^{N}(x_i)^2 - N\bar{x}^2}$$

$$= \frac{\sum_{i=1}^{N}(x_i)^2 \bar{y} - \bar{x}\sum_{i=1}^{N} x_i y_i}{\sum_{i=1}^{N}(x_i - \bar{x})^2} = \frac{\sum_{i=1}^{N}(x_i)^2 \bar{y} - N\bar{x}^2\bar{y}}{\sum_{i=1}^{N}(x_i - \bar{x})^2} - \frac{\bar{x}\sum_{i=1}^{N} x_i y_i + N\bar{x}^2\bar{y}}{\sum_{i=1}^{N}(x_i - \bar{x})^2}$$

$$= \frac{\bar{y}\left(\sum_{i=1}^{N}(x_i)^2 - N\bar{x}^2\right)}{\sum_{i=1}^{N}(x_i - \bar{x})^2} - \frac{\bar{x}\left(\sum_{i=1}^{N} x_i y_i - N\bar{x}\bar{y}\right)}{\sum_{i=1}^{N}(x_i - \bar{x})^2} = \frac{\bar{y}\sum_{i=1}^{N}(x_i - \bar{x})^2}{\sum_{i=1}^{N}(x_i - \bar{x})^2} - \bar{x}\boxed{\frac{\sum_{i=1}^{N}(x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{N}(x_i - \bar{x})^2}}$$

$$= \bar{y} - \bar{x} b_2$$

從Q1結果代入

2) $$Var(b) = \sigma^2 (X'X)^{-1} = \frac{\sigma^2}{N \cdot \sum_{i=1}^{N}(x_i)^2 - (\sum_{i=1}^{N} x_i)^2} \begin{bmatrix} \sum_{i=1}^{N}(x_i)^2 & -\sum_{i=1}^{N} x_i \\ -\sum_{i=1}^{N} x_i & N \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\sigma^2 \sum_{i=1}^{N}(x_i)^2}{N \cdot \sum_{i=1}^{N} x_i^2 - (\sum_{i=1}^{N} x_i)^2} & \frac{-\sigma^2 \sum_{i=1}^{N} x_i}{N \cdot \sum_{i=1}^{N} x_i^2 - (\sum_{i=1}^{N} x_i)^2} \\ \frac{-\sigma^2 \sum_{i=1}^{N} x_i}{N \cdot \sum_{i=1}^{N} x_i^2 - (\sum_{i=1}^{N} x_i)^2} & \frac{\sigma^2 \cdot N}{N \cdot \sum_{i=1}^{N} x_i^2 - (\sum_{i=1}^{N} x_i)^2} \end{bmatrix}$$

$$Var(b_1 | X) = \frac{\sigma^2 \sum_{i=1}^{N}(x_i)^2}{N \cdot \sum_{i=1}^{N} x_i^2 - (\sum_{i=1}^{N} x_i)^2} = \frac{\sigma^2 \sum_{i=1}^{N} x_i^2}{N \sum_{i=1}^{N} x_i^2 - (N\bar{x})^2} = \frac{\sigma^2 \sum_{i=1}^{N} x_i^2}{N(\sum_{i=1}^{N} x_i^2 - N\bar{x}^2)} = \sigma^2 \cdot \left[\frac{\sum_{i=1}^{N} x_i^2}{N(x_i - \bar{x})^2}\right]$$

formula 2.14

$$\mathrm{Var}(b_2|X) = \frac{\sigma^2 \cdot N}{N \cdot \sum_{i=1}^{N} x_i^2 - \left(\sum_{i=1}^{N} x_i\right)^2} = \frac{\sigma^2 N}{N \sum_{i=1}^{N} x_i^2 - (N\bar{x})^2} = \frac{\sigma^2 N}{N\left(\sum_{i=1}^{N} x_i^2 - N\bar{x}^2\right)} = \sigma^2 \left[\frac{N}{N\left(\sum_{i=1}^{N} (x_i - \bar{x})^2\right)}\right]$$

formula 2.15

$$\mathrm{Cov}(b_1, b_2|X) = \frac{-\sigma^2 \sum_{i=1}^{N} x_i}{N \cdot \sum_{i=1}^{N} x_i^2 - \left(\sum_{i=1}^{N} x_i\right)^2} = \frac{-\sigma^2 N \bar{x}}{N \sum_{i=1}^{N} x_i^2 - (N\bar{x})^2} = \frac{-\sigma^2 \cancel{N} \bar{x}}{\cancel{N}\left(\sum_{i=1}^{N} x_i^2 - N\bar{x}^2\right)} = \sigma^2 \left[\frac{-\bar{x}}{N\left(\sum_{i=1}^{N} (x_i - \bar{x})^2\right)}\right]$$

formula 2.16.