8.6 Consider the wage equation

$$WAGE_i = \beta_1 + \beta_2 EDUC_i + \beta_3 EXPER_i + \beta_4 METRO_i + e_i \qquad \text{(XR8.6a)}$$

where wage is measured in dollars per hour, education and experience are in years, and $METRO = 1$ if the person lives in a metropolitan area. We have $N = 1000$ observations from 2013.

**a.** We are curious whether holding education, experience, and *METRO* constant, there is the same amount of random variation in wages for males and females. Suppose $\text{var}(e_i|\mathbf{x}_i, FEMALE = 0) = \sigma_M^2$ and $\text{var}(e_i|\mathbf{x}_i, FEMALE = 1) = \sigma_F^2$. We specifically wish to test the null hypothesis $\sigma_M^2 = \sigma_F^2$ against $\sigma_M^2 \neq \sigma_F^2$. Using 577 observations on males, we obtain the sum of squared OLS residuals, $SSE_M = 97161.9174$. The regression using data on females yields $\hat{\sigma}_F = 12.024$. Test the null hypothesis at the 5% level of significance. Clearly state the value of the test statistic and the rejection region, along with your conclusion.

**b.** We hypothesize that married individuals, relying on spousal support, can seek wider employment types and hence holding all else equal should have more variable wages. Suppose $\text{var}(e_i|\mathbf{x}_i, MARRIED = 0) = \sigma_{SINGLE}^2$ and $\text{var}(e_i|\mathbf{x}_i, MARRIED = 1) = \sigma_{MARRIED}^2$. Specify the null hypothesis $\sigma_{SINGLE}^2 = \sigma_{MARRIED}^2$ versus the alternative hypothesis $\sigma_{MARRIED}^2 > \sigma_{SINGLE}^2$. We add *FEMALE* to the wage equation as an explanatory variable, so that

$$WAGE_i = \beta_1 + \beta_2 EDUC_i + \beta_3 EXPER_i + \beta_4 METRO_i + \beta_5 FEMALE + e_i \qquad \text{(XR8.6b)}$$

Using $N = 400$ observations on single individuals, OLS estimation of (XR8.6b) yields a sum of squared residuals is 56231.0382. For the 600 married individuals, the sum of squared errors is 100,703.0471. Test the null hypothesis at the 5% level of significance. Clearly state the value of the test statistic and the rejection region, along with your conclusion.

**c.** Following the regression in part (b), we carry out the $NR^2$ test using the right-hand-side variables in (XR8.6b) as candidates related to the heteroskedasticity. The value of this statistic is 59.03. What do we conclude about heteroskedasticity, at the 5% level? Does this provide evidence about the issue discussed in part (b), whether the error variation is different for married and unmarried individuals? Explain.

**d.** Following the regression in part (b) we carry out the White test for heteroskedasticity. The value of the test statistic is 78.82. What are the degrees of freedom of the test statistic? What is the 5% critical value for the test? What do you conclude?

**e.** The OLS fitted model from part (b), with usual and robust standard errors, is

$$\widehat{WAGE} = -17.77 + 2.50EDUC + 0.23EXPER + 3.23METRO - 4.20FEMALE$$

| | | | | | |
|---|---|---|---|---|---|
| (se) | (2.36) | (0.14) | (0.031) | (1.05) | (0.81) |
| (robse) | (2.50) | (0.16) | (0.029) | (0.84) | (0.80) |

For which coefficients have interval estimates gotten narrower? For which coefficients have interval estimates gotten wider? Is there an inconsistency in the results?

**f.** If we add *MARRIED* to the model in part (b), we find that its *t*-value using a White heteroskedasticity robust standard error is about 1.0. Does this conflict with, or is it compatible with, the result in (b) concerning heteroskedasticity? Explain.

(a) $\begin{cases} H_0: \sigma_M^2 = \sigma_F^2 \\ H_a: \sigma_M^2 \neq \sigma_F^2 \end{cases}$

$$F = \frac{\hat{\sigma}_M^2}{\hat{\sigma}_F^2} = \frac{\frac{SSE_M}{n_M - k}}{\hat{\sigma}_F^2} = \frac{\frac{97161.9174}{577-4}}{12.024} = 14.10$$

$df_M = 577 - 4 = 573$ , $df_F = (1000 - 577) - 4 = 419$

$F_{0.975}(573, 419) \approx \frac{1}{1.25} \approx 0.80$

$\therefore$ rejection region $\Rightarrow F < 0.80$ or $F > 1.25$

$\because F = 14.10 > 1.25 \quad \therefore$ reject $H_0: \sigma_M^2 = \sigma_F^2 \rightarrow$ heteroskedasticity

(若要合併樣本做regression，須要用heteroskedasticity-robust 的SE做估計)

(b) $\begin{cases} H_0: \sigma_{single}^2 = \sigma_{MARRIED}^2 \\ H_a: \sigma_{MARRIED}^2 > \sigma_{single}^2 \end{cases}$ (右尾檢定).

$$S_M^2 = \frac{SSE_M}{n_M - k} = \frac{100703.0471}{600 - 5} = 169.32$$

$$S_s^2 = \frac{SSE_S}{n_s - k} = \frac{56231.0382}{(1000-600) - 5} = 142.36$$

rejection region $\Rightarrow F > 1.12$

$$F = \frac{S_M^2}{S_s^2} = \frac{169.32}{142.36} = 1.19 > F_{0.95}(595, 395) = 1.12$$

$\Rightarrow$ Reject $H_0$.

(c)

$NR^2$ 檢定 (Breusch-Pagan LM 檢定)

若 heteroskedasticity 為真 ⇒ 符從 $\chi^2$ 分配

$$\begin{cases} H_0: Var(\varepsilon_i) = \sigma^2 & \text{(homoskedastic)} \\ H_a: \text{not all } Var(\varepsilon_i) = \sigma^2 & \text{(heteroskedastic)} \end{cases}$$

$df = 4$ (EDUC. EXPER. METRO. FEMALE)

$\chi^2_{0.95}(4) \doteq 9.49$. $NR^2 = 59.03$

$59.03 > 9.49 \Rightarrow$ reject $H_0$.

(d) White 檢定:

$$\begin{cases} H_0: Var(\varepsilon_i | x_i) = \sigma^2 & \text{homoskedastic} \\ H_a: Var(\varepsilon_i | x_i) = h(x_i) & \text{heteroskedastic} \end{cases}$$

自由度: 項數 $m = 4$ (原變數) $+ 4$ (平方項) $+ C^4_2 = 6$ 交乘項 $= 14$.

$W = 78.82$

$\chi^2_{0.95,14} = 23.7$ ∵ $78.82 > 23.7$, reject $H_0$.

→ The model is heteroskedastic.

(e) wider: EDUC

narrower: EXPER, METRO, FEMALE

That's reasonable. ∵ robust SE 中某些 variables 在原 model 中受 heteroskedasticity 影响較大.

∴ some get wider, some are narrower is reasonable.

(f) 關鍵區別:

① heteroskedasticity 檢定: married 是否影响誤差的變異 → 會, significantly

② 加入 married 變數至 regression: married 是否影响 wage → 不會 ($t=1$)

故兩者不衝突。

**8.16** A sample of 200 Chicago households was taken to investigate how far American households tend to travel when they take a vacation. Consider the model

$$MILES = \beta_1 + \beta_2 INCOME + \beta_3 AGE + \beta_4 KIDS + e$$

*MILES* is miles driven per year, *INCOME* is measured in \$1000 units, *AGE* is the average age of the adult members of the household, and *KIDS* is the number of children.

**a.** Use the data file *vacation* to estimate the model by OLS. Construct a 95% interval estimate for the effect of one more child on miles traveled, holding the two other variables constant.

**b.** Plot the OLS residuals versus *INCOME* and *AGE*. Do you observe any patterns suggesting that heteroskedasticity is present?

**c.** Sort the data according to increasing magnitude of income. Estimate the model using the first 90 observations and again using the last 90 observations. Carry out the Goldfeld–Quandt test for heteroskedastic errors at the 5% level. State the null and alternative hypotheses.

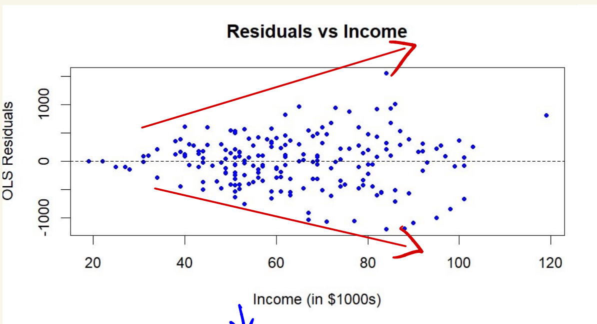
**d.** Estimate the model by OLS using heteroskedasticity robust standard errors. Construct a 95% interval estimate for the effect of one more child on miles traveled, holding the two other variables constant. How does this interval estimate compare to the one in (a)?

**e.** Obtain GLS estimates assuming $\sigma_i^2 = \sigma^2 INCOME_i^2$. Using both conventional GLS and robust GLS standard errors, construct a 95% interval estimate for the effect of one more child on miles traveled, holding the two other variables constant. How do these interval estimates compare to the ones in (a) and (d)?

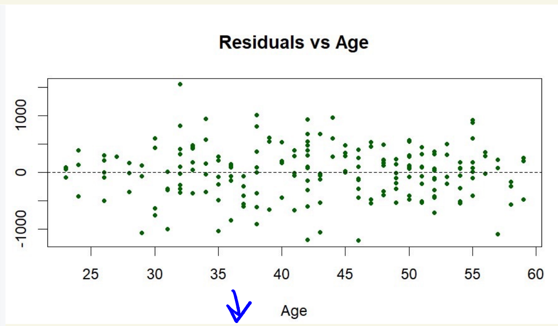
(a) $\widehat{miles} = -391.55 + 14.201\, income + 15.741\, age - 81.826\, kids$

95% C.I. for 1 more child on miles $\Rightarrow$ $[-135.3298, -28.32302]$

(b)




heteroskedaticity




homoskedasticity

(c) $\begin{cases} H_0: \sigma_1^2 = \sigma_2^2 \\ H_a: \sigma_1^2 \neq \sigma_2^2 \end{cases}$

F stat $= 3.1041$

qf(0.95, df1, df2) $= 1.4286$ $\because F > qf$ $\therefore$ reject $H_0$ $\Rightarrow$ heteroskedasticity

(d) C.I. = [-138.75, -24.91] is wider than (a)
(robust)

考慮heteroskedasticity → 估計更保守 → SE增加 → C.I.變寬

(e)

```
GLS（假設 Var(e) ∝ income²）估計值 (kids): -76.81
> cat("95% CI (傳統 GLS 標準誤):", round(ci_gls_conventional, 2), "\n")
95% CI (傳統 GLS 標準誤): -119.89 -33.72
> cat("95% CI (robust GLS 標準誤):", round(ci_gls_robust, 2), "\n")
95% CI (robust GLS 標準誤): -120.97 -32.65
>
```

∵ GLS 其估計較小 → C.I. interval 左移

GLS + robust SE 為最穩健的 C.I.，在 heteroskedasticity 情況下使用

(8.18)

**8.18** Consider the wage equation,

$$\ln(WAGE_i) = \beta_1 + \beta_2 EDUC_i + \beta_3 EXPER_i + \beta_4 EXPER_i^2 + \beta_5 FEMALE_i + \beta_6 BLACK + \beta_7 METRO_i + \beta_8 SOUTH_i + \beta_9 MIDWEST_i + \beta_{10} WEST + e_i$$

where *WAGE* is measured in dollars per hour, education and experience are in years, and *METRO* = 1 if the person lives in a metropolitan area. Use the data file *cps5* for the exercise.

**a.** We are curious whether holding education, experience, and *METRO* equal, there is the same amount of random variation in wages for males and females. Suppose $\text{var}(e_i|\mathbf{x}_i, FEMALE = 0) = \sigma_M^2$ and $\text{var}(e_i|\mathbf{x}_i, FEMALE = 1) = \sigma_F^2$. We specifically wish to test the null hypothesis $\sigma_M^2 = \sigma_F^2$ against $\sigma_M^2 \neq \sigma_F^2$. Carry out a Goldfeld–Quandt test of the null hypothesis at the 5% level of significance. Clearly state the value of the test statistic and the rejection region, along with your conclusion.

**b.** Estimate the model by OLS. Carry out the $NR^2$ test using the right-hand-side variables *METRO*, *FEMALE*, *BLACK* as candidates related to the heteroskedasticity. What do we conclude about heteroskedasticity, at the 1% level? Do these results support your conclusions in (a)? Repeat the test using all model explanatory variables as candidates related to the heteroskedasticity.

**c.** Carry out the White test for heteroskedasticity. What is the 5% critical value for the test? What do you conclude?

**d.** Estimate the model by OLS with White heteroskedasticity robust standard errors. Compared to OLS with conventional standard errors, for which coefficients have interval estimates gotten narrower? For which coefficients have interval estimates gotten wider? Is there an inconsistency in the results?

**e.** Obtain FGLS estimates using candidate variables *METRO* and *EXPER*. How do the interval estimates compare to OLS with robust standard errors, from part (d)?

**f.** Obtain FGLS estimates with robust standard errors using candidate variables *METRO* and *EXPER*. How do the interval estimates compare to those in part (e) and OLS with robust standard errors, from part (d)?

**g.** If reporting the results of this model in a research paper which one set of estimates would you present? Explain your choice.

(a) $F = 1.0508$ $\begin{cases} H_0: \sigma_M^2 = \sigma_F^2 \\ H_a: \sigma_M^2 \neq \sigma_F^2 \end{cases}$

R.R.: $[F < 0.945$ or $F > 1.058)$, ∴ do not reject $H_0$.

無法有充份證據說明在控制性別後，wage的variance有差異。

(b)
(i) $\begin{cases} H_0: \alpha_5 = \alpha_6 = \alpha_7 = 0 \text{ (homoskedasticity)} \\ H_a: \text{not all } \alpha_i = 0 \text{ (Heteroskedasticity)} \end{cases}$

∵ $NR^2 = 23.56 > \chi^2_{0.99,3} = 11.349 \Rightarrow$ reject $H_0$

(ii) all explainable variations: $\begin{cases} H_0: \alpha_i = 0,\ i=1\sim7 \\ H_a: \text{not all } \alpha_i = 0. \end{cases}$

∵ $NR^2 = 109.42 > \chi^2_{0.99,9} = 21.6659$ ∴ We reject $H_0$.

(c) $W = 145.1827 > \chi^2_{0.95,13} = 22.36$ $\begin{cases} H_0: \text{homo} \\ H_a: \text{hetero} \end{cases}$

∴ reject $H_0$.

(d)

| | Coef | Width_OLS | Width_Robust | Narrower_With_Robust |
|---|---|---|---|---|
| (Intercept) | (Intercept) | 0.1259 | 0.1285 | FALSE |
| educ | educ | 0.0069 | 0.0075 | FALSE |
| exper | exper | 0.0051 | 0.0052 | FALSE |
| exper2 | exper2 | 0.0001 | 0.0001 | FALSE |
| female | female | 0.0374 | 0.0372 | TRUE |
| black | black | 0.0664 | 0.0631 | TRUE |
| metro | metro | 0.0482 | 0.0454 | TRUE |
| south | south | 0.0532 | 0.0545 | FALSE |
| midwest | midwest | 0.0553 | 0.0538 | TRUE |
| west | west | 0.0565 | 0.0570 | FALSE |

有些C.I.窄→正常∵針对"部分"variables調整

(e)

| | Coef | Width_Robust | Width_FGLS | FGLS_Narrower |
|---|---|---|---|---|
| (Intercept) | (Intercept) | 0.1285 | 0.1238 | TRUE |
| educ | educ | 0.0075 | 0.0069 | TRUE |
| exper | exper | 0.0052 | 0.0051 | TRUE |
| exper2 | exper2 | 0.0001 | 0.0001 | TRUE |
| female | female | 0.0372 | 0.0372 | TRUE |
| black | black | 0.0631 | 0.0666 | FALSE |
| metro | metro | 0.0454 | 0.0449 | TRUE |
| south | south | 0.0545 | 0.0530 | TRUE |
| midwest | midwest | 0.0538 | 0.0548 | FALSE |
| west | west | 0.0570 | 0.0564 | TRUE |

一次 FGLS：先從 OLS 取得殘差 $\hat{u}_i$: ex: $\log(\hat{u}_i) = \gamma_0 + \gamma z_i + \dots + \varepsilon$.

針对每個 obvs 的 var 進行 GLS 加权迴歸（权重 $= \frac{1}{\hat{\sigma}_i}$）

(f)

```
> print(comparison)
```

| | Coef | Width_OLS_Robust | Width_FGLS | Width_FGLS_Robust |
|---|---|---|---|---|
| (Intercept) | X_star(Intercept) | 0.1285 | 0.1238 | 0.1268 |
| educ | X_stareduc | 0.0075 | 0.0069 | 0.0074 |
| exper | X_starexper | 0.0052 | 0.0051 | 0.0051 |
| exper2 | X_starexper2 | 0.0001 | 0.0001 | 0.0001 |
| female | X_starfemale | 0.0372 | 0.0372 | 0.0370 |
| black | X_starblack | 0.0631 | 0.0666 | 0.0622 |
| metro | X_starmetro | 0.0454 | 0.0449 | 0.0453 |
| south | X_starsouth | 0.0545 | 0.0530 | 0.0542 |
| midwest | X_starmidwest | 0.0538 | 0.0548 | 0.0537 |
| west | X_starwest | 0.0570 | 0.0564 | 0.0568 |

最窄　　　　適中.

(g) I would choose FGLS robust.

兼顧各变數的異質性，且 robust 的 SE 大更具穩健性

| 比較項目 | Goldfeld–Quandt Test | White Test |
|---|---|---|
| 是否需排序變數 | ✅ 是，需要指定一個排序依據變數 | ❌ 不需要 |
| 假設異質來源 | 指定一個變數可能造成變異差異 | 無需指定，可同時考慮多變數平方與交互項 |
| 檢定方式 | F 檢定（比較兩組變異數） | $NR^2$ 法（卡方檢定） |
| 適合何種異質性 | 針對特定變數造成的對稱型異質變異 | 任何形式的異質變異（最通用） |
| 計算複雜度 | 簡單（只跑兩個 OLS 回歸） | 較高（需要建模平方與交互項） |
| 優點 | 解釋直觀、檢定方向明確 | 非常通用，無需指定來源 |
| 缺點 | 只檢測特定變數造成的變異差異 | 可能自由度很高，檢定力下降 |

(卡方檢定).

| 比較項目 | $NR^2$ 檢定（一般） | White 檢定（特例） |
|---|---|---|
| 檢定方法 | $\hat{u}^2$ 回歸某些變數，計算 $nR^2$ | $\hat{u}^2$ 回歸所有變數 + 平方 + 交互項 |
| 是否需平方項 | ❌ 選擇性，可不用 | ✅ 必須 |
| 是否需交互項 | ❌ 選擇性，可不用 | ✅ 必須 |
| 自由度 | 解釋變數個數（彈性） | 較多（通常自由度較高） |
| 檢定力 | 視你放入變數的準確性決定 | 高度一般化，但可能檢定力較弱 |
| 主要用途 | 偵測異質變異數與特定變數的關聯 | 通用檢查異質變異數是否存在 |

例如知道σ(ei)和某個variables的平方程正比

| 比較項目 | GLS（廣義最小平方法） | FGLS（可行廣義最小平方法） |
|---|---|---|
| 誤差變異結構 | **已知（例如：你知道 $\mathrm{Var}(\varepsilon) = \Omega$）** | **未知，需要用 OLS 殘差估計變異結構** |
| 實務可行性 | 通常無法實際應用，因為誤差變異結構很難完全知道 | 實務常用（先用 OLS 拿殘差，再估計變異結構） |
| 適用狀況 | 理論分析、模擬理想條件 | 當資料存在異質變異數或自我相關，但形式未知 |
| 估計過程 | 一步完成 | 分兩步：① OLS 殘差估變異 → ② 做 GLS 加權估計 |
| 效率 | BLUE（線性中最有效） | 在誤差變異估得好時，也接近 BLUE |
| 誤差模型錯誤風險 | 不會錯（因為你已知道誤差結構） | 若估錯誤差變異結構，可能會降低效率或偏誤 |