11.1 Our aim is to estimate the parameters of the simultaneous equations model

$$y_1 = \alpha_1 y_2 + e_1$$
$$y_2 = \alpha_2 y_1 + \beta_1 x_1 + \beta_2 x_2 + e_2$$

We assume that x_1 and x_2 are exogenous and uncorrelated with the error terms e_1 and e_2.

a. Solve the two structural equations for the reduced-form equation for y_2, that is, $y_2 = \pi_1 x_1 + \pi_2 x_2 + v_2$. Express the reduced-form parameters in terms of the **structural parameters** and the reduced-form error in terms of the structural parameters and e_1 and e_2. Show that y_2 is correlated with e_1.

$$y_2 = \alpha_2 y_1 + \beta_1 x_1 + \beta_2 x_2 + e_2$$
$$= \alpha_2(\alpha_1 y_2 + e_1) + \beta_1 x_1 + \beta_2 x_2 + e_2$$
$$= \alpha_1\alpha_2 y_2 + \alpha_2 e_1 + \beta_1 x_1 + \beta_2 x_2 + e_2$$
$$\Rightarrow (1-\alpha_1\alpha_2) y_2 = \alpha_2 e_1 + \beta_1 x_1 + \beta_2 x_2 + e_2$$
$$\Rightarrow y_2 = \underbrace{\frac{\beta_1}{1-\alpha_1\alpha_2}}_{\pi_1} x_1 + \underbrace{\frac{\beta_2}{1-\alpha_1\alpha_2}}_{\pi_2} x_2 + \underbrace{\frac{\alpha_2 e_1 + e_2}{1-\alpha_1\alpha_2}}_{v_2}$$

$$\mathrm{cov}(y_2, e_1) = \mathrm{cov}\left(\frac{\beta_1}{1-\alpha_1\alpha_2} x_1 + \frac{\beta_2}{1-\alpha_1\alpha_2} x_2 + \frac{\alpha_2 e_1 + e_2}{1-\alpha_1\alpha_2}, e_1\right)$$
$$= \frac{\beta_1}{1-\alpha_1\alpha_2}\mathrm{cov}(x_1, e_1) + \frac{\beta_2}{1-\alpha_1\alpha_2}\mathrm{cov}(x_2, e_1) + \mathrm{cov}\left(\frac{\alpha_2 e_1}{1-\alpha_1\alpha_2}, e_1\right) + \mathrm{cov}\left(\frac{e_2}{1-\alpha_1\alpha_2}, e_1\right)$$
$$= 0 + 0 + \frac{\alpha_2}{1-\alpha_1\alpha_2}\mathrm{cov}(e_1, e_1) + 0 = \frac{\alpha_2}{1-\alpha_1\alpha_2}\sigma_{e_1}^2 \neq 0 \text{ if } \alpha_2 \neq 0.$$

b. Which equation parameters are consistently estimated using OLS? Explain.

Both equation (1) (2) have an endogenous variable one the right hand-side. $\Rightarrow$ OLS is biased and inconsistent.
The reduced form equation can be estimated consistently by OLS. Because there is only one exogenous on the right hand side.

c. Which parameters are "identified," in the simultaneous equations sense? Explain your reasoning.

$M = 2 \Rightarrow$ 至少有 $M-1$ 個外生變數被刪除

$y_1 = \alpha_1 y_2 + e_1 \Rightarrow x_1$ and x_2 are absent $\Rightarrow$ identified.

$y_2 = \alpha_2 y_1 + \beta_1 x_1 + \beta_2 x_2 + e_2 \Rightarrow$ No exogenous variable absent $\Rightarrow$ identified.

d. To estimate the parameters of the reduced-form equation for y_2 using the method of moments (MOM), which was introduced in Section 10.3, the two moment equations are

$$N^{-1}\sum x_{i1}(y_2 - \pi_1 x_{i1} - \pi_2 x_{i2}) = 0$$

$$N^{-1}\sum x_{i2}(y_2 - \pi_1 x_{i1} - \pi_2 x_{i2}) = 0$$

Explain why these two moment conditions are a valid basis for obtaining consistent estimators of the reduced-form parameters.

$E(x_{i1}, v_{i1} | x) = E(x_{i2}, v_{i2} | x) = 0$ ($\because x_1, x_2$ are exogenous)

$$E\left(x_{ik}\left(\frac{\alpha_2 e_1 + e_2}{1-\alpha_1\alpha_2}\right) \Big| x\right) = \frac{\alpha_2}{1-\alpha_1\alpha_2} E(x_{ik} e_1 | x) + \frac{1}{1-\alpha_1\alpha_2} E(x_{ik} e_2 | x)$$

$$= 0 + 0 = 0$$

e. Are the MOM estimators in part (d) the same as the OLS estimators? Form the sum of squared errors function for $y_2 = \pi_1 x_1 + \pi_2 x_2 + v_2$ and find the first derivatives. Set these to zero and show that they are equivalent to the two equations in part (d).

$$S(\pi_1, \pi_2 | y, x) = \sum (y_2 - \pi_1 x_1 - \pi_2 x_2)^2$$

$$\frac{\partial S(\pi_1, \pi_2 | y, x)}{\partial \pi_1} = 2\sum (y_2 - \pi_1 x_1 - \pi_2 x_2) x_1 = 0 \Rightarrow \frac{1}{N}\sum (y_2 - \pi_1 x_1 - \pi_2 x_2) x_1 = 0$$

$$\frac{\partial S(\pi_1, \pi_2 | y, x)}{\partial \pi_2} = 2\sum (y_2 - \pi_1 x_1 - \pi_2 x_2) x_2 = 0 \Rightarrow \frac{1}{N}\sum (y_2 - \pi_1 x_1 - \pi_2 x_2) x_2 = 0$$

f. Using $\sum x_{i1}^2 = 1$, $\sum x_{i2}^2 = 1$, $\sum x_{i1}x_{i2} = 0$, $\sum x_{i1}y_{1i} = 2$, $\sum x_{i1}y_{2i} = 3$, $\sum x_{i2}y_{1i} = 3$, $\sum x_{i2}y_{2i} = 4$, and the two moment conditions in part (d) show that the MOM/OLS estimates of π_1 and π_2 are $\hat{\pi}_1 = 3$ and $\hat{\pi}_2 = 4$.

$$\sum (y_2 - \pi_1 x_1 - \pi_2 x_2) x_1 = 0 \Rightarrow \sum (x_1 y_2 - \pi_1 x_1^2 - \pi_2 x_1 x_2) = 0 \Rightarrow 3 - \hat{\pi}_1 \times 1 - \hat{\pi}_2 \times 0 = 0 \Rightarrow \hat{\pi}_1 = 3$$

$$\sum (y_2 - \pi_1 x_1 - \pi_2 x_2) x_2 = 0 \Rightarrow \sum (x_2 y_2 - \pi_1 x_1 x_2 - \pi_2 x_2^2) = 0 \Rightarrow 4 - \hat{\pi}_1 \times 0 - \hat{\pi}_2 \times 1 = 0 \Rightarrow \hat{\pi}_2 = 4$$

g. The fitted value $\hat{y}_2 = \hat{\pi}_1 x_1 + \hat{\pi}_2 x_2$. Explain why we can use the moment condition $\sum \hat{y}_{i2}(y_{i1} - \alpha_1 y_{i2}) = 0$ as a valid basis for consistently estimating α_1. Obtain the IV estimate of α_1.

$$E(y_2 e_1 | x) = E[(\pi_1 x_1 + \pi_2 x_2) e_1 | x] = E[(\pi_1 x_1 + \pi_2 x_2)(y_1 - \alpha_1 y_2) | x] = 0$$

$$\hat{\pi}_1 \xrightarrow{P} \pi_1, \quad \hat{\pi}_2 \xrightarrow{P} \pi_2$$

$$\sum (\hat{\pi}_1 x_1 + \hat{\pi}_2 x_2)(y_1 - \alpha_1 y_2) = \sum \hat{y}_2 (y_1 - \alpha_1 y_2) = 0$$

$$\Rightarrow \sum \hat{y}_2 y_1 - \alpha_1 \sum \hat{y}_2 y_2 = 0 \Rightarrow \hat{\alpha}_1 = \frac{\sum \hat{y}_2 y_1}{\sum \hat{y}_2 y_2} = \frac{\sum (\hat{\pi}_1 x_1 + \hat{\pi}_2 x_2) y_1}{\sum (\hat{\pi}_1 x_1 + \hat{\pi}_2 x_2) y_2} = \frac{\hat{\pi}_1 \sum x_1 y_1 + \hat{\pi}_2 \sum x_2 y_1}{\hat{\pi}_1 \sum x_1 y_2 + \hat{\pi}_2 \sum x_2 y_2} = \frac{18}{25}$$

h. Find the 2SLS estimate of α_1 by applying OLS to $y_1 = \alpha_1 \hat{y}_2 + e_1^*$. Compare your answer to that in part (g).

$$\hat{\alpha}_{1,2SLS} = \frac{\sum \hat{y}_2 y_1}{\sum \hat{y}_2^2}$$

$$\sum \hat{y}_2 \hat{v}_2 = \sum (\hat{\pi}_1 x_1 + \hat{\pi}_2 x_2) \hat{v}_2$$

$$= \hat{\pi}_1 \sum x_1 \hat{v}_2 + \hat{\pi}_2 \sum x_2 \hat{v}_2 = 0$$

$\because \hat{v}_2 = y_2 - \hat{y}_2 \Rightarrow \hat{y}_2 = y_2 - \hat{v}_2$

$\therefore \sum \hat{y}_2^2 = \sum \hat{y}_2 (y_2 - \hat{v}_2) = \sum \hat{y}_2 y_2 - \boxed{\sum \hat{y}_2 \hat{v}_2}$

$$= \sum \hat{y}_2 y_2$$

$\Rightarrow \hat{\alpha}_{1,2SLS} = \dfrac{\sum \hat{y}_2 y_1}{\sum \hat{y}_2 y_2}$ 與(g)相同 #