11.1 Our aim is to estimate the parameters of the simultaneous equations model

$$y_1 = \alpha_1 y_2 + e_1$$

$$y_2 = \alpha_2 y_1 + \beta_1 x_1 + \beta_2 x_2 + e_2$$

We assume that x_1 and x_2 are exogenous and uncorrelated with the error terms e_1 and e_2.

a. Solve the two structural equations for the reduced-form equation for y_2, that is, $y_2 = \pi_1 x_1 + \pi_2 x_2 + v_2$. Express the reduced-form parameters in terms of the **structural parameters** and the reduced-form error in terms of the structural parameters and e_1 and e_2. Show that y_2 is correlated with e_1.

b. Which equation parameters are consistently estimated using OLS? Explain.

c. Which parameters are "identified," in the simultaneous equations sense? Explain your reasoning.

a、 $y_1 = \alpha_1 y_2 + e_1$

$$y_2 = \alpha_2(\alpha_1 y_2 + e_1) + \beta_1 X_1 + \beta_2 X_2 + e_2$$

$$= \alpha_2\alpha_1 y_2 + \alpha_2 e_1 + \beta_1 X_1 + \beta_2 X_2 + e_2$$

$$y_2(1-\alpha_2\alpha_1) = \beta_1 X_1 + \beta_2 X_2 + \alpha_2 e_1 + e_2$$

$$y_2 = \frac{\beta_1}{1-\alpha_1\alpha_2} X_1 + \frac{\beta_2}{1-\alpha_1\alpha_2} X_2 + \frac{\alpha_2 e_1 + e_2}{1-\alpha_1\alpha_2}$$

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b、 y_2與e_1相關，OLS X

y_1受y_2影响，y_2受e_2影响，OLS X

c、 共2个内生变量 ⇒ 各需 2-1个被排除的外生变量

equation 1、排除 x_1, x_2 ⇒ identify

equation 2、無排除 ⇒ not indentify

d. To estimate the parameters of the reduced-form equation for y_2 using the method of moments (MOM), which was introduced in Section 10.3, the two moment equations are

$$N^{-1}\sum x_{i1}(y_2 - \pi_1 x_{i1} - \pi_2 x_{i2}) = 0$$
$$N^{-1}\sum x_{i2}(y_2 - \pi_1 x_{i1} - \pi_2 x_{i2}) = 0$$

Explain why these two moment conditions are a valid basis for obtaining consistent estimators of the reduced-form parameters.

e. Are the MOM estimators in part (d) the same as the OLS estimators? Form the sum of squared errors function for $y_2 = \pi_1 x_1 + \pi_2 x_2 + v_2$ and find the first derivatives. Set these to zero and show that they are equivalent to the two equations in part (d).

f. Using $\sum x_{i1}^2 = 1$, $\sum x_{i2}^2 = 1$, $\sum x_{i1}x_{i2} = 0$, $\sum x_{i1}y_{1i} = 2$, $\sum x_{i1}y_{2i} = 3$, $\sum x_{i2}y_{1i} = 3$, $\sum x_{i2}y_{2i} = 4$, and the two moment conditions in part (d) show that the MOM/OLS estimates of π_1 and π_2 are $\hat{\pi}_1 = 3$ and $\hat{\pi}_2 = 4$.

g. The fitted value $\hat{y}_2 = \hat{\pi}_1 x_1 + \hat{\pi}_2 x_2$. Explain why we can use the moment condition $\sum \hat{y}_{i2}(y_{i1} - \alpha_1 y_{i2}) = 0$ as a valid basis for consistently estimating α_1. Obtain the IV estimate of α_1.

h. Find the 2SLS estimate of α_1 by applying OLS to $y_1 = \alpha_1 \hat{y}_2 + e_1^*$. Compare your answer to that in part (g).

d.
$$N^{-1}\sum X_{i1}(y_2 - \pi_1 X_{i1} - \pi_2 X_{i2}) = 0$$
$$N^{-1}\sum X_{i2}(y_2 - \pi_1 X_{i1} - \pi_2 X_{i2}) = 0$$
$$y_2 = \pi_1 X_1 + \pi_2 X_2 + V_2$$
$$\Rightarrow \sum X_{i1}V_{2i} = 0 \text{ and } \sum X_{i2}V_{2i} = 0$$
$$E(X_{i1}V_{2i}) = E(X_{i2}V_{2i}) = 0$$
$$V_{2i} = \frac{\alpha_2 e_{1i} + e_{2i}}{1-\alpha_1\alpha_2}$$
$$E\left(X_{1i}\cdot\frac{\alpha_2 e_{1i} + e_{2i}}{1-\alpha_1\alpha_2}\right) = \frac{\alpha_2 E(X_i e_{1i}) + E(X_{1i}e_{2i})}{1-\alpha_1\alpha_2} = 0$$
$$\Rightarrow E(X_{2i}V_{2i}) = 0$$

e.
$$S(\pi_1, \pi_2) = \sum(y_{2i} - \pi_1 X_{1i} - \pi_2 X_{2i})^2$$
$$\frac{\partial S}{\partial \pi_1} = -2\sum X_{1i}(y_{2i} - \pi_1 X_{1i} - \pi_2 X_{2i}) = 0$$
$$\frac{\partial S}{\partial \pi_2} = -2\sum X_{2i}(y_{2i} - \pi_1 X_{1i} - \pi_2 X_{2i}) = 0$$

$\Sigma X_{1i}(y_{2i}-\pi_1 X_{1i}-\pi_2 X_{2i})=0$

$\Sigma X_{2i}(y_{2i}-\pi_1 X_{1i}-\pi_2 X_{2i})=0$

OLS and MOM 相同，差在 N^{-1}（邏輯相同）

f.

$$\begin{cases}\Sigma X_{1i}(y_{2i}-\pi_1 X_{1i}-\pi_2 X_{2i})=0\\ \Sigma X_{2i}(y_{2i}-\pi_1 X_{1i}-\pi_2 X_{2i})=0\end{cases}$$

$$\begin{cases}\Sigma X_{1i}y_{2i}-\pi_1\Sigma X_{1i}^2-\pi_2\Sigma X_{1i}X_{2i}=0\\ \Sigma X_{2i}y_{2i}-\pi_1\Sigma X_{2i}X_{1i}-\pi_2\Sigma X_{2i}^2=0\end{cases}$$

$3-\pi_1\cdot 1-\pi_2\cdot 0=0$

$3-\pi_1=0$

$\pi_1=3$

$4-\pi_1\cdot 0-\pi_2\cdot 1=0$

$4-\pi_2=0$

$\pi_2=4$

g. $\Sigma \hat{y}_{2i}(y_{1i}-\alpha_1 y_{2i})=0$

$y_2=\pi_1 X_1+\pi_2 X_2+V_2$

$\hat{\pi}_1=3$, $\hat{\pi}_2=4$

$\hat{y}_{2i}=3X_{1i}+4X_{2i}$

$\Rightarrow \Sigma \hat{y}_{2i}\, y_{1i}=\Sigma(3X_{1i}+4X_{2i})y_{1i}=3\Sigma X_{1i}y_{1i}+4\Sigma X_{2i}y_{1i}$

$=3\cdot 2+4\cdot 3$

$=18$

$$\Rightarrow \sum \hat{y}_{2i}^2 = \sum (3X_{1i}+4X_{2i})^2 = \sum (9X_{1i}^2+24X_{1i}X_{2i}+16X_{2i}^2)$$

$$= 9\sum X_{1i}^2 + 24\sum X_{1i}X_{2i} + 16\sum X_{2i}^2$$

$$= 9\cdot 1 + 24\cdot 0 + 16\cdot 1 = 25$$

$$\Rightarrow \sum \hat{y}_{2i}(y_{1i} - \alpha_1 \hat{y}_{2i}) = 0$$

$$\sum \hat{y}_{2i} y_{1i} - \alpha_1 \sum \hat{y}_{2i}^2 = 0$$

$$\alpha_1 = \frac{\sum \hat{y}_{2i} y_{1i}}{\sum \hat{y}_{2i}^2} = \frac{18}{25} = 0.72$$

h.

皆為0.72

11.16 Consider the following supply and demand model

$$\text{Demand: } Q_i = \alpha_1 + \alpha_2 P_i + e_{di}, \quad \text{Supply: } Q_i = \beta_1 + \beta_2 P_i + \beta_3 W_i + e_{si}$$

where Q is the quantity, P is the price, and W is the wage rate, which is assumed exogenous. Data on these variables are in Table 11.7.

TABLE 11.7 Data for Exercise 11.16

Q	P	W
4	2	2
6	4	3
9	3	1
3	5	1
8	8	3

a. Derive the algebraic form of the reduced-form equations, $Q = \theta_1 + \theta_2 W + v_2$ and $P = \pi_1 + \pi_2 W + v_1$, expressing the reduced-form parameters in terms of the structural parameters.

b. Which structural parameters can you solve for from the results in part (a)? Which equation is "identified"?

c. The estimated reduced-form equations are $\hat{Q} = 5 + 0.5W$ and $\hat{P} = 2.4 + 1W$. Solve for the identified structural parameters. This is the method of **indirect least squares**.

d. Obtain the fitted values from the reduced-form equation for P, and apply 2SLS to obtain estimates of the demand equation.

a. $Q_i = \alpha_1 + \alpha_2 P_i + e_{di} = \beta_1 + \beta_2 P + \beta_3 W + e_{si}$

$\alpha_1 + \alpha_2 P - \beta_1 - \beta_2 P - \beta_3 W - e_s + e_d = 0$

$(\alpha_2 - \beta_2) P = (\beta_1 - \alpha_1) + \beta_3 W + (e_s - e_d)$

$$P = \frac{\beta_1 - \alpha_1}{\alpha_2 - \beta_2} + \frac{\beta_3}{\alpha_2 - \beta_2} W + \frac{e_s - e_d}{\alpha_2 - \beta_2}$$

$\Rightarrow$ 將P代回Q:

$$Q_i = \frac{\alpha_2\beta_1 - \alpha_1\beta_2}{\alpha_2 - \beta_2} + \frac{\alpha_2\beta_3}{\alpha_2 - \beta_2} W + \frac{\alpha_2 e_s - \beta_2 e_d}{\alpha_2 - \beta_2}$$

b.

Demand: $Q = \alpha_1 + \alpha_2 P + e_d$

Supply: $Q = \beta_1 + \beta_2 P + \beta_3 W + e_s$

外生: W, 內生: P、Q $\Rightarrow 2-1=1$

$\Rightarrow$ Demand 排除 W $\Rightarrow$ identify

Supply 無 $\Rightarrow$ not identify

$$\begin{cases} P = \dfrac{\overset{\pi_1}{\beta_1-\alpha_1}}{\alpha_2-\beta_2} + \dfrac{\overset{\pi_2}{\beta_3}}{\alpha_2-\beta_2}W + \dfrac{e_s-e_d}{\alpha_2-\beta_2} \\ Q_i = \dfrac{\overset{\theta_1}{\alpha_2\beta_1-\alpha_1\beta_2}}{\alpha_2-\beta_2} + \dfrac{\overset{\theta_2}{\alpha_2\beta_3}}{\alpha_2-\beta_2}W + \dfrac{\alpha_2 e_s-\beta_2 e_d}{\alpha_2-\beta_2} \end{cases}$$

$$\pi_2 = \frac{\beta_3}{\alpha_2-\beta_2} \Rightarrow \alpha_2-\beta_2 = \frac{\beta_3}{\pi_2}$$

$$\pi_1 = \frac{\beta_1-\alpha_1}{\alpha_2-\beta_2} \Rightarrow \beta_1-\alpha = \pi_1 \cdot \frac{\beta_3}{\pi_2}$$

$$\theta_2 = \frac{\alpha_2\beta_3}{\alpha_2-\beta_2} \Rightarrow \alpha_2-\beta_2 = \frac{\beta_3}{\pi_2}$$

$$\frac{\alpha_2\cancel{\beta_3}}{\cancel{\beta_3}/\pi_2} = \theta_2$$

$$\alpha_2 = \frac{\theta_2}{\pi_2}$$

$$\frac{\beta_3}{\pi_2} = \frac{\theta_2}{\pi_2} - \beta_2$$

$$\beta_2 = \frac{\theta_2}{\pi_2} - \frac{\beta_3}{\pi_2}$$

有$\alpha_1, \alpha_2, \beta_1, \beta_2, \beta_3$，但只有$\theta_1, \theta_2, \pi_1, \pi_2$

僅解出$\alpha_2 = \frac{\theta_2}{\pi_2}$

C.

$$\begin{cases} P = \dfrac{\overset{\pi_1}{\beta_1-\alpha_1}}{\alpha_2-\beta_2} + \dfrac{\overset{\pi_2}{\beta_3}}{\alpha_2-\beta_2}W + \dfrac{e_s-e_d}{\alpha_2-\beta_2} \\ Q_i = \dfrac{\overset{\theta_1}{\alpha_2\beta_1-\alpha_1\beta_2}}{\alpha_2-\beta_2} + \dfrac{\overset{\theta_2}{\alpha_2\beta_3}}{\alpha_2-\beta_2}W + \dfrac{\alpha_2 e_s-\beta_2 e_d}{\alpha_2-\beta_2} \end{cases}$$

$$Q = 5 + 0.5W \Rightarrow \theta_1 = 5,\ \theta_2 = 0.5$$

$$P = 24 + 1W \Rightarrow \pi_1 = 24,\ \pi_2 = 1$$

b小題 $\Rightarrow \alpha_2-\beta_2=\dfrac{\beta_3}{\pi_{22}}$ $=1$

$\alpha_2-\beta_2=\beta_3$

$\theta_2=\dfrac{\alpha_2\beta_3}{\alpha_2-\beta_2}=\dfrac{\alpha_2\beta_3}{\beta_3}=\alpha_2=0.5$

$\pi_1=\dfrac{\beta_1-\alpha_1}{\alpha_2-\beta_2}$

$2.4=\dfrac{\beta_1-\alpha_1}{\alpha_2-\beta_2}$

$\beta_1-\alpha_1=2.4(\alpha_2-\beta_2)=2.4\beta_3$

$\theta_1=\dfrac{\alpha_2\beta_1-\alpha_1\beta_2}{\alpha_2-\beta_2}=5$

$\alpha_2\beta_1-\alpha_1\beta_2=5(\alpha_2-\beta_2)$

代入 $\beta_1-\alpha_1=2.4\beta_3$

$\Rightarrow \alpha_1\underbrace{(0.5-\beta_2)}_{=\beta_3}=3.8\beta_3$

$\Rightarrow \alpha_1=3.8$

(β_1,β_2,β_3 無法求出)

d.

$(Q,P,W)=(4,2,2)(6,4,3)(9,3,1)$
$(3,5,1)(8,8,3)$

compute $\hat{P}$: $2.4+1W$

$\hat{P}=4.4$ 5.4 3.4

$(4,2,2)(6,4,3)(9,3,1)$
$(3,5,1)(8,8,3)$

3.4 5.4

2SLS $\Rightarrow$ ① $\hat{P}_i$ ② $Q_i=\alpha_1+\alpha_2\hat{P}_i+\text{error}$

$$\hat{\alpha}_2=\frac{\Sigma(Q_i-\bar{Q})(\hat{P}_i-\bar{\hat{P}})}{\Sigma(\hat{P}_i-\hat{P})^2},\ \hat{\alpha}_1=\bar{Q}-\hat{\alpha}_2\bar{\hat{P}}$$

$\bar{Q}=6 \quad \bar{\hat{P}}=4.4$

$\Sigma(Q_i-\bar{Q})(\hat{P}_i-\bar{\hat{P}})=2$

$\Sigma(\hat{P}_i-\bar{\hat{P}}_i)^2=4$

$\Rightarrow \hat{\alpha}_1=3.8,\ \hat{\alpha}_2=0.5$, the same as ©