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11.1 Our aim is to estimate the parameters of the simultaneous equations model

$$y_1 = \alpha_1 y_2 + e_1 \quad \text{—— ①}$$
$$y_2 = \alpha_2 y_1 + \beta_1 x_1 + \beta_2 x_2 + e_2 \quad \text{—— ②}$$

We assume that x_1 and x_2 are exogenous and uncorrelated with the error terms e_1 and e_2.

a. Solve the two structural equations for the reduced-form equation for y_2, that is, $y_2 = \pi_1 x_1 + \pi_2 x_2 + v_2$. Express the reduced-form parameters in terms of the **structural parameters** and the reduced-form error in terms of the structural parameters and e_1 and e_2. Show that y_2 is correlated with e_1.

a、將①代入②

$$y_2 = \alpha_2 y_1 + \beta_1 x_1 + \beta_2 x_2 + e_2$$
$$= \alpha_2(\alpha_1 y_2 + e_1) + \beta_1 x_1 + \beta_2 x_2 + e_2$$
$$= \alpha_1\alpha_2 y_2 + \alpha_2 e_1 + \beta_1 x_1 + \beta_2 x_2 + e_2$$
$$\Rightarrow y_2(1-\alpha_1\alpha_2) = \beta_1 x_1 + \beta_2 x_2 + e_2 + \alpha_2 e_1$$
$$\Rightarrow y_2 = \frac{\beta_1}{(1-\alpha_1\alpha_2)} x_1 + \frac{\beta_2}{(1-\alpha_1\alpha_2)} x_2 + \frac{e_2 + \alpha_2 e_1}{(1-\alpha_1\alpha_2)}$$
$$= \pi_1 x_1 + \pi_2 x_2 + v_2$$

The correlation of y_2 and e_1

$$cov(y_2, e_1 | X) = E(y_2, e_1 | X) = E\left[\left(\frac{\beta_1}{1-\alpha_1\alpha_2} x_1 + \frac{\beta_2}{1-\alpha_1\alpha_2} x_2 + \frac{e_2+\alpha_2 e_1}{1-\alpha_1\alpha_2}\right) e_1 \middle| X\right]$$

$$= \underbrace{E\left[\frac{\beta_1}{(1-\alpha_1\alpha_2)} x_1 e_1 \middle| X\right] + E\left[\frac{\beta_2}{(1-\alpha_1\alpha_2)} x_2 e_1 \middle| X\right]} + E\left[\left(\frac{e_2+\alpha_2 e_1}{1-\alpha_1\alpha_2}\right) e_1 \middle| X\right]$$

$$= \underbrace{0 + 0} + E\left[\left(\frac{e_2 e_1 + \alpha_2 e_1^2}{1-\alpha_1\alpha_2}\right) \middle| X\right]$$

$\because$ x's are exogenous and uncorrelated with errors.

Assuming the two equation errors are uncorrelated :

$$cov(y_2, e_1 | X) = E(y_2, e_1 | X) = \frac{E(e_2 e_1 | X) + \alpha_2 E(e_1^2 | X)}{(1-\alpha_1\alpha_2)}$$

$$= \frac{\alpha_2}{(1-\alpha_1\alpha_2)} \sigma_1^2 \neq 0$$ (除非 $\alpha_2 = 0$, 則無 simultaneity 問題。)

b. Which equation parameters are consistently estimated using OLS? Explain.
c. Which parameters are "identified," in the simultaneous equations sense? Explain your reasoning.

b、兩個方程式皆不適用OLS，因為①和②等式右側都同時包含內生變數，使用OLS則會產生biased且inconsistent的結果。

若使用reduced-form方程式則其參數可以用OLS估計，

因為只剩外生變數在等式右側。

c、 $M=2$ ∴至少要有$(M-1)=1$個外生變數在方程式中被省略。

①式中 x_1, x_2 被省略 ∴ identified，可以一致地估計α_1

②式則沒有外生變數有數被省略 ∴ not identified.

d. To estimate the parameters of the reduced-form equation for y_2 using the method of moments (MOM), which was introduced in Section 10.3, the two moment equations are

v_1 誤差項 in reduced-form

$$N^{-1}\sum x_{i1}(y_2 - \pi_1 x_{i1} - \pi_2 x_{i2}) = 0$$

v_2

$$N^{-1}\sum x_{i2}(y_2 - \pi_1 x_{i1} - \pi_2 x_{i2}) = 0$$

> x_i和v_i加權平均為0

Explain why these two moment conditions are a valid basis for obtaining consistent estimators of the reduced-form parameters.

d. x為外生 $\therefore E(x_{i1} v_{i1} | X) = E(x_{i2} v_{i2} | X) = 0$

$$y_2 = \frac{\beta_1}{(1-\alpha_1\alpha_2)} x_1 + \frac{\beta_2}{(1-\alpha_1\alpha_2)} x_2 + \frac{e_2 + \alpha_2 e_1}{(1-\alpha_1\alpha_2)} = \pi_1 x_1 + \pi_2 x_2 + v_2$$

則x's 和 reduced form 的誤差項 v_2 為 uncorrelated：

$$E\left[x_{ik}\left(\frac{e_2+\alpha_2 e_1}{1-\alpha_1\alpha_2}\right) \middle| X\right] = E\left[\frac{1}{(1-\alpha_1\alpha_2)} x_{ik} e_2 \middle| X\right] + E\left[\frac{\alpha_2}{(1-\alpha_1\alpha_2)} x_{ik} e_1 \middle| X\right]$$

$$= 0 + 0 = 0$$

e. Are the MOM estimators in part (d) the same as the OLS estimators? Form the sum of squared errors function for $y_2 = \pi_1 x_1 + \pi_2 x_2 + v_2$ and find the first derivatives. Set these to zero and show that they are equivalent to the two equations in part (d).

e. $y_2 = \pi_1 x_1 + \pi_2 x_2 + v_2$

$v_2 = y_2 - \pi_1 x_1 - \pi_2 x_2$

∴ 最小化誤差函數：

$$S(\pi_1, \pi_2 \mid y, x) = \sum (y_2 - \pi_1 x_1 - \pi_2 x_2)^2$$

對 π_1 偏微並設為 0：

$$\frac{\partial S(\pi_1, \pi_2 \mid y, x)}{\partial \pi_1} = 2 \sum (y_2 - \pi_1 x_1 - \pi_2 x_2) x_1 = 0 \quad — ①$$

對 π_2 偏微並設為 0：

$$\frac{\partial S(\pi_1, \pi_2 \mid y, x)}{\partial \pi_1} = 2 \sum (y_2 - \pi_1 x_1 - \pi_2 x_2) x_2 = 0 \quad — ②$$

若將①和②各除 2，且乘以 N 後則和 MoM 的條件等價。

f. Using $\sum x_{i1}^2 = 1$, $\sum x_{i2}^2 = 1$, $\sum x_{i1} x_{i2} = 0$, $\sum x_{i1} y_{1i} = 2$, $\sum x_{i1} y_{2i} = 3$, $\sum x_{i2} y_{1i} = 3$, $\sum x_{i2} y_{2i} = 4$, and the two moment conditions in part (d) show that the MOM/OLS estimates of π_1 and π_2 are $\hat{\pi}_1 = 3$ and $\hat{\pi}_2 = 4$.

The moment conditions are :

① $N^{-1} \sum x_{i1} (y_2 - \pi_1 x_{i1} - \pi_2 x_{i2}) = 0$

$\Rightarrow \sum x_{i1} y_{i2} - \pi_1 \sum x_{i1}^2 - \pi_2 \sum x_{i1} x_{i2} = 0$

$\Rightarrow 3 - \pi_1 \times 1 - \pi_2 \times 0 = 0$

$\Rightarrow \hat{\pi}_1 = 3$ #

② $N^{-1} \sum x_{i2} (y_2 - \pi_1 x_{i1} - \pi_2 x_{i2}) = 0$

$\Rightarrow \sum x_{i2} y_{i2} - \pi_1 \sum x_{i1} x_{i2} - \pi_2 \sum x_{i2}^2 = 0$

$\Rightarrow 4 - \pi_1 \times 0 - \pi_2 \times 1 = 0$

$\Rightarrow \hat{\pi}_2 = 4$ #

g. The fitted value $\hat{y}_2 = \hat{\pi}_1 x_1 + \hat{\pi}_2 x_2$. Explain why we can use the moment condition $\sum \hat{y}_{i2}(y_{i1} - \alpha_1 y_{i2}) = 0$ as a valid basis for consistently estimating α_1. Obtain the IV estimate of α_1.

$y_1 = \alpha_1 y_2 + e_1$ 要估計 α_1

檢查 $(\pi_1 x_1 + \pi_2 x_2)$ 和 e_1 的關聯性：

$\because E[(\pi_1 x_1 + \pi_2 x_2) e_1 | X] = E[(\pi_1 x_1 + \pi_2 x_2)(y_1 - \alpha_1 y_2) | X] = 0$

The moment condition：

$$N^{-1} \sum (\pi_1 x_{i1} + \pi_2 x_{i2})(y_{i1} - \alpha_1 y_{i2}) = 0$$

當 $n \to \infty$，$\hat{\pi}_1 = \pi_1$，$\hat{\pi}_2 = \pi_2$ 代入 moment condition 得：

$$\sum (\hat{\pi}_{1} x_{i1} + \hat{\pi}_2 x_{i2})(y_{i1} - \alpha_1 y_{i2}) = \sum \hat{y}_{i2}(y_{i1} - \alpha_1 y_{i2}) = 0$$

$$\rightarrow \sum \hat{y}_{i2} y_{i1} - \alpha \sum \hat{y}_{i2} y_{i2} = 0$$

$$\rightarrow \hat{\alpha}_{1,IV} = \frac{\sum \hat{y}_{i2} y_{i1}}{\sum \hat{y}_{i2} y_{i2}} = \frac{\sum (\hat{\pi}_1 x_{i1} + \hat{\pi}_2 x_{i2}) y_{i1}}{\sum (\hat{\pi}_1 x_{i1} + \hat{\pi}_2 x_{i2}) y_{i2}}$$

$$= \frac{\hat{\pi}_1 \sum x_{i1} y_{i1} + \hat{\pi}_2 \sum x_{i2} y_{i1}}{\hat{\pi}_1 \sum x_{i1} y_{i2} + \hat{\pi}_2 \sum x_{i2} y_{i2}}$$

$$= \frac{3 \times 2 + 4 \times 3}{3 \times 3 + 4 \times 4} = \frac{18}{25} \ \#$$

h. Find the 2SLS estimate of α_1 by applying OLS to $y_1 = \alpha_1 \hat{y}_2 + e_1^*$. Compare your answer to that in part (g).

$y_1 = \alpha_1 y_2 + e_1$

1階：$y_2 = \pi_1 x_1 + \pi_2 x_2 + v_2$

2階：$y_1 = \alpha_1 \hat{y}_2 + e_1$

$$\hat{\alpha}_{1,2SLS} = \frac{\sum \hat{y}_{i2} y_{i1}}{\sum \hat{y}_{i2}^2} = \frac{\sum \hat{y}_{i2} y_{i1}}{\sum \hat{y}_{i2} y_{i2}} = \frac{18}{25} \ \#$$

$\because \hat{v}_2 = y_2 - \hat{y}_2$，$\hat{y}_2 = y_2 - \hat{v}_2$

$$\sum \hat{y}_{i2}^2 = \sum \hat{y}_{i2}(y_2 - \hat{v}_2) = \sum \hat{y}_{i2} y_2 - \boxed{\sum \hat{y}_{i2} \hat{v}_2} = \sum \hat{y}_{i2} y_2$$

$$\sum \hat{y}_{i2} \hat{v}_{i2} = \sum (\hat{\pi}_1 x_{i1} + \hat{\pi}_2 x_{i2}) \hat{v}_{i2} = \hat{\pi}_1 \underbrace{\sum x_{i1} \hat{v}_{i2}}_{0} + \hat{\pi}_2 \underbrace{\sum x_{i2} \hat{v}_{i2}}_{0} = 0$$

11.16 Consider the following supply and demand model

內　　內　　　　　　內　　內　　外生

$$\text{Demand: } Q_i = \alpha_1 + \alpha_2 P_i + e_{di}, \quad \text{Supply: } Q_i = \beta_1 + \beta_2 P_i + \beta_3 W_i + e_{si}$$

where Q is the quantity, P is the price, and W is the wage rate, which is assumed exogenous. Data on these variables are in Table 11.7.

TABLE 11.7 Data for Exercise 11.16

Q	P	W
4	2	2
6	4	3
9	3	1
3	5	1
8	8	3

a. Derive the algebraic form of the reduced-form equations, $Q = \theta_1 + \theta_2 W + v_2$ and $P = \pi_1 + \pi_2 W + v_1$, expressing the reduced-form parameters in terms of the structural parameters.

b. Which structural parameters can you solve for from the results in part (a)? Which equation is "identified"?

a. Let Demand = Supply

$$Q_i = \alpha_1 + \alpha_2 P_i + e_{di} = \beta_1 + \beta_2 P_i + \beta_3 W_i + e_{si}$$

$$(\alpha_2 - \beta_2) P_i = (\beta_1 - \alpha_1) + \beta_3 W_i + e_{si} - e_{di}$$

$$P_i = \frac{\beta_1 - \alpha_1}{\alpha_2 - \beta_2} + \frac{\beta_3}{\alpha_2 - \beta_2} W_i + \frac{e_{si} - e_{di}}{\alpha_2 - \beta_2}$$

$$= \theta_1 + \theta_2 W_i + V_2 \ \#$$

將 P_i 代入 Demand Model 求 Q_i

$$Q_i = \alpha_1 + \alpha_2 P_i + e_{di}$$

$$= \alpha_1 + \alpha_2 \left(\frac{\beta_1 - \alpha_1}{\alpha_2 - \beta_2} + \frac{\beta_3}{\alpha_2 - \beta_2} W_i + \frac{e_{si} - e_{di}}{\alpha_2 - \beta_2} \right) + e_{di}$$

$$= \alpha_1 + \left(\frac{\beta_1 - \alpha_1}{\alpha_2 - \beta_2} \right) \alpha_2 + \left(\frac{\beta_3}{\alpha_2 - \beta_2} \right) \alpha_2 \cdot W_i + \left(\frac{e_{si} - e_{di}}{\alpha_2 - \beta_2} \right) \alpha_2 + e_{di}$$

$$= \pi_1 + \pi_2 W + V_1 \ \#$$

外生 : W_i 、

b. $M=2$ ∴ 到要 $(M-1)=1$ 個變數被省略　內生 : Q_i 、P_i

Demand : $Q_i = \alpha_1 + \alpha_2 P_i + e_{di}$ ⇒ 沒有 W_i ∴ identified , α_1, α_2 can be estimated consistently.

Supply : $Q_i = \beta_1 + \beta_2 P_i + \beta_3 W_i + e_{si}$ ⇒ 沒有變數被省略 ∴ not identified

11.16

c. The estimated reduced-form equations are $\hat{Q} = 5 + 0.5W$ and $\hat{P} = 2.4 + 1W$. Solve for the identified structural parameters. This is the method of **indirect least squares**.

d. Obtain the fitted values from the reduced-form equation for P, and apply 2SLS to obtain estimates of the demand equation.

c. $\hat{Q} = 5 + 0.5W$

$\hat{P} = 2.4 + 1W$

代入 Demand :

$$\hat{Q} = 5 + 0.5W = \alpha_1 + \alpha_2(2.4 + 1W)$$

$$\Rightarrow \underline{5} + \underline{0.5W} = \underline{(\alpha_1 + 2.4\cdot\alpha_2)} + \underline{1W\cdot\alpha_2}$$

$$0.5W = \alpha_2 W \qquad \Rightarrow \alpha_2 = 0.5 \ \#$$

$$5 = \alpha_1 + 2.4\alpha_2 \Rightarrow \alpha_1 = 5 - (2.4\times 0.5) = 3.8 \ \#$$

d. $\hat{P} = 2.4 + 1W \quad \bar{\hat{P}} = \frac{22}{5} = 4.4 \ ; \ \bar{Q} = \frac{30}{5} = 6$

	W	$\hat{P}$	$\hat{P} - \bar{\hat{P}}$	Q	$Q - \bar{Q}$
	2	4.4	0	4	-2
	3	5.4	1	6	0
	1	3.4	-1	9	3
	1	3.4	-1	3	-3
	3	5.4	1	8	2
sum		22	2	30	0

$$Q = \alpha_1 + \alpha_2\hat{P} + e_i$$

$$\hat{\alpha}_1 = \frac{\Sigma(\hat{P}_i - \bar{\hat{P}})(Q_i - \bar{Q})}{\Sigma(\hat{P}_i - \bar{\hat{P}})^2} = \frac{0+0-3+3+2}{4} = 0.5 \ \#$$

$$\hat{\alpha}_2 = \bar{Q} - \hat{\alpha}_2\bar{P} = 6 - 0.5\times 4.4 = 3.8 \ \#$$

11.17 Example 11.3 introduces Klein's Model I.

a. Do we have an adequate number of IVs to estimate each equation? Check the necessary condition for the identification of each equation. The necessary condition for identification is that in a system of M equations at least $M - 1$ variables must be omitted from each equation.

b. An equivalent identification condition is that the number of excluded exogenous variables from the equation must be at least as large as the number of included right-hand side endogenous variables. Check that this condition is satisfied for each equation.

c. Write down in econometric notation the first-stage equation, the reduced form, for W_{1t}, wages of workers earned in the private sector. Call the parameters $\pi_1, \pi_2, \ldots$

d. Describe the two regression steps of 2SLS estimation of the consumption function. This is not a question about a computer software command.

e. Does following the steps in part (d) produce regression results that are identical to the 2SLS estimates provided by software specifically designed for 2SLS estimation? In particular, will the t-values be the same?

a. $M=8$

∴ 至少要 $(M-1)=7$ 個外生變數在各方程式中被省略

共有8個外生變數：W_{2t}、P_{t-1}、K_{t-1}、G_t、E_{t-1}、$TIME_t$、TX_t、$X_{1t}\equiv 1$

Consumption Function：

$$CN_t = \alpha_1 + \alpha_2(W_{1t} + W_{2t}) + \alpha_3 P_t + \alpha_4 P_{t-1} + e_{1t}$$

共有6個變數，有 $16-6=10$ 個變數被省略 $\Rightarrow 10>7$ ∴ identified#

Wage Function：

$$W_{1t} = r_1 + r_2 E_t + r_3 E_{t-1} + r_4 TIME_t + e_{3t}$$

共有5個變數，$16-5=11$ 個變數被省略 $\Rightarrow 11>7$ ∴ identified#

Investment Function：

$$I_t = \beta_1 + \beta_2 P_t + \beta_3 P_{t-1} + \beta_4 K_{t-1} + e_{2t}$$

共有5個變數，$16-5=11$ 個變數被省略 $\Rightarrow 11>7$ ∴ identified#

b. CN —
- 右 內生變數：2個
- 外生變數：3個 → 被排除5個　∵ $5>2$ ⇒ satisfied

W_{1t} —
- 右 內生變數：1個
- 外生變數：3個 → 被排除5個　∵ $5>1$ ⇒ satisfied

I_t —
- 右 內生變數：1個
- 外生變數：3個 → 被排除5個　∵ $5>1$ ⇒ satisfied.

c. Write down in econometric notation the first-stage equation, the reduced form, for W_{1t}, wages of workers earned in the private sector. Call the parameters $\pi_1, \pi_2, \ldots$

d. Describe the two regression steps of 2SLS estimation of the consumption function. This is not a question about a computer software command.

e. Does following the steps in part (d) produce regression results that are identical to the 2SLS estimates provided by software specifically designed for 2SLS estimation? In particular, will the t-values be the same?

c、 $W_{1t} = r_1 + r_2 \underline{E_t} + r_3 E_{t-1} + r_4 TIME_t + e_{3t}$

$$= r_1 + r_2 \underline{(CN_t + I_t + (G - W_{2t}))} + r_3 E_{t-1} + r_4 TIME_t + e_{3t}$$

$$W_{1t} = \pi_1 + \pi_2 G_t + \pi_3 W_{2t} + \pi_4 TX_4 + \pi_5 TIME_t + \pi_6 P_{t-1} + \pi_7 K_{t-1} + \pi_8 E_{t-1} + v$$

d、 $CN_t = \alpha_1 + \alpha_2 (W_{1t} + W_{2t}) + \alpha_3 P_t + \alpha_4 P_{t-1} + e_{1t}$

1階迴歸：從(c)的方程式估計出 $\hat{W}_{1t}$ (得到 $W^* = \hat{W}_{1t} + W_{2t}$)

也求出 $\hat{P}_t$

2階迴歸：將 $\hat{W}_{1t}$、$\hat{P}_t$ 代入 Consumption Function

$\Rightarrow CN_t = \alpha_1 + \alpha_2 \hat{W}_{1t} + \alpha_2 W_{2t} + \alpha_3 \hat{P}_t + \alpha_4 P_{t-1} + e_{1t}$

regress CN_t on $\hat{W}_{1t}$、W_{2t}、$\hat{P}_t$、P_{t-1} and constant by OLS.

e、迴歸係數會相同，但 t-value 會不同，

∵ (d) 的標準誤不是正確的 2SLS 標準誤。