Nguyen Quoc Nhan

413707009

HW0505

1.

To derive the reduced-form equation for y_2, we substitute the first equation into the second equation:

$$y_2 = \alpha_2(\alpha_1 y_2 + e_1) + \beta_1 x_1 + \beta_2 x_2 + e_2$$

Expanding this gives:

$$y_2 = \alpha_2 \alpha_1 y_2 + \alpha_2 e_1 + \beta_1 x_1 + \beta_2 x_2 + e_2$$

Rearranging to isolate y_2:

$$y_2 - \alpha_2 \alpha_1 y_2 = \alpha_2 e_1 + \beta_1 x_1 + \beta_2 x_2 + e_2$$
$$y_2(1 - \alpha_2 \alpha_1) = \alpha_2 e_1 + \beta_1 x_1 + \beta_2 x_2 + e_2$$
$$y_2 = \frac{\beta_1 x_1 + \beta_2 x_2 + \alpha_2 e_1 + e_2}{1 - \alpha_2 \alpha_1}$$

This is the reduced-form equation for y_2, which can be written as:

$$y_2 = \pi_1 x_1 + \pi_2 x_2 + v_2$$

Where:

- $\pi_1 = \frac{\beta_1}{1 - \alpha_2 \alpha_1}$,
- $\pi_2 = \frac{\beta_2}{1 - \alpha_2 \alpha_1}$,

- $v_2 = \frac{\alpha_2 e_1 + e_2}{1 - \alpha_2 \alpha_1}$.

Next, we examine whether y_2 is correlated with e_1. The covariance is:

$$\mathrm{Cov}(y_2, e_1) = \mathrm{Cov}(\pi_1 x_1 + \pi_2 x_2 + v_2, e_1)$$

Since x_1 and x_2 are exogenous and uncorrelated with e_1:

$$\mathrm{Cov}(y_2, e_1) = \mathrm{Cov}(v_2, e_1)$$

Substituting the expression for v_2:

$$\mathrm{Cov}(y_2, e_1) = \mathrm{Cov}\left(\frac{\alpha_2 e_1 + e_2}{1 - \alpha_2 \alpha_1}, e_1\right)$$

Since e_2 is uncorrelated with e_1:

$$\mathrm{Cov}(y_2, e_1) = \frac{\alpha_2}{1 - \alpha_2 \alpha_1} \cdot \mathrm{Var}(e_1)$$

As $\mathrm{Var}(e_1) > 0$ and $1 - \alpha_2 \alpha_1 \neq 0$, we conclude that $\mathrm{Cov}(y_2, e_1) \neq 0$. Therefore, y_2 is correlated with e_1.

b.

For OLS to produce consistent estimates, the explanatory variables must be uncorrelated with the error term.

- In the first equation: $y_1 = \alpha_1 y_2 + e_1$,

  y_2 is correlated with e_1, as shown in part (a).

  Therefore, OLS estimation of α_1 will be inconsistent.

- In the second equation: $y_2 = \alpha_2 y_1 + \beta_1 x_1 + \beta_2 x_2 + e_2$,

  y_1 contains e_1 (from the first equation), which affects y_2.

  Thus, y_1 is correlated with e_2, meaning OLS estimation of α_2 will also be inconsistent.

However, β_1 and β_2 can still be consistently estimated because x_1 and x_2 are exogenous.

c.

In a system of M simultaneous equations, at least $M-1$ variables must be excluded from each equation for it to be identified.

Here, $M=2$, so at least 1 variable must be excluded from each equation.

- For the first equation: $y_1 = \alpha_1 y_2 + e_1$,

  Both exogenous variables x_1 and x_2 are excluded.

  Therefore, this equation is identified (over-identified since 2 > $M-1$).

- For the second equation: $y_2 = \alpha_2 y_1 + \beta_1 x_1 + \beta_2 x_2 + e_2$,

  No exogenous variables are excluded.

  Therefore, this equation is not identified (under-identified).

(d) These moment conditions arise from the assumptions that the x's are exogenous. It follows that

$$E\left(x_{i1}v_{i1} \mid \mathbf{x}\right) = E\left(x_{i2}v_{i2} \mid \mathbf{x}\right) = 0$$

From part (a), the reduced form equation for y_2 is

$$y_2 = \frac{\beta_1}{(1-\alpha_1\alpha_2)}x_1 + \frac{\beta_2}{(1-\alpha_1\alpha_2)}x_2 + \frac{e_2+\alpha_2 e_1}{(1-\alpha_1\alpha_2)} = \pi_1 x_1 + \pi_2 x_2 + v_2$$

The reduced form error is uncorrelated with the x's because

$$E\left[x_{ik}\left(\frac{e_2+\alpha_2 e_1}{(1-\alpha_1\alpha_2)}\right)\middle|\,\mathbf{x}\right] = E\left[\frac{1}{(1-\alpha_1\alpha_2)}x_{ik}e_2\middle|\,\mathbf{x}\right] + E\left[\frac{\alpha_2}{(1-\alpha_1\alpha_2)}x_{ik}e_1\middle|\,\mathbf{x}\right] = 0+0$$

(e) The sum of squares function, omitting the subscript i for convenience, is $S(\pi_1,\pi_2 \mid \mathbf{y},\mathbf{x}) = \sum(y_2 - \pi_1 x_1 - \pi_2 x_2)^2$. The first derivatives are

$$\frac{\partial S(\pi_1,\pi_2 \mid \mathbf{y},\mathbf{x})}{\partial \pi_1} = 2\sum(y_2 - \pi_1 x_1 - \pi_2 x_2)x_1 = 0$$
$$\frac{\partial S(\pi_1,\pi_2 \mid \mathbf{y},\mathbf{x})}{\partial \pi_2} = 2\sum(y_2 - \pi_1 x_1 - \pi_2 x_2)x_2 = 0$$

Divide these equations by 2, and multiply the moment equations by N to see that they are equivalent.

(f) The moment conditions are

$$N^{-1}\sum x_{i1}\left(y_2 - \pi_1 x_{i1} - \pi_2 x_{i2}\right) = 0$$
$$N^{-1}\sum x_{i2}\left(y_2 - \pi_1 x_{i1} - \pi_2 x_{i2}\right) = 0$$

Multiplying these out we have

$$\sum x_{i1}y_{i2} - \pi_1 \sum x_{i1}^2 - \pi_2 \sum x_{i1}x_{i2} = 0$$
$$\sum x_{i2}y_{i2} - \pi_1 \sum x_{i1}x_{i2} - \pi_2 \sum x_{i2}^2 = 0$$

Inserting the given values, we have

$$3 - \hat{\pi}_1 = 0 \Rightarrow \hat{\pi}_1 = 3$$
$$4 - \hat{\pi}_2 = 0 \Rightarrow \hat{\pi}_2 = 4$$

(g) The first structural equation is $y_1 = \alpha_1 y_2 + e_1$, so that

$$E\left[\left(\pi_1 x_1 + \pi_2 x_2\right)e_1 \mid \mathbf{x}\right] = E\left[\left(\pi_1 x_1 + \pi_2 x_2\right)\left(y_1 - \alpha_1 y_2\right) \mid \mathbf{x}\right] = 0$$

The empirical analog of this moment condition is

$$N^{-1}\sum\left(\pi_1 x_{i1} + \pi_2 x_{i2}\right)\left(y_{i1} - \alpha_1 y_{i2}\right) = 0$$

If we knew π_1 and π_2 we could solve this moment condition for an estimator of α_1. While we do not know these parameters we can consistently estimate them from the reduced form equations. In large samples the consistent estimators converge to the true parameter values,

$$\text{plim } \hat{\pi}_1 = \pi_1 \text{ and plim } \hat{\pi}_2 = \pi_2$$

In a sense, having consistent estimators of parameters is "just as good as" knowing the parameter values. Replacing the unknowns by their estimates in the empirical moment condition we have

$$\sum\left(\hat{\pi}_1 x_{i1} + \hat{\pi}_2 x_{i2}\right)\left(y_{i1} - \alpha_1 y_{i2}\right) = \sum \hat{y}_{i2}\left(y_{i1} - \alpha_1 y_{i2}\right) = 0$$

So that

$$\sum \hat{y}_{i2}y_{i1} - \alpha_1 \sum \hat{y}_{i2}y_{i2} = 0 \Rightarrow \hat{\alpha}_{1,IV} = \frac{\sum \hat{y}_{i2}y_{i1}}{\sum \hat{y}_{i2}y_{i2}}$$

Inserting the values, we find

$$\hat{\alpha}_{1,IV} = \frac{\sum \hat{y}_{i2}y_{i1}}{\sum \hat{y}_{i2}y_{i2}} = \frac{\sum\left(\hat{\pi}_1 x_{i1} + \hat{\pi}_2 x_{i2}\right)y_{i1}}{\sum\left(\hat{\pi}_1 x_{i1} + \hat{\pi}_2 x_{i2}\right)y_{i2}} = \frac{\hat{\pi}_1 \sum x_{i1}y_{i1} + \hat{\pi}_2 \sum x_{i2}y_{i1}}{\hat{\pi}_1 \sum x_{i1}y_{i2} + \hat{\pi}_2 \sum x_{i2}y_{i2}} = \frac{18}{25}$$

(h) The least squares estimator of the simple regression model with no intercept is given in Exercise 2.4. Applying that result here, and substituting $\hat{y}_2$ for x and y_1 for y, we have

$$\hat{\alpha}_{1,2SLS} = \frac{\sum \hat{y}_{i2} y_{i1}}{\sum \hat{y}_{i2}^2}$$

To show that the equations are equivalent, recall that $\hat{v}_2 = y_2 - \hat{y}_2 \Rightarrow \hat{y}_2 = y_2 - \hat{v}_2$ and therefore

$$\sum \hat{y}_{i2}^2 = \sum \hat{y}_{i2}\left(y_2 - \hat{v}_2\right) = \sum \hat{y}_{i2} y_2 - \sum \hat{y}_{i2}\hat{v}_2 = \sum \hat{y}_{i2} y_2$$

The term

$$\sum \hat{y}_{i2}\hat{v}_{i2} = \sum \left(\hat{\pi}_1 x_{i1} + \hat{\pi}_2 x_{i2}\right)\hat{v}_{i2} = \hat{\pi}_1 \sum x_{i1}\hat{v}_{i2} + \hat{\pi}_2 \sum x_{i2}\hat{v}_{i2} = 0$$

because $\sum x_{i1}\hat{v}_{i2} = 0$ and $\sum x_{i2}\hat{v}_{i2} = 0$. This is a fundamental property of OLS that is illustrated in Exercises 2.3(f) and 2.4(g).

16

Part (a): Deriving Reduced-Form Equations

The model is given as:

- **Demand equation:** $Q_i = \alpha_1 + \alpha_2 P_i + e_{di}$
- **Supply equation:** $Q_i = \beta_1 + \beta_2 P_i + \beta_3 W_i + e_{si}$

At equilibrium, demand equals supply:

$$\alpha_1 + \alpha_2 P_i + e_{di} = \beta_1 + \beta_2 P_i + \beta_3 W_i + e_{si}$$

Rearranging to solve for P_i:

$$P_i(\alpha_2 - \beta_2) = \beta_1 - \alpha_1 + \beta_3 W_i + e_{si} - e_{di}$$
$$P_i = \frac{\beta_1 - \alpha_1 + \beta_3 W_i + e_{si} - e_{di}}{\alpha_2 - \beta_2}$$

This gives the reduced-form equation for P_i:

This gives the reduced-form equation for Q_i:

$$Q_i = \theta_1 + \theta_2 W_i + v_{2i}$$

Where:

- $\theta_1 = \frac{\alpha_2\beta_1 - \alpha_1\beta_2}{\alpha_2 - \beta_2}$
- $\theta_2 = \frac{\alpha_2\beta_3}{\alpha_2 - \beta_2}$
- $v_{2i} = \frac{\alpha_2 e_{si} - \beta_2 e_{di}}{\alpha_2 - \beta_2}$

$$P_i = \pi_1 + \pi_2 W_i + v_{1i}$$

Where:

- $\pi_1 = \frac{\beta_1 - \alpha_1}{\alpha_2 - \beta_2}$
- $\pi_2 = \frac{\beta_3}{\alpha_2 - \beta_2}$
- $v_{1i} = \frac{e_{si} - e_{di}}{\alpha_2 - \beta_2}$

Next, substitute this expression for P_i into the demand equation:

$$Q_i = \alpha_1 + \alpha_2 \left[\frac{\beta_1 - \alpha_1 + \beta_3 W_i + e_{si} - e_{di}}{\alpha_2 - \beta_2}\right] + e_{di}$$

Simplifying:

$$Q_i = \frac{\alpha_2\beta_1 - \alpha_1\beta_2}{\alpha_2 - \beta_2} + \left[\frac{\alpha_2\beta_3}{\alpha_2 - \beta_2}\right] W_i + \frac{\alpha_2 e_{si} - \beta_2 e_{di}}{\alpha_2 - \beta_2}$$

Part (b): Identifying Structural Parameters

From the reduced-form parameters, we can identify α_2:

$$\alpha_2 = \frac{\theta_2}{\pi_2} = \frac{\frac{\alpha_2\beta_3}{\alpha_2 - \beta_2}}{\frac{\beta_3}{\alpha_2 - \beta_2}} = \alpha_2$$

The **demand equation** is identified because:

- The wage rate W appears in the supply equation but not in the demand equation.
- This satisfies the order condition for identification (at least $M - 1$ variables must be excluded).

The **supply equation** is not identified because:

- There are no exogenous variables that appear in the demand equation but not in the supply equation.

Part (c): Indirect Least Squares

Given the estimated reduced-form equations:

- $\hat{Q} = 5 + 0.5W$
- $\hat{P} = 2.4 + 1W$

We can identify α_2:

$$\alpha_2 = \frac{\theta_2}{\pi_2} = \frac{0.5}{1} = 0.5$$

For α_1 (the intercept in the demand equation):

$$\alpha_1 = 5 - 0.5(2.4) = 5 - 1.2 = 3.8$$

Thus, the identified demand equation is:

$$Q = 3.8 + 0.5\hat{P} + e_d$$

Part (d)

Step 1: Calculate fitted values from the reduced-form equation for P

P =2.4+1W

For each observation in the dataset, calculate the fitted values of ▯P using this equation.

Step 2: Apply 2SLS to estimate the demand equation

The demand equation is: $Q = \alpha_1 + \alpha_2 P + e_d$

For 2SLS, we replace P with P̂ in the second stage: $Q = \alpha_1 + \alpha_2 \hat{P} + e_d$

The 2SLS estimate of the demand equation is:
$Q = 3.8 + 0.5\hat{P} + e_d$

This matches the result obtained in part (c) using the indirect least squares (ILS) method. This confirms that, when applied correctly, both 2SLS and ILS yield consistent and identical estimates for the identified equation

17.

In the system, there are M=8 equations, meaning each equation must omit at least

M−1=7 variables to satisfy the necessary condition for identification. The system contains a total of 16 variables.

Consumption Equation:

This equation includes 6 variables and omits 10 variables. Since it omits more than the required 7 variables, the necessary condition is satisfied.

Investment Equation:

This equation includes 5 variables and omits 11 variables. As it omits more than 7 variables, the necessary condition is satisfied.

Private Sector Wage Equation:

This equation includes 5 variables and omits 11 variables. Again, since it omits more than 7 variables, the necessary condition is satisfied.

Thus, all three equations satisfy the necessary condition for identification.

(b) Exogenous and Endogenous Variables

Consumption Equation:

This equation has 2 endogenous variables on the right-hand side (RHS) and excludes 5 exogenous variables.

Investment Equation:

This equation has 1 endogenous variable on the RHS and excludes 5 exogenous variables.

Private Sector Wage Equation:

Similarly, this equation has 1 endogenous variable on the RHS and omits 5 exogenous variables.

(c) $$W_{1t} = \pi_1 + \pi_2 G_t + \pi_3 W_{2t} + \pi_4 TX_t + \pi_5 TIME_t + \pi_6 P_{t-1} + \pi_7 K_{t-1} + \pi_8 E_{t-1} + v$$

(d) Obtain fitted values $\hat{W}_{1t}$ from the estimated reduced form equation in part (c) and similarly obtain $\hat{P}_t$. Create $W_t^* = \hat{W}_{1t} + W_{2t}$. Regress CN_t on W_t^*, $\hat{P}_t$ and P_{t-1} plus a constant by OLS.

(e) The coefficient estimates will be the same. The t-values will not be because the standard errors in part (d) are not correct 2SLS standard errors.