11.6.1 Problems

11.1 Our aim is to estimate the parameters of the simultaneous equations model

$$y_1 = \alpha_1 y_2 + e_1$$
$$y_2 = \alpha_2 y_1 + \beta_1 x_1 + \beta_2 x_2 + e_2$$

We assume that x_1 and x_2 are exogenous and uncorrelated with the error terms e_1 and e_2.

a. Solve the two structural equations for the reduced-form equation for y_2, that is, $y_2 = \pi_1 x_1 + \pi_2 x_2 + v_2$. Express the reduced-form parameters in terms of the **structural parameters** and the reduced-form error in terms of the structural parameters and e_1 and e_2. Show that y_2 is correlated with e_1.

b. Which equation parameters are consistently estimated using OLS? Explain.

c. Which parameters are "identified," in the simultaneous equations sense? Explain your reasoning.

(1) $y_1 = \alpha_1 y_2 + e_1$ — ①

$y_2 = \alpha_2 y_1 + \beta_1 x_1 + \beta_2 x_2 + e_2$ — ②

①代入② $\Rightarrow y_2 = \alpha_2(\alpha_1 y_2 + e_1) + \beta_1 x_1 + \beta_2 x_2 + e_2$

$\Rightarrow y_2 = \alpha_1\alpha_2 y_2 + \alpha_2 e_2 + \beta_1 x_1 + \beta_2 x_2 + e_2$

$\Rightarrow (1-\alpha_1\alpha_2) y_2 = \alpha_2 e_1 + \beta_1 x_1 + \beta_2 x_2 + e_2$

$$\Rightarrow y_2 = \underbrace{\boxed{\frac{\beta_1}{1-\alpha_1\alpha_2}}}_{\pi_1} x_1 + \underbrace{\boxed{\frac{\beta_2}{1-\alpha_1\alpha_2}}}_{\pi_2} x_2 + \underbrace{\boxed{\frac{\alpha_2 e_1 + e_2}{1-\alpha_1\alpha_2}}}_{v_2}$$

y_2 中包含 e_1 所以相關，所以在①中估計 α_1 會有endogeneity的問題

b. ∵① $y_1 = \alpha_1 y_2 + e_1$ 有內生性 ∴ α_1 会有bias和 inconsistent

y_1 ∵含有 y_2，y_2 和 e_1 有関 ∴ inconsistent

但∵ x_1, x_2 是外生変数 ∴ β_1, β_2 可以用OLS估計並 consistent

c. ① 缺席2 exogenous variables → identified

② 缺席0 exogenous variables → not identified

d. To estimate the parameters of the reduced-form equation for y_2 using the method of moments (MOM), which was introduced in Section 10.3, the two moment equations are

$$N^{-1}\sum x_{i1}(y_2 - \pi_1 x_{i1} - \pi_2 x_{i2}) = 0$$
$$N^{-1}\sum x_{i2}(y_2 - \pi_1 x_{i1} - \pi_2 x_{i2}) = 0$$

Explain why these two moment conditions are a valid basis for obtaining consistent estimators of the reduced-form parameters.

e. Are the MOM estimators in part (d) the same as the OLS estimators? Form the sum of squared errors function for $y_2 = \pi_1 x_1 + \pi_2 x_2 + v_2$ and find the first derivatives. Set these to zero and show that they are equivalent to the two equations in part (d).

d. $\frac{1}{n}\sum x_{1i}(y_{2i} - \pi_1 x_{1i} - \pi_2 x_{2i}) = 0$. ∵ x_1, x_2 是外生變數，表示 $E(x_1 \cdot v_2) = 0$, $E(x_2, v_2) = 0$

$\frac{1}{n}\sum x_{2i}(y_{2i} - \pi_1 x_{1i} - \pi_2 x_{2i}) = 0$ 　根據LLN

$\frac{1}{n}\sum z_i(y_i - x_i'\beta) \xrightarrow{p} E(z_i(y_i - x_i'\beta))$

所以解出的 $\hat{\pi}_1, \hat{\pi}_2$ 会是consistent估計值

e. OLS: $S(\pi_1, \pi_2) = \sum_{i=1}^{n} (y_{2i} - \pi_1 x_{1i} - \pi_2 x_{2i})^2$

(1) $\frac{\partial}{\partial \pi_1} S = \sum_{i=1}^{n} -2x_{1i}(y_{2i} - \pi_1 x_{1i} - \pi_2 x_{2i}) = 0$, $\sum x_{1i}(y_{2i} - \pi_1 x_{1i} - \pi_2 x_{2i}) = 0$

↓

(2) $\frac{\partial}{\partial \pi_2} S = \sum x_{2i}(y_{2i} - \pi_1 x_{1i} - \pi_{2i}) = 0$ → 同乘 $\frac{1}{N}$ 会和OLS first order 相同

故2個估計的 $\hat{\pi}_1, \hat{\pi}_2$ 会相同

f. Using $\sum x_{i1}^2 = 1$, $\sum x_{i2}^2 = 1$, $\sum x_{i1}x_{i2} = 0$, $\sum x_{i1}y_{1i} = 2$, $\sum x_{i1}y_{2i} = 3$, $\sum x_{i2}y_{1i} = 3$, $\sum x_{i2}y_{2i} = 4$, and the two moment conditions in part (d) show that the MOM/OLS estimates of π_1 and π_2 are $\hat{\pi}_1 = 3$ and $\hat{\pi}_2 = 4$.

g. The fitted value $\hat{y}_2 = \hat{\pi}_1 x_1 + \hat{\pi}_2 x_2$. Explain why we can use the moment condition $\sum \hat{y}_{i2}(y_{i1} - \alpha_1 y_{i2}) = 0$ as a valid basis for consistently estimating α_1. Obtain the IV estimate of α_1.

h. Find the 2SLS estimate of α_1 by applying OLS to $y_1 = \alpha_1 \hat{y}_2 + e_1^*$. Compare your answer to that in part (g).

f. 展開(1)： $\sum x_{1i}y_{2i} - \pi_1 \sum x_{1i}^2 - \pi_2 \sum x_{1i}x_{2i} = 0$

$3 - \pi_1(1) - \pi_2(0) = 0.$

$\pi_1 = 3$ 同理, $\pi_2 = 4$.

g.

(IV)

↓

x_1, x_2

$E(\hat{y}_2 e_1) = 0 \Rightarrow E(\hat{y}_2(y_1 - \alpha_1 y_2)) = 0.$

$\Rightarrow E(\pi_1 x_1 + \pi_2 x_2)(y_1 - \alpha_1 y_2) = 0$

$\frac{1}{N}\sum y_{2i}(y_{1i} - \alpha_1 y_{2i}) = 0 \Rightarrow \sum \hat{y}_{2i} y_{1i} - \alpha_1 \sum \hat{y}_{2i} y_{2i} = 0 \Rightarrow \hat{\alpha}_1 = \frac{\sum \hat{y}_{2i} y_{1i}}{\sum \hat{y}_{2i} y_{2i}}$

$\sum \hat{y}_2 y_1 = \sum(\hat{\pi}_1 x_{1i} + \hat{\pi}_2 x_{2i}) y_{1i}$

$= \hat{\pi}_1 \sum x_{1i} y_{1i} + \hat{\pi}_2 \sum x_{2i} y_{1i}$

$= 3(2) + 4(3) = 18$

$\sum \hat{y}_2 y_2 = \hat{\pi}_1 \sum x_1 y_2 + \hat{\pi}_2 \sum x_2 y_2$

$= 3(3) + 4(4) = 25$

$\hat{\alpha}_1 = \frac{18}{25} = 0.72$

h.

(2SLS)

$y_1 = \alpha_1 \hat{y}_2 + e^* \Rightarrow \hat{\alpha}_1 = \frac{\sum \hat{y}_2 y_1}{\boxed{\sum \hat{y}_2^2}}$ → 証 $\sum \hat{y}_2^2 = \sum \hat{y}_2 y_2$

$\sum \hat{y}_2^2 = \sum \hat{y}_2 (y_2 - \hat{v}_2) = \sum \hat{y}_2 y_2 - \sum \hat{y}_2 \hat{v}_2 = \sum \hat{y}_2 y_2$ → 解釋變數和error正交,相乘=0.

∵ $\sum \hat{y}_2 \hat{v}_2 = \sum(\pi_1 x_1 + \pi_2 x_2)\hat{v}_2 = \pi_1 \sum x_1 \hat{v}_2 + \pi_2 \sum x_2 \hat{v}_2 = 0$

故 $\hat{\alpha}_1 = \frac{\sum \hat{y}_2 y_1}{\sum \hat{y}_2 y_2}$

故(g)的估計和此題的2SLS估計是相等的

11.16 Consider the following supply and demand model

$$\text{Demand: } Q_i = \alpha_1 + \alpha_2 P_i + e_{di}, \quad \text{Supply: } Q_i = \beta_1 + \beta_2 P_i + \beta_3 W_i + e_{si}$$

where Q is the quantity, P is the price, and W is the wage rate, which is assumed exogenous. Data on these variables are in Table 11.7.

TABLE 11.7 Data for Exercise 11.16

Q	P	W
4	2	2
6	4	3
9	3	1
3	5	1
8	8	3

Avg: $30/5=6$ $\quad 22/5=4.4$

a. Derive the algebraic form of the reduced-form equations, $Q = \theta_1 + \theta_2 W + v_2$ and $P = \pi_1 + \pi_2 W + v_1$, expressing the reduced-form parameters in terms of the structural parameters.

b. Which structural parameters can you solve for from the results in part (a)? Which equation is "identified"?

c. The estimated reduced-form equations are $\hat{Q} = 5 + 0.5W$ and $\hat{P} = 2.4 + 1W$. Solve for the identified structural parameters. This is the method of **indirect least squares**.

d. Obtain the fitted values from the reduced-form equation for P, and apply 2SLS to obtain estimates of the demand equation.

(a) $\alpha_1 + \alpha_2 P_i + e_{di} = \beta_1 + \beta_2 P_i + \beta_3 W_i + e_{si}$

$\Rightarrow (\alpha_2 - \beta_2) P_i = \beta_1 - \alpha_1 + \beta_3 W_i - e_{di} + e_{si}$

$$\Rightarrow P_i = \underbrace{\frac{\beta_1 - \alpha_1}{\alpha_2 - \beta_2}}_{\pi_1} + \underbrace{\frac{\beta_3}{\alpha_2 - \beta_2}}_{\pi_2} W_i + \underbrace{\frac{e_{si} - e_{di}}{\alpha_2 - \beta_2}}_{v_i}$$

(b)

$$Q = \alpha_1 + \alpha_2 P_i + e_{di}$$
$$= \alpha_1 + \alpha_2(\pi_1 + \pi_2 W + v_1) + e_{di}$$
$$= \underbrace{\alpha_1 + \alpha_2\pi_1}_{\theta_1} + \underbrace{\alpha_2\pi_2}_{\theta_2} W + \underbrace{\alpha_2 v_1 + e_{di}}_{v_{2i}}$$

$$\theta_1 = \alpha_1 + \alpha_2\pi_1 = \alpha_1 + \alpha_2 \cdot \frac{\beta_1 - \alpha_1}{\alpha_2 - \beta_2} = \frac{\cancel{\alpha_1\alpha_2} - \alpha_1\beta_2 + \alpha_2\beta_1 - \cancel{\alpha_1\alpha_2}}{\alpha_2 - \beta_2}$$
$$= \frac{\alpha_2\beta_1 - \alpha_1\beta_2}{\alpha_2 - \beta_2}$$

$$\theta_2 = \alpha_2\pi_2 = \alpha_2 \cdot \frac{\beta_3}{\alpha_2 - \beta_2}$$

$$v_{2i} = \alpha_2 v_i + e_{di} = \alpha_2 \cdot \frac{e_{si} - \cancel{e_{di}}}{\alpha_2 - \beta_2} + \frac{\cancel{e_{di}\alpha_2} - e_{di}\beta_2}{\alpha_2 - \beta_2}$$
$$= \frac{\alpha_2 e_{si} - \beta_2 e_{di}}{\alpha_2 - \beta_2}$$

(c) $\hat{Q} = 5 + 0.5W \qquad \theta_1 = 5 \quad \theta_2 = 0.5$

$\hat{P} = 2.4 + 1W \qquad \pi_1 = 2.4 \quad \pi_2 = 1$

$$\theta_2 = \alpha_2 \pi_2 \Rightarrow \underline{\underline{\alpha_2}} = \frac{\theta_2}{\pi_2} = \frac{0.5}{1} = \underline{\underline{0.5}}$$

$$\theta_1 = \alpha_1 + \alpha_2 \pi_1 \Rightarrow \underline{\underline{\alpha_1}} = \theta_1 - \alpha_2 \pi_1 = 5 - 0.5(2.4) = \underline{\underline{3.8}}$$

$\therefore Q_i = \alpha_1 + \alpha_2 P_i + e_{di}$

$$\begin{cases} \beta_1 - \alpha_1 = \pi_1(\alpha_2 - \beta_2) \\ \beta_3 = \pi_2(\alpha_2 - \beta_2) \end{cases}$$

$$\beta_1 - 3.8 = 2.4(0.5 - \beta_2) \Rightarrow \beta_1 = 5 - 2.4\beta_2$$

$$\beta_3 = 0.5 - \beta_2$$ (Indirect Least Square)

(d)

$\hat{P} = 2.4 + W \qquad \bar{P} = 4.4 \quad \bar{Q} = 6$

W	$\hat{P}$	$\hat{P} - \bar{P}$	$Q - \bar{Q}$
2	4.4	0	-2
3	5.4	1	0
1	3.4	-1	3
1	3.4	-1	-3
3	5.4	1	2

$Q = \alpha_1 + \alpha_2 \hat{P} + e_i$

$$\hat{\alpha}_2 = \frac{\Sigma(\hat{P}_i - \bar{P}_i)(Q_i - \bar{Q})}{\Sigma(\hat{P}_i - \bar{P}_i)^2} = \frac{1}{2}.$$

$$\hat{\alpha}_1 = \bar{Q} - \hat{\alpha}_2 \bar{P} = 6 - \frac{1}{2}(4.4) = 3.8$$

OLS: $\hat{Q} = 3.8 + 0.5P$

11.17 Example 11.3 introduces Klein's Model I.

a. Do we have an adequate number of IVs to estimate each equation? Check the necessary condition for the identification of each equation. The necessary condition for identification is that in a system of M equations at least $M - 1$ variables must be omitted from each equation.

b. An equivalent identification condition is that the number of excluded exogenous variables from the equation must be at least as large as the number of included right-hand side endogenous variables. Check that this condition is satisfied for each equation.

c. Write down in econometric notation the first-stage equation, the reduced form, for W_{1t}, wages of workers earned in the private sector. Call the parameters $\pi_1, \pi_2, \ldots$

d. Describe the two regression steps of 2SLS estimation of the consumption function. This is not a question about a computer software command.

e. Does following the steps in part (d) produce regression results that are identical to the 2SLS estimates provided by software specifically designed for 2SLS estimation? In particular, will the

The first equation is a consumption function, in which aggregate consumption in year t, CN_t, is related to total wages earned by all workers, W_t. Total wages are divided into wages of workers earned in the private sector, W_{1t}, and wages of workers earned in the public sector, W_{2t}, so that total wages $W_t = W_{1t} + W_{2t}$. Private sector wages W_{1t} are endogenous and determined within the structure of the model, as we will see below. Public sector wages W_{2t} are exogenous. In addition, consumption expenditures are related to nonwage income (profits) in the current year, P_t, which are endogenous, and profits from the previous year, P_{t-1}. Thus, the consumption function is

$$CN_t = \alpha_1 + \alpha_2\left(W_{1t} + W_{2t}\right) + \alpha_3 P_t + \alpha_4 P_{t-1} + e_{1t} \qquad (11.17)$$

include 6

Now refer back to equation (5.44) in Section 5.7.3. There we introduced the term **contemporaneously uncorrelated** to describe the situation in which an explanatory variable observed at time t, x_{tk} is uncorrelated with the random error at time t, e_t. In the terminology of Chapter 10, the variable x_{tk} is **exogenous** if it is contemporaneously uncorrelated with the random error e_t. And the variable x_{tk} is **endogenous** if it is contemporaneously correlated with the random error e_t. In the consumption equation, W_{1t} and P_t are endogenous and contemporaneously correlated with the random error e_t. On the other hand, wages in the public sector, W_{2t}, are set by public authority and are assumed exogenous and uncorrelated with the current period random error e_{1t}. What about profits in the previous year, P_{t-1}? They are **not** correlated with the random error occurring one year later. Lagged endogenous variables are called **predetermined variables** and are treated just like exogenous variables.

The second equation in the model is the investment equation. Net investment, I_t, is specified to be a function of current and lagged profits, P_t and P_{t-1}, as well as the capital stock at the end of the previous year, K_{t-1}. This lagged variable is predetermined and treated as exogenous. The investment equation is

$$I_t = \beta_1 + \beta_2 P_t + \beta_3 P_{t-1} + \beta_4 K_{t-1} + e_{2t} \qquad (11.18)$$

include 5

Finally, there is an equation for wages in the private sector, W_{1t}. Let $E_t = CN_t + I_t + (G_t - W_{2t})$, where G_t is government spending. Consumption and investment are endogenous and government spending and public sector wages are exogenous. The sum, E_t, total national product minus public sector wages, is endogenous. Wages are taken to be related to E_t and the predetermined variable E_{t-1}, plus a time trend variable, $TIME_t = YEAR_t - 1931$, which is exogenous. The wage equation is

$$W_{1t} = \gamma_1 + \gamma_2 E_t + \gamma_3 E_{t-1} + \gamma_4 TIME_t + e_{3t} \qquad (11.19)$$

include 5

Because there are eight endogenous variables in the entire system there must also be eight equations. Any system of M endogenous variables must have M equations to be complete. In addition to the three equations (11.17)–(11.19), which contain five endogenous variables, there are five other definitional equations to complete the system that introduce three further endogenous variables. In total, there are eight exogenous and predetermined variables, which can be used as IVs. The exogenous variables are government spending, G_t, public sector wages, W_{2t}, taxes, TX_t, and the time trend variable, $TIME_t$. Another exogenous variable is the constant term, the "intercept" variable in each equation, $X_{1t} \equiv 1$. The predetermined variables are lagged profits, P_{t-1}, the lagged capital stock, K_{t-1}, and the lagged total national product minus public sector wages, E_{t-1}.

$M - 1 = 8 - 1 = 7$

(a) absent $M-1$

CN_t: 10 > 7

I_t: 11 > 7

W_{1t}: 11 > 7

satisfied → identified

(b) exclude exogenous variables $\geq$ RHS endogenous variables

$5 \geq 2$

$5 \geq 1$

$5 \geq 1$

(c) $W_{1t} = \pi_1 + \pi_2 G_1 + \pi_3 W_{2t} + \pi_4 TX_t + \pi_5 TIME + \pi_6 P_{t-1} + \pi_7 K_{t-1} + \pi_8 E_{t-1} + V$

W_{1t}, P_t are endogenous

(d) 先從(c)式中取得 $\hat{W}_{1t}$ 值，令 $W_{1t}^* = \hat{W}_{1t} + W_{2t}$，再使用相同的方式取得 $\hat{P}_t$

跑 CN 的迴歸式，其中RHS的變數為 $W_t^*, \hat{P}_t, P_{t-1}$
(OLS)

(e) 係數會相同，但t-value會不同