11.1 Our aim is to estimate the parameters of the simultaneous equations model

$$① \quad y_1 = \alpha_1 y_2 + e_1$$

$$② \quad y_2 = \alpha_2 y_1 + \beta_1 x_1 + \beta_2 x_2 + e_2$$

We assume that x_1 and x_2 are exogenous and uncorrelated with the error terms e_1 and e_2.

a. Solve the two structural equations for the reduced-form equation for y_2, that is, $y_2 = \pi_1 x_1 + \pi_2 x_2 + v_2$. Express the reduced-form parameters in terms of the **structural parameters** and the reduced-form error in terms of the structural parameters and e_1 and e_2. Show that y_2 is correlated with e_1.

① 代入 ②

$$\Rightarrow Y_2 = \alpha_2(\alpha_1 Y_2 + e_1) + \beta_1 X_1 + \beta_2 X_2 + e_2$$

$$\Rightarrow Y_2 - \alpha_1\alpha_2 Y_2 = \alpha_2 e_1 + \beta_1 X_1 + \beta_2 X_2 + e_2$$

$$\Rightarrow Y_2 = \frac{\beta_1}{(1-\alpha_1\alpha_2)} X_1 + \frac{\beta_2}{(1-\alpha_1\alpha_2)} X_2 + \frac{e_2 + \alpha_2 e_1}{(1-\alpha_1\alpha_2)} = \pi_1 X_1 + \pi_2 X_2 + V_2$$

※

$$Cov(e_1, Y_2 \mid x) = E\left[(Y_2 - E(Y_2 \mid x))(e_1 - E(e_1 \mid x))\right]$$

$$(\because E(e_1 \mid x) = 0) \Rightarrow = E(Y_2 e_1 \mid x) = E\left[\frac{\beta_1}{1-\alpha_1\alpha_2} X_1 e_1 \mid x\right] + E\left[\frac{\beta_2}{1-\alpha_1\alpha_2} X_2 e_1 \mid x\right] + E\left[\left(\frac{e_2 + \alpha_2 e_1}{1-\alpha_1\alpha_2}\right) e_1 \mid x\right]$$

$$= 0 + 0 + E\left[\frac{e_2 e_1 + \alpha_2 e_1^2}{1-\alpha_1\alpha_2} \mid x\right]$$

$$= \frac{E[e_2 e_1 \mid x] + \alpha_2 E[e_1^2 \mid x]}{1-\alpha_1\alpha_2} = \frac{\alpha_2}{1-\alpha_1\alpha_2}\sigma_1^2 \neq 0 \quad (\text{unless } \alpha_2 = 0)$$

※

b. Which equation parameters are consistently estimated using OLS? Explain.

Neither of the two structural equations is a reduced-form.

Both equations have endogenous variables on the right-hand side,

so using OLS would be inconsistent and biased.

※

c. Which parameters are "identified," in the simultaneous equations sense? Explain your reasoning.

$M = 2$, it must excluded at least $M-1=1$ exogenous variable

in ① $\Rightarrow$ excludes x_1 and x_2 $\Rightarrow$ indentified

in ② $\Rightarrow$ excludes 0 $\Rightarrow$ not identified

$\Rightarrow$ possible to estimate α_1 consistently. ※

d. To estimate the parameters of the reduced-form equation for y_2 using the method of moments (MOM), which was introduced in Section 10.3, the two moment equations are

$$N^{-1}\sum x_{i1}\left(y_2 - \pi_1 x_{i1} - \pi_2 x_{i2}\right) = 0$$

$$N^{-1}\sum x_{i2}\left(y_2 - \pi_1 x_{i1} - \pi_2 x_{i2}\right) = 0$$

Explain why these two moment conditions are a valid basis for obtaining consistent estimators of the reduced-form parameters.

To ensure OLS consistency,

① $N^{-1}\sum X_{i1} V_{i2} = 0 \Leftarrow E(X_{i1}V_{i2} | X) = 0$

② $N^{-1}\sum X_{i2} V_{i2} = 0 \Leftarrow E(X_{i2}V_{i2} | X) = 0$

then OLS is consistent.

$\because$ X_1, X_2 are uncorrelated with e_2

$\Rightarrow$ X_1, X_2 also uncorrelated with the reduced form error term V_2

From the structural system, we derive $Y_2 = \pi_1 X_1 + \pi_2 X_2 + V_2$

where $\pi_1 = \dfrac{\beta_1}{1-\alpha_1\alpha_2}$, $\pi_2 = \dfrac{\beta_2}{1-\alpha_1\alpha_2}$ $V_2 = \dfrac{e_2+\alpha_2 e_1}{1-\alpha_1\alpha_2}$ From (a)

$$\Rightarrow E\left[X_{ik}\left(\frac{e_2+\alpha_2 e_1}{1-\alpha_1\alpha_2}\right) \Big| X\right] = E\left[\frac{1}{1-\alpha_1\alpha_2} X_{ik} e_2 \Big| X\right] + E\left[\frac{\alpha_2}{1-\alpha_1\alpha_2} X_{ik} e_1 \Big| X\right] = 0 + 0 = 0$$

※

e. Are the MOM estimators in part (d) the same as the OLS estimators? Form the sum of squared errors function for $y_2 = \pi_1 x_1 + \pi_2 x_2 + v_2$ and find the first derivatives. Set these to zero and show that they are equivalent to the two equations in part (d).

OLS - SSE $\Rightarrow S(\pi_1, \pi_2 | Y, X) = \sum (Y_2 - \pi_1 X_1 - \pi_2 X_2)^2$

$$\begin{cases} \Rightarrow \dfrac{\partial S}{\partial \pi_1} = 2\sum (Y_2 - \pi_1 X_1 - \pi_2 X_2) X_1 = 0 \\ \Rightarrow \dfrac{\partial S}{\partial \pi_2} = 2\sum (Y_2 - \pi_1 X_1 - \pi_2 X_2) X_2 = 0 \end{cases} \quad \div 2 = (d) \times N$$

f. Using $\sum x_{i1}^2 = 1$, $\sum x_{i2}^2 = 1$, $\sum x_{i1}x_{i2} = 0$, $\sum x_{i1}y_{1i} = 2$, $\sum x_{i1}y_{2i} = 3$, $\sum x_{i2}y_{1i} = 3$, $\sum x_{i2}y_{2i} = 4$, and the two moment conditions in part (d) show that the MOM/OLS estimates of π_1 and π_2 are $\hat{\pi}_1 = 3$ and $\hat{\pi}_2 = 4$.

Two moment conditions :

① $N^{-1} \sum X_{i1} (Y_2 - \pi_1 X_{i1} - \pi_2 X_{i2}) = 0$

② $N^{-1} \sum X_{i2} (Y_2 - \pi_1 X_{i1} - \pi_2 X_{i2}) = 0$

$\sum X_{i1} Y_{2i} = \pi_1 \sum X_{i1}^2 + \pi_2 \sum X_{i1} X_{i2} \Rightarrow \hat{\pi}_1 = 3$

$\sum X_{i2} Y_{2i} = \pi_1 \sum X_{i1} X_{i2} + \pi_2 \sum X_{i2}^2 \Rightarrow \hat{\pi}_2 = 4$

g. The fitted value $\hat{y}_2 = \hat{\pi}_1 x_1 + \hat{\pi}_2 x_2$. Explain why we can use the moment condition $\sum \hat{y}_{i2}(y_{i1} - \alpha_1 y_{i2}) = 0$ as a valid basis for consistently estimating α_1. Obtain the IV estimate of α_1.

To estimate the structural equation $Y_1 = \alpha_1 Y_2 + e_1$, we use 2SLS. Since $\hat{Y}_2$ is generated from exogenous variables X_1 and X_2, it is uncorrelated with the structural error term e_1. This satisfies the orthogonality condition required for consistent condition.

Use the moment condition $\sum \hat{Y}_{2i} (Y_i - \alpha_1 \hat{Y}_{2i}) = 0 \Rightarrow \alpha_1 = \dfrac{\sum \hat{Y}_{2i} Y_i}{\sum \hat{Y}_{2i}^2}$

Substituting $\hat{Y}_{2i} = \hat{\pi}_1 X_{i1} + \hat{\pi}_2 X_{i2}$, $\alpha_1 = \dfrac{\hat{\pi}_1 \sum X_{i1} Y_{1i} + \hat{\pi}_2 \sum X_{i2} Y_{1i}}{\hat{\pi}_1 \sum X_{i1} \hat{Y}_{2i} + \hat{\pi}_2 \sum X_{i2} \hat{Y}_{2i}} = \dfrac{3\times2+4\times3}{3\times3+4\times4} = \dfrac{18}{25}$

h. Find the 2SLS estimate of α_1 by applying OLS to $y_1 = \alpha_1 \hat{y}_2 + e_1^*$. Compare your answer to that in part (g).

$$\hat{\alpha}_{1,\,2SLS} = \frac{\sum \hat{y}_{i2} y_{i1}}{\sum \hat{y}_{i2}^2}$$

$$\sum \hat{y}_2^2 = \sum \hat{y}_2 (y_2 - \hat{v}_2) = \sum \hat{y}_2 y_2 - \cancel{\sum \hat{y}_2 \hat{v}_2}^{\,0} = \sum \hat{y}_2 y_2$$

$\Rightarrow$ 2SLS estimate of $\hat{\alpha}_1 = \hat{\alpha}$ (by moment condition) ☆

11.16 Consider the following supply and demand model

$$\text{Demand: } Q_i = \alpha_1 + \alpha_2 P_i + e_{di}, \quad \text{Supply: } Q_i = \beta_1 + \beta_2 P_i + \beta_3 W_i + e_{si}$$

where Q is the quantity, P is the price, and W is the wage rate, which is assumed exogenous. Data on these variables are in Table 11.7.

TABLE 11.7 Data for Exercise 11.16

Q	P	W
4	2	2
6	4	3
9	3	1
3	5	1
8	8	3

a. Derive the algebraic form of the reduced-form equations, $Q = \theta_1 + \theta_2 W + v_2$ and $P = \pi_1 + \pi_2 W + v_1$, expressing the reduced-form parameters in terms of the structural parameters.

$$\alpha_1 + \alpha_2 P_i + e_{di} = \beta_1 + \beta_2 P_i + \beta_3 W_i + e_{si}$$

$$\Rightarrow P_i = \frac{\beta_1 - \alpha_1}{\alpha_2 - \beta_2} + \frac{\beta_3}{\alpha_2 - \beta_2} W_i + \frac{e_{si} - e_{di}}{\alpha_2 - \beta_2} = \pi_1 + \pi_2 W + V_1$$

$$\pi_1 = \frac{\beta_1 - \alpha_1}{\alpha_2 - \beta_2}, \quad \pi_2 = \frac{\beta_3}{\alpha_2 - \beta_2}, \quad V_1 = \frac{e_{si} - e_{di}}{\alpha_2 - \beta_2}$$

$$Q_i = \alpha_1 + \alpha_2 (\pi_1 + \pi_2 W + V_1) + e_{di} = \frac{\alpha_2 \beta_1 - \alpha_1 \beta_2}{\alpha_2 - \beta_2} + \frac{\alpha_2 \beta_3}{\alpha_2 - \beta_2} W + \frac{\alpha_2 e_{si} - \beta_2 e_{di}}{\alpha_2 - \beta_2}$$

$$= \theta_1 + \theta_2 W + V_2$$

where $\theta_1 = \dfrac{\alpha_2 \beta_1 - \alpha_1 \beta_2}{\alpha_2 - \beta_2}$, $\theta_2 = \dfrac{\alpha_2 \beta_3}{\alpha_2 - \beta_2}$, $V_2 = \dfrac{\alpha_2 e_{si} - \beta_2 e_{di}}{\alpha_2 - \beta_2}$

b. Which structural parameters can you solve for from the results in part (a)? Which equation is "identified"?

① α_1, α_2

② $M=2, \quad M-1=1$

Demand $\Rightarrow$ identified (1)

Supply $\Rightarrow$ not identified (0)

c. The estimated reduced-form equations are $\hat{Q} = 5 + 0.5W$ and $\hat{P} = 2.4 + 1W$. Solve for the identified structural parameters. This is the method of **indirect least squares**.

$\hat{P} = 2.4 + W = \pi_1 + \pi_2 w \qquad \hat{\pi}_1 = 2.4, \quad \hat{\pi}_2 = 1$

$\hat{Q} = 5 + 0.5W = \theta_1 + \theta_2 W \qquad \hat{\theta}_1 = 5 \qquad \hat{\theta}_2 = 0.5$

$\Rightarrow \quad 5 + 0.5W = \alpha_1 + \alpha_2(2.4 + W) = \alpha_1 + 2.4\alpha_2 + \alpha_2 W$

$$\Rightarrow \begin{cases} 5 = \alpha_1 + 2.4\alpha_2 \\ \alpha_2 = 0.5 \end{cases}$$

$\Rightarrow \quad \alpha_1 = 3.8, \quad \alpha_2 = 0.5$

d. Obtain the fitted values from the reduced-form equation for P, and apply 2SLS to obtain estimates of the demand equation.

$\hat{P}_i = 2.4 + W_i$

$Q_i = \alpha_1 + \alpha_2 \hat{P}_i + e_i$

$$\alpha_2 = \frac{\Sigma(\hat{P}_i - \bar{\hat{P}})(Q_i - \bar{Q})}{\Sigma(\hat{P}_i - \bar{\hat{P}})^2} = \frac{2}{4} = 0.5$$

$\bar{Q}_i = \alpha_1 + 0.5\bar{\hat{P}}_i, \qquad \alpha_1 = 6 - 0.5 \times 0.44 = 3.8$

$\Rightarrow \hat{Q} = 3.8 + 0.5\hat{P}$