11.1

Our aim is to estimate the parameters of the simultaneous equations model

$$y_1 = \alpha_1 y_2 + e_1$$
$$y_2 = \alpha_2 y_1 + \beta_1 x_1 + \beta_2 x_2 + e_2$$

We assume that x_1 and x_2 are exogenous and uncorrelated with the error terms e_1 and e_2.

a. Solve the two structural equations for the reduced-form equation for y_2, that is, $y_2 = \pi_1 x_1 + \pi_2 x_2 + v_2$. Express the reduced-form parameters in terms of the <u>structural parameters</u> and the reduced-form error in terms of the structural parameters and e_1 and e_2. Show that y_2 is correlated with e_1.

Ans.

$(1) \quad y_1 = \alpha_1 y_2 + e_1 \quad ; \quad (2) \quad y_2 = \alpha_2 y_1 + \beta_1 x_1 + \beta_2 x_2 + e_2$

將 (1) 代入 (2)，可得出：

$$\begin{aligned} y_2 &= \alpha_2 y_1 + \beta_1 x_1 + \beta_2 x_2 + e_2 \\ &= \alpha_2(\alpha_1 y_2 + e_1) + \beta_1 x_1 + \beta_2 x_2 + e_2 \\ &= \alpha_1\alpha_2 y_2 + \alpha_2 e_1 + \beta_1 x_1 + \beta_2 x_2 + e_2 \end{aligned}$$

整理：

$$(1 - \alpha_1\alpha_2) y_2 = \beta_1 x_1 + \beta_2 x_2 + \alpha_2 e_1 + e_2$$

因此，y_2 的 reduced-form 為：

$$y_2 = \frac{\beta_1}{1-\alpha_1\alpha_2} x_1 + \frac{\beta_2}{1-\alpha_1\alpha_2} x_2 + \frac{\alpha_2}{1-\alpha_1\alpha_2} e_1 + \frac{1}{1-\alpha_1\alpha_2} e_2 = \pi_1 x_1 + \pi_2 x_2 + v_2$$

其中：

$$\pi_1 = \frac{\beta_1}{1-\alpha_1\alpha_2}, \qquad \pi_2 = \frac{\beta_2}{1-\alpha_1\alpha_2}, \qquad v_2 = \frac{\alpha_2}{1-\alpha_1\alpha_2} e_1 + \frac{1}{1-\alpha_1\alpha_2} e_2$$

因為 v_2 中含有 e_1，且 α_2 不為零（假設模型有意義）而 e_1 是 y_1 的誤差項，所以 y_2 與 e_1 有相關性，y_2 是內生變數（endogenous variable）。

y_2 與 e_1 的條件共變異數不為 0：

$$\begin{aligned} \text{cov}(y_2, e_1 \mid \mathbf{x}) &= E(y_2 e_1 \mid \mathbf{x}) \\ &= E\left[\left(\pi_1 x_1 + \pi_2 x_2 + \frac{e_2 + \alpha_2 e_1}{1-\alpha_1\alpha_2}\right) e_1 \middle| \mathbf{x}\right] \\ &= \pi_1 x_1 E(e_1 \mid \mathbf{x}) + \pi_2 x_2 E(e_1 \mid \mathbf{x}) + \frac{E(e_2 e_1 \mid \mathbf{x}) + \alpha_2 E(e_1^2 \mid \mathbf{x})}{1-\alpha_1\alpha_2} \\ &= 0 + 0 + \frac{0 + \alpha_2\sigma_1^2}{1-\alpha_1\alpha_2} \\ &= \frac{\alpha_2\sigma_1^2}{1-\alpha_1\alpha_2} \end{aligned}$$

只要 $\alpha_2 \neq 0$，上式即不為 0，表示 y_2 與 e_1 有相關性，存在內生性問題。

b. Which equation parameters are consistently estimated using OLS? Explain.

Ans.

在這個模型中，兩條方程式都無法直接用 OLS 得到一致估計值。

第一個方程式 $y_1 = \alpha_1 y_2 + e_1$ 中的解釋變數 y_2 與誤差項 e_1 相關，因此 OLS 無法一致估計 α_1。

第二個方程式 $y_2 = \alpha_2 y_1 + \beta_1 x_1 + \beta_2 x_2 + e_2$ 中解釋變數 y_1 也受 y_2 影響（ y_1 是內生的），因此 OLS 也無法一致估計 α_2。

c. Which parameters are "identified," in the simultaneous equations sense? Explain your reasoning.

Ans.

y_1 和 y_2 各有一條方程式，共有 M = 2 個方程式，因此每個方程式要被識別，至少需要有 M−1 = 1 個外生變數被排除。

方程式 (1) 中，外生變數 x_1 和 x_2 都被排除。排除的外生變數數量為 2，大於所需的 M−1 = 1，因此方程式 (1) "identified"。

方程式 (2) 中，所有外生變數 x_1 和 x_2 都被包含，沒有任何外生變數被排除。排除的外生變數數量為 0，小於所需的 M−1 = 1，因此方程式 (2) "not identified"。

結論：方程式 (1) 中的參數 α_1 是 已識別的，可以一致估計。方程式 (2) 中的參數 $\alpha_2, \beta_1, \beta_2$ 是 未識別的，無法一致估計。

d. To estimate the parameters of the reduced-form equation for y_2 using the method of moments (MOM), which was introduced in Section 10.3, the two moment equations are

$$N^{-1} \sum x_{i1}(y_2 - \pi_1 x_{i1} - \pi_2 x_{i2}) = 0$$

$$N^{-1} \sum x_{i2}(y_2 - \pi_1 x_{i1} - \pi_2 x_{i2}) = 0$$

Explain why these two moment conditions are a valid basis for obtaining consistent estimators of the reduced-form parameters.

Ans.

x_1 和 x_2 是外生的，與模型中的隨機誤差項 v_2 不相關，所以條件期望值為 0：$E(x_{i1}v_{i2} \mid \mathbf{x}) = E(x_{i2}v_{i2} \mid \mathbf{x}) = 0$

由 (a)：

$$y_2 = \frac{\beta_1}{1-\alpha_1\alpha_2}x_1 + \frac{\beta_2}{1-\alpha_1\alpha_2}x_2 + \frac{e_2+\alpha_2 e_1}{1-\alpha_1\alpha_2} = \pi_1 x_1 + \pi_2 x_2 + v_2\text{，其中 } v_2 = \frac{e_2+\alpha_2 e_1}{1-\alpha_1\alpha_2}\text{。}$$

將 $v_2 = \dfrac{e_2+\alpha_2 e_1}{1-\alpha_1\alpha_2}$ 代入條件期望值：

$$\begin{aligned}
E\left[x_{ik}v_2 \mid \mathbf{x}\right] &= E\left[x_{ik}\cdot\left(\frac{e_2+\alpha_2 e_1}{1-\alpha_1\alpha_2}\right)\middle| \mathbf{x}\right] \\
&= E\left[\frac{1}{1-\alpha_1\alpha_2}x_{ik}e_2 \middle| \mathbf{x}\right] + E\left[\frac{\alpha_2}{1-\alpha_1\alpha_2}x_{ik}e_1 \middle| \mathbf{x}\right] \\
&= \frac{1}{1-\alpha_1\alpha_2}E(x_{ik}e_2 \mid \mathbf{x}) + \frac{\alpha_2}{1-\alpha_1\alpha_2}E(x_{ik}e_1 \mid \mathbf{x}) \\
&= 0 + 0 = 0
\end{aligned}$$

因為 x_{ik} 是外生的，與 e_1 和 e_2 不相關，所以 $E(x_{ik}e_2 \mid \mathbf{x}) = 0$ 、 $E(x_{ik}e_1 \mid \mathbf{x}) = 0$。

當樣本量趨於無限大時，這些樣本矩條件會收斂到對應的總體矩條件 $E(x_1 v_2 \mid \mathbf{x}) = E(x_2 v_2 \mid \mathbf{x}) = 0$，使得矩條件成立，可以正確估計 reduced-form 的參數 π_1, π_2。

e. Are the MOM estimators in part (d) the same as the OLS estimators? Form the sum of squared errors function for $y_2 = \pi_1 x_1 + \pi_2 x_2 + v_2$ and find the first derivatives. Set these to zero and show that they are equivalent to the two equations in part (d).

Ans.

reduced–form 方程式：$y_2 = \pi_1 x_1 + \pi_2 x_2 + v_2$

OLS 的目標是最小化誤差平方和（sum of squared errors）：$S(\pi_1,\pi_2) = \sum_{i=1}^{N}(y_{2i} - \pi_1 x_{i1} - \pi_2 x_{i2})^2$

對 π_1, π_2 分別取一階導數，並令導數為零以求解：

$$\frac{\partial S}{\partial \pi_1} = -2\sum_{i=1}^{N} x_{i1}(y_{2i} - \pi_1 x_{i1} - \pi_2 x_{i2}) = 0 \quad ; \quad \frac{\partial S}{\partial \pi_2} = -2\sum_{i=1}^{N} x_{i2}(y_{2i} - \pi_1 x_{i1} - \pi_2 x_{i2}) = 0$$

兩式除以 –2 後，得到的條件如下：

$$\sum_{i=1}^{N} x_{i1}(y_{2i} - \pi_1 x_{i1} - \pi_2 x_{i2}) = 0 \quad ; \quad \sum_{i=1}^{N} x_{i2}(y_{2i} - \pi_1 x_{i1} - \pi_2 x_{i2}) = 0$$

比較第 (d) 小題中 method of moments（MOM）所使用的兩個矩條件，除了常數項 N^{-1} 外，OLS 的一階條件與 MOM 的矩條件完全相同。由於常數項不影響等式為零的條件，因此這兩組條件是等價的。

f. Using $\sum x_{i1}^2 = 1,\ \sum x_{i2}^2 = 1,\ \sum x_{i1}x_{i2} = 0,\ \sum x_{i1}y_{1i} = 2,\ \sum x_{i1}y_{2i} = 3,\ \sum x_{i2}y_{1i} = 3,\ \sum x_{i2}y_{2i} = 4$, and the two moment conditions in part (d) show that the MOM/OLS estimates of π_1 and π_2 are $\hat{\pi}_1 = 3$ and $\hat{\pi}_2 = 4$.

Ans.

展開第一個條件 $\sum_{i=1}^{N} x_{i1}(y_{2i} - \pi_1 x_{i1} - \pi_2 x_{i2}) = 0$：$\sum x_{i1}y_{2i} - \pi_1 \sum x_{i1}^2 - \pi_2 \sum x_{i1}x_{i2} = 0$

代入數值得：$3 - \hat{\pi}_1 \cdot 1 - \hat{\pi}_2 \cdot 0 = 0$，可解出：$\hat{\pi}_1 = 3$

展開第二個條件 $\sum_{i=1}^{N} x_{i2}(y_{2i} - \pi_1 x_{i1} - \pi_2 x_{i2}) = 0$：$\sum x_{i2}y_{2i} - \pi_1 \sum x_{i1}x_{i2} - \pi_2 \sum x_{i2}^2 = 0$

代入數值得：$4 - \hat{\pi}_1 \cdot 0 - \hat{\pi}_2 \cdot 1 = 0$，可解出：$\hat{\pi}_2 = 4$

因此，我們得到 MOM（或 OLS）估計值為：$\hat{\pi}_1 = 3$、$\hat{\pi}_2 = 4$

g. The fitted value $\hat{y}_2 = \hat{\pi}_1 x_1 + \hat{\pi}_2 x_2$. Explain why we can use the moment condition $\sum \hat{y}_{i2}(y_{i1} - \alpha_1 y_{i2}) = 0$ as a valid basis for consistently estimating α_1. Obtain the IV estimate of α_1.

Ans.

結構方程式 $y_1 = \alpha_1 y_2 + e_1$，但 y_2 是內生變數（見 11.1a），不能直接用 OLS。

不過，$\hat{y}_2 = \hat{\pi}_1 x_1 + \hat{\pi}_2 x_2$ 是 y_2 的外生預測值（reduced–form 預測值），僅由 x_1 和 x_2 組成，與 e_1 不相關。

因此，可以把 $\hat{y}_2$ 當作工具變數來估計 α_1。

此時的矩條件為：$\sum_{i=1}^{N} \hat{y}_{i2}(y_{i1} - \alpha_1 y_{i2}) = 0$ 　整理可得：$\sum \hat{y}_{i2}y_{i1} - \alpha_1 \sum \hat{y}_{i2}^2 = 0 \ \Rightarrow\ \hat{\alpha}_1 = \dfrac{\sum \hat{y}_{i2}y_{i1}}{\sum \hat{y}_{i2}y_{i2}}$

代入值：$\hat{y}_2 = \hat{\pi}_1 x_1 + \hat{\pi}_2 x_2 = 3x_1 + 4x_2,\ \sum x_{i1}y_{1i} = 2,\ \sum x_{i2}y_{1i} = 3,\ \sum x_{i1}y_{2i} = 3,\ \sum x_{i2}y_{2i} = 4,$

計算分子：$\sum \hat{y}_{i2}y_{i1} = \sum (3x_{i1} + 4x_{i2})y_{1i} = 3\sum x_{i1}y_{1i} + 4\sum x_{i2}y_{1i} = 3 \cdot 2 + 4 \cdot 3 = 18$

計算分母：$\sum \hat{y}_{i2}y_{i2} = \sum (3x_{i1} + 4x_{i2})y_{i2} = 3\sum x_{i1}y_{i2} + 4\sum x_{i2}y_{i2} = 3 \cdot 3 + 4 \cdot 4 = 25$

因此，IV 估計量為：$\hat{\alpha}_1 = \dfrac{\sum \hat{y}_{i2}y_{i1}}{\sum \hat{y}_{i2}y_{i2}} = \dfrac{18}{25} = 0.72$

h. Find the 2SLS estimate of α_1 by applying OLS to $y_1 = \alpha_1 \hat{y}_2 + e_1^*$. Compare your answer to that in part (g).

Ans.

在一個 沒有截距 的簡單迴歸模型中，OLS 估計量為：$\hat{\alpha}_1 = \dfrac{\sum x_i y_i}{\sum x_i^2}$

在 2SLS 設定下，將 x 換成 $\hat{y}_2$、y 換成 y_1 ，得到：$\hat{\alpha}_{1,\text{2SLS}} = \dfrac{\sum \hat{y}_{i2} y_{i1}}{\sum \hat{y}_{i2}^2}$

使用 $\hat{y}_{i2}$ 作為工具變量（instrumental variable, IV），因為 $\hat{y}_{i2}$ 是由外生變量 x_{i1} 和 x_{i2} 構成的，與模型的誤差無關。這樣可以解決同時方程模型中內生性問題，得到一致的 α_1 估計值。

證明等價性：

目標：證明 $\hat{\alpha}_{1,\text{2SLS}} = \dfrac{\sum \hat{y}_{i2} y_{i1}}{\sum \hat{y}_{i2}^2} = \hat{\alpha}_{1,IV} = \dfrac{\sum \hat{y}_{i2} y_{i1}}{\sum \hat{y}_{i2} y_{i2}}$，因此需證明 $\sum \hat{y}_{i2}^2 = \sum \hat{y}_{i2} y_{i2}$

由於殘差為：$\hat{v}_2 = y_2 - \hat{y}_2 \quad \Rightarrow \quad \hat{y}_2 = y_2 - \hat{v}_2$

代入 $\hat{y}_2 = y_2 - \hat{v}_2$，得出 分母：$\sum \hat{y}_{i2}^2 = \sum \hat{y}_{i2}(y_{i2} - \hat{v}_{i2}) = \sum \hat{y}_{i2} y_{i2} - \sum \hat{y}_{i2} \hat{v}_{i2}$

由於 $\sum \hat{y}_{i2} \hat{v}_{i2} = \sum (\hat{\pi}_1 x_{i1} + \hat{\pi}_2 x_{i2}) \hat{v}_{i2} = \hat{\pi}_1 \sum x_{i1} \hat{v}_{i2} + \hat{\pi}_2 \sum x_{i2} \hat{v}_{i2} = 0$

因 OLS 的基本性質：回歸變數 x_{i1}, x_{i2} 與殘差 $\hat{v}_{i2}$ 正交（即：$\sum x_{i1} \hat{v}_{i2} = 0, \ \sum x_{i2} \hat{v}_{i2} = 0$）

因此 分母：$\sum \hat{y}_{i2}^2 = \sum \hat{y}_{i2} y_{i2} - \sum \hat{y}_{i2} \hat{v}_{i2} = \sum \hat{y}_{i2} y_{i2} - 0 = \sum \hat{y}_{i2} y_{i2}$

故得證 $\hat{\alpha}_{1,\text{2SLS}} = \dfrac{\sum \hat{y}_{i2} y_{i1}}{\sum \hat{y}_{i2}^2} = \dfrac{\sum \hat{y}_{i2} y_{i1}}{\sum \hat{y}_{i2} y_{i2}} = \hat{\alpha}_{1,IV}$

可知在給定的模型設定和使用的工具變數下，2SLS和IV方法在數學上是等價的。