Recall $Y \sim N(X\beta, \sigma^2 I)$. Because $b = (X'X)^{-1}X'Y$ is an affine transformation for Y and hence a normal distribution.

The expectation:

$$E(b) = E((X'X)^{-1}X'Y) = E((X'X)^{-1}X'(X\beta + e)) = \beta + 0 = \beta,$$

The variance:

$$\begin{aligned} \operatorname{var}(b) &= \operatorname{var}\left((X'X)^{-1}X'Y\right) \\ &= (X'X)^{-1}X'\operatorname{var}(Y)((X'X)^{-1}X')' \\ &= ((X'X)^{-1}X')(\sigma^2 I)X(X'X)^{-1} \\ &= \sigma^2(X'X)^{-1}. \end{aligned}$$

As a result, we prove

$$b \sim N(\beta, \sigma^2(X'X)^{-1}),$$

Q2: Let $K=2$, show that cov(b1, b2) in p. 30 of slides in Ch 5 reduces to the formula in (2.14) - (2.16).

$$X = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_N \end{bmatrix}, \quad X'X = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_N \end{bmatrix} \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_N \end{bmatrix} = \begin{bmatrix} N & \Sigma x_i \\ \Sigma x_i & \Sigma x_i^2 \end{bmatrix}$$

$$(X'X)^{-1} = \frac{1}{N\Sigma x_i^2 - (\Sigma x_i)^2} \begin{bmatrix} \Sigma x_i^2 & -\Sigma x_i \\ -\Sigma x_i & N \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\underbrace{ad-bc}_{\text{行列式}}} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\operatorname{var}(b) = \sigma^2 (X'X)^{-1} = \sigma^2 \cdot \frac{1}{N\Sigma x_i^2 - (\Sigma x_i)^2} \begin{bmatrix} \Sigma x_i^2 & -\Sigma x_i \\ -\Sigma x_i & N \end{bmatrix}$$

$$\operatorname{var}(b_1) = \sigma^2 \cdot \frac{\Sigma x_i^2}{N\Sigma x_i^2 - (\Sigma x_i)^2}, \quad \operatorname{var}(b_2) = \sigma^2 \cdot \frac{1}{\Sigma x_i^2 - (\Sigma x_i)^2}$$

$$\operatorname{cov}(b_1, b_2) = \sigma^2 \cdot \frac{-\Sigma x_i}{N\Sigma x_i^2 - (\Sigma x_i)^2} = \sigma^2 \cdot \frac{-N\bar{x}}{N\Sigma x_i^2 - (N\bar{x})^2} = \sigma^2 \cdot \frac{-\bar{x}}{\Sigma x_i^2 - N\bar{x}^2}$$

Equation (2.16): $\operatorname{cov}(b_1, b_2) = \sigma^2 \cdot \dfrac{-\bar{x}}{\Sigma (x_i - \bar{x})^2}$

Equation (2.16) 分母: $\Sigma (x_i - \bar{x})^2 = \Sigma (x_i^2 - 2x_i\bar{x} + \bar{x}^2) = \Sigma x_i^2 - 2\bar{x}\Sigma x_i + N\bar{x}^2$

$$= \Sigma x_i^2 - 2\bar{x}(N\bar{x}) + N\bar{x}^2 = \Sigma x_i^2 - N\bar{x}^2$$

∴ Equation (2.16) 可改寫成:

$$\operatorname{cov}(b_1, b_2) = \sigma^2 \cdot \frac{-\bar{x}}{\Sigma (x_i - \bar{x})^2} = \sigma^2 \cdot \frac{-\bar{x}}{\Sigma x_i^2 - N\bar{x}^2}$$ 故得證

同理, Equation (2.14)(2.15) 可改寫成:

$$\operatorname{var}(b_1) = \sigma^2 \frac{\Sigma x_i^2}{N\Sigma (x_i - \bar{x})^2} = \sigma^2 \frac{\Sigma x_i^2}{\Sigma x_i^2 - N\bar{x}^2}$$ 故得證

$$\operatorname{var}(b_2) = \frac{\sigma^2}{N\Sigma (x_i - \bar{x})^2} = \frac{\sigma^2}{\Sigma x_i^2 - N\bar{x}^2}$$ 故得證

$$\operatorname{var}(b_1|\mathbf{x}) = \sigma^2 \left[\frac{\sum x_i^2}{N\sum (x_i - \bar{x})^2}\right] \tag{2.14}$$

$$\operatorname{var}(b_2|\mathbf{x}) = \frac{\sigma^2}{\sum (x_i - \bar{x})^2} \tag{2.15}$$

$$\operatorname{cov}(b_1, b_2|\mathbf{x}) = \sigma^2 \left[\frac{-\bar{x}}{\sum (x_i - \bar{x})^2}\right] \tag{2.16}$$