5.6 Suppose that, from a sample of 63 observations, the least squares estimates and the corresponding estimated covariance matrix are given by

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} \qquad \widehat{\operatorname{cov}}(b_1, b_2, b_3) = \begin{bmatrix} 3 & -2 & 1 \\ -2 & 4 & 0 \\ 1 & 0 & 3 \end{bmatrix}$$

Using a 5% significance level, and an alternative hypothesis that the equality does not hold, test each of the following null hypotheses:

a. $\beta_2 = 0$
b. $\beta_1 + 2\beta_2 = 5$
c. $\beta_1 - \beta_2 + \beta_3 = 4$

a. null hypothesis: $H_0: \beta_2 = 0$

alternative hypothesis $H_1: \beta_2 \neq 0$

$$t = \frac{b_2 - 0}{se(b_2)} = \frac{3}{\sqrt{4}} = 1.5 < 2.0003 = qt(0.975, 60)$$

$\Rightarrow$ fail to reject H_0.

b. null hypothesis: $\beta_1 + 2\beta_2 = 5$

alternative hypothesis: $\beta_1 + 2\beta_2 \neq 5$

$b_1 + 2b_2 = 2 + 2 \times 3 = 8$

$$se(b_1 + 2b_2) = \sqrt{var(b_1) + 4\,var(b_2) + 2 \times 1 \times 2 \times cov(b_1, b_2)} = \sqrt{3 + 4 \times 4 + 4 \times (-2)} = \sqrt{11}$$

$$t = \frac{(b_1 + 2b_2) - 5}{se(b_1 + 2b_2)} = \frac{8-5}{\sqrt{11}} = \frac{3}{\sqrt{11}} = 0.905 < 2.0003$$

$\Rightarrow$ fail to reject H_0

c. null hypothesis: $H_0: \beta_1 - \beta_2 + \beta_3 = 4$

alternative hypothesis: $H_1\ \beta_1 - \beta_2 + \beta_3 \neq 4$.

$$se(b_1-b_2+b_3) = \sqrt{var(b_1)+Var(b_2)+Var(b_3)-2Cov(b_1,b_2)-2Cov(b_2,b_3)+2Cov(b_1,b_3)}$$

$$= \sqrt{3+4+3-2\times(-2)-2\times0+2\times1} = \sqrt{16} = 4$$

$$t = \frac{-2-4}{4} = -1.5 > -2 = qt(0.025, 60)$$

$\Rightarrow$ fail to reject H_0.

5.31 Each morning between 6:30 AM and 8:00 AM Bill leaves the Melbourne suburb of Carnegie to drive to work at the University of Melbourne. The time it takes Bill to drive to work (*TIME*), depends on the departure time (*DEPART*), the number of red lights that he encounters (*REDS*), and the number of trains that he has to wait for at the Murrumbeena level crossing (*TRAINS*). Observations on these variables for the 249 working days in 2015 appear in the file *commute5*. *TIME* is measured in minutes. *DEPART* is the number of minutes after 6:30 AM that Bill departs.

a. Estimate the equation

$$TIME = \beta_1 + \beta_2 DEPART + \beta_3 REDS + \beta_4 TRAINS + e$$

Report the results and interpret each of the coefficient estimates, including the intercept β_1.

b. Find 95% interval estimates for each of the coefficients. Have you obtained precise estimates of each of the coefficients?

c. Using a 5% significance level, test the null hypothesis that Bill's expected delay from each red light is 2 minutes or more against the alternative that it is less than 2 minutes.

d. Using a 10% significance level, test the null hypothesis that the expected delay from each train is 3 minutes against the alternative that it is not 3 minutes.

e. Using a 5% significance level, test the null hypothesis that Bill can expect a trip to be at least 10 minutes longer if he leaves at 7:30 AM instead of 7:00 AM, against the alternative that it will not be 10 minutes longer. (Assume other things are equal.)

f. Using a 5% significance level, test the null hypothesis that the expected delay from a train is at least three times greater than the expected delay from a red light against the alternative that it is less than three times greater.

g. Suppose that Bill encounters six red lights and one train. Using a 5% significance level, test the null hypothesis that leaving Carnegie at 7:00 AM is early enough to get him to the university on or before 7:45 AM against the alternative that it is not. [Carry out the test in terms of the expected time $E(TIME|\mathbf{X})$ where $\mathbf{X}$ represents the observations on all explanatory variables.]

h. Suppose that, in part (g), it is imperative that Bill is not late for his 7:45 AM meeting. Have the null and alternative hypotheses been set up correctly? What happens if these hypotheses are reversed?

(a)

```
Call:
lm(formula = time ~ depart + reds + trains, data = commute5)

Residuals:
     Min       1Q   Median       3Q      Max
-18.4389  -3.6774  -0.1188   4.5863  16.4986

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)  20.8701     1.6758  12.454  < 2e-16 ***
depart        0.3681     0.0351  10.487  < 2e-16 ***
reds          1.5219     0.1850   8.225 1.15e-14 ***
trains        3.0237     0.6340   4.769 3.18e-06 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 6.299 on 245 degrees of freedom
Multiple R-squared:  0.5346,    Adjusted R-squared:  0.5289
F-statistic: 93.79 on 3 and 245 DF,  p-value: < 2.2e-16
```

Holding other variables constant, for every minute Bill departs later than 6:30, Time increase by 0.3681 minutes.

Holding other variables constant, one more red lights that he encounters, Time increase by 1.5219 minutes.

Holding other variables constant, one more train he has to wait, Time increase by 3.0237 minutes.

If he depart at 6:30, doesn't encounter any red light and doesn't have to wait any train, he will spend 20.8701 minutes to drive to work.

b. Yes, we have obtained precise estimates of each of the coefficients.

```
> #5.31.b
> confint(table, level = 0.95)
                 2.5 %    97.5 %
(Intercept) 17.5694018 24.170871
depart       0.2989851  0.437265
reds         1.1574748  1.886411
trains       1.7748867  4.272505
```

however the trains has a relatively wide confidence interval, suggesting greater uncertainty in its estimate.

c.

$$\begin{cases} H_0: \beta_3 \geq 2 \\ H_1: \beta_3 < 2 \end{cases} \qquad t = \frac{1.5219 - 2}{0.1850} = -2.5843 < -1.6511$$

reject the null hypothesis that the time expected delay from each red light is 2 minutes or more.

d.

$$\begin{cases} H_0: \beta_4 = 3 \\ H_1: \beta_4 \neq 3 \end{cases} \qquad t = \frac{3.0237 - 3}{0.6340} = 0.0374 < 1.6511$$

we fail to reject H0 that the time delay from waiting each train is 3 minutes.

e.

$$\begin{cases} H_0: 60\beta_2 - 30\beta_2 \geq 10 \\ H_1: 60\beta_2 - 30\beta_2 < 10 \end{cases} \rightarrow \begin{cases} H_0: \beta_2 \geq \frac{1}{3} \\ H_0: \beta_2 < \frac{1}{3} \end{cases} \qquad t = \frac{0.3681 - \frac{1}{3}}{0.0351} = 0.99057 > -1.6511$$

→ we fail to reject H0. that the time expected delay at least 10 minutes longer if he leaves at 7:30 AM instead of 7:00 AM.

f.

$$\begin{cases} H_0: \beta_4 \geq 3\beta_3 \\ H_1: \beta_4 < 3\beta_3 \end{cases} \qquad t = \frac{3.0237 - 3 \times 1.5219}{0.844992} = -1.8249 < -1.651$$

reject H0, so the expected time delay from train is less than three times greater than waiting from a red light.

g. $\begin{cases} H_0: \beta_1 + 30\beta_2 + 6\beta_3 + \beta_4 \le 45 \\ H_1: \beta_1 + 30\beta_2 + 6\beta_3 + \beta_4 > 45 \end{cases}$

$$t = \frac{44.062924 - 45}{0.539268} = -1.926 < 1.6511$$

$\Rightarrow$ reject H_0 that Bill encounters six red lights and one train and leaving Carnegie at 7:00 AM is early enough to arrive university at 7:45 AM.

h. we always want to reject H_0, since Bill isn't late for his 7:45 AM meeting.

$\begin{cases} H_0: \beta_1 + 30\beta_2 + 6\beta_3 + \beta_4 > 45 \\ H_1: \beta_1 + 30\beta_2 + 6\beta_3 + \beta_4 \le 45 \end{cases}$ $\quad t = -1.926 < -1.6511$

reject H_0 $\Rightarrow$ so Bill will arrive university on time.

5.33 Use the observations in the data file *cps5_small* to estimate the following model:

$$\ln(WAGE) = \beta_1 + \beta_2 EDUC + \beta_3 EDUC^2 + \beta_4 EXPER + \beta_5 EXPER^2 + \beta_6 (EDUC \times EXPER) + e$$

a. At what levels of significance are each of the coefficient estimates "significantly different from zero"?

b. Obtain an expression for the marginal effect $\partial E[\ln(WAGE)|EDUC, EXPER]/\partial EDUC$. Comment on how the estimate of this marginal effect changes as *EDUC* and *EXPER* increase.

c. Evaluate the marginal effect in part (b) for all observations in the sample and construct a histogram of these effects. What have you discovered? Find the median, 5th percentile, and 95th percentile of the marginal effects.

d. Obtain an expression for the marginal effect $\partial E[\ln(WAGE)|EDUC, EXPER]/\partial EXPER$. Comment on how the estimate of this marginal effect changes as *EDUC* and *EXPER* increase.

e. Evaluate the marginal effect in part (d) for all observations in the sample and construct a histogram of these effects. What have you discovered? Find the median, 5th percentile, and 95th percentile of the marginal effects.

f. David has 17 years of education and 8 years of experience, while Svetlana has 16 years of education and 18 years of experience. Using a 5% significance level, test the null hypothesis that Svetlana's expected log-wage is equal to or greater than David's expected log-wage, against the alternative that David's expected log-wage is greater. State the null and alternative hypotheses in terms of the model parameters.

g. After eight years have passed, when David and Svetlana have had eight more years of experience, but no more education, will the test result in (f) be the same? Explain this outcome?

h. Wendy has 12 years of education and 17 years of experience, while Jill has 16 years of education and 11 years of experience. Using a 5% significance level, test the null hypothesis that their marginal effects of extra experience are equal against the alternative that they are not. State the null and alternative hypotheses in terms of the model parameters.

i. How much longer will it be before the marginal effect of experience for Jill becomes negative? Find a 95% interval estimate for this quantity.

(a).

```
Call:
lm(formula = log(wage) ~ educ + I(educ^2) + exper + I(exper^2) +
    I(educ * exper), data = cps5_small)

Residuals:
    Min      1Q  Median      3Q     Max
-1.6628 -0.3138 -0.0276  0.3140  2.1394

Coefficients:
                  Estimate Std. Error t value Pr(>|t|)
(Intercept)      1.038e+00  2.757e-01   3.764 0.000175 ***
educ             8.954e-02  3.108e-02   2.881 0.004038 **
I(educ^2)        1.458e-03  9.242e-04   1.578 0.114855
exper            4.488e-02  7.297e-03   6.150 1.06e-09 ***
I(exper^2)      -4.680e-04  7.601e-05  -6.157 1.01e-09 ***
I(educ * exper) -1.010e-03  3.791e-04  -2.665 0.007803 **
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 0.4638 on 1194 degrees of freedom
Multiple R-squared:  0.3227,	Adjusted R-squared:  0.3198
F-statistic: 113.8 on 5 and 1194 DF,  p-value: < 2.2e-16
```

All coefficient estimates are significantly different from zero at the 1% level of significance, except from $EDUC^2$, even at 10% level.

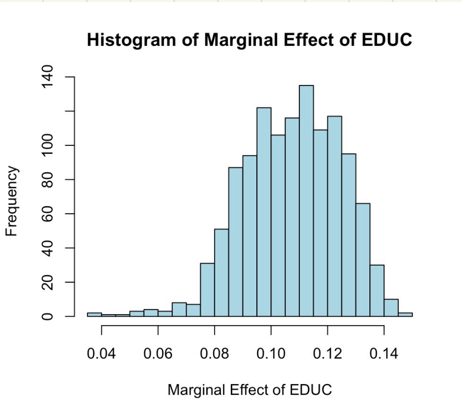
(b). $$\frac{\partial E[\ln(WAGE)\mid EDUC, EXPER]}{\partial EDUC} = \beta_2 + 2\beta_3 EDUC + \beta_6 EXPER$$

$$= 0.08954 + 2 \times 0.001458 \times EDUC + (-0.0010) \times EXPER.$$

EDUC ↑ → marginal effect ↑

EXPER ↑ → marginal effect ↓

(c).



The distribution is approximately symmetric and bell-shaped.

The effects are positive throughout, meaning more education consistenely increases expected log wage for all individuals.

5%	50%	95%
0.08008187	0.10843125	0.13361880

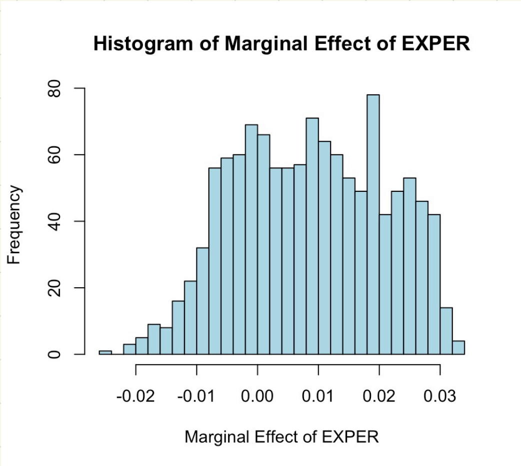
(d). $$\frac{\partial E[\ln(WAGE)|EDUC, EXPER]}{\partial EXPER} = \beta_4 + 2\beta_5 EXPER + \beta_6 EDUC$$

$$= 0.04488 + 2\times(-0.000468)\times EXPER + (-0.00101)\times EDUC$$

EXPER ↑ ⇒ marginal effect ↓

EDUC ↑ ⇒ marginal effect ↓

(e)




```
> quantile(cps5_small$marginal_effect_exper, probs = c(0.05, 0.5, 0.95))
          5%          50%          95%
-0.010376212  0.008418878  0.027931151
```

Most marginal effects are positive, only a small portion may experience negative.

(f). $H_0: \beta_1 + 17\beta_2 + 17^2\beta_3 + 8\beta_4 + 8^2\beta_5 + 8\times17\beta_6 \le \beta_1 + 16\beta_2 + 16^2\beta_3 + 18\beta_4 + 18^2\beta_5 + 18\times16\beta_6.$

$H_1: \beta_1 + 17\beta_2 + 17^2\beta_3 + 8\beta_4 + 8^2\beta_5 + 8\times17\beta_6 > \beta_1 + 16\beta_2 + 16^2\beta_3 + 18\beta_4 + 18^2\beta_5 + 18\times16\beta_6.$

$$\begin{cases} H_0: \beta_2 + 33\beta_3 - 10\beta_4 - 260\beta_5 - 152\beta_6 \le 0 \\ H_1: \beta_2 + 33\beta_3 - 10\beta_4 - 260\beta_5 - 152\beta_6 > 0 \end{cases}$$

→ fail to reject H_0. that Svetlana's expected log-wage is equal to or greater that David's.

$$t = \frac{-0.03588 - 0}{0.021489} = -1.6697 < 1.646$$

(g). $H_0: \beta_1 + 17\beta_2 + 17^2\beta_3 + 16\beta_4 + 16^2\beta_5 + 16\times17\beta_6 \le \beta_1 + 16\beta_2 + 16^2\beta_3 + 26\beta_5 + 26\times16\beta_6$

$H_1: \beta_1 + 17\beta_2 + 17^2\beta_3 + 16\beta_4 + 16^2\beta_5 + 16\times17\beta_6 > \beta_1 + 16\beta_2 + 16^2\beta_3 + 26\beta_5 + 26\times16\beta_6$

$$\begin{cases} H_0: \beta_2 + 33\beta_3 - 10\beta_4 - 420\beta_5 - 144\beta_6 \le 0 \\ H_1: \beta_2 + 33\beta_3 - 10\beta_4 - 420\beta_5 - 144\beta_6 > 0 \end{cases}$$

$$t = \frac{0.03092 - 0}{0.01499} = 2.0627 > 1.646$$

$\Rightarrow$ reject H_0 that Svetlana's expected log-wage isn't equal to or greater than David's after 8 years.

(h) marginal effect $= \beta_4 + 2\beta_5 EXPER + \beta_6 EDUC$

$H_0: \beta_4 + 2\times17\beta_5 + 12\beta_6 = \beta_4 + 2\times11\beta_5 + 16\beta_6$

$H_1: \beta_4 + 2\times17\beta_5 + 12\beta_6 \ne \beta_4 + 2\times11\beta_5 + 16\beta_6$

$$\Rightarrow \begin{cases} H_0: 12\beta_5 - 4\beta_6 = 0 \\ H_1: 12\beta_5 - 4\beta_6 \ne 0 \end{cases}$$

$$t = \frac{-0.001575}{0.0015334} = -1.02713 > -1.962$$

$\Rightarrow$ fail to reject that their marginal effect are equal.

(i). $\beta_4 + 2\times\beta_5\times(11+x) + 16\beta_6 = 0$

$$x = \frac{-\beta_4 - 16\beta_6}{2\beta_5} - 11 = 19.67706.$$

$$se = \sqrt{\left(\frac{-1}{2\beta_5}\right)^2 Var(\beta_4) + \left(\frac{\beta_4+16\beta_6}{2\beta_5^2}\right)^2 Var(\beta_5) + \left(\frac{-8}{\beta_5}\right)^2 Var(\beta_6) + 2\left(\frac{-1}{2\beta_5}\right)\left(\frac{\beta_4+16\beta_6}{2\beta_5^2}\right) Cov(\beta_4,\beta_5) + 2\times\left(\frac{\beta_4+16\beta_6}{2\beta_5^2}\right)\left(\frac{-8}{\beta_5}\right) Cov(\beta_2,\beta_3) + 2\times\left(\frac{-1}{2\beta_5}\right)\left(\frac{-8}{\beta_5}\right) Cov(\beta_1,\beta_3)} = 1.8957$$

confidence interval $\Rightarrow 19.6771 \pm t\times 1.8957 \Rightarrow [15.958, 23.396]$