Trajectory Tracking and Auto Balancing for a Motorcycle

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***Abstract*—Our project is aiming at building a dynamic model for an abstract motorcycle and implement sliding mode controller to it. The SMC should be able generate a control law for the torque on the rear wheel and control the steering handle to drive the bicycle to move along a desired trajectory with designated speed. The control law is robust to different initial state and error on linear velocity and steering angle.**

I. Introduction

A single-track vehicle, such as bicycle or motorcycle, is a kind of vehicle that leaves a track on the ground when moving forward. Bicycle is common to see and widely used in transportation and recreation. It is a tricky system. Since with a relatively simple construction, the bicycle can achieve a wide range variety of motions. From many videos about acrobatic bicycle and mountain bike cycling, it can be seen that a bicycle can move fast and flexibly to a target position under certain control of human by applying torque to the pedal, steering and lifting the handle, and moving the center of gravity. Although a bicycle does not have a stable equilibrium point and have little or even no lateral stability when stationary, it can achieve stable control with continuous input moving forward. In this paper, we want to build a dynamic model for a bicycle, and propose corresponding control law to make it following a certain trajectory.

The existing literature have different dynamic model and corresponding control policy to realize steering control. In [2], authors consider the existence of lateral sliding velocity and the tire models to build the dynamic model for the bicycle. The dynamic model is obtained through a constrained Lagrange modeling approach. The control inputs for the proposed motorcycle dynamics are the front wheel steering angles and the angular velocities for the front and rear wheels.

In their later paper [3], they also include the driver's mass regarded as a unknown nonlinear disturbance which can have a roll angle. They propose a nonlinear disturbance observer(NDOB) to estimate the human disturbance, and adapt the hierarchical sliding-mode control (SMC) approach to realize stable control, and improve the performance of SMC when the system is stationary (velocities are zero).

In paper [4], to prove the gyroscopic precession of the front wheel and wheel contact trailing like a caster behind the steer axis are not necessary for self-stability, they built a model with extra counter-rotating wheels (canceling the wheel spin angular momentum) and with its front wheel ground-contact forward of the steer axis (making the trailing distance negative) when laterally disturbed from rolling straight this bicycle automatically recovers to upright travel. And its result showed that different design parameters (front mass location, the steer axis tilt, etc.) can result in stability in complex ways.

In paper [5], they built a model based on Whipple's bicycle (1899) which consisted of four rigid bodies: a rear wheel R, a front wheel F, a rear frame B with the rider rigidly attached, and a front frame H with the handlebar and fork assembly (figure 1). The shape and mass were generally distributed among the whole frame. Differed from Whipple, they allowed for thickness in the inertial properties, and they neglected the motion of the rider relative to the frame, structural compliances and dampers, friction between joints and true models with compliance and slip.The model included all the sharply defined rigid body effects and neglected less well-defined things while modeling. And most latter control were based on this.

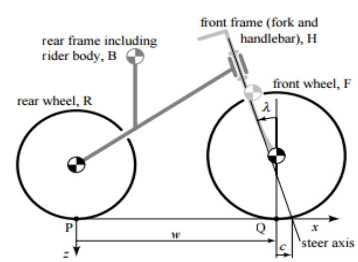

Fig. 1. Bicycle dynamic benchmark.

II. Project Goals

The goal of this project is to realize the auto balance and trajectory tracking of an abstract bicycle. This can be realized by 3 parts: building the dynamic model, proposing a controller, and verifying in a simulation environment.

Building the dynamic model for the bicycle requires us to figure out the physical model of the bicycle. The relationship between the length and angle within the bicycle components and with respect to the ground should be determined. The differential equations between the time invariant input and the current state, also the time invariant input and the state derivative should be established.

The control law should be able to keep the bicycle balance and drive it following a desirable trajectory. The given trajectory can be some reasonable route realizing obstacle-avoidance or reach a certain position. States need to be following including the yaw angle and planar positions of the rear wheel.

The simulation environment is used for the desirable trajectory and the necessary components of the bicycle visualization. Internally, it should calculate the state variables and outputs at any time given an initial state and a control law. The pose of the rear wheel (outputs) compared to the desirable trajectory should be visualized. If time allows, the motion of every components (two wheel, frame, etc.) of the bicycle under the control may also be shown.

For the further study, we want to include the scenario that the bicycle can jump off from a stage, which means it can have a impulse on the front wheel, and will rotate in a constant angular velocity in the air. This can enable the bicycle to adapt to complicated terrain. Our project can also be extended by utilize the roll angle to make the motion more efficient, or include the disturbance from the passengers.

III. Problem Statement

Given a trajectory of the pose (x, y, θ) of the motorcycle, design a controller for inputs of the driving torque and steering angle to make it mimic the desirable trajectory. The controller should also bound the roll angle α to 0 to keep the motorcycle balanced.

IV. Methods

The dynamics model is derived based on Euler-Lagrangian Method. Based on the

The setup of control law and the simulation are both based on the dynamic model. We can use the method in paper [6] to build the model with a big table of bicycle parameters. Then we can give the kinematics of the planar motion and roll dynamics of the bicycle robot. Next, we can consider the roll angle's equilibrium and define the implicit function. Next we can assume the initial point and the final point with initial and final time, so the curve can be represented by a parameterization. Following, we can choose 3rd order polynomials of time to parameterize the track and decide their coefficients. As we have constraint conditions in the initial and final points, we can give the proper equations and using PSO algorithm to get the optimal solution.

Consider the nonlinearity and heavily coupling condition of the motorcycle dynamics, we apply Sliding Model Control (SMC) to our system.

For the control part, a robust control will be applied, auto balance is going to be realized through set-joint control, as if one of the state, roll angle φ, should track 0 and not exceed a certain bilateral boundary. For the position of the outputs, we will use PID control to follow a curve function. The simulation will be based on MATLAB to draw a stick model for animation and a plot to show how good the real trajectory is. The trajectory under disturbance and with different initial state will also be shown. As the bicycle should not fall down and must follow the desired planar trajectory, We can design the controller under following steps, first, designing a balancing controller for roll dynamics. Second, designing a controller for tracking the desired planar trajectory. Finally, designing a controller for tracking with balance by balancing the roll dynamics around its equilibrium manifold, which it can be result from the former step. By stepping these, we can gradually get a proper controller for the whole system.

V. Motorcycle Dynamic Model

The derivation of our dynamics model are mostly based on [1] which is consistent with other literature. Some place in literature is very ambiguous, we derive a more clear explanation.

A. Assumptions on the Model

- Both front and rear wheels are considered to have negligible inertia moments, mass, radius, and width.
- The wheels are considered to roll with neither lateral nor longitudinal slip.
- The motorcycle body frame is considered as a point mass m.
- The rigid frame of the motorcycle is assumed to be symmetric about a plane containing rear wheel.
- The steer-axis is assumed to fixed in the plane of symmetry, and perpendicular to the flat ground when motorcycle is upright.

The simplified motorcycle model is illustrated in Fig.2.

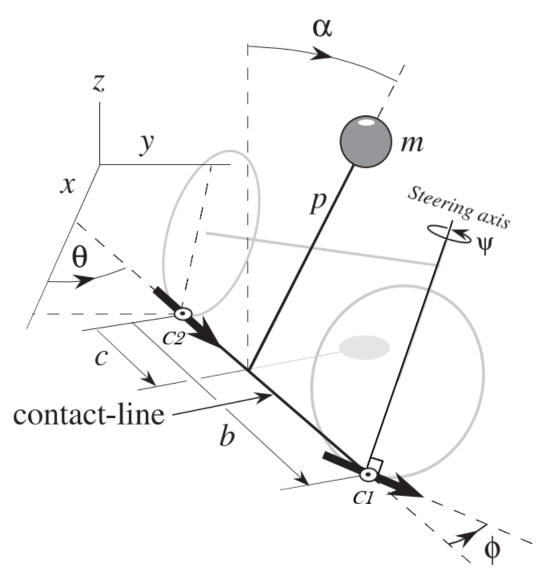

Fig. 2. Schematic of the motorcycle.

B. Variable Specifications

The contact point of a wheel is the point where the wheel contacts the ground. Observing from the world frame $O_0X_0Y_0Z_0$, the position of the bicycle (x, y) is defined as

m Motorcycle mass
L Motorcycle wheelbase
p Original height of motorcycle center of mass
b Horizontal distance between front wheel and real wheel contact points
R Motorcycle turning radius
α Roll (camber) angle of the frame (rear wheel)
θ Yaw angle of the motorcycle frame (rear wheel)
ϕ Steering angle
ψ Front wheel direction angle (effective steering angle)

the contact point of rear wheel, which is the output variable we want to control. θ is also an output, but we don't assign desirable trajectory for it.

Under a normal pose of the motorcycle, we can simplify the model by just assume the contact points of both wheels are on a segment line of length $b = L$ starting at point (x, y) in the direction of θ.

The roll angle α is defined as the angle between the line from the center of the rear wheel to contact point C_2 and the plane $X_0O_0Y_0$, $\alpha \in (-pi/2, pi/2)$. The steering angle ψ is the angle between the front wheel plane and the rear wheel plane, $\psi \in (-pi/2, pi/2)$. ϕ is the angle between the intersecting line between the front wheel plane and the ground and C_1C_2.

C. Constraint Specifications

Constraints are listed here. The requirement should be that the bicycle is not fall down, which means $|\alpha| < \delta$, where δ is a given angle less than 90 degree. And two wheels must always have and only one contact point respectively with the ground. L should keep constant.

D. Derivation of Lagrangian and Dynamics Model

1) Generalized Coordinates: Consider that the generalized coordinates of the motorcycle are $(x, y, \theta, \alpha, \sigma)$, where (x, y), θ, α are defined. For roll angle α, we define the frame of the motorcycle tilting right from the vertical plane is positive, so that for steering shaft angle ψ, the motorcycle turning left is the positive direction. σ is defined as the curvature of the trajectory of the rear wheel contact point C_2, here denote the radius of the trajectory as R, so we can calculate σ as

$$\sigma := \frac{1}{R} = \frac{tan\phi}{L} = \frac{1}{b} tan\phi \tag{1}$$

From the geometry of the front wheel steering mechanism, we can find ψ and ϕ are related by

$$tan\phi cos\alpha = tan\psi \tag{2}$$

So from Eq.1 we have

$$b\sigma \cos\alpha = tan\psi \tag{3}$$

2) Generalized Velocities: Corresponding to the generalized coordinates is a set of generalized velocities $(\dot{x}, \dot{y}, \dot{\theta}, \dot{\alpha}, \dot{\sigma})$. Denote the component of the velocity of the rear wheel along the contact-line direction as v_r, and the component of the velocity of the rear wheel contacts perpendicular to the contact-line and parallel to the ground plane as $v_\perp$. There is a relationship among $\dot{x}$, $\dot{y}$, $v_r$and$v_\perp$ through a rotation by θ,

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix} \begin{bmatrix} v_r \\ v_\perp \end{bmatrix} \tag{4}$$

All the velocities are indicated in Fig3. So we get an alternative set of generalized velocities for the motorcycle $(v_r, v_\perp, \dot{\theta}, \dot{\alpha}, \dot{\sigma})$.

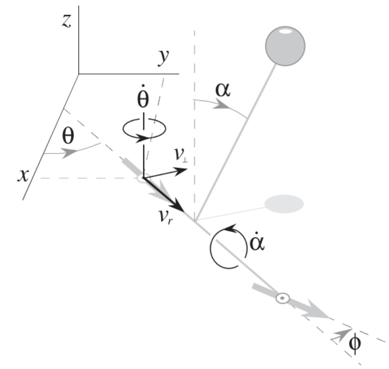

Fig. 3. Generalized velocities.

3) Generalized Forces: Our motorcycle model riderless and driven by two torque generators under automatic control. Denote the reaction force that the ground exerts on the motorcycle at the contact point C_2 as τ^r, and the other torque generator which is a virtual toque that associated with steering variable σ as τ^σ. There is also a gravity force mg acting on the mass point m of the motorcycle. Under the assumption of no slip exists in the wheel motion, we can get the dynamic constraint on the wheels

$$\begin{bmatrix} \dot{\theta} \\ v_\perp \end{bmatrix} + \begin{bmatrix} 0 & -\sigma & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\alpha} \\ v_\perp \\ \dot{\sigma} \end{bmatrix} = 0 \tag{5}$$

We can see that Eq.5 implies

$$\begin{cases} \dot{\theta} = \sigma v_r \\ v_\perp = 0 \end{cases} \tag{6}$$

For gravity mg, the height change Δh_m of the mass point m can be calculated as

$$\Delta h_m = p(1 - cos\alpha) \tag{7}$$

4) Lagrangian Equations: The Lagrangian for the motorcycle is constructed from the kinetic energies associated with the point mass m and the steering axis, as well as the potential energy of point mass m. Consider the constraint Eq.5,6, we can get a constrained Lagrangian for the motorcycle

$$\begin{aligned} L = & -mgp\cos\alpha + \frac{1}{2}J(\alpha,\sigma)\dot{\sigma}^2 + \frac{m}{2}[(v_r + p\sigma v_r \sin\alpha)^2 \\ & + p^2\dot{\alpha}^2\sin^2\alpha + (c\sigma v_r - p\dot{\alpha}\cos\alpha)^2] \end{aligned} \tag{8}$$

,

where $J(\alpha,\sigma)$ is an equivalent moment of inertia of the front wheel about the steering axis. In order to get simplified Lagrangian equations, we assume the term associated with $J(\alpha,\sigma)$ in the equations is negligible. Because for this motorcycle model, it is difficult to control the torque exerts on the front wheel. As stated above we assume that the steering variable σ is directly controlled, which means that we can control the acceleration of the steering of the front wheel directly from tracking any desired $\sigma(t)$ trajectory. For an automatically controlled motorcycle this assumption is a reasonable approximation as long as $|\ddot{\sigma}|$ is sufficiently small. As a result, we denote $\dot{\sigma}$ as the input w^{σ}.

So we get the simplified Lagrangian equations

$$M(r)\begin{bmatrix}\ddot{\alpha}\\ \dot{v_r}\end{bmatrix}=K(r)+B(r)\begin{bmatrix}\dot{\sigma}\\ \tau^r\end{bmatrix} \tag{9}$$

where

$$M(\alpha,\sigma)=\begin{bmatrix}p^2 & -cp\sigma\cos\alpha\\ -cp\sigma\cos\alpha & 1+(c^2+p^2\sin^2\alpha)\sigma^2+2p\sigma\sin\alpha\end{bmatrix} \tag{10}$$

$$K(\alpha,\dot{\alpha},\sigma,v_r)=\begin{bmatrix}gp\sin\alpha+(1+p\sigma\sin\alpha)p\sigma v_r^2\cos\alpha\\ -(1+p\sigma\sin\alpha)2p\sigma v_r\dot{\alpha}\cos\alpha-cp\sigma\dot{\alpha}^2\sin\alpha\end{bmatrix} \tag{11}$$

$$B(\alpha,\sigma,v_r)=\begin{bmatrix}cpv_r\cos\alpha & 0\\ -(c^2\sigma+p\sin\alpha(1+p\sigma\sin\alpha))v_r & \frac{1}{m}\end{bmatrix} \tag{12}$$

For further decoupling, we added a new feedback based on eq.9 and get a new model:

$$M(\alpha,\sigma)\begin{bmatrix}\ddot{\alpha}\\ \dot{v_r}\\ \dot{\sigma}\end{bmatrix}=K(\alpha,\dot{\alpha},\sigma,v_r)+B(\alpha,\sigma,v_r)\begin{bmatrix}\dot{\sigma}\\ \tau^r\end{bmatrix} \tag{13}$$

where

$$M(\alpha,\sigma)=\begin{bmatrix}p^2 & -cp\sigma\cos\alpha & 0\\ -cp\sigma\cos\alpha & 1+(c^2+p^2\sin^2\alpha)\sigma^2+2p\sigma\sin\alpha & 0\\ 0 & 0 & 1\end{bmatrix} \tag{14}$$

$$K(\alpha,\dot{\alpha},\sigma,v_r)=\begin{bmatrix}gp\sin\alpha+(1+p\sigma\sin\alpha)p\sigma v_r^2\cos\alpha\\ -(1+p\sigma\sin\alpha)2p\sigma v_r\dot{\alpha}\cos\alpha-cp\sigma\dot{\alpha}^2\sin\alpha\\ 0\end{bmatrix} \tag{15}$$

$$B(\alpha,\sigma,v_r)=\begin{bmatrix}cpv_r\cos\alpha & 0\\ -(c^2\sigma+p\sin\alpha(1+p\sigma\sin\alpha))v_r & \frac{1}{m}\\ 1 & 0\end{bmatrix} \tag{16}$$

VI. Sliding Mode Control Design

The SMC method applied to our model are derived below.

Consider a multi-input and multi-output nonlinear system as follows:

$$\begin{aligned}y^{(n)}=&f(y^{(n-1)},\cdots,\dot{y},y,t)+\Delta f(y^{(n-1)},\cdots,\dot{y},y,t)+\\ &b(y^{(n-1)},\cdots,\dot{y},y,t)u+d(t)\end{aligned} \tag{17}$$

where $y\in R^m$ is the output, $u\in R^m$ is the input, $f\in R^m$ as well as $b\in R^{m\times m}$ $(rank(b)=m)$ are nonlinear state matrices we have already known, Δf is the uncertainty of the controlled object and $d(t)$ is the external disturbance.

Let $x_1=y$, $x_2=\dot{y},\cdots,x_n=y^{(n-1)}$,so we get a new format for Eq.17

$$\begin{cases}\dot{x_1}= & x_2\\ & \vdots\\ \dot{x_n}= & f(X,t)+\Delta f(X,t)+b(X,t)u+d(t)\end{cases} \tag{18}$$

where $X=[x_1^T\ x_2^T\cdots x_n^T]^T=[y^T\ \dot{y}^T\cdots y^{(n-1)T}]^T$.

Applying to our motorcycle dynamic model eq.13 we have

$$x_1=\begin{bmatrix}\alpha\\ \int v_r\\ \int\sigma\end{bmatrix},x_2=\begin{bmatrix}\dot{\alpha}\\ v_r\\ \sigma\end{bmatrix},x_3=\begin{bmatrix}\ddot{\alpha}\\ \dot{v_r}\\ \dot{\sigma}\end{bmatrix} \tag{19}$$

The uncertainty $\Delta f(X,t)$ and external disturbance $d(t)$ satisfies the following conditions:

$$\begin{aligned}&|\ \Delta f(X,t)\ |\le F(X,t)\\ &|\ d(t)\ |\le D(t)\end{aligned} \tag{20}$$

where $F(X,t)$ and $D(t)$ are non-negative.

A. Switching Surface Design

In order to achieve tracking desired system state $X_d=[x_{1d}^T\ x_{2d}^T\cdots x_{nd}^T]^T=[x_{1d}^T\ \dot{x_{1d}}^T\cdots x_{1d}^{(n-1)T}]^T$ from current system state $X=[x_1^T\ x_2^T\cdots x_n^T]^T=[x_1^T\ \dot{x_1}^T\cdots x_1^{(n-1)T}]^T$ in finite time, we define the error vector as

$$E=X-X_d=[e^T\ \dot{e}^T\cdots e^{(n-1)T}]^T \tag{21}$$

where $e=x_1-x_{1d}=[e_1\ e_2\cdots e_m]^T$.

The equation for the sliding surface we designed is

$$\sigma(X,t)=CE-W(t) \tag{22}$$

where $C=[C_1\ \ C_2\cdots C_n]$ is a matrix, whose element $C_i=(c_{i1},c_{i2},\cdots,c_{im})$, $c_{ij}(i=1,2,\cdots,n;j=1,2,\cdots,m)$ is positive constant, and $W(t)=CP(t)$, $P(t)=[p(t)^T\ \dot{p}(t)^T\cdots p^{(n-1)}(t)^T]^T$.

Let $p(t)=[p_1(t)\ p_2(t)\cdots p_m(t)]^T$, and $p_i(t)$ satisfies the following assumption

Assumption: $p_i(t)\ :\ R_+\ \rightarrow\ R$, $p_i(t)\ \in\ C^n[0,\infty)$, $\dot{p_i},\cdots,p_i^{(n)}\ \in\ L^{\infty}$. For arbitrary constant T, $p_i(t)$ is bounded in time interval $[0,T]$, and $p_i(0)=e_i(0),\dot{p_i}(0)=\dot{e_i}(0),\cdots,p_i^{(n)}(0)=e_i^{(n)}(0)$. $C^n[0,\infty)$ is denoted as all the n order differentiable continuous functions on $[0, \infty)$,where $i = 1, 2, \cdots, m$. Define the function $p_i(t)$ as

$$p_i(t) = \begin{cases} \sum\limits_{k=0}^{n} \frac{1}{k!} e_i(0)^{(k)} t^k + \\ \sum\limits_{j=0}^{n} (\sum\limits_{l=0} \frac{a_{jl}}{T^{j-l+n+1}} e_i(0)^{(l)}) \cdot t^{j+n+l} \\ when\ 0 \le t \le T \\ 0 \\ when\ t > T \end{cases} \tag{23}$$

where a_{ij} can be calculated from the conditions in the assumption.

B. Terminal Sliding Mode Controller Design

From Eq.22 we can get

$$\begin{aligned} \dot{\sigma}(X,t) =& C\dot{E} - C\dot{P}(t) \\ =& C \cdot [\dot{e}^T \ddot{e}^T \cdots e^{(n)T}]^T - C \cdot [\dot{p}(t)^T \ddot{p}(t)^T \cdots p^{(n)}(t)^T]^T \\ =& C_n[f(X,t) + \Delta f(X,t) + b(X,t)u + d(t) \\ & - x_{(1d)}^{(n)} - p(t)^{(n)}] + \sum_{k=1}^{n-1} C_k[e^k - p(t)^k] \end{aligned} \tag{24}$$

So we design the Lyapunov function as

$$V = \frac{1}{2}\sigma^T \sigma \tag{25}$$

which means

$$\begin{aligned} \dot{V} =& \sigma^T \dot{\sigma} \\ =& \sigma^T C_n \{ f(X,t) - x_{id}^{(n)} - p(t)^{(n)} \\ & + C_n^{-1} \sum_{k=1}^{n-1} C_k [e^{(k)} - p(t)^{(k)}] \} \\ & + \sigma^T C_n b(X,t) u + \sigma^T C_n [\Delta f(X,T) + d(t)] \\ \le& \sigma^T C_n \times \{ f(X,t) - x_{1d}^{(n)} - p(t)^{(n)} \\ & + C_{-1} \sum_{k=1}^{n-1} C_k [e^{(n)} - p(t)^{(k)}] \} \\ & + \sigma^T C_n b(X,t) u + \parallel \sigma^T C_n \parallel \parallel \Delta f(X,t) + d(t) \parallel \end{aligned} \tag{26}$$

And we can design the controller as

$$\begin{aligned} u(t) =& - b(X,t)^{-1} \{ f(X,t) - {x_{1d}}^n - p(t)^n \\ & + C_n^{-1} \sum_{k=1}^{n-1} C_k (e^k - p(t)^k) \} \\ & - b(X,t)^{-1} \frac{C_n^T \sigma}{||C_n^T \sigma||} \{ F(X,t) + D(t) + K \} \end{aligned} \tag{27}$$

among which K is a positiAccording to *Assumption* and Eq.22,we can get

$$\begin{aligned} \sigma(X,0) =& CE(0) - W(0) \\ =& C[E(0) - P(0)] \\ =& C\{ [e(0)^T \dot{e}(0)^T \cdots e(0)^{(n-1)T}]^T \\ & - [p(0)^T \dot{p}(0)^T \cdots p(0)^{(n-1)T}]^T \} \\ =& 0 \end{aligned} \tag{28}$$

So now the initial state of system is on the sliding surface, which eliminates the reaching step of sliding mode control, only remains the sliding step. This method ensures the global robustness and stability of the closed-loop system.

Now that the system has the property of global robustness, which means $\sigma(X,t) = 0$ or $E(t) = P(t)$, the system can achieve $E(t) = 0$ by selecting $P(T) = 0$ in the Terminal sliding surface, so that the tracking error can converge to zero within the finite time T.

C. Application on Motorcycle Model

As for the system we got before, we can write it as

$$\begin{aligned} \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix} \ddot{x} = \\ \begin{bmatrix} K_1 \\ K_2 \\ K_3 \end{bmatrix} + \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix} u \end{aligned} \tag{29}$$

Among which

$$x = \begin{bmatrix} x_{11} \\ x_{12} \\ x_{13} \end{bmatrix} \tag{30}$$

$$u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \tag{31}$$

and

$$m = M^{-1} = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{21} & m_{22} & m_{33} \end{bmatrix} \tag{32}$$

So it can be written as

$$\begin{aligned} \ddot{x} = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} \begin{bmatrix} K_1 \\ K_2 \\ K_3 \end{bmatrix} \\ + \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \end{aligned} \tag{33}$$

So we can write it into the form of integral chain

$$\begin{cases} \dot{x_{11}} = & x_{12} \\ \dot{x_{12}} = & m_{11}K_1 + m_{12}K_2 + m_{13}K_3 + (m_{11}B_{11} + m_{12}B_{21} \\ & +m_{13}B_{31})u_1 + (m_{11}B_{12} + m_{12}B_{22} + m_{13}B_{32})u_2 \\ & +(m_{11}B_{13} + m_{12}B_{23} + m_{13}B_{33})u_3 \\ \dot{x_{21}} = & x_{22} \\ \dot{x_{22}} = & m_{21}K_1 + m_{22}K_2 + m_{23}K_3 + (m_{21}B_{11} + m_{22}B_{21} \\ & +m_{23}B_{31})u_1 + (m_{21}B_{12} + m_{22}B_{22} + m_{23}B_{32})u_2 \\ & +(m_{21}B_{13} + m_{22}B_{23} + m_{23}B_{33})u_3 \\ \dot{x_{31}} = & x_{32} \\ \dot{x_{32}} = & m_{31}K_1 + m_{32}K_2 + m_{33}K_3 + (m_{31}B_{11} + m_{32}B_{21} \\ & +m_{33}B_{31})u_1 + (m_{31}B_{12} + m_{32}B_{22} + m_{33}B_{32})u_2 \\ & +(m_{31}B_{13} + m_{32}B_{23} + m_{33}B_{33})u_3 \end{cases} \tag{34}$$

So it's easy to see that

$$f(x,t)=\left[\begin{array}{c} m_{11}K_1+m_{12}K_2+m_{13}K_3\\ m_{21}K_1+m_{22}K_2+m_{23}K_3\\ m_{31}K_1+m_{32}K_2+m_{33}K_3\end{array}\right] \tag{35}$$

$$b(x,t)=\left[\begin{array}{ccc} m_{11} & m_{12} & m_{13}\\ m_{21} & m_{22} & m_{23}\\ m_{31} & m_{32} & m_{33}\end{array}\right]\left[\begin{array}{ccc} B_{11} & B_{12} & B_{13}\\ B_{21} & B_{22} & B_{23}\\ B_{31} & B_{32} & B_{33}\end{array}\right] \tag{36}$$

where $\Delta f=0, d(t)=0$.

$$C_1=diag(c_{11},c_{12})=\left[\begin{array}{ccc} 4 & 0 & 0\\ 0 & 4 & 0\\ 0 & 0 & 4\end{array}\right] \tag{37}$$

$$C_2=diag(c_{21},c_{22})=\left[\begin{array}{ccc} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{array}\right] \tag{38}$$

$$C=[C_1,C_2,C_3]=\left[\begin{array}{ccccccccc} 4 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0\\ 0 & 4 & 0 & 0 & 1 & 0 & 0 & 1 & 0\\ 0 & 0 & 4 & 0 & 0 & 1 & 0 & 0 & 1\end{array}\right] \tag{39}$$

From equation(18), we can get the sliding surface is

$$\begin{aligned}\sigma(X,t)&=\left[\begin{array}{c}\sigma_1\\ \sigma_2\\ \sigma_3\end{array}\right]\\ &=\left[\begin{array}{c} 4e_1+\dot{e_1}-4p_1-\dot{p_1}\\ 4e_2+\dot{e_2}-4p_2-\dot{p_2}\\ 4e_3+\dot{e_3}-4p_3-\dot{p_3}\end{array}\right]\end{aligned} \tag{40}$$

As for the second-order system, $m=3$, $n=2$, so from equation(19) we can get $p_i(t)(i=1,2,3)$ is

$$p_i(t)=\begin{cases}\sum_{k=0}^{2}\frac{1}{k!}e_i(0)^{(k)}t^k+\sum_{j=0}^{2}\left(\sum_{l=0}^{2}\frac{a_{jl}}{T^{j-l+3}}e_i(0)^{(l)}\right)t^{j+3}\\ \text{when } 0\le t\le T\\ 0\\ \text{when } t>T\end{cases} \tag{41}$$

$$\begin{aligned}p_1(t)=&e_1(0)+\dot{e_1}(0)t+\frac{1}{2}\ddot{e_1}(0)t^2+(\frac{a_{00}}{T^3}e_1(0)+\frac{a_{01}}{T^2}\dot{e_1}(0)\\ &+\frac{a_{02}}{T}\ddot{e_1}(0))t^3+(\frac{a_{10}}{T^4}e_1(0)+\frac{a_{11}}{T^3}\dot{e_1}(0)+\frac{a_{12}}{T^2}\ddot{e_1}(0))t^4\\ &+(\frac{a_{20}}{T^5}e_1(0)+\frac{a_{21}}{T^4}\dot{e_1}(0)+\frac{a_{22}}{T^3}\ddot{e_1}(0))t^5\end{aligned} \tag{42}$$

$$\begin{aligned}\dot{p_1}(t)=&\dot{e_1}(0)+\ddot{e_1}(0)t+3(\frac{a_{00}}{T^3}e_1(0)+\frac{a_{01}}{T^2}\dot{e_1}(0)+\frac{a_{02}}{T}\ddot{e_1}(0))t^2\\ &+4(\frac{a_{10}}{T^4}e_1(0)+\frac{a_{11}}{T^3}\dot{e_1}(0)+\frac{a_{12}}{T^2}\ddot{e_1}(0))t^3+5(\frac{a_{20}}{T^5}e_1(0)\\ &+\frac{a_{21}}{T^4}\dot{e_1}(0)+\frac{a_{22}}{T^3}\ddot{e_1}(0))t^4\end{aligned} \tag{43}$$

$$\begin{aligned}\ddot{p_1}(t)=&\ddot{e_1}(0)+6(\frac{a_{00}}{T^3}e_1(0)+\frac{a_{01}}{T^2}\dot{e_1}(0)+\frac{a_{02}}{T}\ddot{e_1}(0))t+12\\ &(\frac{a_{10}}{T^4}e_1(0)+\frac{a_{11}}{T^3}\dot{e_1}(0)+\frac{a_{12}}{T^2}\ddot{e_1}(0))t^2+20(\frac{a_{20}}{T^5}e_1(0)\\ &+\frac{a_{21}}{T^4}\dot{e_1}(0)+\frac{a_{22}}{T^3}\ddot{e_1}(0))t^3\end{aligned} \tag{44}$$

$$\begin{aligned}p_2(t)=&e_2(0)+\dot{e_2}(0)t+\frac{1}{2}\ddot{e_2}(0)t^2+(\frac{a_{00}}{T^3}e_2(0)+\frac{a_{01}}{T^2}\dot{e_2}(0)\\ &+\frac{a_{02}}{T}\ddot{e_2}(0))t^3+(\frac{a_{10}}{T^4}e_2(0)+\frac{a_{11}}{T^3}\dot{e_2}(0)+\frac{a_{12}}{T^2}\ddot{e_2}(0))t^4\\ &+(\frac{a_{20}}{T^5}e_2(0)+\frac{a_{21}}{T^4}\dot{e_2}(0)+\frac{a_{22}}{T^3}\ddot{e_2}(0))t^5\end{aligned} \tag{45}$$

$$\begin{aligned}\dot{p_2}(t)=&\dot{e_2}(0)+\ddot{e_2}(0)t+3(\frac{a_{00}}{T^3}e_2(0)+\frac{a_{01}}{T^2}\dot{e_2}(0)+\frac{a_{02}}{T}\ddot{e_2}(0))t^2\\ &+4(\frac{a_{10}}{T^4}e_2(0)+\frac{a_{11}}{T^3}\dot{e_2}(0)+\frac{a_{12}}{T^2}\ddot{e_2}(0))t^3+5(\frac{a_{20}}{T^5}e_2(0)\\ &+\frac{a_{21}}{T^4}\dot{e_2}(0)+\frac{a_{22}}{T^3}\ddot{e_2}(0))t^4\end{aligned} \tag{46}$$

$$\begin{aligned}\ddot{p_2}(t)=&\ddot{e_2}(0)+6(\frac{a_{00}}{T^3}e_2(0)+\frac{a_{01}}{T^2}\dot{e_2}(0)+\frac{a_{02}}{T}\ddot{e_2}(0))t+12\\ &(\frac{a_{10}}{T^4}e_2(0)+\frac{a_{11}}{T^3}\dot{e_2}(0)+\frac{a_{12}}{T^2}\ddot{e_2}(0))t^2+20(\frac{a_{20}}{T^5}e_2(0)\\ &+\frac{a_{21}}{T^4}\dot{e_2}(0)+\frac{a_{22}}{T^3}\ddot{e_2}(0))t^3\end{aligned} \tag{47}$$

$$\begin{aligned}p_3(t)=&e_3(0)+\dot{e_3}(0)t+\frac{1}{2}\ddot{e_3}(0)t^2+(\frac{a_{00}}{T^3}e_3(0)+\frac{a_{01}}{T^2}\dot{e_3}(0)\\ &+\frac{a_{02}}{T}\ddot{e_3}(0))t^3+(\frac{a_{10}}{T^4}e_3(0)+\frac{a_{11}}{T^3}\dot{e_3}(0)+\frac{a_{12}}{T^2}\ddot{e_3}(0))t^4\\ &+(\frac{a_{20}}{T^5}e_3(0)+\frac{a_{21}}{T^4}\dot{e_3}(0)+\frac{a_{22}}{T^3}\ddot{e_3}(0))t^5\end{aligned} \tag{48}$$

$$\begin{aligned}\dot{p_3}(t)=&\dot{e_3}(0)+\ddot{e_3}(0)t+3(\frac{a_{00}}{T^3}e_3(0)+\frac{a_{01}}{T^2}\dot{e_3}(0)+\frac{a_{02}}{T}\ddot{e_3}(0))t^2\\ &+4(\frac{a_{10}}{T^4}e_3(0)+\frac{a_{11}}{T^3}\dot{e_3}(0)+\frac{a_{12}}{T^2}\ddot{e_3}(0))t^3+5(\frac{a_{20}}{T^5}e_3(0)\\ &+\frac{a_{21}}{T^4}\dot{e_3}(0)+\frac{a_{22}}{T^3}\ddot{e_3}(0))t^4\end{aligned} \tag{49}$$

$$\begin{aligned}\ddot{p_3}(t)=&\ddot{e_3}(0)+6(\frac{a_{00}}{T^3}e_3(0)+\frac{a_{01}}{T^2}\dot{e_3}(0)+\frac{a_{02}}{T}\ddot{e_3}(0))t+12\\ &(\frac{a_{10}}{T^4}e_3(0)+\frac{a_{11}}{T^3}\dot{e_3}(0)+\frac{a_{12}}{T^2}\ddot{e_3}(0))t^2+20(\frac{a_{20}}{T^5}e_3(0)\\ &+\frac{a_{21}}{T^4}\dot{e_1}(0)+\frac{a_{22}}{T^3}\ddot{e_1}(0))t^3\end{aligned} \tag{50}$$

When t=T, $p_i(t)=0$, $\dot{p_i}(t)=0$, $\ddot{p_i}(t)=0$, so

$$p_1(t) = e_1(0) + \dot{e_1}(0)T + \frac{1}{2}\ddot{e_1}(0)T^2 + (\frac{a_{00}}{T^3}e_1(0) + \frac{a_{01}}{T^2}\dot{e_1}(0) + \frac{a_{02}}{T}\ddot{e_1}(0))T^3 + (\frac{a_{10}}{T^4}e_1(0) + \frac{a_{11}}{T^3}\dot{e_1}(0) + \frac{a_{12}}{T^2}\ddot{e_1}(0))T^4 + (\frac{a_{20}}{T^5}e_1(0) + \frac{a_{21}}{T^4}\dot{e_1}(0) + \frac{a_{22}}{T^3}\ddot{e_1}(0))T^5$$
$$= e_1(0)[1 + a_{00} + a_{10} + a_{20}] + T\dot{e_1}(0)[1 + a_{01} + a_{11} + a_{21}] + T^2\ddot{e_1}(0)[\frac{1}{2} + a_{02} + a_{12} + a_{22}]$$
$$= 0 \tag{51}$$

$$\dot{p_1}(T) = \dot{e_1}(0) + \ddot{e_1}(0)T + 3(\frac{a_{00}}{T^3}e_1(0) + \frac{a_{01}}{T^2}\dot{e_1}(0) + \frac{a_{02}}{T}\ddot{e_1}(0))T^2 + 4(\frac{a_{10}}{T^4}e_1(0) + \frac{a_{11}}{T^3}\dot{e_1}(0) + \frac{a_{12}}{T^2}\ddot{e_1}(0))T^3 + 5(\frac{a_{20}}{T^5}e_1(0) + \frac{a_{21}}{T^4}\dot{e_1}(0) + \frac{a_{22}}{T^3}\ddot{e_1}(0))T^4$$
$$= \frac{e_1(0)}{T}[0 + 3a_{00} + 4a_{10} + 5a_{20}] + \dot{e_1}(0)[1 + 3a_{01} + 4a_{11} + 5a_{21}] + T\ddot{e_1}(0)[1 + 3a_{02} + 4a_{12} + 5a_{22}]$$
$$= 0 \tag{52}$$

$$\ddot{p_1}(T) = \ddot{e_1}(0) + 6(\frac{a_{00}}{T^3}e_1(0) + \frac{a_{01}}{T^2}\dot{e_1}(0) + \frac{a_{02}}{T}\ddot{e_1}(0))T + 12(\frac{a_{10}}{T^4}e_1(0) + \frac{a_{11}}{T^3}\dot{e_1}(0) + \frac{a_{12}}{T^2}\ddot{e_1}(0))T^2 + 20(\frac{a_{20}}{T^5}e_1(0) + \frac{a_{21}}{T^4}\dot{e_1}(0) + \frac{a_{22}}{T^3}\ddot{e_1}(0))T^3$$
$$= \frac{e_1(0)}{T^2}[0 + 6a_{00} + 12a_{10} + 20a_{20}] + \frac{\dot{e_1}(0)}{T}[0 + 6a_{01} + 12a_{11} + 20a_{21}] + \ddot{e_1}(0)[1 + 6a_{02} + 12a_{12} + 20a_{22}]$$
$$= 0 \tag{53}$$

And $p_2(T), \dot{p_2}(T), \ddot{p_2}(T), p_3(T), \dot{p_3}(T), \ddot{p_3}(T)$ is the same. So the sufficient conditions of the above three equation $p_1(T) = 0, \dot{p_1}(T) = 0, \ddot{p_1}(T) = 0$ is that

$$\begin{cases} 1 + a_{00} + a_{10} + a_{20} = 0 \\ 1 + a_{01} + a_{11} + a_{21} = 0 \\ \frac{1}{2} + a_{02} + a_{12} + a_{22} = 0 \end{cases} \tag{54}$$

$$\begin{cases} 3a_{00} + 4a_{10} + 5a_{20} = 0 \\ 1 + 3a_{01} + 4a_{11} + 5a_{21} = 0 \\ 1 + 3a_{02} + 4a_{12} + 5a_{22} = 0 \end{cases} \tag{55}$$

$$\begin{cases} 6a_{00} + 12a_{10} + 20a_{20} = 0 \\ 6a_{01} + 12a_{11} + 20a_{21} = 0 \\ 1 + 6a_{02} + 12a_{12} + 20a_{22} = 0 \end{cases} \tag{56}$$

By splitting the former three equation sets and assembles the same parameters, we can get

$$\begin{cases} a_{00} + a_{10} + a_{20} = -1 \\ 3a_{00} + 4a_{10} + 5a_{20} = 0 \\ 6a_{00} + 12a_{10} + 20a_{20} = 0 \end{cases} \tag{57}$$

$$\begin{cases} a_{01} + a_{11} + a_{21} = -1 \\ 3a_{01} + 4a_{11} + 5a_{21} = -1 \\ 6a_{01} + 12a_{11} + 20a_{21} = 0 \end{cases} \tag{58}$$

$$\begin{cases} a_{02} + a_{12} + a_{22} = -\frac{1}{2} \\ 3a_{02} + 4a_{12} + 5a_{22} = -1 \\ 6a_{02} + 12a_{12} + 20a_{22} = -1 \end{cases} \tag{59}$$

And solve these equation sets, we can get

$$\begin{cases} a_{00} = -10 \\ a_{10} = 15 \\ a_{20} = -6 \\ a_{01} = -6 \\ a_{11} = 8 \\ a_{21} = -3 \\ a_{02} = -1.5 \\ a_{12} = 1.5 \\ a_{22} = -0.5 \end{cases} \tag{60}$$

So substitute these parameters into equations we can get

$$p_1(t) = \begin{cases} e_1(0) + \dot{e_1}(0)t + \frac{1}{2}\ddot{e_1}(0)t^2 - (\frac{10}{T^3}e_1(0) + \frac{6}{T^2}\dot{e_1}(0) \\ + \frac{3}{2T}\ddot{e_1}(0))t^3 + (\frac{15}{T^4}e_1(0) + \frac{8}{T^3}\dot{e_1}(0) + \frac{3}{2T^2}\ddot{e_1}(0))t^4 \\ -(\frac{6}{T^5}e_1(0) + \frac{3}{T^4}\dot{e_1}(0) + \frac{1}{2T^3}\ddot{e_1}(0))t^5 \\ \text{when } 0 \le t \le T \\ 0 \\ \text{when } t > T \end{cases} \tag{61}$$

$$p_2(t) = \begin{cases} e_2(0) + \dot{e_2}(0)t + \frac{1}{2}\ddot{e_2}(0)t^2 - (\frac{10}{T^3}e_2(0) + \frac{6}{T^2}\dot{e_2}(0) \\ + \frac{3}{2T}\ddot{e_2}(0))t^3 + (\frac{15}{T^4}e_2(0) + \frac{8}{T^3}\dot{e_2}(0) + \frac{3}{2T^2}\ddot{e_2}(0))t^4 \\ -(\frac{6}{T^5}e_2(0) + \frac{3}{T^4}\dot{e_2}(0) + \frac{1}{2T^3}\ddot{e_2}(0))t^5 \\ \text{when } 0 \le t \le T \\ 0 \\ \text{when } t > T \end{cases} \tag{62}$$

$$p_3(t) = \begin{cases} e_3(0) + \dot{e_3}(0)t + \frac{1}{2}\ddot{e_3}(0)t^2 - (\frac{10}{T^3}e_3(0) + \frac{6}{T^2}\dot{e_3}(0) \\ + \frac{3}{2T}\ddot{e_3}(0))t^3 + (\frac{15}{T^4}e_3(0) + \frac{8}{T^3}\dot{e_3}(0) + \frac{3}{2T^2}\ddot{e_3}(0))t^4 \\ -(\frac{6}{T^5}e_3(0) + \frac{3}{T^4}\dot{e_3}(0) + \frac{1}{2T^3}\ddot{e_3}(0))t^5 \\ \text{when } 0 \le t \le T \\ 0 \\ \text{when } t > T \end{cases} \tag{63}$$

Assume the position position signal is $X_{11d} = 0, X_{22d} = 0$, the controller should be

$$u(t) = -\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + b(X,t)^{-1}\frac{\sigma}{||\sigma||}[F(X,t) + K] \tag{64}$$

and in this equation

$$u_1 = -b(X,t)^{-1}\{f(1) - \ddot{x_{11d}} - \ddot{p_1}(t) + 4[x_{12} - \dot{x_{11d}} - \dot{p_1}(t)]\} \tag{65}$$

$$u_2 = -b(X,t)^{-1}\{f(2) - \ddot{x_{21d}} - \ddot{p_2}(t) + 4[x_{22} - \dot{x_{21d}} - \dot{p_2}(t)]\} \tag{66}$$

$$u_3 = -b(X,t)^{-1}\{f(3) - \ddot{x_{31d}} - \ddot{p_3}(t) + 4[x_{32} - \dot{x_{31d}} - \dot{p_3}(t)]\} \tag{67}$$

VII. Simulation and Result

In the simulation part, we have four kinds of variable. First kind is constant, whose value stays the same or can be regraded as invariant under the simulation condition. $b = L$ is an example. Second is state variable, which should have initial value, and change according to other variables and inputs. This kind includes α, v_r, σ, etc. The third kind is control inputs, which are guaranteed to following their own desired trajectories without time delay. This includes τ_r and $\dot{\sigma}$. Fourth one is output, which will be evaluated by being compared to desirable trajectories that are given in advance. They are selected from or combined by state variables. These are x, y and θ, but in our problem we use $\int \sigma$ and $\int v_r$ equivalently.

A. Discussion about wheelbase

In many paper, it is very unclear about whether the value of b is consistent. Here we will demonstrate $b \approx L$ through calculation. Given a certain value of α and ψ, we can calculate the relative configuration the motorcycle. To start with, we can imagine the motorcycle is roll α first, then steer ψ, and rotate w.r.t. the axle of the real wheel for a certain angle ϵ to make the front wheel contact with the ground with one and only one point.

The first transformation, $T_1^0(x,y,\theta)$, is from the world frame to the C_2.

The second is $T_2^1(\alpha)$,the frame rolls for α.

The third one is $T_3^2(\epsilon)$, translate from C_2 to the center of the rear wheel, move forward for L and pitch for ϵ.

The fourth one is $T_4^3(\psi)$, steer for ψ.

Any point in the frame 4, the front wheel frame, can be transferred into the world frame. The only unknown variable is ϵ. According to the constraint, the transfered coordinates' height of the contact point C_1 should equal to zero. C_1 in frame 4 can only exist in the down side of the plane, so we can represent it as equation 68.

$$T_5^0 \begin{bmatrix} c \\ 0 \\ -\sqrt{r^2 - c^2} \end{bmatrix} = \begin{bmatrix} x_{C1} \\ y_{C1} \\ 0 \end{bmatrix} \tag{68}$$

We get two solutions $c_1(\epsilon)$ and $c_2(\epsilon)$, and we will solve the equation

$$c_1(\epsilon) - c_2(\epsilon) = 0$$

to get the right pitch angle, and substitute in the whole transformation matrix. The real C_1 is actually not on the "contact line", but a little bit excursion to the side of roll. The fake C_1 is actually the point between two lines. The first is the intersecting line of the rear wheel plane and the ground. The second line goes througth real $C2$ and has a slope of $\tan\phi$. The excursion is too small so that we can ignore it.

B. Simulation Setup

Simulink version 8.9 (R2017a) has been used to set up the simulation environment. The whole system is divided into several parts which are shown below.

The calculation process is discussed here. Based on the current state of variable α, $\dot{\alpha}$, σ, v_r, we can use equations 10, 11 and 12 to calculate the coefficient matrices of equation 9. And then generate the control input ω^σ and τ^r, the acceleration of α and path length can be gotten.

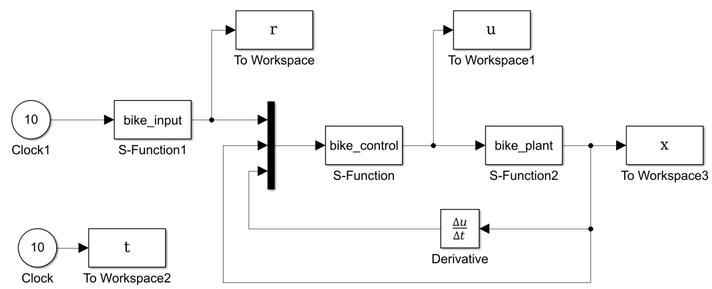

Fig. 4. The highest level of Simulink

C. Simulation Result

For the first experiment, we want to see how motorcycle move under a constant control (free fall). Set $b = L = 1m$, $r = 0.33m$, $m = 40kg$, $p = 1m$, $c = 0.5m$, $v_{r0} = 1m/s$, $g = 9.8m/s^2$ and $\psi = 10°$. Figure 5 shows the motion under a constant controller.

For the second experiment, we want the motorcycle to follow a sine wave trajectory with a constant speed. Set $v_{rd} = 7m/s$, and $y_d = sin(t)$. $T_f = 10s$. Then we can calculate

$$\dot{x}_d = \sqrt{v_{rd}^2 - \dot{y}_d^2} = \sqrt{49 - cos^2(t)}$$

$$\dot{\theta}_d = arctan\frac{\dot{y}_d}{\dot{x}_d}$$

$$\int v_{rd}$$

$$\int \sigma_d = \int \frac{\dot{\theta}_d}{v_{rd}}$$

Fig.6 to fig.10 show the state variables under SMC and corresponding input variables.

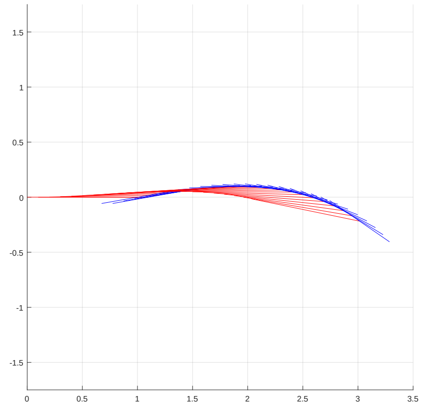

Fig. 5. In this picture, the red lines are the contact lines of the motorcycle at different time. The blue line is the projected steering angle of the front wheel. The time step is set to be $dt = 0.1s$, and the terminating time $T_f = 10s$. The $\dot{\sigma}$ is set to a constant equals to $-0.3m^{-1}$. We can see the motorcycle falls down at $t = 2.1s$.

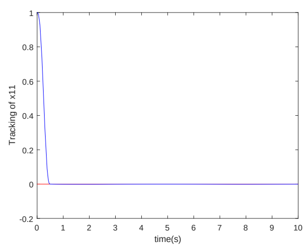

Fig. 6. The roll angle trajectory controlled by SMC. At first the motorcycle leans to the right, and is dragged back to zero less than 1 second.

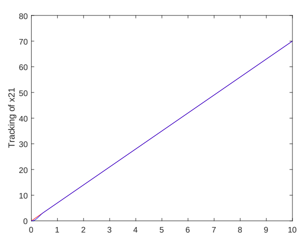

Fig. 7. $\int v_r$, which is the distance. At first the motorcycle falls behind. Later it keeps up the desired trajectory.

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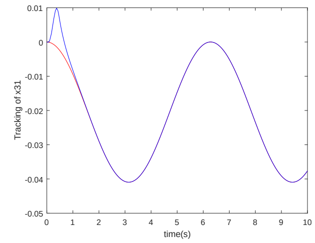

Fig. 8. The $\int \sigma$ controlled by SMC, which represents the accumulation of ”steering” in time. At first the motorcycle should turn right to follow a sine wave trajectory starting from zero point, so it should start to turn right, where σ is negative. But actually it is turning left where σ is positive. So We can see the controller want to decrease the integral of σ to negative. As the trajectory is periodic, the $\int \sigma$ is also periodic.

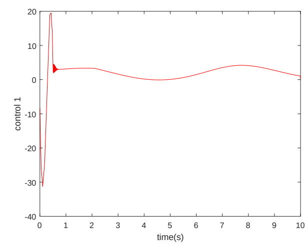

Fig. 9. $\dot{\sigma}$ generated by SMC controller. As $\sigma = \frac{1}{b} tan\phi$, the controller first accelerate the front wheel from truing left to right, and decrease the negative angular velocity to make it follow the desired trajectory.

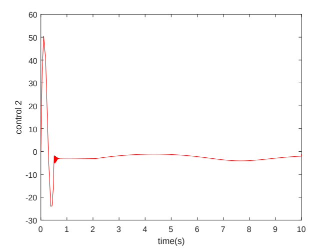

Fig. 10. τ_r generated by SMC controller. First the controller accelerate the speed to catch up with the trajectory, and decrease the speed to follow it.

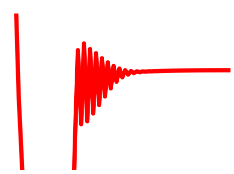

Fig. 11. Shaking of τ. Zoom in the figure of 10, we can see the control input shakes when it switches from large-scale adjustment (approaches to sliding surface) to gentle time-variant value (slide on sliding surface).

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