University of Toronto
CSC418 Computer Graphics

CSC418 A1 Part A

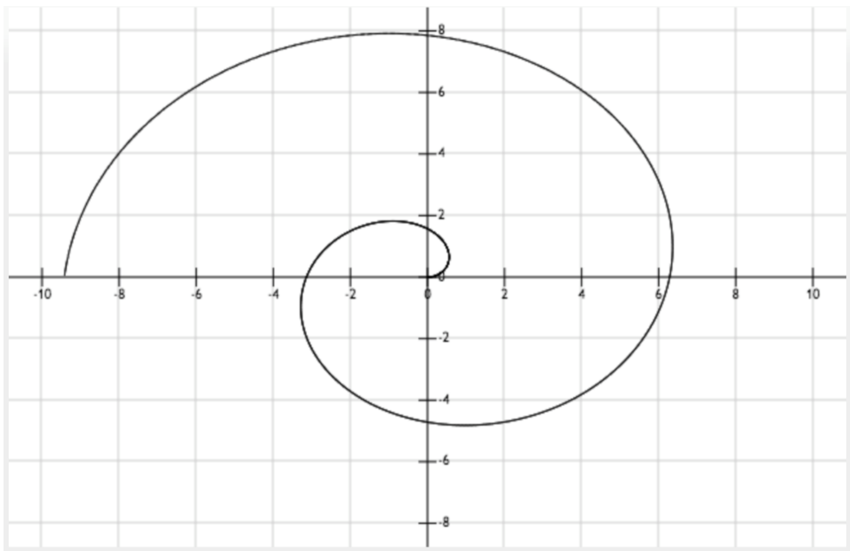
XIARUI HUANG

1001117695

1. A 2D curve can be described parametrically as follows.

$$x(t) = t\ cost$$
$$y(t) = t\ sint$$
$$0 < t < 3\pi$$

a)



b) The tangent vectors are the derivative of the given parametric function.

x component of the tangent vector:

$$\frac{dx(t)}{dt} = cost - t\ sint$$

y component of the tangent vector:

$$\frac{dy(t)}{dt} = sint + t\ cost$$

Therefore, the tangent vectors of this 2D curve can be represented as $\mathrm{T}(t) = (\mathrm{cost} - \mathrm{t\ sint},\ \mathrm{sint} + \mathrm{t\ cost})$.

The normal vectors N(t), are perpendicular/orthogonal to tangent vectors, $\mathrm{T(t)} * N(t) = 0$.

By following this property, we can easily get the normal vector $\mathrm{N(t)} = (\mathrm{sint} + \mathrm{t\ cost},\ \mathrm{t\ sint} - \mathrm{cost})$.

2. Two transformations $f1$ and $f2$ commute when $f1 \circ f2 = f2 \circ f1$. For each pair of transformations below, specify whether or not they commute. In each case, your solution can either be a derivation that proves /disproves commutativity, or if $f1$ and f $f2$ do not commute, a specific counter-example.

   a) Assume there exists two different translations $f1$ and $f2$ as below, where $t_{x1} \neq tx_2$ *and* $t_{y1} \neq t_{y2}$.

$$f1 = \begin{bmatrix} 1 & 0 & t_{x1} \\ 0 & 1 & t_{y1} \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad f2 = \begin{bmatrix} 1 & 0 & t_{x2} \\ 0 & 1 & t_{y2} \\ 0 & 0 & 1 \end{bmatrix}$$

$$f = f1 * f2 = \begin{bmatrix} 1 & 0 & t_{x1} \\ 0 & 1 & t_{y1} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & t_{x2} \\ 0 & 1 & t_{y2} \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & t_{x2} + t_{x1} \\ 0 & 1 & t_{y2} + t_{y1} \\ 0 & 0 & 1 \end{bmatrix}$$

$$f' = f2 * f1 = \begin{bmatrix} 1 & 0 & t_{x2} \\ 0 & 1 & t_{y2} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & t_{x1} \\ 0 & 1 & t_{y1} \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & t_{x1} + t_{x2} \\ 0 & 1 & t_{y1} + t_{y2} \\ 0 & 0 & 1 \end{bmatrix}$$

$$= f$$

By the above calculation, two different translation in two dimensions maintain the property of commutativity $f1 \circ f2 = f2 \circ f1$.

   b) Assume there exists two different rotations $R(\theta)$ *and* $R(\varphi)$ where $\theta \neq \varphi$.

$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad \text{and} \quad R(\varphi) = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}$$

$$R = R(\theta) * R(\varphi) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\theta)\cos(\varphi) - \sin(\theta)\sin(\varphi) & -\cos(\theta)\sin(\varphi) - \sin(\theta)\cos(\varphi) \\ \sin(\theta)\cos(\varphi) + \cos(\theta)\sin(\varphi) & -\sin(\theta)\sin(\varphi) + \cos(\theta)\cos(\varphi) \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\theta+\varphi) & -\sin(\theta+\varphi) \\ \sin(\theta+\varphi) & \cos(\theta+\varphi) \end{bmatrix}$$

$$R' = R(\varphi) * R(\theta) = \begin{bmatrix} \cos\varphi & -sin\varphi \\ sin\varphi & cos\varphi \end{bmatrix} \begin{bmatrix} \cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\varphi)\cos(\theta) - \sin(\varphi)\sin(\theta) & -\cos(\varphi)\sin(\theta) - \sin(\varphi)\cos(\theta) \\ \sin(\varphi)\cos(\theta) + \cos(\varphi)\sin(\theta) & -\sin(\varphi)\sin(\theta) + \cos(\varphi)\, cos(\theta) \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\theta+\varphi) & -\sin(\theta+\varphi) \\ \sin(\theta+\varphi) & \cos(\theta+\varphi) \end{bmatrix} = R$$

Therefore, two different rotations are commutative.

c) Assume there exists two transformations, one is rotation $R(\theta)$ and another one is uniform scaling S.

$$R(\theta) = \begin{bmatrix} \cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix} \quad \text{and} \quad S = \begin{bmatrix} \mathrm{S}_1 & 0 \\ 0 & S_1 \end{bmatrix}$$

$$f = R(\theta) * S = \begin{bmatrix} \cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix} \begin{bmatrix} \mathrm{S}_1 & 0 \\ 0 & S_1 \end{bmatrix}$$

$$= S_1 \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= S_1 \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$f' = S * R(\theta) = \begin{bmatrix} \mathrm{S}_1 & 0 \\ 0 & S_1 \end{bmatrix} \begin{bmatrix} \cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix}$$

$$= S_1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$= S_1 \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$= f$$

Therefore, rotation and uniform scaling can be commutative.

d) Assume there exists two transformations, one is rotation $R(\theta)$ and another one is non-uniform scaling S, where $S_x \neq S_y$.

$R(\theta) = \begin{bmatrix} \cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix}$ and $S = \begin{bmatrix} S_x & 0 \\ 0 & S_y \end{bmatrix}$

$$f = R(\theta) * S = \begin{bmatrix} \cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix} \begin{bmatrix} S_x & 0 \\ 0 & S_y \end{bmatrix}$$

$$= \begin{bmatrix} S_x\cos\theta & -S_y\sin\theta \\ S_x\sin\theta & S_y\cos\theta \end{bmatrix}$$

$$f' = S * R(\theta) = \begin{bmatrix} S_x & 0 \\ 0 & S_y \end{bmatrix} \begin{bmatrix} \cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} S_x\cos\theta & -S_x\sin\theta \\ S_y\sin\theta & S_y\cos\theta \end{bmatrix} \neq f$$

Therefore, a rotation and a non-uniform scaling is not commutative.

3.Points (−1, 0), (0, 1), (0, 0), (1, 1) will be mapped to points (6.3, 2.3), (6.3, 3.7), (7.7, 2.3), (7.7, 3.7) by an affine transformation.

a) By connecting adjacent points, we can easily get a parallelogram. Do the same process of transformed points, it forms a square. Assume there exists an Affine Transformation M. By educational guessing, M can be decomposed into a shear followed by scale followed by translation. Then M can be rewriting as M=T*S*K, where T stands for translation, S stands for scaling, K stands for shearing.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = M \begin{bmatrix} x \\ y \end{bmatrix} \qquad (1)$$

Since the transformed pattern is a square and we want to shear the parallelogram to a rectangle first.

$$\begin{bmatrix} x_k \\ y_k \end{bmatrix} = \begin{bmatrix} 1 & m \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \qquad (2)$$

We derive the equation as below:

$$x_k = x + m$$

$$y_k = y$$

In order to transform to a square, the angle should be 90 degree, which is equivalent to transform the point (0,1) to point (-1,1), and (1,1) to (0,1).

By substituting point (0,1) and (-1,-1) in above equations: we can easily get $m = -1$

$$\mathrm{K} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

Then the pattern after shearing becomes a unit square. And the scaling factor can be easily calculated, just one side length of large square.

$$s = 7.7 - 6.3 = 1.4$$

$$\mathrm{S} = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} = \begin{bmatrix} 1.4 & 0 \\ 0 & 1.4 \end{bmatrix}$$

Assume the simple translation matrix can be interpreted as below:

$$T = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

Then M can be interpreted as follow:

$$M = T * S * K = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1.4 & 0 \\ 0 & 1.4 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1.4 & -1.4 & 0 \\ 0 & 1.4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1.4 & -1.4 & t_x \\ 0 & 1.4 & t_y \\ 0 & 0 & 1 \end{bmatrix} \quad (3)$$

Then (1) can become

$$\begin{bmatrix} x' \\ y' \end{bmatrix} \equiv \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1.4 & -1.4 & t_x \\ 0 & 1.4 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \quad (4)$$

We derived the equation as below:

$$x' = 1.4x - 1.4y + t_x \quad (5)$$

$$y' = 1.4y + t_y \quad (6)$$

By substituting any given transformed points, (6.3,3.7), etc. We can get

$$t_x = 7.7 \quad t_y = 2.3$$

Therefore

$$M = \begin{bmatrix} 1.4 & -1.4 & 7.7 \\ 0 & 1.4 & 2.3 \\ 0 & 0 & 1 \end{bmatrix}$$

b) Under this transformation, substituting points into equation (5)&(6) to get the mapped points, where $t_x = 7.7\ and\ t_y = 2.3$.

$$x' = 1.4 * (-3) - 1.4 + 7.7 = 2.1$$

$$y' = 1.4 + 2.3 = 3.7$$

Therefore, the point (-3,1) under the transformation can be mapped to the point (2.1,3.7).

4. Decompose the following 2D affine transformation into a translation followed by a scale followed by a rotation. Simply write the steps of how to calculate them.

$$\begin{bmatrix} 0 & 1.5 & 4.5 \\ -1.5 & 0 & -1.5 \\ 0 & 0 & 1 \end{bmatrix}$$

Assume this 2D affine transformation matrix called M.

As mentioned in the problem, M= R*S*T, where R is a rotation, S is a scale T is translation.

$$\mathrm{T} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \qquad \mathrm{S} = \begin{bmatrix} S_x & 0 & 0 \\ 0 & S_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad R(\theta) = \begin{bmatrix} \cos\theta & -sin\theta & 0 \\ sin\theta & cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Since M can be derived as R*(S*T).

$$P = S * \mathrm{T} = \begin{bmatrix} S_x & 0 & 0 \\ 0 & S_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} S_x & 0 & S_x t_x \\ 0 & S_y & S_y t_y \\ 0 & 0 & 1 \end{bmatrix}$$

$$M = R * \mathrm{P} = \begin{bmatrix} \cos\theta & -sin\theta & 0 \\ sin\theta & cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} S_x & 0 & S_x t_x \\ 0 & S_y & S_y t_y \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} S_x \cos\theta & -\mathrm{S_y} sin\theta & S_x t_x cos\theta - S_y t_y sin\theta \\ S_x sin\theta & S_y cos\theta & S_x t_x sin\theta + S_y t_y cos\theta \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1.5 & 4.5 \\ -1.5 & 0 & -1.5 \\ 0 & 0 & 1 \end{bmatrix}$$

By looking the calculation above, we can get the several relationships shown in below.

1) $S_x \cos\theta = 0$
2) $S_y \cos\theta = 0$
3) $S_x sin\theta = -1.5$
4) $-\mathrm{S_y} sin\theta = 1.5$
5) $S_x t_x cos\theta - S_y t_y sin\theta = 4.5$
6) $S_x t_x sin\theta + S_y t_y cos\theta = -1.5$

By using 1 and 4 relationships, we can simplify 5 into $1.5t_y = 4.5$, which can get $t_y = 3$

Similarly, by using 2 and 3 relationships, we can simplify 6 into $-1.5t_x = -1.5$, which can get $t_x = 1$

Meanwhile, we can use relation 3 and 4, such as $\frac{S_y sin\theta}{S_x sin\theta} = 1$.

Therefore, we can easily get the relationship between S_y and S_x, where $S_x = S_y \neq 0$.

Since $S_x = S_y \neq 0$, from relation 1) $S_x \cos\theta = 0$, we can easily calculate the angle θ by computing $\cos\theta = 0$. Thus $\theta = 90° = \frac{\pi}{2}$.

According to the calculated angle, we can find that $S_x = S_y = -1.5$

Therefore, by the calculation above, the above transformations can be shown as blow. It need to be mentioned in a scaling matrix, it could be many optional answers.

$$\mathrm{T} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \qquad \mathrm{S} = \begin{bmatrix} -1.5 & 0 & 0 \\ 0 & -1.5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R\left(\frac{\pi}{2}\right) = \begin{bmatrix} \cos\frac{\pi}{2} & -sin\frac{\pi}{2} & 0 \\ sin\frac{\pi}{2} & cos\frac{\pi}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$