Theorem (Progress) : Suppose e is a well-typed term. Then either e is a value or else there is e' with $e \to e'$.

```
;; Typing judgment for c-b-v cons
[(⊢_e Γ e_1 T_1)
   (⊢_e Γ e_2 (List T_1))
   --------------------------------- "T-CONS"
   (⊢_e Γ (cons e_1 e_2) (List T_1))]

;; Eval-context for c-b-v cons
   (E-value ...
           (cons E-value e)
           (cons v E-value)
           ...
```

proof. By induction on a derivation of $e : T$.

(other cases are skipped)

Case T-CONS:

$e = \text{cons } e_1 e_2 : \text{List } T$

$e_1 : T, e_2 : \text{List } T$

By induction hypothesis, either e_1 is a value or there exists e' such that $e \to e'$. Similarly, e_2 is either a value or steps to e_2'

If e_1 is a value and e_2 is a value, e is a value by:

```
  (v .... (cons v v))
```

If e_1 is a value and e_2 steps to e_2', e steps to $e' = \text{cons } e_1 e_2'$ by eval-context:

```
(E-value ...
        (cons v E-value)
        ...)
```

If e_1 steps to e_1', then e steps to cons $e_1' e_2$ by eval-context:

```
(E-value ...
```

```
        (cons E-value e)
        ...)
```

We showed in each case, e is either a value or $e \to e'$. $\square$

Theorem (Preservation) : If $e : T$ and $e \to e'$, then $e' : T$.

proof. By induction on derivation of $e : T$.

(other cases are skipped)

Case T-CONS:

$e = \text{cons } e_1 e_2 : \text{List } T$

$e_1 : T, e_2 : \text{List } T$

If the last rule in the derivation is **T_CONS**, then we know from the form of this rule that e must have the form cons $e_1 e_2$ for some e_1 and e_2.

We must also have subderivations with conclusions $e_1 : T$ and $e_2 : \text{List } T$. Now looking at the eval-context, we find that there are two rules by which $e \to e'$ can be derived.

```
    (E-value ...
            (cons E-value e)
            (cons v E-value)
            ...
```

- cons E-value e

Then cons $e_1 e_2 \to$ cons $e_1' e_2$. We have a subderivation of the original typing whose conclusion is $e_1 : T$. We can apply the I.H. to this, obtaining $e_1' : T$. Combining this with the facts that $e_2 : \text{List } T$, we can apply the **T-CONS** rule that cons $e_1' e_2 : \text{List } T$.

- cons v E-value

Then cons $e_1 e_2 \to$ cons $e_1 e_2'$ and e_1 is a value. We have a subderivation of the original typing that $e_2 : \text{List } T$. By the I.H. $e_2' : \text{List } T$. Combining this with the facts that $e_1 : T$, we conclude that cons $e_1 e_2' : \text{List } T$. $\square$