Review I(Slides 20 - 69)

Logics,Sets,Numbers

"Without them, mathematics will fall apart... "

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VV186 - Honors Mathmatics II

About Me

Schedule

- RC: Thursday 16:00 - 17:40
- OH: Thursday 17:40 - 19:40

Any questions or advice ? Please contact me !

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Something more about me:

- Name: Heyinong
- Nickname: Kulu (A long story behind)
- JI SSTIA Minister
- Love music, love piano. Enjoy everything a normal student in University may enjoy.

Statement

Frequently asked questions: What's the difference between statement and a statement frame ? What's a predicate ? First, recall the definition...

Statements: Sentence that's T or F definitely.

Eg: A: $1 > 0$ (T)

B: Dino (F).

- statement
  - true statement
  - false statement
- statement structure
  - quantifier
  - statement frame/predicate
  - specific value

When there's a variable in a statement:

Eg: $A(x): x^3 > 0$.

Quantifier $\underbrace{\forall x > 0 . A(x)}_{T}$

or

Specific Value x 1

Easy... but be careful!

Definitions? Examples? Notations?

Induction: $A(n) \Rightarrow A(n+1)$ $\begin{cases} A(1) \\ A(-1) \end{cases}$ True Start--

Logical Operation

Priority : Eg: $A \vee \neg B \vee C$
$A \vee \neg(B \vee C)$

Question: Are you familiar with their truth tables and notations?

$(x_1 \wedge x_2 \wedge x_3 \cdots \wedge x_n)$

$(y_1 \vee y_2 \vee y_3 \cdots \vee y_n)$

Type	Logical Operation		Priority
unary	$\neg$	Negation	1
binary	$\wedge$	Conjunction	2
	$\vee$	Disjunction	3
	$\Rightarrow$	Implication	4
	$\Leftrightarrow$	Equivalence	5

$3 > 1$

- compound statement
  - *tautology*
  - *contradiction*
  - *contingency*

Assign the variables with all possibilities of values.

A	B	$A \vee B$
T	T	T
F	F	F

Tautology

Q&A: How to interpret implication : $A \Rightarrow B$?

$(P \wedge Q) \rightarrow P$

$\Leftrightarrow \neg(P \wedge Q) \vee P \Leftrightarrow \neg P \vee \neg Q \vee P$

TRUTH TABLE

		$A \Rightarrow B$
T	T	T
T	F	F
F	T	T
F	F	T

- A implies B / if A then B / A only if B
- $A \Rightarrow B \equiv \neg(A \wedge \neg B) \equiv \neg A \vee B$ (Proof by Contradiction)
- vacuous truth
  - pink elephants could fly!
  - more examples?
  - A vacuous truth may have the form similar to :
    - ★ $A \Rightarrow B$,where A is false
    - ★ $\forall x, A(x) \Rightarrow B(x)$, where $\forall x, \neg A(x)$
    - ★ $\forall x \in A, B(x)$, where A is an empty set.

Interpret it! $A \Rightarrow B$ A is true then B is true

So $(A \wedge \neg B)$ is False.

$\neg(A \wedge \neg B)$, TRUE.

Observe: for $A \Rightarrow B$:

if A is False , $A \Rightarrow B$ is always TRUE.

if B is TRUE , $A \Rightarrow B$ is always TRUE.

Q&A: What's the difference between **equivalence($\Leftrightarrow$)** and **logically equivalent($\equiv$)**?

Related but Different !

- equivalence ($\Leftrightarrow$) is just one of the binary logical operators. It inputs two statements and returns a single statement. It can be **either true or false**.
  Conventionally, we can just understand it as "if and only if".
- logically equivalent ($\equiv$) tells the relationship between two statements. $A \equiv B$ if they have same truth value for all possible combinations of truth values assigned to all variables appearing in A and B. i.e. $A \Leftrightarrow B$ is a tautology.

$A \equiv B$

A	B	$A \Leftrightarrow B$
T	T	T
T	F	T

Truth Table

How to prove logically equivalent? Use truth table !

- How to use a truth table?
  - Understand the problem is about
  - Cover all the possible situations
  - e.g. Prove ***de Morgen rules*** for statements and sets:

$$\neg(A \wedge B) \equiv (\neg A) \vee (\neg B) \qquad \neg(A \wedge B) \equiv (\neg A) \vee (\neg B)$$

$$(A \cap B)^c = A^c \cup B^c \qquad (A \cup B)^c = A^c \cap B^c$$

How to prove a logical equivalence? Truth table !
How to prove two sets P=Q? (I)$P \subset Q$ (II)$Q \subset P$

Observation: Sets and statements are similar.
Why are ***de Morgen rules*** so important?

★ switch between $\vee$ and $\wedge$ using $\neg$. (Frequent in Exercises)

$A: \{x: P(x)\}$ $B: \{x: Q(x)\}$. $\underline{M: \{x: \forall x\}}$.

$$A \cap B: \{x: P(x) \land Q(x)\}.$$

$$(A \cap B)^c: \{x: \neg(P(x) \land Q(x))\}.$$

$A^c \cup B^c$

$$A^c: \{x: \neg P(x)\}$$

$$B^c: \{x: \neg Q(x)\}.$$

$$\underline{(A \cap B)^c = A^c \cup B^c.}$$

$$\Longleftrightarrow \neg(P(x) \land Q(x)) \equiv (\neg P(x)) \lor (\neg Q(x))$$

$$\neg(A \wedge B) \equiv (\neg A) \vee (\neg B)$$

columns

A	B	$\neg A$	$\neg B$	$A \wedge B$	$\neg(A \wedge B)$	$(\neg A) \vee (\neg B)$
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

Relations

$A \Rightarrow B.$

Those two relations are useful for our mindset when facing an exercise.

- Proof by Contradiction $A \quad \neg B.$ two conditions.

$$(A \Rightarrow B) \equiv \neg(A \wedge \neg B)$$

We just have one condition A at first, through proof by contradiction, we have two conditions: A and B. So you have one more condition to play with.

- Proof by Contraposition

$$(A \Rightarrow B) \equiv (\neg B \Rightarrow \neg A)$$

Through contraposition, we don't get addition conditions, but the mindset is changed. If it's easier to use B than A, you can try thinking about contraposition.

An exercise from assignment

Exercise 1.4

Suppose that a truth table in n propositional variables is specified. Show that a compound proposition with this truth table can be formed by taking the disjunction of conjunctions of the variables or their negations, with one conjunction for each combination of values for which the compound proposition is true. The resulting compound proposition is said to be in *disjunctive normal form*.

For example, the table with the two propositional variables A and B,

A	B	$f(A,B)$
T	T	T
T	F	F
F	T	T
F	F	F

has the disjunctive normal form $f(A,B) = (A \wedge B) \vee (\neg A \wedge B)$.
(2 Marks)

An example to illustrate DNF and CNF

Just an example.

$$((p \vee q) \Rightarrow r) \Rightarrow p$$

$\Leftrightarrow (\neg(p \vee q) \vee r) \Rightarrow p$, Using $(A \Rightarrow B \equiv \neg A \vee B)$ here.

$\Leftrightarrow \neg(\neg(p \vee q) \vee r) \vee p$, Using $(A \Rightarrow B \equiv \neg A \vee B)$ here.

$\Leftrightarrow ((p \vee q) \wedge \neg r) \vee p$, Using $(\neg(A \wedge B) \equiv (\neg A \wedge \neg B))$ here.

$\Leftrightarrow (p \vee q) \wedge (\neg r \vee p)$, CNF Using $(A \wedge B) \vee C \equiv (A \vee C) \wedge (B \vee C)$

$\Leftrightarrow (p \wedge \neg r) \vee (q \wedge \neg r) \vee p$, DNF Using $(A \vee B) \wedge C \equiv (A \wedge C) \vee (B \wedge C)$

$$((p \vee q) \wedge \neg r) \vee p \equiv ((p \vee q) \vee p) \wedge (\neg r \vee p).$$

DNF and CNF

$(\wedge \cdots \wedge) \vee \cdots \vee (\wedge \cdots \wedge)$

$(\vee \cdots \vee) \wedge \cdots \wedge (\vee \cdots \vee)$.

- Definition for DNF and CNF
  - Disjunctive Normal Form (DNF): A disjunction of one or more conjunctions of one or more variables or their negations.
  - Conjunctive Normal Form (CNF): A conjunction of one or more disjunctions of one or more variables or their negations.
- Principal Disjunctive Normal Form & Principal Conjunctive Normal Form

Logical Quantifiers

Logical Quantifiers		
Sign	Type	Interpretation
$\forall$	universal	for any; for all
$\exists$	existential	there exist; there is some
$\forall \ldots \forall \ldots$	nesting quantifier	for all ...for all ...
$\exists \ldots \exists \ldots$	nesting quantifier	there exists ...(such that) there exist ...
$\forall \ldots \exists \ldots$	nesting quantifier	for any ..., there exists ...
$\exists \ldots \forall \ldots$	nesting quantifier	there exists ...(such that) for any ...
...	...	...

Quick Check: $\exists y: y+x^2>0$. $\forall x$

- Hanging Quantifier, be careful with the order. (In Slide 39)

is equivalent to ① $\exists y \forall x,\ y+x^2>0$ ✓

or ② $\forall x \exists y.\ y+x^2>0$. ✗

Order Matters when quantifiers are different

$$\begin{cases} \forall x\, \exists y : x+y>0 \text{ is a true statement. ①} \\ \exists x\, \forall y : x+y>0 \text{ is a false statement. ②} \end{cases}$$

Exercise 1.3

Explain in your own words the difference between the statements

$$\underset{0\in\mathbb{Q}}{\exists}\ \underset{a\in\mathbb{Q}}{\forall}\ a+0=0+a=a \qquad \text{and} \qquad \underset{a\in\mathbb{Q}}{\forall}\ \underset{0\in\mathbb{Q}}{\exists}\ a+0=0+a=a.$$

(4 Marks)

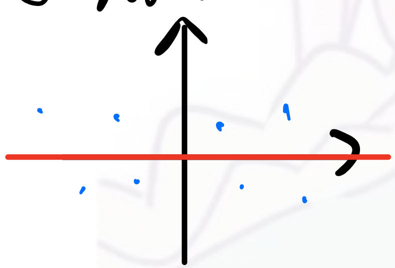
$\forall x\, \forall y$

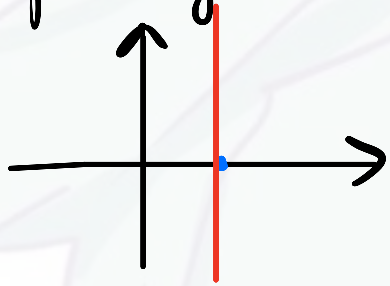
$\forall y\, \forall x$

$\forall x, y.$

① For all x. pick a $y = 1-x$, $x+y>0$.

② No matter what x, pick $y = -x-1$. $x+y<0$





$\exists x\, \exists y \Leftrightarrow \exists y\, \exists x.$

Negation for logical quantifiers

Use quantifiers to rewrite the following definition of covergence:

Let $(a_n)_{n\in\mathbb{N}}$ be a real sequence. If for some fixed $a \in \mathbb{R}$, for any $\varepsilon > 0$, there is an $N \in \mathbb{N}$, such that for all $n > N$, $|a_n - a| < \varepsilon$, then we say (a_n) converges to a.

What's the negation of this statement?

Steps to write out a negation of a statement:
(I) Transfer every $\forall$ to $\exists$ and transfer every $\exists$ to $\forall$
(II) Take the negation of predicates

Let $(a_n)_{n\in\mathbb{N}}$ be a real sequence. If for some fixed $a \in \mathbb{R}$, for any $\varepsilon > 0$, there is an $N \in \mathbb{N}$, such that for all $n > N$, $|a_n - a| < \varepsilon$, then we say (a_n) converges to a.

$\forall \varepsilon > 0$. $\exists N \in \mathbb{N}$, $\forall n > N$. $|a_n - a| < \varepsilon$.

$\varepsilon \leq 0$ X

$n \leq N$ X

$\exists \varepsilon > 0$, $\forall N \in \mathbb{N}$, $\exists n > N$ $|a_n - a| \geq \varepsilon$.

Diverge.

Functionally Complete

plug in all possibilities of values into the statement (TRUTH TABLE)

Exercise 1.5

A collection of logical operators is called *functionally complete* if every compound proposition is logically equivalent to a compound proposition involving only these logical operators.

i) Show that $\{\wedge, \vee, \neg\}$ is a functionally complete collection of logical operators. (Hint: use the disjunctive normal form.)
(1 Mark)

ii) Show that $\{\wedge, \neg\}$ is a functionally complete collection of logical operators. (Hint: use a de Morgan law.)
(1 Mark)

iii) The *Scheffer stroke* $|$ is a logical operation defined by

$$A \mid B :\equiv \neg(A \wedge B).$$

(In computer science, it is known as the NAND operation.[2]) Write down the truth table for the Scheffer stroke.
(1 Mark)

iv) Prove that $\{|\}$ is a functionally complete collection of logical operators.
(3 Marks)

v) Show that $|$ is commutative but not associative, i.e., $A \mid B \equiv B \mid A$ but $(A \mid B) \mid C \not\equiv A \mid (B \mid C)$.
(2 Marks)

2. Scheffer stroke (|) is also called "NAND". Similarly, Peirce arrow (↓) is called "NOR". $P \downarrow Q$ is true iff: P and Q are both false.
Please use "↓" only to construct the statement $P \Rightarrow Q$.

$P \Rightarrow Q$ is False iff: P True and Q false.

P	Q	$P \downarrow Q$
T	T	F
T	F	F
F	F	T
F	T	F

$P \downarrow Q \equiv \neg(P \vee Q)$. ①

$P \downarrow P \equiv \neg P$ ②

$P \Rightarrow Q \equiv \neg P \vee Q$

$P \vee Q \equiv (P \downarrow Q) \downarrow (P \downarrow Q)$.

$\neg P \vee Q \equiv ((P \downarrow P) \downarrow Q) \downarrow ((P \downarrow P) \downarrow Q)$.

Optional Questions

For one line in TRUTH TABLE.

Take a line

A_1	A_2	------	A_n	$f(A_1 \cdots A_n)$.
T	T	----	T	F.

Mathmetical Induction.

Last step:

1. How to show that $\{\wedge, \vee\}$ is **NOT** functionally complete? $F \wedge \cancel{A_n} \Rightarrow F$.

2. Scheffer stroke (|) is also called "NAND". Similarly, Peirce arrow ($\downarrow$) is called "NOR". $P \downarrow Q$ is true iff: P and Q are both false. Please use "$\downarrow$" only to construct the statement $P \Rightarrow Q$.

After n calculations, we could get F.

1° $n=1$ F.x

2° $n=K$ steps. F.

$n=K+1$ F.

Each time we calculate.

The number of operators in c s ↓.

Sets

- What is a set? $\{1, \{1\}, \text{Tree}\}$.
- How to interpret a set $\{1, 2, 3, 4, 3, 4\}$? The elments can be the same ?
  $S = \{x : x=1 \text{ or } x=2 \text{ or } x=3 \text{ or } x=4\}$.
- Common Type of Set
  - Empty set: $\emptyset := \{x : x \neq x\}$
  - Total set
  - Subset
  - Proper subset
  - Power set(finite)
    $\{\phi\}$. $\{\phi, \{\phi\}\}$.
    Question: What's the power set for $\emptyset$ and for $\{\emptyset\}$?
- Cardinality(finite)

$P(x) : x > 0$.

Quick check:

Let $X = \{x : P(x)\}$. Is $P(x)$ a statement? Predicate.

Operations on Sets

Let

$$A := \{1, 2\} \quad B := \{2, 3\} \quad M := \{1, 2, 3, 4, 5\}$$

Please recall the operations and calculate !

$A = \{x : P(x)\}$ $B = \{x : Q(x)\}$

Set Operations		Answer.
$A \cup B$	Union	$\{1, 2, 3\}$.
$A \cap B$	Intersection	$\{2\}$.
$A \setminus B$	Difference	$\{1\}$.
$A^c := M \setminus A$	Complement	$\{3, 4, 5\}$.

$\{x : P(x) \wedge \neg Q(x)\}$

- The notation $A - B$ is also used for $A \setminus B$ and $\bar{A}$ for A^c

Ordered Pairs

Asked in OH. why $\{a,\{b\}\}$ is not well-defined.

- What is an ordered pair? $(\{\{1\}\}, 1) = (\{1\}, \{1\})$
  - Property: (a,b)=(c,d)⇔ $(a = c) \wedge (b = d)$
  - Define ordered pairs using sets: (a,b):=$\{\{a\}, \{a, b\}\}$ Prove the property !
  - (a,b,c)? n-tuple? Recursive Definition !

$(a,b,c) = ((a,b), c)$.
$(a',b',c') = ((a',b'), c')$.

$\{\{a\},\{a,b\}\} = \{\{c\},\{c,d\}\}$

1° if $a=b$ then. $\{\{a\},\{a,b\}\} = \{\{a\}\} \Rightarrow \{c\} = \{c,d\} = \{a\}$

$c=d=a=b$.

2° if $a \neq b$. Check cardinality. $\begin{cases}\{a\}=\{c\}\\ \{a,b\}=\{c,d\}\end{cases}$

- Concept of *Cartesian product.*

$x \in R$ $y \in R$.

$$A \times B := \{(a, b) : a \in A, b \in B\}$$
$$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$$

$f(x,y) = x+y$.

$f: \mathbb{R}^2 \longrightarrow \mathbb{R}$.

Frequently used for denoting a domain of a function. $\downarrow$ $R \times R$.

Natural Number

Every time we study an operation in a algebra system, we start from studying those properties.

$\mathbb{N}$ $+$ $\cdot$

$(S, *)$ e

for all $x \in S$.

$e * x = x * e = x$.

Three properties for sum:

- $a + (b + c) = (a + b) + c$ (Associativity)
- $a + 0 = 0 + a = a$ (Existence of a neutral element)
- $a + b = b + a$ (Commutativity)

Four properties for product:

- $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ (Associativity)
- $a \cdot 1 = 1 \cdot a = a$ (Existence of a neutral element)
- $a \cdot b = b \cdot a$ (Commutativity)
- $a \cdot (b + c) = a \cdot b + a \cdot c$ (Distributivity)

$e_1 \neq e_2$.

$e_1 = e_1 \cdot e_2 = e_2$ contradict.

Remark: If the neutral element exists, it is unique. Why?

DIY in class: if $a \neq 0$ and $ab = ac$, then b=c.

Rational Number. inverse $b = a^{-1} ab = a^{-1} ac = c$

Mathematical Induction

Mathematical Induction is useful because when we don't have enough conditions to play with, it provides us with a condition.

The goal is to show that statement frame A(n) is true for all $n \in N$ with $n \geq n_0$ for some $n_0 \in \mathbb{N}$. Mathematical induction works by establishing two statements:

1° Easy Cases
2° Build Relations

Mathematical Induction I
(I) $A(n_0)$ is true.
(II) A(n+1) is true whenever A(n) is true for $n \geq n_0$.

Mathematical Induction II
(I) $A(n_0)$ is true.
(II) A(n+1) is true whenever A(k) is true for all $n_0 \leq k \leq n$.

Mathematical Induction I Exercise

Try to write a formal proof !

Let $a, b \in \mathbb{R}$, prove that $|a+b|^n \leq 2^{n-1}(|a|^n + |b|^n), n \in \mathbb{N}$

1° Base case let $n=1$ $|a+b| \leq |a| + |b|$

Triangle inequality.

2° Assume that the inequality holds true when $n=k$, Consider $n=k+1$. By Assumption. $(a+b)^k \leq 2^{k-1}(|a|^k + |b|^k)$

So we have $|a+b|^{k+1} \leq |a+b| \cdot 2^{k-1}(|a|^k + |b|^k)$

$\leq (|a|+|b|) \cdot 2^{k-1} \cdot (|a|^k + |b|^k)$.

It remains to be proved that: $(|a|+|b|)(|a|^k + |b|^k) \leq 2(|a|^{k+1} + |b|^{k+1})$

$\Leftrightarrow |a|^{k+1} + |b|^{k+1} - |a|^k|b| - |a||b|^k \geq 0 \Leftrightarrow (|a|^k - |b|^k)(|a| - |b|) \geq 0$.

1° $|a| > |b|$ / 2° $|a| < |b|$

Mathematical Induction II Exercise

Try to write a formal proof !

The Fibonacci sequence is defined as follows:

$$a_1 = 1$$

$$a_2 = 1$$

$$a_n = a_{n-1} + a_{n-2}; n > 2$$

Prove that:

$$a_n = \frac{(\frac{1+\sqrt{5}}{2})^n - (\frac{1-\sqrt{5}}{2})^n}{\sqrt{5}}$$

1° $n=1$, $n=2$

2° $n_0 \leq k \leq n$

$a_{n+1} = a_{n-1} + a_n$

Reference

- Exercises from 2021–Vv186 TA-Ni Yinchen .
- Exercises from 2021–Vv186 TA-Huang Yue .
- Learn from 2021–Vv186 TA-Ding Zizhao
- Learn from 2021–Vv186 TA-Ma Tianyi
- Learn from 2021–Vv186 TA-Sun Meng

Thanks & Have Fun !