Review VIII

Integral

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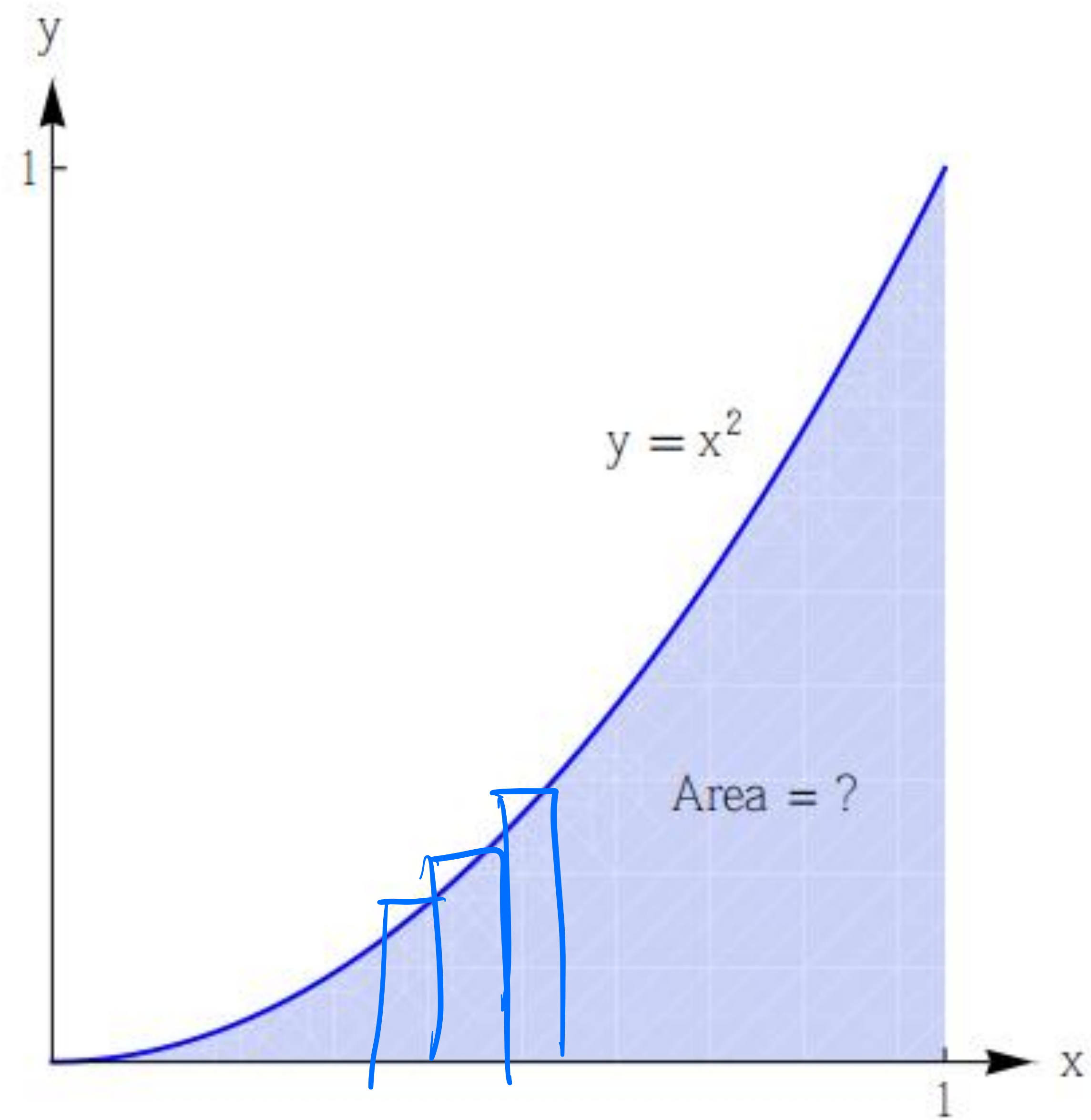
VV186 - Honors Mathmatics II


y
1
$y = x^2$
Area = ?
x
1

Our first step is to give a proper definition of a "piecewise constant" function and the area "under" its graph.

4.1.1. Definition. A ***partition P*** of an interval $[a, b] \subset \mathbb{R}$ is a tuple of numbers $P = (a_0, \ldots, a_n)$ such that

$$a = a_0 < a_1 < \ldots < a_n = b.$$

4.1.2. Definition. A function $\varphi \colon [a, b] \to \mathbb{R}$ is called a ***step function with respect to a partition*** $\boldsymbol{P = (a_0, \ldots, a_n)}$ if there exist numbers $y_i \in \mathbb{R}$, $i = 1, \ldots, n$, such that

$$\varphi(t) = y_i \qquad \text{whenever } a_{i-1} < t < a_i \tag{4.1.1}$$

for $i = 1, \ldots, n$. We denote the set of all step functions by $\mathrm{Step}([a, b])$

4.1.5. Definition. Let P be a partition of an interval $[a, b]$. We call a partition R of $[a, b]$ a ***sub-partition*** of P if $R \supset P$, where we naturally regard P and R as sets instead of tuples.

4.1.6. Remark. If $\varphi\colon [a, b] \to \mathbb{R}$ is a step function with respect to a partition P of $[a, b]$, then it as also a step function with respect to any sub-partition R of P.

4.1.7. Proposition. If φ, ψ are step functions on $[a, b]$ and $\lambda, \mu \in \mathbb{R}$, then $\lambda\varphi + \mu\psi$ is a step function on $[a, b]$.

4.1.8. Remark. This also shows that $\mathrm{Step}([a, b])$ is a vector space.

$\varphi \quad a_0 \quad a_1 \quad -- \quad a_n$

$\psi \quad b_0 \quad b_1 \quad -- \quad b_n.$

4.1.9. Definition and Theorem. Let $\varphi\colon [a,b] \to \mathbb{R}$ be a step function of the form (4.1.1) with respect to some partition P. Then

$$I_P(\varphi) := (a_1 - a_0)y_1 + \cdots + (a_n - a_{n-1})y_n \qquad (4.1.2)$$

is independent of the choice of the partition P. We define the ***integral*** of f as

$$\int_a^b \varphi := I_P(\varphi)$$

for any partition P with respect to which φ is a step function.

4.1.11. Definition. Let $I \subset \mathbb{R}$ be an interval. We say that a function $f\colon I \to \mathbb{R}$ is ***bounded*** if

$$\|f\|_\infty := \sup_{x\in I} |f(x)| < \infty. \qquad (4.1.5)$$

4.1.13. Definition. A function $f \in L^\infty([a, b])$ is said to be ***regulated*** if for any $\varepsilon > 0$ there exists a step function φ such that

$$\sup_{x\in[a,b]} |f(x) - \varphi(x)| < \varepsilon. \tag{4.1.6}$$

4.1.15. Theorem. The continuous functions on the interval $[a, b]$ are regulated, i.e., $C([a, b]) \subset \operatorname{Reg}([a, b])$.

4.1.18. Definition and Theorem. Let $f \in \operatorname{Reg}([a, b])$ and (φ_n) a sequence in $\operatorname{Step}([a, b])$ converging uniformly to f. Then the ***regulated integral*** of f, defined by

$$\int_a^b f := \lim_{n\to\infty} \int_a^b \varphi_n \tag{4.1.8}$$

exists and does not depend on the choice of (φ_n).

4.1.16. Definition. Let $I \subset \mathbb{R}$ be an interval, $\{a_1, \ldots, a_n\} \subset I$ a finite set of points and $f : I \to \mathbb{C}$ a function such that

(i) $f : I \setminus \{a_1, \ldots, a_n\} \to \mathbb{C}$ is continuous and

(ii) For all a_i, $i = 1, \ldots, n$, both

$$\lim_{x \nearrow a_i} f(x) \qquad \text{and} \qquad \lim_{x \searrow a_i} f(x)$$

exist.

Then f is said to be ***piecewise continuous***. We denote the set of piecewise continuous functions on I by $\mathrm{PC}(I)$.

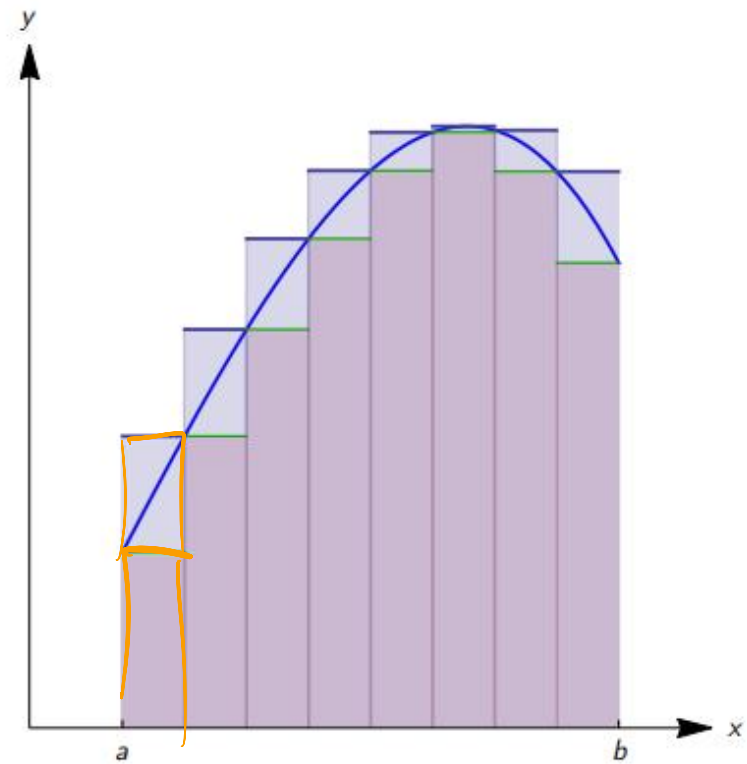
4.1.17. Theorem. The piecewise continuous functions on the interval $[a, b]$ are regulated, i.e., $\mathrm{PC}([a, b]) \subset \mathrm{Reg}([a, b])$.

$1 \quad x = \frac{1}{n}$

0

In contrast, we will now try to approximate the area under f, ***without attempting to approximate f at all***.

We will do this by considering step functions that are larger than f (i.e., $\varphi(x) \geq f(x)$ for all x) and step functions that are smaller than f ($\psi(x) \leq f(x)$ for all x), as illustrated at right.



$$f : [0,1] \to \mathbb{R}, \qquad f(x) = \begin{cases} 1 & \text{if } x = 1/n \text{ for some } n \in \mathbb{N} \setminus \{0\}, \\ 0 & \text{otherwise.} \end{cases}$$

4.1.22. Definition. Let $[a,b] \subset \mathbb{R}$ be a closed interval and f a bounded real function on $[a,b]$. Let $\mathscr{U}_f$ denote the set of all step functions u on $[a,b]$ such that $u \geq f$ and $\mathscr{L}_f$ the set of all step functions v on $[a,b]$ such that $v \leq f$. The function f is then said to be ***Darboux-integrable*** if

$$\underline{I}(f) = \sup_{v \in \mathscr{L}_f} \int_a^b v = \inf_{u \in \mathscr{U}_f} \int_a^b u = \overline{I}(f).$$

In this case, the ***Darboux integral of f***, $\int_a^b f$, is defined to be this common value.

4.1.26. Theorem. If $f \in \operatorname{Reg}([a,b])$, then f is Darboux-integrable and the Darboux integral coincides with the regulated integral.

A ***tagged partition*** (P, Ξ) on an interval $[a,b]$ consists of a partition $P = \{x_0, \dots, x_n\}$ together with numbers $\Xi = \{\xi_1, \dots, \xi_n\}$ such that each $\xi_k \in [x_{k-1}, x_k]$. The ***mesh size*** of P is defined as

$x_k - x_{k-1}$

$$m(P) := \max_{k=1,\dots,n} (x_k - x_{k-1})$$

A step function $\varphi \in \operatorname{Step}([a,b])$ for a function $f \in L^\infty([a,b])$ with respect to (P, Ξ) is given by

$$\varphi(x) = f(\xi_k) \qquad \text{for } x_{k-1} < x < x_k, \qquad k = 1, \dots, n,$$

where as usual $\varphi(x_k)$ can be defined in an arbitrary manner. The sum

$$\int_a^b \varphi := \sum_{k=1}^n f(\xi_k)(x_k - x_{k-1}) \tag{4.1.12}$$

is then called a ***Riemann sum*** for f.

4.1.28. Definition. Let $[a, b] \subset \mathbb{R}$ be a closed interval and f a bounded real function on $[a, b]$. Then f is Riemann-integrable with integral

$$\int_a^b f \in \mathbb{R}$$

if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for any tagged partition (P, Ξ) on $[a, b]$ with mesh size $m(P) < \delta$

$$\left| \sum_{k=1}^{n} f(\xi_k)(x_k - x_{k-1}) - \int_a^b f \right| < \varepsilon.$$

4.1.29. Theorem. A bounded real function is Riemann-integrable on $[a, b]$ if and only if it is Darboux-integrable. Moreover, the values of the integrals coincide.

Properties of the Integral

P532

If the area between "upper" and "'lower" step functions can be made as small as desired, we will have successfully approximated the area of f without necessarily needing to approximate f. We will now formalize these ideas.

We have proven (or can easily check) that the regulated integral has the following properties:

(i) The integral is linear, i.e., $\int(\lambda f + \mu g) = \lambda \int f + \mu \int g$ for $\lambda, \mu \in \mathbb{R}$;

(ii) The integral is positive, i.e., if $f > 0$ on $[a, b]$, then $\int f > 0$;

(iii) The integral of a constant function $f = c$, $c \in \mathbb{R}$, on $[a, b]$ is given by $\int f = c \cdot (b - a)$;

(iv) The integral does not depend on the values of a function on finite sets. If $f(x) = 0$ for all but a finite set of $x \in [a, b]$, then $\int f = 0$.

4.2.3. Lemma. Let $f \in C([a, b], \mathbb{C})$ and F be a primitive of f. Then

$$\int_a^b f = F(b) - F(a).$$

$$\int_1^2 x\,dx. \qquad \frac{x^2}{2}\Big|_1^2 = \frac{2^2}{2} - \frac{1^2}{2} = \frac{3}{2}.$$

$$\left(\frac{x^2}{2}\right)' = x.$$

Integration Method

www. integral- calculator. com.

Method0: Symmetry

Suppose $f(x)$ is an odd integrable function, then

$$\int_{-a}^{a} f(x)dx = 0$$

Exercises: For $a > 0$, calculate

$$\int_{-a}^{a} \frac{\cos(x)}{1 + even(x)^{odd(x)}} dx$$

where $even(x)$ is a continuous strictly positive even function, and $odd(x)$ is an odd function

$$\int_{-a}^{a} \frac{\cos(x)}{1 + even(x)^{odd(x)}} dx$$

$f(x)$

$$\int_0^a \cos x \, dx.$$

$$= \sin x \Big|_0^a = \sin a.$$

$$f(x) + f(-x) = \frac{\cos x}{1 + even(x)^{odd(x)}} + \frac{\overset{\cos x}{\overset{/\!/}{\cos(-x)}}}{1 + even(x)^{-odd(x)}}$$

$$= \frac{1 + 1 + even^{odd} + even^{-odd}}{1 + 1 + even^{odd} + even^{-odd}} \cos x$$

$$= \cos x.$$

Integration Method

Method1: Recite!

Common indefinite integrals include:

- $\int x^{\alpha} dx = \frac{x^{\alpha+1}}{\alpha+1} + C$
- $\int e^x dx = e^x + C$
- $\int \frac{1}{x} dx = \ln(|x|) + C$
- $\int \cos x \, dx = \sin x + C$
- $\int \sin x \, dx = -\cos x + C$
- $\int \frac{1}{1+x^2} dx = arctan(x) + C$
- $\int \ln(x) = ?$ $x\ln x - x$.

$\int \tan x \, dx$. $-\ln|\cos x|$.

For more complex integrals, we need other theorems to help us evaluate them.

Exercise

Calculte the following integrals:

- $$\int \frac{2}{\sqrt{x^3}} dx$$

- $$\int \frac{1}{x^2 + 6x + 5} dx$$

$f(x)$

$2x^{-\frac{3}{2}}$

$(x^{-\frac{1}{2}})' = -\frac{1}{2} x^{-\frac{3}{2}}$

$(-4) \cdot x^{-\frac{1}{2}} + C.$

Comment. Partial fraction is sometimes powerful!

$\frac{1}{(x+1)(x+5)} = \left(\frac{1}{x+1} - \frac{1}{x+5}\right) \times \frac{1}{4}.$

$\ln|x+1|.$ $\ln|x+5|.$

$\frac{1}{4} \ln\left|\frac{x+1}{x+5}\right| + C.$

Integration Method

4.2.4. Substitution Rule. Let $f \in \text{Reg}([\alpha, \beta])$ and $g: [a, b] \to [\alpha, \beta]$ continuously differentiable. Then

$$\int_a^b (f \circ g)(x) g'(x)\, dx = \int_{g(a)}^{g(b)} f(y)\, dy.$$

Method2: Substitution!

- Let $u = g(x)$.
- Compute $du = g'(x)$.
- Substitute $g(x) = u$ and $g'(x) = du$. **At this moment, only u, no x !!!**
- Calculate the above integral of u, it should be easier.
- Replace u by $g(x)$ to get the result with x.
- For definite integral, pay attention to the range.

Demo

$u = \sin x.$ $\frac{du}{dx} = \cos x$ $\frac{\sin^2 x}{2} + C.$

$du = dx \cdot \cos x.$

$\frac{1}{2}\int \sin 2x \, dx$ $-\frac{\cos 2x}{2}$

$$\int_0^{\frac{\pi}{2}} \sin(x)\cos(x)dx \Rightarrow \int u \cdot du = \frac{u^2}{2} + C.$$

$$\int_0^{\frac{\pi}{2}} \sin x \cos x \, dx \overset{u=\sin x}{=} \int_0^1 u \, du = \frac{u^2}{2}\Big|_0^1 = \frac{1}{2} - 0 = \frac{1}{2}.$$

① symmetry ② substitute.

$$\int_0^{\pi} \sin(x\cos\theta)\sin^{2n}\theta \, d\theta = 0.$$

x and n are constants.

Sin cos tan.

$$\int_0^{\pi} \sin(x\cos\theta)\sin^{2n}\theta \, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \sin(x\cos\theta)\sin^{2n}\theta \, d\theta + \int_{\frac{\pi}{2}}^{\pi} \sin(x\cos\theta)\sin^{2n}\theta \, d\theta$$

$t = \pi - \theta.$

$$= \int_0^{\frac{\pi}{2}} \sin(x\cos\theta)\sin^{2n}\theta \, d\theta + \int_0^{\frac{\pi}{2}} \sin(x\cos(\pi-\theta))\sin^{2n}(\pi-\theta) \, d\theta.$$

$$= \int_0^{\frac{\pi}{2}} \left(\sin(x\cos\theta)\sin^{2n}\theta - \sin(x\cos\theta)\sin^{2n}\theta\right) d\theta$$

$$= \int_0^{\frac{\pi}{2}} 0 \, d\theta = 0.$$

Exercise

Calculte the following integrals:

- $$\int_{-1}^{3} \sqrt{9 - x^2}\,dx$$
- $$\int \tan(x)\,dx$$ $-\ln|\cos x|$
- $$\int \frac{x}{3x^2 + 6x + 10}\,dx$$
- $$\int \frac{e^{4x}}{1 + e^{2x}}\,dx$$

$$\int_{-1}^{3}\sqrt{9-x^2}\,dx$$

$$\frac{1}{2}x\sqrt{9-x^2} - 9\arctan\frac{\sqrt{9-x^2}}{x+3} + C.$$

$$x = 3\sin\alpha. \qquad \frac{dx}{d\alpha} = 3\cos\alpha.$$

$$dx = 3\cos\alpha\, d\alpha.$$

$$\int_{-\arcsin\frac{1}{3}}^{\frac{\pi}{2}} 3\cos\alpha\cdot 3\cos\alpha\, d\alpha$$

$$= 9\int_{-\arcsin\frac{1}{3}}^{\frac{\pi}{2}} \frac{1+\cos 2\alpha}{2}\, d\alpha.$$

$$= 9\cdot\left(\frac{\alpha}{2}+\frac{\sin 2\alpha}{4}\right)\Big|_{-\arcsin\frac{1}{3}}^{\frac{\pi}{2}}.$$

$$= \sqrt{2} + 9\arctan\sqrt{2}$$

$$\int \frac{x}{3x^2+6x+10}\,dx = \int \frac{x}{3(x+1)^2+7}\,dx \qquad u = x+1$$

$$= \int \frac{u-1}{3u^2+7}\,du = \underbrace{\int \frac{u}{3u^2+7}\,du}_{A} - \underbrace{\int \frac{1}{3u^2+7}\,du}_{B}.$$

$$\frac{1}{6}\ln(3u^2+7) \qquad \arctan\triangle.$$

$$V = \sqrt{3}u., \quad V^2 = 3u \qquad \frac{dv}{du} = \sqrt{3}.$$

$$\frac{1}{\sqrt{3}}\int \frac{dv}{V^2+7}$$

4.2.6. Example. We want to calculate $\int \frac{dx}{\lambda^2 + x^2}$ for $\lambda \neq 0$. We write

$$\int \frac{dx}{\lambda^2 + x^2} = \int \frac{1}{\lambda^2} \frac{dx}{1 + (x/\lambda)^2}.$$

We now set $f(x) = 1/(1 + x^2)$, $g(x) = y = x/\lambda$. Then $g'(x) = 1/\lambda$, so we have

$$\begin{aligned}
\int \frac{dx}{\lambda^2 + x^2} &= \frac{1}{\lambda} \int \frac{1}{\lambda} \frac{dx}{1 + (x/\lambda)^2} \\
&= \frac{1}{\lambda} \int g'(x) f(g(x))\, dx = \frac{1}{\lambda} \int f(y)\, dy \Big|_{y=g(x)} \\
&= \frac{1}{\lambda} \int \frac{dy}{1 + y^2} \Big|_{y=x/\lambda} = \frac{1}{\lambda} \arctan y \Big|_{y=x/\lambda} = \frac{1}{\lambda} \arctan \frac{x}{\lambda}.
\end{aligned}$$

$\arctan x \quad \frac{1}{x^2+1}$

$$\int \frac{e^{4x}}{1 + e^{2x}} dx$$

$e^{2x} = u.$

$\frac{du}{dx} = 2e^{2x} = 2u.$

$dx = \frac{du}{2u}.$

$$\begin{aligned}
&\int \frac{u^2}{1+u} \frac{du}{2u} \\
&= \frac{1}{2} \int \frac{u}{1+u} du. \\
&= \frac{1}{2} \int 1 - \frac{1}{1+u} du \\
&= \frac{1}{2} (u - \ln|1+u|) \\
&= \frac{1}{2} (e^{2x} - \ln|1+e^{2x}|) + C.
\end{aligned}$$

Integral Method

Method3: Integration by part!

For definite integral:

$$\int_a^b f'(x)g(x)dx = f(x)g(x)\Big|_a^b - \int_a^b f(x)g'(x)dx$$

For indefinite integral:

$$\int f'(x)g(x)dx = f(x)g(x) - \int f(x)g'(x)dx$$

Demo:

$$\int x\sin(x)dx$$

$$= -\int x\cdot(\cos x)'\,dx = -x\cos x + \underbrace{\int \cos x\,dx}_{\sin x.}$$

$$\sin x - x\cos x + C.$$

Exercise

- $$\int \arctan x \cdot dx$$
- $$\int x^2 e^{-x} dx$$
- $$\int (\ln x)^2 dx$$
- $$\int_0^{\frac{\pi}{2}} \sin^n x \, dx$$

$$\int \arctan x \cdot dx$$

$$\int \ln x \, dx.$$

$$= x\ln x - \int x \cdot \frac{1}{x} dx$$

$$= x\ln x - x.$$

$$= \int (x')\arctan x \cdot dx.$$

$$= x\arctan x - \int x \cdot \frac{1}{1+x^2} dx.$$

$$\frac{1}{2}\ln(1+x^2)$$

$$\int x^2 e^{-x} dx = \int x^2(-e^{-x})' dx$$

$$(e^{-x})' = -e^{-x}.$$

$$= -x^2 e^{-x} - \int 2x \cdot (-e^{-x}) dx.$$

$$\int x e^{-x} dx.$$

$$= \int x (-e^{-x})' dx$$

$$= -xe^{-x} + \underbrace{\int e^{-x} dx}_{-e^{-x}}.$$

$$\int (\ln x)^2 dx$$

$$\int (\ln x) \cdot (x\ln x - x)' \, dx.$$

$$= (x\ln x - x)\ln x - \int \frac{1}{x} \cdot (x\ln x - x)\, dx$$

$$\int (\ln x - 1)\, dx$$

$$= x\ln x - x - x$$

$$= x\ln x - 2x + C.$$

$$I_n = \int_0^{\frac{\pi}{2}} \sin^n x \, dx$$

$$= \int_0^{\frac{\pi}{2}} \sin^{n-1} x \cdot (-\cos x)' \, dx$$

$$= \underbrace{-\sin^{n-1} x \cdot \cos x \Big|_0^{\frac{\pi}{2}}}_{0} + \int_0^{\frac{\pi}{2}} (\sin^{n-1} x)' \cos x \, dx.$$

$$= \int_0^{\frac{\pi}{2}} (n-1) \sin^{n-2} x (1 - \sin^2 x) \, dx.$$

$$= \int_0^{\frac{\pi}{2}} (n-1) \sin^{n-2} x \, dx - \int_0^{\frac{\pi}{2}} (n-1) \sin^n x \, dx.$$

$$I_n = (n-1) I_{n-2} - (n-1) I_n$$

$$I_n = \frac{n-1}{n} I_{n-2}$$

$$I_0 = \int_0^{\frac{\pi}{2}} dx = \frac{\pi}{2}$$

$$I_1 = \int_0^{\frac{\pi}{2}} \sin x \, dx = -\cos x \Big|_0^{\frac{\pi}{2}} = 1$$

$$I_n = \begin{cases} \frac{(n-1)!!}{n!!} \cdot \frac{\pi}{2} & 2 \mid n \\ \frac{(n-1)!!}{n!!} & 2 \nmid n \end{cases}$$

$$I_2 = \frac{1}{2} I_0$$

$$I_4 = \frac{3}{4} I_2 = \frac{3}{4} \times \frac{1}{2} I_0$$

$$I_3 = \frac{2}{3} I_1$$

$$I_5 = \frac{4}{5} I_3$$

$$\int_0^{\pi} \cos^{2m}\theta \cdot \sin^{2n}\theta \, d\theta = \frac{(2m)!}{2^{2m} \cdot m!} \frac{(2n)!}{2^{2n} \cdot n!} \frac{\pi}{(n+m)!} \quad (n, m \in N)$$

Reference

- 2021-Vv186 TA-Niyinchen

End

Thanks!