RC 4(Slides 255 - 307)

Differentiation

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VV186 - Honors Mathmatics II

Differentiation – An Introduction

① $L(a+b) = La + Lb$.

② $L(k \cdot a) = k \cdot La$.

In order to investigate a function's derivative, we should first take a close look of **Linear** **map**. $L: R \longrightarrow R$

$x \to 2x$

① additive ② homogeneous.

$2(x_1+x_2) = 2x_1+2x_2$

$2(kx) = k \cdot 2x$.

Definition : A linear map on $\mathbb{R}$ is a function given by :

$$L : \mathbb{R} \to \mathbb{R}, \qquad L(x) = \alpha x, \alpha \in \mathbb{R}$$

Clearly, such a function has lots of good properties, which made our discussion becomes easier.

In this perspective, we would like to approximate any functions which we are interested in by a linear map. And if such linear map exists, we say this function is differentiable.

Differentiation – An Introduction

Translating into mathematical language...

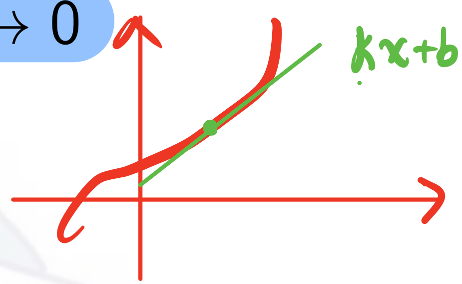
Definition : Let $\Omega \subseteq \mathbb{R}$ be a set and $x \in \mathrm{int}\Omega$. Moreover, Let $f : \Omega \to \mathbb{R}$ be a real function. Then we say f is **differentiable** if there exists a linear map L_x such that for all sufficiently small $h \in \mathbb{R}$,

derivative.

differentiation.

$$f(x+h) = f(x) + L_x(h) + o(h) \quad \text{as } h \to 0$$

$L_x(h) \to k \cdot h.$



This linear map is **unique**, if it exists.

We call L_x "the derivative of f at x". If f is differentiable at all points of some open set $U \subseteq \Omega$, we say f is differentiable on U.

3.1.5. Theorem. Let Ω be a set, $x \in \Omega$ an interior point and $f : \Omega \to \mathbb{R}$ a function that is differentiable at x with derivative $L_x = f'(x)$. Then

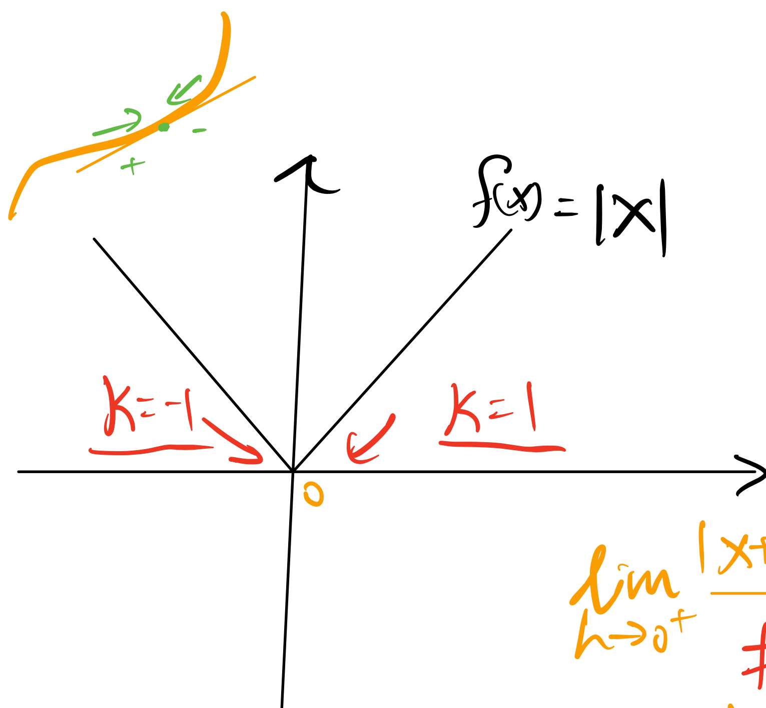
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}. \tag{3.1.7}$$

Furthermore, if the limit in (3.1.7) exists for an interior point $x \in \Omega$, then f is differentiable at x and the derivative is given by (3.1.7).

$f'(x) = L_x$

$[a, b]$

$f'(b)$ X



$$\lim_{h\to 0^+} \frac{|x+h| - |x|}{h} \neq \lim_{h\to 0^-} \frac{|x+h| - |x|}{h}$$

$$\lim_{h\to 0^+} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0^-} \frac{f(x+h) - f(x)}{h}.$$

$f'(x)$.

Derivative

x^2 $f'(x) = 2x$ Lh_1 $g(h_1)$

Common misunderstandings:

$L: h \longrightarrow 2xh$

$$L_x \text{ is a number for a fixed } x \in \Omega, \text{ because } L_x = \alpha.$$

① additive $2x(h_1+h_2) = 2xh_1 + 2xh_2$ ②

$L(h_1+h_2) = Lh_1 + Lh_2$

L_x is **not a number**, but a **linear map**, or one can say "linear function", so it essentially is a function. $L_x \cdot h = \alpha \cdot h$ (for some α) doesn't mean $L_x = \alpha$.

To see this, one can consider a function given by

$$f(x) = 2x$$

,which doesn't mean $f = 2$.

Linear Map

- A more general case.

Derivative

$L_x : h \longrightarrow 4x^3 h.$

① $4x^3(h_1+h_2) = 4x^3h_1 + 4x^3h_2$

② $4x^3(kh) = k \cdot 4x^3h.$

Common misunderstandings:

$$\text{For } f(x) = x^4,\ f'(x) = 4x^3, \text{ so } L_x \text{ may not be linear}$$

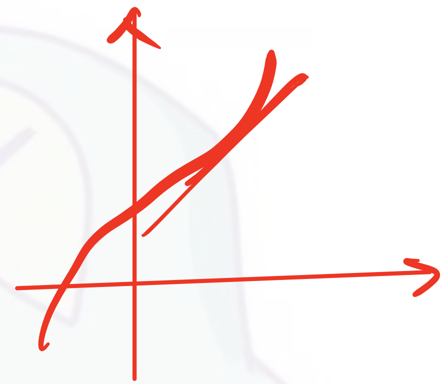
You are confusing "derivative at a point" with "function that gives derivative". At certain point x, $4x^3$ is just a number in $\mathbb{R}$. Using our notation for L_x(or $f'(x)$), we can express L_x as

$$L_x(\cdot) = 4x^3(\cdot)$$

, the variable of L_x is not x, so L_x is **linear** for its input $(\cdot)$

Given a differentiable function $f : \Omega \to \mathbb{R}$, the function that gives a derivative can be denoted by $L : (\Omega \to \mathbb{R}) \to (\Omega \to \mathbb{R})$, $L(\cdot)(x) = L_x(\cdot)$. **It is a function that maps function to function.**

Derivative



$\mathbb{R} \to \mathbb{R}$

$\mathbb{R}^2 \to \mathbb{R}$.

Common misunderstandings:

$f'(x)$

The derivative of f at x is a line passing through $(x, f(x))$

Although it is usually a good idea to sketch something to help you to understand some mathematical concepts, but you always need to aware of the essential reason why such a graph make sense.

The derivative of f at x is a function, not a graph. We simply use the graph to illustrate our function sometimes, in this case($\mathbb{R}$), it will be a straight line, but in other case, it can be more complicated.

Rules of Differentiation

We now assume both f and g are differentiable functions, then:

- $(f+g)'(x) = f'(x) + g'(x)$
- $(f \cdot g)'(x) = f'(x)g(x) + f(x)g'(x)$
- $(f \circ g)'(x) = f'(g(x))g'(x)$ chain rule.
- $(\frac{f}{g})'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$

Exercise

$\underline{x^k = k \cdot x^{k-1}}$

$(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$

$f'(x) = 6x^5$

$= \frac{1}{3} g(x)^{-\frac{2}{3}} \cdot \frac{6x(x^2+1)-(3x^2+1)(2x)}{(x^2+1)^2}$

$g'(x) = 2+10x$

$\frac{1}{3}\left(\frac{3x^2+1}{x^2+1}\right)^{-\frac{2}{3}}$

1. Practical calculation is really important! Please calculate the derivatives of the following functions.

- $f(x) = (2x+5x^2)^6$
- $\frac{\sqrt{x}}{x+1}$
- $\sqrt[3]{\frac{3x^2+1}{x^2+1}}$

$g(x) = 2x+5x^2$

$f(x) = x^6$

$(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$

$x^{\frac{1}{3}} = \frac{1}{3} x^{-\frac{2}{3}}$

$f(x) = \sqrt[3]{x}$

$g(x) = \frac{3x^2+1}{x^2+1}$

$x^{\frac{1}{2}}$

$= 6 g(x)^5 \cdot g'(x)$

$= 6 \cdot (2x+5x^2)^5 \cdot (10x+2)$

$\left(\frac{\sqrt{x}}{x+1}\right)' = \frac{(\sqrt{x})'(x+1) - (x+1)'\sqrt{x}}{(x+1)^2} = \frac{\frac{1}{2\sqrt{x}}(x+1) - \sqrt{x}}{(x+1)^2}$

not relative to x.

2. More general Cases! Please calculate following functions' derivative. (Suppose g' always exists and doesn't vanish)

i. $f(x) = g(x \cdot g(a))$ $f'(x) = g'(x \cdot g(a)) \cdot \underline{(x \cdot g(a))'} = g'(x \cdot g(a)) \cdot g(a)$.

ii. $f(x) = g(x + g(x)) + \frac{1}{g(x)}$ $g(x)$

iii. $f(x) = \underline{g(x)(x-a)} = g'(x)(x-a) + \underline{g(x)(x-a)'}$

(ii) $f'(x) = g'(x+g(x)) \cdot (x+g(x))' + g(x)^{-1}$

$(1+g'(x).)$ $\underline{\frac{1}{-g(x)^2} \cdot g'(x)}$

Inverse Function Theorem

Let I be an open interval and let $f : I \to \mathbb{R}$ be differentiable and strictly monotonic. Then the inverse map $f^{-1} : f(I) \to I$ exists and is differentiable at all points $y \in f(I)$ for which $f'(f^{-1}(y)) \neq 0$.

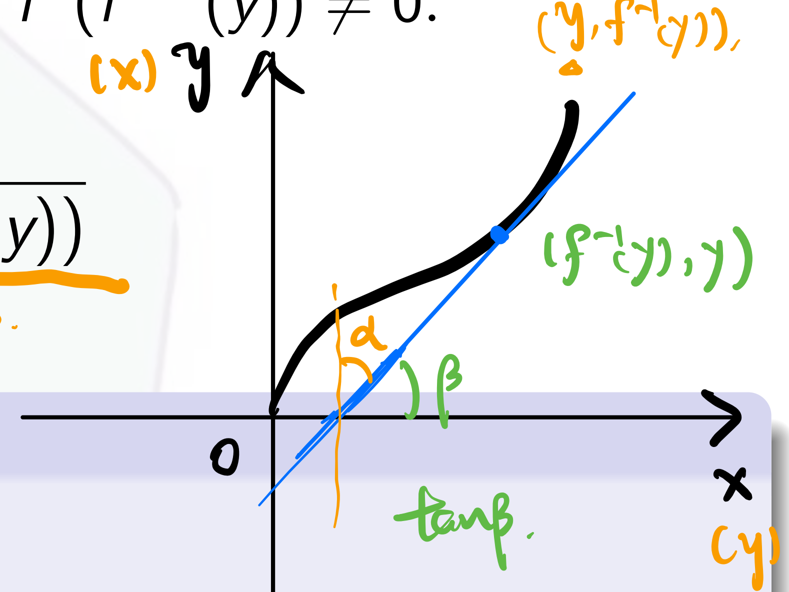
$(\sin x)' = \cos x$

$(\cos x)' = -\sin x$

$(\tan x)' = \dfrac{1}{\cos^2 x}$

$$(f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))}$$

$\tan\alpha$ $\tan\beta$



$\tan\alpha \cdot \tan\beta = 1$

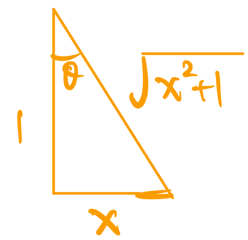
Demo

$\left(\dfrac{\sin x}{\cos x}\right)' = \dfrac{\cos x \cdot \cos x + \sin x \cdot \sin x}{(\cos x)^2}$

- Calculate $(\arctan x)'$
- Calculate $(\arcsin x)'$
- Calculate $(\arccos x)'$

$f(x) = \arctan x.$ $\quad x \longrightarrow \arctan x.$

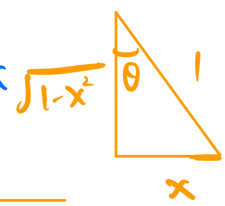
$f^{-1}(x) = \tan x \,.\, \frac{1}{\cos^2 x}$ $\quad \arctan x \longrightarrow x$



$$(\arctan x)' = \left(\frac{1}{\cos^2(\arctan x)}\right)^{-1} = \cos^2(\arctan x) = \frac{1}{x^2+1}$$

$f(x) = \arcsin x$ $\quad x \longrightarrow \arcsin x$

$f^{-1}(x) = \sin x \quad \cos x$ $\quad \arcsin x \longrightarrow x$



$$(\arcsin x)' = (\cos(\arcsin x))^{-1} = \frac{1}{\sqrt{1-x^2}}$$

$f(x) = \arccos x$ $\quad x \longrightarrow \arccos x$

$f^{-1}(x) = \cos x, \; -\sin x$ $\quad \arccos x \longrightarrow x$

$$(\arccos x)' = (-\sin(\arccos x))^{-1} = \frac{1}{-\sqrt{1-x^2}}$$

L'Hopital's Rule

$$\lim_{x\searrow b}\frac{f(x)}{g(x)}=\lim_{x\searrow b}\frac{f'(x)}{g'(x)}\text{ , if }\lim_{x\searrow b}\frac{f(x)}{g(x)}=\frac{0}{0}\text{ or }\frac{\infty}{\infty}\text{ and }\lim_{x\searrow b}\frac{f'(x)}{g'(x)}\text{ exists.}$$

What is wrong?

$\longrightarrow -2$ $-\infty$

$$\lim_{x\to1}\frac{x^3-x-2}{x^2-3x+2}=\lim_{x\to1}\frac{3x^2-1}{2x-3}=\lim_{x\to1}\frac{6x}{2}=3$$

$\longrightarrow 0.$

Put a gun on your head: do write down the word "L'Hopital" !

L'Hopital's Rule Exercise

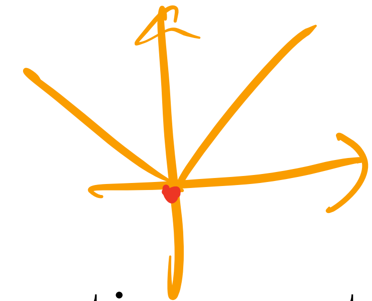
Calculate: $\lim_{x\to 0}(sinx)^x$ 0^0 ∞^∞ $0\cdot\infty$.

$(\ln x)' = \frac{1}{x}$

$e^{\ln(sinx)^x} = e^{\underbrace{x\ln sinx}_{\to 0}} \to 1$

$\lim_{x\to 0}\frac{\ln sinx}{\frac{1}{x}} = \lim_{x\to 0}\frac{\frac{cosx}{sinx}}{-\frac{1}{x^2}} = \lim_{x\to 0}\frac{-x^2}{tanx} = \lim_{x\to 0}\frac{-2x \to 0}{\frac{1}{cos^2x} \to 1} = 0.$

Application of Differentiation



We list some useful Results and Theorems.

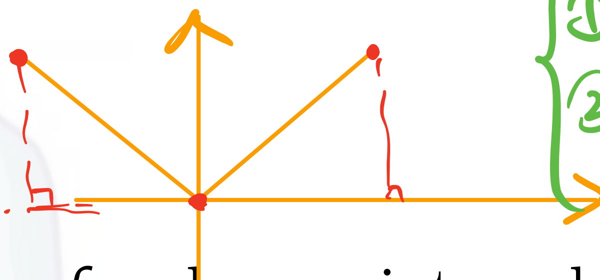
1. If a real function is differentiable at x, then it is continuous at x.
2. Hierarchy of local smoothness.
   1. Arbitrary function
   2. Function continuous at x
   3. Function differentiable at x
   4. Function continuously differentiable at x
   5. Function twice differentiable at x
   6. ...

$f(x) = x^2$

$f'(x) = 2x$

$f'(x)$

Application of Differentiation



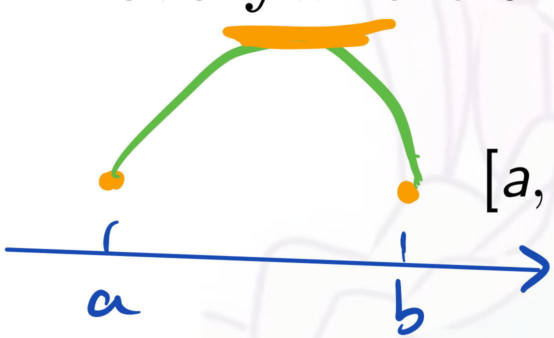
① $f' = 0$
② [ ] boundary
③ not differentiable

Result and Theorems. $f(x)$ (a, b).

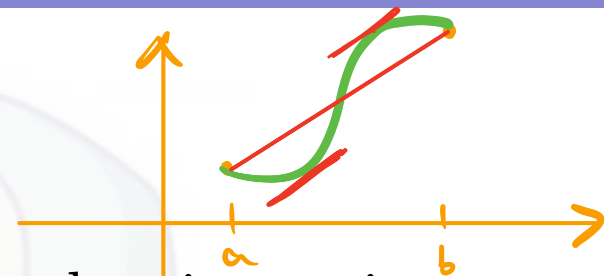
3. Let f be a function and $(a, b) \subseteq \text{dom } f$ and open interval. If $x \in (a, b)$ is a maximum(or minimum) point of $f \subseteq (a, b)$ and if f is differentiable at x, then $f'(x) = 0$.
   [a,b]

4. Let f be a function and $[a, b] \subseteq \text{dom } f$. Assume that f is differentiable on (a, b) and $f(a) = f(b)$. Then there is a number $x \in (a, b)$ such that $f'(x) = 0$. Rolle Theorem.

   Comment. We need the requirement that f is **differentiable everywhere** on (a, b). Otherwise, a counterexample can be:



$$[a, b] = [0, 2], \quad \begin{cases} f(x) = x & x \in [0, 1] \\ f(x) = 2 - x & x \in (1, 2] \end{cases}$$



Application of Differentiation

Result and Theorems.

5. Let $[a, b] \subseteq \operatorname{dom} f$ be a function that is continuous on $[a, b]$ and differentiable on (a, b). Then there exists a number $x \in (a, b)$ such that $f'(x) = \frac{f(b)-f(a)}{b-a}$. mean value theorem.

B_ε $(x-\varepsilon, x+\varepsilon)$ $-\uparrow 0 \downarrow$

6. Let f be a real function and $x \in \operatorname{dom} f$ such that $f'(x) = 0$. If $f''(x) > 0$, then f has a local minimum at x, if $f''(x) < 0$, then f has a local maximum at x.

$f'(x)\uparrow$

$f'(x) = 0$. x is called a critical point.

$f'(x) > 0$. $f(x)\uparrow$

Comment

$f'(x) < 0$. $f(x)\downarrow$.

$\geq$ $\leq$

The case in which $f''(x) = 0$ is more complicated, different conditions may occur.

global minimum/maximum. ① find all local

Example 1: $f'(x) = x^2$. Example 2: $f'(x) = x^3$.

As you can see from example 2, f may not even have a local extremum if $f''(x) = 0$.

$f^{(n)}$ odd / even. min/max.

Application of Differentiation

Result and Theorems.

7. Let f be a twice differentiable function on an open set $\Omega \subseteq \mathbb{R}$. If f has a local minimum at some point $a \in \Omega$, then $f''(a) \geq 0$.

Proof :

Suppose f has a local minimum at a. If $f''(a) < 0$, then f would also have a local maximum at a. Thus, f would be constant in some interval containing a. So $f''(a) = 0$. But this contradicts to our assumption.

Comment. An analogous statement is : If f has a local maximum at some point $a \in \Omega$, then $f''(a) \leq 0$.

Exercise

3. This exercise aims to show that differentiation can also be used to prove sequential results. Recall the inequality (see also review 2)

$$|a+b|^n \leq 2^{n-1}(|a|^n + |b|^n) \quad \text{"="} \ |a|=|b|.$$

$$(|a|+|b|)^n \leq 2^{n-1}(|a|^n+|b|^n) \quad (*).$$

Now try to use differentiable function to prove it.

$$(1+t)^n \leq 2^{n-1}(1+(t)^n).$$

$$f(t) = 2^{n-1}(1+t^n) - (1+t)^n \geq 0.$$

$t<1 \Rightarrow f'(t) \leq 0.$

$t=1$ $f(1)$ min. $f'(t)=0 \Rightarrow t=1$

$t>1 \Rightarrow f'(t) \geq 0$

Exercise

(2min) Hint: Mean Value Theorem.

target.

$\forall \varepsilon > 0.\ \exists \delta > 0,\ \forall |x_1 - x_2| < \delta,\ |\arctan x_1 - \arctan x_2| < \varepsilon$

$(\arctan x)' = \frac{1}{1+x^2}.$

Prove that : arctanx is uniformly continuous in $(-\infty, \infty)$

$x_1 < x_2.\quad \exists t \in (x_1, x_2).\ f'(t) = \frac{f(x_1) - f(x_2)}{x_1 - x_2}.$

$\exists t \in (x_1, x_2),\quad \frac{1}{1+t^2} = \frac{\arctan x_2 - \arctan x_1}{x_2 - x_1}.$

$|\underbrace{\arctan x_2 - \arctan x_1}_{>0}| = |\underbrace{(x_2 - x_1)}_{>0} \frac{1}{1+t^2}| \leq |x_2 - x_1|.$

$\forall \varepsilon$ let. $\delta < \varepsilon.$

Exercise

3min. Hint: Rolle Theorem

4. Let f be a function and $[a, b] \subseteq \operatorname{dom} f$. Assume that f is differentiable on (a, b) and $f(a) = f(b)$. Then there is a number $x \in (a, b)$ such that $f'(x) = 0$. Rolle Theorem.

Prove that if

$$\frac{a_0}{1} + \frac{a_1}{2} + \cdots + \frac{a_n}{n+1} = 0$$

Then $a_0 + a_1 x + \cdots + a_n x^n = 0$ for some $x \in [0, 1]$

exist.

$$(x^k)' = k \cdot x^{k-1}$$

$$\frac{a_i}{i+1} x^{i+1} \leftarrow a_i x^i$$

$$f'(x) = a_0 + a_1 x + \cdots + a_n x^n = 0$$

$$\Downarrow$$

$$f(x) = a_0 x + \frac{a_1}{2} x^2 + \cdots + \frac{a_n}{n+1} x^{n+1}$$

$$\begin{cases} f(0) = 0, \\ f(1) = 0. \end{cases}$$

$\exists t \in [0,1]$ $f'(t) = 0$.

Exercise

3 min

$f''(t) = \underbrace{f'(t)g(t)}_{=0} + \underbrace{f(t)}_{>0} > 0.$

$f(t) > 0$. max. $f'(t) = 0$.

$f''(t) \le 0 \Leftarrow$ local maximum

0 a t b

[a,b]. closed set + continuous $\Rightarrow$ max min exist

Suppose that f satisfies $f'' + f'g - f = 0$ for some function g. Prove that if f is 0 at two distinct points, then f is 0 on the interval between them.

$\exists t$ $f(t) < 0$.

$f(a) = 0$, $f(b) = 0$., $a < b$

proof by contradiction:

1° Suppose $\exists t$. $f(t) > 0$. then because ↓. max exists, and >0. let $f(t)$ be the maximum. then. $f'(t) = 0$ plug in $f'' + f'g - f = 0 \Rightarrow f''_{(t)} = f'_{(t)}g + f_{(t)} = 0 + f_{(t)} > 0$. ②

Conclusion3

① and ② contradicts.

However, for a local maximum. $f''(t) \le 0$. ①

Conclusion7

2° Suppose $\exists t$ $f(t) < 0$.

Exercise

Suppose $f: [0, n], n \in \mathbb{N}$ is a continuous function, and is differentiable on $(0, n)$. Furthermore, assume that

$$f(0) + f(1) + \cdots + f(n-1) = n, \;\; f(n) = 1$$

Show that there must exist $c \in (0, n)$ such that $f'(c) = 0$.

prove existence + derivative 0

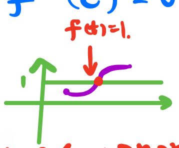
$\Rightarrow$ think about Rolle Theorem

$\Rightarrow$ Target. exist t. $f(t) = 1$? if $f(t) = f(n) = 1$. $\exists\, c \in (t, n)$, $f'(c) = 0$.

suppose there's no t, such that $f(t) = 1$.

$f(0) + f(1) + \cdots + f(n-1) = n$. There must exist some i, j $\in \{1, 2, \cdots n-1\}$. $f(i) > 1$ and $f(j) < 1$.

otherwise all > 1 or all < 1 $\Rightarrow$ sum $> n$ or $< n$.



intermediate theorem.

Exercise

$\forall \varepsilon > 0$. let $\delta < (\frac{1}{k} \cdot \varepsilon)^{1/\alpha}$ fix $x \in \Omega$.

$\forall y \in \Omega$. $|x-y| < \delta \Rightarrow |T(x) - T(y)| \leq k \cdot (x-y)^{\alpha} < \varepsilon$

In this exercise, we would like to give a deeper investigation of Lipschitz condition. If a real function $T : \Omega \to \mathbb{R}$ satisfies

$$|T(x) - T(y)| \leq k \cdot |x - y|^{\alpha}$$

for any $x, y \in \Omega$, we say T satisfies "Lipschitz condition of order α".

1. Show that if $\alpha > 0$, then T is continuous.
2. Show that if $\alpha > 1$, then T is a constant function, i.e.,

$$\left| \frac{T(x) - T(y)}{x - y} \right| \leq k \cdot |x-y|^{\alpha - 1} \qquad \underset{C \in \mathbb{R}}{\exists} T(x) = C$$

the slope.

$\Rightarrow$ when $y \to x$. $\left| \frac{T(x) - T(y)}{x-y} \right| \to 0$. $f'(x) = 0$. for all $x \in \Omega$.

So T = constant.

Reference

- Exercises from 2021-Vv186 TA-Niyinchen.

End

Thanks