Honors Mathematics II
Functions of a Single Variable

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Welcome to MATH186 !

- Please read the Course Description, which has been uploaded to the Files section on the Canvas course site.
- My office is Room 441c in in the Longbin Building. Feel free to drop in during my office hours (announced on Canvas) or just whenever you find me there.
- You can also make an appointment by email or write to me with any questions. My email is horst@sjtu.edu.cn
- The Teaching Assistants for this course will provide recitation classes, office hours, and help with grading.

Office Hours / Piazza

In addition to being available in office hours, the TAs and I will be answering course-related questions on Piazza. Please also create an account such that your name in pinyin is visible.

It is possible to send private messages on Piazza, but most messages should be public so that everyone can see them and the responses or respond themselves. Feel free to answer other students' questions!

Please do not post anonymously unless you have a good reason. Don't be shy!

Please post messages in English only.

Here is the sign-up link:

https://piazza.com/sjtu.org/fall2022/math1860j

Remote Video Client: Zhumu

Please download an “international” Zhumu client here:

https://zhumu.com/download-intl

(Note that this is a different client from the one that is offered by default on the main page.)

Please create an account using your SJTU email address and make sure that your alias is visible in roman transliteration.

Links for joining our classes by video will be published on Canvas. You are required to keep these links confidential and to not share them with any other JI student or anyone else.

An effort will be made to provide recordings of the classes for later viewing. However, it can not be guaranteed that every class will be recorded perfectly and your attendance is strongly encouraged.

Coursework

- There will be weekly coursework (assignments) throughout the term.
- You will be randomly assigned into ***assignment groups*** of three students; you are expected to collaborate within each group and hand in a single, common solution paper to each coursework.
- Each group must achieve 60% of the total coursework points by the end of the term in order to obtain a passing grade for the course. However, the assignment points have no effect on the course grade.
- Please hand in your coursework on time, by the date given on each set of course work. Late work will not be accepted unless you come to me personally and I find your explanation for the lateness acceptable.
- You can be deducted up to 10% of the awarded marks for an assignment if you fail to write neatly and legibly.
- Further details can be found in the course description.

Use of Wikipedia and Other Sources; Honor Code Policy

When faced with a particularly difficult problem, you may want to refer to other textbooks or online sources such as Wikipedia. Here are a few guidelines:

- Outside sources may treat a similar sounding subject matter at a much more advanced or a much simpler level than this course. This means that explanations you find are much more complicated or far too simple to help you. For example, when looking up the "induction axiom" you may find many high-school level explanations that are not sufficient for our problems; on the other hand, wikipedia contains a lot of information relating to formal logic that is far beyond what we are discussing here.
- If you do use any outside sources to help you solve a homework problem, you are not allowed to just copy the solution; this is considered a violation of the Honor Code.

Use of Wikipedia and Other Sources; Honor Code Policy

- The correct way of using outside sources is to understand the contents of your source and then to write in your own words and without referring back to the source the solution of the problem. Your solution should differ in style significantly from the published solution. If you are not sure whether you are incorporating too much material from your source in your solutions, then you must cite the source that you used.
- You may and are required to collaborate freely with other students in your assignment group. However, you may not communicate at all about concrete coursework with students from other groups. However, discussing general questions regarding the lecture contents with any other student is of course fine and encouraged.

  Do not show or explain your solutions to any student outside your assignment group.

Use of Wikipedia and Other Sources; Honor Code Policy

In this course, the following actions are examples of violations of the Honor Code ("another student" means a student outside your assignment group):

- Showing another student your written solution to a problem.
- Sending a screenshot of your solution via QQ, email or other means to another student.
- Showing another student the written solution of a third student; distributing some student's solution to other students.
- Viewing another student's written solution.
- Copying your solution in electronic form (LaTeX source, PDF, JPG image etc.) to the computer hardware (flash drive, hard disk etc.) of another student. Having another student's solution in electronic form on your computer hardware.

If you have any questions regarding the application of the Honor Code, please contact me or any of the TAs.

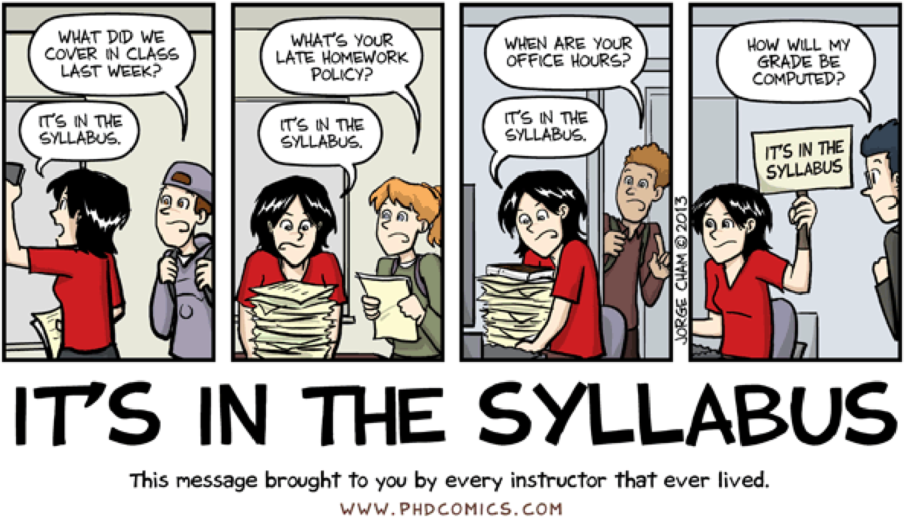
Grading Policy

- The grade will be composed of the course work and the exams as follows:
  - Surveys and Course Evaluation: 3 points
  - Course Outcome Quizzes: 7 points
  - First midterm exam: 30 points
  - Second midterm exam: 30 points
  - Final exam: 30 points
- The actual grading scale will usually be based on the top approximately 6-12% of students receiving a grade of A+, with the following grades determined by (mostly) fixed point increments.
- Apart from this normalization, the grade distribution is up to you! If (for example) all students obtain many points in the exams, I am happy to see everyone receive a grade of A. Students are primarily evaluated with respect to a fixed point scale, not with respect to each other.

Course Grade Example – MATH1860J in Fall 2021

Points	Grade	No. of students	% of students	Percentile
0	F	3	2.4%	0.0%
35	D	4	3.2%	2.4%
40	C-	4	3.2%	5.6%
45	C	9	7.2%	8.8%
50	C+	14	11.2%	16.0%
55	B-	14	11.2%	27.2%
60	B	17	13.6%	38.4%
65	B+	18	14.4%	52.0%
70	A-	18	14.4%	66.4%
75	A	14	11.2%	80.8%
80	A+	10	8.0%	92.0%

More Info: Syllabus (a.k.a. Course Description)



What is Mathematics?

Mathematics is about certainty.

Mathematics is about modeling.

Mathematics is about thinking.

Course Focus

> *Please forget everything you have learned in school; you haven't learned it.*
>
> *Please keep in mind everywhere the corresponding portions of your school work; you haven't actually forgotten them.*

Edmund Landau, Foundations of Analysis, 1929

The course will focus on functions of one variable, in particular

- Foundations: Set Theory, Logic, Numbers, Algebraic Structures.
- Functions, Limits and Continuity,
- Differentiation
- Integration

Course Topics: Foundations

1. Basic Concepts in Logic
2. Basic Concepts in Set Theory
3. The Natural Numbers and Mathematical Induction
4. The Rational Numbers
5. Roots, Bounds & Real Numbers
6. Complex Numbers

Course Topics: Functions, Convergence and Continuity

7. Functions and Maps
8. Sequences
9. Real Functions
10. Limits of Functions
11. Continuous Functions

Course Topics: Derivatives, Vector Spaces and Series

12. Differentiation of Real Functions
13. Properties of Differentiable Real Functions
14. Vector Spaces
15. Sequences of Real Functions
16. Series
17. Real and Complex Power Series
18. The Exponential Function
19. The Trigonometric Functions

Course Topics: Integrals and Applications

20. Notions of Integration

21. Practical Integration

22. Applications of Integration

Notation

The notation used in these notes should be fairly well-known from school; some symbols will be defined explicitly to make sure that they are understood unambiguously (for example, the summation symbol $\sum$).

A particular convention needs some explaination: when an equality is a definition (and not a statement that can be proven), the equal symbol will sometimes have a colon added on the side of the defined quantity. For example,

$$A \cup B := \{x : x \in A \text{ or } x \in B\}$$

indicates that that the union $A \cup B$ is being defined through this equality. Another example might involve

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2} =: a_n,$$

indicating that the number a_n is defined to be $n(n+1)/2$.

Notation

On the other hand, it would be wrong to write

$$1 + 2 + \cdots + n := \frac{n(n+1)}{2}$$

because the objects on the left and the right are already defined and the equality is a statement that can be proven; it does not define anything.

Part I

Foundations

Propositional Logic, Statements

Reference L. Loomis and S. Sternberg, ***Advanced Calculus***, Sections 0.1-0.4. Freely available here (clickable link):

http://www.math.harvard.edu/~shlomo/docs/Advanced_Calculus.pdf.

A ***statement*** (also called a ***proposition***) is anything we can regard as being either ***true*** or ***false***. We do not define here what the words "statement", "true" or "false" mean. This is beyond the purview of mathematics and falls into the realm of philosophy. Instead, we apply the principle that "we know it when we see it."

We will generally not use examples from the "real world" as statements. The reason is that in general objects in the real world are much to loosely define for the application of strict logic to make any sense. For example, the statement "It is raining." may be considered true by some people ("Yes, raindrops are falling out of the sky.") while at the same time false by others ("No, it is merely drizzling.") Furthermore, important information is missing (Where is it raining? When is it raining?).

Statements

1.1.1. Examples.

- "$3 > 2$" is a ***true statement***.
- "$x^3 > 0$" is not a statement, because we can not decide whether it is true or not.
- "the cube of any number is strictly positive (> 0)" is a ***false statement***.

The last example can be written using a ***statement variable*** x:

- "For any number x, $x^3 > 0$"

The first part of the statement is a ***quantifier*** ("for any number x"), while the second part is called a ***statement frame*** or ***predicate*** ("$x^3 > 0$").

A statement frame becomes a statement (which can then be either true or false) when the variable takes on a specific value; for example, $1^3 > 0$ is a true statement and $0^3 > 0$ is a false statement.

We will treat quantifiers and predicates in more detail later.

Working with Statements

We will denote statements by capital letters such as $A, B, C, \ldots$ and statement frames by symbols such as $A(x)$ or $B(x, y, z)$ etc.

1.1.2. Examples.

- A: 4 is an even number.
- B : $2 > 3$.
- $A(n)$: $1 + 2 + 3 + \ldots + n = n(n + 1)/2$.

We will now introduce logical operations on statements. The simplest possible type of operation is a ***unary operation***, i.e., it takes a statement A and returns a statement B.

1.1.3. Definition. Let A be a statement. Then we define the ***negation of A***, written as $\neg A$, to be the statement that is true if A is false and false if A is true.

Negation

1.1.4. Example. If A is the statement A: $2 > 3$, then the negation of A is $\neg A$: $2 \not> 3$.

We can describe the action of the unary operation $\neg$ through the following table:

A	$\neg A$
T	F
F	T

If A is true (T), then $\neg A$ is false (F) and vice-versa. Such a table is called a ***truth table***.

We will use truth tables to define all our operations on statements.

Conjunction

The next simplest type of operations on statements are ***binary operations***. The have two statements as arguments and return a single statement, called a ***compound statement***, whose truth or falsehood depends on the truth or falsehood of the original two statements.

1.1.5. Definition. Let A and B be two statements. Then we define the ***conjunction*** of A and B, written $A \wedge B$, by the following truth table:

A	B	$A \wedge B$
T	T	T
T	F	F
F	T	F
F	F	F

The conjunction $A \wedge B$ is spoken "A and B." It is true only if both A and B are true, false otherwise.

Disjunction

1.1.6. Definition. Let A and B be two statements. Then we define the ***disjunction*** of A and B, written $A \vee B$, by the following truth table:

A	B	$A \vee B$
T	T	T
T	F	T
F	T	T
F	F	F

The disjunction $A \vee B$ is spoken "A or B." It is true only if either A or B is true, false otherwise.

1.1.7. Example.

- Let A: $2 > 0$ and B: $1 + 1 = 1$. Then $A \wedge B$ is false and $A \vee B$ is true.
- Let A be a statement. Then the compound statement "$A \vee (\neg A)$" is always true, and "$A \wedge (\neg A)$" is always false.

Proofs using Truth Tables

How do we prove that "$A \vee (\neg A)$" is an always true statement? We are claiming that $A \vee (\neg A)$ will be a true statement, regardless of whether the statement A is true or not. To prove this, we go through all possibilities using a truth table:

A	$\neg A$	$A \vee (\neg A)$
T	F	T
F	T	T

Since the column for $A \vee (\neg A)$ only lists T for "true," we see that $A \vee (\neg A)$ is always true. A logical operator that always yields true statements is called a ***tautology***.

Correspondingly, the truth table for $A \wedge (\neg A)$ is:

A	$\neg A$	$A \wedge (\neg A)$
T	F	F
F	T	F

A logical operator that always yields false statements is called a ***contradiction***.

Implication

1.1.8. Definition. Let A and B be two statements. Then we define the ***implication*** of B and A, written $A \Rightarrow B$, by the following truth table:

A	B	$A \Rightarrow B$
T	T	T
T	F	F
F	T	T
F	F	T

We read "$A \Rightarrow B$" as "A implies B," "if A, then B" or "A only if B".

The fact that F $\Rightarrow$ T is a true statement may seem a little strange. Before we go into the reason for this, note that Definition 1.1.8 is a ***definition***! That means, we don't have to justify it in any way, there is nothing to prove, it is just a notation for a certain logical operator. Hence, we could leave it at that and give no further explanation. However, it would be useful and appropriate to ***motivate*** why we are defining the implication in this way.

Implication

To illustrate why the implication is defined as it is, it is useful to look at a specific example:

$$\text{For all real numbers } x, \qquad x > 0 \Rightarrow x^3 \geq 0 \tag{1.1.1}$$

It is hoped that the reader will agree that this should be a true statement. In order to verify this, we would need to verify the truth of the statement obtained from the predicate "$x > 0 \Rightarrow x^3 \geq 0$" for all real numbers x.

We shall see later that a real number can be either strictly positive, strictly negative or zero (cf. the trichotomy law on Slide 74).

- If x is a strictly positive number, both statements are true, so we have $\mathrm{T} \Rightarrow \mathrm{T}$.
- If x is a strictly negative number, both statements are false, so we have $\mathrm{F} \Rightarrow \mathrm{F}$.
- If x is zero, the first statement is false but the second statement is true, so we have $\mathrm{F} \Rightarrow \mathrm{T}$.

Equivalence

Hence, if we want (1.1.1) to be a true statement, then we need to define $A \Rightarrow B$ as being true in all of the three cases above. We next introduce yet another, important, logical operator.

1.1.9. Definition. Let A and B be two statements. Then we define the ***equivalence*** of A and B, written $A \Leftrightarrow B$, by the following truth table:

A	B	$A \Leftrightarrow B$
T	T	T
T	F	F
F	T	F
F	F	T

We read "$A \Leftrightarrow B$" as "A is equivalent to B" or "A if and only if B". Some textbooks abbreviate "if and only if" by "iff."

If A and B are both true or both false, then they are equivalent. Otherwise, they are not equivalent. In propositional logic, "equivalence" is the closest thing to the "equality" of arithmetic.

Equivalence

On the one hand, logical equivalence is strange; two statements A and B do not need to have anything to do with each other to be equivalent. For example, the statements "$2 > 0$" and "$100 - 1 = 99$" are both true, so they are equivalent.

1.1.10. Definition. Two compound statements A and B are called ***logically equivalent*** if their equivalence is a tautology, i.e., $A \Leftrightarrow B$ is always true. We then write $A \equiv B$.

1.1.11. Example. The two ***de Morgan rules*** are the tautologies

$$\neg(A \vee B) \Leftrightarrow (\neg A) \wedge (\neg B), \qquad \neg(A \wedge B) \Leftrightarrow (\neg A) \vee (\neg B).$$

To indicate that they are tautologies, we write

$$\neg(A \vee B) \equiv (\neg A) \wedge (\neg B), \qquad \neg(A \wedge B) \equiv (\neg A) \vee (\neg B).$$

These logical equivalencies will be proved in the assignments.

Contraposition

An important logical equivalence is the ***contrapositive***,

$$(A \Rightarrow B) \equiv (\neg B \Rightarrow \neg A).$$

For example, for any number x, the statement "$x > 0 \Rightarrow x^3 > 0$" is equivalent to "$x^3 \not> 0 \Rightarrow x \not> 0$." This principle is used in proofs by contradiction.
We prove the contrapositive using a truth table:

A	B	$\neg A$	$\neg B$	$\neg B \Rightarrow \neg A$	$A \Rightarrow B$	$(A \Rightarrow B) \Leftrightarrow (\neg B \Rightarrow \neg A)$
T	T	F	F	T	T	T
T	F	F	T	F	F	T
F	T	T	F	T	T	T
F	F	T	T	T	T	T

Rye Whiskey

The following song is an old Western-style song, called “Rye Whiskey” and performed by Tex Ritter in the 1930’s and 1940’s.

If the ocean was whiskey and I was a duck,
I’d swim to the bottom and never come up.

But the ocean ain’t whiskey, and I ain’t no duck,
So I’ll play jack-of-diamonds and trust to my luck.

For it’s whiskey, rye whiskey, rye whiskey I cry.
If I don’t get rye whiskey I surely will die.

The lyrics make sense (at least as much as song lyrics generally do).

Rye Whiskey

One can use de Morgan's rules and the contrapositive to re-write the song lyrics as follows

If I never reach bottom or sometimes come up,
Then the ocean's not whiskey, or I'm not a duck.

But my luck can't be trusted, or the cards I'll not buck,
So the ocean is whiskey or I am a duck.

For it's whiskey, rye whiskey, rye whiskey I cry.
If my death is uncertain, then I get whiskey (rye).

These lyrics seem to be logically equivalent to the original song, but are just humorous nonsense. This again illustrates clearly why it is futile to apply mathematical logic to everyday language.

This example is due to (clickable link) W. P. Cooke, The American Mathematical Monthly, Vol. 76, No. 9 (Nov., 1969), p. 1051.

Logical Quantifiers

In the previous examples we have used predicates $A(x)$ with the words "for all x." This is an instance of a ***logical quantifier*** that indicates for which x a predicate $A(x)$ is to be evaluated to a statement.

In order to use quantifiers properly, we clearly need a universe of objects x which we can insert into $A(x)$ (a ***domain*** for $A(x)$). This leads us immediately to the definition of a ***set***. We will discuss set theory in detail later. For the moment it is sufficient for us to view a set as a "collection of objects" and assume that the following sets are known:

- the set of natural numbers $\mathbb{N}$ (which includes the number 0),
- the set of integers $\mathbb{Z}$,
- the set of real numbers $\mathbb{R}$,
- the empty set $\emptyset$ (also written $\varnothing$ or $\{\}$) that does not contain any objects.

If M is a set containing x, we write $x \in M$ and call x an ***element*** of M.

Logical Quantifiers

There are two types of quantifiers:

- the ***universal quantifier***, denoted by the symbol $\forall$, read as "for all" and
- the ***existential quantifier***, denoted by $\exists$, read as "there exists."

1.1.12. Definition. Let M be a set and $A(x)$ be a predicate. Then we define the quantifier $\forall$ by

$$\mathop{\forall}_{x\in M} A(x) \quad \Leftrightarrow \quad A(x) \text{ is true for all } x \in M$$

We define the quantifier $\exists$ by

$$\mathop{\exists}_{x\in M} A(x) \quad \Leftrightarrow \quad A(x) \text{ is true for at least one } x \in M$$

We may also write $\forall x \in M \colon A(x)$ instead of $\mathop{\forall}_{x\in M} A(x)$ and similarly for $\exists$.

Logical Quantifiers

We may also state the domain before making the statements, as in the following example.

1.1.13. Examples. Let x be a real number. Then

- $\forall x\colon x > 0 \Rightarrow x^3 > 0$ is a true statement;
- $\forall x\colon x > 0 \Leftrightarrow x^2 > 0$ is a false statement;
- $\exists x\colon x > 0 \Leftrightarrow x^2 > 0$ is a true statement.

Sometimes mathematicians put a quantifier at the end of a statement frame; this is known as a ***hanging quantifier***. Such a hanging quantifier will be interpreted as being located just before the statement frame:

$$\exists y\colon y + x^2 > 0 \qquad\qquad \forall x$$

is equivalent to $\exists y \forall x\colon y + x^2 > 0$.

Contraposition and Negation of Quantifiers

We do not actually need the quantifier $\exists$ since

$$\begin{aligned} \underset{x\in M}{\exists} A(x) \quad &\Leftrightarrow \quad A(x) \text{ is true for at least one } x \in M \\ &\Leftrightarrow \quad A(x) \text{ is not false for all } x \in M \\ &\Leftrightarrow \quad \neg \underset{x\in M}{\forall} (\neg A(x)) \end{aligned} \tag{1.1.2}$$

The equivalence (1.1.2) is called ***contraposition of quanifiers***. It implies that the negation of $\exists x \in M \colon A(x)$ is equivalent to $\forall x \in M \colon \neg A(x)$. For example,

$$\neg \Big(\exists x \in \mathbb{R} \colon x^2 < 0\Big) \qquad \Leftrightarrow \qquad \forall x \in \mathbb{R} \colon x^2 \not< 0.$$

Conversely,

$$\neg \left(\forall x \in M \colon A(x)\right) \qquad \Leftrightarrow \qquad \exists x \in M \colon \neg A(x).$$

Vacuous Truth

If the domain of the universal quantifier $\forall$ is the empty set $M = \varnothing$, then the statement $\forall x \in M\colon A(x)$ is defined to be true regardless of the predicate $A(x)$. It is then said that $A(x)$ is ***vacuously true***.

1.1.14. Example. Let M be the set of real numbers x such that $x = x + 1$. Then the statement

$$\mathop{\forall}_{x \in M} x > x$$

is true.

This convention reflects the philosophy that a universal statement is true unless there is a counterexample to prove it false. While this may seem a strange point of view, it proves useful in practice.

This is similar to saying that "All pink elephants can fly." is a true statement, because it is impossible to find a pink elephant that can't fly.

Nesting Quantifiers

We can also treat predicates with more than one variable as shown in the following example.

1.1.15. Examples. In the following examples, x, y are taken from the real numbers.

- $\forall x \forall y \colon x^2 + y^2 - 2xy \geq 0$ is equivalent to $\forall y \forall x \colon x^2 + y^2 - 2xy \geq 0$. Therefore, one often writes $\forall x, y \colon x^2 + y^2 - 2xy \geq 0$.
- $\exists x \exists y \colon x + y > 0$ is equivalent to $\exists y \exists x \colon x + y > 0$, often abbreviated to $\exists x, y \colon x + y > 0$.
- $\forall x \exists y \colon x + y > 0$ is a true statement.
- $\exists x \forall y \colon x + y > 0$ is a false statement.

As is clear from these examples, the order of the quantifiers is important if they are different.

Naive Set Theory: Sets via Predicates

We want to be able to talk about "collections of objects"; however, we will be unable to strictly define what an "object" or a "collection" is (except that we also want any collection to qualify as an "object"). The problem with "naive" set theory is that any attempt to make a formal definition will lead to a contradiction - we will see an example of this later. However, for our practical purposes we can live with this, as we won't generally encounter these contradictions.

We indicate that an object (called an ***element***) x is part of a collection (called a ***set***) X by writing $x \in X$. We characterize the elements of a set X by some predicate P:

$$x \in X \qquad \Leftrightarrow \qquad P(x). \tag{1.2.1}$$

We write such a set X in the form $X = \{x : P(x)\}$.

Notation for Sets

We define the empty set $\varnothing := \{x \colon x \neq x\}$. The empty set has no elements, because the predicate $x \neq x$ is never true.

We may also use the notation $X = \{x_1, x_2, \dots, x_n\}$ to denote a set. In this case, X is understood to be the set

$$X = \{x \colon (x = x_1) \vee (x = x_2) \vee \cdots \vee (x = x_n)\}.$$

We will frequently use the convention

$$\{x \in A \colon P(x)\} = \{x \colon x \in A \wedge P(x)\}$$

1.2.1. Example. The set of even positive integers is

$$\{n \in \mathbb{N} \colon \underset{k \in \mathbb{Z}}{\exists} n = 2k\}$$

Subsets and Equality of Sets

If every object $x \in X$ is also an element of a set Y, we say that X is a subset of Y, writing $X \subset Y$; in other words,

$$X \subset Y \qquad \Leftrightarrow \qquad \forall x \in X : x \in Y.$$

We say that $X = Y$ if and only if $X \subset Y$ and $Y \subset X$.

We say that X is a ***proper subset*** of Y if $X \subset Y$ but $X \neq Y$. In that case we write $X \underset{\neq}{\subset} Y$.

Some authors write $\subseteq$ for $\subset$ and $\subset$ for $\underset{\neq}{\subset}$. Pay attention to the convention used when referring to literature.

Examples of Sets and Subsets

1.2.2. Examples.

1. For any set X, $\varnothing \subset X$. Since $\varnothing$ does not contain any elements, the domain of the statement $\forall y \in \varnothing \colon y \in X$ is empty. Therefore, it is vacuously true and hence $\varnothing \subset X$.
2. Consider the set $A = \{a, b, c\}$ where a, b, c are arbitrary objects, for example, numbers. The set

$$B = \{a, b, a, b, c, c\}$$

is equal to A, because it satisfies $A \subset B$ and $B \subset A$ as follows:

$$x \in A \quad \Leftrightarrow \quad (x = a) \vee (x = b) \vee (x = c) \quad \Leftrightarrow \quad x \in B.$$

Therefore, neither order nor repetition of the elements affects the contents of a set.

If $C = \{a, b\}$, then $C \subset A$ and in fact $C \subsetneq A$. Setting $D = \{b, c\}$ we have $D \subsetneq A$ but $C \not\subset D$ and $D \not\subset C$.

Power Set and Cardinality

If a set X has a finite number of elements, we define the ***cardinality*** of X to be this number, denoted by $\#X$, $|X|$ or $\operatorname{card} X$.

We define the ***power set***

$$\mathscr{P}(M) := \{A \colon A \subset M\}.$$

Here the elements of the set $\mathscr{P}(M)$ are themselves sets; $\mathscr{P}(M)$ is the "set of all subsets of M." Therefore, the statements

$$A \subset M \qquad \text{and} \qquad A \in \mathscr{P}(M)$$

are equivalent.

1.2.3. Example. The power set of $\{a, b, c\}$ is

$$\mathscr{P}(\{a, b, c\}) = \{\varnothing, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}\}.$$

The cardinality of $\{a, b, c\}$ is 3, the cardinality of the power set is $|\mathscr{P}(\{a, b, c\})| = 8$.

Operations on Sets

If $A = \{x\colon P_1(x)\}$, $B = \{x\colon P_2(x)\}$ we define the ***union***, ***intersection*** and ***difference*** of A and B by

$$A \cup B := \{x\colon P_1(x) \vee P_2(x)\}, \qquad A \cap B := \{x\colon P_1(x) \wedge P_2(x)\},$$
$$A \setminus B := \{x\colon P_1(x) \wedge (\neg P_2(x))\}.$$

Let $A \subset M$. We then define the ***complement*** of A by

$$A^{\mathrm{c}} := M \setminus A.$$

If $A \cap B = \emptyset$, we say that the sets A and B are ***disjoint***.

Occasionally, the notation $A - B$ is used for $A \setminus B$ and A^{c} is sometimes denoted by $\overline{A}$.

1.2.4. Example. Let $A = \{a, b, c\}$ and $B = \{c, d\}$. Then

$$A \cup B = \{a, b, c, d\}, \qquad A \cap B = \{c\}, \qquad A \setminus B = \{a, b\}.$$

Operations on Sets

The laws for logical equivalencies immediately lead to several rules for set operations. For example, de Morgan's laws (see Example 1.1.11) for $\wedge$ and $\vee$ imply

- $(A \cup B)^c = A^c \cap B^c$,
- $(A \cap B)^c = A^c \cup B^c$,

Other such rules are, for example,

- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
- $(A \cup B) \setminus C = (A \setminus C) \cup (B \setminus C)$
- $(A \cap B) \setminus C = (A \setminus C) \cap (B \setminus C)$
- $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$
- $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$

Some of these will be proved in the recitation class and the assignments.

Operations on Sets

Occasionally we will need the following notation for the union and intersection of a finite number $n \in \mathbb{N}$ of sets:

$$\bigcup_{k=0}^{n} A_k := A_0 \cup A_1 \cup A_2 \cup \dots \cup A_n,$$
$$\bigcap_{k=0}^{n} A_k := A_0 \cap A_1 \cap A_2 \cap \dots \cap A_n.$$

This notation even extends to $n = \infty$, but needs to be properly defined:

$$x \in \bigcup_{k=0}^{\infty} A_k \quad :\Leftrightarrow \quad \underset{k \in \mathbb{N}}{\exists}\, x \in A_k,$$
$$x \in \bigcap_{k=0}^{\infty} A_k \quad :\Leftrightarrow \quad \underset{k \in \mathbb{N}}{\forall}\, x \in A_k.$$

Operations on Sets

In particular,

$$\bigcap_{k=0}^{\infty} A_k \subset \bigcup_{k=0}^{\infty} A_k.$$

1.2.5. Example. Let $A_k = \{0, 1, 2, \dots, k\}$ for $k \in \mathbb{N}$. Then

$$\bigcup_{k=0}^{\infty} A_k = \mathbb{N}, \qquad \bigcap_{k=0}^{\infty} A_k = \{0\}.$$

To see the first statement, note that $\mathbb{N} \subset \bigcup_{k=0}^{\infty} A_k$ since $x \in \mathbb{N}$ implies $x \in A_x$ implies $x \in \bigcup_{k=0}^{\infty} A_k$. Furthermore, $\bigcup_{k=0}^{\infty} A_k \subset \mathbb{N}$ since $x \in \bigcup_{k=0}^{\infty} A_k$ implies $x \in A_k$ for some $k \in \mathbb{N}$ implies $x \in \mathbb{N}$.

For the second statement, note that $\bigcap_{k=0}^{\infty} A_k \subset \mathbb{N}$. Now $0 \in A_k$ for all $k \in \mathbb{N}$. Thus $\{0\} \subset \bigcap_{k=0}^{\infty} A_k$. On the other hand, for any $x \in \mathbb{N} \setminus \{0\}$ we have $x \notin A_{x-1}$ whence $x \notin \bigcap_{k=0}^{\infty} A_k$.

Ordered Pairs

References Loomis/Sternberg, ***Advanced Calculus***, Chapter 0.6.
See also Spivak, ***Calculus***, Appendix to Chapter 3.

A set does not contain any information about the order of its elements, e.g.,

$$\{a, b\} = \{b, a\}.$$

Thus, there is no such a thing as the "first element of a set". However, sometimes it is convenient or necessary to have such an ordering. This is achieved by defining an ***ordered pair***, denoted by

$$(a, b)$$

and having the property that

$$(a, b) = (c, d) \qquad \Leftrightarrow \qquad (a = c) \wedge (b = d).$$

Cartesian Product of Sets

There are (at least) two ways of defining an ordered pair as a set:

(i) $(a, b) := \{\{a\}, \{a, b\}\}$ or

(ii) $(a, b) := \{\{1, a\}, \{2, b\}\}$.

The first definition does not need the natural numbers and uses only set theory, but both are of course equivalent.

If A, B are sets and $a \in A$, $b \in B$, then we denote the set of all ordered pairs by

$$A \times B := \{(a, b) \colon a \in A,\ b \in B\}.$$

$A \times B$ is called the ***cartesian product*** of A and B.

In this manner we can also define an ***ordered triple*** (a, b, c) or, more generally, an ordered ***n-tuple*** $(a_1, \ldots, a_n)$ and the n-fold cartesian product $A_1 \times \cdots \times A_n$ of sets A_k, $k = 1, \ldots, n$.

If we take the cartesian product of a set with itself, we may abbreviate it using exponents, e.g.,

$$\mathbb{N}^2 := \mathbb{N} \times \mathbb{N}.$$

Problems in Naive Set Theory

If one simply views sets as arbitrary "collections" and allows a set to contain arbitrary objects, including other sets, then fundamental problems arise. We first illustrate this by an analogy:

Suppose a library contains not only books but also catalogs of books, i.e., books listing other books. For example, there might be a catalog listing all mathematics books in the library, a catalog listing all history books, etc. Suppose that there are so many catalogs, that you are asked to create catalogs of catalogs, i.e., catalogs listing other catalogs. In particular, you are asked to create the following:

(i) A catalog of all catalogs in the library. This catalog lists all catalogs contained in the library, so it must of course also list itself.

(ii) A catalog of all catalogs that list themselves. Does this catalog also list itself?

(iii) A catalog of all catalogs that do not list themselves. Does this catalog also list itself?

The Russel Antinomy

In the previous analogy, we can view "catalogs" as "sets" and being "listed in a catalog" as "being an element of a set". Then we have

(i) The set of all sets must have itself as an element.

(ii) The set of all sets that have themselves as elements may or may not contain itself. (This may be decidable by adding some rule to set theory.)

(iii) It is not decidable whether the "set of all sets that do not have themselves as elements" has itself as an element.

The Russel Antinomy

Formally, this paradox is known as the ***Russel antinomy:***

1.2.6. Russel Antinomy. The predicate $P(x)\colon x \notin x$ does not define a set $A = \{x \colon P(x)\}$.

Proof.

If $A = \{x \colon x \notin x\}$ were a set, then we should be able to decide for any set y whether $y \in A$ or $y \notin A$. We show that for $y = A$ this is not possible because either assumption leads to a contradiction:

(i) Assume $A \in A$. Then $P(A)$ by (1.2.1), i.e., $A \notin A$. $\lightning$

(ii) Assume $A \notin A$. Then $\neg P(A)$ by (1.2.1), therefore $A \in A$. $\lightning$

Since we cannot decide whether $A \in A$ or $A \notin A$, A can not be a set. $\square$

The Russel Antinomy

There are several examples in classical literature and philosophy of the Russel antimony and other“paradoxes”:

1. A person says: “This sentence is a lie.” Is he lying or telling the truth? (An example of a sentence which is not a logical statement, since a truth value can not be assigned to it.)
2. In a mountain village, there is a barber. Some villagers shave themselves (always) while the others never shave themselves. The barber shaves those and only those villagers that never shave themselves. Who shaves the barber? (An illustration of the Russel antimony.)

Russel Antinomy

We will simply ignore the existence of such contradictions and build on naive set theory. There are further paradoxes (antinomies) in naive set theory, such as ***Cantor's paradox*** and the ***Burali-Forti paradox***. All of these are resolved if naive set theory is replaced by a ***modern axiomatic set theory*** such as ***Zermelo-Fraenkel set theory.***

Further Information:

- ***Set Theory***, Stanford Encyclopedia of Philosophy, http://plato.stanford.edu/entries/set-theory/
- P.R. Halmos, ***Naive Set Theory***,Aavilable here: http://link.springer.com/book/10.1007/978-1-4757-1645-0
- T. Jech, ***Set Theory: The Third Millennium Edition, Revised and Expanded***, Available here: http://link.springer.com/book/10.1007/3-540-44761-X

The Natural Numbers

We denote the set of natural numbers by $\mathbb{N}$, i.e.,

$$\mathbb{N} = \{0, 1, 2, 3, 4, \ldots\}.$$

The natural numbers can be ***constructed*** rigorously from set theory, but we will not do so here.

We assume that the familiar operations of addition and multiplication are known for the natural numbers, i.e., for any two natural numbers $a, b \in \mathbb{N}$ we can define the natural number $c = a + b \in \mathbb{N}$ called the ***sum*** of a and b. The addition has the following properties (here $a, b, c \in \mathbb{N}$):

1. $a + (b + c) = (a + b) + c$ (***Associativity***)
2. $a + 0 = 0 + a = a$ (***Existence of a neutral element***)
3. $a + b = b + a$ (***Commutativity***)

Multiplication

Similarly, we can define ***multiplication***, where $a \cdot b \in \mathbb{N}$ is called the ***product*** of a and b. We have the following properties (here $a, b, c \in \mathbb{N}$):

1. $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ (***Associativity***)
2. $a \cdot 1 = 1 \cdot a = a$ (***Existence of a neutral element***)
3. $a \cdot b = b \cdot a$ (***Commutativity***)

We also have a property that essentially states that addition and multiplication are ***consistent***,

$$a \cdot (b + c) = a \cdot b + a \cdot c. \qquad \text{(Distributivity)}$$

Note that we are not able to define subtraction or division for all natural numbers.

Notation for Addition and Multiplication

For any numbers $a_1, a_2, \dots, a_n$ we define the notation

$$a_1 + a_2 + \cdots + a_n =: \sum_{j=1}^{n} a_j =: \sum_{1 \le j \le n} a_j$$

and

$$a_1 \cdot a_2 \cdots a_n =: \prod_{j=1}^{n} a_j =: \prod_{1 \le j \le n} a_j.$$

(We will use this notation for rational and real numbers a_k; see the following sections.) For $n \in \mathbb{N}$ we define

$$0! := 1 \qquad \text{and} \qquad n! := n \cdot (n-1)! \quad \text{for } n > 0.$$

This is an example of a ***recursive definition***.

Powers and Ordering

We are also able to define the exponential, "intuitively" written as

$$a^b = \underbrace{a \cdot a \cdot \ldots \cdot a}_{b \text{ times}} \qquad \text{for } a, b \in \mathbb{N}$$

by setting $a^0 := 1$ and $a^n := a \cdot a^{n-1}$ (another recursive definition). We note that

$$a^{b+c} = a^b \cdot a^c \qquad \text{and} \qquad (a^b)^c = a^{b \cdot c}.$$

We can imbue the natural numbers with an ***ordering***, denoted by $<$. We say that $a < b$ if there exists some $c \in \mathbb{N}^* := \mathbb{N} \setminus \{0\}$ such that $b = a + c$. We write $a > b$ if $b < a$.

We define the symbol $\le$ to mean "$<$ or $=$", so

$$a \le b \qquad \Leftrightarrow \qquad (a < b \vee a = b).$$

We write $a \ge b$ if $b \le a$.

Mathematical Induction

Reference Spivak, Chapter 2.

Often one wants to show that some statement frame $A(n)$ is true for all $n \in \mathbb{N}$ with $n \geq n_0$ for some $n_0 \in \mathbb{N}$. Mathematical induction works by establishing two statements:

(I) $A(n_0)$ is true.

(II) $A(n+1)$ is true whenever $A(n)$ is true for $n \geq n_0$, i.e.,

$$\underset{\substack{n \in \mathbb{N} \\ n \geq n_0}}{\forall} (A(n) \Rightarrow A(n+1))$$

Note that (II) does not make a statement on the situation when $A(n)$ is false; it is permitted for $A(n+1)$ to be true even if $A(n)$ is false.

The principle of mathematical induction now claims that $A(n)$ is true for all $n \geq n_0$ if (I) and (II) are true. This follows from the fifth Peano axiom (the induction axiom).

Binomial Formula

In practice, one first establishes $A(n_0)$, then shows that $A(n+1)$ is true, assuming that $A(n)$ holds.

As an example, we will prove the binomial formula:

$$A(n): \quad (a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k},$$

where $a, b \in \mathbb{R}$, $n \in \mathbb{N}$ and

$$\binom{n}{k} := \frac{n!}{k!(n-k)!}, \qquad n! := \prod_{k=1}^{n} k = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n, \qquad 0! := 1.$$

We will need the relation

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}. \tag{1.3.1}$$

Binomial Formula

We start by showing that $A(0)$ is true. We see that

$$(a+b)^0 = 1, \qquad \text{and} \qquad \sum_{k=0}^{0} \binom{0}{k} a^k b^{0-k} = \binom{0}{0} a^0 b^{0-0} = 1.$$

Hence $A(0)$ is true, since both sides of the equation are equal to unity.

We now consider $A(n+1)$:

$$(a+b)^{n+1} = (a+b)(a+b)^n = (a+b) \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} \tag{1.3.2}$$

We have inserted the statement $A(n)$ in (1.3.2), since we want to show that $A(n+1)$ is true ***under the condition that A(n) is true***.

Binomial Formula

Applying some algebraic manipulations, we obtain

$$\begin{aligned}(a+b)^{n+1} &= (a+b)\sum_{k=0}^{n}\binom{n}{k}a^k b^{n-k}\\ &= \sum_{k=0}^{n}\binom{n}{k}a^{k+1}b^{n-k}+\sum_{k=0}^{n}\binom{n}{k}a^k b^{n+1-k}\\ &= \sum_{j=1}^{n+1}\binom{n}{j-1}a^j b^{n+1-j}+\sum_{k=0}^{n}\binom{n}{k}a^k b^{n+1-k}\end{aligned}$$

where we have set $j=k+1$.

Binomial Formula

We can now rename the index back to k and split off some terms of the sums:

$$\begin{aligned}(a+b)^{n+1} &= a^{n+1} + \sum_{k=1}^{n} \binom{n}{k-1} a^k b^{n+1-k} + b^{n+1} + \sum_{k=1}^{n} \binom{n}{k} a^k b^{n+1-k} \\ &= a^{n+1} + b^{n+1} + \sum_{k=1}^{n} \left(\binom{n}{k-1} + \binom{n}{k} \right) a^k b^{n+1-k}\end{aligned}$$

Now we apply (1.3.1):

$$\begin{aligned}(a+b)^{n+1} &= a^{n+1} + b^{n+1} + \sum_{k=1}^{n} \binom{n+1}{k} a^k b^{n+1-k} \\ &= \sum_{k=0}^{n+1} \binom{n+1}{k} a^k b^{n+1-k},\end{aligned}$$

which is just $A(n+1)$. Hence the binomial formula holds for all $n \in \mathbb{N}$.

Basic Properties of Numbers - Addition

Reference Spivak, Chapter 1.

We assume that the set of rational numbers (fractions)

$$\mathbb{Q} = \left\{ \frac{p}{q} \colon p, q \in \mathbb{Z} \ \wedge \ q \neq 0 \right\}$$

is known and has been introduced from set theory. We recapitulate the basic structural properties of the rational numbers. For addition we have

associativity $\quad \underset{a,b,c\in\mathbb{Q}}{\forall} \quad a + (b + c) = (a + b) + c \qquad$ **(P1)**

neutral element $\quad \underset{0\in\mathbb{Q}}{\exists} \ \underset{a\in\mathbb{Q}}{\forall} \quad a + 0 = 0 + a = a \qquad$ **(P2)**

inverse element $\quad \underset{a\in\mathbb{Q}}{\forall} \ \underset{-a\in\mathbb{Q}}{\exists} \quad (-a) + a = a + (-a) = 0 \qquad$ **(P3)**

commutativity $\quad \underset{a,b\in\mathbb{Q}}{\forall} \quad a + b = b + a. \qquad$ **(P4)**

Multiplication

For multiplication, we have a similar set of properties

associativity	$\underset{a,b,c\in\mathbb{Q}}{\forall} \quad a\cdot(b\cdot c)=(a\cdot b)\cdot c,$	**(P5)**
neutral element	$\underset{\substack{1\in\mathbb{Q}\\1\neq 0}}{\exists}\ \underset{a\in\mathbb{Q}}{\forall} \quad a\cdot 1=1\cdot a=a,$	**(P6)**
inverse element	$\underset{\substack{a\in\mathbb{Q}\\a\neq 0}}{\forall}\ \underset{a^{-1}\in\mathbb{Q}}{\exists} \quad a\cdot a^{-1}=a^{-1}\cdot a=1,$	**(P7)**
commutativity	$\underset{a,b\in\mathbb{Q}}{\forall} \quad a\cdot b=b\cdot a.$	**(P8)**

With these properties we can prove that if $a\neq 0$ and $a\cdot b=a\cdot c$, then $b=c$. Furthermore,if $a\cdot b=0$, then either $a=0$ or $b=0$.

Combining Addition and Multiplication

These twice four properties for addition and multiplication are basically independent of each other (the only connection was that $0 \neq 1$). The following property ensures that the two operations "work well together",

$$\text{distributivity} \qquad \underset{a,b,c\in\mathbb{Q}}{\forall} \quad a\cdot(b+c)=a\cdot b+a\cdot c. \qquad \textbf{(P9)}$$

It is this relation that allows us to prove that $a-b=b-a$ only if $a=b$, and it is also the law of distributivity that we employ to do elementary multiplications.

The Positive Rational Numbers

We assume that we know what a strictly positive rational number is, i.e., that there exists a set P with the property that for every number a one and only one of the following holds:

(i) $a = 0$,

(ii) $a \in P$,

(iii) $-a \in P$.

This is known as the ***trichotomy law*** **(P10)** for the rational numbers. We also assume that the set of positive numbers is closed under addition and multiplication:

If a and b are in P, then $a + b$ is in P, **(P11)**

If a and b are in P, then $a \cdot b$ is in P. **(P12)**

The Ordering Relation

We then write

$$P = \{p \in \mathbb{Q} : p > 0\}.$$

so that $p > 0$ is defined by $p \in P$. Furthermore, for two rational numbers we can then define

$$a > b \quad :\Leftrightarrow \quad a - b > 0 \quad \Leftrightarrow \quad a - b \in P.$$

We say that we have an ***ordering relation*** on the set of rational numbers. We also write

$$a \geq b \quad :\Leftrightarrow \quad a > b \;\vee\; a = b.$$

The existence of the set P hence allows us to define what $>$ means and the properties (P11) and (P12) ensure that the ordering relation is compatible with multiplication and addition.

The Absolute Value

We define the ***absolute value*** (sometimes called the ***modulus***) $|a|$ of a rational number a by

$$|a| = \begin{cases} a & \text{if } a \geq 0, \\ -a & \text{if } a < 0. \end{cases}$$

An important basic result is the ***triangle equality***:

1.4.1. Triangle Inequality. For all rational numbers $a, b \in \mathbb{Q}$, we have

$$|a + b| \leq |a| + |b|. \tag{1.4.1}$$

Proof.
We prove the triangle inequality by considering all possible cases for a and b according to the trichotomy law (P10):

- $a = 0,\ b \in \mathbb{Q}$. If $a = 0$ we have $a + b = b$ and $|a| = a = 0$, so $|a| + |b| = |b|$. Both sides of (1.4.1) are then equal to $|b|$, so the inequality holds.

The Triangle Inequality

Proof.

We prove the triangle inequality by considering all possible cases for a and b according to the trichotomy law:

- $a > 0,\ b > 0$. In this case, $a + b > 0$ by (P11) so $|a + b| = a + b$. Furthermore, $|a| = a$ and $|b| = b$, so both sides of (1.4.1) are equal to $a + b$.
- $a < 0,\ b < 0$. In this case, $a + b < 0$ by (P11) (why?) so $|a + b| = -a - b$. Furthermore, $|a| = -a$ and $|b| = -b$, so both sides of (1.4.1) are equal to $-a - b$.
- $a < 0,\ b > 0,\ a + b \geq 0$. In this case, $|a| + |b| = b - a$ and $|a + b| = b + a$. Since $a < 0$ and $-a > 0$ we have

  $$|a + b| = b + a < b + 0 = b < b + (-a) = b - a = |a| + |b|$$

  and the inequality holds.

The case $a < 0,\ b > 0,\ a + b < 0$ is left to the reader! $\square$

The Triangle Inequality

As a corollary of the triangle inequality we obtain the ***reverse triangle inequality***:

$$|a+b| \geq \big||a|-|b|\big| \qquad \text{for all } a, b \in \mathbb{Q}. \tag{1.4.2}$$

For the proof, it is sufficient to apply the triangle inequality as follows:

$$|b| = |a+b-a| \leq |a+b| + |a|.$$

Rearranging gives

$$|a+b| \geq |b| - |a|.$$

Interchanging a and b then yields (1.4.2).

The Rational Numbers are Insufficient

While the rational numbers are closed with respect to addition and multiplication, and furthermore are large enough to include "inverse elements" $-a$ and a^{-1} for these operations, they are still "incomplete" in more than one sense:

1. Algebraically: there is no inverse element for operation of squaring. Given $b \in \mathbb{Q}$, it may not be possible to find $a \in \mathbb{Q}$ such that $a^2 = b$.
2. Analytically: Given a sequence of rational numbers that come closer and closer to each other, there may not be a "limiting value" in $\mathbb{Q}$. (We will discuss this later.)
3. Set theoretically: Given a bounded subset of $\mathbb{Q}$, there may not be a least upper or greatest lower bound for this set.

We will discuss i) and iii) in more detail now, and come back to ii) later, when we construct a suitable expansion of the rational numbers.

The Square Root Problem

Reference Spivak, end of Chapter 2.

1.5.1. Theorem. There exists no number $a \in \mathbb{Q}$ such that $a^2 = 2$.

Proof.
Assume that $a = p/q \in \mathbb{Q}$ has the property that $(p/q)^2 = 2$, where $p, q \in \mathbb{N}$. We further assume that p and q have no common divisor greater than 1. Then

$$p^2 = 2q^2,$$

so p^2 is even. This implies that p is even, so $p = 2k$ for some $k \in \mathbb{N}$. But then

$$2q^2 = 4k^2 \qquad \Rightarrow \qquad q^2 = 2k^2 \text{ is even} \qquad \Rightarrow \qquad q \text{ is even.}$$

Thus p and q are both divisible by 2. $\lightning$ $\square$

Bounded Subsets of $\mathbb{Q}$

Reference Spivak, beginning of Chapter 8.

1.5.2. Definition. We say that a set $U \subset \mathbb{Q}$ is ***bounded*** if there exists a constant $c \in \mathbb{Q}$ such that

$$|x| \leq c \qquad \text{for all } x \in U.$$

If this is not the case, we say that U is ***unbounded***.

Numbers c_1 and c_2 such that

$$c_1 \leq x \leq c_2 \qquad \text{for all } x \in U$$

are called ***lower*** and ***upper bounds*** for U, respectively.

We say that a set $U \in \mathbb{Q}$ is ***bounded above*** if there exists an upper bound for U, and ***bounded below*** if there exists a lower bound for U.

Bounded Subsets of $\mathbb{Q}$

1.5.3. Examples.

- The set $U = \{1/n \in \mathbb{Q} : n \in \mathbb{N}^*\}$ is bounded since $|x| \le 1$ for all $x \in U$. Furthermore, the number -1 is a lower bound and 1 is an upper bound. Of course, there are many more upper and lower bounds.
- The set $\mathbb{N} \subset \mathbb{Q}$ is bounded below (lower bound: 0), but not bounded above. It is unbounded.

Maxima and Minima of Sets

1.5.4. Definition. Let $U \subset \mathbb{Q}$ be a subset of the rational numbers. We say that a number $x_1 \in U$ is the ***minimum*** of U if

$$x_1 \le x, \qquad \text{for all } x \in U$$

and we write $x_1 =: \min U$.

Similarly, we say that $x_2 \in U$ is the ***maximum*** of U if

$$x_2 \ge x, \qquad \text{for all } x \in U$$

and write $x_2 =: \max U$.

A set that is not bounded below has no minimum, and a set that is not bounded above has no maximum. However, even a bounded set does not need to have a maximum or a minimum.

Greatest Lower and Least Upper Bounds of Sets

1.5.5. Definition. We say that an upper bound $c_2 \in \mathbb{Q}$ of a set $U \subset \mathbb{Q}$ is the ***least upper bound*** or ***supremum*** of U if no $c \in \mathbb{Q}$ with $c < c_2$ is an upper bound. We then write $c_2 =: \sup U$.

We say that a lower bound $c_1 \in \mathbb{Q}$ is the ***greatest lower bound*** or ***infimum*** of U if no $c \in \mathbb{Q}$ with $c > c_1$ is a lower bound. We then write $c_1 =: \inf U$.

1.5.6. Example. The set $U = \{1/n \in \mathbb{Q} : n \in \mathbb{N}^*\}$ has

$$\begin{aligned} &\inf U = 0, && \sup U = 1, \\ &\min U \text{ does not exist}, && \max U = 1. \end{aligned}$$

In general, if $\max U$ exists, then $\sup U = \max U$, and if $\min U$ exists, then $\inf U = \min U$.

Existence of Suprema and Infima

Not every bounded set in $\mathbb{Q}$ has an infimum or supremum: consider

$$U = \{x \in \mathbb{Q} : 2 < x^2 \leq 4 \wedge x > 0\}$$

It is clear that $\sup U = \max U = 2$, but the minimum and even the infimum do not exist (you will prove this in the assignments).

On the other hand, if we add a postulate to our previous axioms, we can widen the rational numbers to include solutions of $x^2 = 2$.

The Real Numbers

We define the set of real numbers $\mathbb{R}$ as the smallest extension of the rational numbers $\mathbb{Q}$ such that the following property holds:

> **(P13)** *If $A \subset \mathbb{R}$, $A \neq \emptyset$ is bounded above, then there exists a least upper bound for A in $\mathbb{R}$.*

We call all real numbers that are not rational ***irrational numbers***.

We will not immediately construct such a set of objects, but do so a little later in this lecture. For now, we stipulate that such a set exists, and we call it the set of ***real numbers*** $\mathbb{R}$. Furthermore, we can add and multiply elements of $\mathbb{R}$ and in general all properties (P1)-(P12) remain valid for elements of $\mathbb{R}$.

Note that (P13) is sufficient to guarantee the existence of greatest lower bounds; we need but consider $-A = \{x \in \mathbb{R} \colon -x \in A\}$ instead of A. The least upper bound of $-A$ will yield the negative of the greatest lower bound of A.

The Square Root Problem

We can now solve the square root problem:

1.5.7. Theorem. For every real number $x > 0$, there exists a unique positive real number y solving the equation $y^2 = x$.

Proof.
The proof is in two parts: existence and uniqueness, i.e., we show separately that i) there exists a solution and ii) the solution is unique. Often, it is easier to show uniqueness than existence. Also, the uniqueness proof may yield helpful results that can be used for the (generally more difficult) existence proof. Therefore, one often starts with proving the uniqueness.

Assume that there are two numbers $y_1 > y_2 > 0$ such that $y_1^2 = y_2^2 = x$. Then

$$0 = y_1^2 - y_2^2 = \underbrace{(y_1 - y_2)}_{>0}\underbrace{(y_1 + y_2)}_{>0} > 0 \lightning$$

The Square Root Problem

Proof (continued).
We now show the existence of a solution to the equation $y^2 = x$. Let $M := \{t \in \mathbb{R} \colon t > 0 \wedge t^2 > x\}$. Then M is non-empty and bounded from below, so by (P13) it has a greatest lower bound $y = \inf M$. We will establish that $y^2 = x$ by showing that $y^2 > x$ and $y^2 < x$ lead to contradictions. This strategy essentially uses the trichotomy law (P10).

Assume that $y^2 > x$, so $y > \frac{x}{y}$. Then

$$0 < \left(y - \frac{x}{y}\right)^2 = y^2 - 2y\frac{x}{y} + \left(\frac{x}{y}\right)^2,$$

so that

$$y^2 + \left(\frac{x}{y}\right)^2 > 2y\frac{x}{y} = 2x.$$

The Square Root Problem

Proof (continued).
Set $s := \frac{1}{2}(y + \frac{x}{y})$. Then

$$s^2 = \frac{1}{4}\left(y^2 + 2\frac{x}{y}y + \left(\frac{x}{y}\right)^2\right) > \frac{1}{4}(2x + 2x) = x$$

so $s \in M$. However,

$$s = \frac{1}{2}(y + \frac{x}{y}) < \frac{1}{2}(y + y) = y = \inf M \lightning$$

The proof that $y^2 < x$ leads to a contradiction is simpler and will be discussed in the assignments. $\square$

Therefore, the real numbers include all square roots of positive numbers; we denote the ***positive*** solution to $y^2 = x$ by $y = \sqrt{x}$. Similarly, we write the positive solution to $y^n = x$ as $y = \sqrt[n]{x}$.

Subsets of the Real Numbers

We introduce some notation we will have occasion to use often later.

1.5.8. Definition. Let $a, b \in \mathbb{R}$ with $a < b$. Then we define the following special subsets of $\mathbb{R}$, which we call ***intervals***:

$$\begin{aligned}
[a, b] &:= \{x \in \mathbb{R} \colon (a \le x) \wedge (x \le b)\} = \{x \in \mathbb{R} \colon a \le x \le b\}, \\
[a, b) &:= \{x \in \mathbb{R} \colon (a \le x) \wedge (x < b)\} = \{x \in \mathbb{R} \colon a \le x < b\}, \\
(a, b] &:= \{x \in \mathbb{R} \colon (a < x) \wedge (x \le b)\} = \{x \in \mathbb{R} \colon a < x \le b\}, \\
(a, b) &:= \{x \in \mathbb{R} \colon (a < x) \wedge (x < b)\} = \{x \in \mathbb{R} \colon a < x < b\}.
\end{aligned}$$

Furthermore, for any $a \in \mathbb{R}$ we set

$$\begin{aligned}
[a, \infty) &:= \{x \in \mathbb{R} \colon x \ge a\}, & (a, \infty) &:= \{x \in \mathbb{R} \colon x > a\}, \\
(-\infty, a] &:= \{x \in \mathbb{R} \colon x \le a)\}, & (-\infty, a) &:= \{x \in \mathbb{R} \colon x < a\}
\end{aligned}$$

Finally, we set $(-\infty, \infty) := \mathbb{R}$.

Subsets of the Real Numbers

1.5.9. Definition. We call an interval I

- ***open*** if $I = (a, b)$, $a \in \mathbb{R}$ or $a = -\infty$, $b \in \mathbb{R}$ or $b = \infty$;
- ***closed*** if $I = [a, b]$, $a, b \in \mathbb{R}$;
- ***half-open*** if is neither open or closed.

We often denote intervals by capital letters I or J. If we say "Let I be an open interval..." we mean "Let $I \subset \mathbb{R}$ be a set such that there exist numbers $a \in \mathbb{R}$ or $a = -\infty$ and $b \in \mathbb{R}$ or $b = \infty$ such that $I = (a, b)$..."

Subsets of the Real Numbers

More generally, we have the following classification of points with respect to a set:

1.5.10. Definition. Let $A \subset \mathbb{R}$ be any set.

(i) We call $x \in \mathbb{R}$ an ***interior point of A*** if there exists some $\varepsilon > 0$ such that the interval $(x - \varepsilon, x + \varepsilon) \subset A$. The set of interior points of A is denoted by $\operatorname{int} A$.

(ii) We call $x \in \mathbb{R}$ an ***exterior point of A*** if there exists some $\varepsilon > 0$ such that the interval $(x - \varepsilon, x + \varepsilon) \cap A = \emptyset$.

(iii) We call $x \in \mathbb{R}$ a ***boundary point of A*** if for every $\varepsilon > 0$ $(x - \varepsilon, x + \varepsilon) \cap A \neq \emptyset$ and $(x - \varepsilon, x + \varepsilon) \cap A^{\mathrm{c}} \neq \emptyset$. The set of boundary points of A is denoted by ∂A.

(iv) We call $x \in \mathbb{R}$ an ***accumulation point of A*** if for every $\varepsilon > 0$ the interval $(x - \varepsilon, x + \varepsilon) \cap A \setminus \{x\} \neq \emptyset$.

Subsets of the Real Numbers

1.5.11. Example. For the interval $A = [0, 1)$,

- $\operatorname{int} A = (0, 1)$,
- Any $x \in \mathbb{R} \setminus [0, 1]$ is an exterior point,
- $\partial A = \{0, 1\}$,
- Any $x \in [0, 1]$ is an accumulation point.

1.5.12. Example. For the set $A = \{x \in \mathbb{R} \colon x = \frac{1}{n}, n \in \mathbb{N} \setminus \{0\}\}$,

- $\operatorname{int} A = \emptyset$,
- Any $x \in \mathbb{R} \setminus (A \cup \{0\})$ is an exterior point,
- $\partial A = A \cup \{0\}$,
- Only $x = 0$ is an accumulation point.

Subsets of the Real Numbers

We can hence define general open and closed sets:

1.5.13. Definition.

(i) A set $A \subset \mathbb{R}$ is called ***open*** if all points of A are interior points, i.e., if

$$A = \operatorname{int} A.$$

(ii) A set $A \subset \mathbb{R}$ is called ***closed*** if $\mathbb{R} \setminus A$ is open.

(iii) The set

$$\overline{A} := A \cup \partial A$$

is called the ***closure*** of A. It can be shown that $\overline{A}$ is the smallest closed set that contains A.

The Complex Numbers

Reference Spivak, Chapter 24.

Although we have added the square roots (solutions to $y^2 - x = 0$, $x > 0$) to the rational numbers, and also many other elements through the addition of (P13), we have still not quite solved the problem of solutions of quadratic equations: there is no solution y to $y^2 + 1 = 0$ in the real numbers. (This is expressed by saying that $\mathbb{R}$ is ***algebraically incomplete***.)

We again need to extend the real numbers to include solutions to equations such as $y^2 = -1$. We follow the familiar procedure of considering pairs of real numbers and define the ***complex numbers*** $\mathbb{C}$ as

$$\mathbb{C} = \{(a, b) : a, b \in \mathbb{R}\} = \mathbb{R}^2$$

From the point of view of sets, there is no difference between $\mathbb{C}$ and $\mathbb{R}^2$. We consider the real numbers $\mathbb{R}$ as a subset of $\mathbb{C}$, writing $(x, 0) \in \mathbb{C}$ for $x \in \mathbb{R}$.

The Complex Numbers

We define addition in $\mathbb{C}$ through

$$(a_1, b_1) + (a_2, b_2) := (a_1 + a_2, b_1 + b_2),$$

so in particular the sum of two real numbers is again a real number.

We next define multiplication in an apparently strange way:

$$(a_1, b_1) \cdot (a_2, b_2) := (a_1 a_2 - b_1 b_2, a_1 b_2 + a_2 b_1).$$

However, it turns out that this multiplication satisfies the axioms (P5)-(P8), and also (P9) together with addition. The neutral element is $(1, 0)$ (as expected), but the inverse element is a bit more complicated.

The Complex Numbers

For $z = (a, b) \in \mathbb{C} \setminus \{0\}$ the multiplicative inverse is given by

$$z^{-1} = \left(\frac{a}{a^2 + b^2}, \frac{-b}{a^2 + b^2} \right),$$

i.e., $zz^{-1} = (1, 0)$.

With the previously defined multiplication we can find a solution to the equation $z^2 = -1$:

$$(0, 1)^2 = (0, 1) \cdot (0, 1) = (0 \cdot 0 - 1 \cdot 1, 0 \cdot 1 + 0 \cdot 1) = (-1, 0).$$

Fundamental Theorem of Algebra

In fact, it is possible to prove that $\mathbb{C}$ is algebraically closed:

1.6.1. Fundamental Theorem of Algebra. The expression is said to be a $p(x) = a_0 + a_1 x + \cdots + a_n x^n$ be a ***polynomial of degree n***. There exist numbers $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ such that

$$p(x) = (x - \lambda_1)(x - \lambda_2) \cdots (x - \lambda_n). \tag{1.6.1}$$

We say that the λ_k are the ***roots*** or ***zeroes*** of p. The representation (1.6.1) is called the ***factorization*** of $p(x)$.

If exactly m of the λ_k in (1.6.1) are equal to the same number $\lambda \in \mathbb{C}$, then m is said to be the ***multiplicity*** of the root λ.

We will give a proof of this result in Vv286 next year.

The Complex Numbers

We further define multiplication of complex numbers by real numbers in the following way:

$$\lambda \cdot (a, b) = (\lambda a, \lambda b), \qquad \lambda \in \mathbb{R},\ (a, b) \in \mathbb{C}.$$

Note that is multiplication together with addition in $\mathbb{C}$ is distributive.

We can then split any complex number into two components:

$$(a, b) = (a, 0) + (0, b) = a \cdot (1, 0) + b \cdot (0, 1).$$

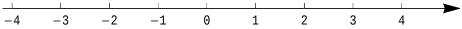
The pair $(1, 0) \in \mathbb{C}$ corresponds to $1 \in \mathbb{R}$, while the pair $(0, 1) \in \mathbb{C}$ is often denoted by the letter i. Hence we usually write

$$\mathbb{C} \ni z = (a, b) = a \cdot 1 + b \cdot i = a + bi, \qquad a, b \in \mathbb{R},$$

where $a =: \operatorname{Re} z$ is called the ***real part*** and $b =: \operatorname{Im} z$ the ***imaginary part*** of a complex number $z = a + bi$.

Visualizing Real Numbers

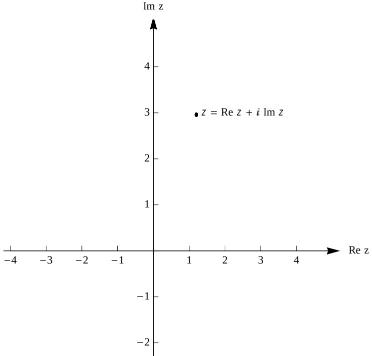
We visualize real numbers by drawing a single line, with an arrow pointing to the right, in the direction of increasing positive numbers:



We will identify the set of real numbers with this line so that a single real number corresponds to a “point” on the line. This is of course problematic; in how far this is justified, which assumptions enter into this identification and which precise axioms need to be stated falls into the purview of ***analytic geometry***. We will not discuss this further in this lecture, but simply assume a correspondence between points on a line and the real numbers.

Visualizing Complex Numbers

We visualize complex numbers by drawing two perpendicular lines ("axes"), with arrows pointing to the right and upward. each representing the real numbers. A complex number $z = a + bi = \operatorname{Re} z + (\operatorname{Im} z) \cdot i$ is represented by a point in the plane whose orthogonal projection onto the horizontal axis gives $\operatorname{Re} z$, and whose orthogonal projection onto the vertical axis gives $\operatorname{Im} z$.



The Complex Numbers

It can be shown that it is not possible to define a set P with properties (P10)-(P12) on $\mathbb{C}$; hence there exists no ordering relation "$>$" for the complex numbers.

1.6.2. Notation. Whenever we write $x > 0$, we automatically assume that $x \in \mathbb{R}$.

1.6.3. Definition. We define the ***modulus*** or ***absolute value*** of a complex number $z = a + bi$ by

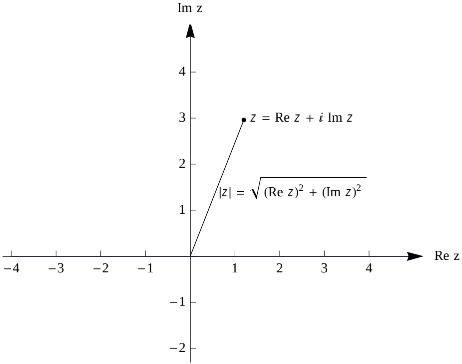
$$|z| = \sqrt{a^2 + b^2} = \sqrt{z \cdot \bar{z}}.$$

Here $\bar{z} = a - bi$ is called the ***complex conjugate*** of z.

This definition of the absolute value is called an ***extension*** of the absolute value from the real numbers to the complex numbers.

The Complex Numbers

The complex modulus has a geometric interpretation as ***distance***:



In fact, the geometric distance between two complex numbers z_1 and z_2 can is

$$\varrho(z_1, z_2) = |z_1 - z_2|.$$

The Complex Numbers

Although we cannot define the ordering $<$ on the complex numbers, we can nevertheless define the concept of bounded sets in $\mathbb{C}$.

1.6.4. Definition. Let $z_0 \in \mathbb{C}$. Then we define the ***open ball of radius $R > 0$ centered at z_0*** by

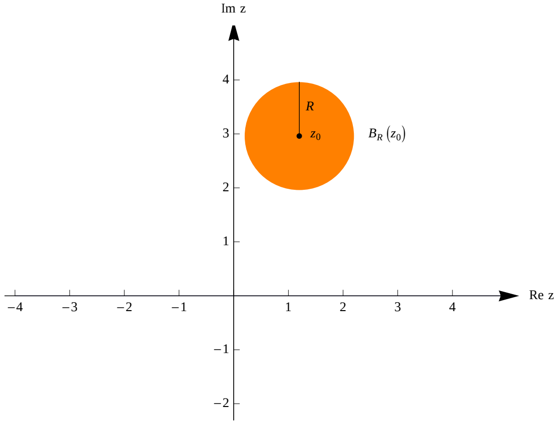
$$B_R(z_0) := \{z \in \mathbb{C} : |z - z_0| < R\}.$$

We say that a set $\Omega \subset \mathbb{C}$ is ***bounded*** if there exists some $R > 0$ such that $\Omega \subset B_R(0)$.

Of course, it makes no sense to talk of upper or lower bounds of sets.

The Complex Numbers

The open ball of radius R about $z_0 \in \mathbb{C}$ has a geometric interpretation as a ***neighbourhood***:



Part II

Functions, Convergence and Continuity

Functions

Reference Spivak, ***Calculus***, Chapter 3.

Our goal is to put the concept of a ***function*** on a sound mathematical footing. From school, we imagine a function to be an ***assignment*** of values, e.g.,

$$x \mapsto x^2$$

where to every number x we associate the square of this number. This concept of a function is very ***active***: we take a number x (input) and return a number x^2 (output). We are ***doing something*** with x. Of course, we have to specify which numbers x are acceptable as input (we call this the ***domain*** of the function) and in what set the output lies (this set is sometimes called the ***codomain***).

We also need to make sure that every input value yields exactly one output value. In summary, a function should be an object to which we associate

(i) a domain and co-domain
(ii) a unique assignment

Definition

Note that the above is mostly a "wishlist" - we still have no proper definition of what a function actually is. It turns out that defining a "unique assignment" is very difficult in mathematics. The idea of "doing something" does not lend itself to contradiction-free definitions. In modern mathematics, ***active*** concepts are always replaced by ***passive*** or ***static*** concepts that do not involve "actions". The definition of a function is the first, basic example of this.

In the following definition, we will replace the idea of an assignment with the idea of a set. Essentially, the assignment

$$x \mapsto x^2$$

is replaced with the set of all pairs

$$(x, x^2).$$

Definition

2.1.1. Definition. Let X and Y be sets and let P be a predicate with domain $X \times Y$. Let

$$f := \{(x, y) \in X \times Y : P(x, y)\}$$

and assume that P has the property that

$$\underset{(x_1,y_1)\in f}{\forall} \; \underset{(x_2,y_2)\in f}{\forall} \quad x_1 = x_2 \;\Rightarrow\; y_1 = y_2. \tag{2.1.1}$$

Let

$$\operatorname{dom} f := \Big\{x \in X : \underset{y\in Y}{\exists} : (x, y) \in f\Big\}$$

Then we say that f is a ***function from*** $\boldsymbol{\operatorname{dom} f}$ ***to*** $\boldsymbol{Y}$. The set $\operatorname{dom} f \subset X$ is called the ***domain*** of f and Y is called the ***codomain*** of f. We also define the ***range*** of f by

$$\operatorname{ran} f := \Big\{y \in Y : \underset{x\in X}{\exists} : (x, y) \in f\Big\}.$$

Examples

2.1.2. Example. Let $X, Y = \mathbb{Z}$ and $P(x, y)\colon x^2 = y$. Then

$$f = \{(x, y) \in \mathbb{Z}^2 \colon y = x^2\} = \{(0, 0), (-1, 1), (1, 1), (-2, 4), (2, 4), \ldots\}.$$

The function f has domain

$$\operatorname{dom} f = \Big\{x \in \mathbb{Z} \colon \underset{y \in \mathbb{Z}}{\exists} \colon y = x^2\Big\} = \mathbb{Z}$$

and the range satisfies

$$\operatorname{ran} f = \Big\{y \in \mathbb{Z} \colon \underset{x \in \mathbb{Z}}{\exists} \colon y = x^2\Big\} \subset \mathbb{N}.$$

By contrast, the predicate $Q(x, y)\colon y^2 = x$ does not give rise to a function from $\mathbb{Z}$ to $\mathbb{Z}$ because

$$g = \{(x, y) \in \mathbb{Z}^2 \colon y^2 = x\} = \{(0, 0), (1, -1), (1, 1), (4, 2), (4, -2), \ldots\}$$

does not satisfy (2.1.1): $(4, 2)$ and $(4, -2)$ are both in g, but $2 \neq -2$.

Functions as Assignments

The previous example illustrates how our "wishlist" for the properties of functions is fulfilled by Definition 2.1.1. Having a formal definition is going to be useful whenever we need to check whether an object actually is a function or not.

In practice, however, we will still think of a function as an assignment (albeit implemented technically through a static set of pairs). In the expression

$$f = \{(x, y) \in X \times Y : P(x, y)\}$$

the predicate P is not arbitrary, but for every $x \in \operatorname{dom} f$ there must ***exist*** a ***unique*** value y for which $P(x, y)$ is true. We denote this value by $f(x)$ (here the letter f is used in a slightly different way; it does not denote a set), i.e., we write $y = f(x)$. (Thus, $P(x, f(x))$ is a tautology.)

We can think of a function as an assignment of $f(x) \in Y$ to certain $x \in X$ such that $P(x, f(x))$ is a tautology, where we are given some sets X, Y and a predicate with domain $X \times Y$.

Notation

The idea of a function as an assignment, while mathematically difficult to handle, is very intuitive, and in calculus we mostly think of functions in this way. Therefore, we use the following notation to denote a function f with domain $\Omega \subset X$ and range contained in Y:

$$f\colon \Omega \to Y, \qquad\qquad x \mapsto f(x).$$

The above is the traditional notation in calculus for a function f. It contains all necessary information: the domain of f and the assignment which makes $P(x, f(x))$ a tautology. (Note the different shapes of the arrows; they are also traditional.) An alternative notation is

$$f\colon \Omega \to Y, \qquad\qquad y = f(x).$$

Another word for function is also ***map*** or ***mapping***.

Notation

2.1.3. Example. Instead of writing

$$f = \{(x, y) \in \mathbb{Z}^2 \colon y = x^2\}$$

we write

$$f \colon \mathbb{Z} \to \mathbb{Z}, \qquad x \mapsto x^2$$

or

$$f \colon \mathbb{Z} \to \mathbb{Z}, \qquad f(x) = x^2.$$

Compositions of Functions

If X, Y, Z are sets, $\Omega \subset X$, $\Sigma \subset Y$, and two maps $f\colon \Omega \to Y$, $g\colon \Sigma \to Z$ are given such that $\operatorname{ran} f \subset \Sigma = \operatorname{dom} g$, then we define the ***composition***

$$g \circ f\colon \Omega \to Z, \qquad\qquad x \mapsto g(f(x)).$$

2.1.4. Example. Consider the function $f\colon \mathbb{Z} \to \mathbb{Z}$, $f(x) = 1 + x^2$ and $g\colon \mathbb{Z} \setminus \{0\} \to \mathbb{Q}$, $g(x) = 1/x$. Then $\operatorname{ran} f \subset \mathbb{N} \setminus \{0\} \subset \operatorname{dom} g$, so we can define the composition

$$g \circ f\colon \mathbb{Z} \to \mathbb{Q}, \qquad\qquad x \mapsto \frac{1}{1 + x^2}.$$

Sequences - Definitions

Reference Spivak, ***Calculus***, Chapter 22.

We will first discuss a relatively elementary type of map, the so-called ***(infinite) sequences***. A sequence is simply a map from the natural to the real or complex numbers. In fact, we define

2.2.1. Definition. Let $\Omega \subset \mathbb{N}$. A map $\Omega \to \mathbb{R}$ is called a real sequence, a map $\Omega \to \mathbb{C}$ a complex sequence.

2.2.2. Notation. We denote the values of a sequence by a_n, i.e., a sequence maps $n \mapsto a_n$, $n \in \Omega \subset \mathbb{N}$. Instead of $\{(n, a_n) \colon n \in \Omega\}$ we denote a sequence by any of the following,

$$(a_n)_{n \in \Omega} = (a_n) = a_0, a_1, a_2, \ldots .$$

The values a_n are called ***terms*** of the sequence.

Sequences - Examples

2.2.3. Examples.

- The first terms of the sequence of even numbers, $(a_n) = (2n)$ are

$$(a_n) = 0, 2, 4, 6, 8, 10, \ldots$$

  The definition $a_n = 2n$ is called an ***explicit definition*** of the sequence.
- The ***Fibonacci sequence*** is defined by $a_0 = a_1 = 1$, $a_n = a_{n-1} + a_{n-2}$. This is called a ***recursive definition***.
- A sequence that takes on only two values in an alternating manner is called an ***alternatng sequence***, e.g. $a_n = (-1)^n$.
- The above are all ***integer sequences*** since their values are integers. A non-integer sequence is, for example, $(a_n)_{n\in\mathbb{N}\setminus\{0\}} = (1/n)_{n\in\mathbb{N}\setminus\{0\}}$.

Sequences - Convergence

2.2.4. Definition. The real or complex sequence $(a_n)\colon \Omega \to X$, $\Omega \subset \mathbb{N}$, $X = \mathbb{R}$ or $\mathbb{C}$, is said to ***converge with limit $a \in X$*** if

$$\underset{\varepsilon>0}{\forall}\quad \underset{N\in\mathbb{N}}{\exists}\quad \underset{n>N}{\forall}\quad |a_n - a| < \varepsilon.$$

We then write "$\lim_{n\to\infty} a_n = a$" or "$a_n \to a$ as $n \to \infty$."

A sequence that does not converge (to any limit) is called ***divergent***.

2.2.5. Remark. Definition 2.2.4 can alternatively be formulated as

$$\lim_{n\to\infty} a_n = a \qquad :\Leftrightarrow \qquad \underset{\varepsilon>0}{\forall}\ \underset{N\in\mathbb{N}}{\exists}\ \underset{n>N}{\forall}\ a_n \in B_\varepsilon(a). \tag{2.2.1}$$

where

$$B_\varepsilon(a) = \{z \in X \colon |a - z| < \varepsilon\}, \qquad \varepsilon > 0, \qquad a \in X,$$

for $X = \mathbb{R}$ or $\mathbb{C}$ is called an ***ε-neighbourhood*** of a. We hence say, "For sufficiently large n, a_n is arbitrarily close to a."

Sequences - Examples

2.2.6. Theorem. The sequence (a_n), $a_n = \frac{1}{n}$, $n \in \mathbb{N} \setminus \{0\}$, converges towards $a = 0$.

Proof.
We need to show that for any $\varepsilon > 0$ we can find an $N \in \mathbb{N}$ such that for all $n > N$ we have

$$|a_n - a| = \left|\frac{1}{n} - 0\right| = \frac{1}{n} < \varepsilon.$$

We note that $\frac{1}{n} < \varepsilon \Leftrightarrow n > \frac{1}{\varepsilon}$. Therefore, given $\varepsilon > 0$, we choose any $N \in \mathbb{N}$ with $N > 1/\varepsilon$. Then for all $n > N$ we have $n > 1/\varepsilon$ and hence $1/n < \varepsilon$. Thus $|a_n - 0| = a_n < \varepsilon$ for all $n > N$, so by the definition of convergence we have

$$\lim_{n\to\infty} \frac{1}{n} = 0.$$

□

Sequences - Examples

2.2.7. Example. The sequence (a_n), $a_n = 2n + 1$, $n \in \mathbb{N}$, does not converge.

Proof.
Let a be any real number. We will show that $a_n \not\to a$. This is equivalent to showing that

$$\underset{\varepsilon>0}{\exists} \quad \underset{N\in\mathbb{N}}{\forall} \quad \underset{n>N}{\exists} \quad |a_n - a| \geq \varepsilon. \tag{2.2.2}$$

We will take $\varepsilon = 1$ and show that for all $N \in \mathbb{N}$ there exists an $n > N$ such that $|a_n - a| \geq 1$. Let N be fixed and choose $n > \max\{N, |a|\}$. Then

$$|a_n - a| \geq \big||a_n| - |a|\big| = \big|2n + 1 - |a|\big| \geq \big|1 + |a|\big| \geq \varepsilon,$$

which is what we wanted to show. □

Sequences - Divergence to Infinity

2.2.8. Definition. A real sequence (a_n) is called ***divergent to infinity*** if

$$\underset{C>0}{\forall} \quad \underset{N\in\mathbb{N}}{\exists} \quad \underset{n>N}{\forall} \quad a_n > C.$$

It is called ***divergent to minus infinity*** if

$$\underset{C<0}{\forall} \quad \underset{N\in\mathbb{N}}{\exists} \quad \underset{n>N}{\forall} \quad a_n < C.$$

2.2.9. Notation. We write "$\lim_{n\to\infty} a_n = \infty$" or "$a_n \to \infty$ as $n \to \infty$" in the first case, "$\lim_{n\to\infty} a_n = -\infty$" or "$a_n \to -\infty$ as $n \to \infty$" in the second case.

Sequences - Examples

2.2.10. Examples.

- The sequence (a_n), $a_n = 2n + 1$ diverges to infinity.
- Any sequence that diverges to infinity or minus infinity does not converge.
- The sequence (a_n), $a_n = (-1)^n$ does not converge and does not diverge to infinity or minus infinity.

Sequences - General Results

2.2.11. Definition. A sequence is bounded if its range is a bounded set.

The range of a sequence is of course

$$\operatorname{ran}(a_n) = \{a_n \colon n \in \mathbb{N}\}.$$

Since

$$\sup\{a_0, a_1, \dots\} \le \sup_{n\in\mathbb{N}} |a_n| \qquad \text{and} \qquad |\inf\{a_0, a_1, \dots\}| \le \sup_{n\in\mathbb{N}} |a_n|,$$

it follows that a sequence is bounded if and only if $\sup_{n\in\mathbb{N}} |a_n| < \infty$.

Sequences - General Results

2.2.12. Lemma. A convergent sequence is bounded.

Proof.
Let $a_n \to a$. Then there exists some N such that $|a_n - a| < 1$ for all $n > N$. By the triangle inequality, this means that

$$|a_n| = |a_n - a + a| \le |a_n - a| + |a| < 1 + |a|$$

for $n > N$. It follows that

$$|a_n| \le \max\{|a_1|, \dots, |a_N|, |a| + 1\} =: C$$

for all $n \in \mathbb{N}$, so $\sup|a_n| \le C < \infty$ and the range is bounded. $\square$

Sequences - General Results

The following result looks harmless, even trivial, but is of great importance for applications, proofs and later generalizations.

2.2.13. Lemma.

1. The sequence (a_n) converges to a if and only if the sequence $(b_n) := (a_n - a)$ converges to zero, i.e.,
$$a_n \to a \qquad \Leftrightarrow \qquad a_n - a \to 0$$
2. The sequence (a_n) converges to 0 if and only if the sequence $(b_n) = (|a_n|)$ converges to zero, i.e.,
$$a_n \to 0 \qquad \Leftrightarrow \qquad |a_n| \to 0$$

Proof.

1. $|a_n - a| < \varepsilon \quad \Leftrightarrow \quad |(a_n - a) - 0| < \varepsilon.$
2. $||a_n| - 0| < \varepsilon \quad \Leftrightarrow \quad |a_n| < \varepsilon \quad \Leftrightarrow \quad |a_n - 0| < \varepsilon$ $\square$

Sequences - General Results

2.2.14. Lemma. A convergent sequence has precisely one limit.

Obviously a convergent sequence has a limit; the statement of the lemma is that it can not have more than one limit.

Proof.
Let $a \neq b$ and assume that $a_n \to a$ and $a_n \to b$. Then for every $\varepsilon > 0$ there exists some $N(\varepsilon)$ such that $|a_n - a|, |a_n - b| < \varepsilon/2$ for all $n > N(\varepsilon)$. Let $\varepsilon = |a-b|/2$ and choose $n > N(|a-b|/2)$. Then

$$|a-b| = |a - a_n + a_n - b| \leq |a - a_n| + |a_n - b| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \frac{|a-b|}{2},$$

which is a contradiction. □

Sequences - General Results

2.2.15. Proposition. Let (a_n) and (b_n) be convergent real or complex sequences with $a_n \to a$ and $b_n \to b$ for some $a, b \in \mathbb{C}$. Then

1. $\lim(a_n + b_n) = a + b$,
2. $\lim(a_n \cdot b_n) = ab$,
3. $\lim \frac{a_n}{b_n} = a/b$ if $b \neq 0$.

We will prove the first statement, the second statement will be part of your homework, while the last statement will be discussed in the recitation class.

Sequences - General Results

Proof of 1.

Let $a_n \to a$, $b_n \to b$. We will show that $a_n + b_n \to a + b$. Fix any $\varepsilon > 0$. Then for some $N_a \in \mathbb{N}$ for all $n > N_a$ we have

$$|a_n - a| < \frac{\varepsilon}{2}.$$

Similarly, for some $N_b \in \mathbb{N}$ we have $|b_n - b| < \varepsilon/2$ for all $n > N_b$. Define $N := \max\{N_a, N_b\}$. Then for all $n > N$ we have $|a_n - a| < \varepsilon/2$ and $|b_n - b| < \varepsilon/2$. Furthermore,

$$|a_n + b_n - (a + b)| \le |a_n - a| + |b_n - b| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \tag{2.2.3}$$

Hence for any $\varepsilon > 0$ we have found some N such that (2.2.3) holds for all $n > N$. $\square$

Sequences - Some Applications

One of the most useful tools for showing the convergence of a sequence is the so-called ***Squeze Theorem***:

2.2.16. Theorem. Let (a_n), (b_n) and (c_n) be real sequences with $a_n < c_n < b_n$ for sufficiently large n. Suppose that $\lim a_n = \lim b_n = a$. Then (c_n) converges and $\lim c_n = a$.

We use the phrase "$(\ldots)$ for sufficiently large n" here to say "there exists some $N_0 \in \mathbb{N}$ such that for all $n > N_0$ $(\ldots)$." This is a common expression in mathematics and will occur later in other contexts.

Sequences - Some Applications

Proof of the Squeeze Theorem.

Fix $\varepsilon > 0$. Then there exists a sufficiently large $N \in \mathbb{N}$ such that

$$a - \varepsilon < a_n \qquad \text{and} \qquad b_n < a + \varepsilon \qquad \text{for all } n > N.$$

Since $a_n < c_n < b_n$ for sufficiently large n, this implies

$$a - \varepsilon < c_n < a + \varepsilon \qquad \Leftrightarrow \qquad |c_n - a| < \varepsilon$$

for all $n > N$. □

Sequences - Some Applications

2.2.17. Corollary. Let (r_n) be a real sequence with $\lim r_n = 0$ and let (z_n) be a complex sequence with

$$|z_n| < r_n \qquad \text{for sufficiently large } n.$$

Then $z_n \to 0$.

This follows from the squeeze theorem with $a_n = 0$, $b_n = r_n$ and $c_n = |z_n|$ together with Lemma 2.2.13, part 2.

Sequences - Some Applications

2.2.18. Proposition. Let $q \in \mathbb{C}$, $|q| < 1$. Then $\lim_{n\to\infty} q^n = 0$.

For the proof, we will need a consequence of ***Bernoulli's Inequality***, which will be discussed in the assignments:

2.2.19. Lemma. For $x > -1$ and $n \in \mathbb{N}$,

$$(1 + x)^n > nx.$$

The proof of Bernoulli's inequality (by induction) is part of your exercises.

Sequences - Some Applications

2.2.20. Proposition. Let $q \in \mathbb{C}$, $|q| < 1$. Then $\lim_{n\to\infty} q^n = 0$.

Proof.

The case $q = 0$ is trivial, so we now assume $0 < |q| < 1$. Then

$$\frac{1}{|q|} =: 1 + \tau \qquad \text{for some } \tau > 0.$$

By Bernoulli's inequality, $\left(\frac{1}{|q|}\right)^n = (1+\tau)^n > n\tau$. Therefore,

$$|q^n| = |q|^n < \frac{1}{n\tau}.$$

The sequence $\left(\frac{1}{n\tau}\right)$ converges to 0 as $n \to \infty$, so by Corollary 2.2.17 we have $q^n \to 0$ as $n \to \infty$. $\square$

Complex Sequences

2.2.21. Lemma. Let (z_n) be a complex sequence, $z_n = x_n + iy_n$ and let $z = x + iy \in \mathbb{C}$. Then

$$z_n \to z \qquad \Longleftrightarrow \qquad (x_n \to x \quad \text{and} \quad y_n \to y).$$

Proof.
We use that

$$\max\{a, b\} \le \sqrt{a^2 + b^2} \le |a| + |b|$$

(see this by squaring both sides) and

$$|z_n - z| = |x_n - x + i(y_n - y)| = \sqrt{(x_n - x)^2 + (y_n - y)^2},$$

which gives

$$\max\{|x_n - x|, |y_n - y|\} \le |z_n - z| \le |x_n - x| + |y_n - y|.$$

Complex Sequences

Proof (continued).

It is then clear that

$$|z_n - z| < \varepsilon \quad \Rightarrow \quad |x_n - x| < \varepsilon \ \wedge \ |y_n - y| < \varepsilon$$

and

$$|x_n - x| < \frac{\varepsilon}{2} \ \wedge \ |y_n - y| < \frac{\varepsilon}{2} \quad \Rightarrow \quad |z_n - z| < \varepsilon.$$

From this and the definition of convergence, the Lemma follows. □

Real Sequences

As we have seen, the convergence of complex sequences can be reduced to the convergence of real sequences. This is very important, because one of the most useful techniques for showing convergence use the order relation "$<$", which is defined for real, but not complex numbers.

2.2.22. Definition. A real sequence (a_n) is called

- ***increasing*** if $a_n \leq a_{n+1}$ for all $n \in \mathbb{N}$.
- ***decreasing*** if $a_n \geq a_{n+1}$ for all $n \in \mathbb{N}$.
- ***strictly increasing*** if $a_n < a_{n+1}$ for all $n \in \mathbb{N}$.
- ***strictly decreasing*** if $a_n > a_{n+1}$ for all $n \in \mathbb{N}$.
- ***monotonic*** if it is either increasing or decreasing.

2.2.23. Notation. If x_n is increasing (decreasing) and $x_n \to x$, we write $x_n \nearrow x$ ($x_n \searrow x$).

Real Sequences - Boundedness & Monotonicity

2.2.24. Theorem. Every monotonic and bounded (real) sequence (a_n) is convergent. More precisely,

$$\begin{aligned} &a_n \nearrow \sup\{a_n \colon n \in \mathbb{N}\} && \text{if } (a_n) \text{ is increasing,} \\ &a_n \searrow \inf\{a_n \colon n \in \mathbb{N}\} && \text{if } (a_n) \text{ is decreasing,} \end{aligned}$$

We will show the first statement only.

Proof.
Let (a_n) be an increasing and bounded sequence. Then the range

$$A := \operatorname{ran}(a_n) = \{a_n \colon n \in \mathbb{N}\}$$

is non-empty and bounded. By property (P13) it follows that the least upper bound

$$\sup A =: \sigma < \infty$$

exists.

Order relations for Sequences in $\mathbb{R}$

Proof (continued).

For every $\varepsilon > 0$ there exists some number N such that $\sigma - \varepsilon < a_N$. (If this were untrue, then $\sigma - \varepsilon$ would be an upper bound for X, contradicting the definition of σ as least upper bound.) Since the sequence is increasing, it follows that

$$\sigma - \varepsilon < a_N \leq a_n \leq \sigma$$

for all $n \geq N$. But this means that $\sigma - a_n < \varepsilon$ for all $n > N$, so $a_n \to \sigma$. The case where (a_n) is decreasing is dealt with analogously. □

Note that the proof has essentially used not only the ordering "$<$" (which is defined for rational and real numbers), but also property (P13), which is only defined for real numbers. The statement of the theorem is false for rational numbers.

General Sequences - Subsequences

2.2.25. Definition. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence and let $(n_k)_{k\in\mathbb{N}}$ be a strictly increasing sequence of natural numbers. Then the composition

$$(a_{n_k})_{k\in\mathbb{N}} := (a_n)_{n\in\mathbb{N}} \circ (n_k)_{k\in\mathbb{N}} = (a_{n_1}, a_{n_2}, a_{n_3}, \ldots)$$

is called a subsequence of (a_n).

2.2.26. Example. If $(a_n)_{n\in\mathbb{N}} = (a_0, a_1, a_2, \ldots)$ is a sequence, then $(a_{2k}) = (a_0, a_2, a_4, \ldots)$ is a subsequence of (a_n), since the map $k \mapsto 2k$ (the sequence $(2k)_{k\in\mathbb{N}}$) is strictly increasing.

General Sequences - Subsequences

2.2.27. Definition. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence and $A \subset \mathbb{N}$ any infinite set of natural numbers. We define the subsequence

$$(a_n)_{n\in A}$$

to be the composition of (a_n) and the sequence (n_k), where $n_0 = \min A$ and $n_{k+1} > n_k$, $n_k \in A$, for all $k \in \mathbb{N}$ (n_k is the kth-smallest element of A).

2.2.28. Example. If $(a_n)_{n\in\mathbb{N}}$ is a sequence and if we denote the set of even natural numbers by E, then

$$(a_{2k})_{k\in\mathbb{N}} = (a_n)_{n\in E}.$$

General Sequences - Subsequences

2.2.29. Lemma. Let (a_n) be a convergent sequence with limit a. Then any subsequence of (a_n) is convergent with the same limit.

Proof.
Let (a_{n_k}) be a subsequence of (a_n). Then we need to show that

$$\underset{\varepsilon>0}{\forall}\ \underset{K\in\mathbb{N}}{\exists}\ \underset{k>K}{\forall}\ |a_{n_k}-a|<\varepsilon.$$

We know that

$$\underset{\varepsilon>0}{\forall}\ \underset{N\in\mathbb{N}}{\exists}\ \underset{n>N}{\forall}\ |a_n-a|<\varepsilon.$$

Fix $\varepsilon>0$ and choose a corresponding N. Since the sequence (n_k) is strictly increasing, there exists some $K\in\mathbb{N}$ such that $n_k>N$ for all $k>K$. $\square$

Monotonic Subsequences of Real Sequences

2.2.30. Lemma. Every real sequence has a monotonic subsequence.

Proof.
Let the sequence (a_n) be given and define

$$A := \{n \in \mathbb{N} \colon a_n \geq a_k \quad \text{for all } k > n\}.$$

Now if A is not finite, the subsequence

$$(a_n)_{n \in A}$$

is decreasing and we have found a monotonic subsequence.

Monotonic Subsequences of Real Sequences

Proof (continued).
If

$$A = \{n \in \mathbb{N} \colon a_n \geq a_k \quad \text{for all } k > n\}.$$

is finite, then there exists some N such that no number $n > N$ belongs to A. But that means that

$$\underset{n>N}{\forall} \ \underset{l>n}{\exists} \quad a_l > a_n. \tag{2.2.4}$$

Fix any number $n_0 > N$. Next, in (2.2.4) set $n = n_0$ and choose some $l > n_0$ such that $a_l > a_{n_0}$. Denote this l by n_1. Then, in (2.2.4) set $n = n_1$ and choose some $l > n_1$ such that $a_l > a_{n_1}$. This procedure can be repeated to find a sequence of increasing numbers $n_0, n_1, n_2, \ldots$ such that $(a_{n_k})_{k\in\mathbb{N}}$ is a strictly increasing subsequence of $(a_n)_{n\in\mathbb{N}}$. $\square$

Accumulation Points

2.2.31. Definition. Let (a_n) be a sequence. Then a number a is called an ***accumulation point*** of (a_n) if

$$\underset{\varepsilon>0}{\forall}\ \underset{N\in\mathbb{N}}{\forall}\ \underset{n>N}{\exists}\ |a_n - a| < \varepsilon.$$

2.2.32. Remark. If a sequence converges, then the limit is the only accumulation point. Prove this explicitly for yourself!

2.2.33. Examples.

1. The sequence $(-1)^n - 1/n$ has accumulation points 1 and -1, but does not converge.
2. The sequence (a_n) given by

$$a_n = \begin{cases} 0 & n \text{ even} \\ n & n \text{ odd} \end{cases}$$

has the accumulation point 0, but does not converge.

Accumulation Points

2.2.34. Lemma. A number a is an accumulation point of a sequence (a_n) if and only if there exists a subsequence of (a_n) that converges to a.

Proof.

($\Leftarrow$) This case follows immediately from the definitions - do it yourself!

($\Rightarrow$) We assume that a is an accumulation point of $(a_n) = (a_1, a_2, a_3, \ldots)$. By the definition of an accumulation point, there exists some a_k such that $|a - a_k| < \varepsilon := 1$; let $n_1 = k$. Again, for $\varepsilon = 1/2$ and $N = n_1$ there exists some $l > n_1$ such that $|a - a_l| < 1/2$; let $n_2 = l$. By choosing $\varepsilon = 1, 1/2, 1/3, 1/4, \ldots$ we hence iteratively find natural numbers n_k, $k \in \mathbb{N}$, such that (n_k) is a strictly increasing sequence and

$$|a_{n_k} - a| < \frac{1}{k}, \qquad\qquad k = 1, 2, 3, \ldots.$$

It follows that (a_{n_k}) is a subsequence of (a_n) that converges to a. $\square$

Theorem of Bolzano-Weierstraß

We can now prove the following fundamental result:

2.2.35. Theorem of Bolzano-Weierstraß. Every bounded real sequence has an accumulation point.

Proof.
Let (a_n) be a bounded sequence of real numbers. By Lemma 2.2.30, this sequence has a monotonic subsequence, which is of course also bounded. By Theorem 2.2.24, this subsequence converges, so by Lemma 2.2.34, (a_n) has an accumulation point. $\square$

Generalizing Convergence

As previously hinted, the concept of sequences and convergence only requires two things:

- A sequence of objects, realized by a map $(a_n)\colon \mathbb{N} \to M$, where M is some arbitrary set and
- The concept of "closeness," so we can define convergence. We want to express mathematically that a is a limit of (a_n) if a_n is arbitrarily close to a for sufficiently large n.

The first concept is already quite clear, and we don't need to elaborate. We simply define a sequence in a set M as a map $(a_n)\colon \mathbb{N} \to M$.

Generalizing Convergence

For the second concept there are various generalizations, where the most general ideas are part of the mathematical field of ***Topology***, which we will not go into here. We will instead discuss a generalization of the neighborhoods

$$B_\varepsilon(x) = \{y \in M : |x - y| < \varepsilon\}$$

where we replace $|x - y|$ by a general "distance function", called a ***metric***.

A metric will associate to two points $x, y \in M$ their distance, which should be a positive real number. Hence it will be a function defined not on M, but on the set of pairs $M \times M$, and it should satisfy certain conditions (such as being positive).

We also want the metric to be symmetric (the distance from x to y should equal the distance from y to x), we want the distance from x to y to be 0 if and only if $x = y$, and we want a form of the triangle inequality to hold.

Generalizing Convergence

We hence allow any function that satisfies these conditions to be called a distance function, or metric. Formally, we state the following definition:

2.2.36. Definition. Let M be a set. A map $\varrho\colon M \times M \to \mathbb{R}$ is called a ***metric*** if

(i) $\varrho(x, y) \geq 0$ for all $x, y \in M$ and $\varrho(x, y) = 0$ if and only if $x = y$.

(ii) $\varrho(x, y) = \varrho(y, x)$

(iii) $\varrho(x, z) \leq \varrho(x, y) + \varrho(y, z)$.

The pair (M, ϱ) is then called a ***metric space***.

2.2.37. Examples.

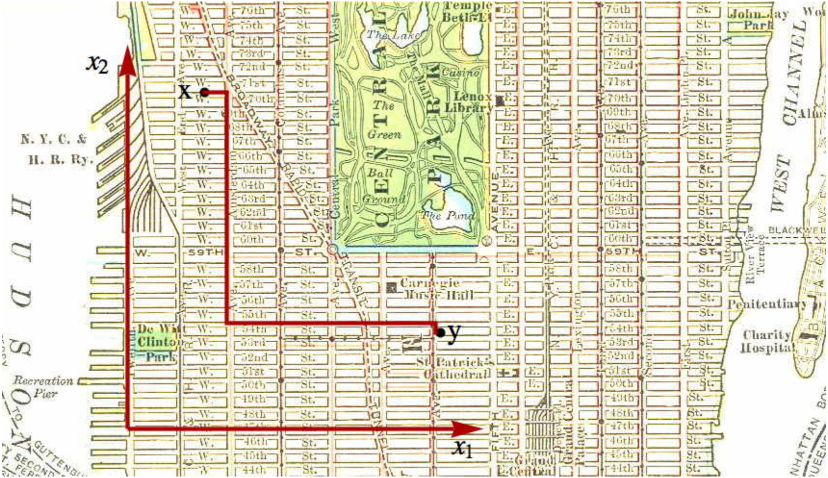
(i) $M = (0, 1] \subset \mathbb{R}$, $\varrho(x, y) = |x - y|$,

(ii) $M = \mathbb{C}$, $\varrho(z, w) = \max\{|\operatorname{Re} z - \operatorname{Re} w|, |\operatorname{Im} z - \operatorname{Im} w|\}$,

(iii) $M = \mathbb{C}$, $\varrho(z, w) = |\operatorname{Re} z - \operatorname{Re} w| + |\operatorname{Im} z - \operatorname{Im} w|$

Examples of Metric Spaces

2.2.38. Example. For $M = \mathbb{C}$ the ***New York City metric*** is given by $\varrho(x, y) = |x_1 - y_1| + |x_2 - y_2|$, where $x = x_1 + ix_2$ and $y = y_1 + iy_2$.



Extracted from Hammonds 1910 map, "New York City and vicinity." 1910. Wikipedia. Wikimedia Foundation. Web. 20 August 2013

Generalizing Convergence

We can hence define convergence of sequences in metric spaces (M, ϱ), where convergence of a sequence $(a_n)\colon \mathbb{N} \to M$ is determined by (2.2.1),

$$\lim_{n\to\infty} a_n = a \qquad :\Leftrightarrow \qquad \underset{\varepsilon>0}{\forall}\ \underset{N\in\mathbb{N}}{\exists}\ \underset{n>N}{\forall}\ a_n \in B_\varepsilon(a).$$

where

$$B_\varepsilon(a) = \{y \in M \colon \varrho(y, a) < \varepsilon\}, \qquad \varepsilon > 0, \qquad a \in M.$$

All results that do not use the properties of the real or complex numbers then also apply to sequences in (M, ϱ), we just need to replace $|x - y|$ everywhere with $\varrho(x, y)$.

Generalizing Convergence

In particular, we can define boundedness.

2.2.39. Definition. Let M be a set and (a_n) be any sequence in M. Fix $x \in M$. Then (a_n) is called bounded if

$$\underset{R>0}{\exists} \ \underset{n\in\mathbb{N}}{\forall} \ a_n \in B_R(x)$$

Note that this definition does not depend on the point x; if (a_n) is bounded for some choice of x, then it is also bounded if some other $y \in M$ is used instead. We say that boundedness is ***well-defined***.

Cauchy Sequences

We now want to study sequences in metric spaces whose elements "group close together", without necessarily converging. In particular, we want to characterize sequences, where the distance between the elements becomes smaller and smaller. Such sequences are called ***Cauchy squences***.

2.2.40. Definition. A sequence (a_n) in a metric space (M, ϱ) is called a ***Cauchy sequence*** if

$$\underset{\varepsilon>0}{\forall} \; \underset{N\in\mathbb{N}}{\exists} \; \underset{m,n>N}{\forall} \; \varrho(a_m, a_n) < \varepsilon.$$

2.2.41. Remark. Every convergent sequence is a Cauchy sequence. This follows from

$$\varrho(a_n, a_m) \le \varrho(a_n, a) + \varrho(a_m, a).$$

where $a_n \to a$. However, not every Cauchy sequence converges.

Cauchy Sequences

2.2.42. Lemma. Every Cauchy sequence in a metric space (M, ϱ) is bounded.

Proof.
Let (a_n) be a Cauchy sequence. We show that (a_n) is bounded. For $\varepsilon = 1$ we can find some N such that $\varrho(a_N, a_n) < 1$ for all $n > N$. Let $x \in M$ be some fixed point. By the triangle inequality,

$$\varrho(a_n, x) \leq \varrho(a_n, a_N) + \varrho(a_N, x) < 1 + \varrho(a_N, x) \qquad \text{for } n > N$$

and

$$\varrho(a_n, x) \leq \max\{\varrho(a_1, x), \ldots, \varrho(a_N, x), \varrho(a_N, x) + 1\} =: C,$$

hence $a_n \in B_{C+1}(x)$ for all $n \in \mathbb{N}$ and (a_n) is bounded. $\square$

Cauchy Sequences

2.2.43. Theorem. Every Cauchy sequence in $\mathbb{R}$ with the metric

$$\varrho\colon \mathbb{R}\times\mathbb{R}\to\mathbb{R}, \qquad\qquad \varrho(x,y)=|x-y|,$$

is convergent.

Proof.
Let (a_n) be a Cauchy sequence, so in particular (a_n) is bounded. By the theorem of Bolzano-Weierstraß, (a_n) has a convergent subsequence (a_{n_k}) whose limit we denote by a, i.e., $a_{n_k}\to a$.

We now prove that $a_n\to a$. Let $\varepsilon>0$ be fixed. Then we can fix some N such that $|a_n-a_m|<\varepsilon/2$ for $n,m>N$. Due to the convergence of the subsequence, we can also find some $n_k>N$ such that $|a_{n_k}-a|<\varepsilon/2$. Then for any $m>N$ we have

$$|a_m-a|\le|a_m-a_{n_k}|+|a_{n_k}-a|<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon. \qquad \square$$

Complete Metric Spaces

Similarly, one can show that every complex Cauchy sequence converges. The property that every Cauchy sequence converges is a very nice property of the metric space (M, ϱ), i.e., it depends on the metric ϱ as well as on the set M.

In fact, this property is so useful, that we give it a special name,

2.2.44. Definition. A metric space (M, ϱ) is called ***complete*** if every Cauchy sequence converges in M.

2.2.45. Examples. Let $\varrho(x, y) = |x - y|$. Then

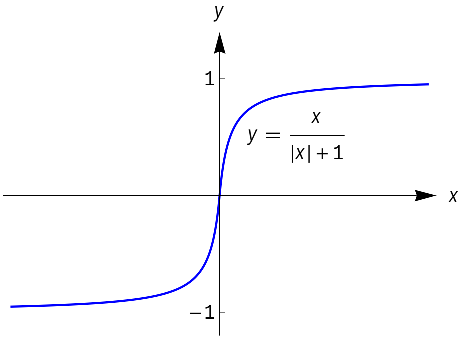
1. The spaces $(\mathbb{R}, \varrho)$ and $(\mathbb{C}, \varrho)$ are complete. (See recitation class for $(\mathbb{C}, \varrho)$.)
2. The space $([0, 1), \varrho)$ is incomplete. (Take $a_n = 1 - 1/n$.)
3. The space $(\mathbb{Q}, \varrho)$ is incomplete. (See exercises.)

Complete Metric Spaces

2.2.46. Example. Consider the metric $\varrho\colon \mathbb{R}\times\mathbb{R}\to\mathbb{R}$ defined by

$$\varrho(x,y) = \left|\frac{x}{1+|x|} - \frac{y}{1+|y|}\right| .$$

It can easily be checked that ϱ is actually a metric; the graph pictured below helps to see why $\varrho(x,y)=0$ if and only if $x=y$, the symmetry and triangle inequality are clear from the definition.



Complete Metric Spaces

Now we claim that $\mathbb{R}$ with this metric is incomplete. In fact, take the sequence (a_n) given by $a_n = n$. We will first show that this is a Cauchy sequence. Consider the sequence (b_n) in $\mathbb{R}$ with the usual metric given by

$$b_n = \frac{n}{1+n}.$$

Since $b_n \to 1$ as $n \to \infty$ it is clear that (b_n) is a Cauchy sequence in $\mathbb{R}$ with the usual metric, i.e., for any $\varepsilon > 0$ there exists an $N > 0$ such that for all $n, m > N$

$$\left| \frac{m}{1+m} - \frac{n}{1+n} \right| < \varepsilon.$$

But the left-hand side is just $\varrho(a_m, a_n)$, so we see that (a_n) is a Cauchy sequence in $(\mathbb{R}, \varrho)$. However, it is easily seen that (a_n) does not converge to any number a in $(\mathbb{R}, \varrho)$, so we have found a non-convergent Cauchy sequence in $(\mathbb{R}, \varrho)$ and the metric space is hence incomplete.

The Real Numbers

Reference Spivak, ***Calculus***, 4th ed., Chapter 28, Problem 1.

Our aim is to construct the ***completion*** of an incomplete metric space by "adding" all the limits of Cauchy sequences to the space. We will illustrate how to do this with the rational numbers; the general procedure is analogous.

Given $\mathbb{Q}$, we may consider the set of all sequences in $\mathbb{Q}$ that converge to a limit. Denote this set by $\mathrm{Conv}(\mathbb{Q})$. Each sequence $(a_n) \in \mathrm{Conv}(\mathbb{Q})$ is associated uniquely to a number $a \in \mathbb{Q}$, namely its limit. We can now say that two sequences are equivalent if they have the same limit, i.e.,

$$(a_n) \sim (b_n) \qquad :\Leftrightarrow \qquad \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n. \tag{2.2.5}$$

(This is called an ***equivalence relation***.)

Construction of the Real Numbers

We denote the set of all sequences with the same limit as a sequence (a_n) by $[(a_n)]$. Such a set is called a ***(equivalence) class*** and the set of all classes is denoted $\mathrm{Conv}(\mathbb{Q})/\sim$.

Since each rational number is represented by a class (why?) we see that the rational numbers may be identified with the set of all classes of convergent sequences:

$$\mathbb{Q} \simeq \mathrm{Conv}(\mathbb{Q})/\sim .$$

This is just a formal way of saying that the set of rational numbers corresponds to the set of all convergent sequences of rational numbers, if any two sequences with the same limit are considered equivalent. Any rational number corresponds to exactly one class of sequences and vice-versa.

Construction of the Real Numbers

We can now consider a larger class of sequences, that of Cauchy sequences of rational numbers, denoted by $\mathrm{Cauchy}(\mathbb{Q})$. Since every convergent sequence is a Cauchy sequence, $\mathrm{Conv}(\mathbb{Q}) \subset \mathrm{Cauchy}(\mathbb{Q})$.

Furthermore, we say that two Cauchy sequences are equivalent not if they have the same limit (because they might not converge) but rather if their difference converges to zero:

$$(a_n) \sim (b_n) \qquad :\Leftrightarrow \qquad \lim_{n\to\infty}(a_n - b_n) = 0. \tag{2.2.6}$$

Of course, (2.2.6) is equivalent to (2.2.5) for convergent sequences. We now have the larger set

$$\mathrm{Cauchy}(\mathbb{Q})/\sim \;\supset\; \mathrm{Conv}(\mathbb{Q})/\sim \;\simeq\; \mathbb{Q}$$

Construction of the Real Numbers

The set $\text{Cauchy}(\mathbb{Q})/\sim$ incorporates the rational numbers and by its construction every Cauchy sequence (a_n) in $\text{Cauchy}(\mathbb{Q})/\sim$ has a limit, namely precisely the object represented by the class $[(a_n)]$. We write

$$\mathbb{R} := \text{Cauchy}(\mathbb{Q})/\sim$$

and call this set the ***real numbers***.

It can be shown that all the operations of the rational numbers can be extended to $\mathbb{R}$ and that properties (P1) – (P12) continue to hold. Furthermore, in the set $\mathbb{R}$ property (P13) holds. (This can be shown by relating least upper bounds to Cauchy sequences; the details are left to you!)

Construction of the Real Numbers

2.2.47. Example. Every rational number has a finite decimal representation. We can think of a real number as having an "infinite decimal representation." (More details on this later.) For example, the sequence

$$1,\ 1.4,\ 1.41, \dots$$

may converge to $\sqrt{2}$ if the following terms are chosen appropriately.

This "infinite decimal representation" is just the way that real numbers are introduced in middle school. As another example, the sequences

$$(a_n) := (0.4,\ 0.49,\ 0.499,\ 0.4999,\ 0.49999, \dots)$$

and

$$(b_n) := (0.5,\ 0.5,\ 0.5,\ 0.5,\ 0.5, \dots)$$

are equivalent in the sense of (2.2.6), since

$$|a_n - b_n| = 10^{-(n+1)} \xrightarrow{n\to\infty} 0.$$

Hence, 0.499999... and 0.5 are considered to represent the same number.

Real Functions

Reference Spivak, ***Calculus***, Chapters 3,4.

In the previous section, we have considered real and complex sequences, i.e., functions defined on a subset of the natural numbers. Now, we will investigate ***real functions***, i.e., functions defined on a subset of $\mathbb{R}$ with real values, i.e., with range in the real numbers. Recall that we use the notation

$$f\colon \Omega \to \mathbb{R}, \qquad\qquad x \mapsto f(x).$$

where $\Omega \subset \mathbb{R}$ is the domain of f.

Real Functions

IMPORTANT: Remember that we denote a function by the symbol f, and the values of the function at a some point x by $f(x)$. It is important not to confuse these two notations; f is a function (set of pairs), while $f(x)$ is a (real) number.

Sometimes we will say (e.g.) "the function $f(x) = x^2$" or even "the function x^2", but this is just short for the correct formulation "the function f with values given by $f(x) = x^2$."

In this context, we consider the function f to have the maximal possible domain, in other words,

$$\operatorname{dom} f = \{x \in \mathbb{R} \colon f(x) \text{ makes sense as a real number}\}.$$

Graphs

In order to specify a unique function we always need to give

(i) the domain Ω and the space Y containing the range,

(ii) the association ("mapping") $x \mapsto f(x)$.

Although a function is actually defined through a set of pairs of elements, sometimes one prefers to think of a function as being given by (i) and (ii) above, and denotes by

$$\Gamma(f) = \{(x, f(x)) \in X \times Y : x \in \operatorname{dom} f\}$$

the ***graph*** of the function f.

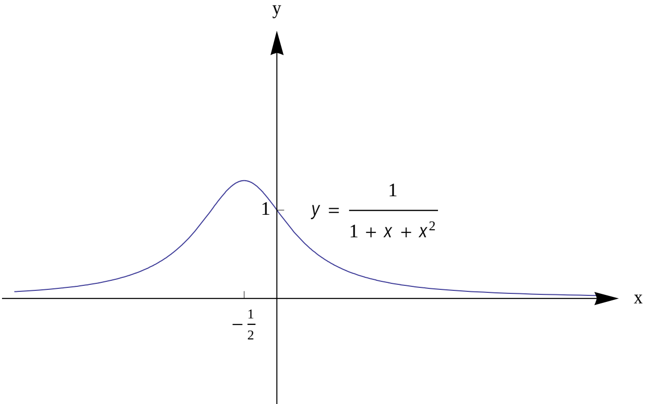
If $\Omega \subset \mathbb{R}$ and $Y = \mathbb{R}$, we visualize the graph through the drawing of two perpendicular axes, the horizontal of which represents the real numbers that encompass the domain, the vertical representing the reals including the range. In this way, a point $(x, y) \in \mathbb{R}^2$ is represented by a point in the "coordinate system" of these graphs.

Graphs

Plotting the graph of a function can be very useful for a quick overvew of its properties. Compare the information conveyed by

$$f : \mathbb{R} \to \mathbb{R}, \qquad\qquad f(x) = \frac{1}{1 + x + x^2}$$

and



Graphs

In order to arrive at such a plot, we need to analyze the properties of the function. We are interested in finding

- the intercepts with the axes,
- any "jumps" or "breaks" in the graph,
- points where the graph approaches a maximum or minimum value,
- values which are approached by the graph as $|x|$ becomes very large.

Much of the rest of this term's lecture will focus on such an analysis. For complicated functions, we will literally need "every trick in the book."

Polynomial Functions

An important class of functions is constituted b the ***polynomial functions***, which have the form

$$\operatorname{dom} f = \mathbb{R}, \qquad f(x) = \sum_{k=0}^{n} c_k x^k, \qquad n \in \mathbb{N}, \qquad c_1, \ldots, c_n \in \mathbb{R}.$$

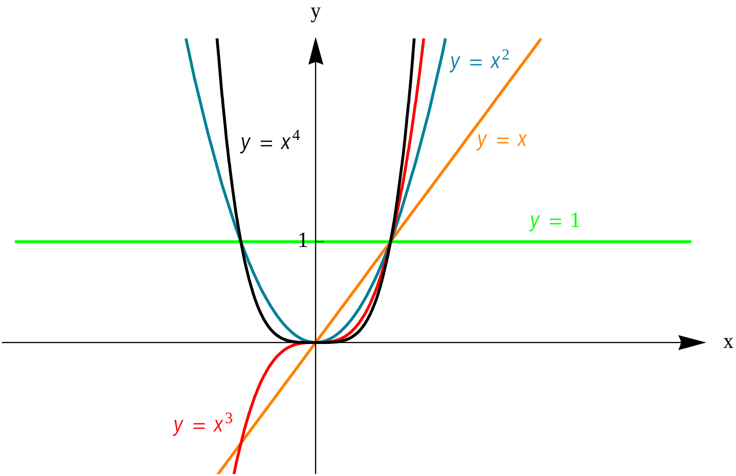
The largest k with $k \neq 0$ in the sum is called the ***degree*** of the polynomial.

Polynomial functions of degree 1 are straight lines, and can be uniquely determined by fixing two points through which they pass:
Let (x_1, y_1) and (x_2, y_2) be two points in the plane $\mathbb{R}^2$. Then the straight line through these two points is given by

$$f(x) = \frac{x - x_2}{x_1 - x_2} y_1 + \frac{x - x_1}{x_2 - x_1} y_2 = \frac{y_1 - y_2}{x_1 - x_2}(x - x_2) + y_2$$

Power Functions

Polynomial functions of the form $f(x) = x^n$, $n \in \mathbb{N}$, are also called ***power functions*** or ***monomials*** and can best be visualized through their graphs:

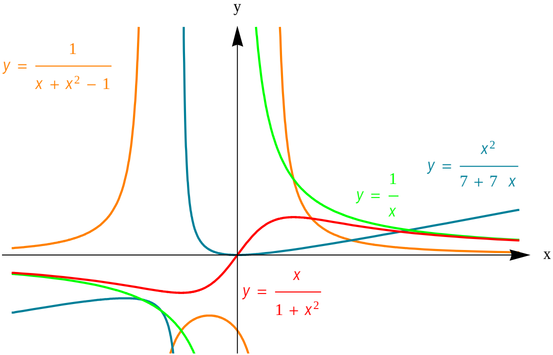


Rational Functions

Quotients of polynomial functions are called ***rational functions***. They have the form

$$f(x) = \frac{p(x)}{q(x)}, \qquad\qquad p,\ q \text{ are polynomial functions.}$$

and have domain $\operatorname{dom} f = \{x \in \mathbb{R} \colon q(x) \neq 0\}$.



Piecewise Functions

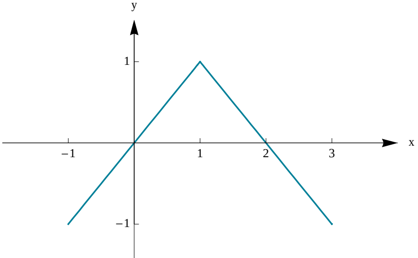
We can define functions in a piecewise way by writing

$$f(x) = \begin{cases} g_1(x) & P_1(x) \\ \vdots & \vdots \\ g_n(x) & P_n(x) \end{cases}$$

where $P_1, \dots, P_n$ are predicates $g_1, \dots, g_n$ are suitable functions with $\operatorname{dom} g_k = \{x \in \mathbb{R} \colon P_k(x)\}$. Generally we require that for any given $x \in \mathbb{R}$ at most one of the predicates is true. The domain of f is then given by $\operatorname{dom} f = \{x \in \mathbb{R} \colon \underset{1 \le k \le n}{\exists} P_k(x)\}$.

2.3.1. Example.

$$f(x) = \begin{cases} x, & -1 \le x < 1, \\ 2 - x, & 1 \le x < 3, \end{cases}$$



$\operatorname{dom} f = [-1, 3)$.

Periodic Functions

A function $f\colon \mathbb{R} \to \mathbb{R}$ such that there exists some $p \in \mathbb{R}$ such that

$$f(x+p) = f(x) \qquad\qquad \text{for all } x \in \mathbb{R}$$

is called ***periodic*** with ***period p***.

We can construct a periodic function as follows: First, we define the ***floor*** and ***ceiling*** of a number $x \in \mathbb{R}$ as

$$\lfloor x \rfloor := \max\{n \in \mathbb{Z} \colon n \le x\}, \qquad \lceil x \rceil := \min\{n \in \mathbb{Z} \colon n \ge x\},$$

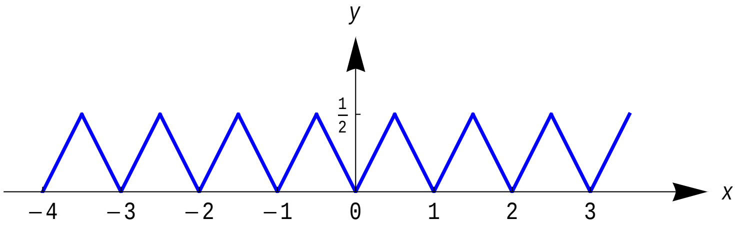
respectively. We then define the "distance to the nearest integer" function, the function

$$\{\,\cdot\,\}\colon \mathbb{R} \to [0, 1/2], \qquad \{x\} := \min\{|x - \lceil x \rceil|, |x - \lfloor x \rfloor|\}. \tag{2.3.1}$$

(Here we have used a dot to indicate the position of the argument x when evaluating $\{\,\cdot\,\}$ at $x \in \mathbb{R}$.) Clearly, $\{x+1\} = \{x\}$ for all x, so $\{\,\cdot\,\}$ is periodic with period $p = 1$.

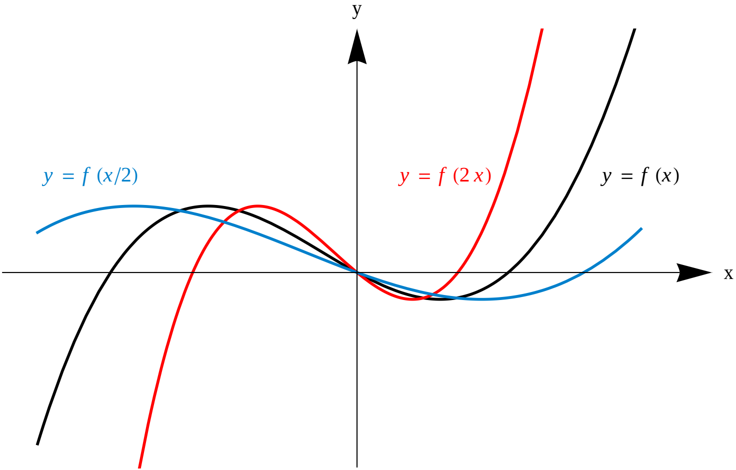
Periodic Functions

The graph of the function $\{\cdot\}$ is sketched below:



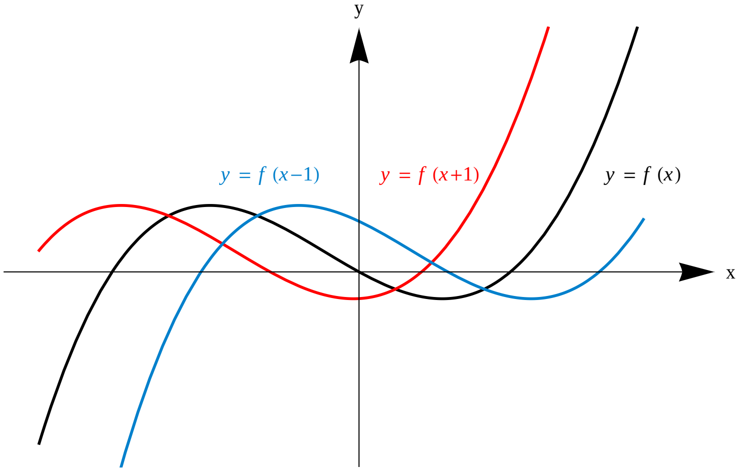
Manipulating Functions

Given a function $f\colon \mathbb{R} \to \mathbb{R}$, we can stretch and translate the graph of f horizontally by multiplying its argument with a number or adding a constant to the argument:



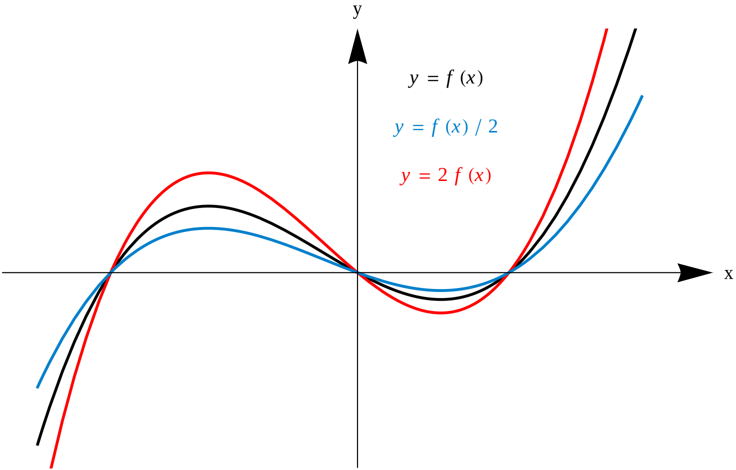
Manipulating Functions

Given a function $f\colon \mathbb{R} \to \mathbb{R}$, we can stretch and translate the graph of f horizontally by multiplying its argument with a number or adding a constant to the argument:



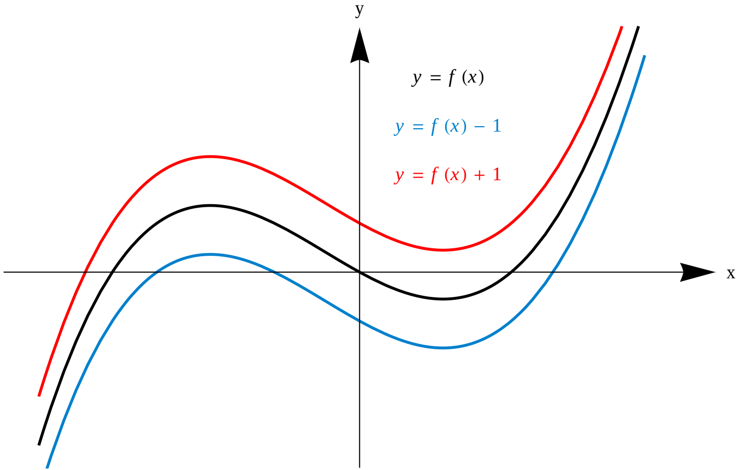
Manipulating Functions

Given a function $f : \mathbb{R} \to \mathbb{R}$, we can stretch and translate the graph of f vertically by multiplying its argument with a number or adding a constant to the value of f:



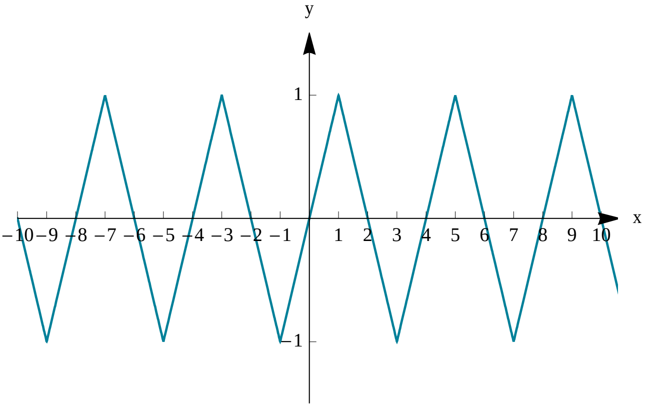
Manipulating Functions

Given a function $f\colon \mathbb{R} \to \mathbb{R}$, we can stretch and translate the graph of f vertically by multiplying its argument with a number or adding a constant to the value of f:



Periodic Functions

Given the function $\{\,\cdot\,\}$ of (2.3.1), we would like to define the function ψ having the graph shown below:



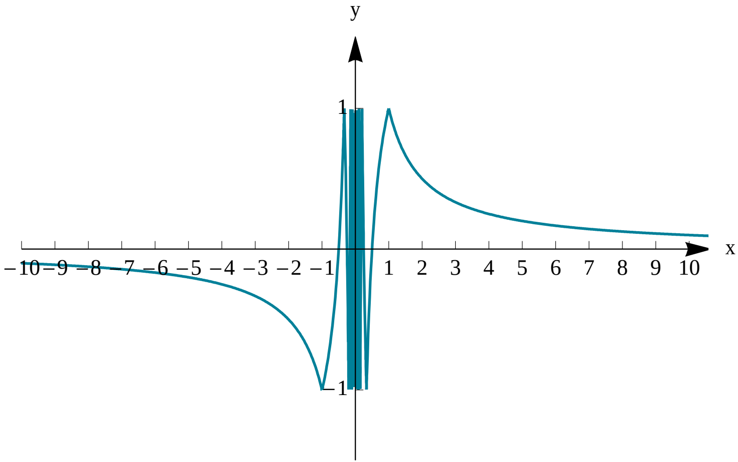
Find an expression for $\psi(x)$ using $\{\,\cdot\,\}$!

Weird Functions

Recall that we have defined the composition of maps. Now the function $\varphi\colon \mathbb{R}\setminus\{0\} \to \mathbb{R}$, $\varphi(x) = 1/x$ has an interesting property: it maps the interval $(0,1)$ to the interval $(1,\infty)$ bijectively, and the interval $(1,\infty)$ to $(0,1)$ bijectively. Furthermore, $f(1) = 1$. Therefore, the action of φ on the positive real numbers can be considered as a "mirroring" at $x = 1$. The action on the negative real numbers is similar.

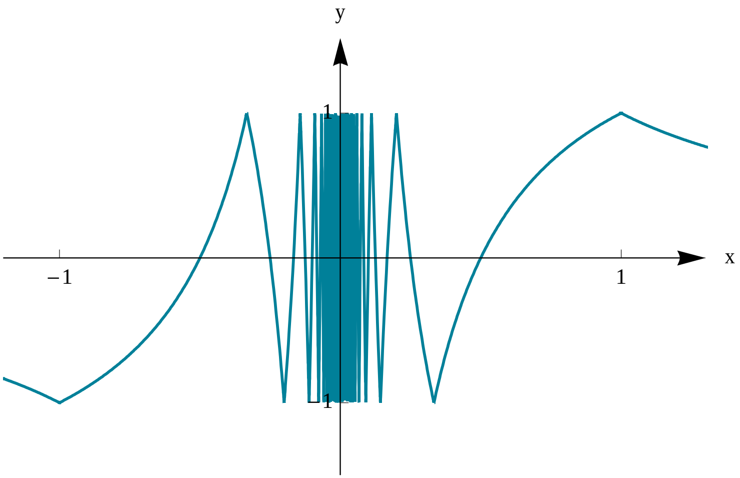
If we compose a function g with φ, the graph of $g \circ \varphi$ shows this mirroring. We illustrate by taking the periodic function ψ that we defined on the previous slide.

Weird Functions

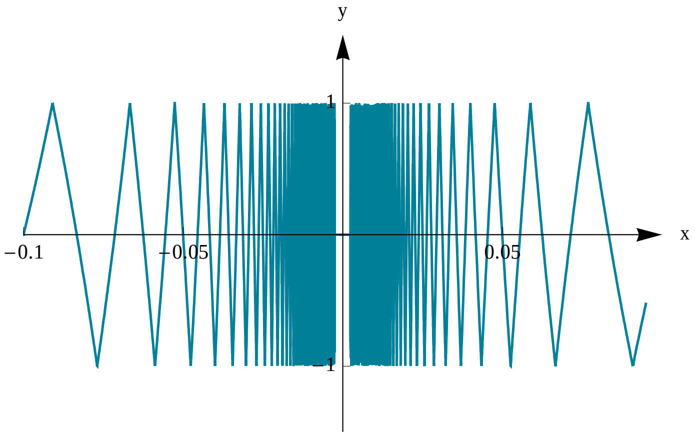


The function $\psi \circ \varphi$, $\operatorname{dom} \psi \circ \phi = \mathbb{R} \setminus \{0\}$, graphed above, is not periodic any more. Something strange seems to go on near the origin.

Weird Functions



Weird Functions



The infinitely many peaks in $(1, \infty)$ of ψ have been crammed into the interval $(0, 1)$ by the composition. It becomes impossible to give a precise rendering of the graph near $x = 0$.

Weird Functions

Strange functions such as the last one can be constructed quite easily. For example, it is impossible to give any precise idea of the graph of

$$f(x) = \begin{cases} 1, & x \text{ rational}, \\ 0, & x \text{ irrational}. \end{cases}$$

Parity and Monotonicity

2.3.2. Definition. Let $\Omega \subset \mathbb{R}$ and $f\colon \Omega \to \mathbb{R}$. The f is said to be

- ***even*** if $f(-x) = f(x)$ for all $x \in \Omega$.
- ***odd*** if $f(-x) = -f(x)$ for all $x \in \Omega$.
- ***increasing*** if $f(x) \geq f(y)$ for all $x > y$, $x, y \in \Omega$.
- ***strictly increasing*** if $f(x) > f(y)$ for all $x > y$, $x, y \in \Omega$.
- ***decreasing*** if $f(x) \leq f(y)$ for all $x > y$, $x, y \in \Omega$.
- ***strictly decreasing*** if $f(x) < f(y)$ for all $x > y$, $x, y \in \Omega$.
- ***monotonic*** if f is either decreasing or increasing.
- ***strictly monotonic*** if f is either strictly decreasing or strictly increasing.

Limit of a Function

Taking our inspiration from sequences, we would like to formalize the concept of the limit of a function. In other words, we would like to be able to describe the behaviour of a function f near some point x_0.

In the case of sequences, the only interesting case was $x_0 = \infty$. While we are ultimately interested in limits of functions also for $x_0 \in \mathbb{R}$, this case can be generalized to functions of a real variable in a straightforward manner:

2.4.1. Definition. Let f be a real- or complex-valued function defined on a subset of $\mathbb{R}$ that includes some interval (a, ∞), $a \in \mathbb{R}$. Then f converges to $L \in \mathbb{C}$ as $x \to \infty$, written

$$\lim_{x\to\infty} f(x) = L \qquad :\Leftrightarrow \qquad \underset{\varepsilon>0}{\forall}\ \underset{C>0}{\exists}\ \underset{x>C}{\forall}\ |f(x) - L| < \varepsilon. \tag{2.4.1}$$

The limit of a function as $x \to -\infty$ is defined similarly.

Limit of a Function

2.4.2. Example. We have

$$\lim_{x\to\infty}\frac{1}{x}=0, \qquad \lim_{x\to-\infty}\frac{1}{x}=0,$$

The analogy of the previous definition to that for sequences is clear. However, if we want to extend the concept of a limit to $x_0 \in \mathbb{R}$, we need to look at the essence of this definition:

> *We preset how close we want $f(x)$ to be to L ("$\underset{\varepsilon>0}{\forall}$"). Then if x is close enough to $x_0 = \infty$ ("$\underset{C>0}{\exists}\ \underset{x>C}{\forall}$") we can achieve $|f(x) - L| < \varepsilon$.*

If we want to extend this to $x_0 \in \mathbb{R}$, we merely have to find another way of saying "if x is close enough to $x_0 \in \mathbb{R}$".

Limit of a Function

This is easily done; we express "if x is close enough to $x_0 \in \mathbb{R}$" by saying "if there exists some $\delta > 0$ such that $|x - x_0| < \delta$." We want this to hold for all x, except possibly for $x = x_0$. (Hence, "near x_0" does not mean "at x_0!"

We also want f to be defined "near x_0," so we assume that x_0 is an accumulation point of its domain. We can then finally write down our definition for the limit of a function at $x_0 \in \mathbb{R}$:

2.4.3. Definition. Let f be a real- or complex-valued function defined on a subset $\Omega \subset \mathbb{R}$ and let x_0 be an accumulation point of Ω. Then the limit of f as $x \to x_0$ is equal to $L \in \mathbb{C}$, written

$$\lim_{x \to x_0} f(x) = L \quad :\Leftrightarrow \quad \forall_{\varepsilon > 0} \ \exists_{\delta > 0} \ \forall_{x \in \Omega \setminus \{x_0\}} \ |x - x_0| < \delta \Rightarrow |f(x) - L| < \varepsilon.$$

Limit of a Function

2.4.4. Examples.

1. $\lim\limits_{x\to 0} \frac{x+1}{2x+1} = 1,$
2. $\lim\limits_{x\to 1} \sqrt{x} = 1.$

As for sequences, the main method for finding limits is not directly by the definition, but rather through the following theorem:

2.4.5. Theorem. Let f and g be real- or complex-valued functions and x_0 an accumulation point of $\operatorname{dom} f \cap \operatorname{dom} g$ such that $\lim\limits_{x\to x_0} f(x)$ and $\lim\limits_{x\to x_0} g(x)$ exist. Then

1. $\lim\limits_{x\to x_0} (f(x) + g(x)) = \lim\limits_{x\to x_0} f(x) + \lim\limits_{x\to x_0} g(x)$,
2. $\lim\limits_{x\to x_0} (f(x) \cdot g(x)) = \big(\lim\limits_{x\to x_0} f(x)\big)\big(\lim\limits_{x\to x_0} g(x)\big),$
3. $\lim\limits_{x\to x_0} \dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x\to x_0} f(x)}{\lim\limits_{x\to x_0} g(x)}$ if $\lim_{x\to x_0} g(x) \neq 0.$

These statements remain true if $x_0 = \pm\infty$.

One-sided Limits

Occasionally, it may be useful to consider ***one-sided limits*** for some $x_0 \in \mathbb{R}$:

2.4.6. Definition. Let f be a real- or complex-valued function defined on a subset $\Omega \subset \mathbb{R}$ and let x_0 be an accumulation point of Ω.

Then the limit of f as x converges to x_0 from above is equal to $L \in \mathbb{C}$,

$$\lim_{x \searrow x_0} f(x) = L \quad :\Leftrightarrow \quad \underset{\varepsilon>0}{\forall}\ \underset{\delta>0}{\exists}\ \underset{x \in \Omega \setminus \{x_0\}}{\forall} \quad 0 < x - x_0 < \delta \Rightarrow |f(x) - L| < \varepsilon.$$

Analogously, the limit of f as x converges to x_0 from below is equal to $L \in \mathbb{C}$,

$$\lim_{x \nearrow x_0} f(x) = L \quad :\Leftrightarrow \quad \underset{\varepsilon>0}{\forall}\ \underset{\delta>0}{\exists}\ \underset{x \in \Omega \setminus \{x_0\}}{\forall} \quad 0 < x_0 - x < \delta \Rightarrow |f(x) - L| < \varepsilon.$$

2.4.7. Remark. Clearly, $f(x) \to L$ as $x \to x_0$ if and only if $f(x) \to L$ as $x \searrow x_0$ and $f(x) \to L$ as $x \nearrow x_0$.

Limits of Functions

In a similar manner to our definition for sequences, we can also define divergence to plus or minus infinity. The precise formulation is left as an exercise. We give some concrete examples

2.4.8. Examples.

1. $\lim\limits_{x\to 0} \frac{1}{x}$ does not exist; $\lim\limits_{x\searrow 0} \frac{1}{x} = \infty$, $\lim\limits_{x\nearrow 0} \frac{1}{x} = -\infty$. Note that $1/x$ is not defined at $x = 0$.
2. Let $f\colon \mathbb{R} \to \mathbb{R}$ be defined as

$$f(x) = \begin{cases} \frac{x}{1+x^2}, & -\infty < x \le 0, \\ 1 + x, & 0 < x < \infty. \end{cases}$$

Then $\lim\limits_{x\searrow 0} f(x) = 1$, $\lim\limits_{x\nearrow 0} f(x) = 0$ and $\lim\limits_{x\to x_0} f(x)$ exists for all $x_0 \neq 0$. Note that $f(0) = 0$.

Limits of Functions using Sequences

There is a close relation between the limit of a function and the limits of sequences:

2.4.9. Theorem. Let f be a real- or complex-valued function defined on a subset $\Omega \subset \mathbb{R}$ and let x_0 be an accumulation point of Ω. Then

$$\lim_{x\to x_0} f(x) = L \qquad \Leftrightarrow \qquad \underset{\substack{(a_n)\\ a_n\in\Omega\setminus\{x_0\}}}{\forall} \left(a_n \xrightarrow{n\to\infty} x_0 \Rightarrow f(a_n) \xrightarrow{n\to\infty} L\right)$$

A similar result holds for $x_0 = \pm\infty$.

This remarkable result means that the limits of functions can be described entirely through sequences.

Limits of Functions using Sequences

Proof.

($\Rightarrow$) Assume that $\lim_{x\to x_0} f(x) = L$. Let (a_n) be a sequence such that $a_n \to x_0$. We will show that then $f(a_n) \to L$. Fix $\varepsilon > 0$. Then we choose a $\delta > 0$ such that

$$|x - x_0| < \delta \Rightarrow |f(x) - L| < \varepsilon.$$

Furthermore, we can choose an $N \in \mathbb{N}$ such that $|a_n - x_0| < \delta$ for $n > N$. Then for $n > N$, $|f(a_n) - L| < \varepsilon$, so $f(a_n) \to L$.

Limits of Functions using Sequences

Proof (continued).

($\Leftarrow$) Assume that $f(a_n) \to L$ for all sequences (a_n) with $a_n \to x_0$. We prove that then $f(x) \to L$ as $x \to x_0$ by ***contraposition***. Assume that $f(x) \not\to L$ as $x \to x_0$. Then

$$\begin{aligned} &\underset{\varepsilon>0}{\exists}\ \underset{\delta>0}{\forall}\ \underset{x\in\Omega}{\exists}\ |x - x_0| < \delta \not\Rightarrow |f(x) - L| < \varepsilon \\ \Leftrightarrow\ &\underset{\varepsilon>0}{\exists}\ \underset{\delta>0}{\forall}\ \underset{x\in\Omega}{\exists}\ |x - x_0| < \delta \wedge |f(x) - L| \geq \varepsilon \end{aligned}$$

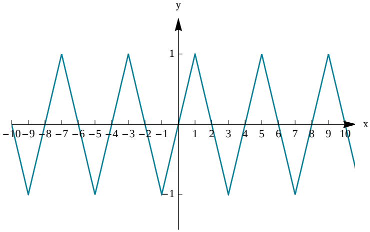
Fix $\varepsilon > 0$. Then for every $n \in \mathbb{N}^*$ there exists a number x_n such that

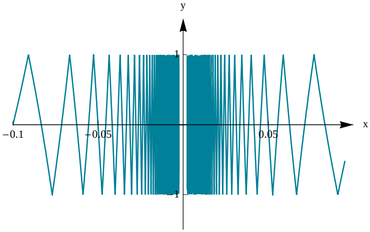
$$|x_n - x_0| < \frac{1}{n} \wedge |f(x_n) - L| \geq \varepsilon.$$

Clearly the sequence (x_n) converges to x_0, but $f(x_n)$ cannot converge to L. $\square$

Limits of Functions using Sequences

2.4.10. Example. Let ψ denote the periodic function we studied earlier, as shown in the left graph below. We want to study $\lim_{x\to 0} \psi(1/x)$, shown below right.





We suspect that the limit does not exist. We consider the sequence (x_n), $x_n = 1/(4n+1)$. Then $x_n \to 0$ and $\psi(1/x_n) = \psi(4n+1) = 1 \to 1$ as $n \to \infty$.

For (y_n) given by $y_n = 1/(4n+3)$ we also have $y_n \to 0$ but $\psi(1/y_n) = \psi(4n+3) = -1 \to -1$. Therefore $\lim_{x\to 0} \psi(1/x)$ does not exist.

Limits of Functions using Sequences

2.4.11. Example. Now consider the function f given by $f(x) = x\psi(1/x)$. We want to show that

$$\lim_{x\to 0} x\psi(1/x) = 0.$$

Let (x_n) be any sequence with $x_n \to 0$ as $n \to \infty$. Note that

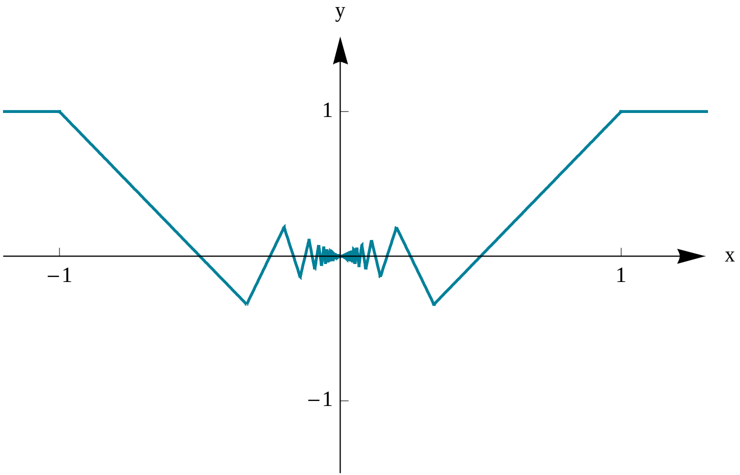
$$|f(x_n)| = |x_n| \cdot |\psi(1/x_n)| \le |x_n| \xrightarrow{n\to\infty} 0,$$

since $|\psi(x)| \le 1$ for all $x \in \mathbb{R}$. It follows that

$$\lim_{n\to\infty} |f(x_n)| = 0,$$

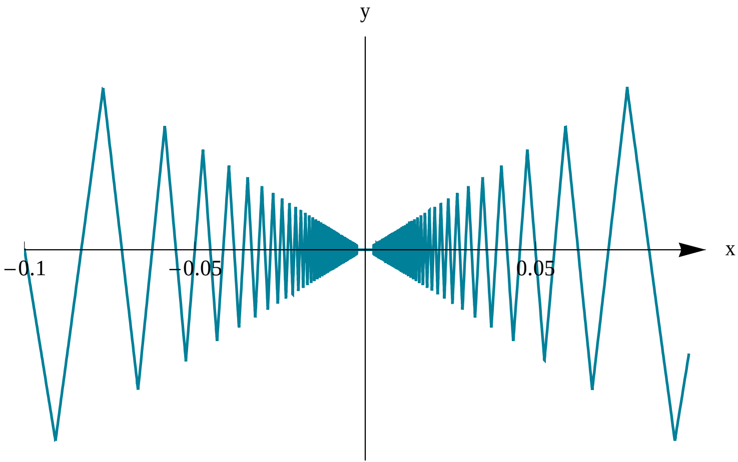
so $\lim\limits_{x\to 0} f(x) = 0$.

Limits of Functions using Sequences



The function f given by $f(x) = x\psi(1/x)$; note that it is constant for $|x| > 1$.

Limits of Functions using Sequences



Behavior near $x = 0$.

Behavior of Functions

Hence limits serve to characterize the behavior of functions near points. However, our current techniques do not give us very detailed information.For example, consider the functions $f, g, h\colon \mathbb{R} \to \mathbb{R}$,

$$f(x) = \sqrt{x^2+1}, \qquad g(x) = x, \qquad h(x) = x^2.$$

We expect f and g to behave "similarly" as $x \to \infty$, but h to have "much larger" values. However, at this point we can only say that the individual limits do not exist and the functions diverge to infinity.

Similarly, x^2 should be "much smaller" than x as $x \to 0$, even though both limits vanish.

The main tools for characterizing such behavior are known as ***Landau symbols***. There are two different "kinds": big-O and little-O. We first introduce big-Oh.

The Big-O Landau Symbol

2.4.12. Definition. Let f, ϕ be real- or complex-valued functions defined on a subset $\Omega \subset \mathbb{R}$ and let x_0 be an accumulation point of Ω. We say that

$$f(x) = O(\phi(x)) \qquad \text{as } x \to x_0$$

if and only if

$$\underset{C>0}{\exists}\ \underset{\varepsilon>0}{\exists}\ \underset{x\in\Omega}{\forall} \quad |x - x_0| < \varepsilon \quad \Rightarrow \quad |f(x)| \le C|\phi(x)| \tag{2.4.2}$$

2.4.13. Example. We have $x + x^2 = O(x)$ as $x \to 0$: taking $\varepsilon = 1$ and $C = 2$, for all $x \in \mathbb{R}$,

$$|x - 0| < 1 \quad \Rightarrow \quad |x + x^2| \le |x| + |x|^2 = \underbrace{(1 + |x|)}_{<2}|x|.$$

2.4.14. Remark. "$f(x) = O(\phi(x))$ as $x \to x_0$" can be interpreted as "$|f(x)|$ is not significantly larger than $|\phi(x)|$ when x is near x_0."

The Big-O Landau Symbol

We can make an analogous definition for $x \to \infty$,

2.4.15. Definition. Let f, ϕ be real- or complex-valued functions defined on a subset $\Omega \subset \mathbb{R}$ containing the interval (L, ∞) for some $L \in \mathbb{R}$. We say that

$$f(x) = O(\phi(x)) \qquad\qquad \text{as } x \to \infty$$

if and only if

$$\underset{C>0}{\exists}\ \underset{M>L}{\exists} \quad x > M \quad \Rightarrow \quad |f(x)| \le C|\phi(x)|$$

The statement that $f(x) = O(\phi(x))$ as $x \to -\infty$ is defined similarly.

2.4.16. Example. We have $x + x^2 = O(x^2)$ as $x \to \infty$: taking $M = 1$ and $C = 2$, for all $x \in \mathbb{R}$,

$$|x| > 1 \quad \Rightarrow \quad |x + x^2| \le |x| + |x|^2 < 2|x|^2.$$

The Little-O Landau Symbol

2.4.17. Definition. Let f, ϕ be real- or complex-valued functions defined on a subset $\Omega \subset \mathbb{R}$ and let x_0 be an accumulation point of Ω. We say that

$$f(x) = o(\phi(x)) \qquad\qquad \text{as } x \to x_0$$

if and only if

$$\underset{C>0}{\forall}\ \underset{\varepsilon>0}{\exists}\ \underset{x\in\Omega\setminus\{x_0\}}{\forall} \quad |x - x_0| < \varepsilon \quad \Rightarrow \quad |f(x)| < C|\phi(x)| \qquad (2.4.3)$$

2.4.18. Example. We have $x^2 = o(x)$ as $x \to 0$: for any $C > 0$ we can take $\varepsilon = C$, so that for all $x \in \mathbb{R}$,

$$|x - 0| < \varepsilon = C \quad \Rightarrow \quad |x^2| = |x| \cdot |x| < C|x|.$$

2.4.19. Remark. "$f(x) = o(\phi(x))$ as $x \to x_0$" can be interpreted as "$|f(x)|$ is much smaller than $|\phi(x)|$ when x is near x_0."

The Little-O Landau Symbol

We can make an analogous definition for $x \to \infty$,

2.4.20. Definition. Let f, ϕ be a real- or complex-valued functions defined on a subset $\Omega \subset \mathbb{R}$ containing the interval (L, ∞) for some $L \in \mathbb{R}$. We say that

$$f(x) = o(\phi(x)) \qquad\qquad \text{as } x \to \infty$$

if and only if

$$\underset{C>0}{\forall} \ \underset{M>L}{\exists} \quad x > M \quad \Rightarrow \quad |f(x)| < C|\phi(x)|$$

The statement that $f(x) = o(\phi(x))$ as $x \to -\infty$ is defined similarly.

2.4.21. Example. We have $1/x^2 = o(1/x)$ as $x \to \infty$: For any $C > 0$ take $M > 1/C$. Then, for all $x \in \mathbb{R}$,

$$x > M = 1/C \quad \Rightarrow \quad \frac{1}{x^2} \le C\frac{1}{x}$$

More Examples for Landau Symbol Notation

Some more examples are given below:

2.4.22. Examples.

(i) $\dfrac{1}{x^2} = O\left(\dfrac{1}{x}\right)$ as $x \to \infty$,

(ii) $\dfrac{1}{x} = O\left(\dfrac{1}{x^2}\right)$ as $x \to 0$,

(iii) $\dfrac{1}{x^2+x} = O\left(\dfrac{1}{x}\right)$ as $x \to 0$,

(iv) $\dfrac{1}{x^2+x} = O\left(\dfrac{1}{x}\right)$ as $x \to \infty$,

(v) $\dfrac{1}{x^2+x} = o\left(\dfrac{1}{x^3}\right)$ as $x \to 0$,

(vi) $\dfrac{1}{x^2+x} = o(1)$ as $x \to \infty$.

Landau Symbols using Limits

In many practical cases, the Landau symbols can be calculated using limits:

2.4.23. Theorem. Let f, ϕ be a real- or complex-valued functions defined on a subset $\Omega \subset \mathbb{R}$ and let x_0 be an accumulation point of Ω. If $x_0 \in \Omega$, we require $|\phi(x_0)| > 0$. Suppose that exists some $C \geq 0$ such that

$$\lim_{x \to x_0} \frac{|f(x)|}{|\phi(x)|} = C. \tag{2.4.4}$$

Then $f(x) = O(\phi(x))$ as $x \to x_0$.

Proof.
Let f, ϕ, I and x_0 be given and suppose that (2.4.4) holds for some $C \geq 0$. Then

$$\underset{\varepsilon>0}{\forall} \; \underset{\delta>0}{\exists} \; \underset{x \in \Omega \setminus \{x_0\}}{\forall} \; |x - x_0| < \delta \Rightarrow \left| \frac{|f(x)|}{|\phi(x)|} - C \right| < \varepsilon.$$

Landau Symbols using Limits

Proof (continued).

Choose $\varepsilon = 1$. Then there exists some $\delta > 0$ such that $|x - x_0| < \delta$, $x \neq x_0$, implies

$$\big||f(x)| - C|\phi(x)|\big| < |\phi(x)| \qquad \text{for all } x \in \Omega \setminus \{x_0\},$$

i.e.,

$$|f(x)| < (C+1)|\phi(x)|.$$

for $x \neq x_0$. To prove an analogous inequality for $x = x_0$, note that $\phi(x_0)$ is assumed to have a non-zero values, so $|f(x_0)| \le C' \cdot |\phi(x_0)|$ for some $C' > 0$. Hence, $f(x) = O(\phi(x))$ as $x \to x_0$. □

2.4.24. Theorem. Let f, ϕ be a real- or complex-valued functions defined on an interval $I \subset \mathbb{R}$ and let $x_0 \in \overline{I}$. Then

$$\lim_{x \to x_0} \frac{|f(x)|}{|\phi(x)|} = 0 \qquad \Leftrightarrow \qquad f(x) = o(\phi(x)) \text{ as } x \to x_0. \tag{2.4.5}$$

This result will be proven in the assignments.

Usage of Landau Symbols

The notation $f(x) = O(\phi(x))$ as $x \to x_0$ is used to describe an absolute size relation near x_0 ***and at x_0***, to the effect that f is not significantly larger than ϕ. For such size comparisons, it is useful to have the domain of the inequality (2.4.2) encompass the point x_0 also.

On the other hand, when using little-o, the convention is that only the behaviour near x_0 should enter into the statement that f is much smaller than ϕ, so (2.4.3) does not encompass the point x_0.

This is what is responsible for the different statements of Theorems 2.4.23 and 2.4.24. In fact, often the little-o symbol is defined in terms of the limit in (2.4.5) while the big-o symbol is defined in terms of (2.4.2); see http://mathworld.wolfram.com/LandauSymbols.html.

Note also that the article on Landau symbols on Wikipedia is at https://en.wikipedia.org/wiki/Big_O_notation and written exclusively from the point of view of computer science - little-o, so important in mathematics, is hardly mentioned at all.

Some Remarks on Landau Symbols

More properly, "$O(\phi(x))$ as $x \to x_0$" denotes a set: the set of all functions f such that for some $C \geq 0$ and some $\varepsilon > 0$, $|f(x)| < C|\phi(x)|$ for all $x \in (x_0 - \varepsilon, x_0 + \varepsilon)$. A better notation for this set might be $O_{x_0}(\phi)$ and we could (perhaps more correctly) write

$$f \in O_{x_0}(\phi) \qquad \text{instead of} \qquad f(x) = O(\phi(x)) \quad \text{as } x \to x_0.$$

However, the second notation has become standard among physicists and mathematicians, and we write expressions like

$$(x+1)^4 = 1 + 4x + O(x^2) \quad \text{meaning} \quad (x+1)^4 - (1+4x) = O(x^2)$$

as $x \to 0$. This is often more intuitive than writing

$$(x+1)^4 - (1+4x) \in O_0(x^2) \quad \text{or} \quad (x+1)^4 = 1 + 4x \bmod O_0(x^2).$$

Some Remarks on Landau Symbols

We do sacrifice the symmetry of the "=" sign, however. For example,

$$O(x^3) = O(x^2) \quad \text{as } x \to 0$$

means "any function f such that $|f(x)| < C|x|^3$ for all $|x| < \varepsilon$ for some ε, C also satisfies $|f(x)| < C'|x|^2$ for all $|x| < \varepsilon'$ for some ε', C'." Clearly,

$$O(x^2) \stackrel{?}{=} O(x^3) \quad \text{as } x \to 0 \qquad \text{is false}.$$

Landau symbols can be combined with each other and with functions. For example, as $x \to 0$,

$$c \cdot O(x^n) = O(x^n), \qquad x^n O(x^m) = O(x^{n+m}),$$
$$O(x^n) + O(x^m) = O(x^{\min(n,m)}), \qquad O(x^n)O(x^m) = O(x^{n+m}).$$

Further examples and applications will be given in the assignments.

Continuity

We have extensively discussed limits of functions, and also discovered that the values of a function at some point need not have anything to do with the behavior of the function near that point.

However, we would of course like to assume that "reasonable" functions behave in a consistent manner on their domain. We will therefore give a name to this reasonableness:

2.5.1. Definition. Let $\Omega \subset \mathbb{R}$ be any set and $f\colon \Omega \to \mathbb{R}$ be a function defined on Ω. Let $x_0 \in \Omega$. We say that f is ***continuous at x_0*** if

$$\lim_{x \to x_0} f(x) = f(x_0).$$

If $U \subset \Omega$, we say that f is ***continuous on U*** if f is continuous at every $x_0 \in U$.

We say that f is ***continuous on its domain***, or simply ***continuous***, if f is continuous at every $x_0 \in \Omega$.

Continuity

2.5.2. Remark. For a function f to be continuous at a point x_0, three conditions have to be fulfilled:

(i) f needs to have a limit at x_0 ($\lim_{x \to x_0} f(x)$ must exist);
(ii) f needs to be defined at x_0 ($f(x_0)$ must exist);
(iii) the value of f must coincide with its limit ($f(x_0) = \lim_{x \to x_0} f(x)$).

2.5.3. Examples.

1. The zig-zag function $f(x) = \psi(1/x)$ is not continuous at $x = 0$ because the limit $\lim_{x \to 0} f(x)$ does not exist (see Example 2.4.10).
2. The function $f(x) = x\psi(1/x)$ is not continuous at $x = 0$ because f is not defined at $x = 0$.
3. The function

$$\widetilde{f}(x) = \begin{cases} x\psi(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

is continuous at $x = 0$ (see Example 2.4.11).

Continuity

We can use the results that we have obtained previously concerning limits of sequences and functions to express the continuity of f at x_0 in equivalent ways:

2.5.4. Theorem. Let $\Omega \subset \mathbb{R}$ be any set and $f\colon \Omega \to \mathbb{R}$ be a function defined on Ω. Let $x_0 \in \Omega$. Then the following are equivalent:

1. f is continuous at x_0;
2. for any real sequence (a_n) with $a_n \to x_0$, $\lim_{n\to\infty} f(a_n) = f(x_0)$;
3. $\underset{\varepsilon>0}{\forall}\ \underset{\delta>0}{\exists}\ \underset{x\in\operatorname{dom} f}{\forall} : |x - x_0| < \delta \quad \Rightarrow \quad |f(x) - f(x_0)| < \varepsilon.$

In fact, since we have discussed sequences in metric spaces, and we have an idea of distance in metric spaces, we can define continuity easily for maps between metric spaces. This is left to the exercises.

One-Sided Continuity

Occasionally it will be useful to consider one-sided continuity.

2.5.5. Definition. Let $\Omega \subset \mathbb{R}$ be any set and $f\colon \Omega \to \mathbb{R}$ be a function defined on Ω. Let $x_0 \in \Omega$. We say that f is ***continuous from the left or from below*** at x_0 if

$$\lim_{x \nearrow x_0} f(x) = f(x_0).$$

We say that f is ***continuous from the right or from above*** at x_0 if

$$\lim_{x \searrow x_0} f(x) = f(x_0).$$

Of course, a function is continuous at a point if and only if it is continuous both from above and from below. One-sided continuity can also be characterized using monotonic sequences; the precise formulation is left to you!

Continuous Extension

2.5.6. Definition. Let $\Omega \subset \mathbb{R}$ be any set and $\widetilde{\Omega} \supset \Omega$. Suppose that $f : \Omega \to \mathbb{R}$ and $\widetilde{f} : \widetilde{\Omega} \to \mathbb{R}$ are continuous functions. If $\widetilde{f}(x) = f(x)$ for all $x \in \Omega$, we say that $\widetilde{f}$ is a ***continuous extension*** of f to $\widetilde{\Omega}$.

2.5.7. Examples.

1. Let $f : [0, \infty) \to \mathbb{R}$, $f(x) = \sqrt{x}$ and $\widetilde{f} : \mathbb{R} \to \mathbb{R}$, $\widetilde{f}(x) = \sqrt{|x|}$. Then $\widetilde{f}$ is a continuous extension of f.
2. Let $f : \mathbb{R} \setminus \{1\} \to \mathbb{R}$, $f(x) = (x^2 - 1)/(x - 1)$ and $\widetilde{f} : \mathbb{R} \to \mathbb{R}$, $\widetilde{f}(x) = x + 1$. Then $\widetilde{f}$ is a continuous extension of f.

2.5.8. Remark. Suppose that $\Omega \subset \mathbb{R}$, $x_0 \in \Omega$ and $f : \Omega \setminus \{x_0\} \to \mathbb{R}$ is continuous and has the property that $\lim\limits_{x \to x_0} f(x)$ exists. Then $\widetilde{f} : \Omega \to \mathbb{R}$,

$$\widetilde{f}(x) = \lim_{y \to x} f(y), \qquad x \in \Omega,$$

defines the unique continuous extension of f to Ω.

Continuity

While continuous functions may still exhibit some curious behavior (we will study examples later), nevertheless continuity is sufficient as a hypothesis for many theorems. In fact, calculus may be regarded as the branch of mathematics studying continuous functions.

For real functions f and g we can define their sum $f+g$ and product $f \cdot g$ by

$$\begin{aligned} &f+g\colon (\operatorname{dom} f)\cap(\operatorname{dom} g)\to\mathbb{R}, && (f+g)(x) := f(x)+g(x),\\ &f\cdot g\colon (\operatorname{dom} f)\cap(\operatorname{dom} g)\to\mathbb{R}, && (f\cdot g)(x) := f(x)\cdot g(x). \end{aligned}$$

We thus define the sum/product of two functions by describing how $f+g$ and $f\cdot g$ act on every $x \in (\operatorname{dom} f)\cap(\operatorname{dom} g)$. This type of definition is called a ***pointwise*** definition.

Continuity

2.5.9. Theorem. Let f and g be two real functions and $x \in (\operatorname{dom} f) \cap (\operatorname{dom} g)$. Assume that both f and g are continuous at x. Then

(i) $f + g$ is continuous at x and

(ii) $f \cdot g$ is continuous at x.

Furthermore, if $g(x) \neq 0$, the function h defined by $h(x) = 1/g(x)$ is continuous at x.

Moreover, if f and g are real functions with $x \in \operatorname{dom} g$, $g(x) \in \operatorname{dom} f$, g is continuous at x and f is continuous at $g(x)$, then the composition $f \circ g$ is continuous at x.

Continuity

Proof.

The first three assertions follow immediately from the corresponding theorem for limits. We will prove the last assertion concerning the composition of f and g.

Let $\varepsilon > 0$. Since f is continuous at $g(x)$, we know that there exists some $\delta > 0$ such that for all $y \in \operatorname{dom} f$

$$|g(x) - y| < \delta \quad \Rightarrow \quad |f(g(x)) - f(y)| < \varepsilon.$$

We fix some such $\delta > 0$. Now since g is continuous at x, there exists some $\widetilde{\delta} > 0$ such that for all $z \in \operatorname{dom} g$

$$|x - z| < \widetilde{\delta} \quad \Rightarrow \quad |g(x) - g(z)| < \delta.$$

Recall $z \in \operatorname{dom}(f \circ g)$ if $z \in \operatorname{dom} g$ and $g(z) \in \operatorname{dom} f$. Hence for all $z \in \operatorname{dom}(f \circ g)$, if $|x - z| < \widetilde{\delta}$, then $|f(g(x)) - f(g(z))| < \varepsilon$. This proves that $f \circ g$ is continuous. $\square$

Continuity

In fact, we have a stronger version of the last statement that does not presume the continuity of f.

2.5.10. Theorem. Let f, g be real functions such that $\lim_{x \to x_0} g(x) = L$ exists and f is continuous at $L \in \operatorname{dom} f$. Then

$$\lim_{x \to x_0} f(g(x)) = f(L).$$

The proof is similar to the previous proof and will be treated in your recitation classes next week.

Continuity

The following lemma is useful in many proofs and also gives a first idea of how powerful continuity is.

2.5.11. Lemma. Let $\Omega \subset \mathbb{R}$ be some set, $f\colon \Omega \to \mathbb{R}$ a function that is continuous at some point $x_0 \in \Omega$ and assume that $f(x_0) > 0$. Then there exists a $\delta > 0$ such that $f(x) > 0$ for all $x \in (x_0 - \delta, x_0 + \delta) \cap \Omega$.

Proof.
Since f is continuous at x_0, for $\varepsilon = f(x_0)/2$ we can find a $\delta > 0$ such that for all $x \in \Omega$

$$\begin{aligned} |x - x_0| < \delta &\Rightarrow |f(x) - f(x_0)| < f(x_0)/2 \\ &\Rightarrow f(x) > f(x_0)/2 > 0. \end{aligned}$$

□

The Bolzano Intermediate Value Theorem

This is the first of three important theorems that enable us to obtain hard information on the behavior of continuous functions.

2.5.12. Theorem. Let $a < b$ and $f : [a, b] \to \mathbb{R}$ be a continuous function with $f(a) < 0 < f(b)$. Then there exists some $x \in [a, b]$ such that $f(x) = 0$.

Proof.
We consider

$$A := \{x \in [a, b] \colon f(y) < 0 \text{ for } y \in [a, x]\}.$$

Clearly A is bounded ($a \le x \le b$ for all $x \in A$) and non-empty ($a \in A$). Therefore, A has a least upper bound $a \le \alpha \le b$.

We claim that in fact $\alpha < b$. Note that $f(b) > 0$. By Lemma 2.5.11, we know that there exists some $\delta > 0$ such that $f(x) > 0$ for all $x \in (b - \delta, b]$. Therefore, $b - \delta$ is an upper bound for A and hence $\alpha < b$.

The Bolzano Intermediate Value Theorem

Proof (continued).

Similarly, there exists a δ' such that $f(x) < 0$ for all $x \in [a, a+\delta')$, so α is at least as large as $a+\delta'$. It follows that $a < \alpha < b$.

We will now show that $f(\alpha) = 0$, by proving that $f(\alpha) > 0$ and $f(\alpha) < 0$ lead to contradictions.

First, suppose $f(\alpha) < 0$. Then by Lemma 2.5.11, there exists a $\delta > 0$ such that $f(x) < 0$ for $x \in (\alpha-\delta, \alpha+\delta)$. Since $\alpha = \sup A$, for this δ there exists some $x_0 \in A$ such that $\alpha - \delta < x_0 < \alpha$. (Otherwise, $\alpha - \delta$ would be an upper bound for A, giving a contradiction.) This means that $f(x) < 0$ for all $x \in [a, x_0]$. Since $f(x) < 0$ for $x \in (\alpha-\delta, \alpha+\delta)$ and $x_0 > \alpha - \delta$, it follows that $f(x) < 0$ for $x \in [a, \alpha+\delta)$. But this contradicts the assumption that α is an upper bound for A.

The proof for $f(\alpha) > 0$ is analogous and left to you! $\square$

The Bolzano Intermediate Value Theorem

An immediate corollary is the following useful result:

2.5.13. Bolzano Intermediate Value Theorem. Let $a < b$ and $f : [a, b] \to \mathbb{R}$ be a continuous function. Then for $y \in [\min\{f(a), f(b)\}, \max\{f(a), f(b)\}]$ there exists some $x \in [a, b]$ such that $y = f(x)$.

Proof.
If $f(a) = f(b)$, the statement is trivially true. Suppose that $f(a) \leq f(b)$. Given y, we apply Theorem 2.5.12 to the function $g : [a, b] \to \mathbb{R}$, $g(x) = f(x) - y$. If $f(a) > f(b)$, we apply Theorem 2.5.12 to the function given by $g(x) = f(a + b - x) - y$. □

A Fixed Point Theorem

There are many applications of the intermediate value theorem. Most involve choosing the function f to which the theorem is applied in a smart way. We give one such result here:

2.5.14. Theorem. Let $f : [a, b] \to \mathbb{R}$ be a continuous function with $\operatorname{ran} f \subset [a, b]$. Then f has a fixed point, i.e., there exists some $x \in [a, b]$ such that $f(x) = x$.

This is an example of a ***fixed point theorem***. (In fact, it is a special case of ***Brouwer's fixed point theorem***.) Such results are extremely useful for proving existence and/or uniqueness of very general objects in mathematics.

One of the most basic fixed point theorems is called the ***contraction mapping principle*** or ***Banach's fixed point theorem*** and is treated in the assignments.

A Fixed Point Theorem

Proof.

We assume that $f(a) > a$ and $f(b) < b$, for otherwise a and b would be fixed points. Then $f(a) - a > 0$ and $f(b) - b < 0$. Consider the function $g(x) = f(x) - x$.

Since $g(a) > 0$ and $g(b) < 0$, it follows that for some $x_0 \in [a, b]$

$$0 = g(x_0) = f(x_0) - x_0 \quad \Leftrightarrow \quad f(x_0) = x_0,$$

so x_0 is a fixed point for f. $\square$

Boundedness of Continuous Functions

It turns out that continuity of a function implies its boundedness on closed and bounded domains; we will study this more closely.

2.5.15. Lemma. Let $\Omega \subset \mathbb{R}$ be some set and $f : \Omega \to \mathbb{R}$ a function that is continuous at some point $x_0 \in \Omega$. Then there exists a $\delta > 0$ such that f is bounded above on $(x_0 - \delta, x_0 + \delta) \cap \Omega$.

Basically, this lemma says that $\sup\{f(x) : x \in B_\delta(x_0)\}$ exists for some $\delta > 0$.

Proof.
Since f is continuous at x_0, for $\varepsilon = 1$ we can find a $\delta > 0$ such that for all $x \in \Omega$

$$\begin{aligned} |x - x_0| < \delta &\Rightarrow |f(x) - f(x_0)| < 1 \\ &\Rightarrow |f(x)| < |f(x_0)| + 1. \end{aligned}$$

□

Boundedness of Continuous Functions

We can now prove a more general theorem.

2.5.16. Proposition. Let $a < b$ and $f : [a, b] \to \mathbb{R}$ be a continuous function. Then f is bounded above.

By this result, $\sup\{f(x) : x \in [a, b]\}$ exists. Proposition 2.5.16 also implies that f is bounded below and hence bounded on $[a, b]$.

Proof.
We define

$$A := \{x \in [a, b] : f \text{ is bounded above on } [a, x]\}.$$

We would like to show that $b \in A$, proving the theorem. Clearly A is bounded ($a \le x \le b$ for all $x \in A$) and non-empty ($a \in A$). Therefore, A has a least upper bound $a \le \alpha \le b$.

Boundedness of Continuous Functions

Proof (continued).
We claim that in fact $\alpha = b$. Suppose that $\alpha < b$. By Lemma 2.5.15, we know that there exists some $\delta > 0$ such that f is bounded above on $(\alpha - \delta, \alpha + \delta)$. Since α is the least upper bound of A, there exists some $x_0 \in A$ such that $\alpha - \delta < x_0 < \alpha$. We conclude that f is bounded above on $[a, \alpha + \delta)$ and hence $\alpha + \delta/2 \in A$. This gives a contradiction, hence $\alpha = b$.

We have shown that $\sup A = b$, i.e., f is bounded above on $[a, x]$ for every $x < b$. But that this does not automatically imply that f is bounded above on $[a, b]$! For this, we need to check that actually $\max A$ exists and equals b. But since there exists a $\delta > 0$ such that f is bounded above on $(b - \delta, b]$, we can conclude that f is bounded above on $[a, b]$ and we are finished. $\square$

Boundedness of Continuous Functions

The previous result can be significantly strengthened.

2.5.17. Theorem. Let $a < b$ and $f: [a, b] \to \mathbb{R}$ be a continuous function. Then there exists a $y \in [a, b]$ such that $f(x) \le f(y)$ for all $x \in [a, b]$.

Hence $\max\{f(x): x \in [a, b]\}$ exists. Colloquially, we say that "a continuous function attains its maximum".

Proof.
We define

$$A := \{y \in \mathbb{R}: y = f(x),\ x \in [a, b]\}.$$

By Proposition 2.5.16, A is bounded and A is clearly non-empty ($f(a) \in A$). Therefore, A has a least upper bound α such that $\alpha \ge f(x)$ for all $x \in [a, b]$. We will show that $\alpha = f(y)$ for some $y \in [a, b]$.

Boundedness of Continuous Functions

Proof (continued).

We argue by contradiction: assume $\alpha \neq f(y)$ for all $y \in [a, b]$. Then

$$g\colon [a,b] \to \mathbb{R}, \qquad g(x) = \frac{1}{\alpha - f(x)}$$

is continuous on $[a, b]$ but not bounded above, as we shall see.

Since α is the least upper bound of A, for every $\varepsilon > 0$ there exists an $x \in [a, b]$ such that $\alpha - f(x) < \varepsilon$. But then for this x, $g(x) > 1/\varepsilon$. Since we can make ε as small as desired, g becomes as large as desired, so g is not bounded above. But this contradicts Proposition 2.5.16. $\square$

Inverse Functions

We want to say a few words about the inverse of a function. First, we take the opportunity to introduce some new terms:

2.5.18. Definition. Let $\Omega, \widetilde{\Omega} \subset \mathbb{R}$ and $f\colon \Omega \to \widetilde{\Omega}$ a function. We say that f is

- ***injective*** if $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ for all $x_1, x_2 \in \Omega$;
- ***surjective*** if for every $y \in \widetilde{\Omega}$ there exists an $x \in \Omega$ such that $f(x) = y$ (i.e., if $\operatorname{ran} f = \widetilde{\Omega}$);
- ***bijective*** if for every $y \in \widetilde{\Omega}$ there exists a ***unique*** $x \in \Omega$ such that $f(x) = y$ (i.e., f is injective and surjective).

For $f\colon \Omega \to \widetilde{\Omega}$ we would like to define the ***inverse function of f***, denoted by f^{-1} as

$$f^{-1}\colon \widetilde{\Omega} \to \Omega, \qquad\qquad f(x) \mapsto x.$$

If f^{-1} exists, we say that f is ***invertible*** on Ω.

Inverse Functions

We have defined f as the set

$$f := \{(x, f(x)) \colon x \in \Omega\}$$

with the property (2.1.1) that $(x, y_1) = (x, y_2)$ implies $y_1 = y_2$.

If f is ***injective***, we know that $(x_1, y) = (x_2, y)$ implies $x_1 = x_2$. Switching the entries, this is the same as saying that $(y, x_1) = (y, x_2)$ implies $x_1 = x_2$. It follows that

$$f^{-1} := \{(f(x), x) \colon x \in \Omega\}$$

has the property (2.1.1). If f is ***surjective***, we can write

$$\begin{aligned} f^{-1} &= \{(f(x), x) \colon f(x) \in \widetilde{\Omega}\} \\ &= \{(y, f^{-1}(y)) \colon y \in \widetilde{\Omega}\} \end{aligned}$$

and f^{-1} is a function. Hence, if f is bijective, then f^{-1} exists.

Inverse Functions

It is not difficult to show the converse, so that a function is invertible if and only if it is bijective. We would like to find a criterion for bijectivity. If f is strictly monotonic (either increasing or decreasing) on an interval I then, in particular, $f(x) \neq f(y)$ for $x \neq y$. This should imply that f is injective and hence bijective for a suitably chosen codomain.

2.5.19. Theorem. Let $a, b \in \mathbb{R} \cup \{-\infty\} \cup \{\infty\}$ such that $a < b$. Let $f\colon (a, b) \to \mathbb{R}$ be strictly increasing and continuous. Then

$$\alpha := \lim_{x \searrow a} f(x) \geq -\infty, \qquad \beta := \lim_{x \nearrow b} f(x) \leq \infty,$$

exist and $f\colon (a, b) \to (\alpha, \beta)$ is bijective. Furthermore, the inverse function f^{-1} is also continuous and strictly increasing and, furthermore,

$$\lim_{y \searrow \alpha} f^{-1}(y) = a, \qquad \lim_{y \nearrow \beta} f^{-1}(y) = b. \tag{2.5.1}$$

Inverse Functions

Below we will give a sketch of the proof. Here, "sketch" means that some details are left to you to work out.

Sketch of Proof.

First, we need to show that α and β exist. To prove the existence of α, we need to consider whether or not f is bounded below near a. In the first case, α will be a finite number, in the second case, $\alpha = -\infty$. The details to prove the existence of the limits are left to you. A similar discussion applies to β.

Next, we show that f can not take on any value that is not in (α, β). This is done by contradiction: suppose, for instance, that $f(x_0) > \beta$ for some $x_0 < b$. Then there must exist some $x_1 > x_0$ such that $|f(x_1) - \beta| < \varepsilon := (f(x_0) - \beta)/2$ and hence $f(x_1) < f(x_0)$. But this contradicts the fact that f is increasing. Hence, $f\colon (a, b) \to (\alpha, \beta)$.

Due to the intermediate value theorem, f takes on every value in (α, β) at least once [why?], so f is surjective.

Inverse Functions

Sketch of Proof (continued).

Since f is strictly increasing it is also injective. Hence f is bijective and the inverse function $f^{-1}\colon (\alpha, \beta) \to (a, b)$ exists. We denote it by g.

For $y_1, y_2 \in (\alpha, \beta)$ we claim that

$$y_1 < y_2 \quad \Rightarrow \quad g(y_1) < g(y_2).$$

This is equivalent to showing that

$$g(y_1) \geq g(y_2) \qquad \Rightarrow \qquad y_1 \geq y_2$$

Now suppose that y_1, y_2 are given with $g(y_1) \geq g(y_2)$. Then

$$f(g(y_1)) \geq f(g(y_2))$$

because f is increasing. But since $f(g(y)) = y$, this implies that $y_1 \geq y_2$. Hence g is strictly increasing.

Inverse Functions

Sketch of Proof (continued).

We now prove that g is continuous from the left. Fix $y_0 \in (\alpha, \beta)$ and choose $x_0 \in (a, b)$ such that $f(x_0) = y_0$. Let (a_n) be an increasing sequence with $a_n \nearrow y_0$. Then the sequence $(g(a_n))$ is increasing [why?] and bounded by x_0, since

$$a_n \leq y_0 \qquad \Rightarrow \qquad g(a_n) \leq x_0$$

where we have used that g is increasing. In fact, x_0 is the least upper bound of $g(a_n)$ [why?] and so, by Theorem 2.2.24, $g(a_n) \to g(y_0)$. This proves that g is continuous from the left. In a similar manner, we can prove that g is continuous from the right.

It remains to show (2.5.1). This is left to you! $\square$

Inverse Functions

We have seen that a monotonic function on an interval is bijective; the following result gives a converse to that statement.

2.5.20. Theorem. Let $I \subset \mathbb{R}$ be an interval and $\widetilde{\Omega} \subset \mathbb{R}$ a set. If $f : I \to \widetilde{\Omega}$ is continuous and bijective, then f is strictly monotonic on I.

Proof.
Let $a_0, b_0 \in I$ with $a_0 < b_0$ be given. Then either

i) $f(b_0) - f(a_0) > 0$ or ii) $f(b_0) - f(a_0) < 0$

We will show that if i) is true, then i) holds for any other $a_1, b_1 \in I$ with $a_1 < b_1$, i.e., that f is strictly increasing. The proof for the case ii) is similar and left to the reader.

Inverse Functions

Proof (continued).

For $0 \le t \le 1$ define

$$x_t := (1-t)a_0 + ta_1, \qquad\qquad y_t := (1-t)b_0 + tb_1.$$

Then $x_0 = a_0$ and $x_1 = a_1$ and $x_t \in [a_0, a_1]$. An analogous statement is true for y_t. Thus $x_t, y_t \in \operatorname{dom} f$ for all $t \in [0, 1]$.

Since $a_0 < b_0$ and $a_1 < b_1$, we have

$$x_t < y_t \qquad\qquad \text{for } t \in [0.1].$$

We define the function $g\colon [0,1] \to \mathbb{R}$, $g(t) = f(y_t) - f(x_t)$. Since g is a composition of continuous functions, it is continuous on $[0, 1]$.

Moreover, $g(t) \neq 0$ for all $t \in [0, 1]$ since $x_t < y_t$ and f is bijective. Therefore, either $g > 0$ or $g < 0$ on $[0, 1]$. But $g(0) > 0$ by i), so $g(t) > 0$ for all $t \in [0, 1]$. This implies that i) also holds for $a_1 < b_1$. $\square$

Image and Pre-Image of Sets

We take this opportunity to define some notation for future use:

2.5.21. Definition. Let $\Omega \subset \mathbb{R}$ and $f\colon \Omega \to \mathbb{R}$. Then for any $A \subset \Omega$ and $B \subset \operatorname{ran} f$

$$f(A) := \Big\{ y \in \mathbb{R} \colon \underset{x \in A}{\exists} f(x) = y \Big\}$$

is called the ***image of A*** and

$$f^{-1}(B) := \Big\{ x \in \Omega \colon \underset{y \in B}{\exists} f(x) = y \Big\}$$

is called the ***pre-image of B***. Note that the symbol $f^{-1}(B)$ makes sense whether or not f is invertible.

2.5.22. Example. Let $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$. Then

$$f([-1, 2]) = [0, 4] \qquad \text{and} \qquad f^{-1}([1, 4]) = [-2, -1] \cup [1, 2].$$

Continuity in Intervals

Up to now we have mainly discussed continuity at single points. Recall that a function f was said to be continuous on an interval I if it is continuous at all points of the interval, i.e., f is continuous on $I \subset \operatorname{dom} f$ if and only if

$$\underset{x \in I}{\forall}\ \underset{\varepsilon > 0}{\forall}\ \underset{\delta > 0}{\exists}\ \underset{y \in I}{\forall} |x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon.$$

Note that we can exchange the first two quantifiers without changing the expression, so we obtain

$$\underset{\varepsilon > 0}{\forall}\ \underset{x \in I}{\forall}\ \underset{\delta > 0}{\exists}\ \underset{y \in I}{\forall} |x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon.$$

Here the $\delta > 0$ essentially depends on the point $x \in I$; given $\varepsilon > 0$, we would be able to choose a different $\delta > 0$ depending on x.

Continuity in Intervals

If, however, a function f is "well-behaved" enough such that we can use the same $\delta > 0$ for all $x \in I$, i.e., if in fact

$$\underset{\varepsilon>0}{\forall}\ \underset{\delta>0}{\exists}\ \underset{x\in I}{\forall}\ \underset{y\in I}{\forall} |x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon.$$

we would want to give the function a special attribute.

2.5.23. Definition. Let $I \subset \mathbb{R}$ be an interval and $f\colon \Omega \to \mathbb{R}$ a function with $I \subset \Omega$. Then f is called ***uniformly continuous on I*** if and only if

$$\underset{\varepsilon>0}{\forall}\ \underset{\delta>0}{\exists}\ \underset{x,y\in I}{\forall} |x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon.$$

Clearly, if f is uniformly continuous on I, then f is also continuous on I.

Uniform Continuity on Closed Intervals

2.5.24. Theorem. Let $f\colon \Omega \to \mathbb{R}$ a function with $I = [a, b] \subset \Omega$. If f is continuous on $[a, b]$ then f is also uniformly continuous on $[a, b]$.

Proof.
Suppose that f is not uniformly continuous on I. Then

$$\underset{\varepsilon>0}{\exists}\ \underset{\delta>0}{\forall}\ \underset{x,y\in I}{\exists} \quad |x-y|<\delta \quad \wedge \quad |f(x)-f(y)|\geq \varepsilon.$$

Denote this ε by ε_0. Then for each $\delta = 1/n$ there exist numbers $x_n, y_n \in I$ such that

$$|x_n - y_n| < \frac{1}{n} \quad \wedge \quad |f(x_n) - f(y_n)| \geq \varepsilon_0.$$

We obtain two sequences (x_n) and (y_n) that are bounded (why?). By the theorem of Bolzano-Weierstraß (x_n) has a convergent subsequences (x_{n_k}), say with limit ξ. Consider now the sequence (y_{n_k}). It is also bounded and hence has a convergent subsequence $(y_{n_{k_j}})$, say with limit η.

Uniform Continuity on Closed Intervals

Proof (continued).

Since (x_{n_k}) converges to ξ, so does the subsequence $(x_{n_{k_j}})$. It follows that we have subsequences $(x_{n_{k_j}})$ and $(y_{n_{k_j}})$, converging to ξ and η, respectively. We will now drop the subscript j, writing simply (x_{n_k}) and (y_{n_k}) for these sequences. Since I is closed, both ξ and η must lie in I.

By construction, $|x_{n_k} - y_{n_k}| < \frac{1}{n_k}$, so we see that $\xi = \eta$. However, then

$$x_{n_k} \to \xi \ \wedge \ y_{n_k} \to \xi \quad \wedge \quad |f(x_{n_k}) - f(y_{n_k})| \geq \varepsilon_0 \not\to 0.$$

and so f is not continuous at ξ. This implies that f is not continuous on I and the theorem is proved by contraposition. □

Outlook

Continuity is an important basic concept in calculus, and so it should not be surprising that there are several generalizations and further developments of continuity. As an example, consider the following result:

2.5.25. Corollary. Let $\Omega \subset \mathbb{R}$ and $f\colon \Omega \to \mathbb{R}$ continuous. Suppose that $I \subset \Omega$ is a closed interval. Then the image

$$f(I) = \Big\{ y \in \mathbb{R} \colon \underset{x \in I}{\exists} f(x) = y \Big\}$$

is also a closed interval.

Proof.
The continuity of f on I implies that there exist $\alpha := \inf_{x \in I} f(x)$ and $\beta := \sup_{x \in I} f(x)$ so that $f(I) \subset [\alpha, \beta]$. Furthermore, by Theorem 2.5.17, $\alpha, \beta \in f(I)$. Then by the Intermediate Value Theorem 2.5.13, $[\alpha, \beta] \subset f(I)$, so $f(I) = [\alpha, \beta]$. $\square$

Outlook

Now a closed interval is a special case of the following:

2.5.26. Definition. A closed and bounded set $K \subset \mathbb{R}$ is called ***compact***.

It is possible to prove the following:

2.5.27. Theorem. Let $K \subset \mathbb{R}$ be a compact set and $f\colon K \to \mathbb{R}$ be continuous. Then f is uniformly continuous on K, f is bounded and has a maximum and minimum. Furthermore, the image $f(K)$ is also compact.

This result will be proven in the next term, where it will be very useful to us in the proof of various generalizations of Theorems 2.5.17 and 2.5.24. It turns out that if we want to consider functions of several variables, the higher-dimensional analogue of a closed interval is a compact set and the basic results that we have proven here carry over to continuous functions on compact sets.

First Midterm Exam

The preceding material completes the first third of the course material. It encompasses everything that will be the subject of the First Midterm Exam.

The exam date and time will be announced on Canvas.

No calculators or other aids will be permitted during the exam. A sample exam with solutions has been uploaded to Canvas. Please study it carefully, including the instructions on the cover page.

Part III

Differential Calculus in One Variable

Finer Analysis of Functions

We have seen that even functions whose definitions appear quite straightforward can exhibit a lot of complexity in their behavior. When we first started introducing functions, our goal was to develop mathematical methods to describe and analyze this behavior. In this respect, the definition of limits and Landau symbols was a relatively modest step - we were describing the values of the function.

Of course, the behavior of a function is not just described by its numerical values. For example, the graphs of the functions $f(x) = x^2$ and $g(x) = 3x^2 + x - 4$ look quite different at $x = 1$, even though $f(1) = g(1) = 1$.

The next step in analyzing a function is therefore to look beyond the actual values to the ***change of the values***: Is the function monotonic in a neighborhood of a certain point? Do the values attain a maximum or a minimum? Or is the situation more complex than that?

Linear Approximations

In order to answer (some of) these questions, it is useful to consider the simplest non-constant type of function, the ***linear map***

$$L\colon \mathbb{R} \to \mathbb{R}, \qquad L(h) = \alpha \cdot h \qquad \text{for } \alpha \in \mathbb{R}.$$

Observing the graph, it is immediately obvious whether a linear map is increasing, strictly increasing, constant or (strictly) decreasing.

We hence aim to describe the behavior of an arbitrary function f at a point x_0 by comparing it with a linear map L. In fact, we want to be able to use a linear map as a ***linear approximation*** to f near x_0. It will turn out that if such an approximation exists it is unique and we can hence speak of a ***linearisation*** of f.

We will consider a basic example before giving the general definition.

Linear Approximation of the Square Root

Problem: Calculate $\sqrt{9.15}$ approximately.

Approach: $\sqrt{9} = 3$ is known, write $9.15 = 9 + 0.15$, where 0.15 is considered "small". Generalize: $x = 9$ and $h = 0.15$.

Goal: Find a ***linear approximation*** of the square root $\sqrt{x+h}$ near $\sqrt{x}$.

What form does such a linear approximation take? We aim for

$$\sqrt{x+h} \approx \sqrt{x} + L_x \cdot h \tag{3.1.1}$$

where $L_x \in \mathbb{R}$ is a suitable constant that in general depends on x. Equation (3.1.1) is called a ***linear*** approximation because we use $L_x \cdot h$ (a linear function of h) to approximate $\sqrt{x+h}$.

Our goal is now to find L_x, which will in general depend on x (but is independent of h).

Linear Approximations

We now keep $x > 0$ fixed throughout the discussion. Instead of the approximate identity (3.1.1) we write

$$\sqrt{x+h} = \sqrt{x} + L_x \cdot h + r(x,h) \tag{3.1.2}$$

where $r(x,h)$ is some remainder function which should become small when h approaches zero. Let us formally solve for L_x:

$$\begin{aligned}
& \sqrt{x+h} = \sqrt{x} + L_x \cdot h + r(x,h) \\
\Leftrightarrow \quad & \sqrt{x+h} - \sqrt{x} - r(x,h) = L_x \cdot h \\
\Leftrightarrow \quad & L_x = \frac{\sqrt{x+h}-\sqrt{x}}{h} - \frac{r(x,h)}{h}.
\end{aligned} \tag{3.1.3}$$

Finding L

Since L is independent of h, we can let h approach zero on the right-hand side of (3.1.3). Let us require that

$$\frac{r(x,h)}{h} \to 0 \qquad \text{as } h \to 0. \tag{3.1.4}$$

The requirement (3.1.4) means that the remainder $r(x,h) = o(h)$ as $h \to 0$, so the remainder is "much smaller" than the linear approximation.

Then

$$L_x = \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}.$$

Linear Approximation of the Square Root

Let us find L_x explicitly. For this, we use the following argument:

$$\begin{aligned}
\frac{\sqrt{x+h}-\sqrt{x}}{h} &= \frac{(\sqrt{x+h}-\sqrt{x})(\sqrt{x+h}+\sqrt{x})}{h(\sqrt{x+h}+\sqrt{x})} \\
&= \frac{x+h-x}{h(\sqrt{x+h}+\sqrt{x})} \\
&= \frac{1}{\sqrt{x+h}+\sqrt{x}}
\end{aligned}$$

Since $x > 0$, we can now easily see that

$$L_x = \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} = \frac{1}{\lim\limits_{h\to 0}(\sqrt{x+h}+\sqrt{x})} = \frac{1}{2\sqrt{x}}$$

Linear Approximations

We have shown that

$$\sqrt{x+h} = \sqrt{x} + \frac{h}{2\sqrt{x}} + o(h)$$

To answer our original question:

$$\sqrt{9.15} \approx \sqrt{9} + \frac{0.15}{2\sqrt{9}} = 3.025.$$

This is actually a good approximation. The exact (rounded) value is $\sqrt{9.15} = 3.024897$.

We can generalize this example simply by substituting an arbitrary function f for the square root. However, it turns out that not all functions can be approximated linearly at every point; this is a very nice property of a function and hence deserves an explicit definition.

Differentiability

3.1.1. Definition. Let $\Omega \subset \mathbb{R}$ be a set, $x \in \Omega$ an interior point of Ω and $f\colon \Omega \to \mathbb{R}$ a real function. Then we say that f is ***differentiable*** at x if there exists a linear map L_x such that for all sufficiently small $h \in \mathbb{R}$

$$f(x+h) = f(x) + L_x(h) + o(h) \qquad \text{as } h \to 0. \tag{3.1.5}$$

We say that f is differentiable on some open set $U \subset \Omega$ if f is differentiable at every point of U.

We say that f is differentiable if its domain is an open set and f is differentiable at every point of the domain.

3.1.2. Lemma. For given f and x the linear map L_x in (3.1.5) is unique.

Differentiability

The map $L_x\colon \mathbb{R} \to \mathbb{R}$ is a linear map, i.e., $L_x(h) = \alpha h$ for some $\alpha \in \mathbb{R}$. It quickly becomes tedious to use two different symbols to describe the same map, so we commit some slight ***abuse of notation***: we will use the symbol L_x to denote

1. the function $L_x\colon \mathbb{R} \to \mathbb{R}$ as well as
2. the number $\alpha \in \mathbb{R}$ such that $L_x(h) = \alpha \cdot h$.

This means that we write

$$L_x\colon \mathbb{R} \to \mathbb{R}, \qquad\qquad L_x(h) = L_x \cdot h.$$

This "double meaning" of L_x as a function and as a number may be confusing at first, but will later seem very natural.

Since by Lemma 3.1.2 the map L_x is unique, we may define the ***derivative of f at x***, denoted by $f'(x)$, by

$$f'(x) = L_x.$$

The Derivative

Proof of Lemma 3.1.2.

Assume that two maps L_x and M_x exist such that

$$f(x+h) = f(x) + L_x(h) + o(h) = f(x) + M_x(h) + o(h) \qquad \text{as } h \to 0.$$

Then

$$L_x(h) - M_x(h) = (L_x - M_x)h = o(h) \qquad \text{as } h \to 0. \tag{3.1.6}$$

But this mean that for all $C > 0$ there exists some $\varepsilon > 0$ such that for $|h| < \varepsilon$

$$|(L_x - M_x)h| < C|h| \qquad \text{or, equivalently,} \qquad |L_x - M_x| < C.$$

But this is possible only if $L_x = M_x$. □

Examples of Differentiable Functions

3.1.3. Examples.

(i) $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = c$ has derivative $f'(x) = 0$ for any $x \in \mathbb{R}$.

(ii) $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = x$ has derivative $f'(x) = 1$ for any $x \in \mathbb{R}$.

(iii) $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$ has derivative $f'(x) = 2x$ for any $x \in \mathbb{R}$.

(iv) Let $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = x^n$, $n \in \mathbb{N}$. By the binomial formula,

$$(x+h)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} h^k = x^n + nx^{n-1}h + o(h) \qquad \text{as } h \to 0.$$

It follows that

$$f'(x) = nx^{n-1}.$$

Examples of Differentiable Functions

3.1.4. Lemma. The derivative of the function $f\colon \mathbb{R}\setminus\{0\} \to \mathbb{R}$, $f(x) = 1/x$, is given by

$$f'(x) = -\frac{1}{x^2}.$$

Proof.
Since $(1+x)(1-x) = 1-x^2$, we have

$$\frac{1}{1+x} = 1 - x + \frac{x^2}{1+x} = 1 - x + o(x) \qquad \text{as } x \to 0.$$

Now

$$\frac{1}{x+h} = \frac{1}{x}\frac{1}{1+h/x} = \frac{1}{x}\left(1 - \frac{h}{x} + o(h)\right) = \frac{1}{x} - \frac{1}{x^2}h + o(h)$$

as $h \to 0$. This implies that

$$\left(\frac{1}{x}\right)' = -\frac{1}{x^2}$$

□

Geometric Interpretation of the Derivative

Geometrically, a linear approximation to a function can be visualised as a tangent line to the graph of the function, where the slope of the tangent is equal to the derivative. In order to obtain the tangent to a graph at a point geometrically, one considers secants through the fixed point and points on the graph progressively approaching the first.

Let $(x_1, f(x_1))$ and $(x_2, f(x_2))$ be two points on the graph of a function f. Then the secant line through these two points is given by

$$S(x; x_1, x_2) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}(x - x_1) + f(x_1).$$

If we write $x_2 = x_1 + h$, the the secant line can be written as

$$S(x; x_1, x_1 + h) = \frac{f(x_1 + h) - f(x_1)}{h}(x - x_1) + f(x_1).$$

Geometric Interpretation of the Derivative

We can use the slope, $\frac{f(x_1+h)-f(x_1)}{h}$, to obtain an equivalent definition of the derivative.

3.1.5. Theorem. Let Ω be a set, $x \in \Omega$ an interior point and $f\colon \Omega \to \mathbb{R}$ a function that is differentiable at x with derivative $L_x = f'(x)$. Then

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}. \qquad (3.1.7)$$

Furthermore, if the limit in (3.1.7) exists for an interior point $x \in \Omega$, then f is differentiable at x and the derivative is given by (3.1.7).

Geometric Interpretation of the Derivative

Proof.

Assume first that f is differentiable at x. We will show that (3.1.7) holds. Since

$$f(x+h) = f(x) + f'(x)\cdot h + o(h) \qquad\qquad \text{as } h \to 0,$$

we have

$$\begin{aligned}\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} &= \lim_{h\to 0}\frac{f(x)+f'(x)\cdot h + o(h) - f(x)}{h}\\ &= \lim_{h\to 0}\frac{f'(x)\cdot h + o(h)}{h} = f'(x) + \lim_{h\to 0}\frac{o(h)}{h}\end{aligned}$$

Since $o(h)$ denotes a function g such that $g(h)/h \to 0$ as $h \to 0$, we see that

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = f'(x).$$

Geometric Interpretation of the Derivative

Proof (continued).

Now let

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = L_x. \tag{3.1.8}$$

We will show that then $f(x+h) = f(x) + L_x \cdot h + o(h)$, proving that f is differentiable at x and that $f'(x) = L_x$. From (3.1.8) it follows that

$$\frac{f(x+h)-f(x)}{h} = L_x + r(x,h),$$

where the remainder $r(x,h) \to 0$ as $h \to 0$. We can rearrange this equation to yield

$$f(x+h) = f(x) + L_x h + hr(x,h),$$

where $hr(x,h)/h \to 0$ as $h \to 0$. But this means $hr(x,h) = o(h)$. $\square$

Physical Interpretation of the Derivative

In physics one is often interested in the dependence of one physical quantity on another, for example the dependence of position on time, or of electrical conductivity on temperature. Let us consider the first case, where the distance x travelled by a point mass depends on time t.

Suppose at time t_1 the point mass has travelled the distance $x_1 = x(t_1)$ and at time $t_2 > t_1$ it has travelled the distance $x_2 = x(t_2)$. Then the ***average speed*** in the interval $[t_1, t_2]$ is given by

$$\frac{\text{distance travelled}}{\text{time taken}} = \frac{x(t_2) - x(t_1)}{t_2 - t_1}.$$

The ***instantaneous speed v*** at any given time t is defined by the limit,

$$v(t) = \lim_{\Delta t \to 0} \frac{x(t + \Delta t) - x(t)}{\Delta t}$$

if the limit exists. This is just the derivative of the function x.

Physical Interpretation of the Derivative

Therefore, the speed is given by

$$v(t) = x'(t)$$

and we say that it is the ***rate of change*** of the distance. Similarly, the rate of change of the velocity is called the ***acceleration***,

$$a(t) = v'(t),$$

and the rate of change of the acceleration is called the ***jerk***,

$$j(t) = a'(t).$$

Hence, the acceleration is the derivative of the derivative of the position. We are thus led naturally to introduce the concept of higher derivatives.

Higher Derivatives

Consider a differentiable real function f. Then for every point $x \in \operatorname{dom} f$ we can find precisely one real number $f'(x)$. Thus we can regard the map

$$f' \colon \operatorname{dom} f \to \mathbb{R}, \qquad\qquad x \mapsto f'(x)$$

as a real function, which we call the ***derivative of f***. The function f' may itself be differentiable; in this case we use the notation

$$(f')'(x) = f''(x)$$

to denote the derivative of f' at x, and call $f''(x)$ the ***second derivative of f at x***. If $f''(x)$ exists, we say that f is ***twice differentiable at x***, and if f' is differentiable, we say that f is ***twice differentiable***.

This can of course be developed further, so that we say that f is n times differentiable at f if

$$\underbrace{(\dots((f')')'\dots)'}_{n \text{ times}}(x)$$

exists.

Higher Derivatives

We would hence have

$$x'(t) = v(t), \qquad x''(t) = a(t), \qquad x'''(t) = j(t)$$

for the velocity, acceleration and jerk, respectively.

3.1.6. Notation. For the first three derivatives, one writes f', f'' and f''', respectively. For higher derivatives, one writes

$$f^{(4)}, f^{(5)}, f^{(6)}, \ldots$$

In physics, for time-dependent quantities only, a slightly different notation going back to Newton is often used. There one denotes the first three derivatives by dots instead of primes, i.e.,

$$\dot{x}(t) = v(t), \qquad \ddot{x}(t) = a(t), \qquad \dddot{x}(t) = j(t)$$

These dots are only used for the first three derivatives, because in mechanics higher derivatives very rarely occur (for instance, Newton's equations of motion often only contain first and second derivatives).

Smoothness

Geometrically, a function f is differentiable at x if we can find a tangent to the graph of f at $(x, f(x))$, i.e., if the limit of the slopes of secants exists. It is easy to find an example of a function where this is not the case at a single point: the function

$$f(x) = |x|$$

has derivative

$$f'(x) = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases}$$

but the derivative is not defined for $x = 0$. In fact,

$$\lim_{h \searrow 0} \frac{|0 + h| - |0|}{h} = \lim_{h \searrow 0} \frac{h}{h} = 1$$

while

$$\lim_{h \nearrow 0} \frac{|0 + h| - |0|}{h} = \lim_{h \searrow 0} \frac{-h}{h} = -1$$

so $f'(0) = \lim_{h \to 0} \frac{|0+h|-|0|}{h}$ does not exist.

Smoothness

The fact that $f(x) = |x|$ is not differentiable at 0 is clearly visible by the sharp corner in the graph. As a first result we can note:

3.1.7. Remark. A continuous function is not necessarily differentiable.

However, if a function is differentiable, it is necessarily continuous:

3.1.8. Lemma. Let f be a function that is differentiable at some $x \in \operatorname{dom} f$. Then f is continuous at x.

Proof.
Since f is differentiable at x, we can write

$$f(x+h) = f(x) + f'(x)h + o(h).$$

In order to show that f is continuous at x we need to show that $\lim_{y\to x} f(y) = f(x)$.

Smoothness

Proof (continued).
We can instead write

$$\lim_{y\to x} f(y) = \lim_{h\to 0} f(x+h)$$

(why?) Now

$$\lim_{h\to 0} f(x+h) = \lim_{h\to 0} f(x) + \lim_{h\to 0} f'(x)h + \lim_{h\to 0} o(h)$$

if all the limits on the right exist. Since $\lim_{h\to 0} \frac{o(h)}{h} = 0$ we have

$$\lim_{h\to 0} o(h) = \lim_{h\to 0} h\frac{o(h)}{h} = \lim_{h\to 0} h \lim_{h\to 0} \frac{o(h)}{h} = 0.$$

Hence $\lim\limits_{h\to 0} f(x+h) = \lim\limits_{h\to 0} f(x)$ so f is continuous. □

Smoothness

This shows that the differentiable functions are a "special" or "nice" subset of the continuous functions. We often say that they are "smoother" than general continuous functions, alluding to the fact that they cannot have any sharp edges.

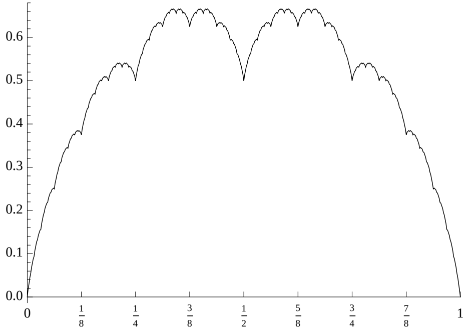
Of course this can be generalized to higher derivatives; a function that is twice differentiable will be "smoother" than a merely once differentiable function. We make one obvious remark:

3.1.9. Remark. If f is differentiable at x, f' is not necessarily continuous at x. A function whose derivative is also continuous (at x) is called ***continuously differentiable*** (at x).

We will later give an example of a function that is differentiable, but not continuously differentiable at $x = 0$.

Smoothness

We have looked at an example of a function that is not differentiable at a single point; there seems to be no reason why there should not be a function that is continuous but not differentiable at any point. An example of such a function is the self-similar ***Van der Waerden*** function, pictured below.



We will later construct a modified version of this function (see Slides 444 onwards).

Smoothness

The van der Waerden function is defined as an infinite sum of simple continuous functions. We will only later be able to study the definition of this function and prove its properties.

We hence have a certain hierarchy of smoothness at a point $x \in \mathbb{R}$:

- arbitrary functions
- functions continuous at x
- functions differentiable at x
- functions continuously differentiable at x
- functions twice differentiable at x
- ...

Differentiation

As usual, we will not always calculate derivatives by hand; rather we will calculate some derivatives of basic functions and then obtain the derivatives of more complicated functions using general theorems.

3.1.10. Lemma. Let f, g be functions on $\mathbb{R}$, $x \in \operatorname{dom} f \cap \operatorname{dom} g$ and assume that f and g are both differentiable at x. Then

$$(f+g)'(x) = f'(x) + g'(x).$$

Furthermore,

$$(\lambda f)'(x) = \lambda f'(x) \qquad\qquad \text{for } \lambda \in \mathbb{R}.$$

This result is a simple consequence of the definitions.

Differentiation

Using Lemma 3.1.10 we can now calculate the derivative of any polynomial. If $p(x) = \sum_{k=0}^{n} c_k x^k$, then

$$p'(x) = \sum_{k=1}^{n} c_k k x^{k-1}.$$

The Product Rule

We may also find the derivative of the product of two functions:

3.1.11. Lemma. Let f and g be real functions with $x \in \operatorname{dom} f \cap \operatorname{dom} g$ such that f and g are differeniable at x. Then

$$(f \cdot g)'(x) = f'(x)g(x) + g'(x)f(x).$$

Proof.
We just write out the definitions and use what we know about Landau symbols:

$$\begin{aligned}
(f \cdot g)(x+h) &= f(x+h)g(x+h) \\
&= \big(f(x) + f'(x)h + o(h)\big)\big(g(x) + g'(x)h + o(h)\big) \\
&= f(x)g(x) + \big(f'(x)g(x) + g'(x)f(x)\big)h + f'(x)g'(x)h^2 \\
&\qquad + o(h)[f'(x)h + g'(x)h + o(h)] \\
&= f(x)g(x) + \big(f'(x)g(x) + g'(x)f(x)\big)h + o(h)
\end{aligned}$$

□

The Chain Rule

In order to calculate the derivatives of more complicated functions, we need to find a formula for the derivative of the composition of functions. This result is commonly called the ***chain rule***.

3.1.12. Theorem. Let f, g be real functions and $x \in \mathbb{R}$ such that $x \in \operatorname{dom} g$, $g(x) \in \operatorname{dom} f$ are interior points of the domains of g and f, respectively. Assume further that g is differentiable at x and f is differentiable at $g(x)$. Then

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x).$$

The Chain Rule

Proof.

By our assumed differentiability of g at x we have

$$(f \circ g)(x+h) = f(g(x+h)) = f(g(x) + g'(x)h + o(h))$$

as $h \to 0$. Writing $H = g'(x)h + o(h)$ and using the differentiability of f at $g(x)$, we have

$$(f \circ g)(x+h) = f(g(x) + H) = f(g(x)) + f'(g(x))H + o(H).$$

Now

$$H = g'(x)h + o(h) = O(h) + O(h) = O(h)$$

as $h \to 0$, so $o(H) = o(O(h)) = o(h)$ as $h \to 0$ (as proven the assignments).

The Chain Rule

Proof (continued).

We hence have

$$\begin{aligned}(f \circ g)(x+h) &= f(g(x)) + f'(g(x))H + o(h) \\ &= f(g(x)) + f'(g(x))(g'(x)h + o(h)) + o(h) \\ &= f(g(x)) + f'(g(x))g'(x)h + o(h)\end{aligned}$$

as $h \to 0$, so $(f \circ g)(x) = f'(g(x))g'(x)$. □

Using the chain rule, we can now calculate the derivative of many more functions, such as $f(x) = x^{-n}$, $n \in \mathbb{N}$ or general quotients:

3.1.13. Lemma. Let f and g be real functions with $x \in \operatorname{dom} f \cap \operatorname{dom} g$ such that f and g are differentiable at x and $g(x) \neq 0$. Then

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - g'(x)f(x)}{g(x)^2}.$$

The Quotient Rule

Proof.

Let $h(x) = 1/x$. Then $\dfrac{f}{g} = f \cdot (h \circ g)$. Using the product rule,

$$\left(\frac{f}{g}\right)'(x) = f'(x) \cdot (h \circ g)(x) + f(x)(h \circ g)'(x)$$

Now by Lemma 3.1.4 we have $h'(x) = -1/x^2$, and by the chain rule we know that

$$(h \circ g)'(x) = h'(g(x))g'(x),$$

so

$$\begin{aligned}\left(\frac{f}{g}\right)'(x) &= f'(x)\frac{1}{g(x)} + f(x)\left(-\frac{1}{g(x)^2}\right)g'(x) \\ &= \frac{f'(x)g(x) - g'(x)f(x)}{g(x)^2}\end{aligned}$$

$\square$

The Inverse Function Theorem

A simple consequence of the quotient rule is that $(x^{-n})' = (-n)x^{-n-1}$ for $n \in \mathbb{N}$, $n \geq 1$. In fact, we can now differentiate all rational functions (quotients of polynomials). However, there are still some functions we can not yet differentiate, such as the square root function $f(x) = \sqrt{x}$.

Since the square root function is defined as the inverse of $g(x) = x^2$ for $x > 0$, it is perhaps more fruitful to consider derivatives of inverse functions in general, rather than trying to obtain the derivative of every function with a rational exponent separately.

3.1.14. Theorem. Let I be an open interval and let $f \colon I \to \mathbb{R}$ be differentiable and strictly monotonic. Then the inverse map $f^{-1} = g \colon f(I) \to I$ exists and is differentiable at all points $y \in f(I)$ for which $f'(g(y)) \neq 0$. Furthermore,

$$g'(y) = \frac{1}{f'(g(y))}.$$

The Inverse Function Theorem

Proof.

Let $y = f(x)$ and $y + H = f(x+h)$. Then

$$\begin{aligned} y + H = f(x+h) &= f(x) + f'(x)h + o(h) \\ &= y + f'(g(y))h + o(h) \\ &= y + (f'(g(y)) + o(1))h \end{aligned}$$

as $h \to 0$. Assuming that $f'(g(y)) \neq 0$, this gives

$$h = \frac{1}{f'(g(y)) + o(1)} H \qquad \text{as } h \to 0.$$

Note that $h = g(y+H) - g(y)$ and g is continuous, so $h \to 0$ if $H \to 0$. By the continuity of g we then have

$$h = \frac{1}{f'(g(y)) + o(1)} H = O(H) \qquad \text{as } H \to 0.$$

The Inverse Function Theorem

Proof (continued).

This allows us to use

$$y + H = f(x+h) = f(x) + f'(x)h + o(h) = y + f'(g(y))h + o(h)$$

as $h \to 0$ to deduce

$$\begin{aligned} h &= \frac{1}{f'(g(y))}H + o(h) && \text{as } h \to 0, \\ &= \frac{1}{f'(g(y))}H + o(O(H)) && \text{as } H \to 0, \\ &= \frac{1}{f'(g(y))}H + o(H) && \text{as } H \to 0. \end{aligned}$$

Then, as $H \to 0$,

$$g(y+H) = x + h = g(y) + \frac{1}{f'(g(y))}H + o(H)$$

□

The Inverse Function Theorem

The main difficulty in the proof of the previous theorem was showing that f^{-1} was differentiable; if we know this, then the derivative is computed from the chain rule, because

$$y = f(g(y)),$$

so

$$1 = y' = f'(g(y))g'(y)$$

and hence $g'(y) = 1/f'(g(y))$.

We are now able to differentiate all algebraic functions and their inverses. For example,

$$(\sqrt{x})' = \frac{1}{2\sqrt{x}}.$$

In general, we can show that $(x^p)' = px^{p-1}$ for all rational numbers p.

Leibniz Notation

Next to the notation f' for the derivative of f there is also a notation going back to Leibniz, which writes

$$f' = \frac{dy}{dx}.$$

This is of course inspired by taking the derivative as a "differential quotient"

$$f'(x_0) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

where $\Delta x = x - x_0$ and $\Delta y = f(x) - f(x_0)$. Leibniz notation has proven to be extremely versatile and flexible, and has become extremely popular in mathematics.

Leibniz Notation

There are some fine points one should be aware of:

1. Leibniz notation does not differentiate between $f'(x)$ and f'; both can be written as $\dfrac{df}{dx}$.
2. Leibniz himself regarded df and dx as independent infinitesimal elements, called ***differentials*** and $\dfrac{df}{dx}$ as the ***differential quotient***. For the moment, we are not going to treat these individually, but only in the form of a fraction.

There is a way around the first problem: we can use the notation $|_{x_0}$, read "at x_0," to denote a function evaluated at a specific point. For example,

$$f(x_0) = f|_{x_0} = f(x)|_{x=x_0}$$

are all notations for the same number. Hence

$$f'(x_0) = \frac{df}{dx}\bigg|_{x_0}$$

can be used to indicate the derivative evaluated at a specific point.

Leibniz Notation

While it at first might look like this ambiguity of df/dx (does it indicate a function or the value of a function?) is a disadvantage, it turns out that precisely this double meaning can be very useful to write out formulas efficiently. Furthermore, in advanced mathematics such a double meaning is often built into the statement, and a notation that refects this is very useful.

In Leibniz notation, the chain rule for $f \circ g$ becomes

$$\frac{df}{dx} = \frac{df}{dg} \cdot \frac{dg}{dx},$$

and the inverse function theorem can be written

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}.$$

While these formulae are useful mnemonics, they clearly need to be interpreted correctly!

Leibniz Notation

Another advantage of Leibniz notation is that it becomes natural to write

$$\frac{df}{dx} = \frac{d}{dx} f,$$

where $\dfrac{d}{dx}$ is a so-called differential operator that takes a function and returns its derivative. For example,

$$\frac{d}{dx}(4x^3) = 12x^2.$$

For higher derivatives, we write

$$f'' = \frac{d^2 f}{dx^2}, \qquad f''' = \frac{d^3 f}{dx^3}, \qquad f^{(n)} = \frac{d^n f}{dx^n}.$$

This allows us to write

$$\frac{d}{dx}\left(\frac{d}{dx} x^5\right) = \frac{d^2}{dx^2} x^5 = 20x^3.$$

Extrema of Real Functions

Now that we know how to differentiate and have established the basic notation, we can at last treat our basic objective: the analysis of the behavior of functions. We begin by looking at extrema, i.e., maxima and minima of functions.

3.2.1. Definition. Let f be a function and $\Omega \subset \operatorname{dom} f$. Then $x \in \Omega$ is called a (global) ***maximum point for f on Ω*** if

$$f(x) \geq f(y) \qquad\qquad \text{for all } y \in \Omega.$$

The number $f(x)$ is called the ***maximum value*** of f on Ω.

A minimum point/minimum value is defined similarly.

We will use derivatives to characterize maximum and minimum points.

Extrema of Real Functions

3.2.2. Theorem. Let f be a function and $(a, b) \subset \operatorname{dom} f$ an open interval. If $x \in (a, b)$ is a maximum (or minimum) point for f on (a, b) and if f is differentiable at x, then $f'(x) = 0$.

Proof.
We will consider the case of a maximum point only. Clearly, the geometric interpretation of the derivative is well-suited to this theorem: we want to show that the tangent at the maximum point is horizontal, i.e., has slope zero.

Assume that f has a maximum at x. For sufficiently small h, $x + h \in (a, b)$, so

$$f(x + h) - f(x) \le 0.$$

Extrema of Real Functions

Proof (continued).

If $h > 0$, then

$$\frac{f(x+h) - f(x)}{h} \le 0$$

and hence

$$\lim_{h \searrow 0} \frac{f(x+h) - f(x)}{h} \le 0.$$

Similarly,

$$\lim_{h \nearrow 0} \frac{f(x+h) - f(x)}{h} \ge 0.$$

But since f is differentiable at x, these limits must be equal, so they must both be zero. $\square$

Extrema of Real Functions

3.2.3. Definition. Let f be a real function and $\Omega \subset \operatorname{dom} f$. Then $x \in \Omega$ is called a ***local maximum (minimum) point for f on Ω*** if there exists some $\varepsilon > 0$ such that x is a maximum (minimum) point for f on $\Omega \cap B_\varepsilon(x)$.

3.2.4. Remark. If f is defined on an open interval (a, b) and has a local maximum (minimum) at $x \in (a, b)$, and if f is differentiable at x, then $f'(x) = 0$.

3.2.5. Definition. A function f is said to have a ***critical point*** at $x \in \operatorname{dom} f$ if $f'(x) = 0$. The value $f(x)$ is then called a ***critical value*** of f.

Therefore, when finding the (local and global) maxima and minima of a function on a set $\Omega \subset \operatorname{dom} f$, we need to check

1. the critical points of f,
2. the boundary points of Ω,
3. interior points of Ω where f is not differentiable.

Rolle's Theorem

A basic result which will allow us to develop our analysis further is the following, named after the French mathematician Rolle.

3.2.6. Rolle's Theorem. Let f be a real function and $a < b \in \mathbb{R}$ such that $[a, b] \in \operatorname{dom} f$. Assume that f is continuous on $[a, b]$ and differentiable on (a, b) and that $f(a) = f(b)$. Then there exists a number $x \in (a, b)$ such that $f'(x) = 0$.

Proof.

Since f is continuous, it will have a maximum and a minimum value on $[a, b]$. If either is in (a, b), then f' vanishes there by Theorem 3.2.2, so we are finished. Assume that both the maximum and the minimum are at a or b. Since $f(a) = f(b)$, it follows that f is constant, so $f'(x) = 0$ for any x in (a, b). $\square$

The Mean Value Theorem

Rolle's theorem may not look spectacular, but it already contains the proof of the powerful mean value theorem:

3.2.7. Mean Value Theorem. Let f be a real function and $a < b \in \mathbb{R}$ such that $[a, b] \in \operatorname{dom} f$. Assume that f is continuous on $[a, b]$ and differentiable on (a, b). Then there exists a number $x \in (a, b)$ such that

$$f'(x) = \frac{f(b) - f(a)}{b - a}.$$

Proof.
Let

$$h(x) = f(x) - \frac{f(b) - f(a)}{b - a}(x - a).$$

Then $h(a) = h(b) = f(a)$ and by Rolle's theorem there exists some $x \in (a, b)$ such that $h'(x) = 0$.

Since $h'(x) = f'(x) - \dfrac{f(b) - f(a)}{b - a}$, this proves the theorem. $\square$

The Mean Value Theorem

We can now prove a seemingly "obvious" statement, which, however, requires the powerful mean value theorem.

3.2.8. Corollary. Let f be a real function and $I \subset \operatorname{dom} f$. Assume that $f' = 0$ on I. Then f is constant on I.

Proof.
Let $a, b \in I$, $a < b$. Then there exist some $x \in (a, b)$ such that

$$f'(x) = \frac{f(b) - f(a)}{b - a}.$$

But since $f' = 0$ on I, it follows that $f(a) = f(b)$. □

3.2.9. Corollary. Let f, g be real functions and $I \subset \operatorname{dom} f \cap \operatorname{dom} g$. Assume that $f' = g'$ on I. Then there exists some $c \in \mathbb{R}$ such that $f = g + c$.

The Mean Value Theorem

The following corollary is of great importance for analyzing the behavior of functions:

3.2.10. Corollary. Let f be a real function and $I \subset \operatorname{dom} f$. Assume that $f' > 0$ on I. Then f is strictly increasing on I. If $f' < 0$ on I, f is strictly decreasing on I.

Proof.
Consider the case where $f' > 0$. Let $a, b \in I$, $a < b$. Then there exist some $x \in (a, b)$ such that

$$f'(x) = \frac{f(b) - f(a)}{b - a} > 0.$$

Since $b > a$ this implies $f(b) > f(a)$. $\square$

Maxima and Minima

3.2.11. Theorem. Let f be a real function and $x \in \operatorname{dom} f$ such that $f'(x) = 0$. If $f''(x) > 0$, then f has a local minimum at x; if $f''(x) < 0$, then f has a local maximum at x.

Proof.
We know that

$$f'(x+h) = f'(x) + f''(x)h + o(h) = \big(f''(x) + o(1)\big)h \qquad \text{as } h \to 0$$

Assume that $f''(x) > 0$. Then for sufficiently small h,

$$\begin{aligned} f'(x+h) &> 0 && \text{if } h > 0, \\ f'(x+h) &< 0 && \text{if } h < 0. \end{aligned}$$

This means that f is strictly increasing to the right of x and strictly decreasing to the left of x. Thus f has a minimum at x. □

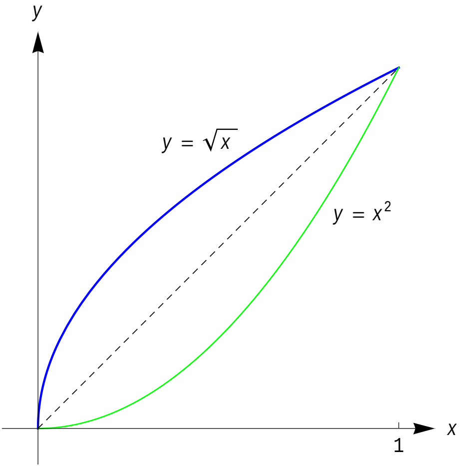
Maxima and Minima

3.2.12. Theorem. Let f be a real function and $x \in \operatorname{dom} f$ such that $f''(x)$ exists. If f has a local minimum at x, then $f''(x) \geq 0$; if f has a local maximum at x, then $f''(x) \leq 0$.

Proof.
Suppose f has a minimum at x. If $f''(x) < 0$, then f also has a maximum at x. But this means that f is constant, therefore $f''(x) = 0$. ↯ $\square$

Convexity and Concavity

We have seen how to find the extrema of real functions and also know how to determine intervals of monotonicity. However, an increasing function may exhibit one of two fundamentally different shapes:



The graph of $f(x) = x^2$ stays below the secant joining the origin with the point $(1, 1)$, while the graph of $g(x) = \sqrt{x}$ stays above this line.

In fact, if we select any two points on the graph of f and join them with a line, the graph of f will remain below these points. A corresponding statement is true for g.

Convexity and Concavity

These observations motivate the following definition:

3.2.13. Definition. Let $\Omega \subset \mathbb{R}$ be any set and $I \subset \Omega$ an interval. A function $f : \Omega \to \mathbb{R}$ is called ***strictly convex*** on I if for all $a, x, b \in I$ with $a < x < b$,

$$\frac{f(x) - f(a)}{x - a} < \frac{f(b) - f(a)}{b - a}.$$

We say that f is ***strictly concave*** on I if $-f$ is strictly convex. If we replace "$<$" by "$\leq$" above, f is simply called convex and $-f$ concave.

3.2.14. Theorem. Let $f : I \to \mathbb{R}$ be be strictly convex on I and differentiable at $a, b \in I$. Then

(i) For any $h > 0$ ($h < 0$) such that $a + h \in I$, the graph of f over the interval $(a, a + h)$ lies below the secant line through the points $(a, f(a))$, $(a + h, f(a + h))$.

(ii) The graph of f over all of I lies above the tangent line through the point $(a, f(a))$.

(iii) If $a < b$, then $f'(a) < f'(b)$.

Convexity and Concavity

Proof.

(i) Let $a < x < a + h$, $h > 0$. The secant line through the points $(a, f(a))$ and $(a + h, f(a + h))$ is given by

$$\psi_{a,a+h} \colon \mathbb{R} \to \mathbb{R}, \qquad \psi_{a,a+h}(t) = f(a) + m_{a,a+h}(t - a),$$

while the line through the points $(a, f(a))$ and $(x, f(x))$ is

$$\psi_{a,x} \colon \mathbb{R} \to \mathbb{R}, \qquad \psi_{a,x}(t) = f(a) + m_{a,x}(t - a)$$

with

$$m_{a,a+h} = \frac{f(a + h) - f(a)}{h}, \qquad m_{a,x} = \frac{f(x) - f(a)}{x - a}.$$

Since f is strictly convex, $m_{a,x} < m_{a,a+h}$. This implies

$$f(x) = \psi_{a,x}(x) < \psi_{a,a+h}(x).$$

Convexity and Concavity

Proof.

(ii) We show that the graph of f to the right of a lies above the tangent line through a. A similar argument can be used for the graph of f to the left of a. Let $0 < h_1 < h_2$. Then, by the convexity of f,

$$\frac{f(a+h_1)-f(a)}{h_1} < \frac{f(a+h_2)-f(a)}{h_2}$$

It follows that the function

$$g(h) = \frac{f(a+h)-f(a)}{h}$$

is strictly decreasing as $h \searrow 0$. Since $\lim\limits_{h\to 0} g(h) = f'(a)$ we see that

$$f'(a) < \frac{f(a+h)-f(a)}{h} =: m_{a,a+h} \qquad \text{for } h > 0. \tag{3.2.1}$$

Convexity and Concavity

Proof (continued).

Consider the secant line through the points $(a, f(a))$ and $(a+h, f(a+h))$, $h > 0$, given by

$$\psi_{a,a+h} \colon \mathbb{R} \to \mathbb{R}, \qquad \psi_{a,a+h}(t) = f(a) + m_{a,a+h}(t-a).$$

The tangent line through the point $(a, f(a))$ is given by

$$T_a \colon [a, a+h] \to \mathbb{R}, \qquad T_a(t) = f(a) + f'(a)(t-a).$$

Since $f'(a) < m_{a,a+h}$ by (3.2.1) it follows that $T_a(x) < \psi_{a,h}(x)$ for $a < x \le a+h$. In particular,

$$f(a+h) = \psi_{a,a+h}(a+h) > T_a(a+h).$$

Since this is true for any $h > 0$, the graph of f to the right of a lies above the tangent line through a.

Convexity and Concavity

Proof (continued).

(iii) By (ii), we know that the graph of the function stays above the tangent at any point. In particular, $f(b)$ is above the tangent line through $(a, f(a))$, so

$$f'(a) < m_{a,b} = \frac{f(b) - f(a)}{b - a}.$$

Similarly, considering the tangent line through $(b, f(b))$ we have that $f(a)$ is above this line, so

$$f'(b) > m_{b,a} = \frac{f(a) - f(b)}{a - b} = \frac{f(b) - f(a)}{b - a}.$$

This implies $f'(a) < f'(b)$. □

Convexity and Concavity

3.2.15. Lemma. Let I be an open interval, $f\colon I \to \mathbb{R}$ differentiable and f' strictly increasing. If $a, b \in I$, $a < b$, and $f(a) = f(b)$, then $f(x) < f(a) = f(b)$ for all $x \in (a, b)$.

Proof.
We first show that $f(x) \le f(a)$ for all $x \in (a, b)$. Suppose that there exists some $x \in (a, b)$ with $f(x) > f(a)$. Then there exists some point $x_0 \in (a, b)$ such that $f(x_0)$ is a maximum of f for $[a, b]$. Consequently, by Theorem 3.2.2, $f'(x_0) = 0$. However, the Mean Value Theorem 3.2.7 for the interval $[a, x_0]$ shows the existence of some $x_1 \in (a, x_0)$ such that

$$f'(x_1) = \frac{f(x_0) - f(a)}{x_0 - a} > 0.$$

Since $x_1 < x_0$ but $f'(x_1) > f'(x_0)$, we obtain a contradiction to the assumption that f' is strictly increasing.

Convexity and Concavity

Proof (continued).
Now suppose there exists some $x_0 \in (a, b)$ with $f(x_0) = f(a)$. Since $f(x) \le f(a)$ for all $x \in [a, b]$, x_0 is a local maximum point for f and so $f'(x_0) = 0$.

Since f' is strictly increasing, f can not be constant on $[a, x_0]$. Thus, there exists an $x_1 \in (a, x_0)$ with $f(x_1) < f(x_0)$. Applying the Mean Value Theorem 3.2.7 to the interval $[x_1, x_0]$ we find an $x_2 \in (x_1, x_0)$ such that

$$f'(x_2) = \frac{f(x_0) - f(x_1)}{x_0 - x_1} > 0.$$

This again contradicts the assumption that f' is increasing on I. $\square$

The most important result in applications is the following:

3.2.16. Theorem. Let I be an open interval, $f : I \to \mathbb{R}$ differentiable and f' strictly increasing. Then f is strictly convex.

Convexity and Concavity

Proof.

Let $a, b \in I$ with $a < b$. Set $g\colon I \to \mathbb{R}$,

$$g(x) = f(x) - \frac{f(b) - f(a)}{b - a}(x - a).$$

Then g' is strictly increasing and $g(a) = g(b) = f(a)$. By Lemma 3.2.15,

$$g(x) < f(a) \qquad\qquad \text{for } x \in (a, b).$$

This implies

$$f(x) - \frac{f(b) - f(a)}{b - a}(x - a) < f(a)$$

or

$$\frac{f(x) - f(a)}{x - a} < \frac{f(b) - f(a)}{b - a}$$

for $x \in (a, b)$. This shows that f is convex. $\square$

Curve Sketching

We now have all the tools we need to sketch the graph of a given function. In orer to perform a sketch, we first need to perform a ***curve discussion***, where we explore the behavior of the curve at all points of its domain. In particular, we are interested in

1. the domain and range;
2. the continuity and behavior near points of discontinuity;
3. the behavior as $x \to \pm\infty$; in particular ***asymptotes*** (straight lines such that the values of the function approach the points on the line as $x \to \pm\infty$).;
4. local and global extrema;
5. intervals where the function is increasing, decreasing or constant;
6. ***inflection points***, where the second derivative changes sign;
7. other remarkable features of the curve.

Curve Sketching

The object of a sketch of a curve is to visualize the curve for the reader, and to convey all relevant information regarding the curve. Knowing what to include in a sketch is just as important as knowing what ***not*** to include.

First, here are some general guidelines:

- Decide what area of the coordinate system you will need (for the graph of a function, consider its domain and range). In most graphs, you should include the origin $(0, 0)$ of the coordinate system.
- Make sure your sketch is large enough; generally, it should be as wide as the page on which you are writing, and at least 8 cm high!
- Use a ruler to draw the axes. If the sketch includes the origin of the coordinate system, the axes should intersect there.
- Add an arrowhead pointing right to the right end of the horizontal axis (called the abscissa), and an arrowhead pointing upwards to the top end of the vertical axis (called the ordinate).

Curve Sketching

- Label the axes (generally, "x" for the abscissa and "y" for the ordinate, but you can use any other suitable variables), writing the labels under the horizontal arrowhead and on the left of the vertical arrowhead.
- You do not need to label the origin $(0, 0)$. It is understood that the axes intersect at $(0, 0)$. Only if in your sketch the axes do ***not*** intersect at the origin should you label the intersection.
- Sketch the curve so that all the space provided by the axes is used; for a graph of a function, the maximum should be near the top of the ordinate and the minimum should be near the bottom of the ordinate. The domain of the function should be clearly visible in the sketch. If the graph continues for $x \to \pm\infty$, then draw the curve until you are just above the ends of the abscissa - do not draw a curve further than the horizontal or vertical axes, and do not stop drawing before the ends are reached.

Curve Sketching

- Label the curve, e.g., "$y = f(x)$", "f". You may use little arrows to indicate the curve.
- Mark characteristic points of the curve: intersections with the axes, extrema, inflection points, other interesting points. Generally, it is sufficient to mark these points on the axes; only in rare cases should you resort to labeling a point (x, y) directly on the graph.
- If you do not have exact values for some characteristic points, indicate them through suitable symbols (e.g., $x_1, x_2, \ldots$) and explain these symbols briefly at the bottom of the graph.

Curve Sketching

- Indicate asymptotes through dashed lines, and label horizontal and vertical asymptotes on the axes of the graph. A slant asymptote should be labeled as "$y = mx + b$" (perhaps using an arrow) in the same way the graph of the function is labeled. If a horizontal asymptote lies on the abscissa ($y = 0$), or a vertical asymptote lies on the ordinate ($x = 0$), then you should not try to draw the asymptote, but rather indicate through words and an arrow that there is an asymptote on these lines.
- Title your sketch (e.g., "Graph of the function f").

Curve Sketching

Things you should generally ***not*** do:

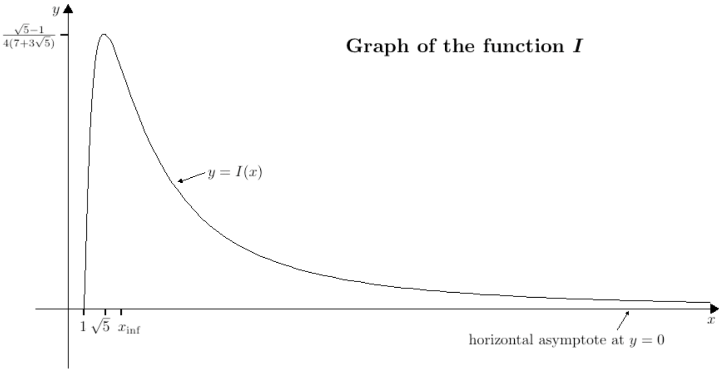
- Mark points on the axes that do not correspond to characteristic points of the graph; there is really no need to give a "scale" on the axes.
- Draw lines inside the graph; only use dashed lines for asymptotes. If you feel that you need to connect points on the axes with points on the graph, then your sketch is not clear enough!

All of the above points are guidelines; sometimes it may be necessary to break them and to do things differently. However, if you do so, be sure you have a ***very*** good reason!

Curve Sketching

As an example, let us sketch the function

$$I\colon [1,\infty) \to \mathbb{R}, \qquad I(x) = \frac{x-1}{(1+x)^2(3+x)}.$$



x_{inf} denotes the x-coordinate of the inflection point

The Cauchy Mean Value Theorem

3.2.17. Theorem. Let f, g be real functions and $[a, b] \subset \operatorname{dom} f \cap \operatorname{dom} g$. If f and g are continuous on $[a, b]$ and differentiable on (a, b), then there exists an $x \in (a, b)$ such that

$$(f(b) - f(a))g'(x) = (g(b) - g(a))f'(x).$$

Proof.
We apply Rolle's Theorem to

$$h(x) = f(x)(g(b) - g(a)) - g(x)(f(b) - f(a)).$$

□

L'Hôpital's Rule

3.2.18. Theorem. Let f and g be real functions and $b \in \overline{\operatorname{dom} f \cap \operatorname{dom} g}$ with

$$\lim_{x \searrow b} f(x) = 0 \qquad \text{and} \qquad \lim_{x \searrow b} g(x) = 0.$$

Suppose that there exists a $\delta > 0$ such that f and g are defined and differentiable on the interval $(b, b+\delta)$ and $g'(x) \neq 0$ for all $x \in (b, b+\delta)$. Suppose further that the limit $\lim_{x \searrow b} f'(x)/g'(x) =: L$ exists. Then

$$\lim_{x \searrow b} \frac{f(x)}{g(x)} = L.$$

For the following proof, we will change the definition of f and g to ensure that $f(b) = g(b) = 0$. This ensures that f and g are continuous at b, but does not change any of the statements of the theorem.

L'Hôpital's Rule

Proof.

Let f and g be given as in the theorem with $f(b) = g(b) = 0$. Furthermore, choose $\delta > 0$ small enough. Then for all $x \in (b, b + \delta)$ we have $g(x) \neq 0$ by the mean value theorem. (Otherwise, $g' = 0$ somewhere in (b, x), which is not allowed.)

Fix $\varepsilon > 0$. Choose $\delta > 0$ (perhaps decreasing the choice of δ above) such that

$$\left| \frac{f'(\xi)}{g'(\xi)} - L \right| < \varepsilon \qquad \text{for all } \xi \in (b, b + \delta).$$

Choose $x \in (b, b + \delta)$. Applying the Cauchy mean value theorem, there is a number α_x in (b, x) such that

$$\frac{f(x)}{g(x)} = \frac{f'(\alpha_x)}{g'(\alpha_x)}.$$

L'Hôpital's Rule

Proof (continued).
But then

$$\left|\frac{f(x)}{g(x)} - L\right| < \varepsilon \qquad\qquad \text{for all } x \in (b, b+\delta).$$

Thus $f(x)/g(x) \to L$ as $x \searrow b$. $\square$

L'Hôpital's Rule

An analogous version of l'Hopital's rule applies when considering limits at infinity:

3.2.19. Theorem. Let f and g be real functions such that for some $C > 0$ the interval $(C, \infty) \subset \operatorname{dom} f \cap \operatorname{dom} g$ and

$$\lim_{x\to\infty} f(x) = 0 \qquad \text{and} \qquad \lim_{x\to\infty} g(x) = 0.$$

Suppose further that f and g are defined and differentiable on the interval (C, ∞) and $g'(x) \neq 0$ for all $x \in (C, \infty)$. Suppose further that the limit $\lim\limits_{x\to\infty} f'(x)/g'(x) =: L$ exists. Then

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L.$$

L'Hôpital's Rule

Proof.

Fix $\varepsilon > 0$. By assumption, we can find (and then fix) a $C > 0$ such that for all $y > C$

$$\left|\frac{f'(x)}{g'(x)} - L\right| < \varepsilon \qquad \text{for } x > y.$$

This implies, in particular, that $g'(x) \neq 0$ for $x > C$. Choose any $x > y$. Then $g(x) \neq g(y)$, for otherwise, by Rolle's theorem, we would have $g'(z) = 0$ for some $z \in (y, x)$. Furthermore, for any $y > C$ and $x > y$ we can apply the Cauchy mean value theorem to the interval $[y, x]$ to find some $z \in (y, x)$ such that

$$\frac{f(y) - f(x)}{g(y) - g(x)} = \frac{f'(z)}{g'(z)}.$$

L'Hôpital's Rule

Proof (continued).

We deduce that for all $y > C$

$$\left|\frac{f(y)-f(x)}{g(y)-g(x)} - L\right| < \varepsilon \qquad \text{for } x > y.$$

The right-hand side does not depend on x, so we can let $x \to \infty$ and see that

$$\left|\frac{f(y)}{g(y)} - L\right| \le \varepsilon \qquad \text{for } y > C.$$

This proves that $\lim\limits_{x\to\infty} \frac{f(x)}{g(x)} = L$. □

Further versions of l'Hôspital's theorem will be proved in the recitation classes.

Vector Spaces

We have now developed many tools to analyze the functions that we know, which essentially comprise

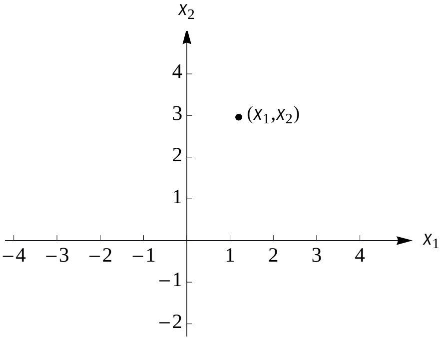
1. algebraic functions (quotients of polynomials),
2. their inverse functions (root functions) and
3. piece-wisely defined functions (e.g., $f(x) = |x|$).

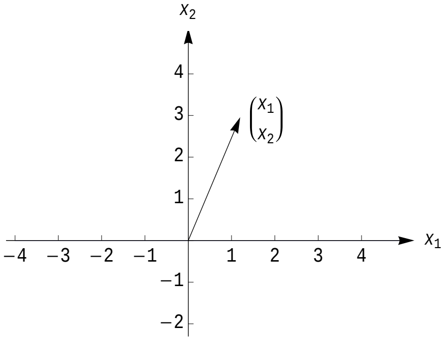
Of course there are many more functions out there! However, in order to find and properly define more functions, we need to mathematically organize the "set of all functions" in such a way that we can prove theorems on classes of functions and are able, for example, to consider sequences of functions.

We have previously added the concept of ***distance*** to a set by defining a metric space (M, ϱ) consisting of a set M and a map $\varrho\colon M \times M \to \mathbb{R}$. However, this does not give us enough structure for our purposes, as we also need to be able to at least add elements of M. Thus, we will introduce a more suitable setting for our functions.

Vector Spaces

Our model for the concept of a ***vector space*** is based on $\mathbb{R}^2$. From the point of view of mathematics, $\mathbb{R}^2$ is simply the set of pairs (x_1, x_2) where $x_1, x_2 \in \mathbb{R}$. Every pair can also be interpreted as a point in the plane, which we also call $\mathbb{R}^2$. However, from a physical point of view we can identify (x_1, x_2) with an arrow joining the origin to (x_1, x_2).





Vector Spaces

The classical (physicist's) notation for a vector in $\mathbb{R}^2$ is

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

We will keep the idea of writing $\binom{x_1}{x_2}$ instead of (x_1, x_2) when it is useful to do so, but we will drop the arrow in $\vec{x}$ and simply write x. It will always be clear from context whether $x \in \mathbb{R}$ or $x \in \mathbb{R}^2$.

Vector Spaces

In physics, vectors are used to indicate ***directional quantities*** such as velocity or forces. In this context, ***position*** is also assumed to be directional. As is well-known (from school-level physics), vectors can be added and one can consider multiples of vectors, as follows:

$$x + y = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \end{pmatrix}, \qquad \lambda \cdot x = \lambda \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \lambda x_1 \\ \lambda x_2 \end{pmatrix},$$

where $x, y \in \mathbb{R}^2$ and $\lambda \in \mathbb{R}$.

However, there is no straightforward way to multiply two vectors in $\mathbb{R}^2$ and obtain a new vector in $\mathbb{R}^2$. And for physical as well as mathematical applications, that is often not necessary!

Implementing the ideas of the previous discussion, we want to define a vector space as a set of elements ("vectors") that can be added together or multiplied by a (real or complex) number to obtain new elements of the set.

Vector Spaces

3.3.1. Definition. A triple $(V, +, \cdot)$ is called a ***real vector space*** (or ***real linear space***) if

1. V is any set;
2. $+ \colon V \times V \to V$ is a map (called addition) with the following properties:
   - $(u + v) + w = u + (v + w)$ for all $u, v, w \in V$ (***associativity***),
   - $u + v = v + u$ for all $u, v \in V$ (***commutativity***),
   - there exists an element $e \in V$ such that $v + e = v$ for all $v \in V$ (***existence of a neutral element***),
   - for every $v \in V$ there exists an element $-v \in V$ such that $v + (-v) = e$;
3. $\cdot \colon \mathbb{R} \times V \to V$ is a map (called scalar multiplication) with the following properties:
   - $1 \cdot u = u$ for all $u \in V$,
   - $\lambda \cdot (u + v) = \lambda \cdot u + \lambda \cdot v$ for all $\lambda \in \mathbb{R}$, $u, v \in V$,
   - $(\lambda + \mu) \cdot u = \lambda \cdot u + \mu \cdot u$ for all $\lambda, \mu \in \mathbb{R}$, $u \in V$,
   - $(\lambda\mu) \cdot u = \lambda \cdot (\mu \cdot u)$ for all $\lambda, \mu \in \mathbb{R}$, $u \in V$.

Examples of Vector Spaces

If we replace $\mathbb{R}$ with $\mathbb{C}$, we say that $(V, +, \cdot)$ is a ***complex vector (or linear) space***.

We will often simply write V instead of $(V, +, \cdot)$ if the operations $\cdot$ and $+$ are understood.

3.3.2. Examples.

1. The set of n-tuples (or n-dimensional vectors) $\mathbb{R}^n = \{x = (x_1, \ldots, x_n) \colon x_1, \ldots, x_n \in \mathbb{R}\}$ is a real vector space with addition defined by

$$\begin{aligned} x + y &= (x_1, \ldots, x_n) + (y_1, \ldots, y_n) \\ &:= (x_1 + y_1, \ldots, x_n + y_n), \qquad\qquad x, y \in \mathbb{R}^n, \end{aligned}$$

and scalar multiplication defined by

$$\lambda x = \lambda \cdot (x_1, \ldots, x_n) := (\lambda x_1, \ldots, \lambda x_n), \qquad \lambda \in \mathbb{R},\ x \in \mathbb{R}^n.$$

We will sometimes abbreviate $(x_1, \ldots x_n)$ by writing $(x_k)_{k=1}^n$.

Examples of Vector Spaces

2. The set $\mathbb{C}^n = \{z = (z_1, \dots, z_n) : z_1, \dots, z_n \in \mathbb{C}\}$ is a complex vector space with addition defined by

$$w + z := (w_1 + z_1, \dots, w_n + z_n), \qquad w, z \in \mathbb{C}^n, \tag{3.3.1}$$

and scalar multiplication defined by

$$\lambda z := (\lambda z_1, \dots, \lambda z_n), \qquad \lambda \in \mathbb{C},\ z \in \mathbb{C}^n.$$

3. The set $\mathbb{C}^n$ is a real vector space if we define addition as in (3.3.1) and scalar multiplication

$$\lambda z := (\lambda z_1, \dots, \lambda z_n), \qquad \lambda \in \mathbb{R},\ z \in \mathbb{C}^n.$$

Examples of Vector Spaces

4. The set of real polynomials of degree less than or equal to n,

$$\mathscr{P}_n = \Big\{ p\colon \mathbb{R} \to \mathbb{R} \colon p(x) = \sum_{k=0}^{n} a_k x^k, \quad a_0, \ldots, a_n \in \mathbb{R} \Big\}$$

is a real vector space with pointwise addition and scalar multiplication:

$$(p+q)(x) := p(x) + q(x), \qquad (\lambda p)(x) := \lambda p(x).$$

for $p, q \in \mathscr{P}_n$, $\lambda \in \mathbb{R}$.

5. More generally, for any set M, the set of maps $f\colon M \to \mathbb{R}$ $(\mathbb{C})$ is a real (real or complex) vector space with pointwise addition and scalar multiplication

$$(f+g)(x) := f(x) + g(x), \qquad (\lambda f)(x) := \lambda f(x),$$

for $f, g\colon M \to \mathbb{R}$, $\lambda \in \mathbb{R}$ $(\mathbb{C})$.

Examples of Vector Spaces

6. Hence for $M = \mathbb{N}$ the set of all real (complex) sequences $(a_n)_{n\in\mathbb{N}}$ is a real (real or complex) vector space.
7. The set of all functions $f\colon \mathbb{R} \to \mathbb{R}$ is a real vector space.
8. For any set $\Omega \subset \mathbb{R}$ the set of all continuous functions $f\colon \Omega \to \mathbb{R}$ is a real vector space. We denote this set by $C(\Omega, \mathbb{R})$ or just $C(\Omega)$.
9. For any open set $\Omega \subset \mathbb{R}$ and $k \in \mathbb{N}$ the set of all k times continuously differentiable functions $f\colon \Omega \to \mathbb{R}$ is a real vector space. We denote this set by $C^k(\Omega, \mathbb{R})$ or just $C^k(\Omega)$.
10. For any open set $\Omega \subset \mathbb{R}$ we define

$$C^\infty(\Omega, \mathbb{R}) = C^\infty(\Omega) = \{f\colon \Omega \to \mathbb{R}\colon f \in C^k(\Omega, \mathbb{R}) \text{ for all } k \in \mathbb{N}\},$$

which is also a real vector space.

Vector Spaces

3.3.3. Remarks. It is easy to prove that the neutral element $e \in V$ has the properties

$$\underset{\lambda \in \mathbb{R}\ (\mathbb{C})}{\forall} \lambda \cdot e = e, \qquad\qquad \underset{x \in V}{\forall} 0 \cdot x = e.$$

Instead of $e \in V$, we will often denote the neutral element by $0 \in V$, using the same symbol as for $0 \in \mathbb{R}$. This will simplify notation and generally not cause any confusion.

There can only be one neutral element in V: if $0, 0' \in V$ both have the property that $v + 0 = v = v + 0'$, then $0 = 0'$.

Let $-v$ be the inverse of $v \in V$, i.e., $v + (-v) = 0$. Then $-v = (-1) \cdot v$.

If $\lambda \cdot v = 0$ for some $v \in V$ and $\lambda \in \mathbb{R}\,(\mathbb{C})$, then either $v = 0$ or $\lambda = 0$.

Subspaces

In the above examples, it seems clear that some vector spaces are in a sense "contained" in others. For example, any polynomial of degree $\leq n$ is also a polynomial of degree $\leq n+1$, so $\mathscr{P}_n \subset \mathscr{P}_{n+1}$ as a set. Furthermore, if we apply the addition and scalar multiplication defined in $\mathscr{P}_{n+1}$ to the elements in $\mathscr{P}_n$, then these elements remain in $\mathscr{P}_n$ and, furthermore, this addition and scalar multiplication acts in the same way as the corresponding operations defined in $\mathscr{P}_n$. This motivates the following definition:

3.3.4. Definition. Let $(V, +, \cdot)$ be a real or complex vector space. If $U \subset V$ and $(U, +, \cdot)$ is also a vector space, then we say that $(U, +, \cdot)$ is a ***subspace*** of $(V, +, \cdot)$.

Subspaces

3.3.5. Example. The set $L = \{(x_1, x_2) \in \mathbb{R}^2 \colon x_1 + x_2 = 0\}$ gives a vector space $(L, +, \cdot)$ with the usual component-wise addition and scalar multiplication. We first need to check that $+$ is actually a map $L \times L \to L$, i.e., that $x + y \in L$ if $x, y \in L$. Note that

$$x + y = (x_1 + y_1, x_2 + y_2)$$

so we need to check that $(x_1 + y_1) + (x_2 + y_2) = 0$. Since

$$(x_1 + y_1) + (x_2 + y_2) = (x_1 + x_2) + (y_1 + y_2) = 0 + 0 = 0,$$

this is the case. Furthermore, the addition is associative and commutative for $x, y \in \mathbb{R}^2$, so it will also have these properties for $x, y \in L$. The neutral element is $e = (0, 0) \in L$, and the inverse element to x is $-x = (-x_1, -x_2) \in L$ (because $-x_1 + (-x_2) = -(x_1 + x_2) = -0 = 0$). In a similar manner, we can check that scalar multiplication is a map $\mathbb{R} \times L \to L$ and satisfies all conditions to ensure that $(L, +, \cdot)$ is a vector space.

Since $L \subset \mathbb{R}^2$, $(L, +, \cdot)$ is a subspace of $(\mathbb{R}^2, +, \cdot)$

Subspaces

Fortunately, we don't always have to go to as much trouble as we did in the previous example to verify that a vector space is a subspace of another vector space.

3.3.6. Lemma. Let $(V, +, \cdot)$ be a real (complex) vector space and $U \subset V$. If $u_1 + u_2 \in U$ for $u_1, u_2 \in U$ and $\lambda u \in U$ for all $u \in U$ and $\lambda \in \mathbb{R}$ $(\mathbb{C})$, then $(U, +, \cdot)$ is a subspace of $(V, +, \cdot)$.

Proof.
We consider the case of a real vector space. Assume that $u_1 + u_2 \in U$ for $u_1, u_2 \in U$ and $\lambda u \in U$ for all $u \in U$ and $\lambda \in \mathbb{R}$. Commutativity and associativity of $+$ are assured since they have these properties for all elements of V. Furthermore, $e = 0 \cdot v$ for any $v \in V$, so taking some $u \in U$, we have $e = 0 \cdot u \in U$ by our hypotheses. In the same way, $(-1) \cdot u = -u$ for any $u \in U$. Thus U contains the neutral element and inverse elements for every $u \in U$. All other properties of the maps $+$ and $\cdot$ are inherited from their properties in $(V, +, \cdot)$. $\square$

Subspaces

Just as we write V for $(V, +, \cdot)$ when the operations are understood, we also write $U \subset V$ to indicate a subspace U of V.

3.3.7. Examples.

1. $C(\mathbb{R}) \supset C^1(\mathbb{R}) \supset \cdots \supset C^\infty(\mathbb{R})$.
2. $\mathscr{P}_n \subset C^\infty(\mathbb{R})$ for any $n \in \mathbb{N}$.
3. Denote the set of bounded sequences by

$$\ell^\infty := \{(a_n)_{n=0}^\infty \colon \sup_{n\in\mathbb{N}} |a_n| < \infty\}$$

and the set of sequences converging to 0 by

$$c_0 := \{(a_n)_{n=0}^\infty \colon \lim_{n\to\infty} a_n = 0\}.$$

Both of these sets become vector spaces with component-wise addition and scalar multiplication, and $c_0 \subset \ell^\infty$ as vector spaces.

Normed Vector Spaces

Now that we have introduced the concept of a vector space, we want to define on vector spaces something akin to the modulus $|x|$ for $x \in \mathbb{R}$ or the euclidean length of a vector in $\mathbb{R}^2$,

$$|x| = |(x_1, x_2)| = \sqrt{x_1^2 + x_2^2}.$$

3.3.8. Definition. Let V be a real (complex) vector space. Then a map $\|\cdot\|\colon V \to \mathbb{R}$ is said to be a ***norm*** if for all $u, v \in V$ and all $\lambda \in \mathbb{R}$ $(\mathbb{C})$,

1. $\|v\| \geq 0$ for all $v \in V$ and $\|v\| = 0$ if and only if $v = 0$,
2. $\|\lambda \cdot v\| = |\lambda| \cdot \|v\|$,
3. $\|u + v\| \leq \|u\| + \|v\|$.

The pair $(V, \|\cdot\|)$ is called a ***normed vector space*** or a ***normed linear space***.

Clearly, any normed vector space can also be considered as a metric space: it is easy to check that

$$\varrho(x, y) := \|x - y\|$$

satisfies all the requirements of a metric.

Normed Vector Spaces

3.3.9. Examples.

1. $\mathbb{R}^n$ with $\|x\|_2 = \left(\sum_{j=1}^{n} x_i^2\right)^{1/2}$,
2. $\mathbb{R}^n$ with $\|x\|_p = \left(\sum_{j=1}^{n} |x_i|^p\right)^{1/p}$ for any $p \in \mathbb{N} \setminus \{0\}$,
3. $\mathbb{R}^n$ with $\|x\|_\infty = \max_{1 \le k \le n} |x_k|$,
4. ℓ^∞ with $\|(a_n)\|_\infty = \sup_{n \in \mathbb{N}} |a_n|$,
5. c_0 with $\|(a_n)\|_\infty = \sup_{n \in \mathbb{N}} |a_n|$,
6. $C([a, b])$, $[a, b] \subset \mathbb{R}$, with $\|f\|_\infty = \sup_{x \in [a,b]} |f(x)|$.

Sequences in Vector Spaces

Since every normed vector space is also a metric space, we can consider sequences in normed vector spaces without stating any new definitions. However, it may be useful to recall the definitions for the case of normed vector spaces.

In the following, we assume that $(V, \|\cdot\|)$ is a normed vector space.

A ***sequence in a vector space V*** is a map $(a_n)\colon \mathbb{N} \to V$. We say that (a_n) converges to $a \in V$ if

$$\underset{\varepsilon>0}{\forall}\ \underset{N\in\mathbb{N}}{\exists}\ \underset{n>N}{\forall} : \|a_n - a\| < \varepsilon.$$

3.3.10. Example. Let $V = \mathbb{R}^2$ with $\|x\| = \max\{|x_1|, |x_2|\}$. Then the sequence given by

$$a_n = \left(\frac{n^2}{2+n^2}, \frac{1}{n}\right) \xrightarrow{n\to\infty} (1, 0).$$

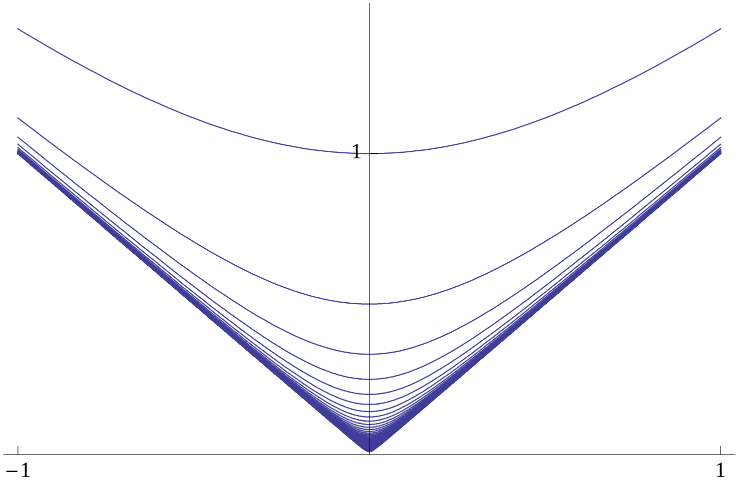
Sequences of Functions

We will interest us in particular in ***sequences of functions***. Thus we will consider a sequence (f_n), where for every $n \in \mathbb{N}$, f_n is a function.As an example, take the sequence of continuous functions (f_n) given by

$$f_n \colon [-1, 1] \to \mathbb{R}, \qquad f_n(x) = \sqrt{\frac{1}{(n+1)^2} + x^2}.$$

For every $n \in \mathbb{N}$, $f_n \in C([-1, 1])$ (f_n is continuous on $\mathbb{R}$). We endow $C(\mathbb{R})$ with the norm $\sup_{x \in [-1,1]} |f(x)|$. Then $f_n \to f$ as $n \to \infty$, where $f(x) = |x|$. We shall see and prove this below.

Sequences of Functions



Sequences of Functions

Now it seems clear that $f_n \to f$, $f(x) = |x|$, as stated. Mathematically, there are two distinct points of view:

1. ***pointwise convergence***: For every $x \in [-1, 1]$,

$$f_n(x) \xrightarrow{n\to\infty} f(x) \qquad :\Leftrightarrow \qquad |f_n(x) - f(x)| \xrightarrow{n\to\infty} 0$$

2. ***uniform convergence***: Each f_n is an element of the normed vector space $C([-1, 1])$, and so is f. Then

$$f_n \xrightarrow{n\to\infty} f \qquad :\Leftrightarrow \qquad \|f_n - f\|_\infty \xrightarrow{n\to\infty} 0.$$

The second approach is more natural: we are considering a sequence of functions f_n, not values of functions $f_n(x)$, so we should consider the properties of f_n over all of $[-1, 1]$, not just separately those of $f_n(x)$ for every $x \in [-1, 1]$.

Sequences of Functions

It is obvious that uniform convergence implies pointwise convergence for all x. In our example, we will prove that $\sqrt{1/(n+1)^2 + x^2} \to |x|$ uniformly for $x \in [-1, 1]$, i.e,

$$\|f_n - f\|_\infty = \sup_{x\in[-1,1]} \Big|\sqrt{1/(n+1)^2 + x^2} - |x|\Big| \xrightarrow{n\to\infty} 0.$$

Note that we can take the supremum over $[0, 1]$ instead of $[-1, 1]$, since $f_n(-x) = f_n(x)$ and $|-x| = |x|$. Furthermore, it is sufficient to show that

$$\begin{aligned}&\Big(\sup_{x\in[-1,1]} \Big|\sqrt{1/(n+1)^2 + x^2} - |x|\Big|\Big)^2 \\ &= \sup_{x\in[-1,1]} \Big|\sqrt{1/(n+1)^2 + x^2} - |x|\Big|^2 \xrightarrow{n\to\infty} 0.\end{aligned}$$

Sequences of Functions

Now

$$\Big|\sqrt{1/(n+1)^2+x^2}-|x|\Big|^2=\frac{1}{(n+1)^2}+\underbrace{2x^2-2x\sqrt{x^2+1/(n+1)^2}}_{<0\text{ if }x>0}.$$

It follows that the supremum is attained at $x=0$ and

$$\|f_n-f\|_\infty^2=\sup_{x\in[-1,1]}\Big|\sqrt{1/(n+1)^2+x^2}-|x|\Big|^2=\frac{1}{(n+1)^2}\xrightarrow{n\to\infty}0.$$

Thus $f_n\to f$ uniformly (or as a sequence in $(C([-1,1]),\|\cdot\|_\infty)$).

Sequences of Functions

Summarizing, we can say that a sequence of continuous functions f_n that converges ***uniformly*** in an interval $[a, b]$ to some function f also converges ***pointwise*** to f:

$$\sup_{x\in[a,b]} |f_n(x) - f(x)| \xrightarrow{n\to\infty} 0 \qquad \Rightarrow \qquad \forall_{x\in[a,b]} |f_n(x) - f(x)| \xrightarrow{n\to\infty} 0$$

In general, we make the following definition:

3.4.1. Definition. Let $\Omega \subset \mathbb{R}$ and (f_n) be a sequence of functions $f_n \colon \Omega \to \mathbb{C}$. We say that the sequence (f_n) converges pointwise to the function $f \colon \Omega \to \mathbb{C}$ if

$$\forall_{x\in\Omega} |f_n(x) - f(x)| \xrightarrow{n\to\infty} 0.$$

If f is the pointwise limit of (f_n), we say that (f_n) converges ***uniformly*** to f on Ω if

$$\sup_{x\in\Omega} |f_n(x) - f(x)| \xrightarrow{n\to\infty} 0.$$

Sequences of Functions

A sequence of continuous functions f_n that converges pointwise to some function f does not need to converge uniformly to f.

3.4.2. Example. The sequence (f_n),

$$f_n \colon [0,1] \to \mathbb{R}, \qquad f_n(x) = \begin{cases} 1 - nx, & 0 \le x \le 1/n, \\ 0, & \text{otherwise}, \end{cases}$$

converges to

$$f \colon [0,1] \to \mathbb{R}, \qquad f(x) = \begin{cases} 1, & x = 0, \\ 0, & \text{otherwise}, \end{cases}$$

pointwise, but not uniformly, as we now show.

Sequences of Functions

Clearly, for every $x \in (0,1]$ there exists an $N \in \mathbb{N}$ such that $f_n(x) = 0$ for all $n > N$, (Just take $N = 1/x$.) Furthermore, $f_n(0) = 1$ for all n. Hence $f_n(x) \to f(x)$ for all $x \in [0,1]$.

However,

$$\sup_{x\in[0,1]} |f_n(x) - f(x)| \geq |f_n(1/(2n)) - f(1/(2n))| = |1 - n/(2n) - 0| = \frac{1}{2},$$

so $\sup\limits_{x\in[0,1]} |f_n(x) - f(x)| \not\to 0$ as $n \to \infty$.

In general, when considering sequences of functions you should first find the pointwise limit and then test whether the convergence is uniform.

Sequences of Functions

In the previous example, the sequence of continuous functions f_n converged to the discontinuous function pointwise, but not uniformly. This no accident. In fact, a uniformly convergent sequence of continuous functions will always converge to a continuous function:

3.4.3. Theorem. Let $[a, b] \subset \mathbb{R}$ be a closed interval. Let (f_n) be a sequence of continuous functions defined on $[a, b]$ such that $f_n(x)$ converges to some $f(x) \in \mathbb{R}$ as $n \to \infty$ for every $x \in [a, b]$. If the sequence (f_n) converges uniformly to the thereby defined function $f : [a, b] \to \mathbb{R}$, then f is continuous.

Sequences of Functions

Proof.

We need to show that f is continuous for all $x \in [a, b]$. We will here deal only with $x \in (a, b)$; the cases $x = a$ and $x = b$ are left to you.

Let $x \in (a, b)$. We will show that for any $\varepsilon > 0$ there exists a $\delta > 0$ such that $|h| < \delta$ implies $|f(x+h) - f(x)| < \varepsilon$ (for h so small that $x + h \in (a, b)$). Fix $\varepsilon > 0$. Then there exists some $N \in \mathbb{N}$ such that

$$\|f_n - f\|_\infty = \sup_{x \in [a,b]} |f_n(x) - f(x)| < \frac{\varepsilon}{3}.$$

for all $n > N$. Choose some such $n \in \mathbb{N}$. Since each f_n is continuous on $[a, b]$, there exists some $\delta > 0$ such that $|h| < \delta$ implies

$$|f_n(x) - f_n(x+h)| < \frac{\varepsilon}{3}.$$

Sequences of Functions

Proof (continued).

Then for $|h| < \delta$ we have

$$\begin{aligned} |f(x+h) - f(x)| &\le |f(x+h) - f_n(x+h)| + |f_n(x+h) - f_n(x)| \\ &\quad + |f_n(x) - f(x)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

□

3.4.4. Theorem. Let $[a, b] \subset \mathbb{R}$ be a closed interval and $C([a, b])$ the vector space of continuous functions on $[a, b]$, endowed with the metric

$$\varrho(f, g) = \|f - g\|_\infty = \sup_{x \in [a,b]} |f(x) - g(x)|.$$

Then the metric space $(C([a, b]), \varrho)$ is complete, i.e., every Cauchy sequence in the space converges.

Completeness of $C([a, b])$

Proof.

Let (f_n) be a Cauchy sequence in $C([a, b])$. We will show that $\lim_{n\to\infty} f_n(x)$ exists for every $x \in [a, b]$. First, by definition, for every $\varepsilon > 0$ we have

$$\sup_{x\in[a,b]} |f_n(x) - f_m(x)| < \varepsilon$$

for n, m sufficiently large. But then, for every fixed $x \in [a, b]$, we have

$$|f_n(x) - f_m(x)| < \varepsilon$$

for n, m sufficiently large. This implies that for every $x \in [a, b]$ the sequence or real numbers $(f_n(x))$ is Cauchy. Since the real numbers are complete, $(f_n(x))$ converges. Hence we can define the pointwise limit.

$$f(x) := \lim_{n\to\infty} f_n(x) \qquad \text{for every } x \in [a, b].$$

Completeness of $C([a, b])$

Proof (continued).

We now show that the convergence of (f_n) to f is uniform: fix $\varepsilon > 0$ and choose N so that

$$\sup_{x\in[a,b]} |f_n(x) - f_m(x)| < \varepsilon \qquad\qquad \text{for } n, m > N.$$

Then for $n > N$ we have

$$\sup_{x\in[a,b]} |f(x) - f_n(x)| = \sup_{x\in[a,b]} \lim_{m\to\infty} |f_m(x) - f_n(x)|.$$

For any $p > N$ we have

$$|f_p(x) - f_n(x)| \le \sup_{m>N} |f_m(x) - f_n(x)|. \tag{3.4.1}$$

Completeness of $C([a, b])$

Proof (continued).

We can let $p \to \infty$ in (3.4.1) and obtain

$$\lim_{m\to\infty} |f_m(x) - f_n(x)| \le \sup_{m>N} |f_m(x) - f_n(x)|.$$

This implies that for all $n > N$

$$\begin{aligned}\sup_{x\in[a,b]} |f(x) - f_n(x)| &\le \sup_{x\in[a,b]} \sup_{m>N} |f_m(x) - f_n(x)| \\ &= \sup_{m>N} \sup_{x\in[a,b]} |f_m(x) - f_n(x)| < \sup_{m\ge N} \varepsilon = \varepsilon,\end{aligned}$$

so the convergence is uniform. By Theorem 3.4.3, f is continuous, so $f \in C([a, b])$. $\square$

Sequences of Functions

We will be especially interested in a particular type of sequence of functions, called ***power series***. For example, we would like to find the limit of (f_n) where

$$f_0(x) = 1, \quad f_1(x) = 1 + x, \quad f_2(x) = 1 + x + \frac{x^2}{2!}, \ldots, \quad f_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!}.$$

Sequences which are formed by continually adding summands are called ***series***, and we will have to start with studying their basic theory.

Convergence

A series is "a sequence together with the wish to sum."

3.5.1. Definition. Let (a_n) be a sequence in a normed vector space $(V, \|\cdot\|)$. Then we say that (a_n) is ***summable*** with sum $s \in V$ if

$$\lim_{n\to\infty} s_n = s, \qquad\qquad s_n := \sum_{k=0}^{n} a_k.$$

We call s_n the ***nth partial sum*** of (a_n). We use the notation

$$\sum_{k=0}^{\infty} a_k \qquad \text{or simply} \qquad \sum a_k \tag{3.5.1}$$

to denote not only s, but also the "procedure of summing the sequence (a_n)." We call (3.5.1) an ***infinite series*** and we say that the series converges if (a_n) is summable. If (s_n) does not converge, we say that $\sum a_k$ diverges.

The Geometric Series

3.5.2. Lemma. The geometric series $\sum_{k=0}^{\infty} z^k$, $z \in \mathbb{C}$, converges to $\dfrac{1}{1-z}$ if $|z| < 1$ and diverges if $|z| > 1$.

Proof.
From

$$(1-z)\sum_{k=0}^{n} z^k = \sum_{k=0}^{n} z^k - \sum_{k=1}^{n+1} z^k = 1 - z^{n+1}$$

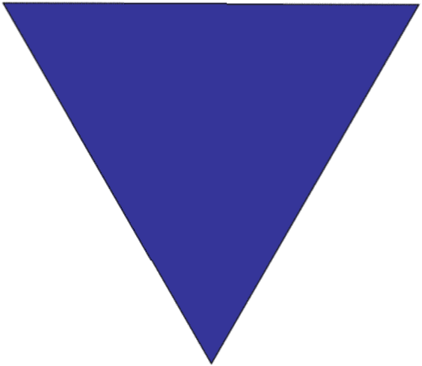
we see that

$$\sum_{k=0}^{n} z^k = \frac{1-z^{n+1}}{1-z}. \tag{3.5.2}$$

As $n \to \infty$, $|z|^{n+1}$ will diverge if $|z| > 1$ and converge to zero if $|z| < 1$. $\square$

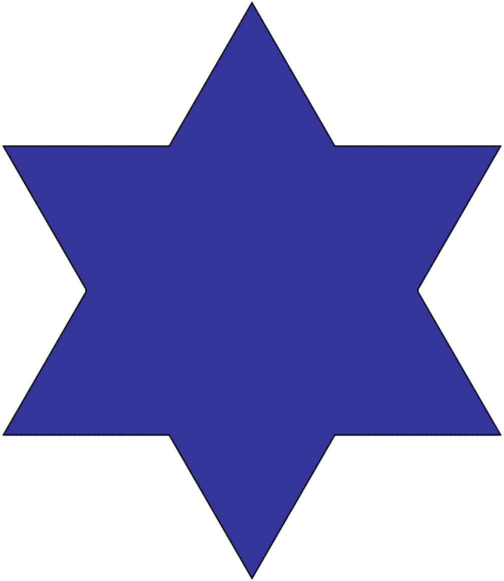
The Koch Snowflake

3.5.3. Example. The Koch snowflake is a simple example of a ***fractal curve***. It is constructed as follows: one starts with an equilateral triangle:

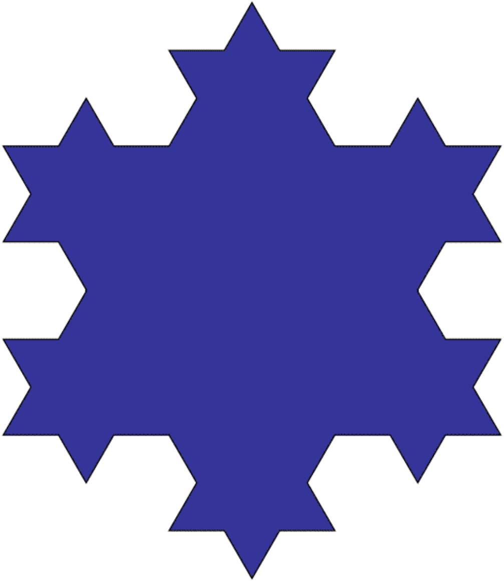


One then successively adds an equilateral triangle to the middle third of every line segment.

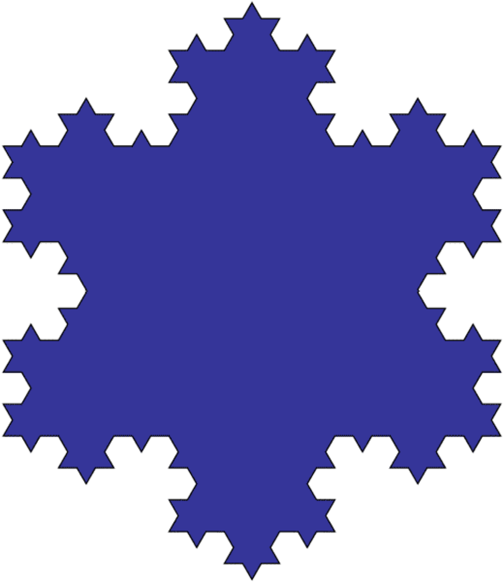
The Koch Snowflake - First Iteration



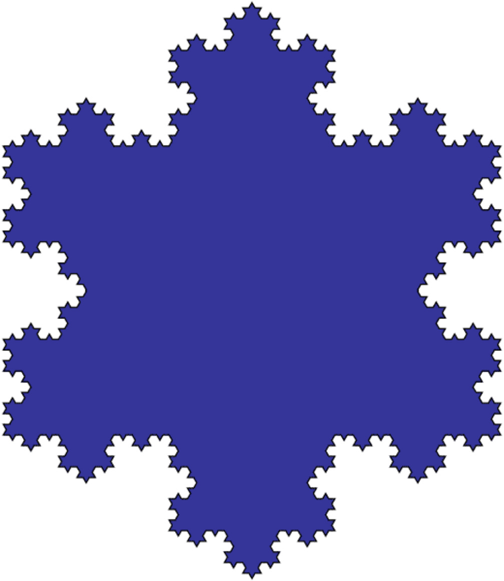
The Koch Snowflake - Second Iteration



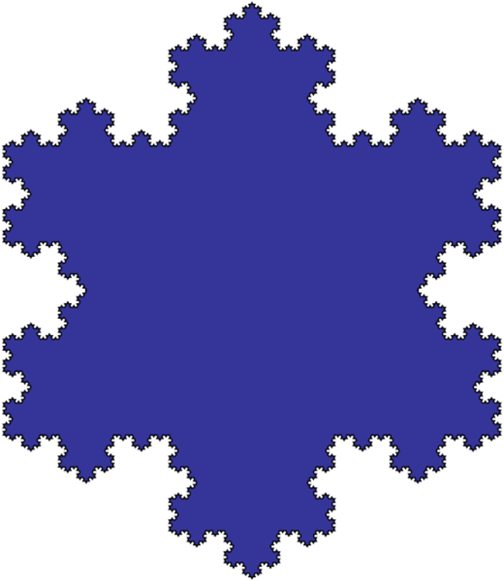
The Koch Snowflake - Third Iteration



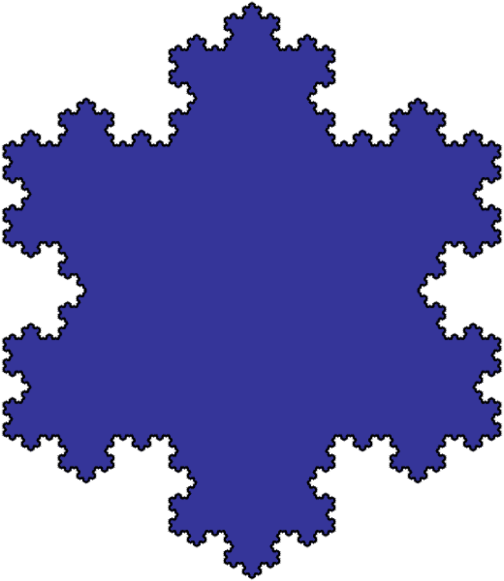
The Koch Snowflake - Fourth Iteration



The Koch Snowflake - Fifth Iteration



The Koch Snowflake - Sixth Iteration



The Koch Snowflake

The Koch snowflake is now defined as the limit as $n \to \infty$ of the nth iteration. The fractal nature of the curve becomes apparent when one tries to "zoom in" to a segment to analyze its structure:

This property is called ***self-similarity***: any arbitrarily small segment of the Koch curve contains all the structural information of the entire curve.

We are now interested in the total line length l_n and the area A_n circumscribed by the nth iteration of the Koch snowflake.

Length of the Koch Snowflake

It is easy to see that in every iteration the length of the curve increases by $1/3$: effectively, one adds $1/3$ of the length of each individual line segment, so

$$l_{n+1} = \frac{4}{3} l_n.$$

Hence, if the initial triangle has side length l_0,

$$l_n = \left(\frac{4}{3}\right)^n l_0$$

When $n \to \infty$, the length of the nth iteration diverges to infinity, so the Koch curve has infinite length.

Area enclosed by the Koch Snowflake

Note that an equilateral triangle of side length l has area $A = (\sqrt{3}/4)l^2$. To the area of the initial triangle we add that of three triangles of length $l_0/3$, giving

$$A_0 = \frac{\sqrt{3}}{4} l_0^2, \qquad\qquad A_1 = \frac{\sqrt{3}}{4} l_0^2 + 3 \cdot \frac{\sqrt{3}}{4} \left(\frac{l_0}{3} \right)^2 .$$

The first iteration consists of 12 line segments and thereafter, the nth iteration will have 4 times as many as the previous iteration. There will be a total of $3 \cdot 4^n$ line segments in the nth iteration and the length of each segment is $l_0/3^n$. This means that the nth iteration, $n \geq 1$, will have an area of

$$A_n = \frac{\sqrt{3}}{4} l_0^2 + \sum_{k=1}^{n} 3 \cdot 4^{k-1} \frac{\sqrt{3}}{4} \left(\frac{l_0}{3^k} \right)^2 = \frac{\sqrt{3}}{4} l_0^2 \left(1 + \frac{3}{4} \sum_{k=1}^{n} \left(\frac{4}{9} \right)^k \right)$$

Area enclosed by the Koch Snowflake

We have

$$A_n = \frac{3}{4} A_0 \left(\frac{1}{3} + \sum_{k=0}^{n} \left(\frac{4}{9} \right)^k \right).$$

This is a geometric series, and when $n \to \infty$, we find that

$$\lim_{n \to \infty} A_n = \frac{3}{4} A_0 \left(\frac{1}{3} + \frac{1}{1 - 4/9} \right) = \frac{8}{5} A_0$$

Hence the Koch snowflake has $8/5$ times the area of the original triangle. A finite area is hereby enclosed by a curve of infinite length.

The theory of fractals owes much to the ideas of Benoit Mandelbrot, who popularized the famous ***Mandelbrot fractal***. Fractals play an important role in chaos theory and non-linear dynamics.

The Cauchy Criterion and its Consequences

We now return to the general theory of series. In particular, we want to develop techniques to analyze the geometric series when $|z| = 1$.

3.5.4. Cauchy Criterion. Let $\sum a_k$ be a series in a **complete** vector space $(V, \|\cdot\|)$. Then

$$\begin{aligned}
\sum a_k \text{ converges} \quad &\Leftrightarrow \quad (s_n)_{n\in\mathbb{N}} \text{ converges, } s_n = \sum_{k=0}^{n} a_k \\
&\Leftrightarrow \quad (s_n) \text{ is Cauchy} \\
&\Leftrightarrow \quad \underset{\varepsilon>0}{\forall}\ \underset{N\in\mathbb{N}}{\exists}\ \underset{m>n>N}{\forall}\ \|s_m - s_n\| < \varepsilon \\
&\Leftrightarrow \quad \underset{\varepsilon>0}{\forall}\ \underset{N\in\mathbb{N}}{\exists}\ \underset{m>n>N}{\forall}\ \Big\|\sum_{k=n+1}^{m} a_k\Big\| < \varepsilon
\end{aligned}$$

The Cauchy Criterion and its Consequences

3.5.5. Corollary. If the series $\sum_{k=0}^{\infty} a_k$ converges, then the sequence $a_k \to 0$ as $k \to \infty$. (Take $m = n + 1$ in the Cauchy Criterion.)

3.5.6. Corollary. If the series $\sum_{k=0}^{\infty} a_k$ converges, then the sequence (A_n) given by

$$A_n := \sum_{k=n}^{\infty} a_k$$

converges to 0 as $n \to \infty$. (Let $m \to \infty$ in the Cauchy Criterion.)

3.5.7. Example. The harmonic series $\sum\limits_{k=1}^{\infty} \frac{1}{k}$ is divergent. (Since $|s_{2n} - s_n| \geq 1/2$, the sequence of partial sums (s_n) is not Cauchy.)

The p-Series

We can extend the reasoning used to show the divergence of the harmonic series to study the so-called ***p*-series**,

$$\sum_{k=1}^{\infty} \frac{1}{k^p}, \qquad\qquad p \in \mathbb{R}.$$

For $p \le 0$ the series diverges by Corollary 3.5.5, because it is not the sum of a sequence converging to zero.

3.5.8. Lemma. Denote by $s_n(p)$ the nth partial sum of the p-series. For $p > 0$,

$$\frac{2^p - 1}{2^p} + \frac{2}{2^p} s_n(p) < s_{2n}(p) < 1 + \frac{2}{2^p} s_n(p)$$

The p-Series

Proof.

By definition,

$$\begin{aligned} s_{2n}(p) &= 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{(2n)^p} \\ &= 1 + \Big(\frac{1}{2^p} + \frac{1}{4^p} + \cdots + \frac{1}{(2n)^p}\Big) + \Big(\frac{1}{3^p} + \frac{1}{5^p} + \cdots + \frac{1}{(2n-1)^p}\Big) \\ &> 1 + \frac{1}{2^p} s_n(p) + \Big(\frac{1}{4^p} + \frac{1}{6^p} + \cdots + \frac{1}{(2n)^p}\Big) \end{aligned}$$

where we have used that $p > 0$ in the last step. Thus

$$s_{2n}(p) > 1 + \frac{1}{2^p} s_n(p) - \frac{1}{2^p} + \frac{1}{2^p} s_n(p) = \frac{2^p - 1}{2^p} + \frac{2}{2^p} s_n(p).$$

The p-Series

Proof (continued).
Similarly,

$$\begin{aligned} s_{2n}(p) &= 1 + \Big(\frac{1}{2^p} + \frac{1}{4^p} + \dots + \frac{1}{(2n)^p}\Big) + \Big(\frac{1}{3^p} + \frac{1}{5^p} + \dots + \frac{1}{(2n-1)^p}\Big) \\ &< 1 + \frac{1}{2^p} s_n(p) + \Big(\frac{1}{2^p} + \frac{1}{4^p} + \dots + \frac{1}{(2n-2)^p}\Big) + \frac{1}{(2n)^p} \\ &< 1 + \frac{2}{2^p} s_n(p) \end{aligned}$$

□

We can now state our main result:

3.5.9. Theorem. The p-series $\sum\limits_{k=0}^{\infty} \frac{1}{k^p}$ converges when $p > 1$ and diverges when $p \le 1$.

The p-Series

Proof.

We only need to consider the case $p > 0$. Assume first that $0 < p \le 1$ and that the p-series converges, i.e., $\lim_{n\to\infty} s_n(p) =: s(p)$ exists. Then the estimate of Lemma 3.5.8 gives

$$1 + \frac{1}{2^{p-1}} s(p) - \frac{1}{2^p} = \frac{2^p - 1}{2^p} + \frac{2}{2^p} s(p) \le s(p).$$

and hence

$$0 < \frac{2^p - 1}{2^p} \le \frac{2^p - 2}{2^p} s(p) \le 0$$

which is a contradiction. Thus the series diverges for $p \le 1$. Now for $p > 1$ we have

$$s_n(p) < s_{2n}(p) < 1 + 2^{1-p} s_n(p),$$

so $s_n(p) < 1/(1 - 2^{1-p})$. Since $(s_n(p))$ is increasing, the sequence of partial sums converges. $\square$

Absolute Convergence

3.5.10. Example. The geometric series $\sum_{k=0}^{\infty} z^k$, $z \in \mathbb{C}$, diverges if $|z| = 1$.

To see this, note that

$$\lim_{k\to\infty} |z^k| = \lim_{k\to\infty} |z|^k = 1,$$

so the sequence z^k does not converge to zero. By Corollary 3.5.5, the series can not converge.

3.5.11. Definition. A series $\sum a_k$ in a normed vector space $(V, \|\cdot\|)$ is called ***absolutely convergent*** if $\sum \|a_k\|$ converges.

A sequence (a_k) in a normed vector space $(V, \|\cdot\|)$ is said to be ***absolutely summable*** if $\sum a_k$ converges absolutely.

3.5.12. Theorem. An absolutely convergent series $\sum a_k$ in a **complete** vector space $(V, \|\cdot\|)$ is convergent.

Absolute Convergence

Proof.

Let (a_k) be a sequence with partial sum

$$s_n = \sum_{k=0}^{n} a_k .$$

Then

$$\|s_n - s_m\| = \Big\| \sum_{k=m+1}^{n} a_k \Big\| \le \sum_{k=m+1}^{n} \|a_k\|. \tag{3.5.3}$$

Since $\sum a_k$ converges absolutely, $\sum \|a_k\|$ is Cauchy, so for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $\sum_{k=m+1}^{n} \|a_k\| < \varepsilon$ for all $n, m > N$. But by (3.5.3), this is also true for $\|s_n - s_m\|$. Hence the sequence (s_n) is Cauchy and converges because $(V, \|\cdot\|)$ is complete. Thus $\sum a_k$ converges. $\square$

Absolute Convergence

The concept of absolute convergence is useful in many contexts, such as

(i) ***Alternating series***: A series of real numbers of the form $\sum_{k=0}^{\infty}(-1)^k a_k$ with $a_k > 0$ for all k is called ***alternating***. The series will converge absolutely if

$$\sum_{k=0}^{\infty} |(-1)^k a_k| = \sum_{k=0}^{\infty} a_k$$

converges.

(ii) ***Complex series***: The series $\sum_{k=0}^{\infty} z_k$ with $z_k = a_k + ib_k$ converges absolutely if the series

$$\sum_{k=0}^{\infty} |z_k| = \sum_{k=0}^{\infty} \sqrt{a_k^2 + b_k^2}$$

converges.

Absolute Convergence

(iii) ***Series of functions***: The series $\sum_{k=0}^{\infty} f_k$ of continuous functions $f_k \in C(I, \mathbb{C})$, $I = [a, b] \subset \mathbb{R}$ a closed interval, converges absolutely if the series

$$\sum_{k=0}^{\infty} \|f_k\|_\infty = \sum_{k=0}^{\infty} \sup_{x \in I} |f(x)|$$

converges.

Since the real numbers are complete, absolute convergence in (i) implies convergence of the alternating series to some real number. Similarly, absolute convergence of the complex sequence in (ii) implies convergence to some complex number. The most interesting case from the point of view of calculus is (iii): since $(C(I, \mathbb{C}), \|\cdot\|_\infty)$ is complete by Theorem 3.4.4, absolute convergence implies convergence to some continuous function $f \in C(I, \mathbb{C})$.

Series of Functions

3.5.13. Example. Let (f_n) be the sequence of functions

$$f_n\colon \left[-\frac{1}{2}, \frac{1}{2}\right] \to \mathbb{R}, \qquad\qquad f_n(x) = x^n.$$

Clearly, $f_n \in C([-\frac{1}{2}, \frac{1}{2}])$ for all $n \in \mathbb{N}$. We want to find out if (f_n) is summable, i.e., if the series

$$\sum_{n=0}^{\infty} f_n \tag{3.5.4}$$

converges to a function $f \in C([-\frac{1}{2}, \frac{1}{2}])$. This would be the case if the series (3.5.4) were to converge absolutely. We have

$$\sum_{n=0}^{\infty} \|f_n\|_\infty = \sum_{n=0}^{\infty} \sup_{-1/2 \le x \le 1/2} |x^n| = \sum_{n=0}^{\infty} \frac{1}{2^n} = 2 < \infty.$$

Hence (3.5.4) converges absolutely and therefore also converges to a function $f \in C([-1/2, 1/2])$.

Series of Functions

In this simple example we can actually calculate the pointwise limit of the series:

$$f(x) = \sum_{n=0}^{\infty} f_n(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

for all $x \in [-\frac{1}{2}, \frac{1}{2}]$. In many applications, however, we will not be able to calculate the pointwise limit so easily. We will return to a discussion of limits of series of functions below.

3.5.14. Lemma. Let $(V, \|\cdot\|)$ be a complete normed vector space and $\sum a_k$ an absolutely convergent series. Then

$$\Big\| \sum_{k=0}^{\infty} a_k \Big\| \le \sum_{k=0}^{\infty} \|a_k\|. \tag{3.5.5}$$

The "Infinite" Triangle Inequality

Proof.

The triangle inequality yields, for any $n \in \mathbb{N}$,

$$\Big\| \sum_{k=0}^{n} a_k \Big\| \le \sum_{k=0}^{n} \|a_k\| \le \sum_{k=0}^{\infty} \|a_k\|. \tag{3.5.6}$$

Since the series converges absolutely, it also converges and

$$\Big\| \sum_{k=0}^{\infty} a_k \Big\| = \Big\| \lim_{n\to\infty} \sum_{k=0}^{n} a_k \Big\| = \lim_{n\to\infty} \Big\| \sum_{k=0}^{n} a_k \Big\|$$

since the norm is continuous (i.e., $b_n \to b$ implies $\|b_n\| \to \|b\|$. Why?). Since the limit exists, the estimate (3.5.6) implies (3.5.5). □

The Comparison Test

In the rest of this section we will use the phrase "for sufficiently large k" to mean "for $k > k_0$ for some $k_0 \in \mathbb{N}$". For example, the statement "$1/k < 0.001$ for sufficiently large k" should be interpreted in this way.

Many of the following criteria apply to series of positive real numbers. They are therefore suitable for establishing the ***absolute convergence*** of a series.

3.5.15. Comparison Test. Let (a_k) and (b_k) be real-valued sequences with $0 \le a_k \le b_k$ for sufficiently large k. Then

$$\sum b_k \text{ converges} \qquad \Rightarrow \qquad \sum a_k \text{ converges.}$$

3.5.16. Remark. In applications one also often uses the contrapositive: if $0 \le a_k \le b_k$ for sufficiently large k, then

$$\sum a_k \text{ diverges} \qquad \Rightarrow \qquad \sum b_k \text{ diverges.}$$

The Comparison Test

Proof.

Let (a_k) and (b_k) be positive real sequences and denote by

$$S_n := \sum_{k=0}^{n} a_k, \qquad s_n := \sum_{k=0}^{n} b_k, \qquad n \in \mathbb{N},$$

their partial sums. Since the series $\sum b_k$ converges, the sequence (s_n) converges to $\sum_{k=0}^{\infty} b_k < \infty$ as $n \to \infty$.

Since $a_k \le b_k$ for $k > k_0$ form some $k_0 \in \mathbb{N}$, the partial sums of (a_k) are bounded by

$$S_n = \sum_{k=0}^{n} a_k \le C + \sum_{k=k_0}^{n} b_k \le C + \sum_{k=0}^{\infty} b_k$$

for some $C > 0$. Hence (S_n) is bounded above. Since (S_n) is obviously increasing, the sequence converges. This means that $\sum a_k$ is convergent. $\square$

The Comparison Test

3.5.17. Example. The series

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k2^k} \tag{3.5.7}$$

converges. To see this, we note that

$$a_k := \left| \frac{(-1)^k}{k2^k} \right| = \frac{1}{k}\frac{1}{2^k} \le \left(\frac{1}{2}\right)^k =: b_k.$$

Since the geometric series $\sum_{k=0}^{\infty} z^k$ converges if $z = 1/2$, we know that $\sum_{k=0}^{\infty} b_k$ converges. By the comparison test, $\sum_{k=0}^{\infty} a_k$ converges, so (3.5.7) is absolutely convergent. Since the real numbers are complete, it also converges.

The Comparison Test

3.5.18. Example. The series

$$\sum_{k=1}^{\infty} \frac{k+2}{k(k+1)} \tag{3.5.8}$$

diverges. To see this, we note that

$$b_k := \frac{k+2}{k(k+1)} \geq \frac{1}{k} =: a_k.$$

Since the harmonic series $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges, so does $\sum_{k=1}^{\infty} \frac{k+2}{k(k+1)}$. (See Remark 3.5.16.)

Many criteria for absolute convergence can be derived from the comparison test. We will now study a few examples.

The Weierstraß M-test

3.5.19. Weierstraß M-test. Let $\Omega \subset \mathbb{R}$ and (f_k) be a sequence of functions defined on Ω, $f_k \colon \Omega \to \mathbb{C}$, satisfying

$$\sup_{x \in \Omega} |f_k(x)| \le M_k, \qquad\qquad k \in \mathbb{N} \tag{3.5.9}$$

for a sequence of real numbers (M_k). Suppose that $\sum M_k$ converges. Then the limit

$$f(x) := \sum_{k=0}^{\infty} f_k(x) \qquad\qquad \text{exists for every } x \in \Omega.$$

Furthermore, the sequence (F_n) of partial sums

$$F_n(x) = \sum_{k=0}^{n} f_k(x)$$

converges uniformly to f.

The Weierstraß M-test

Proof.

For every $x \in \Omega$, $|f_k(x)| \le M_k$, so $\sum_{k=0}^{\infty} |f_k(x)|$ converges by the comparison test. Thus, for every $x \in \Omega$, the series of numbers $\sum_{k=0}^{\infty} f_k(x)$ converges absolutely. Since the complex numbers are complete, we deduce that

$$f(x) := \sum_{k=0}^{\infty} f_k(x) \qquad \text{exists for every } x \in \Omega.$$

For any $x \in \Omega$,

$$\begin{aligned} |f(x) - F_n(x)| &= \left| f(x) - \sum_{k=0}^{n} f_k(x) \right| = \left| \sum_{k=n+1}^{\infty} f_k(x) \right| \\ &\le \sum_{k=n+1}^{\infty} |f_k(x)| \le \sum_{k=n+1}^{\infty} M_k. \end{aligned}$$

The Weierstraß M-test

Proof (continued).

Hence,

$$\sup_{x\in\Omega}\left|f(x)-\sum_{k=0}^{n} f_k(x)\right| \le \sum_{k=n+1}^{\infty} M_k.$$

By Corollary 3.5.6, the right-hand side converges to zero as $n\to\infty$, so we have

$$\sup_{x\in\Omega}\left|f(x)-\sum_{k=0}^{n} f_k(x)\right| \xrightarrow{n\to\infty} 0.$$

This shows the uniform convergence. $\square$

Uniform Convergence of Series of Functions

3.5.20. Definition. Let $\Omega \subset \mathbb{R}$ and (f_k) be a sequence of functions defined on Ω, $f_k \colon \Omega \to \mathbb{C}$. We say that the series

$$\sum_{k=0}^{\infty} f_k$$

converges uniformly (to a function $f \colon \Omega \to \mathbb{C}$) if the sequence of partial sums $F_n = \sum_{k=0}^{n} f_k$ converges uniformly to f.

Uniform Convergence of Series of Functions

3.5.21. Example. Consider the sequence of monomials (f_n),

$$f_n\colon \left[-\frac{1}{2},\frac{1}{2}\right] \to \mathbb{R}, \qquad f_n(x)=\frac{1}{n}x^n, \qquad n\in\mathbb{N}\setminus\{0\}.$$

Since

$$\sup_{x\in[-\frac{1}{2},\frac{1}{2}]} |f_n(x)| = \frac{1}{n2^n} < \frac{1}{2^n} =: M_n$$

and $\sum M_n$ converges, by the Weierstraß M-test the series $\sum f_k$ converges uniformly to

$$f\colon \left[-\frac{1}{2},\frac{1}{2}\right] \to \mathbb{R}, \qquad f(x)=\sum_{n=1}^{\infty}\frac{1}{n}x^n.$$

At this time we are not able to find a more explicit expression for f. However, by Theorem 3.4.3 we know that f is a continuous function on $[-1/2,1/2]$.

The Root Test

A further consequence of the Comparison Test 3.5.15 is the following, quite fundamental criterion for convergence:

3.5.22. Root Test. Let $\sum a_k$ be a series of positive real numbers $a_k \geq 0$.

(i) Suppose that there exists a $q < 1$ such that

$$\sqrt[k]{a_k} \leq q \qquad \text{for all sufficiently large } k.$$

Then $\sum a_k$ converges.

(ii) Suppose that

$$\sqrt[k]{a_k} \geq 1 \qquad \text{for all sufficiently large } k.$$

Then $\sum a_k$ diverges.

The Root Test

Proof.

(i) Suppose that we have a $q < 1$ such that $\sqrt[k]{a_k} \le q$ for all $k \ge k_0$. Then

$$a_k \le q^k \qquad \text{for } k \ge k_0.$$

Since the geometric series $\sum_{k=k_0}^{\infty} q^k$ converges for $q < 1$, the series $\sum_{k=k_0}^{\infty} a_k$ converges by the Comparison Test 3.5.15.

(ii) Suppose that $\sqrt[k]{a_k} \ge 1$ for all $k \ge k_0$. Then $a_k \ge 1$, so $a_k \not\to 0$ as $k \to \infty$. By Corollary 3.5.5, the series $\sum a_k$ can not converge. $\square$

3.5.23. Remark. Note that the existence of a $q < 1$ so that $\sqrt[k]{a_k} < q$ is crucial; this is not the same as requiring $\sqrt[k]{a_k} < 1$.

The Root Test

3.5.24. Example. The series

$$\sum_{k=0}^{\infty} kz^k$$

converges absolutely if $|z| < 1$ and diverges if $|z| > 1$. This is easily seen from

$$\sqrt[k]{|kz^k|} = \sqrt[k]{k} \cdot |z|.$$

Since $\sqrt[k]{k} \to 1$ as $k \to \infty$, we see that for sufficiently large k,

$$\sqrt[k]{|kz^k|} > 1 \qquad \text{if } |z| > 1$$

and

$$\sqrt[k]{|kz^k|} < \frac{|z|+1}{2} < 1 \qquad \text{if } |z| < 1.$$

The Root Test

3.5.25. Example. The root test does not always yield a result: Consider the series

$$\sum_{k=1}^{\infty} \frac{1}{k} \qquad \text{and} \qquad \sum_{k=1}^{\infty} \frac{1}{k^2}.$$

In both cases, $\sqrt[k]{a_k} < 1$ for all k, but there does not exist a constant $q < 1$ such that $\sqrt[k]{a_k} < q$. We have seen that the harmonic series

$$\sum_{k=1}^{\infty} \frac{1}{k}$$

diverges, while the ***Euler series***

$$\sum_{k=1}^{\infty} \frac{1}{k^2}$$

converges.

The Root Test using Limits

3.5.26. Root Test. Let a_k be a sequence of positive real numbers $a_k \geq 0$. Then

$$\overline{\lim_{k\to\infty}} \sqrt[k]{a_k} < 1 \qquad \Rightarrow \qquad \sum_{k=0}^{\infty} a_k \qquad \text{converges,}$$

$$\overline{\lim_{k\to\infty}} \sqrt[k]{a_k} > 1 \qquad \Rightarrow \qquad \sum_{k=0}^{\infty} a_k \qquad \text{diverges.}$$

3.5.27. Remarks.

(i) No statement is possible if $\overline{\lim_{k\to\infty}} \sqrt[k]{a_k} = 1$.

(ii) If $\lim_{k\to\infty} \sqrt[k]{a_k}$ exists, it equals $\overline{\lim_{k\to\infty}} \sqrt[k]{a_k}$. This will be the case in many applications.

The Root Test using Limits

Proof.

Recall that for a bounded sequence (x_k)

$$\overline{\lim_{k\to\infty}} x_k := \lim_{n\to\infty} \sup\{x_n, x_{n+1}, x_{n+2}, \dots\}.$$

Suppose that $\overline{\lim_{k\to\infty}} \sqrt[k]{a_k} =: p < 1$. Then there exists an $N \in \mathbb{N}$ such that

$$\left|\sup\{\sqrt[n]{a_n},\ \sqrt[n+1]{a_{n+1}}, \dots\} - p\right| < \frac{1-p}{2} \qquad \text{for all } n \geq N.$$

This implies

$$\sup\left\{\sqrt[n]{a_n},\ \sqrt[n+1]{a_{n+1}}, \dots\right\} < p + \frac{1-p}{2} =: q < 1$$

for all $n \geq N$, so the series converges by the Root Test 3.5.22. This proves the first statement. The proof of the second statement is analogous. □

The Ratio Test

The Root Test is extremely powerful, but sometimes quite difficult to handle. A more convenient criterion for convergence is provided by the ratio test:

3.5.28. Ratio Test. Let $\sum a_k$ be a series of strictly positive real numbers $a_k > 0$.

(i) Suppose that there exists a $q < 1$ such that

$$\frac{a_{k+1}}{a_k} \le q \qquad \text{for all sufficiently large } k.$$

Then $\sum a_k$ converges.

(ii) Suppose that

$$\frac{a_{k+1}}{a_k} \ge 1 \qquad \text{for all sufficiently large } k.$$

Then $\sum a_k$ diverges.

The Ratio Test

Proof.

(i) Suppose that we have a $q < 1$ such that $a_{k+1}/a_k \le q$ for all $k \ge k_0$. Then

$$a_k \le q a_{k-1} \le q^2 a_{k-2} \le a_{k_0} \cdot q^{k-k_0} \qquad \text{for } k \ge k_0.$$

Since the geometric series $\sum_{k=k_0}^{\infty} q^k$ converges for $q < 1$, the series $\sum_{k=k_0}^{\infty} a_k$ converges by the Comparison Test 3.5.15.

(ii) Suppose that $\frac{a_k}{a_{k-1}} \ge 1$ for all $k \ge k_0$. Then $a_k \ge a_{k-1}$, so $a_k \not\to 0$ as $k \to \infty$. By Corollary 3.5.5, the series $\sum a_k$ can not converge. $\square$

The Ratio Test

3.5.29. Example. Consider the series

$$\sum_{n=0}^{\infty} \frac{4^n}{n!}.$$

Applying the root test would be quite tedious, because we don't know how $\sqrt[n]{n!}$ behaves. However,

$$\frac{4^{n+1}}{(n+1)!}\frac{n!}{4^n} = \frac{4}{n} \le \frac{4}{5}$$

if $n \ge 5$, so the series converges.

The ratio test is easier to apply than the root test. However, it is formally not as powerful: there are series whose convergence or divergence can not be determined using the ratio test, but where the root test shows the convergence. An example will be given in the assignments.

The Ratio Test using Limits

3.5.30. Ratio Test. Let (a_k) be a sequence of strictly positive real numbers $a_k > 0$. Then

$$\varlimsup_{k\to\infty} \frac{a_{k+1}}{a_k} < 1 \qquad \Rightarrow \qquad \sum_{k=0}^{\infty} a_k \qquad \text{converges,}$$

$$\varliminf_{k\to\infty} \frac{a_{k+1}}{a_k} > 1 \qquad \Rightarrow \qquad \sum_{k=0}^{\infty} a_k \qquad \text{diverges.}$$

The proof is left to you; note the ***inferior limit*** in the condition for divergence!

The Ratio Comparison Test

We will now try to derive a finer version of the ratio test that will enable us to distinguish the divergence of the harmonic series from the convergence of the Euler series. A first step is the following criterion:

3.5.31. Ratio Comparison Test. Let (a_k) and (b_k) be sequences of strictly positive real numbers $a_k, b_k > 0$. Suppose that $\sum b_k$ converges. If

$$\frac{a_{k+1}}{a_k} \le \frac{b_{k+1}}{b_k} \qquad \text{for sufficiently large } k,$$

then $\sum a_k$ converges.

The Ratio Comparison Test

Proof.

The hypothesis $\frac{a_{k+1}}{a_k} \le \frac{b_{k+1}}{b_k}$ is equivalent to

$$\frac{a_{k+1}}{b_{k+1}} \le \frac{a_k}{b_k}.$$

Hence the sequence (a_k/b_k) is decreasing and positive (since $a_k, b_k > 0$ for all k).Thus (a_k/b_k) converges and is therefore bounded: there exists some $M > 0$ such that

$$\frac{a_k}{b_k} < M \quad \Leftrightarrow \quad a_k < M \cdot b_k.$$

Since $\sum b_k$ converges by assumption, so does $\sum (Mb_k)$. But then $\sum a_k$ converges by the Comparison Test 3.5.15. $\square$

Raabe's Test

The following is now a finer version of the ratio test:

3.5.32. Raabe's Test. Let $\sum a_k$ be a series of positive real numbers $a_k \geq 0$. Suppose that there exists a number $p > 1$ such that

$$\frac{a_{k+1}}{a_k} \leq 1 - \frac{p}{k} \qquad \text{for sufficiently large } k.$$

Then the series $\sum a_k$ converges.

3.5.33. Example. The series

$$\sum_{n=1}^{\infty} \frac{(2n)!}{4^n (n+1)! n!}$$

is convergent by Raabe's test.

Raabe's Test

Proof.

We first observe that if $p > 1$ and $0 \le x < 1$, then

$$1 - px \le (1 - x)^p. \tag{3.5.10}$$

At $x = 0$ we have equality, while the derivative of the left-hand side is less than the derivative of the right-hand side for $x \in (0, 1)$. We now apply (3.5.10) with $x = 1/k$ to our hypothesis: for sufficiently large k,

$$\frac{a_{k+1}}{a_k} \le 1 - \frac{p}{k} \le (1 - 1/k)^p = \left(\frac{k-1}{k}\right)^p = \frac{b_{k+1}}{b_k},$$

where $b_k := 1/(k-1)^p$. Since $\sum b_k$ converges, it follows from the Ratio Comparison Test 3.5.31 that $\sum a_k$ converges. $\square$

Absolutely and Conditionally Convergent Series

All of the tests that we have discussed essentially test series for absolute convergence. In complete vector spaces, this implies ordinary convergence. But what is more, absolutely convergent series have properties that series that converge without being absolutely convergent do not have.

3.5.34. Definition. A series in a normed vector space $(V, \|\cdot\|)$ is called conditionally convergent if it is convergent, but not absolutely convergent.

3.5.35. Example. We will see later that $\sum(-1)^k \frac{1}{k}$ is conditionally convergent.

Rearrangements of Terms in Series

For the proof of the following two theorems we refer to the text book (***Spivak***, Chapter 23, Theorems 7 & 8).

3.5.36. Theorem. Assume that $\sum a_k$ is an absolutely convergent series in a complete normed vector space. If the summands of the series are rearranged, the new series $\sum b_j$, $b_j = a_{k(j)}$, $k\colon \mathbb{N} \to \mathbb{N}$ bijective, converges absolutely with the same sum as $\sum a_k$.

In particular, if $\sum a_k$ and $\sum b_k$ are absolutely convergent, then $\sum (a_k + b_k)$ is absolutely convergent and equal to $\sum a_k + \sum b_k$.

Contrast this with the following result:

3.5.37. Theorem. Let $\sum a_k$ be a conditionally convergent series of real numbers. Then for any $\alpha \in \mathbb{R}$ there exists a rearrangement $b_j = a_{k(j)}$, $k\colon \mathbb{N} \to \mathbb{N}$ bijective, of $\sum a_k$ such that $\sum b_j = \alpha$.

The Leibniz Theorem

For this reason, we are mostly interested in absolutely convergent series. However, we give one result concerning conditionally convergent series.

3.5.38. Leibniz Theorem. Let $\sum \alpha_k$ be a complex series whose partial sums are bounded but need not converge. Let (a_k) be a decreasing convergent sequence with limit zero, $a_k \searrow 0$. Then the series

$$\sum \alpha_k a_k \qquad \text{converges.}$$

A typical application of the Leibniz Theorem 3.5.38 are alternating series, for which $\alpha_k = (-1)^k$. In particular, the Leibniz series

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$$

converges by the Leibniz Theorem.

Summation by Parts

3.5.39. Lemma. Let (a_k) and (b_k) denote complex sequences. Then for $m > n$

$$\sum_{k=n+1}^{m} a_k(b_k - b_{k-1}) = a_{m+1}b_m - a_{n+1}b_n - \sum_{k=n+1}^{m} (a_{k+1} - a_k)b_k. \tag{3.5.11}$$

Proof.

$$\begin{aligned}\sum_{k=n+1}^{m} a_k(b_k - b_{k-1}) &= \sum_{k=n+1}^{m} a_k b_k - \sum_{k=n+1}^{m} a_k b_{k-1} \\ &= \sum_{k=n+1}^{m} a_k b_k - \sum_{k=n}^{m-1} a_{k+1} b_k \\ &= \sum_{k=n+1}^{m} (a_k - a_{k+1})b_k + a_{m+1}b_m - a_{n+1}b_n \qquad \square\end{aligned}$$

The Leibniz Theorem

Proof of Theorem 3.5.38.

Denote by (s_n) the sequence of partial sums of (α_k),

$$s_n = \sum_{k=0}^{n} \alpha_k.$$

For $k \geq 1$, $\alpha_k = s_k - s_{k-1}$. Hence, applying (3.5.11),

$$\begin{aligned}\sum_{k=0}^{n} \alpha_k a_k &= \sum_{k=1}^{n} (s_k - s_{k-1}) a_k + \alpha_0 a_0 \\ &= \alpha_0 a_0 + s_n a_{n+1} - a_1 \alpha_0 - \sum_{k=1}^{n} (a_{k+1} - a_k) s_k\end{aligned}$$

We will show that the limit as $n \to \infty$ of the right-hand side exists.

The Leibniz Theorem

Proof of Theorem 3.5.38 (continued).

Since (a_k) is decreasing, $a_k - a_{k+1} > 0$ and therefore

$$\sum_{k=1}^{n} |a_{k+1} - a_k| = \sum_{k=1}^{n} (a_k - a_{k+1}) = a_1 - a_{n+1}$$

for all $n \in \mathbb{N} \setminus \{0\}$. We further assume $|s_k| \leq M$ for some $M > 0$. Thus

$$|(a_{k+1} - a_k)s_k| \leq M|a_{k+1} - a_k|.$$

Since $\sum |a_{k+1} - a_k|$ converges, the Comparison Test 3.5.15 implies that $\sum (a_{k+1} - a_k)s_k$ is absolutely convergent. Since $\mathbb{C}$ is complete, the series $\sum (a_{k+1} - a_k)s_k$ converges.

Furthermore, since $|s_k| \leq M$, $s_n a_{n+1} \to 0$ as $n \to \infty$. $\square$

The Cauchy Product and Convolution of Sequences

3.5.40. Theorem. Let $\sum a_k$ and $\sum b_k$ be absolutely convergent series. Then the ***Cauchy product*** $\sum c_k$ given by

$$c_k := \sum_{i+j=k} a_i b_j$$

converges absolutely and $\sum c_k = \left(\sum a_k\right)\left(\sum b_k\right)$.

3.5.41. Remark. If $a = (a_k)$ and $b = (b_k)$ are two absolutely summable sequences, the sequence

$$a * b := (c_k), \qquad c_k := \sum_{i+j=k} a_i b_j,$$

is called the ***convolution*** of a and b.

The Cauchy Product and Convolution of Sequences

Proof.

Let

$$A_n := \sum_{k=0}^{n} a_k, \qquad B_n := \sum_{k=0}^{n} b_k, \qquad C_n := \sum_{k=0}^{n} c_k.$$

We will show that $\lim\limits_{n\to\infty} |C_n - A_n B_n| = 0$. Now

$$|C_n - A_n B_n| = \Bigl| \sum_{k=0}^{n} \sum_{i+j=k} a_i b_j - \sum_{i,j=0}^{n} a_i b_j \Bigr| \le \sum_{\substack{i+j>n \\ 0\le i,j\le n}} |a_i b_j| =: R_n$$

If $i + j > n$, at least one of i and j will be greater than $n/2$. Thus

$$R_n \le \sum_{n/2\le i\le n} |a_i| \sum_{j=0}^{n} |b_j| + \sum_{i=0}^{n} |a_i| \sum_{n/2\le j\le n} |b_j|$$

The Cauchy Product and Convolution of Sequences

Proof (continued).

We have

$$R_n \le \underbrace{\sum_{n/2\le i} |a_i|}_{\to 0} \cdot \underbrace{\sum_{j=0}^{n} |b_j|}_{\text{bounded}} + \underbrace{\sum_{i=0}^{n} |a_i|}_{\text{bounded}} \cdot \underbrace{\sum_{n/2\le j} |b_j|}_{\to 0}$$

$$\to 0 \quad \text{as } n \to \infty.$$

This implies that $\sum c_k = \left(\sum a_k\right)\left(\sum b_k\right)$. Furthermore, $\sum c_k$ is absolutely convergent, because

$$\sum |c_k| \le \left(\sum |a_k|\right)\left(\sum |b_k|\right) < \infty.$$

□

Series of Monomials

In this section we would like to study expressions of the type

$$f(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots$$

where $z \in \mathbb{C}$ and (a_k) is a sequence of complex numbers. At first, it might seem that we should simply write

$$f(z) = \sum_{k=0}^{\infty} a_k z^k \tag{3.6.1}$$

and then discuss the convergence of the series: the partial sums on the right-hand side of (3.6.1) are polynomials and defined on all of $\mathbb{C}$, and the Weierstraß M-test would allow us to check whether the series converges.

However, for most sequences (a_k), the series will only converge if the domain of the z is restricted to a subset of $\mathbb{C}$. This makes the discussion of convergence a little bit more subtle.

Formal Power Series

3.6.1. Definition. For any complex sequence (a_k), the expression

$$\sum_{k=0}^{\infty} a_k z^k, \qquad \text{or simply} \qquad \sum a_k z^k \tag{3.6.2}$$

is called a ***formal power series*** or just ***power series***.

We can think fo a power series as a sequence of coefficients (a_k) that we would like to multiply with z^k and sum, at least for certain values of z. However, until we determine for which z this is possible, (3.6.2) is just a formal symbol with no meaning beyond listing the sequence (a_k).

3.6.2. Remark. We will sometimes use notation such as

$$\sum_{k=1}^{\infty} a_k z^{k-1}, \qquad \text{to mean} \qquad \sum_{k=0}^{\infty} a_{k+1} z^k \tag{3.6.3}$$

in the obvious manner.

Convergence of Power Series

3.6.3. Definition. Let $\sum a_k z^k$ be a power series and $z_0 \in \mathbb{C}$. Then the power series is said to be ***(absolutely) convergent at z_0*** if the series (of numbers)

$$\sum_{k=0}^{\infty} a_k z_0^k$$

converges (absolutely).

Given a power series, we are interested in finding the set of all $z \in \mathbb{C}$ for which the series converges.

Addition and Multiplication of Power Series

If $\sum a_k z^k$ and $\sum b_k z^k$ are two formal power series that both converge absolutely at $z_0 \in \mathbb{C}$, we have

$$\sum_{k=0}^{\infty} a_k z_0^k + \sum_{k=0}^{\infty} b_k z_0^k = \sum_{k=0}^{\infty} (a_k + b_k) z_0^k.$$

Independent of whether or not they converge, it therefore makes sense to define

$$\sum_{k=0}^{\infty} a_k\, z^k + \sum_{k=0}^{\infty} b_k\, z^k := \sum_{k=0}^{\infty} (a_k + b_k)\, z^k$$

for formal power series $\sum a_k\, z^k$ and $\sum b_k\, z^k$.

Addition and Multiplication of Power Series

The Cauchy product is well-suited for multiplication of power series: if $\sum a_k z^k$ and $\sum b_k z^k$ both converge absolutely at $z_0 \in \mathbb{C}$, then

$$\Big(\sum_{k=0}^{\infty} a_k\, z_0^k\Big)\Big(\sum_{k=0}^{\infty} b_k\, z_0^k\Big) = \sum_{k=0}^{\infty} \sum_{i+j=k} a_i z_0^i b_j z_0^j = \sum_{k=0}^{\infty}\Big(\sum_{i+j=k} a_i b_j\Big) z_0^k$$

converges absolutely. Hence, we define the product of formal power series $\sum a_k\, z^k$ and $\sum b_k\, z^k$ by

$$\Big(\sum_{k=0}^{\infty} a_k\, z^k\Big)\Big(\sum_{k=0}^{\infty} b_k\, z^k\Big) := \Big(\sum_{k=0}^{\infty} (a * b)_k\, z^k\Big),$$

where $*$ denotes the convolution of sequences.

Radius of Convergence

3.6.4. Definition and Theorem. Let $\sum a_k z^k$ be a complex power series. Then there exists a unique number $\varrho \in [0, \infty) \cup \{\infty\}$ such that

(i) $\sum a_k z^k$ is absolutely convergent at $z_0 \in \mathbb{C}$ if $|z_0| < \varrho$,

(ii) $\sum a_k z^k$ diverges at $z_0 \in \mathbb{C}$ if $|z_0| > \varrho$.

This number is called the ***radius of convergence*** of the power series.

In particular, ϱ is given by ***Hadamard's formula***,

$$\varrho = \begin{cases} \dfrac{1}{\overline{\lim} \sqrt[k]{|a_k|}}, & 0 < \overline{\lim\limits_{k\to\infty}} \sqrt[k]{|a_k|} < \infty, \\ 0, & \overline{\lim\limits_{k\to\infty}} \sqrt[k]{|a_k|} = \infty, \\ \infty, & \overline{\lim\limits_{k\to\infty}} \sqrt[k]{|a_k|} = 0. \end{cases} \tag{3.6.4}$$

Radius of Convergence

3.6.5. Remark. No information is given about the convergence of $\sum a_k z^k$ at z_0 if $|z_0| = \varrho$. In this case the series may converge or diverge, depending on the point z_0.

3.6.6. Example. The formal power series $\sum_{k=0}^{\infty} \frac{(-1)^k}{k} z^k$ has radius of convergence $\varrho = 1$. The series converges for $z_0 = 1$ and diverges for $z_0 = -1$. Other values of z_0 with $|z_0| = 1$ can be checked individually.

Radius of Convergence

Proof of Theorem 3.6.4.

It is sufficient to prove that ϱ defined by Hadamard's formula (3.6.4) has the properties (i) and (ii) (Clearly, no other number can have these properties.)

Let $\sum a_k z^k$ be a convergent power series with $\varrho \in (0, \infty)$ given by (3.6.4). (The cases $\varrho = 0$ and $\varrho = \infty$ are left to the reader.) Let $z_0 \in \mathbb{C}$ be given.

To check the convergence of $\sum a_k z^k$ at z_0, we use the root test and calculate

$$\varlimsup_{k\to\infty} \sqrt[k]{|a_k z_0^k|} = |z_0| \varlimsup_{k\to\infty} \sqrt[k]{|a_k|} = \frac{|z_0|}{\varrho}.$$

If $|z_0| < \varrho$, then the quotient is strictly less than 1 and the series diverges. If $|z_0| > \varrho$, the series diverges. This verifies properties (i) and (ii). $\square$

Radius of Convergence

3.6.7. Lemma. Let $\sum_{k=0}^{\infty} a_k z^k$ be a complex power series with radius of convergence ϱ. Then the series $\sum_{k=1}^{\infty} k a_k z^{k-1}$ has the same radius of convergence ϱ.

Proof.
We note that for any $z_0 \in \mathbb{C}$,

$$\sum_{k=1}^{\infty} k a_k z_0^{k-1} \quad \text{converges}$$

if and only if

$$z_0 \sum_{k=1}^{\infty} k a_k z_0^{k-1} = \sum_{k=1}^{\infty} k a_k z_0^{k} \quad \text{converges.}$$

Therefore, we show that the radius of convergence of the power series $\sum k a_k z^k$ equals ϱ.

Radius of Convergence

Proof (continued).

Suppose that $0 < \varrho < \infty$. (The cases $\varrho = 0$ and $\varrho = \infty$ are left to the reader.) Then

$$\varrho = \frac{1}{\overline{\lim\limits_{k\to\infty}} \sqrt[k]{|a_k|}}.$$

Suppose that $|z| < \varrho$. Then

$$1 > \frac{|z|}{\varrho} = \overline{\lim_{k\to\infty}} \sqrt[k]{|a_k||z|^k} = \overline{\lim_{k\to\infty}} \sqrt[k]{|ka_k z^k|/k}.$$

This implies that there exists a $q < 1$ such that

$$\sqrt[k]{|ka_k z^k|/k} \le q \qquad \text{and, hence,} \qquad |ka_k z^k| \le kq^k$$

for sufficiently large k. By Example 3.5.24 the series $\sum kq^k$ converges, so the Comparison Test 3.5.15 shows that $\sum ka_k z^k$ converges absolutely.

Uniform Convergence of Power Series

Proof (continued).

Similarly, if $|z| > \varrho$, then $\overline{\lim\limits_{k\to\infty}} \sqrt[k]{|ka_k z^k|/k} > 1$ and $|ka_k z^k| > kp^k$ for some $p \geq 1$. This shows that $\sum ka_k z^k$ diverges. Therefore, the radius of convergence of $\sum ka_k z^k$ must be ϱ. $\square$

Recall from Definition 3.5.20 that a series $\sum_{k=0}^{\infty} f_k$ of continuous functions $f_k \colon \Omega \to \mathbb{C}$ converges uniformly to a function f if

$$\sup_{x\in\Omega}\left|f(x) - \sum_{k=0}^{n} f_k(x)\right| \xrightarrow{n\to\infty} 0.$$

In that case, the function f is continuous by Theorem 3.4.3. We now apply these general results to power series, where

$$f_k(x) = a_k x^k.$$

Uniform Convergence of Power Series

3.6.8. Lemma. If $\sum a_k z^k$ is a complex power series with radius of convergence ϱ, then for any $R < \varrho$ the series converges uniformly on $B_R(0) = \{z : |z| < R\}$.

Proof.
Let $0 < R < \varrho$ be fixed. Then

$$\sup_{z \in B_R(0)} |a_k z^k| < |a_k| R^k =: M_k$$

Now

$$\sum_{k=0}^{\infty} M_k = \sum_{k=0}^{\infty} |a_k| R^k$$

converges, because the point $R < \varrho$ lies within the radius of convergence of the series and the series convergences absolutely within its radius of convergence. By the Weierstraß M-test 3.5.19, the series $\sum a_k z^k$ of functions $a_k z^k : B_R(0) \to \mathbb{C}$ converges uniformly on $B_R(0)$. $\square$

Continuity of Power Series

3.6.9. Corollary. A power series $\sum_{k=0}^{\infty} a_k z^k$ with radius of convergence ϱ defines a continuous function

$$f\colon B_\varrho(0) \to \mathbb{C}, \qquad\qquad f(z) = \sum_{k=0}^{\infty} a_k z^k.$$

Proof.
Let $z \in \mathbb{C}$ have modulus $|z| < \varrho$. Define $R := |z| + (\varrho - |z|)/2 < \varrho$. Then the power series converges uniformly on $B_R(0)$ and is hence continuous on $B_R(0)$, which includes the point z. $\square$

3.6.10. Example. The power series $\sum_{k=0}^{\infty}(-1)^k x^k$ converges uniformly on $[-R, R]$ for any $0 < R < 1$. However, it does not converge uniformly on $(-1, 1)$. (Why?) However, the series defines a continuous function on the interval $(-1, 1)$.

Differentiability of Power Series

3.6.11. Remark. The definition of the derivative can be extended to functions of a single complex variable; in particular, if $\Omega \subset \mathbb{C}$ is an open set, we say that a function $f\colon \Omega \to \mathbb{C}$ is differentiable at $z \in \Omega$ if there exists a number $f'(z) \in \mathbb{C}$ such that

$$f(z+h) = f(z) + f'(z)h + o(h), \qquad h \to 0.$$

3.6.12. Theorem. The power series $\sum a_k z^k$ with radius of convergence ϱ defines a differentiable function $f\colon B_\varrho(0) \to \mathbb{C}$. Furthermore,

$$f'(z_0) = \sum k a_k z_0^{k-1} \qquad \text{for } z_0 \in B_\varrho(0), \tag{3.6.5}$$

where the series on the right has the same radius of convergence ϱ.

Differentiability of Power Series

Proof.

The statement that the series (3.6.5) has the same radius of convergence as f was established in Lemma 3.6.7. For any $z \in B_\varrho(0)$ we define

$$g(z) := \sum_{k=1}^{\infty} k a_k z^{k-1}$$

and set

$$f(z) = S_N(z) + E_N(z), \quad S_N(z) := \sum_{k=0}^{N} a_k z^k, \quad E_N(z) := \sum_{k=N+1}^{\infty} a_k z^k.$$

Fix $z_0 \in B_r(0)$ with $r < \varrho$. We want to show that for any $\varepsilon > 0$ there exists a $\delta > 0$ so that

$$\left| \frac{f(z_0 + h) - f(z_0)}{h} - g(z_0) \right| < \varepsilon \qquad \text{whenever } |h| < \delta.$$

Differentiability of Power Series

Proof (continued).

This is enough to show that f is differentiable at z_0 and $f'(z_0) = g(z_0)$.
Fix $\varepsilon > 0$ and choose some δ such that $|z_0 + h| < r$ if $h \in B_\delta(z_0)$.

$$\begin{aligned}\frac{f(z_0+h)-f(z_0)}{h} - g(z_0) = \bigg(&\frac{S_N(z_0+h)-S_N(z_0)}{h} - S_N'(z_0)\bigg) \\ &+ (S_N'(z_0)-g(z_0)) + \frac{E_N(z_0+h)-E_N(z_0)}{h}\end{aligned}$$

We will show that each of the three terms on the right can be made small so that the modulus of the difference on the left is less than ε. We start with

$$\left|\frac{E_N(z_0+h)-E_N(z_0)}{h}\right| \le \sum_{k=N+1}^{\infty} \frac{|a_k|}{|h|}\left|(z_0+h)^k - z_0^k\right|.$$

Differentiability of Power Series

Proof (continued).

Using that for $k \in \mathbb{N} \setminus \{0\}$ and $a, b \in \mathbb{C}$,

$$a^k - b^k = (a - b) \sum_{i=0}^{k-1} a^i b^{k-1-i}$$

and $|z_0|, |z_0 + h| < r$, we have

$$\left| \frac{E_N(z_0 + h) - E_N(z_0)}{h} \right| \le \sum_{k=N+1}^{\infty} k|a_k| r^{k-1}. \tag{3.6.6}$$

Since the series $\sum k|a_k|r^k$ converges, we can choose N_1 large enough so that the right-hand side of (3.6.6) is less than $\varepsilon/3$ for $N > N_1$.

Differentiability of Power Series

Proof (continued).
Since $\lim_{N\to\infty} S_N'(z_0) = g(z_0)$ we can also find $N_2 > 0$ such that

$$|S_N'(z_0) - g(z_0)| < \frac{\varepsilon}{3}$$

for $N > N_2$. Let us now fix $N > \max(N_1, N_2)$. Finally, because S_N is differentiable (a polynomial), we can find a $\delta > 0$ small enough so that

$$\left| \frac{S_N(z_0 + h) - S_N(z_0)}{h} - S_N'(z_0) \right| < \frac{\varepsilon}{3}$$

for $|h| < \delta$. □

Some Applications

An application of our theoretical developments on power series is the ability to define a ***decimal expansion*** for any real number $a \in \mathbb{R}$. We want to write

$$a = a_0.a_1a_2a_3\ldots$$

where $a_0 \in \mathbb{Z}$ and $a_1, a_2, \ldots \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. We realize this by defining

$$a = a_0 + \sum_{k=1}^{\infty} a_k 10^{-k}.$$

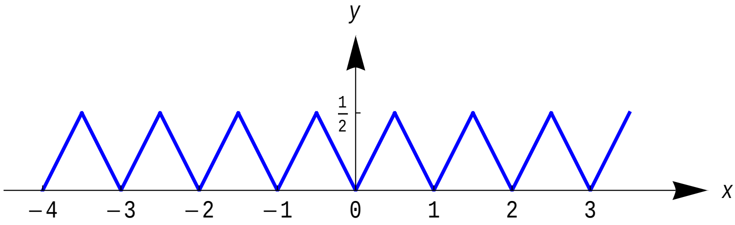
How this can be used as a definition for the set of real numbers is sketched out in Chapter 29, Exercise 2 of the textbook.

The Van der Waerden Function

We will now construct the continuous and nowhere-differentiable Van der Waerden function. We use the "distance to the nearest integer" function defined in (2.3.1),

$$\{\,\cdot\,\}\colon \mathbb{R} \to [0, 1/2], \qquad \{x\} := \min\{|x - \lceil x \rceil|, |x - \lfloor x \rfloor|\}.$$

The function is sketched below:

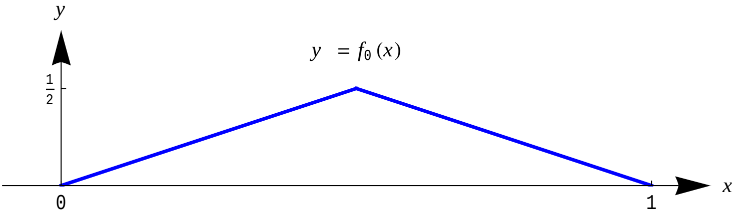


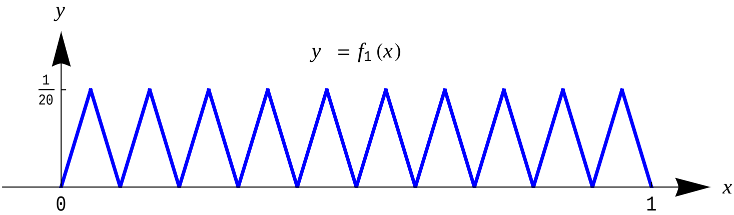
The Van der Waerden Function

We now define a sequence of functions $(f_n)_{n\in\mathbb{N}}$ by

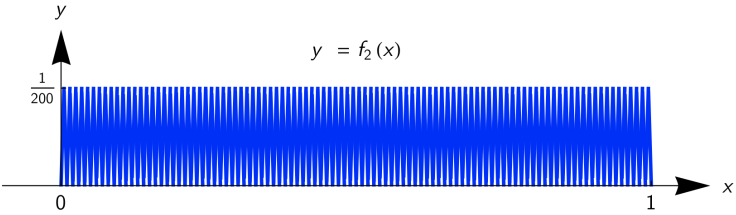
$$f_n(x) := 10^{-n}\{10^n x\}.$$

The graphs of the first few functions are plotted below:





The Van der Waerden Function



We then set

$$W(x) := \sum_{n=0}^{\infty} f_n(x).$$

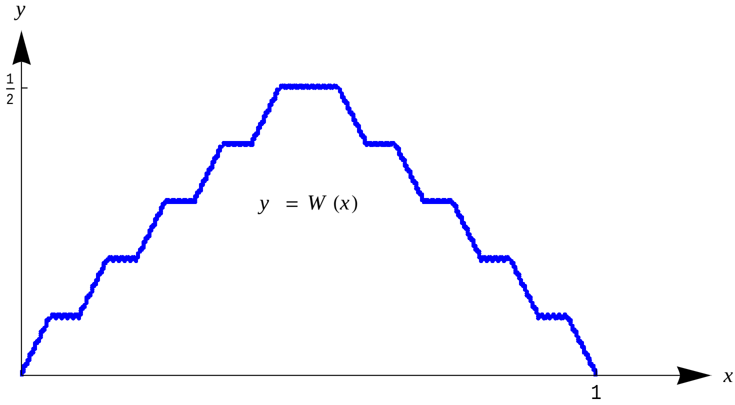
From the definition of the f_n it is immediate that

$$\sup_{x\in\mathbb{R}} |f_n(x)| = \frac{1}{2} \cdot 10^{-n}$$

so by the Weierstraß M-Test 3.5.19 we see that W defines a continuous real function on $\mathbb{R}$.

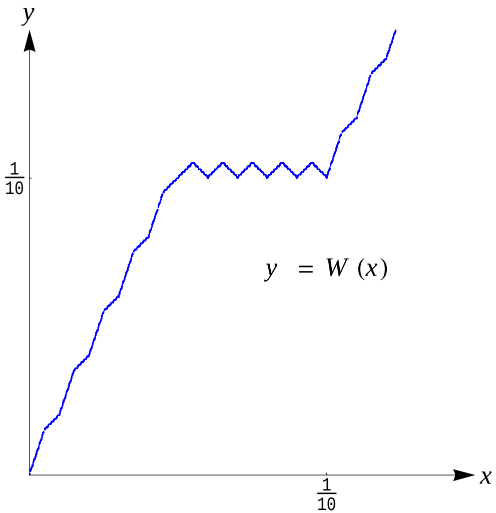
The Van der Waerden Function

An approximation to the function W is plotted below:



The Van der Waerden Function

A closer look at the graph on the interval $(0, 1/10)$ shows the fractal nature of the function:



The Van der Waerden Function

3.6.13. Theorem. The Van der Waerden function

$$W\colon \mathbb{R} \to \mathbb{R}, \qquad\qquad W(x) = \sum_{n=0}^{\infty} 10^{-n}\{10^n x\}$$

is continuous but not differentiable at any point of its domain.

Proof.
Due to the periodicity of W, it suffices to show the assertion for points $a \in (0,1]$. We have already established the continuity of W, so it remains to show that W is not differentiable anywhere. We will do this by fixing $a \in (0,1]$ and exhibiting a sequence (h_m) with $h_m \to 0$ as $m \to \infty$ such that

$$\lim_{m\to\infty} \frac{W(a+h_m) - W(a)}{h_m} \tag{3.6.7}$$

does not exist.

The Van der Waerden Function

Proof (continued).
Suppose that a has the decimal expansion

$$a = 0.a_1 a_2 a_3 \ldots$$

where $a_m \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$, $m \in \mathbb{N} \setminus \{0\}$. We then define

$$h_m := \begin{cases} -10^{-m} & \text{if } a_m = 4 \text{ or } a_m = 9, \\ 10^{-m} & \text{otherwise.} \end{cases}$$

Then

$$\frac{W(a + h_m) - W(a)}{h_m} = \sum_{n=0}^{\infty} \pm 10^{m-n} (\{10^n(a + h_m)\} - \{10^n a\}) \qquad (3.6.8)$$

where the "$\pm$" indicates a "$+$" or "$-$" sign, depending on m.

The Van der Waerden Function

Proof (continued).

The series in (3.6.8) is actually finite, because for $n \geq m$ the term $10^n h_m \in \mathbb{N}$ and then $\{10^n(a + h_m)\} = \{10^n a\}$. When $n < m$ we have

$$10^n a = A + 0.a_{n+1} a_{n+2} \ldots a_{m-1} a_m a_{m+1} \ldots, \tag{3.6.9}$$

$$10^n(a + h_m) = A + 0.a_{n+1} a_{n+2} \ldots a_{m-1}(a_m \pm 1) a_{m+1} \ldots \tag{3.6.10}$$

where $A \in \mathbb{N}$ and the "$\pm$" again depends on m. Our choice of the minus sign when $a_m = 9$ is essential to ensure that (3.6.8) holds.

Now suppose that the non-integer part on the right of (3.6.9), $10^n a - A$, is less than $1/2$. Then also the non-integer part on the right of (3.6.10) will be less than $1/2$. (For this to be true even when $m = n + 1$, it is essential that we choose the minus sign for h_m when $a_m = 4$.) It follows that

$$\{10^n(a + h_m)\} - \{10^n a\} = \pm 10^{n-m} \qquad \text{when } n < m. \tag{3.6.11}$$

The Van der Waerden Function

Proof (continued).

It is not difficult to see that (3.6.11) also holds when $10^n a - A \geq 1/2$. We hence have

$$10^{m-n}(\{10^n(a+h_m)\} - \{10^n a\}) = \pm 1 \qquad \text{when } n < m.$$

Then (3.6.8) becomes

$$\frac{W(a+h_m) - W(a)}{h_m} = \sum_{n=0}^{m} \pm 1 =: S_m.$$

Now the sequence (S_m) satisfies

$$|S_{m+1} - S_m| = 1,$$

for all $m \in \mathbb{N}$. Thus, it is not a Cauchy sequence and can not converge. It follows that the limit (3.6.7) does not exist. $\square$

The Exponential Function

We can now introduce one of the most fundamentally important functions in calculus:

3.7.1. Definition. We define the ***exponential function***

$$\exp\colon \mathbb{C} \to \mathbb{C}, \qquad \exp z := \sum_{n=0}^{\infty} \frac{z^n}{n!}. \tag{3.7.1}$$

We similarly define its restriction to the real numbers,

$$\exp\colon \mathbb{R} \to \mathbb{R}, \qquad \exp x := \sum_{n=0}^{\infty} \frac{x^n}{n!}. \tag{3.7.2}$$

The series (3.7.1) converges for all $z \in \mathbb{C}$ ($\varrho = \infty$), so it defines a continuous function for all $z \in \mathbb{C}$ ($x \in \mathbb{R}$).

We will now investigate the various properties of the exponential function.

The Exponential Function

The ***functional equation*** for the exponential function is

$$(\exp x)(\exp y) = \exp(x+y), \qquad\qquad x, y \in \mathbb{C},$$

which can be shown using the Cauchy product as follows.

$$\begin{aligned}
(\exp x)(\exp y) &= \Big(\sum_{n=0}^{\infty} \frac{x^n}{n!}\Big)\Big(\sum_{m=0}^{\infty} \frac{y^m}{m!}\Big) = \sum_{n=0}^{\infty} \bigg(\Big(\frac{x^k}{k!}\Big) * \Big(\frac{y^k}{k!}\Big)\bigg)_n \\
&= \sum_{n=0}^{\infty} \sum_{l+m=n} \frac{1}{l!m!} x^l y^m \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{l+m=n} \frac{n!}{l!m!} x^l y^m \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} (x+y)^n \\
&= \exp(x+y)
\end{aligned}$$

The Exponential Function

Note that $\exp(0) = 1$ from the series representation

$$\exp(0) = 1 + 0 + \frac{0^2}{2!} + \cdots = 1.$$

We next show that $\exp(z) \neq 0$ for all $z \in \mathbb{C}$. Assume that $\exp(z_0) = 0$ for some $z_0 \in \mathbb{C}$. Then

$$1 = \exp(0) = \exp(z_0 - z_0) = \exp(z_0)\exp(-z_0) = 0 \cdot \exp(-z_0) = 0,$$

which is impossible. Since $\exp(0) = 1$ and exp is continuous, we can deduce from the intermediate value theorem that

$$\exp(x) > 0 \qquad \text{for all } x \in \mathbb{R}.$$

The Exponential Function

We can also calculate the derivative of the exponential function:

$$\exp(x+h) = \exp(x)\exp(h) = \exp(x)(1+h+o(h)) = \exp(x)+\exp(x)h+o(h),$$

so

$$\exp'(x) = \exp(x).$$

In fact, it turns out that exp is the only function satisfying the ***initial value problem***

$$y'(x) = y(x), \qquad\qquad y(0) = 1. \tag{3.7.3}$$

Here the equation on the left is called a ***differential equation***; its solution is a function y such that the derivative of y is equal to y. The second equation is called an ***initial condition***, because it specifies the value of y at for some $x_0 \in \mathbb{R}$.

The Exponential Function

If an initial value problem has a unique solution, it can of course be used to define this solution. Thus we could use (3.7.3) to define exp, if we were able to prove that there exists a unique solution to (3.7.3). Since we have already defined exp, this is not really necessary. Nevertheless, it is instructive to prove the uniqueness of the solution.

Assume that two functions f, g both solve (3.7.3). Then

$$(f(x)\exp(-x))' = (f'(x) - f(x))\exp(-x) = 0 = (g(x)\exp(-x))'$$

Now by Corollary 3.2.9 $f(x)\exp(-x) = g(x)\exp(-x) + C$ for some $C \in \mathbb{R}$. Setting $x = 0$ we see

$$C = f(0)\exp(0) - g(0)\exp(0) = 1 \cdot 1 - 1 \cdot 1 = 0,$$

so $f(x)\exp(-x) = g(x)\exp(-x)$ or (multiplying with $\exp(x)$) $f = g$. Thus exp is the ***only*** function satisfying (3.7.3).

The Exponential Function

The function $\exp\colon \mathbb{R} \to \mathbb{R}$ satisfies

$$\exp'' = \exp' = \exp > 0,$$

so the real exponential function is increasing and convex. There are no local extrema. Furthermore,

$$\exp(x) > 1 + x \qquad\qquad \text{for } x > 0,$$

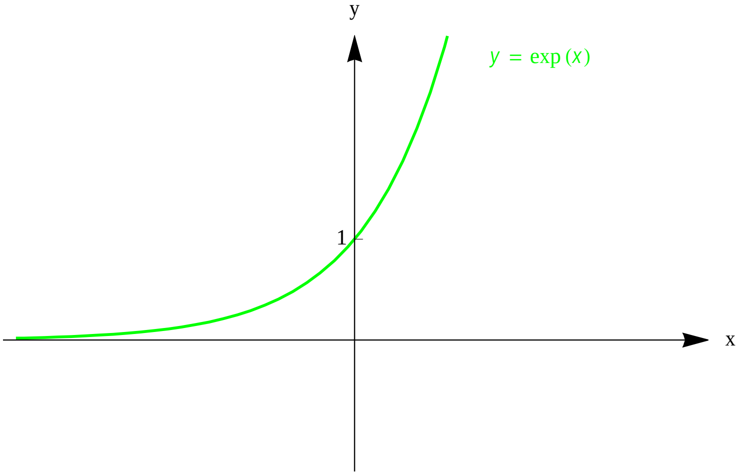
so $\lim\limits_{x\to\infty} \exp(x) = \infty$. Since

$$1 = \exp(0) = \exp(x - x) = \exp(x)\exp(-x) \quad\Leftrightarrow\quad \exp(-x) = 1/\exp(x),$$

we see that $\lim\limits_{x\to-\infty} \exp(x) = \lim\limits_{x\to\infty} \exp(-x) = 0$.

The Exponential Function

We thus obtain the following plot of the exponential function:



Compound Interest

We now analyze a problem that will turn out to have an intrinsic connection to the exponential function and the Euler number. Consider the question of compound interest. If money is left in a bank savings account for one year, the bank will return $(1 + z)$ times the original amount after the year is over. (For example, at 5% interest, $z = 0.05$.)

However, if the money is withdrawn after half a year, tthe bank will return $(1 + z/2)$ times the investment. If this money is then left with the bank for another half-year, the total amount returned is

$$\left(1 + \frac{z}{2}\right)^2 > 1 + z$$

times the original investment. Thus it seems advantageous to withdraw money after the nth part of a year and replace it in the savings account immediately. Repeating this n times, one earns

$$\left(1 + \frac{z}{n}\right)^n$$

times the original investment.

Compound Interest

One would perhaps like to choose n as large as possible. The question arises: will the return approach a limit? The answer is given by the following proposition:

3.7.2. Proposition. Let $z \in \mathbb{C}$. Then $\lim_{n\to\infty}\left(1+\frac{z}{n}\right)^n = \exp z$

In real life, banks have taken this into account, and publish the ***effective annual interest rate***, also called the ***Annual Equivalent Rate (AER)*** which takes into account the compound interest.

Proof.
From the functional equation for the exponential function we have

$$\exp(z) - \left(1+\frac{z}{n}\right)^n = \underbrace{\left(\exp\left(\frac{z}{n}\right)\right)^n}_{=:a^n} - \underbrace{\left(1+\frac{z}{n}\right)^n}_{=:b^n}.$$

Compound Interest

Proof (continued).

Since

$$a^n - b^n = (a-b)\underbrace{\sum_{k=0}^{n-1} a^k b^{n-1-k}}_{=:\varrho}$$

we analyze first $a - b$:

$$a - b = \exp\frac{z}{n} - \left(1 + \frac{z}{n}\right) = \frac{z}{n}\sum_{k=1}^{\infty}\frac{1}{(k+1)!}\left(\frac{z}{n}\right)^k = \frac{z}{n}\cdot o(1)$$

as $n \to \infty$.

Compound Interest

Proof (continued).

Furthermore,

$$
\begin{aligned}
|\varrho| &\le \sum_{k=0}^{n-1} \left|\exp \frac{z}{n}\right|^k \left|1 + \frac{z}{n}\right|^{n-1-k} \\
&\le \sum_{k=0}^{n-1} \left(\exp \frac{|z|}{n}\right)^k \left(1 + \frac{|z|}{n}\right)^{n-1-k} \\
&\le \sum_{k=0}^{n-1} \left(\exp \frac{|z|}{n}\right)^k \left(\exp \frac{|z|}{n}\right)^{n-1-k} \\
&= n \left(\exp \frac{|z|}{n}\right)^{n-1} \\
&\le n \exp(|z|)
\end{aligned}
$$

The Euler Number

Proof (continued).
Hence

$$\left|\exp(z) - \left(1 + \frac{z}{n}\right)^n\right| \le |a - b| \cdot |\varrho| \le \frac{|z|}{n} n \exp(|z|) \cdot o(1) \to 0$$

as $n \to \infty$. □

3.7.3. Definition. The number

$$e := \exp(1)$$

is called the ***Euler number***.

It can be proven that e is irrational and transcendental (not the solution of an algebraic equation). An approximate value is

$$e \approx 2.7182818284590$$

The Euler Number

3.7.4. Lemma. The Euler number is the monotonic limit of the sequence in Proposition 3.7.2, i.e.,

$$\left(1+\frac{1}{n}\right)^n \nearrow e \qquad\qquad \text{as } n \to \infty.$$

Proof.
By the binomial theorem,

$$\begin{aligned}\left(1+\frac{1}{n}\right)^n &= \sum_{k=0}^{n} \frac{n!}{k!(n-k)!n^k} = \sum_{k=0}^{n} \frac{1}{k!}\frac{n}{n}\frac{n-1}{n}\cdots\frac{n-k+1}{n}\\ &= \sum_{k=0}^{n} \frac{1}{k!}\left(1-\frac{1}{n}\right)\cdots\left(1-\frac{k-1}{n}\right)\\ &< \sum_{k=0}^{n+1} \frac{1}{k!}\left(1-\frac{1}{n+1}\right)\cdots\left(1-\frac{k-1}{n+1}\right) = \left(1+\frac{1}{n+1}\right)^{n+1}. \square\end{aligned}$$

The Euler Number

3.7.5. Proposition. For any rational number $q \in \mathbb{Q}$, $e^q = \exp(q)$.

Proof.
We first show that $e^n = \exp(n)$ for $n \in \mathbb{N}$ by induction. For $n = 0$ we have $e^0 = 1 = \exp(0)$. Furthermore,

$$\exp(n+1) = \exp(n) \cdot \exp(1) = e^n \cdot e = e^{n+1}.$$

Moreover, for $n \in \mathbb{N}$ we have $\exp(n)\exp(-n) = 1$, so

$$\exp(-n) = \frac{1}{\exp(n)} = \frac{1}{e^n} =: e^{-n}.$$

Thus the statement holds for all $n \in \mathbb{Z}$.

The Euler Number

Proof (continued).
Now let $x = p/q$ with $p \in \mathbb{Z}$ and $q \in \mathbb{N} \setminus \{0\}$. Then

$$\left(\exp \frac{p}{q}\right)^q = \exp p = e^p$$

and thus

$$\exp \frac{p}{q} = \sqrt[q]{e^p} =: e^{p/q}. \qquad \square$$

Real Exponents, Logarithm

The exponential function is continuous (even C^∞) on $\mathbb{R}$, and for rational x coincides with e^x. We do not yet have a definition for e^x when x is real, so it is logical to define

$$e^x := \exp x \qquad\qquad \text{for } x \in \mathbb{R}$$

We say that $\exp x$ is a ***continuous extension*** of e^x to the real numbers. It is automatically the only such extension.

We have seen that the function $\exp\colon \mathbb{R} \to \mathbb{R}_+$ ($\mathbb{R}_+ := \{x \in \mathbb{R} \colon x > 0\}$) is increasing and hence bijective. Thus there exists an inverse function, which we call the ***(natural) logarithm*** and denote by $\ln\colon \mathbb{R}_+ \to \mathbb{R}$.

3.7.6. Lemma. The logarithm satisfies the functional equation

$$\ln(u \cdot v) = \ln(u) + \ln(v).$$

for all $u, v \in \mathbb{R}_+$

Real Exponents, Logarithm

Proof.

Let $u = e^x$, $v = e^y$. Then

$$\ln(u \cdot v) = \ln(e^x \cdot e^y) = \ln(e^{x+y}) = x + y = \ln(u) + \ln(v).$$ □

In a similar manner we can use the properties of the exponential function to prove

$$\ln \frac{1}{u} = -\ln(u), \qquad \ln(u^{p/q}) = \frac{p}{q}\ln(u) \qquad (p \in \mathbb{Z},\ q \in \mathbb{N} \setminus \{0\}).$$

This motivates the following definition:

3.7.7. Definition. For all $a > 0$ and $x \in \mathbb{R}$ we define

$$a^x := e^{x \ln(a)}.$$

Real Exponents, Logarithm

We immediately obtain

$$\ln a^x = x \ln a, \qquad (a^x)^y = e^{\ln[(a^x)^y]} = e^{yx \ln a} = a^{xy}, \qquad a^{x+y} = a^x a^y$$

and so on for all $a > 0$ and $x, y \in \mathbb{R}$.

3.7.8. Remark. Since

$$e^x = 1 + x + \frac{x^2}{2!} + \dots,$$

we see that $e^x > x^n/n!$ for any n and $x > 0$. In particular, $\lim\limits_{x\to\infty} e^x/x^n = \infty$ for any $n \in \mathbb{N}$:

$$\frac{e^x}{x^n} > \frac{1}{(n+1)!}\frac{x^{n+1}}{x^n} = \frac{x}{(n+1)!} \xrightarrow{x\to\infty} \infty.$$

Real Exponents, Logarithm

3.7.9. Theorem. For any $\alpha > 0$ we have

$$\lim_{x\to\infty} \frac{\ln x}{x^\alpha} = 0, \qquad\qquad \lim_{x\searrow 0} x^\alpha \ln x = 0.$$

Proof.
Letting $y = \ln x$, we have

$$\lim_{x\to\infty} \frac{\ln x}{x^\alpha} = \frac{1}{\alpha}\lim_{y\to\infty} \frac{\alpha y}{e^{\alpha y}} = \frac{1}{\alpha}\lim_{t\to\infty} \frac{t}{e^t} = 0$$

Furthermore, with $y = 1/x$,

$$\lim_{x\searrow 0} x^\alpha \ln x = \lim_{y\to\infty} \frac{1}{y^\alpha} \ln\frac{1}{y} = -\lim_{y\to\infty} \frac{\ln y}{y^\alpha} = 0.$$

□

Real Exponents, Logarithm

3.7.10. Corollary. Since the exponential function is continuous, we see that for $n \in \mathbb{N}$

$$\sqrt[n]{n} = n^{1/n} = e^{\frac{1}{n}\ln n} \xrightarrow{n\to\infty} e^0 = 1.$$

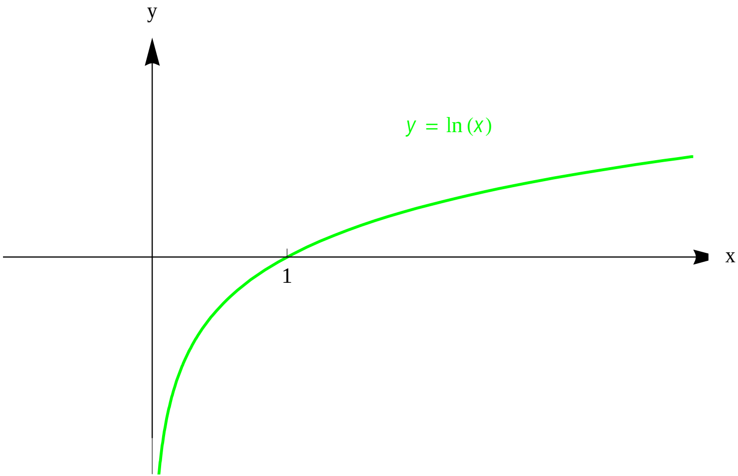
Finally, we note that by the inverse function theorem the logarithm is differentiable on $\mathbb{R}_+$ and

$$(\ln y)' = \frac{1}{(e^x)'|_{x=\ln y}} = \frac{1}{e^x|_{x=\ln y}} = \frac{1}{y}.$$

Hence $\ln \in C^\infty(\mathbb{R}_+)$.

The Logarithm

We thus obtain the following plot of the logarithmic function:



The Trigonometric Functions

We first note that we define

$$e^z := \exp z \qquad \text{for } z \in \mathbb{C}.$$

We then introduce the well-known trigonometric cosine and sine functions $\cos, \sin\colon \mathbb{R} \to \mathbb{R}$ by

$$\cos(x) := \operatorname{Re} e^{ix} = \frac{e^{ix} + e^{-ix}}{2} = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!},$$

$$\sin(x) := \operatorname{Im} e^{ix} = \frac{e^{ix} - e^{-ix}}{2i} = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}.$$

The equation

$$e^{ix} = \cos(x) + i \sin(x)$$

is sometimes called the ***Euler relation***.

The Trigonometric Functions

We can immediately prove some useful properties of the sine and cosine: it follows from the definition that

$$\cos(0) = 1, \quad \sin(0) = 0, \quad \sin^2(x) + \cos^2(x) = |e^{ix}|^2 = e^{ix}e^{-ix} = 1.$$

From

$$\begin{aligned} e^{i(x+y)} &= e^{ix}e^{iy} = (\cos(x) + i\sin(x))(\cos(y) + i\sin(y)) \\ &= (\cos(x)\cos(y) - \sin(x)\sin(y)) + i(\cos(x)\sin(y) + \sin(x)\cos(y)) \\ &= \cos(x+y) + i\sin(x+y) \end{aligned}$$

we obtain the ***addition relations*** for the sine and cosine:

$$\begin{aligned} \cos(x+y) &= \cos(x)\cos(y) - \sin(x)\sin(y), \\ \sin(x+y) &= \cos(x)\sin(y) + \sin(x)\cos(y). \end{aligned}$$

The Trigonometric Functions

We can verify immediately that the sine and cosine functions are differentiable, since

$$\begin{aligned}\cos(x+h) &= \cos(x)\cos(h) - \sin(x)\sin(h)\\ &= \cos(x)(1 - h^2/2! + o(h^2)) - \sin(x)(h - h^3/3! + o(h^3))\\ &= \cos(x) - \sin(x)h + o(h)\end{aligned}$$

and

$$\begin{aligned}\sin(x+h) &= \cos(x)\sin(h) + \sin(x)\cos(h)\\ &= \cos(x)(h - h^3/3! + o(h^3)) + \sin(x)(1 - h^2/2! + o(h^2))\\ &= \sin(x) + \cos(x)h + o(h)\end{aligned}$$

so we have

$$\frac{d}{dx}\cos x = -\sin x, \qquad \frac{d}{dx}\sin x = \cos x$$

The Trigonometric Functions

It follows from these formulas that the sine and cosine functions both satisfy the differential equation

$$y'' + y = 0,$$

where the sine function also satisfies the initial condition $y(0) = 0$ and the cosine function satisfies $y'(0) = 0$. In fact,

$$y(x) = a\cos x + b\sin x \tag{3.8.1}$$

satisfies the general initial value problem

$$y'' + y = 0, \qquad y(0) = a, \qquad y'(0) = b \qquad \text{for any } a, b \in \mathbb{R}. \tag{3.8.2}$$

We will show that (3.8.1) is in fact the only solution to (3.8.2).

The Trigonometric Functions

3.8.1. Lemma. Suppose that $y\colon \mathbb{R} \to \mathbb{R}$ is twice differentiable and that

$$y'' + y = 0, \qquad y(0) = 0, \qquad y'(0) = 0. \tag{3.8.3}$$

Then $y(x) = 0$ for all $x \in \mathbb{R}$.

Proof.
Multiplying (3.8.3) with y', we obtain

$$0 = y'y'' + yy' = \frac{1}{2}((y')^2 + y^2)',$$

so $(y')^2 + y^2$ is constant. Since $y(0) = y'(0) = 0$, it follows that $(y'(x))^2 + y(x)^2 = 0$ for all x, so $y(x) = 0$ for all $x \in \mathbb{R}$. □

The Trigonometric Functions

3.8.2. Theorem. Suppose that $y\colon \mathbb{R} \to \mathbb{R}$ is twice differentiable and that for some $a, b \in \mathbb{R}$

$$y'' + y = 0, \qquad y(0) = a, \qquad y'(0) = b. \tag{3.8.4}$$

Then

$$y(x) = a\cos x + b\sin x.$$

Proof.
Let f be any solution to (3.8.4) and $g(x) = f(x) - a\cos x - b\sin x$.Then g solves

$$g'' + g = 0, \qquad g(0) = 0, \qquad g'(0) = 0.$$

By Lemma 3.8.1, $g(x) = 0$ for all x, so

$$f(x) = a\cos x + b\sin x.$$

□

The Trigonometric Functions

We can hence conclude:

- If $y'' + y = 0$ and $y(0) = 1$, $y'(0) = 0$, then $y(x) = \cos x$,
- If $y'' + y = 0$ and $y(0) = 0$, $y'(0) = 1$, then $y(x) = \sin x$.

This property will be quite useful later.

Now consider the cosine function. We know that $\cos(0) = 1 > 0$ and $\cos^2(x) + \sin^2(x) = 1$. Therefore the cosine function has a local maximum at $x = 0$. Furthermore, at least for small $\varepsilon > 0$, $\cos(x) > 0$ on $[0, \varepsilon)$.

Since $(\sin(x))' = \cos x$, we see that $\sin x$ is strictly increasing on $[0, \varepsilon)$ and $\sin(x) > 0$ on $(0, \varepsilon)$. Let $\delta := \sin(\varepsilon) > 0$.

We will now show that the cosine function decreases at least until it intersects the abscissa, i.e., that it has a positive zero.

The Number π

Let $\delta := \sin(\varepsilon) > 0$. Suppose that there exists an $x_1 > \varepsilon$ such that $\sin(x_1) \le \delta$. Then, by Rolle's theorem, there exists a $\xi \in (\varepsilon, x_1)$ such that $(\sin(\xi))' = \cos(\xi) = 0$. On the other hand, if such an x_1 does not exist, then

$$(\cos(x))' = -\sin(x) < -\delta$$

for all x and the cosine will eventually decrease to zero as well.

Hence there exists at least one $x_0 > 0$ such that $\cos(x_0) = 0$. We define

$$\pi := 2 \cdot \min\{x \in \mathbb{R}_+ \colon \cos(x) = 0\}.$$

(Exercise: why is this a minimum and not just an infimum?) It will turn out that π is an irrational and transcendental number which is identical to the diameter of the unit circle. Numerically,

$$\pi \approx 3.141\,592\,653\,589\,793\,238\,462\,643\,383\,279\,502\,189.$$

The Number π

We thus have $\cos(\pi/2) = 0$ and $|\sin(\pi/2)| = 1$. Since $\sin(0) = 0$ and the sine function is increasing until $\cos(x) = 0$, i.e., for $0 \le x \le \pi/2$, we see that $\sin(\pi/2) = 1$. It follows that

$$\begin{aligned}
\cos(\pi) &= \cos(\pi/2 + \pi/2) = \cos(\pi/2)\cos(\pi/2) - \sin(\pi/2)\sin(\pi/2) = -1, \\
\sin(\pi) &= \cos(\pi/2)\sin(\pi/2) + \sin(\pi/2)\cos(\pi/2) = 0, \\
\cos(2\pi) &= \cos(\pi)\cos(\pi) - \sin(\pi)\sin(\pi) = 1, \\
\sin(2\pi) &= \cos(\pi)\sin(\pi) + \sin(\pi)\cos(\pi) = 0.
\end{aligned}$$

This implies

$$\begin{aligned}
\cos(x + 2\pi) &= \cos(x)\cos(2\pi) - \sin(x)\sin(2\pi) = \cos(x), \\
\sin(x + 2\pi) &= \cos(x)\sin(2\pi) + \sin(x)\cos(2\pi) = \sin(x),
\end{aligned}$$

so the sine and cosine functions are periodic with period 2π. Furthermore,

$$\sin(x + \pi/2) = \cos(x)\sin(\pi/2) + \sin(x)\cos(\pi/2) = \cos x,$$

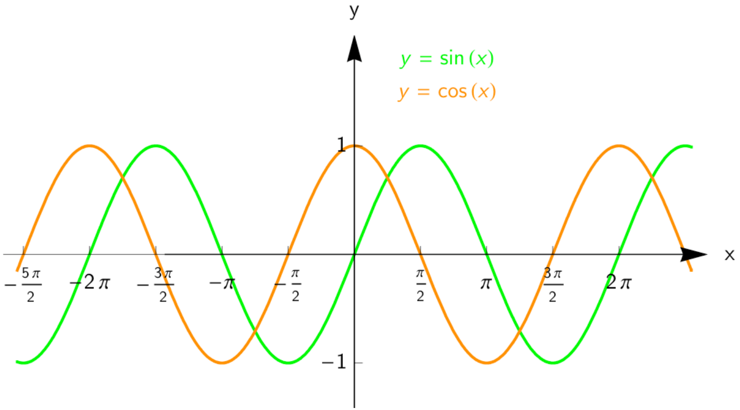
so the functions are simply translations of each other.

The Graph of Sine and Cosine

Lastly, we note that directly from the definition one sees that

$$\sin(-x) = -\sin(x), \qquad \cos(-x) = \cos(x)$$

for all $x \in \mathbb{R}$. Using that the cosine function is concave in $[-\pi/2, \pi/2]$, we are now able to sketch the functions:



Trigonometric Interpretation of Sine and Cosine

Of course, the sine and cosine functions are already familiar from elementary trigonometry, where they were defined in right-angled triangles as

$$\mathrm{COS}\,\theta := \frac{\text{Adjacent side}}{\text{Hypotenuse}}, \qquad \mathrm{SIN}\,\theta := \frac{\text{Opposite side}}{\text{Hypotenuse}}, \qquad 0 < \theta < 90^\circ.$$

In order to prove that $\sin x = \mathrm{SIN}\,\theta$ and $\cos x = \mathrm{COS}\,\theta$ for suitable $x = x(\theta)$, one proceeds as follows:

1. Using geometrical methods, one proves that for fixed length of sides of the triangle,

   $$\lim_{\theta\to 0} \mathrm{COS}\,\theta = 1, \qquad\qquad \lim_{\theta\to 0} \mathrm{SIN}\,\theta = 0,$$

   and thus extends the definition of the geometric sine and cosine to zero accordingly.

Trigonometric Interpretation of Sine and Cosine

2. Using geometrical methods, one proves that
$$\begin{aligned} \mathsf{COS}(\theta+\phi) &= \mathsf{COS}\,\theta\,\mathsf{COS}\,\phi - \mathsf{SIN}\,\theta\,\mathsf{SIN}\,\phi, \\ \mathsf{SIN}(\theta+\phi) &= \mathsf{COS}\,\theta\,\mathsf{SIN}\,\phi + \mathsf{SIN}\,\theta\,\mathsf{COS}\,\phi. \end{aligned}$$
whenever $0 \le \theta, \phi, \theta+\phi < 90^\circ$.
3. Using geometrical methods, one further shows (See ***Spivak***, Ch. 15. Ex. 27) that
$$\lim_{\theta\to 0}\frac{1-\mathsf{COS}\,\theta}{\theta} = 0, \qquad \lim_{\theta\to 0}\frac{\theta-\mathsf{SIN}\,\theta}{\theta} = 0.$$
4. From the above, one deduces $(\mathsf{COS}\,\theta)' = -\mathsf{SIN}\,\theta$ and $(\mathsf{SIN}\,\theta)' = \mathsf{COS}\,\theta$.

Trigonometric Interpretation of Sine and Cosine

6. It follows that $\mathrm{SIN}\,\theta$ and $\mathrm{COS}\,\theta$ satisfy
   - $(\mathrm{SIN}\,\theta)'' + (\mathrm{SIN}\,\theta) = 0$ and $\mathrm{SIN}(0) = 1$, $\mathrm{SIN}'(0) = 0$,
   - $(\mathrm{COS}\,\theta)'' + (\mathrm{COS}\,\theta) = 0$ and $\mathrm{COS}(0) = 0$, $\mathrm{COS}'(0) = 1$.

Since the solution to the corresponding initial value problems are unique, $\mathrm{COS}\,\theta = \cos\theta$, $\mathrm{SIN}\,\theta = \sin\theta$. Now θ is a number between 0 and 90, and

$$\lim_{\theta\to 90^\circ} \mathrm{SIN}\,\theta = 1, \qquad \lim_{\theta\to 90^\circ} \mathrm{COS}\,\theta = 0.$$

A degree θ is defined as the 360th part of the circumference of a circle; this is clearly an arbitrary definition, and we should find what the relationship between θ and x is, where x is scaled so that $\lim\limits_{x\to\pi/2} \sin x = 1$.

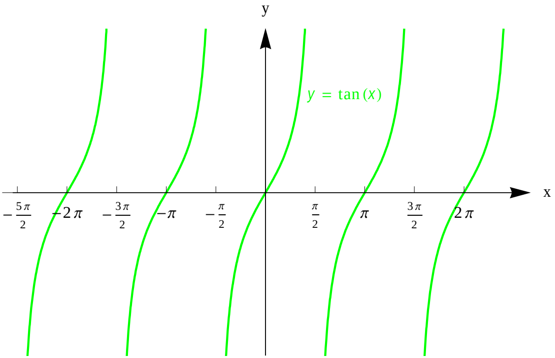
It is clear that $\theta = 360x/(2\pi)$. This is just a linear re-scaling of the coordinate plane to accomodate the “artificial” idea of dividing a circle into 360 parts. We will dispense with this and always assume that our coordinates are not rescaled in such a way.

More Trigonometric Functions

We further define trigonometric functions such as the tangent, cotangent, secant and cosecant by

$$\tan x := \frac{\sin x}{\cos x}, \quad \cot x := \frac{\cos x}{\sin x}, \quad \sec x := \frac{1}{\cos x}, \quad \csc x := \frac{1}{\sin x},$$

Except for the tangent, we will not use these frequently.



Inverse Trigonometric Functions

The inverses to the various trigonometric functions are called the ***arc sine***, ***arc cosine arc tangent*** etc. and denoted

$$\arcsin x, \qquad \arccos x, \qquad \arctan x,$$

respectively. In each case, the ranges of the inverses have to be chosen carefully, as the original functions are not injective for all x. For example, one commonly defines

$$\begin{aligned} \arcsin\colon [-1,1] &\to [-\pi/2, \pi/2], \\ \arccos\colon [-1,1] &\to [0, \pi], \\ \arctan\colon \mathbb{R} &\to (-\pi/2, \pi/2), \end{aligned}$$

but other definitions are of course possible and sometimes useful (in particular, for the arc tangent).

Inverse Trigonometric Functions

We can use the inverse function theorem to calculate the deivatives of the inverse trigonometric functions. For this, we first note that

$$(\cos x)' = -\sin x = -\sqrt{1-\cos^2 x},$$

$$(\sin x)' = \cos x = \sqrt{1-\sin^2 x},$$

$$(\tan x)' = \left(\frac{\sin x}{\cos x}\right)' = \frac{1}{\cos^2 x} = 1 + \tan^2 x.$$

Then

$$(\arccos x)' = \frac{1}{(\cos y)'|_{y=\arccos x}} = \frac{-1}{\sqrt{1-\cos^2 y}|_{y=\arccos x}} = \frac{-1}{\sqrt{1-x^2}},$$

$$(\arcsin x)' = \frac{1}{(\sin y)'|_{y=\arcsin x}} = \frac{1}{\sqrt{1-\sin^2 y}|_{y=\arcsin x}} = \frac{1}{\sqrt{1-x^2}},$$

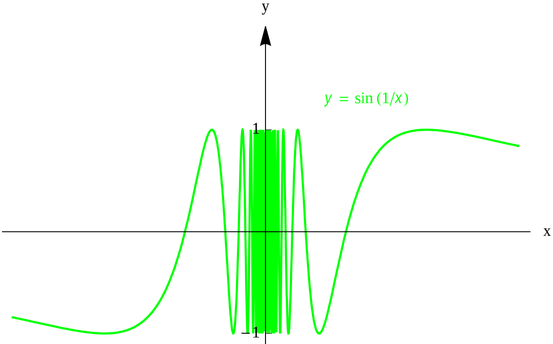
$$(\arctan x)' = \frac{1}{(\tan y)'|_{y=\arctan x}} = \frac{1}{1+\tan^2 y|_{y=\arctan x}} = \frac{1}{1+x^2}.$$

Some Further Properties of the Sine Function

The sine (and cosine) functions have some interesting properties which allow them to be used as study cases for continuity and differentiability. First, note that

$$\lim_{x\to 0} \sin\frac{1}{x}$$

does not exist. This can be proven similarly as for the periodic function ψ studied in the section on continuty.

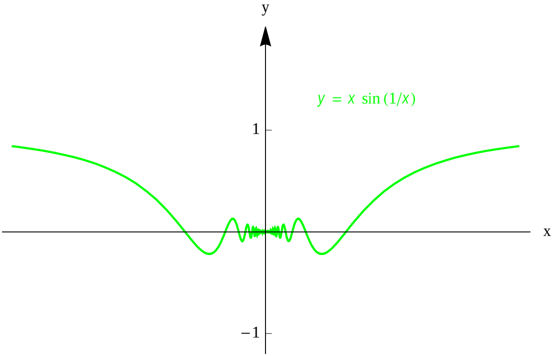


Some Further Properties of the Sine Function

Now it is easy to prove that the function

$$f(x) = \begin{cases} x \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

is continuous at $x = 0$.



Properties of the Sine Function

The function

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

is even differentiable: For $x \neq 0$,

$$f'(x) = 2x \sin \frac{1}{x} - \cos \frac{1}{x}.$$

Furthermore, $f(0 + h) = f(h) = f(0) + o(h)$, so $f'(0) = 0$. However, f' is not continuous at 0, since $\lim\limits_{x \to 0} f'(x)$ does not exist.

We thus finally have an example of a function that is differentiable but not continuously differentiable!

The Most Beautiful Theorem

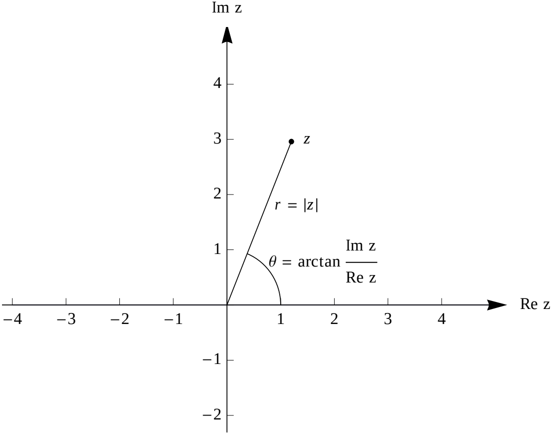
Since $e^{i\theta} = \cos\theta + i\sin\theta$, we observe that $e^{i\pi} = -1$, or

$$e^{i\pi} + 1 = 0.$$

This wonderful formula, which relates the most important constants in calculus with each other has been voted the most beautiful theorem of mathematics in a survey by the magazine ***The Mathematical Intelligencer***.

Polar Coordinates in the Complex Plane

An important observation is that every complex number z (except 0) in the complex plane has a unique representation through the distance of z to the origin and the angle it makes with the positive real axis:



Polar Coordinates in the Complex Plane

Thus every $z \in \mathbb{C} \setminus \{0\}$ uniquely determines a pair (r, θ) given by

$$r = |z| = \sqrt{(\operatorname{Re} z)^2 + (\operatorname{Im} z)^2}, \qquad \theta = \begin{cases} \arctan \frac{\operatorname{Im} z}{\operatorname{Re} z}, & \operatorname{Re} z > 0, \\ \pi + \arctan \frac{\operatorname{Im} z}{\operatorname{Re} z}, & \operatorname{Re} z < 0. \end{cases}$$

Conversely, given $(r, \theta) \in \mathbb{R}_+ \times [0, 2\pi)$ we obtain a unique number

$$z = r \cos \theta + i r \sin \theta = r \cdot e^{i\theta} \in \mathbb{C} \setminus \{0\}.$$

We call this the ***polar representation*** of a complex number z. While the cartesian representation

$$z = a + bi$$

for $a, b \in \mathbb{R}$ is useful for addition and subtraction, the polar representation is more suited to multiplication, division and exponentiation.

Polar Coordinates in the Complex Plane

The product and quotient of two complex numbers $z_1, z_2 \in \mathbb{C} \setminus \{0\}$ can be expressed through their polar representation as

$$z_1 \cdot z_2 = (r_1 e^{i\theta_1})(r_2 e^{i\theta_2}) = r_1 r_2 e^{i(\theta_1+\theta_2)},$$

$$\frac{z_1}{z_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)}.$$

In particular, we see that

$$z^n = r^n e^{in\theta} \qquad \text{for } n \in \mathbb{Z}.$$

This allows us to obtain the nth roots of a complex number, something that was not easily done for complex numbers in cartesian representation.

Polar Coordinates in the Complex Plane

For $0 \neq z = re^{i\theta}$ we clearly have an nth root given by

$$z^{1/n} = r^{1/n} e^{i\theta/n},$$

since $(z^{1/n})^n = z$. We would, however, expect to be able to find n distinct roots. The key to finding them is to note that

$$e^{i(\theta+2\pi)} = \cos(\theta + 2\pi) + i\sin(\theta + 2\pi) = \cos\theta + i\sin\theta = e^{i\theta}.$$

Since $z = re^{i\theta} = re^{i(\theta+2\pi)}$, another root is given by

$$z^{1/n} = r^{1/n} e^{i(\theta/n+2\pi/n)}.$$

Similarly, since $z = re^{i\theta} = re^{i(\theta+k2\pi)}$ for $k \in \mathbb{Z}$, we have n nth roots

$$r^{1/n}e^{i\theta/n}, \quad r^{1/n}e^{i(\theta+2\pi)/n}, \quad r^{1/n}e^{i(\theta+2\cdot 2\pi)/n}, \quad \dots, \quad r^{1/n}e^{i(\theta+(n-1)\cdot 2\pi)/n}.$$

The Hyperbolic Trigonometric Functions

We conclude this part by introducing a final family of functions, the ***hyperbolic trigonometric functions***:

We define the ***hyperbolic sine*** and ***hyperbolic cosine***, $\sinh, \cosh\colon \mathbb{C} \to \mathbb{C}$, by

$$\sinh(x) := \frac{e^x - e^{-x}}{2}, \qquad \cosh(x) := \frac{e^x + e^{-x}}{2}$$

A comparison with the definition of the sine and cosine functions immediately shows that

$$\sinh(ix) = i\sin(x), \qquad \cosh(ix) = \cos x.$$

From the definition, we see that

$$\cosh(x) + \sinh(x) = e^x.$$

The Hyperbolic Trigonometric Functions

Similarly to their trigonometric counterparts, we have

$$\cosh(-x) = \cosh(x), \qquad \sinh(-x) = -\sinh(x),$$

the hyperbolic cosine and sine are even and odd functions, respectively. We also see that

$$\begin{aligned}\cosh^2(x) - \sinh^2(x) &= \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2 \\ &= \frac{1}{4}\left(e^{2x} + 2 + e^{-2x} - e^{2x} + 2 - e^{-2x}\right) \\ &= 1\end{aligned}$$

This last relation explains the origin of the term ***hyperbolic***: if $x = \cosh t$ and $y = \sinh t$, then $x^2 - y^2 = 1$, which is the equation for a hyperbola.

Second Midterm Exam

The preceding material completes the second third of the course material. It encompasses everything that will be the subject of the Second Midterm Exam.

The exam date and time will be announced on Canvas.

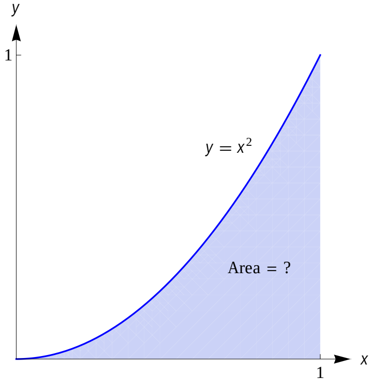
No calculators or other aids will be permitted during the exam. A sample exam with solutions has been uploaded to Canvas. Please study it carefully, including the instructions on the cover page.

Part IV

Integral Calculus in One Variable

Areas under Curves

We now turn to a question that seems unrelated to our previous investigations: How can we determine the area between the graph of a function and the abscissa?

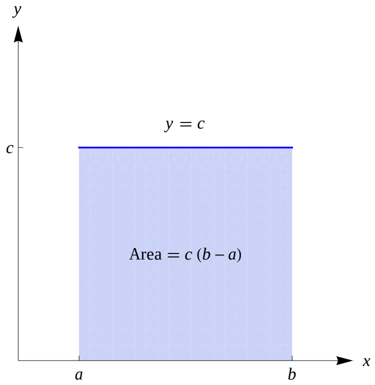


The sketch at left shows the graph of $f\colon [0,1] \to \mathbb{R}$, $f(x) = x^2$. While it is possible to determine the area bounded by the graph through elementary geometrical means, we aim to find a general procedure so that we can discuss general functions, e.g., $f(x) = e^x$.

At the outset, we have not even defined what an "area" is supposed to be, so we must first lay some groundwork.

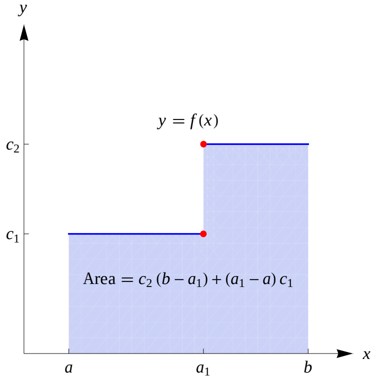
Areas under Curves

Suppose $f: [a, b] \to \mathbb{R}$ is a constant function, i.e., there exists some $c \in \mathbb{R}$ such that $f(x) = c$ for all $x \in [a, b]$. We would then naturally consider the "area bounded by the abscissa and the graph of f" to be equal to $c \cdot (b - a)$:



Areas under Curves

We can generalize this to the case where $f : [a, b] \to \mathbb{R}$ is "piecewise constant":



For our intuitive concept of what the "area" is, the value of $f(a_1)$ is irrelevant: we could have $f(a_1) = c_1$ or $f(a_1) = c_2$ or some completely different value. The "area" should remain the same.

Step Functions

Our first step is to give a proper definition of a "piecewise constant" function and the area "under" its graph.

4.1.1. Definition. A ***partition*** P of an interval $[a, b] \subset \mathbb{R}$ is a tuple of numbers $P = (a_0, \dots, a_n)$ such that

$$a = a_0 < a_1 < \dots < a_n = b.$$

4.1.2. Definition. A function $\varphi\colon [a, b] \to \mathbb{R}$ is called a ***step function with respect to a partition $P = (a_0, \dots, a_n)$*** if there exist numbers $y_i \in \mathbb{R}$, $i = 1, \dots, n$, such that

$$\varphi(t) = y_i \qquad \text{whenever } a_{i-1} < t < a_i \tag{4.1.1}$$

for $i = 1, \dots, n$. We denote the set of all step functions by $\text{Step}([a, b])$

Step Functions

4.1.3. Remarks.

1. We don't care how φ is defined at the points a_i!
2. We call φ simply a step function if there exists some partition P with respect to which it is a step function.

4.1.4. Example. The function

$$\varphi\colon [0,1] \to \mathbb{R}, \qquad \varphi(x) = \begin{cases} 0 & 0 \le x < \frac{1}{2}, \\ \frac{1}{2} & x = \frac{1}{2}, \\ 1 & \frac{1}{2} < x \le 1, \end{cases}$$

is a step function. In particular, it is a step function with respect to $P = (0, \frac{1}{2}, 1)$ or any other suitable partition, e.g., $\widetilde{P} = (0, \frac{1}{2}, \frac{3}{4}, 1)$.

Sum of Step Functions

4.1.5. Definition. Let P be a partition of an interval $[a, b]$. We call a partition R of $[a, b]$ a ***sub-partition*** of P if $R \supset P$, where we naturally regard P and R as sets instead of tuples.

4.1.6. Remark. If $\varphi \colon [a, b] \to \mathbb{R}$ is a step function with respect to a partition P of $[a, b]$, then it as also a step function with respect to any sub-partition R of P.

We can define the sum of two step functions as follows:
Given a step function φ (w.r.t. a partition P) and a step function ψ (w.r.t. a partition Q), we first define the partition $R := P \cup Q$ of $[a, b]$ which contains all points of P and Q, ordered appropriately from smallest to largest.

Then $R \supset P$ and $R \supset Q$, i.e., R is a sub-partition of both P and Q. Hence φ and ψ are step functions with respect to R and we can define $\varphi + \psi$ by adding $\varphi(x)$ and $\psi(x)$ on each interval induced by $R = P \cup Q$.

Sum of Step Functions

Since multiplying step functions by a number does not present any problems, we have proven the following:

4.1.7. Proposition. If φ, ψ are step functions on $[a, b]$ and $\lambda, \mu \in \mathbb{R}$, then $\lambda\varphi + \mu\psi$ is a step function on $[a, b]$.

4.1.8. Remark. This also shows that $\operatorname{Step}([a, b])$ is a vector space.

Integral of a Step Function

4.1.9. Definition and Theorem. Let $\varphi\colon [a,b] \to \mathbb{R}$ be a step function of the form (4.1.1) with respect to some partition P. Then

$$I_P(\varphi) := (a_1 - a_0)y_1 + \cdots + (a_n - a_{n-1})y_n \tag{4.1.2}$$

is independent of the choice of the partition P. We define the ***integral*** of f as

$$\int_a^b \varphi := I_P(\varphi)$$

for any partition P with respect to which φ is a step function.

4.1.10. Remark. It is common to use the notation

$$\int_a^b \varphi(x)\,dx := \int_a^b \varphi.$$

The motivation underlying this notation will remain mysterious for now. However, it is an intuitive notation in the same sense that writing $\lim_{x\to x_0} f(x)$ is more intuitively friendly that $\lim_{x_0} f$.

Integral of a Step Function

Proof.

Let φ be a step function w.r.t. a partition P and also w.r.t. some other partition Q. Let $R = P \cup Q$ be a common sub-partition of both P and Q. We will show that

$$I_P(\varphi) = I_R(\varphi) = I_Q(\varphi).$$

Let $c \in R \setminus P$ and suppose that $a_k < c < a_{k+1}$ for some $k = 1, \ldots, n$. We add c to P, denoting the resulting partition by

$$P_c = (a_0, \ldots, a_k, c, a_{k+1}, \ldots, a_n).$$

Integral of a Step Function

Proof (continued).
Then $\varphi(t) = y_k$ for $a_k < t < a_{k+1}$, so

$$\begin{aligned} I_P(\varphi) &= (a_1 - a_0)y_1 + \ldots + (a_{k+1} - a_k)y_k + \ldots + (a_n - a_{n-1})y_n \\ &= (a_1 - a_0)y_1 + \ldots + (c - a_k)y_k + (a_{k+1} - c)y_k \\ &\quad + \ldots + (a_n - a_{n-1})y_n \\ &= I_{P_c}(\varphi). \end{aligned}$$

Repeating this procedure, we eventually have $I_P(\varphi) = I_R(\varphi)$. In the same way we show that $I_Q(\varphi) = I_R(\varphi)$. $\square$

Properties of the Integral of a Step Function

It is easy to see that

$$\left| \int_a^b \varphi \right| \le (b-a) \sup_{x\in[a,b]} |\varphi(x)|$$

for a step function φ, since, given a partition P,

$$\begin{aligned} |I_P(\varphi)| &\le \sum_{i=1}^n (a_i - a_{i-1}) \cdot |y_i| \le \max_{1\le i\le n} |y_i| \sum_{i=1}^n (a_i - a_{i-1}) \\ &= (b-a) \sup_{x\in[a,b]} |\varphi(x)|. \end{aligned} \tag{4.1.3}$$

Furthermore, it is not difficult to show that

$$\int_a^b (\lambda\varphi + \mu\psi) = \lambda \int_a^b \varphi + \mu \int_a^b \psi \tag{4.1.4}$$

for step functions φ, ψ and $\lambda, \mu \in \mathbb{R}$.

Bounded Functions

4.1.11. Definition. Let $I \subset \mathbb{R}$ be an interval. We say that a function $f : I \to \mathbb{R}$ is ***bounded*** if

$$\|f\|_\infty := \sup_{x \in I} |f(x)| < \infty. \tag{4.1.5}$$

The set of all bounded functions is denoted $L^\infty(I)$.

4.1.12. Remarks.

1. It is easy to check that $L^\infty(I)$ (endowed with point-wise addition and multiplication) is a vector space and that (4.1.5) defines a norm on $L^\infty(I)$.
2. If $I = [a, b]$ is a closed interval, every continuous function is bounded, so $C([a, b]) \subset L^\infty([a, b])$.
3. If $I = [a, b]$ is a closed interval, any step function on $[a, b]$ takes on only a finite number of real values, so is automatically bounded. Hence, $\text{Step}([a, b]) \subset L^\infty([a, b])$.

Area under the Graph of Bounded Functions

We restrict ourselves (for now) to bounded functions on an interval $[a, b]$. The area under the graph of a bounded function (if it exists) can then not be larger than the product of $(a - b)$ and the bound of the function, i.e., it must be finite.

However, not all bounded functions may have an intuitive "area" under their graph. As an example, consider the ***Dirichlet function***

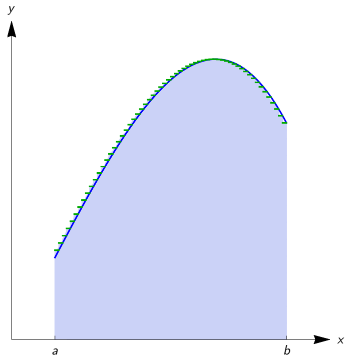
$$\chi\colon [0,1] \to \mathbb{R}, \qquad\qquad \chi(x) = \begin{cases} 1 & \text{for } x \text{ rational,} \\ 0 & \text{for } x \text{ irrational.} \end{cases}$$

It is not clear, what the value of the "area" might be: 1? 0? undefined?

Given such an example, we will have to approach the concept of area very carefully.

Area under the Graph of Bounded Functions

Suppose a function $f \in L^\infty([a, b])$ can be approximated uniformly by step functions, i.e., it is the uniform limit of a sequence of step functions. Then we might hope that the area under the step functions approaches the area under f:



Regulated Functions

It turns out that this approach works for functions that can be approximated uniformly by step functions. We give the class of these functions a specific name:

4.1.13. Definition. A function $f \in L^\infty([a, b])$ is said to be ***regulated*** if for any $\varepsilon > 0$ there exists a step function φ such that

$$\sup_{x\in[a,b]} |f(x) - \varphi(x)| < \varepsilon. \tag{4.1.6}$$

Equivalently, a regulated function f is a function such that there exists a sequence of step functions (φ_n) which converges uniformly to f. (To see this, we simply define φ_n as the step function satisfying (4.1.6) with $\varepsilon = 1/n$.)

Example of a Regulated Function

4.1.14. Example. The function $f\colon [0,1] \to \mathbb{R}$, $f(x) = x^2$ is regulated. To see this, choose the sequence of step functions (φ_n) on $[0,1]$ given by

$$\varphi_n(x) = \frac{k^2}{n^2} \qquad \text{for } \frac{k-1}{n} < x \le \frac{k}{n}, \qquad k = 1, \dots, n,$$

and $\varphi_n(0) = 0$. Then

$$\begin{aligned}
\|\varphi_n - f\|_\infty &= \sup_{x\in[0,1]} |\varphi_n(x) - f(x)| = \max_{k=1,\dots,n} \sup_{\frac{k-1}{n}<x\le\frac{k}{n}} |\varphi_n(x) - f(x)| \\
&= \max_{k=1,\dots,n} \sup_{\frac{k-1}{n}<x\le\frac{k}{n}} \left|\frac{k^2}{n^2} - x^2\right| \\
&= \max_{k=1,\dots,n} \left|\frac{k^2}{n^2} - \frac{(k-1)^2}{n^2}\right| = \max_{k=1,\dots,n} \left|\frac{2k-1}{n^2}\right| = \frac{2n-1}{n^2}.
\end{aligned}$$

It follows that $\|\varphi_n - f\|_\infty \to 0$ as $n \to \infty$, so $f \in \operatorname{Reg}([a,b])$.

Continuous Functions are Regulated

4.1.15. Theorem. The continuous functions on the interval $[a, b]$ are regulated, i.e., $C([a, b]) \subset \text{Reg}([a, b])$.

Proof.
We need to show that any continuous function on the interval $[a, b]$ can be regarded as the uniform limit of step functions. Let $f \in C([a, b])$ be given. By Theorem 2.5.24 f is also uniformly continuous, i.e.,

$$\underset{\varepsilon>0}{\forall}\ \underset{\delta>0}{\exists}\ \underset{x,y\in[a,b]}{\forall} \quad |x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon. \tag{4.1.7}$$

Now fix $\varepsilon > 0$ and choose some $\delta > 0$ according to (4.1.7). Define a partition $P = (a_0, \dots, a_n)$ of $[a, b]$ such that $a_j - a_{j-1} = (a - b)/n < \delta$, $j = 1, \dots, n$, by choosing n large enough. Define $\varphi\colon [a, b] \to \mathbb{R}$ by $\varphi(b) = f(b)$ and

$$\varphi(t) = f(a_{i-1}) \qquad\qquad \text{for } a_{i-1} \le t < a_i,\ i = 1, \dots, n.$$

Continuous Functions are Regulated

Proof (continued).

Then $\varphi \in \operatorname{Step}([a, b])$ and

$$\begin{aligned} \|f - \varphi\|_\infty &= \sup_{t\in[a,b]} |f(t) - \varphi(t)| \\ &= \max_{1\le i\le n} \sup_{t\in[a_{i-1},a_i]} |f(t) - \varphi(t)| \\ &= \max_{1\le i\le n} \sup_{t\in[a_{i-1},a_i]} |f(t) - f(a_{i-1})| < \varepsilon \end{aligned}$$

Thus for every $\varepsilon > 0$ we can find a function φ with $\|f - \varphi\|_\infty < \varepsilon$. $\square$

Piecewise Continuous Functions

4.1.16. Definition. Let $I \subset \mathbb{R}$ be an interval, $\{a_1, \ldots, a_n\} \subset I$ a finite set of points and $f : I \to \mathbb{C}$ a function such that

(i) $f : I \setminus \{a_1, \ldots, a_n\} \to \mathbb{C}$ is continuous and

(ii) For all a_i, $i = 1, \ldots, n$, both

$$\lim_{x \nearrow a_i} f(x) \qquad \text{and} \qquad \lim_{x \searrow a_i} f(x)$$

exist.

Then f is said to be ***piecewise continuous***. We denote the set of piecewise continuous functions on I by $\mathrm{PC}(I)$.

4.1.17. Theorem. The piecewise continuous functions on the interval $[a, b]$ are regulated, i.e., $\mathrm{PC}([a, b]) \subset \mathrm{Reg}([a, b])$.

The proof is left to the assignments.

Integral of a Regulated Function

We now define the integral of a regulated function as follows:

4.1.18. Definition and Theorem. Let $f \in \operatorname{Reg}([a,b])$ and (φ_n) a sequence in $\operatorname{Step}([a,b])$ converging uniformly to f. Then the ***regulated integral*** of f, defined by

$$\int_a^b f := \lim_{n\to\infty} \int_a^b \varphi_n \tag{4.1.8}$$

exists and does not depend on the choice of (φ_n).

Proof.

We first prove that the limit (4.1.8) exists. Since (φ_n) converges to f uniformly, (φ_n) is a Cauchy sequence in $C([a,b])$ with the $\|\cdot\|_\infty$ norm. This implies that the sequence $(\int_a^b \varphi_n)$ is Cauchy, since

$$\left|\int_a^b \varphi_n - \int_a^b \varphi_m\right| = \left|\int_a^b (\varphi_n - \varphi_m)\right| \le (b-a)\|\varphi_n - \varphi_m\|_\infty.$$

by (4.1.4) and (4.1.3).

Integral of a Regulated Function

Proof.

Since the real numbers are complete, this implies that the sequence $(\int_a^b \varphi_n)$ converges. Hence, the limit (4.1.8) exists.

Furthermore, let (φ_n) and (ψ_n) be two sequences of step functions converging uniformly to f. Then

$$\begin{aligned}\left|\int_a^b \varphi_n - \int_a^b \psi_n\right| &= \left|\int_a^b (\varphi_n - \psi_n)\right| \\ &\le (b-a)\|\varphi_n - \psi_n\|_\infty \\ &\le (b-a)\|\varphi_n - f\|_\infty + (b-a)\|\psi_n - f\|_\infty \\ &\xrightarrow{n\to\infty} 0.\end{aligned}$$

This shows that both sequences yield the same limit (4.1.8), i.e., the integral of f does not depend on the choice of the sequence of step functions. $\square$

Example of a Regulated Integral

4.1.19. Example. We have seen in Example 4.1.14 that the function $f\colon [0,1] \to \mathbb{R}$, $f(x) = x^2$ is the uniform limit of the sequence of step functions (φ_n) on $[0,1]$ given by

$$\varphi_n(x) = \frac{k^2}{n^2} \qquad \text{for } \frac{k-1}{n} < x \le \frac{k}{n}, \qquad k = 1, \dots, n,$$

and $\varphi_n(0) = 0$. Now it is easy to see that the integral of the step functions, by Definition 4.1.9, is given by

$$\int_0^1 \varphi_n = \sum_{k=1}^{n} \frac{1}{n} \cdot \frac{k^2}{n^2} = \frac{1}{n^3} \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6n^3}.$$

It follows that

$$\int_0^1 f = \lim_{n\to\infty} \int_0^1 \varphi_n = \frac{1}{3}.$$

Properties of the Regulated Integral

We can verify that properties (4.1.4) and (4.1.3) also hold for the regulated integral. Let $f, g \in \text{Reg}([a, b])$ and $(\varphi_n), (\psi_n)$ be sequences in $\text{Step}([a, b])$ converging uniformly to f and g, respectively. Then

$$\begin{aligned}\left|\int_a^b f\right| = \left|\lim_{n\to\infty}\int_a^b \varphi_n\right| = \lim_{n\to\infty}\left|\int_a^b \varphi_n\right| \le (b-a)\lim_{n\to\infty}\sup_{x\in[a,b]}|\varphi_n(x)| \\ \le (b-a)\sup_{x\in[a,b]}|f(x)|. \qquad (4.1.9)\end{aligned}$$

In the same way,

$$\begin{aligned}\lambda\int_a^b f + \mu\int_a^b g = \lambda\lim_{n\to\infty}\int_a^b \varphi_n + \mu\lim_{n\to\infty}\int_a^b \psi_n = \lim_{n\to\infty}\int_a^b(\lambda\varphi_n + \mu\psi_n) \\ = \int_a^b(\lambda f + \mu g)\end{aligned}$$

since the sequence $(\lambda\varphi_n + \mu\psi_n)$ converges uniformly to $\lambda f + \mu g$.

Properties of the Regulated Integral

We also note that if $a < c < b$, then for $f \in L^\infty([a, b])$,

$$\int_a^b f = \int_a^c f + \int_c^b f. \tag{4.1.10}$$

This can easily be seen by including the point c in any partition of step functions approximating f.

The set of regulated functions is closed and the regulated integral is continuous on this set. More precisely, we have the following result:

4.1.20. Theorem. Let (f_n) be a sequence of regulated functions such that $f_n \to f$ uniformly for some $f \in L^\infty$, i.e., $\|f - f_n\|_\infty \to 0$. Then f is regulated and

$$\int_a^b f_n \xrightarrow{n\to\infty} \int_a^b f. \tag{4.1.11}$$

The proof is left to the assignments.

Non-Regulated Functions

However, not all bounded functions are regulated:

4.1.21. Example. Consider the function

$$f\colon [0,1] \to \mathbb{R}, \qquad f(x) = \begin{cases} 1 & \text{if } x = 1/n \text{ for some } n \in \mathbb{N} \setminus \{0\}, \\ 0 & \text{otherwise.} \end{cases}$$

Now any step function φ must be constant (say, equal to $c \in \mathbb{R}$) in the interval $(0, \delta)$ for some $\delta > 0$. This interval will always contain both points where $x = 1/n$ and points where $x \neq 1/n$, no matter how small δ is chosen. Then

$$\sup_{x \in [0,1]} |f(x) - \varphi(x)| \geq \sup_{x \in (0,\delta)} |f(x) - c| \geq \max\{|1 - c|, |c|\} \geq \frac{1}{2}$$

so that (4.1.6) can not be satisfied. Hence, f is not regulated.

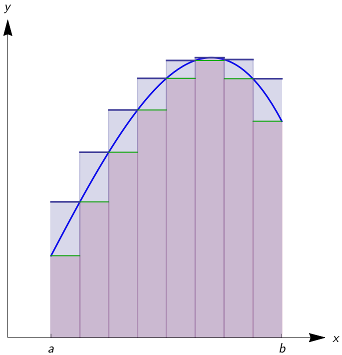
For technical reasons, however, it is desirable to extend the notion of integral to cover even non-regulated functions such as the one above.

Extending the Regulated Integral

Let us attempt another approach at defining the area under a curve: the regulated integral was based on approximating the function f uniformly through step functions and then defining the area under f to be the limit of the areas of step functions.

In contrast, we will now try to approximate the area under f, ***without attempting to approximate f at all***.

We will do this by considering step functions that are larger than f (i.e., $\varphi(x) \geq f(x)$ for all x) and step functions that are smaller than f ($\psi(x) \leq f(x)$ for all x), as illustrated at right.



Properties of the Integral

If the area between "upper" and "'lower" step functions can be made as small as desired, we will have successfully approximated the area of f without necessarily needing to approximate f. We will now formalize these ideas.

We have proven (or can easily check) that the regulated integral has the following properties:

(i) The integral is linear, i.e., $\int(\lambda f + \mu g) = \lambda \int f + \mu \int g$ for $\lambda, \mu \in \mathbb{R}$;

(ii) The integral is positive, i.e., if $f > 0$ on $[a, b]$, then $\int f > 0$;

(iii) The integral of a constant function $f = c$, $c \in \mathbb{R}$, on $[a, b]$ is given by $\int f = c \cdot (b - a)$;

(iv) The integral does not depend on the values of a function on finite sets. If $f(x) = 0$ for all but a finite set of $x \in [a, b]$, then $\int f = 0$.

Suppose there is a class of bounded functions on $[a, b]$ that includes the regulated functions and all those functions for which it is possible to define an integral with the properties (i) - (iv) above.

Upper and Lower Step Functions

Let $f \in L^\infty([a, b])$ be a function for which an integral with properties (i) - (iv) exists. Since f is bounded, there exists a step function u such that $u \geq f$ on $[a, b]$. Thus

$$\int_a^b f \leq \int_a^b u$$

by properties (i) and (ii) (since $u - f > 0$ and $\int_a^b (u - f) = \int_a^b u - \int_a^b f$). Denote by $\mathscr{U}_f$ the set of all such "upper" step functions. Then

$$\int_a^b f \leq \inf_{u \in \mathscr{U}_f} \int_a^b u.$$

This infimum exists because the step functions are bounded below by the lower bound for f, and properties (ii) and (iii) guarantee the existence of a lower bound for the integral. We define the ***upper area*** of f as

$$\overline{I}(f) := \inf_{u \in \mathscr{U}_f} \int_a^b u.$$

Upper and Lower Step Functions

Similarly, if $\mathscr{L}_f$ denotes the set of all step functions v such that $v \leq f$, then

$$\int_a^b f \geq \sup_{v \in \mathscr{L}_f} \int_a^b v =: \underline{I}(f).$$

Thus, if it is possible to define an integral of the bounded function f, then

$$\underline{I}(f) \leq \int_a^b f \leq \overline{I}(f).$$

Even if it is not possible to define an integral of f, we may still define $\mathscr{U}_f$ and $\mathscr{L}_f$ and it remains true that

$$\underline{I}(f) \leq \overline{I}(f).$$

Clearly, for an arbitrary bounded f the supremum and infimum may differ; but if they are equal, then their common value must also be the value of $\int_a^b f$. We can use this to ***define*** $\int_a^b f$ by this common value, if it exists.

The Darboux Integral

4.1.22. Definition. Let $[a, b] \subset \mathbb{R}$ be a closed interval and f a bounded real function on $[a, b]$. Let $\mathscr{U}_f$ denote the set of all step functions u on $[a, b]$ such that $u \geq f$ and $\mathscr{L}_f$ the set of all step functions v on $[a, b]$ such that $v \leq f$. The function f is then said to be ***Darboux-integrable*** if

$$\underline{I}(f) = \sup_{v \in \mathscr{L}_f} \int_a^b v = \inf_{u \in \mathscr{U}_f} \int_a^b u = \overline{I}(f).$$

In this case, the ***Darboux integral of f***, $\int_a^b f$, is defined to be this common value.

4.1.23. Remark. A bounded function $f \in L^\infty([a, b])$ is Darboux-integrable if and only if for every $\varepsilon > 0$ there exist step functions u_ε and v_ε such that $v_\varepsilon \leq f \leq u_\varepsilon$ and

$$\int_a^b u_\varepsilon - \int_a^b v_\varepsilon \leq \varepsilon.$$

Example for the Darboux Integral of a Function

4.1.24. Example. Consider the function $f\colon [0,1] \to \mathbb{R}$, $f(x) = x^2$. For any $n \in \mathbb{N}$, the step function defined by

$$u_n(x) = \frac{k^2}{n^2} \qquad \text{for } \frac{k-1}{n} < x \le \frac{k}{n}, \qquad k = 1, \dots, n,$$

and $u_n(0) = 0$ is an upper step function for f, while the step function

$$v_n(x) = \frac{(k-1)^2}{n^2} \qquad \text{for } \frac{k-1}{n} < x \le \frac{k}{n}, \qquad k = 1, \dots, n,$$

and $u_n(0) = 0$ is a lower step function for f. Then

$$\int_0^1 u_n = \sum_{k=1}^{n} \frac{1}{n} \cdot \frac{k^2}{n^2} = \frac{1}{n^3} \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6n^3},$$

$$\int_0^1 v_n = \sum_{k=1}^{n} \frac{1}{n} \cdot \frac{(k-1)^2}{n^2} = \frac{1}{n^3} \sum_{k=1}^{n-1} k^2 = \frac{n(n-1)(2n-1)}{6n^3}.$$

Example for the Darboux Integral of a Function

Now

$$\inf_{u\in\mathscr{U}_f}\int_0^1 u \le \lim_{n\to\infty}\int_0^1 u_n = \frac{1}{3},$$
$$\sup_{v\in\mathscr{L}_f}\int_0^1 v \ge \lim_{n\to\infty}\int_0^1 v_n = \frac{1}{3}.$$

Since $\inf\limits_{u\in\mathscr{U}_f}\int_0^1 u \ge \sup\limits_{v\in\mathscr{L}_f}\int_0^1 v$, this implies that

$$\inf_{u\in\mathscr{U}_f}\int_0^1 u = \sup_{v\in\mathscr{L}_f}\int_0^1 v = \frac{1}{3}.$$

By the definition of the Darboux integral,

$$\int_0^1 f = 1/3.$$

The Darboux Integral of a Non-Regulated Function

4.1.25. Example. The function

$$f\colon [0,1] \to \mathbb{R}, \qquad f(x) = \begin{cases} 1 & \text{if } x = 1/n \text{ for some } n \in \mathbb{N} \setminus \{0\}, \\ 0 & \text{otherwise.} \end{cases}$$

of Example 4.1.21 is not regulated, but it is Darboux-integrable. To see this, note that $v_0(x) = 0$ defines a lower step function $v_0 \in \mathscr{L}_f$, while for any n,

$$u_n(x) = \begin{cases} 1 & 0 \le x \le 1/n, \\ f(x) & \text{otherwise} \end{cases}$$

defines an upper step function $u_n \in \mathscr{U}_f$. As in the previous example,

$$\inf_{u \in \mathscr{U}_f} \int_0^1 u \le \lim_{n \to \infty} \int_0^1 u_n = \frac{1}{n} = 0$$

and $\sup\limits_{v \in \mathscr{L}_f} \int_0^1 v \ge \int_0^1 v_0 = 0$, yielding the common value $\int_0^1 f = 0$.

A regulated function is Darboux-integrable

4.1.26. Theorem. If $f \in \operatorname{Reg}([a,b])$, then f is Darboux-integrable and the Darboux integral coincides with the regulated integral.

Proof.
Let $f \in \operatorname{Reg}([a,b])$. We will show that for every $\varepsilon > 0$ there exist upper and lower step functions u_ε and v_ε such that $\int_a^b u_\varepsilon - \int_a^b v_\varepsilon \le \varepsilon$.

Fix $\varepsilon > 0$. Since f is regulated, there exists a step function φ such that $\sup\limits_{x \in [a,b]} |f(x) - \varphi(x)| < \varepsilon/[2(b-a)]$. Then

$$u_\varepsilon := \varphi + \frac{\varepsilon}{2(b-a)}, \qquad v_\varepsilon := \varphi - \frac{\varepsilon}{2(b-a)}.$$

are upper and lower step functions for f and

$$\int_a^b u_\varepsilon - \int_a^b v_\varepsilon \le (b-a) \sup_{x \in [a,b]} |u_\varepsilon(x) - v_\varepsilon(x)| = \varepsilon.$$

A regulated function is Darboux-integrable

Proof (continued).

This shows that f is Darboux integrable.

To show that the Darboux integral is equal to the regulated integral, we will show that the limit of the integrals of step functions converging uniformly to f is just the Darboux integral. Let (φ_n) be a sequence of step functions such that

$$\sup_{x\in[a,b]} |f(x) - \varphi_n(x)| \le \frac{1}{2n}.$$

Then

$$u_n := \varphi_n + \frac{1}{n}, \qquad\qquad v_n := \varphi_n - \frac{1}{n}.$$

define upper and lower step functions for f.

A regulated function is Darboux-integrable

Proof (continued).

The Darboux integral satisfies

$$\int_a^b v_n \le \int_a^b f \le \int_a^b u_n.$$

Since

$$\lim_{n\to\infty}\int_a^b v_n = \lim_{n\to\infty}\int_a^b \varphi_n - \lim_{n\to\infty}\frac{b-a}{n} = \lim_{n\to\infty}\int_a^b \varphi_n,$$

$$\lim_{n\to\infty}\int_a^b u_n = \lim_{n\to\infty}\int_a^b \varphi_n + \lim_{n\to\infty}\frac{b-a}{n} = \lim_{n\to\infty}\int_a^b \varphi_n.$$

it follows that

$$\int_a^b f = \lim_{n\to\infty}\int_a^b \varphi_n. \qquad \square$$

A regulated function is Darboux-integrable

Hence, the Darboux integral is an extension of the regulated integral for functions $f : [a, b] \to \mathbb{R}$. We can easily verify that the Darboux integral is bounded and additive:

4.1.27. Proposition. The Darboux integral of Riemann-integrable functions f and g on $[a, b]$ satisfies

$$\left| \int_a^b f \right| \le (b - a) \sup_{x \in [a,b]} |f(x)|,$$

$$\lambda \int_a^b f + \mu \int_a^b g = \int_a^b (\lambda f + \mu g).$$

The proof is left as an exercise.

Tagged Partitions and Riemann Sums

There is another approach to the integral that is most often seen in undergraduate textbooks.

A ***tagged partition*** (P, Ξ) on an interval $[a, b]$ consists of a partition $P = \{x_0, \ldots, x_n\}$ together with numbers $\Xi = \{\xi_1, \ldots, \xi_n\}$ such that each $\xi_k \in [x_{k-1}, x_k]$. The ***mesh size*** of P is defined as

$$m(P) := \max_{k=1,\ldots,n} (x_k - x_{k-1})$$

A step function $\varphi \in \operatorname{Step}([a, b])$ for a function $f \in L^\infty([a, b])$ with respect to (P, Ξ) is given by

$$\varphi(x) = f(\xi_k) \qquad \text{for } x_{k-1} < x < x_k, \qquad k = 1, \ldots, n,$$

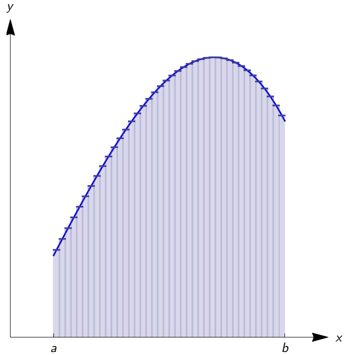
where as usual $\varphi(x_k)$ can be defined in an arbitrary manner. The sum

$$\int_a^b \varphi := \sum_{k=1}^{n} f(\xi_k)(x_k - x_{k-1}) \tag{4.1.12}$$

is then called a ***Riemann sum*** for f.

Riemann Sums

The Riemann sum will hopefully approach the area under the graph of f as the partitions become finer. Below, we have taken partitions of equal size and tags lying in the midpoints of the subintervals, but the subinterval lengths and tag locations should not influence the limit:



Given a function f, the choice of finer and finer partitions gives Riemann sums of more and more values of f, weighted by the mesh size.

In the limit, one may regard a sequence of such Riemann sums as approaching the "sum of all values of f in the interval $[a, b]$."

The Riemann Integral

4.1.28. Definition. Let $[a, b] \subset \mathbb{R}$ be a closed interval and f a bounded real function on $[a, b]$. Then f is Riemann-integrable with integral

$$\int_a^b f \in \mathbb{R}$$

if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for any tagged partition (P, Ξ) on $[a, b]$ with mesh size $m(P) < \delta$

$$\left| \sum_{k=1}^{n} f(\xi_k)(x_k - x_{k-1}) - \int_a^b f \right| < \varepsilon.$$

Definition 4.1.28 is quite difficult to work with in practice, so it is fortunate that the following result holds:

4.1.29. Theorem. A bounded real function is Riemann-integrable on $[a, b]$ if and only if it is Darboux-integrable. Moreover, the values of the integrals coincide.

Equivalence of the Darboux and Riemann Integrals

Proof.

Suppose that f is Riemann integrable. Fix $\varepsilon > 0$. Then there exists a partition $P = (x_0, \dots, x_n)$ such that

$$\left| \sum_{k=1}^{n} f(\xi_k)(x_k - x_{k-1}) - \int_a^b f \right| < \frac{\varepsilon}{4}.$$

for any choice of $\xi_k \in [x_{k-1}, x_k]$. On this partition P we can define upper and lower step functions for f by

$$u(x) := \sup_{\xi \in [x_{k-1}, x_k]} f(\xi), \qquad v(x) := \inf_{\xi \in [x_{k-1}, x_k]} f(\xi)$$

for $x \in (x_{k-1}, x_k)$. By choosing ξ_k appropriately, we can ensure that the corresponding Riemann sum differs from the integral of u by less than $\varepsilon/4$.

Equivalence of the Darboux and Riemann Integrals

Proof (continued).

This ensures that

$$\begin{aligned}\left|\int_a^b u - \int_a^b f\right| &\le \left|\int_a^b u - \sum_{k=1}^n f(\xi_k)(x_k - x_{k-1})\right| \\ &\qquad + \left|\sum_{k=1}^n f(\xi_k)(x_k - x_{k-1}) - \int_a^b f\right| \\ &< \frac{\varepsilon}{2}.\end{aligned}$$

Applying the same argument for the lower step function, we find

$$\left|\int_a^b u - \int_a^b v\right| < \left|\int_a^b u - \int_a^b f\right| + \left|\int_a^b v - \int_a^b f\right| < \varepsilon. \tag{4.1.13}$$

Equivalence of the Darboux and Riemann Integrals

Proof (continued).

Since such step functions u and v can be found for any $\varepsilon > 0$, this shows that f is Darboux-integrable (see Remark 4.1.23). Furthermore, since for any choice of partition

$$\int_a^b v \leq \sum_{k=1}^{n} f(\xi_k)(x_k - x_{k-1}) \leq \int_a^b u$$

we see that the Darboux integral equals the Riemann integral.

The proof that a Darboux-integrable function is Riemann-integrable is left as an exercise. □

4.1.30. Remark. The Riemann integral emphasizes the interpretation of the integral as a sum or series over a "continuous parameter range".

Insufficiency of the Darboux/Riemann Integral

From the definition of a step function, we can immediately see that the regulated integral of a function $f \in L^\infty([a, b])$ such that $f(x) = 0$ at all except a finite number of points in $[a, b]$ vanishes. For example, the function $f(x) = 0$ for $x \neq 0$, $f(0) = 1$ is integrable on $[0, 1]$ and the regulated integral vanishes.

On the other hand, the function

$$f : [0, 1] \to \mathbb{R}, \quad f(x) = \begin{cases} 1 & \text{if } x = 1/n \text{ for some } n \in \mathbb{N} \setminus \{0\}, \\ 0 & \text{otherwise.} \end{cases} \tag{4.1.14}$$

does not have a regulated integral. Since f vanishes "at almost every point of the interval", we would like the integral of f to be defined and to be zero, but the notion of the regulated integral is not sufficient for this. However, we were able to extend the concept of an integral to the Darboux integral and as we have seen, the Darboux integral of f is indeed zero.

Insufficiency of the Darboux/Riemann Integral

However, there are still functions that seem to vanish "always everywhere" but whose Darboux integral is not defined. The basic example is the Dirichlet function, which we have already mentioned earlier.

4.1.31. Example. The Dirichlet function

$$\chi\colon [0,1] \to \mathbb{R}, \qquad \chi(x) = \begin{cases} 1 & \text{for } x \text{ rational}, \\ 0 & \text{for } x \text{ irrational}. \end{cases} \tag{4.1.15}$$

is not Darboux-integrable.

The reason for this is simple: any lower step function v must satisfy $v(x) \le 0$ while an upper step function must satisfy $u(x) \ge 1$ for all x except a finite number of exceptions (at the "jumps" between the steps). Hence,

$$\inf_{u \in \mathscr{U}_\chi} \int_0^1 u \ge 1 > 0 \ge \sup_{v \in \mathscr{L}_\chi} \int_0^1 v$$

and the two bounds can not have a common value.

Extensions of the Darboux/Riemann Integral

There exist extensions of the Darboux and Riemann integrals that allow us to define an integral even of functions such as χ. While we lack the time to go into details on these, two deserve to be mentioned:

(i) The ***Lebesgue integral*** (dating from around 1917) is the most commonly seen extension. It may be regarded as a modification of the Darboux integral in which step functions are replaced with so-called ***simple functions*** that are basically step functions with an "infinite number" of steps.

(ii) The ***gauge integral***, also called the ***Henstock-Kurzweil*** integral, is a fairly recent formulation (developed in the 1980s) that modifies the Riemann integral to require the mesh size of a tagged partition to depend on the specific tag within each subinterval. Hence, in Definition 4.1.28 $m(P) < \delta$ is replaced by the condition $x_{k+1} - x_k < \delta(\xi_k)$ for a ***gauge function*** $\delta\colon [a, b] \to (0, \infty)$.

It turns out that the Henstock-Kurzweil integral is more general than the Lebesgue integral.

Conclusion

In fact, there is no significant benefit in the introduction of the Darboux and Riemann integrals since:

- All functions we will encounter in practice are at least piecewise continuous and therefore regulated.
- The Darboux integral is not general enough to treat a very wide class of functions that includes, e.g., the Dirichlet function. If it were necessary for us to treat such functions, we would need to develop an even more general concept.

The Darboux and Riemann integrals were mainly introduced to illustrate some basic ideas on how to implement the seemingly simple question of defining the "area under a function". In practice, it is entirely sufficient to consider all integrals as referring to the regulated integral.

The Fundamental Theorem of Calculus

The derivation in the previous chapter can also be applied to complex-valued functions with no change in the arguments. We henceforth assume that the regulated integral has been established for functions $f\colon [a,b] \to \mathbb{C}$.

4.2.1. Theorem. Let $f\colon [a,b] \to \mathbb{C}$ be continuous and set

$$F\colon [a,b] \to \mathbb{C}, \qquad\qquad F(x) := \int_a^x f.$$

Then F is differentiable on (a,b) and

$$F'(x) = f(x), \qquad\qquad x \in (a,b).$$

While this theorem can be proved for general regulated functions, we are content with the (relatively simple) proof for continuous functions. It may be immediately applied to piecewise continuous functions by using (4.1.10). In practice, we will not often come across integrals of functions that are not at least piecewise continuous.

The Fundamental Theorem of Calculus

Proof.
Let h be sufficiently small. Then for $x, x+h \in (a,b)$ we have

$$\begin{aligned} F(x+h) &= \int_a^{x+h} f = \int_a^x f(t)\,dt + \int_x^{x+h} f(t)\,dt = F(x) + \int_x^{x+h} f(t)\,dt \\ &= F(x) + hf(x) + \int_x^{x+h} (f(t)-f(x))\,dt. \end{aligned}$$

Now

$$\begin{aligned} \lim_{h\to 0} \frac{1}{|h|}\left|\int_x^{x+h} (f(t)-f(x))\,dt\right| &\le \lim_{h\to 0} \frac{|h|}{|h|} \sup_{x\le t\le x+h} |f(t)-f(x)| \\ &= \lim_{h\to 0} \sup_{x\le t\le x+h} |f(t)-f(x)| = 0 \end{aligned}$$

Hence the integral is $o(h)$ as $h \to 0$ and $F'(x) = f(x)$. $\square$

The Fundamental Theorem of Calculus

4.2.2. Definition. Let $\Omega \subset \mathbb{R}$ be an open set and $f\colon \Omega \to \mathbb{C}$. We say that $F\colon \Omega \to \mathbb{C}$ is a ***primitive***, ***indefinite integral*** or ***anti-derivative*** of f if $F' = f$. We write

$$F = \int f = \int f(t)\,dt.$$

4.2.3. Lemma. Let $f \in C([a,b],\mathbb{C})$ and F be a primitive of f. Then

$$\int_a^b f = F(b) - F(a).$$

This fundamental result establishes that integration can be regarded as a sort of "inverse differentiation."

The Fundamental Theorem of Calculus

Proof.

We note that by Theorem 4.2.1

$$\frac{d}{dx}\Big(\int_a^x f - F(x)\Big) = f(x) - f(x) = 0,$$

so there exists a $c \in \mathbb{C}$ such that

$$\int_a^x f - F(x) = c$$

Letting $x \to a$, we see that $c = -F(a)$. This implies

$$\int_a^x f = F(x) - F(a).$$

Letting $x \to b$ we obtain the result. $\square$

Practical Integration

We can hence immediately calculate various integrals by guessing their primitives. For example, for $\alpha \neq -1$,

$$\int_a^b x^\alpha \, dx = \frac{1}{\alpha+1} x^{\alpha+1}\Big|_{x=b} - \frac{1}{\alpha+1} x^{\alpha+1}\Big|_{x=a} =: \frac{1}{\alpha+1} x^{\alpha+1}\Big|_a^b.$$

For short, we often write the right-hand side as

$$\int_a^b x^\alpha \, dx = \frac{1}{\alpha+1} x^{\alpha+1}\Big|_a^b = \Big[\frac{1}{\alpha+1} x^{\alpha+1}\Big]_a^b.$$

Similarly, we "see" that

$$\int e^{\alpha x} \, dx = \frac{1}{\alpha} e^{\alpha x}, \qquad \alpha \neq 0,$$

$$\int \frac{1}{x} \, dx = \int \frac{dx}{x} = \ln|x|, \qquad x \neq 0.$$

Substitution Rule

However, not all integrals are obvious. For example,

$$\int \ln x\,dx = x\ln x - x, \qquad \int \frac{dx}{\sqrt{1+x^2}} = \operatorname{Arsinh} x.$$

Since we integrate by "inverse differentiation," we should expect some help from the theorems that govern differentiation of products and compositions of functions.

The following theorem is a direct result of the chain rule of differentiation.

4.2.4. Substitution Rule. Let $f \in \operatorname{Reg}([\alpha,\beta])$ and $g\colon [a,b] \to [\alpha,\beta]$ continuously differentiable. Then

$$\int_a^b (f\circ g)(x)g'(x)\,dx = \int_{g(a)}^{g(b)} f(y)\,dy.$$

Substitution Rule

Proof.

Let F be a primitive of f. Then $(F\circ g)' = (f\circ g)g'$ and

$$\begin{aligned}\int_a^b (f\circ g)(x)g'(x)\,dx &= \int_a^b (F\circ g)'\,dx \\ &= (F\circ g)(x)\big|_a^b = F(x)\big|_{g(a)}^{g(b)} \\ &= \int_{g(a)}^{g(b)} f(y)\,dy.\end{aligned}$$

□

4.2.5. Remark. Using the same proof, we see that the substitution rule can be applied to indefinite integrals (primitives):

$$\int (f\circ g)(x)g'(x)\,dx = \int f(y)\,dy\Big|_{y=g(x)}.$$

Substitution Rule

4.2.6. Example. We want to calculate $\int \frac{dx}{\lambda^2 + x^2}$ for $\lambda \neq 0$. We write

$$\int \frac{dx}{\lambda^2 + x^2} = \int \frac{1}{\lambda^2} \frac{dx}{1 + (x/\lambda)^2}.$$

We now set $f(x) = 1/(1 + x^2)$, $g(x) = y = x/\lambda$. Then $g'(x) = 1/\lambda$, so we have

$$\begin{aligned}
\int \frac{dx}{\lambda^2 + x^2} &= \frac{1}{\lambda} \int \frac{1}{\lambda} \frac{dx}{1 + (x/\lambda)^2} \\
&= \frac{1}{\lambda} \int g'(x) f(g(x))\, dx = \frac{1}{\lambda} \int f(y)\, dy \Big|_{y=g(x)} \\
&= \frac{1}{\lambda} \int \frac{dy}{1 + y^2} \Big|_{y=x/\lambda} = \frac{1}{\lambda} \arctan y \Big|_{y=x/\lambda} = \frac{1}{\lambda} \arctan \frac{x}{\lambda}.
\end{aligned}$$

Substitution Rule

The following "short cut" using Leibniz notation works very well with the substitution rule: Set $y = g(x)$, so $g'(x) = \frac{dy}{dx}$. Then we substitute y for $g(x)$ and dy for $g'(x)\,dx$.

4.2.7. Example. We calculate $\int xe^{-x^2}\,dx$. We set $y = -x^2$. Then

$$\frac{dy}{dx} = -2x \qquad \Leftrightarrow \qquad dy = -2x\,dx \qquad \Leftrightarrow \qquad -\frac{1}{2}dy = x\,dx.$$

Thus

$$\int xe^{-x^2}\,dx = -\frac{1}{2}\int e^y\,dy\Big|_{y=-x^2} = -\frac{1}{2}e^y\big|_{y=-x^2} = -\frac{1}{2}e^{-x^2}.$$

Integration by Parts

Analogously, we obtain a technique for integration based on the product rule of differentiation. (Compare also with Lemma 3.5.39!)

4.2.8. Theorem. Let $f, g \in C^1([a,b],\mathbb{C})$. Then

$$\int_a^b f'(x)g(x)\,dx = f(x)g(x)\Big|_a^b - \int_a^b f(x)g'(x)\,dx.$$

Proof.
This follows from the fact that fg is a primitive of $f'g + g'f$. $\square$

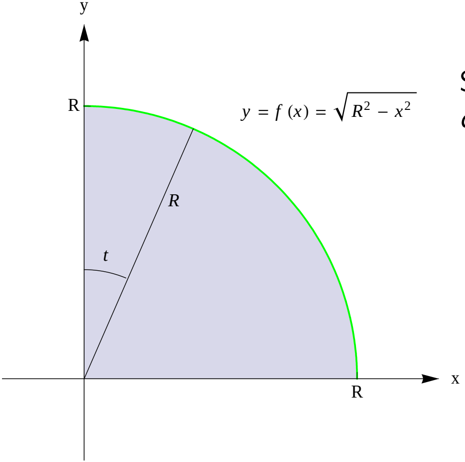
4.2.9. Example.

$$\int \ln x\,dx = \int \underbrace{1}_{f'(x)} \cdot \underbrace{\ln x}_{g(x)}\,dx = \underbrace{x\ln x}_{f(x)g(x)} - \int \underbrace{x}_{f(x)} \cdot \underbrace{\frac{1}{x}}_{g'(x)}\,dx$$

$$= x\ln x - x$$

The Area of a Disc

We now calculate the area of a disc of radius R. In fact, we calculate the area of a quarter-disc by integrating the function f given by $f(x) = \sqrt{R^2 - x^2}$ from 0 to R:



Substituting $x = R\sin t$, $dx = R\cos t\,dt$, we have

$$\begin{aligned}
&\int_0^R \sqrt{R^2 - x^2}\,dx \\
&= \int_0^{\pi/2} \sqrt{R^2 - R^2\sin^2 t}\, R\cos t\,dt \\
&= R^2 \int_0^{\pi/2} \cos^2 t\,dt
\end{aligned}$$

The Area of a Disc

We are now left with calculating $\displaystyle\int_0^{\pi/2} \cos^2 t\,dt$.

- Using integration by parts:

$$\int_0^{\pi/2} \cos^2 t\,dt = \sin t \cos t\Big|_0^{\pi/2} + \int_0^{\pi/2} \sin^2 t\,dt$$
$$= 0 + \int_0^{\pi/2} (1 - \cos^2 t)\,dt = \pi/2 - \int_0^{\pi/2} \cos^2 t\,dt$$
$$\Rightarrow \int_0^{\pi/2} \cos^2 t\,dt = \frac{\pi}{4}$$

- or using trigonometry and substitution:

$$\int_0^{\pi/2} \cos^2 t\,dt = \frac{1}{2}\int_0^{\pi/2} (\cos(2t) + 1)\,dt = \frac{\pi}{4} + \frac{1}{2}\int_0^{\pi/2} \cos(2t)\,dt$$
$$= \frac{\pi}{4} + \frac{1}{4}\int_0^{\pi} \cos(s)\,ds = \frac{\pi}{4} + \frac{1}{4}\sin s\Big|_0^{\pi} = \frac{\pi}{4}$$

The Area of a Disc

Hence the area of a disc of radius R is $4 \cdot \frac{\pi}{4} R^2 = \pi R^2$, as we know from elementary geometry.

Integration is an art, not a skill like differentiation. There are many more "tricks" and "techniques" for calculating integrals, some of which you will encounter in the assignments and recitation classes.

Improper Integrals

An integral

$$\int_a^b f(t)\,dt$$

is called "improper" if

- the domain of integration is unbounded i.e., $a = -\infty$ or $b = \infty$ and/or
- the integrand f is unbounded on (a, b) or otherwise not regulated.

4.2.10. Definition. Assume that $b \leq \infty$ and that $f : [a, b) \to \mathbb{C}$ is regulated on any closed subinterval $[a, x]$, $x < b$. Then

$$\int_a^b f(t)\,dt$$

is called an ***improper integral*** and is said to ***converge*** or ***exist*** if

$$\lim_{x \nearrow b} \int_a^x f(t)\,dt =: I$$

exists.

Improper Integrals

4.2.10. Definition. (continued). The number $I \in \mathbb{C}$ is then called the value of the improper integral and we write

$$I = \int_a^b f(t)\,dt.$$

Similarly, we define

$$\int_a^b f(t)\,dt = \lim_{x \searrow a} \int_x^b f(t)\,dt$$

(if the limit exists) in the case where $b < \infty$. In general, we define the integral

$$\int_a^b f(t)\,dt := \int_a^c f(t)\,dt + \int_c^b f(t)\,dt, \qquad -\infty \le a < b \le \infty$$

for any $c \in \mathbb{R}$ such that both limits on the right exist.

Improper Integrals

4.2.11. Example. Consider the integral $\displaystyle\int_1^\infty \frac{dt}{t^\alpha}$. Then for $\alpha \neq 1$

$$\int_1^x \frac{dt}{t^\alpha} = \frac{1}{1-\alpha}(x^{1-\alpha} - 1) \xrightarrow{x\to\infty} \begin{cases} \frac{1}{\alpha-1} & \text{if } \alpha > 1 \\ \infty & \text{if } \alpha < 1 \end{cases}$$

Thus the integral converges for $\alpha > 1$ and diverges for $\alpha < 1$. We also see that in the case $\alpha = 1$,

$$\int_1^x \frac{dt}{t} = \ln x - \ln 1 \xrightarrow{x\to\infty} \infty,$$

so the integral diverges for $\alpha = 1$.

In the same way we can show that $\displaystyle\int_0^1 \frac{dt}{t^\alpha}$ converges if $\alpha < 1$ and diverges if $\alpha \geq 1$.

Cauchy Property of Functions

We would like to find a criterion for the convergence of improper integrals. As a preliminary step, we establish the following convergence criterion.

4.2.12. Theorem. Let $I \subset \mathbb{R}$ be an interval, $a \in \bar{I} \subset \mathbb{R} \cup \{-\infty, \infty\}$ and $F : I \to \mathbb{C}$. Then the following statements are equivalent:

a) The limit $\lim_{x \to a} F(x)$ exists.

b) The function F satisfies the ***Cauchy property***,

$$\underset{\varepsilon>0}{\forall}\ \underset{\delta>0}{\exists}\ \underset{x,y\in I}{\forall} : |x-a| < \delta \wedge |y-a| < \delta \Rightarrow |F(x) - F(y)| < \varepsilon. \tag{4.2.1}$$

when $a \in \mathbb{R}$, or

$$\underset{\varepsilon>0}{\forall}\ \underset{R>0}{\exists}\ \underset{x,y\in I}{\forall} : \pm x, \pm y > R \Rightarrow |F(x) - F(y)| < \varepsilon. \tag{4.2.2}$$

when $a = \pm\infty$.

Cauchy Property of Functions

Proof.

We will only consider the case $a \in \mathbb{R}$. The case $a = \pm\infty$ is left to you.

- *a)* $\Rightarrow$ *b)* Fix $\varepsilon > 0$. If the limit exists (call it L) we can find $\delta > 0$ such that for all $x, y \in I$

$$|x - a| < \delta \Rightarrow |F(x) - L| < \varepsilon/2.$$

  Then $|F(x) - F(y)| \le |F(x) - L| + |F(y) - L|$ implies b).

- *b)* $\Rightarrow$ *a)* We suppose that (4.2.1) holds and show that $\lim\limits_{x \to a} F(x)$ exists by considering sequences (x_n) with $x_n \to a$.

  Let (x_n) be given and suppose that $x_n \to a$. Then b) implies that $(F(x_n))$ is a complex Cauchy sequence (why? prove this!) and this sequence converges because $\mathbb{C}$ is complete.

Cauchy Property of Functions

Proof (continued).

Next, suppose that (x_n) and (y_n) are two such sequences. Define

$$(z_n) = (x_1, y_1, x_2, y_2, \dots)$$

Then by the same argument as for (x_n), the limit of $F(z_n)$ exists. Since the limit is the only accumulation point of $(F(z_n))$,

$$L := \lim_{n\to\infty} F(z_n) = \lim_{n\to\infty} F(x_n) = \lim_{n\to\infty} F(y_n)$$

so $\lim\limits_{x\to a} F(x)$ exists. $\square$

Cauchy Criterion for Functions

The main application of the Cauchy property is in the study of improper integrals. For example, we have the following corollary of Theorem 4.2.12.

4.2.13. Cauchy Criterion. Let $a \in \mathbb{R}$ and $f\colon [a,\infty) \to \mathbb{R}$ be integrable on every interval $[a,x]$, $x \in \mathbb{R}$. The improper integral

$$\int_a^\infty f(x)\,dx$$

converges if and only if

$$\underset{\varepsilon>0}{\forall}\ \underset{R>0}{\exists}\ \underset{x,y>R}{\forall} \quad \left|\int_x^y f(t)\,dt\right| < \varepsilon.$$

Proof.

Apply Theorem 4.2.12 to $F\colon [a,\infty) \to \mathbb{R}$, $F(x) = \int_a^x f(t)\,dt$, noting that

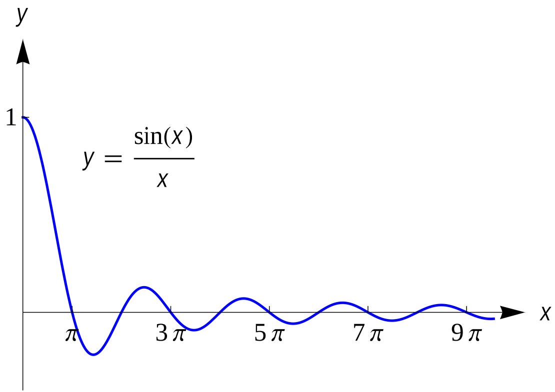
$$F(y) - F(x) = \int_a^y f(t)\,dt - \int_a^x f(t)\,dt = \int_x^y f(t)\,dt. \qquad \square$$

The Dirichlet Integral

4.2.14. Example. The ***Dirichlet integral***

$$\int_0^\infty \frac{\sin x}{x}\,dx \tag{4.2.3}$$

is convergent.



The Dirichlet Integral

To prove that (4.2.3) converges, we consider

$$I(x,y) := \int_x^y \frac{\sin t}{t}\,dt, \qquad x < y, \tag{4.2.4}$$

which we can rewrite as

$$I(x,y) = \int_x^{x+\pi} \frac{\sin t}{t}\,dt - \int_y^{y+\pi} \frac{\sin t}{t}\,dt + \int_{x+\pi}^{y+\pi} \frac{\sin \tau}{\tau}\,d\tau.$$

Substituting $t = \tau - \pi$ in the last integral, we obtain

$$I(x,y) = \int_x^{x+\pi} \frac{\sin t}{t}\,dt - \int_y^{y+\pi} \frac{\sin t}{t}\,dt - \int_x^y \frac{\sin t}{t+\pi}\,dt. \tag{4.2.5}$$

Adding (4.2.4) to (4.2.5), we obtain

$$2I(x,y) = \int_x^{x+\pi} \frac{\sin t}{t}\,dt - \int_y^{y+\pi} \frac{\sin t}{t}\,dt + \pi \int_x^y \frac{\sin t}{t(t+\pi)}\,d\tau. \tag{4.2.6}$$

The Dirichlet Integral

Since $|\sin(t)/t| \le 1/t$ and $|\sin(t)/(t(t+\pi))| \le 1/t^2$ we obtain from (4.2.6) that

$$2|I(x,y)| \le \frac{2\pi}{x} + \pi\int_x^y \frac{dt}{t^2} < \frac{2\pi}{x} + \pi\int_x^\infty \frac{dt}{t^2} = \frac{2\pi}{x} + \frac{\pi}{x}.$$

Fix $\varepsilon > 0$. Then choose $R > 3\pi/(2\varepsilon)$. Then for all $x, y > R$ we have

$$\left|\int_x^y \frac{\sin t}{t}\,dt\right| = |I(x,y)| < \varepsilon.$$

By the Cauchy Criterion 4.2.13, the integral (4.2.3) converges.

4.2.15. Remark. The ***sine cardinal***, sinc: $\mathbb{R} \to \mathbb{R}$,

$$\operatorname{sinc}(x) := \begin{cases} \frac{\sin x}{x} & x \ne 0, \\ 1 & x = 0, \end{cases}$$

plays an important role in signal processing. We will later be able to prove that the value of the integral (4.2.3) is $\pi/2$.

Comparison Test for Improper Integrals

Another important consequence of Theorem 4.2.12 is the following:

4.2.16. Comparison Test. Let $I \subset \mathbb{R}$ and $f : I \to \mathbb{C}$, $g : I \to [0, \infty)$. Suppose that $|f(t)| \le g(t)$ for $t \in I$ and $\int_I g(t)\,dt$ converges. Then $\int_I f(t)\,dt$ also converges.

Proof.
We treat the case $I = [a, b]$ only. Let $F(x) = \int_a^x f$, $G(x) = \int_a^x g$ for $a < x < b$. Then

$$\begin{aligned} |F(x) - F(y)| &= \left| \int_a^x f - \int_a^y f \right| = \left| \int_y^x f \right| \\ &\le \int_y^x |f| \le \int_y^x g = G(x) - G(y). \end{aligned} \tag{4.2.7}$$

Since $\lim\limits_{x \nearrow b} G(x)$ exists, G satisfies the Cauchy criterion (4.2.1). From the estimate (4.2.7), F also satisfies (4.2.1), so $\lim\limits_{x \nearrow b} F(x)$ exists. $\square$

Absolute and Conditional Convergence

4.2.17. Definition. Let $I \subset \mathbb{R}$ and $f : I \to \mathbb{C}$. We say that the improper integral $\int_I f(t)\,dt$ ***converges absolutely*** if the improper integral $\int_I |f(t)|\,dt$ converges.

If $\int_I f(t)\,dt$ converges but is not absolutely convergent, we say that the integral is ***conditionally convergent***.

4.2.18. Remark. An immediate consequence of the Comparison Test 4.2.16 is that every absolutely convergent integral is also convergent.

4.2.19. Example. The Dirichlet integral (4.2.3) is conditionally convergent, i.e.,

$$\int_0^\infty \frac{|\sin x|}{x}\,dx$$

diverges.

Absolute and Conditional Convergence

For any $R > 0$

$$\int_0^R \frac{|\sin x|}{x}\,dx \geq \int_0^{n_R\pi} \frac{|\sin x|}{x}\,dx,$$

where $n_R \in \mathbb{N}$ is the largest integer such that $n_R\pi \leq R$, i.e., $n_R = \lfloor R/\pi \rfloor$. We write

$$\int_0^{n_R\pi} \frac{|\sin x|}{x}\,dx = \sum_{k=1}^{n_R} \int_{(k-1)\pi}^{k\pi} \frac{|\sin x|}{x}\,dx$$

and use

$$\int_{(k-1)\pi}^{k\pi} \frac{|\sin x|}{x}\,dx \geq \frac{1}{k\pi} \int_{(k-1)\pi}^{k\pi} |\sin x|\,dx = \frac{2}{k\pi}$$

to see that

$$\int_0^R \frac{|\sin x|}{x}\,dx \geq \frac{1}{\pi} \sum_{k=1}^{n_R} \frac{1}{k}.$$

Since the harmonic series diverges, it follows that $\lim\limits_{R\to\infty} \int_0^R \frac{|\sin x|}{x}\,dx = \infty$.

Euler Gamma Function

The ***Euler gamma function*** is defined through an improper integral,

$$\Gamma\colon \mathbb{R}_+ \to \mathbb{R}, \qquad \Gamma(t) := \int_0^\infty z^{t-1} e^{-z}\,dz, \qquad t > 0. \tag{4.2.8}$$

The gamma function plays an important role in many parts of mathematics, physics and engineering. For example, standard probabilistic distributions that describe system failure are based on the gamma function (this will be discussed in detail in the course Ve401 Probabilistic Methods in Engineering).

We will first show that the integral (4.2.8) converges absolutely for every $t > 0$. We write

$$\int_0^\infty z^{t-1} e^{-z}\,dz = \int_0^1 z^{t-1} e^{-z}\,dz + \int_1^\infty z^{t-1} e^{-z}\,dz.$$

For $z > 0$ we have $|z^{t-1}e^{-z}| < z^{t-1}$ and for $t > 0$ we know that $t - 1 > -1$ and $\int_0^1 z^{t-1}\,dz$ converges. Hence the first integral converges absolutely.

Euler Gamma Function

Next we consider

$$\int_1^\infty z^{t-1}e^{-z}\,dz. \tag{4.2.9}$$

Recall that $e^x > x^n/n!$ for $x > 0$ and any $n \in \mathbb{N}$. Hence (setting $x = z/2$, $n = \lceil t \rceil = \min\{k \in \mathbb{Z} \colon k > t\}$)

$$z^{t-1}e^{-z/2} = \frac{1}{z^{1-t}e^{z/2}} < \frac{2^{\lceil t \rceil}\lceil t \rceil!}{z^{1-t}z^{\lceil t \rceil}} < C(t) \qquad \text{for } z > 1.$$

Hence $z^{t-1}e^{-z} < C(t)e^{-z/2}$. Since

$$\int_1^\infty e^{-z/2}\,dz = \lim_{x\to\infty}\int_1^x e^{-z/2}\,dz = \lim_{x\to\infty}(2e^{-1/2} - 2e^{-x/2}) = \frac{2}{\sqrt{e}}$$

converges, we see that (4.2.9) and hence $\Gamma(t)$ converges absolutely.

Properties of the Gamma Function

4.2.20. Lemma. For any $t > 0$ we have

$$\Gamma(t+1) = t\Gamma(t).$$

Before we discuss the proof, we note that for improper integrals, the concept of integration by parts generalizes to

$$\int_a^b f'(x)g(x)\,dx = f(x)g(x)\Big|_a^b - \int_a^b f(x)g'(x)\,dx. \tag{4.2.10}$$

where all the integrals are improper and

$$f(x)g(x)\Big|_a^b := \lim_{x\nearrow b} f(x)g(x) - \lim_{x\searrow a} f(x)g(x).$$

All the above limits are of course assumed to exist for (4.2.10) to hold.

Properties of the Gamma Function

Proof of Lemma 4.2.20.

We have

$$\Gamma(t+1) = \int_0^\infty z^t e^{-z}\,dz = -e^{-z}z^t\big|_0^\infty + t\int_0^\infty z^{t-1}e^{-z}\,dz = t\Gamma(t) \quad \square$$

Lemma 4.2.20 implies

$$n! = \Gamma(n+1) \qquad \text{for } n \in \mathbb{N}.$$

The Gamma function is hence a continuous extension of the factorial function to the positive real numbers. It is also the ***only*** continuous extension such that $\ln \circ \Gamma$ is a convex function on $\mathbb{R}_+$. (You will check that Γ is even differentiable in the assignments.)

Euler Gamma Function

The Gamma function is, by its definition, only defined for $t > 0$. However, it turns out that one can find an extension of $\Gamma(t)$ to values $t > -1$, $t \neq 0$, as follows: Set

$$F_1(t) := \frac{\Gamma(t+1)}{t}.$$

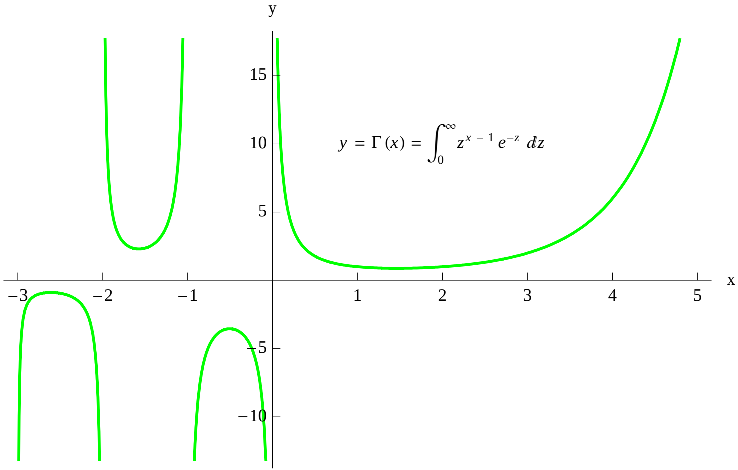
Clearly, F_1 defines a function on $(-1, \infty) \setminus \{0\}$. If $t > 0$, we have $\Gamma(t+1) = t\Gamma(t)$, so

$$F_1(t) = \frac{\Gamma(t+1)}{t} = \frac{t\Gamma(t)}{t} = \Gamma(t)$$

for $t > 0$. Hence F_1 is an extension of Γ to values $t > -1$, $t \neq 0$. In a similar manner, an extension F_2 of F_1 can be found that is defined on $(-2, \infty) \setminus \{-1, 0\}$. This process can be continued indefinitely to achieve an extension of the Gamma function to $\{x \in \mathbb{R} \colon x \neq -n,\ n \in \mathbb{N}\}$. This extension is also denoted $\Gamma(t)$.

Euler Gamma Function

By defining the Gamma function for complex $t \in \mathbb{C}$ it can be proven that the above extension defines a continuous function for all $t \in \mathbb{C}$ except the negative real integers, thereby justifying the definitions of F_1, F_2 etc.



Integrals and Series

From the preceding discussion of improper integrals, it seems that there are obvious analogies between functions and sequences:

$$\begin{array}{rcl} \text{functions } f & \longleftrightarrow & \text{sequences } (a_n) \\ \text{integral } \displaystyle\int_1^\infty f & \longleftrightarrow & \text{series } \displaystyle\sum_{n=1}^\infty a_n \\ \text{Comparison Test 4.2.16} & \longleftrightarrow & \text{Comparison Test 3.5.15} \end{array}$$

and many more. But can we use these analogies in a practical manner to help solve problems?

Integral Test for Series

4.3.1. Integral Test. Let $m \in \mathbb{N}$ and $f\colon [m, \infty) \to [0, \infty)$ be a decreasing function, i.e., $f(x) \ge f(y)$ for $m \le x \le y$, such that $\int_m^x f(t)\,dt$ exists for every $x > m$. Then

$$\int_m^\infty f(x)\,dx < \infty \qquad \Leftrightarrow \qquad \sum_{n=m}^\infty f(n) < \infty. \tag{4.3.1}$$

Proof.
Define the functions

$$\begin{aligned}
g\colon [m, \infty) \to \mathbb{R}, \quad & g(x) = f(n) \quad \text{for } x \in [n-1, n),\ n \in \mathbb{N},\ n \ge m, \\
h\colon [m, \infty) \to \mathbb{R}, \quad & h(x) = f(n-1) \quad \text{for } x \in [n-1, n),\ n \in \mathbb{N},\ n \ge m.
\end{aligned}$$

Integral Test for Series

Proof (continued).

For any $N \in \mathbb{N}$, the functions $g|_{[m,N]}$ and $h|_{[m,N]}$ are just step functions and we find

$$\int_m^N g = \sum_{n=m+1}^{N} f(n), \qquad \int_m^N h = \sum_{n=m+1}^{N} f(n-1) = \sum_{n=m}^{N-1} f(n)$$

Since f is decreasing, we have $g(x) \le f(x) \le h(x)$ for all $x \in [m, \infty)$ and so

$$\sum_{n=m+1}^{N} f(n) = \int_m^N g \le \int_m^N f \le \int_m^N h = \sum_{n=m}^{N-1} f(n)$$

Letting $N \to \infty$, we obtain

$$\sum_{n=m+1}^{\infty} f(n) \le \int_m^{\infty} f \le \sum_{n=m}^{\infty} f(n). \qquad \square$$

Integral Test for Series

4.3.2. Example. The series

$$\sum_{n=1}^{\infty} \frac{\ln(n)}{n^{\alpha}}$$

converges if and only if $\alpha > 1$. Note that $f\colon [1,\infty) \to \mathbb{R}$, $f(x) = x^{-\alpha} \ln x$ is decreasing for $x^{\alpha} > e$:

$$f'(x) = x^{-\alpha-1}(1 - \alpha \ln x)$$

We apply the integral test for $m > e^{1/\alpha}$. Suppose that $\alpha \neq 1$. Integrating by ports,

$$\begin{aligned} \int \frac{\ln x}{x^{\alpha}}\,dx &= \frac{\ln x}{(1-\alpha)x^{\alpha-1}} - \frac{1}{1-\alpha}\int \frac{1}{x^{\alpha}}\,dx \\ &= -\frac{1+(\alpha-1)\ln x}{x^{\alpha-1}(\alpha-1)^2} \end{aligned}$$

so it is easy to see that $\int_3^{\infty} \frac{\ln x}{x^{\alpha}}\,dx$ converges if and only if $\alpha > 1$. The case $\alpha = 1$ can be handled similarly.

Sequences of Functions

Recall from Theorem 4.1.20 that if (f_n) is a sequence of functions $f_n \in C([a, b])$ such that $f_n \xrightarrow[\|\cdot\|_\infty]{n\to\infty} f$ then

$$\int_a^b f_n \xrightarrow{n\to\infty} \int_a^b f.$$

We can apply this to give the following practical result:

4.3.3. Theorem. Let $[a, b] \subset \mathbb{R}$ and (f_n) be a sequence of functions where $f_n \in C^1([a, b])$. Suppose that

$$f_n' \xrightarrow[\|\cdot\|_\infty]{n\to\infty} g \qquad \text{for some } g \in C([a, b])$$

and that for some $x_0 \in [a, b]$ the sequence $(f_n(x_0))$ converges. Then (f_n) converges uniformly to a function $f \in C^1([a, b])$ such that $f' = g$.

Sequences of Functions

Proof.

For $n \in \mathbb{N}$ we have

$$f_n(x) = \int_a^x f_n' + c_n, \qquad\qquad c_n = f_n(a).$$

Let $x = x_0$ as in the theorem. Then

$$c_n = f_n(x_0) - \int_a^{x_0} f_n' \xrightarrow{n\to\infty} \lim_{n\to\infty} f_n(x_0) - \int_a^{x_0} g =: c.$$

Hence for any $x \in [a, b]$ we have

$$f_n(x) \xrightarrow{n\to\infty} \int_a^x g + c =: f(x),$$

so we have pointwise convergence to a function f with $f' = g$.

Sequences of Functions

Proof (continued).

We will show that the convergence is actually uniform:

$$\begin{aligned}\|f_n - f\|_\infty &\le \sup_{x\in[a,b]} \left| \int_a^x f_n' - \int_a^x g \right| + |c_n - c| \\ &\le \sup_{x\in[a,b]} |x-a| \sup_{t\in[a,x]} |f_n'(t) - g(t)| + |c_n - c| \\ &= |b-a| \cdot \|f_n' - g\|_\infty + |c_n - c|.\end{aligned}$$

Since $c_n \to c$ and $f_n' \to g$ uniformly, we are finished. $\square$

Series of Functions

Theorem 4.3.3 can be expressed for series of functions in the following way:

4.3.4. Corollary. Let $[a, b] \subset \mathbb{R}$ and (f_n) be a sequence of functions, $f_n \in C^1([a, b])$, $n \in \mathbb{N}$, such that

$$\sum_{n=0}^{\infty} f_n =: f \tag{4.3.2}$$

converges pointwise on $[a, b]$ and

$$\sum_{n=0}^{\infty} f_n'$$

converges uniformly on $[a, b]$. Then the convergence in (4.3.2) is actually uniform, f is differentiable and

$$f' = \sum_{n=0}^{\infty} f_n'.$$

Differentiation and Integration of Power Series

Let

$$f\colon (-\varrho,\varrho)\to\mathbb{R}, \qquad\qquad f(x)=\sum_{n=0}^{\infty} a_n x^n$$

be the function defined by a power series with radius of convergence ϱ. Recall from Lemma 3.6.8 that a power series converges uniformly within any closed and bounded set within its radius of convergence. Then

(i) f is continuous on $(-\varrho,\varrho)$ by Theorem 3.4.3.

(ii) f is differentiable by Theorem 3.6.12 and

$$\begin{aligned} f'(x) &= \frac{d}{dx}\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty}\frac{d}{dx}(a_n x^n) \\ &= \sum_{n=1}^{\infty} n a_n x^{n-1}, \qquad\qquad x\in(-\varrho,\varrho). \end{aligned}$$

Differentiation and Integration of Power Series

(iii) f may be integrated componentwise by Corollary 4.3.4,

$$\begin{aligned}\int_0^x f(y)\,dy = \int_0^x \Big(\sum_{n=0}^{\infty} a_n y^n\Big)\,dy = \sum_{n=0}^{\infty} \int_0^x (a_n y^n)\,dy \\ = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}, \qquad\qquad x \in (-\varrho, \varrho).\end{aligned}$$

In summary:

> *Within the radius of convergence we can "interchange" differentiation and summation as well as integration and summation of a power series.*

The analogous statements hold for power series "centered at $x_0 \in \mathbb{R}$",

$$f\colon (x_0 - \varrho, x_0 + \varrho) \to \mathbb{R}, \qquad f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n. \tag{4.3.3}$$

Coefficients of Power Series

For the power series

$$f\colon (x_0-\varrho, x_0+\varrho) \to \mathbb{R}, \qquad\qquad f(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n.$$

we have

$$a_0 = f(x_0).$$

Moreover, $f'(x) = 1 \cdot a_1 + 2a_2(x-x_0)^{2-1} + \dots$, so

$$a_1 = \frac{f'(x_0)}{1}$$

and, in general,

$$a_n = \frac{f^{(n)}(x_0)}{n!}, \qquad\qquad n \in \mathbb{N}. \tag{4.3.4}$$

Taylor's Theorem

Hence, a power series

$$f\colon (x_0-\varrho, x_0+\varrho) \to \mathbb{R}, \qquad f(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n$$

has coefficients given by

$$a_n = \frac{f^{(n)}(x_0)}{n!}, \qquad n \in \mathbb{N}.$$

We are now interested in the converse question: If $I \subset \mathbb{R}$ is an interval and $f \in C^\infty(\mathbb{R})$ a smooth function, does the power series

$$\sum_{n=0}^{\infty} a_n (x-x_0)^n, \qquad a_n = \frac{f^{(n)}(x_0)}{n!}, \qquad n \in \mathbb{N}.$$

exist for $x_0 \in I$? Does it converge to $f(x)$ for all x?

Taylor's Theorem

4.3.5. Taylor's Theorem. Let $I \subset \mathbb{R}$ an open interval and $f \in C^k(I)$. Let $x \in I$ and $y \in \mathbb{R}$ such that $x + y \in I$. Then for all $p \le k$,

$$f(x+y) = f(x) + \frac{1}{1!} f'(x) y + \cdots + \frac{1}{(p-1)!} f^{(p-1)}(x) y^{p-1} + R_p \quad (4.3.5)$$

with the remainder term

$$R_p := \int_0^1 \frac{(1-t)^{p-1}}{(p-1)!} f^{(p)}(x+ty) y^p \, dt.$$

Proof.
We prove the theorem by induction in p. For $p = 1$, (4.3.5) becomes

$$f(x+y) - f(x) = \int_0^1 f'(x+ty) \cdot y \, dt = \int_x^{x+y} f'(z) \, dz,$$

which is just the fundamental theorem of calculus.

Taylor's Theorem

Proof (continued).

In order to prove that (4.3.5) for p implies (4.3.5) for $p+1$, we show

$$R_p = \frac{1}{p!} f^{(p)}(x) y^p + R_{p+1}.$$

Integrating by parts, we have

$$\begin{aligned}
R_p &= \int_0^1 \frac{(1-t)^{p-1}}{(p-1)!} f^{(p)}(x+ty) y^p \, dt \\
&= - \frac{(1-t)^p}{p!} f^{(p)}(x+ty) y^p \Big|_{t=0}^{1} + \int_0^1 \frac{(1-t)^p}{p!} \frac{d}{dt} f^{(p)}(x+ty) y^p \, dt \\
&= \frac{1}{p!} f^{(p)}(x) y^p + \int_0^1 \frac{(1-t)^p}{p!} f^{(p+1)}(x+ty) y^{p+1} \, dt \\
&= \frac{1}{p!} f^{(p)}(x) y^p + R_{p+1}.
\end{aligned}$$

□

Lagrange Remainder

If we set $x = x_0$ and $y = x - x_0$ in (4.3.5),

$$f(x) = \sum_{n=0}^{p-1} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + R_p \qquad (4.3.6)$$

for $f \in C^p(I)$, with

$$\begin{aligned} R_p &= \int_0^1 \frac{(1-t)^{p-1}}{(p-1)!} f^{(p)}(tx + (1-t)x_0)\, dt \cdot (x - x_0)^p \\ &= \int_{x_0}^{x} \frac{(x-\tau)^{p-1}}{(p-1)!} f^{(p)}(\tau)\, d\tau \end{aligned} \qquad (4.3.7)$$

This is known as the ***Lagrange form*** of the remainder. The remainder term satisfies the estimate

$$|R_p| \le \frac{1}{(p-1)!} \sup_{y \in [x_0, x]} |f^{(p)}(y)| \cdot |x - x_0|^p \qquad (4.3.8)$$

Analytic Functions and Taylor Series

We will call

$$T_{f;p,x_0}(x) := f(x) - R_{p+1} = \sum_{n=0}^{p} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n \tag{4.3.9}$$

the ***Taylor polynomial of degree p of f at x_0***. The series

$$\lim_{p\to\infty} T_{f;p,x_0}(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n \tag{4.3.10}$$

is called the ***Taylor series of f at x_0***.

4.3.6. Definition. Let $I \subset \mathbb{R}$ be an open interval and $f \in C^\infty(I)$. We say that f is ***real-analytic*** or just ***analytic*** at $x_0 \in I$ if there exists a neighborhood $B_\varepsilon(x_0) = (x_0 - \varepsilon, x_0 + \varepsilon) \subset I$ such that

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

for all $x \in B_\varepsilon(x_0)$.

Taylor Series

4.3.7. Example. Let $f\colon (-1,\infty) \to \mathbb{R}$, $f(x) = 1/(1+x)$. Letting $x_0 = 0$, we have

$$f(x) = \sum_{n=0}^{p-1} \frac{f^{(n)}(0)}{n!} x^n + R_p.$$

We can easily check that

$$f^{(n)}(x) = (1+x)^{-1-n} \cdot (-1)^n n!,$$

so $f^{(n)}(0) = n!(-1)^n$ and

$$f(x) = \sum_{n=0}^{p-1} \frac{f^{(n)}(0)}{n!} x^n + R_p = \sum_{n=0}^{p-1} (-1)^n x^n + R_p.$$

Thus the Taylor polynomial of degree p for f at $x_0 = 0$ is given by

$$T_{f;p,0}(x) = \sum_{n=0}^{p} (-1)^n x^n$$

Taylor Series

We will show that f actually has a Taylor series at $x_0 = 0$,

$$f(x) = \sum_{n=0}^{\infty} (-1)^n x^n, \qquad \text{for } x \in (-1, 1). \tag{4.3.11}$$

First, note that the power series in (4.3.11) has radius of convergence $\varrho = 1$, so the right-hand side is well-defined.

We now prove that the Taylor series actually converges to f, i.e., for any fixed $x \in (-1, 1)$ $R_p \to 0$ as $p \to \infty$. This is easy to show if $x \geq 0$. In that case, we can use the estimate (4.3.8) to see that

$$\begin{aligned} |R_p| &\leq \frac{1}{(p-1)!} \sup_{y \in [x_0, x]} |f^{(p)}(y)| \cdot |x - x_0|^p \\ &= \frac{1}{(p-1)!} \sup_{y \in [0, x]} |1 + y|^{-1-p} p! |x|^p = p|x|^p \xrightarrow{p \to \infty} 0. \end{aligned}$$

Taylor Series

However, if $-1 < x < 0$ we have to use a more careful estimate. In that case, the Lagrange remainder (4.3.7) gives

$$R_p = \int_0^x \frac{(x-\tau)^{p-1}}{(p-1)!} f^{(p)}(\tau)\,d\tau = -p\int_0^{-|x|} \frac{(|x|+\tau)^{p-1}}{(1+\tau)^{p+1}}\,d\tau$$
$$= p\int_0^{|x|} \frac{(|x|-\tau)^{p-1}}{(1-\tau)^{p+1}}\,d\tau$$

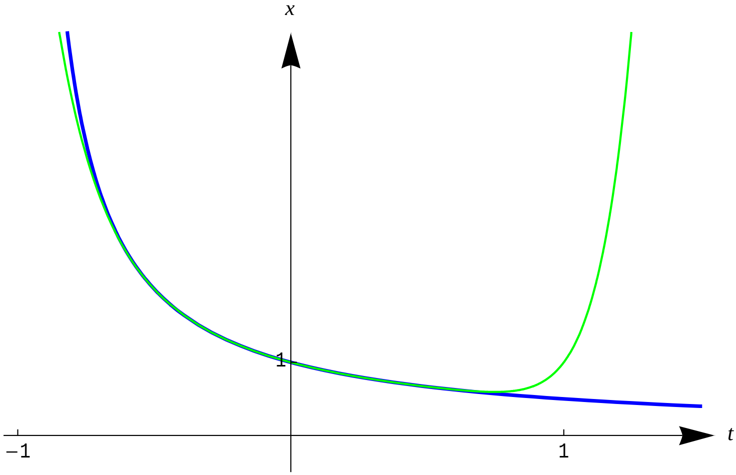
Then

$$|R_p| \le p \sup_{\tau\in[0,|x|]} \left|\frac{|x|-\tau}{1-\tau}\right|^{p-1} \int_0^{|x|} \frac{1}{(1-\tau)^2}\,d\tau$$
$$= p|x|^{p-1}\left|1-\frac{1}{1-|x|}\right| \xrightarrow{p\to\infty} 0.$$

Thus we have proven that for every $x \in (-1,1)$ the formal Taylor series of f actually converges to f. The convergence is uniform in $[-R,R]$ for every $R<1$ (by the general theory of power series).

Taylor Series

A 10-term approximation (green) to $f(x) = 1/(1+x)$ (blue):



Taylor Series

4.3.8. Remark. The Taylor series for the previous example could have been obtained in an easier way: using the geometric series expansion, we have

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-x)^n$$

whenever $|x| < 1$. Since the Taylor series of a function f at x_0 is unique (the coefficients are determined by (4.3.4)) it ***doesn't matter how we obtain a series expansion*** - it is always the Taylor series.

Taylor Series

4.3.9. Remark. In the previous example, the Taylor series converged only for $|x| < 1$. This was not surprising, since the function has a singularity at $x = -1$. However, the function

$$f : \mathbb{R} \to \mathbb{R}, \qquad\qquad f(x) = \frac{1}{1+x^2}$$

has a series expansion at $x_0 = 0$ given by

$$f(x) = \frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-1)^n x^{2n}, \qquad\qquad |x| < 1.$$

The function evidently has no poles in $\mathbb{R}$, but nevertheless the power series only has radius of convergence $\varrho = 1$. In this case, the poles that limit the radius of convergence are hidden in the complex plane, at $x = \pm i$.

Taylor Series

We can use the term-by-term integrability of power series to obtain series for functions where this would otherwise be difficult:

4.3.10. Example. Consider the function

$$f\colon \mathbb{R} \to \mathbb{R}, \qquad f(x) = \arctan x.$$

To find the Taylor series (4.3.10) directly, we would need to calculate derivatives:

$$f'(x) = \frac{1}{1+x^2}, \qquad f''(x) = \frac{-2x}{(1+x^2)^2},$$

$$f'''(x) = \frac{-2}{(1+x^2)^2} + \frac{2\cdot(2x)^2}{(1+x^2)^3}$$

and it is obvious that obtaining higher-order derivatives becomes ever more complicated.

Taylor Series

However, we can calculate

$$\begin{aligned}\arctan x &= \int_0^x \frac{dy}{1+y^2} = \int_0^x \sum_{n=0}^{\infty} (-1)^n y^{2n}\, dy = \sum_{n=0}^{\infty} (-1)^n \int_0^x y^{2n}\, dy \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}.\end{aligned}$$

The radius of convergence of the power series is $\varrho = 1$, and the above equalities hold by Corollary 4.3.4 for all $x \in (-1, 1)$. So far, so good.

However, the situation at $x = 1$ is now quite interesting: both $\arctan 1$ and $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$ converge (the latter by the Leibniz theorem). If they were equal, we would have

$$\arctan 1 = \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + - \dots,$$

which is known as the ***Leibniz series***.

Abel's Limit Theorem

The justification for the Leibniz series is in fact provided by the following theorem, which will be proved in the assignments:

4.3.11. Abel's Limit Theorem. Let $f\colon [0,1) \to \mathbb{R}$ be defined by

$$f(x) = \sum_{k=0}^{\infty} a_k x^k,$$

where $\sum a_k$ converges. Then $\sum a_k x^k$ converges uniformly on $[0,1]$, so $f \in C([0,1])$ and

$$\sum_{k=0}^{\infty} a_k = \lim_{x \nearrow 1} \sum_{k=0}^{\infty} a_k x^k.$$

Abel Summability

Abel's Limit Theorem gives rise to another consideration: if a series $\sum_{k=0}^{\infty} a_k$ converges, then the power series

$$\sum_{k=0}^{\infty} a_k x^k$$

has radius of convergence $\varrho \geq 1$ and certainly

$$\lim_{x \nearrow 1} \sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} a_k.$$

We can hence generalize the concept of summability of sequences, by defining the sum to be the limit of the power series; we define the ***Abel sum***

$$\mathrm{A}-\sum_{n=0}^{\infty} a_n := \lim_{x \nearrow 1} \sum_{n=0}^{\infty} a_n x^n$$

for any complex sequence (a_n) for which the right-hand side exists.

Abel Summability

4.3.12. Remark. If $\sum_{k=0}^{\infty} a_k$ converges,

$$\sum_{k=0}^{\infty} a_k = \mathrm{A}-\sum_{n=1}^{\infty} a_n.$$

But we also have sequences whose series does not converge in the classical sense, such as the following:

4.3.13. Example. (a_n) with $a_n = (-1)^n$. From Example 4.3.7 we know that

$$\mathrm{A}-\sum_{n=0}^{\infty}(-1)^n = \lim_{x\nearrow 1}\sum_{n=0}^{\infty}(-1)^n x^n = \lim_{x\nearrow 1}\frac{1}{1+x} = \frac{1}{2}$$

so the sequence $(1,-1,1,-1,1,\dots)$ is Abel summable with sum $1/2$. We hence write

$$1-1+1-1+1-+\cdots = \frac{1}{2} \tag{4.3.12}$$

where the left-hand side is a classically divergent series.

Abel Summability

In the assignments you will show that

$$\mathrm{A}-\sum_{n=1}^{\infty} n(-1)^{n+1} = 1-2+3-4+5-+\cdots = \frac{1}{4}.$$

These types of divergent series were historically analyzed by Euler and others through naive methods, which we briefly sketch here.

Let us revisit (4.3.12). If we write $S = 1-1+1-1+1-1+-\ldots$ observe that

$$\begin{aligned} 1-S &= 1-(1-1+1-1+1-1+-\ldots) \\ &= 1-1+1-1+1-+\ldots = S \end{aligned}$$

Hence we can deduce that $S = 1-S$ or

$$S = 1-1+1-1+1-1+-\ldots = 1/2.$$

Divergent Series

In a similar way we can tackle a more interesting series: let

$$S = 1 + 2 + 3 + 4 + 5 + 6 + 7 + \dots$$

Then

$$\begin{aligned} -3S = (S - 2 \cdot 2S) &= 1 + 2 + 3 + 4 + \dots - 2(2 + 4 + 6 + 8 + \dots) \\ &= 1 - 2 + 3 - 4 + 5 - 6 + - \dots \end{aligned}$$

We can further write

$$\begin{aligned} -3S &= 1 - 2 + 3 - 4 + 5 - 6 + - \dots \\ &= 1 - (2 - 3 + 4 - 5 + - \dots) \\ &= 1 - (1 - 2 + 3 - 4 + - \dots) - (1 - 1 + 1 - 1 + - \dots) \\ &= 1 + 3S - 1/2 \end{aligned}$$

Together, this gives

$$S = 1 + 2 + 3 + 4 + 5 + 6 + 7 + \dots = -\frac{1}{12}. \tag{4.3.13}$$

Divergent Series

It may seem strange that a divergent series yields a "sum" which is negative. However, mathematicians in the 17$^{\text{th}}$ century did believe that "beyond positive infinity" lay the negative numbers.

It is remarkable that this concept is even relevant in physics today: for example, physical system of atoms can have a "negative (absolute) temperature" and then be "hotter" than any such system with a "positive temperature".

Reference Max-Planck-Gesellschaft. "A temperature below absolute zero: Atoms at negative absolute temperature are the hottest systems in the world." ScienceDaily, 4 Jan. 2013. http://www.sciencedaily.com/releases/2013/01/130104143516.htm.

The actual sum (4.3.13) occurs in quantum field theory and string theory, so that it is not "nonsense" as you may think. Let us try to justify the sum in a more rigorous fashion.

Riemann Zeta Function

It turns out that the series $1 + 2 + 3 + 4 + 5 + 6 + \dots$ is not Abel-summable: the limit

$$\text{A}-\sum_{n=1}^{\infty} n = \lim_{x \nearrow 1} \sum_{n=1}^{\infty} nx^n = \lim_{x \nearrow 1} \frac{x}{(1-x)^2}$$

does not exist! To justify (4.3.13), a more sophisticated approach is necessary.

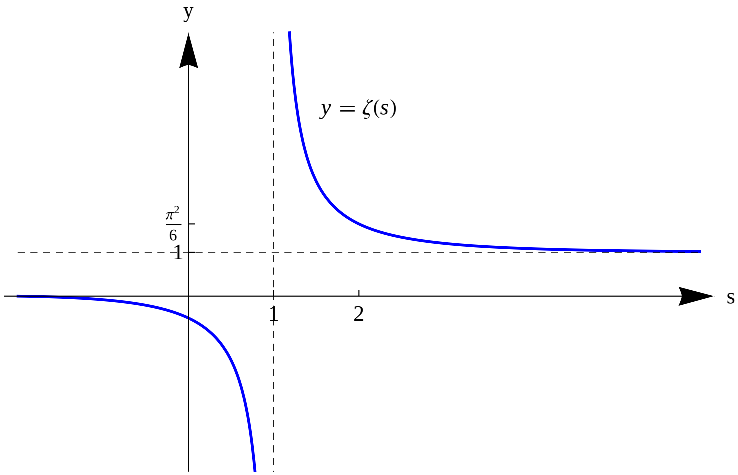
The ***Riemann zeta function*** is defined as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

Note that the right-hand side is just the p-series, so we know that $\zeta(s)$ exists for $s > 1$. However, ζ can be extended to a function defined on $\mathbb{R} \setminus \{1\}$. This extension is analogous to, but more complicated than, the extension of the Euler Gamma function (see Slide 584).

Riemann Zeta Function

We denote the extension by ζ also; a sketch is shown below:



For example, $\zeta(2) = \sum \frac{1}{n^2} = \frac{\pi^2}{6}$ (see the exercises).

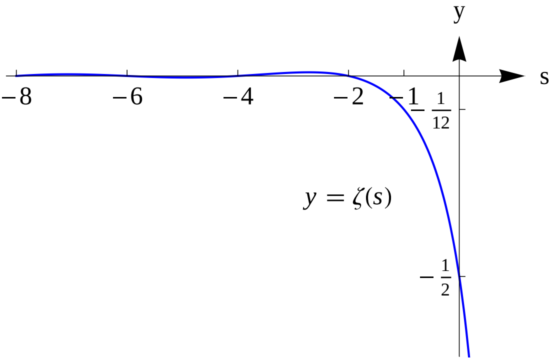
Riemann Zeta Function

Now some divergent series might be considered as special cases of the zeta function. For example,

$$\zeta(0) = \sum_{n=1}^{\infty} \frac{1}{n^0} = \sum_{n=1}^{\infty} 1 = 1 + 1 + 1 + 1 + \dots,$$
$$\zeta(-1) = \sum_{n=1}^{\infty} \frac{1}{n^{-1}} = \sum_{n=1}^{\infty} n = 1 + 2 + 3 + 4 + \dots$$

We see that the series in (4.3.13) is given by $\zeta(-1)$. Let us analyze the graph of the zeta function more closely in this region.

Riemann Zeta Function



Observe that $\zeta(0) = -1/2$ and $\zeta(-1) = -1/12$! Thus, it (4.3.13) does make sense if we regard the summation by extending the zeta function. This summation procedure is called ***zeta function regularization***.

Riemann Zeta Function

In order to calculate $\zeta(-1)$, we need to understand more about the extension of the zeta function. In particular, we would need to prove that the extension of the zeta function satisfies the ***functional equation***

$$\zeta(1-s) = \frac{2\Gamma(s)}{(2\pi)^s} \cos\left(\frac{\pi}{2}s\right)\zeta(s) \tag{4.3.14}$$

where Γ is the Euler Gamma function. If we set $s = 2$ in (4.3.14) we obtain

$$\zeta(-1) = \frac{2}{4\pi^2}\Gamma(2)\cos(\pi)\zeta(2) = -\frac{1}{12}.$$

The proof of (4.3.14) is today usually performed using methods of complex analysis. An elementary (but complicated) proof, retracing Euler's original calculations, is available on SAKAI.

Reference B. Cais, ***Divergent series: why*** $\boldsymbol{1+2+3+\cdots = -1/12}$, retrieved from http://www.math.mcgill.ca/bcais/papers.html

A Deceptively Simple Power Series

Consider the power series

$$\sum_{n=1}^{\infty} \frac{n^n}{n!} z^n. \tag{4.3.15}$$

The radius of convergence can be found from

$$\lim_{n\to\infty} \frac{(n+1)^{n+1} n!}{n^n (n+1)!} = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e$$

(see Proposition 3.7.2), so

$$\varrho = \frac{1}{e}.$$

We are interested in the convergence at $\pm 1/e$. Let us discuss first the series

$$\sum_{n=1}^{\infty} \frac{n^n}{n! e^n}. \tag{4.3.16}$$

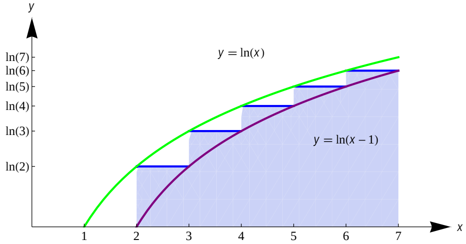
Elementary Estimate on the Factorial

To analyze the convergence of (4.3.16), we need an estimate on the growth of $n!$. We may obtain an elementary estimate as follows: Let

$$S_n = \ln(n!) = \sum_{k=1}^{n} \ln k.$$

Then it is easy to see that

$$\int_1^{n+1} \ln x\,dx > S_n > \int_2^{n+1} \ln(x-1)\,dx = \int_1^{n} \ln x\,dx. \tag{4.3.17}$$



Elementary Estimate on the Factorial

Integrating (4.3.17), we immediately obtain

$$\ln \frac{n^n}{e^{n-1}} < S_n < \ln \frac{(n+1)^{n+1}}{e^n}$$

or, since $S_n = \ln(n!)$,

$$\frac{n^n}{e^{n-1}} < n! < \frac{(n+1)^{n+1}}{e^n}. \tag{4.3.18}$$

Noting that $(1 + 1/n)^n < e$ by Lemma 3.7.4, we have $(n+1)^n < en^n$ and obtain

$$e\left(\frac{n}{e}\right)^n < n! < e(n+1)\left(\frac{n}{e}\right)^n. \tag{4.3.19}$$

Elementary Estimate on the Factorial

4.3.14. Remark. From (4.3.19) and the squeeze theorem, we find

$$\frac{\sqrt[n]{n!}}{n} \xrightarrow{n\to\infty} \frac{1}{e}.$$

This relation is useful when applying the root test to series with factorial coefficients.

We can now analyze the convergence of (4.3.15) at $x = \varrho = 1/e$. The estimate (4.3.19) gives

$$\frac{1}{n!}\left(\frac{n}{e}\right)^n > \frac{1}{n+1}\frac{1}{e}$$

so (4.3.16) diverges by comparison with the harmonic series.

Elementary Estimate on the Factorial

We now turn to the convergence of (4.3.15) at $z = -1/e$, i.e., the convergence of

$$\sum_{n=1}^{\infty} (-1)^n \frac{n^n}{n! e^n}. \tag{4.3.20}$$

We would like to apply the Leibniz criterion to show the convergence of (4.3.20). However, while it is not difficult to show that the sequence (a_n),

$$a_n = \frac{n^n}{n! e^n}$$

is decreasing, it does not follow that it converges to zero. The estimate (4.3.19) only yields $a_n < e$ for all $n \in \mathbb{N}$. Thus, we need a finer estimate on the behavior of $n!$.

Stirling's Formula

We say that two functions $f, g\colon \mathbb{R} \to \mathbb{R}$ are ***asymptotically equal as $x \to \infty$***, writing $f(x) \sim g(x)$ as $x \to \infty$, if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1.$$

This implies that

$$\lim_{x\to\infty} \frac{g(x) - f(x)}{g(x)} = 0,$$

i.e., the relative difference between $f(x)$ and $g(x)$ converges to zero. We now prove a famous result on the asymptotic behavior of the factorial which has important applications in probability theory and statistics.

4.3.15. Stirling's Formula. $n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$ as $n \to \infty$.

Stirling's Formula

Proof.

We will consider the sequence (a_n) given by

$$a_n := \frac{n!}{\sqrt{2n}\left(\frac{n}{e}\right)^n}$$

and first show that it converges to some $C \in \mathbb{R}$. Write

$$b_n := \ln a_n.$$

Then, after some elementary transformations,

$$b_n - b_{n+1} = \frac{2n+1}{2} \ln \frac{n+1}{n} - 1. \tag{4.3.21}$$

We want to show that the right-hand side is positive, so that (b_n) is decreasing.

Stirling's Formula

Proof (continued).

We will work with a series expansion for $\ln \frac{n+1}{n}$, $n \in \mathbb{N} \setminus \{0\}$. The most immediate way to obtain such an expansion would be to use the Taylor series

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + - \dots, \qquad |x| < 1. \qquad (4.3.22)$$

However, this leads to an alternating series, which will not be useful in estimating the RHS of (4.3.21):

$$\ln \frac{n+1}{n} = \ln\left(1 + \frac{1}{n}\right) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \frac{1}{n^k}$$

Instead, we might use that

$$-\ln(1-x) = x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \dots, \qquad |x| < 1. \qquad (4.3.23)$$

Stirling's Formula

Proof (continued).

Then

$$\ln\frac{n+1}{n} = -\ln\frac{n}{n+1} = -\ln\left(1-\frac{1}{n+1}\right) = \sum_{k=1}^{\infty}\frac{1}{k}\frac{1}{(n+1)^k}$$
$$= \frac{1}{n+1} + \sum_{k=2}^{\infty}\frac{1}{k}\frac{1}{(n+1)^k}.$$

However, without additional estimates, this is not sufficient to show that the RHS of (4.3.21) is positive. We need to use a more sneaky expansion. Note that (4.3.22) and (4.3.23) give

$$\ln\frac{1+x}{1-x} = \ln(1+x) - \ln(1-x) = 2\sum_{k=0}^{\infty}\frac{x^{2k+1}}{2k+1}.$$

Stirling's Formula

Proof (continued).

We then have

$$\begin{aligned}\ln\frac{n+1}{n} &= \ln\frac{1+\frac{1}{2n+1}}{1-\frac{1}{2n+1}} = 2\sum_{k=0}^{\infty}\frac{1}{2k+1}\frac{1}{(2n+1)^{2k+1}}\\ &= \frac{2}{2n+1} + 2\sum_{k=1}^{\infty}\frac{1}{2k+1}\frac{1}{(2n+1)^{2k+1}}\end{aligned}$$

Inserting this into (4.3.21), we have

$$b_n - b_{n+1} = \frac{2n+1}{2}\ln\frac{n+1}{n} - 1 = \sum_{k=1}^{\infty}\frac{1}{2k+1}\frac{1}{(2n+1)^{2k}} > 0.$$

It follows that the sequence (b_n) is decreasing.

Stirling's Formula

Proof (continued).

Observe that

$$b_n - b_{n+1} < \sum_{k=1}^{\infty} \frac{1}{(2n+1)^{2k}} = \frac{1}{(2n+1)^2} \frac{1}{1 - \frac{1}{(2n+1)^2}} = \frac{1}{4} \frac{1}{n(n+1)}.$$

Hence,

$$\begin{aligned} b_1 - b_n &= (b_1 - b_2) + (b_2 - b_3) + \cdots + (b_{n-1} - b_n) \\ &< \frac{1}{4} \sum_{k=1}^{n-1} \frac{1}{k(k+1)} \\ &< \frac{1}{4} \sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \frac{1}{4} \sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+1} \right) = \frac{1}{4}. \end{aligned}$$

Stirling's Formula

Proof (continued).

so

$$b_n > b_1 - \frac{1}{4} = \frac{e}{\sqrt{2}} - \frac{1}{4}$$

and (b_n) is bounded below. Hence, (b_n) converges. Since the exponential function is continuous, we have

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} e^{b_n} = e^{\lim\limits_{n\to\infty} b_n},$$

so (a_n) converges, i.e.,

$$C := \lim_{n\to\infty} \frac{n!}{\sqrt{2n}\left(\frac{n}{e}\right)^n}$$

exists.

Stirling's Formula

Proof (continued).

We now use the Wallis product formula,

$$\frac{\pi}{2} = \lim_{n\to\infty} \frac{2^{4n}(n!)^4}{((2n)!)^2(2n+1)}$$

(which is proven in the assignments). Then

$$\begin{aligned}\frac{\pi}{2} &= \lim_{n\to\infty} \frac{2^{4n}}{(2n+1)} \left(\sqrt{2n}\left(\frac{n}{e}\right)^n\right)^4 \left(\frac{n!}{\sqrt{2n}\left(\frac{n}{e}\right)^n}\right)^4 \\ &\quad\times \left(\frac{1}{\sqrt{4n}\left(\frac{2n}{e}\right)^{2n}}\right)^2 \left(\frac{\sqrt{4n}\left(\frac{2n}{e}\right)^{2n}}{(2n)!}\right)^2 \\ &= C^2 \lim_{n\to\infty} \frac{n^2}{n(2n+1)} = \frac{C^2}{2}.\end{aligned}$$

Stirling's Formula

Proof (continued).

It follows that $C = \sqrt{\pi}$ and hence

$$\lim_{n\to\infty} \frac{n!}{\sqrt{2n}\left(\frac{n}{e}\right)^n} = \sqrt{\pi},$$

completing the proof. $\square$

We can now see that there exists a $C > 0$ such that

$$a_n = \frac{n^n}{n!e^n} < \frac{C}{\sqrt{n}}$$

for all n large enough, i.e., the sequence (a_n) converges to zero monotonically. Thus, the power series (4.3.15) converges for $z = -1/e$.

Final Exam

The preceding material completes the final third of the course material. It encompasses everything that will be the subject of the Final Exam.

The exam date and time will be announced on Canvas.

No calculators or other aids will be permitted during the exam. A sample exam with solutions has been uploaded to Canvas. Please study it carefully, including the instructions on the cover page.