Review V(Slides 308-330)

Differentiation

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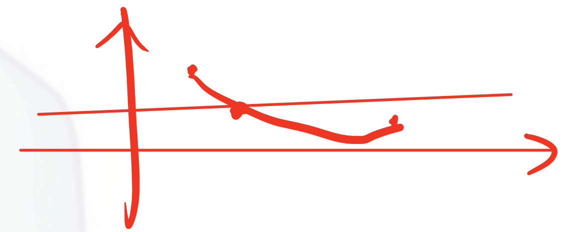
VV186 - Honors Mathmatics II

Exercise (Left in RC4)



if $f(0), f(1) \cdots f(n-1) \neq 1$, $\exists\, i.\ j.$ $f(i) > 1.$ $f(j) < 1.$

Suppose $f\colon [0, n]$, $n \in \mathbb{N}$ is a continuous function, and is differentiable on $(0, n)$. Furthermore, assume that

$f(k) = 1.$

$$f(0) + f(1) + \cdots + f(n-1) = n, \quad f(n) = 1$$

Show that there must exist $c \in (0, n)$ such that $f'(c) = 0$.

Rolle Theorem.

$a \qquad b$ continuous. $\exists\, t \in (a, b)$

$f(a) = f(b).$ differentiable. $f'(t) = 0.$

Exercise (Left in RC4)

In this exercise, we would like to give a deeper investigation of Lipschitz condition. If a real function $T : \Omega \to \mathbb{R}$ satisfies

$$|T(x) - T(y)| \leq k \cdot |x - y|^{\alpha}$$

for any $x, y \in \Omega$, we say T satisfies "Lipschitz condition of order α".

1. Show that if $\alpha > 0$, then T is continuous.
2. Show that if $\alpha > 1$, then T is a constant function, i.e.,

$$\exists_{C \in \mathbb{R}} \quad T(x) = C$$

let $\delta < \left(\frac{\varepsilon}{k}\right)^{\frac{1}{\alpha}}$

Target: $\forall \varepsilon > 0, \forall y, \exists \delta > 0, \forall x \in (y - \delta, y + \delta)$

$|T(x) - T(y)| \leq k \cdot \underbrace{|x - y|}_{< \delta}{}^{\alpha} < \varepsilon.$

$$f'(x) = 0.$$

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \underset{\Delta}{\leq} \frac{K\cdot h^{\alpha}}{h} = K\cdot h^{\underbrace{\alpha-1}_{}>0}$$

$$= 0. \qquad h \longrightarrow 0.$$

Differentiation and Uniformly Continuity

Hint:

3min. Mean value Theorem.

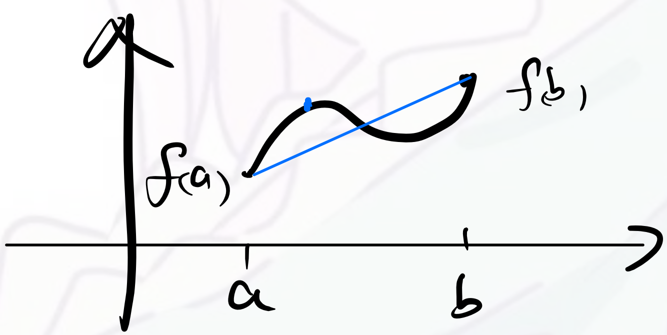
Think: if f(x) uniformly continuous, derivative also bounded?

(1) If the derivative for f(x) is bounded for $x \in (a, b)$, then f(x) is uniformly continuous on f(x). (2) Show that f(x)=sin(x) is uniformly continous. (3) Show that f(x)=arctan(x) is uniformly continuous.

cos x

Mean Value Theorem:

if f(x) is differentiable, $\exists\, t \in (a,b)$, $f'(t) = \dfrac{f(a)-f(b)}{a-b}$.



$$\frac{1}{1+x^2} \leq 1$$

$\forall \varepsilon>0.\ \exists \delta.\ |x_1-x_2|<\delta,\ |f(x_1)-f(x_2)|<\varepsilon.$

$$\left|\frac{f(x_1)-f(x_2)}{x_1-x_2}\right| = f'(t) \leq M$$

$t\in(x_1, x_2).$

$$\delta < \frac{\varepsilon}{M}.$$

$$|f(x_1)-f(x_2)| \leq M\,|\underbrace{x_1-x_2}_{<\delta}| < \varepsilon.$$

$f(x)=\sqrt{x}\quad (0,1).$

$\delta<\varepsilon^2$

$0<x_1<x_2<1$

$$\sqrt{x_2}-\sqrt{x_1} = \frac{x_2-x_1}{\sqrt{x_1}+\sqrt{x_2}} < \frac{x_2-x_1}{\sqrt{x_1}+\sqrt{x_2}}.$$

$$(\sqrt{x})' = (x^{\frac{1}{2}})' \qquad < \frac{x_2-x_1}{\sqrt{x_2-x_1}} < \sqrt{\underbrace{x_2-x_1}_{<\delta}} < \sqrt{\delta} < \varepsilon.$$

$$= \frac{1}{2\sqrt{x}}\quad (0,1).$$

$+\infty.$

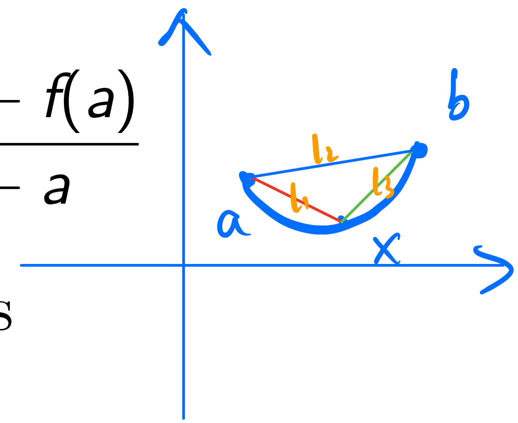
Convexity and Concavity

For further analysis of functions, we would introduce the concept of **Convexity** and **Concavity**.

The definition of these two concepts are as follows.

Let $\Omega \subseteq \mathbb{R}$ be any set and $I \subseteq \Omega$ an interval. A function $f : \Omega \to \mathbb{R}$ is called convex on I if for all

$$x, a, b \in I \text{ with } a < x < b, \ \frac{f(x) - f(a)}{x - a} \leq \frac{f(b) - f(a)}{b - a}$$

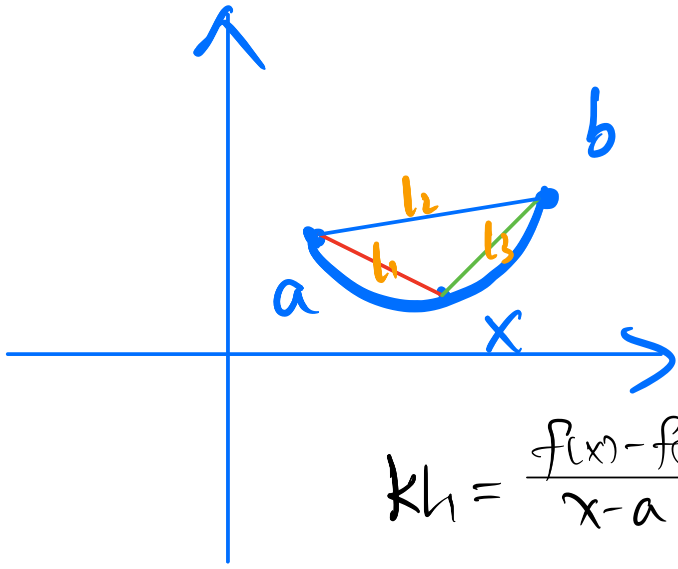


A strictly convex function is a function that satisfies

$$\frac{f(x) - f(a)}{x - a} < \frac{f(b) - f(a)}{b - a}. \tag{1}$$

$Kl_1 < Kl_2 < Kl_3$

We say a function f is concave if $-f$ is convex. We say a function f is strictly concave if $-f$ is strictly convex.



$$k_{l_1} = \frac{f(x)-f(a)}{x-a} \qquad k_{l_3} = \frac{f(b)-f(x)}{b-x}.$$

$$k_{l_1} < k_{l_2} < k_{l_3}$$

$$\frac{b}{a} < \frac{b+d}{a+c} < \frac{d}{c}$$

Convexity and Concavity

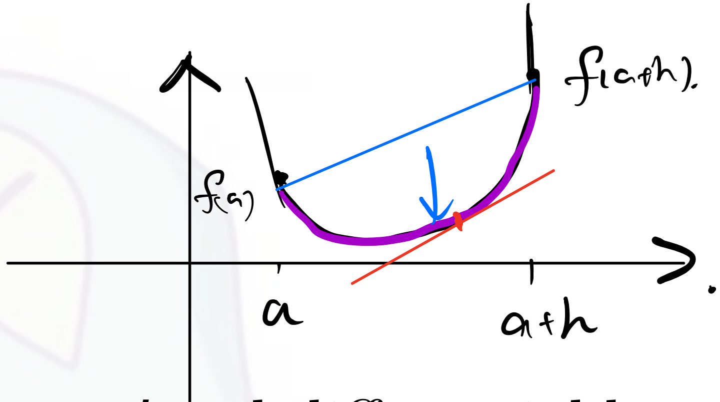
Comment 1.

We often use "−"(minus sign) to define a new definition from an existing one. The benefit is that these two definitions can be strongly related with each other.

Comment 2.

There is a quick way to memorize it... Con**cave**...

Convexity and Concavity



Results/Theorem & Comment

1. Let $f : I \to \mathbb{R}$ be strictly convex on I and differentiable at $a, b \in I$. Then:

   i For any $h > 0 (h < 0)$ such that $a + h \in I$, the graph of f over the interval $(a, a + h)$ lies below the secant line through the points $(a, f(a))$ and $(a + h, f(a + h))$

   ii The graph of f over all I lies above the tangent line through the point $(a, f(a))$

   iii If $a < b$, then $f'(a) < f'(b)$

Draw some pictures to visualize these results!

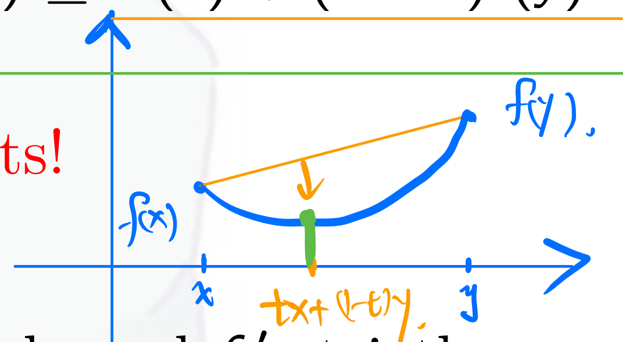
Convexity and Concavity

Results/Theorem & Comment

2. A function $f\colon I \to \mathbb{R}$(I is an interval) is convex if and only if

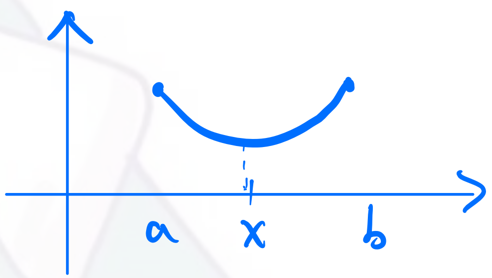
$$\underset{t\in(0,1)}{\forall} \ \underset{x,y\in I}{\forall} \text{ with } x<y,\ f(tx+(1-t)y) \le tf(x)+(1-t)f(y)$$

Draw some pictures to visualize these results!



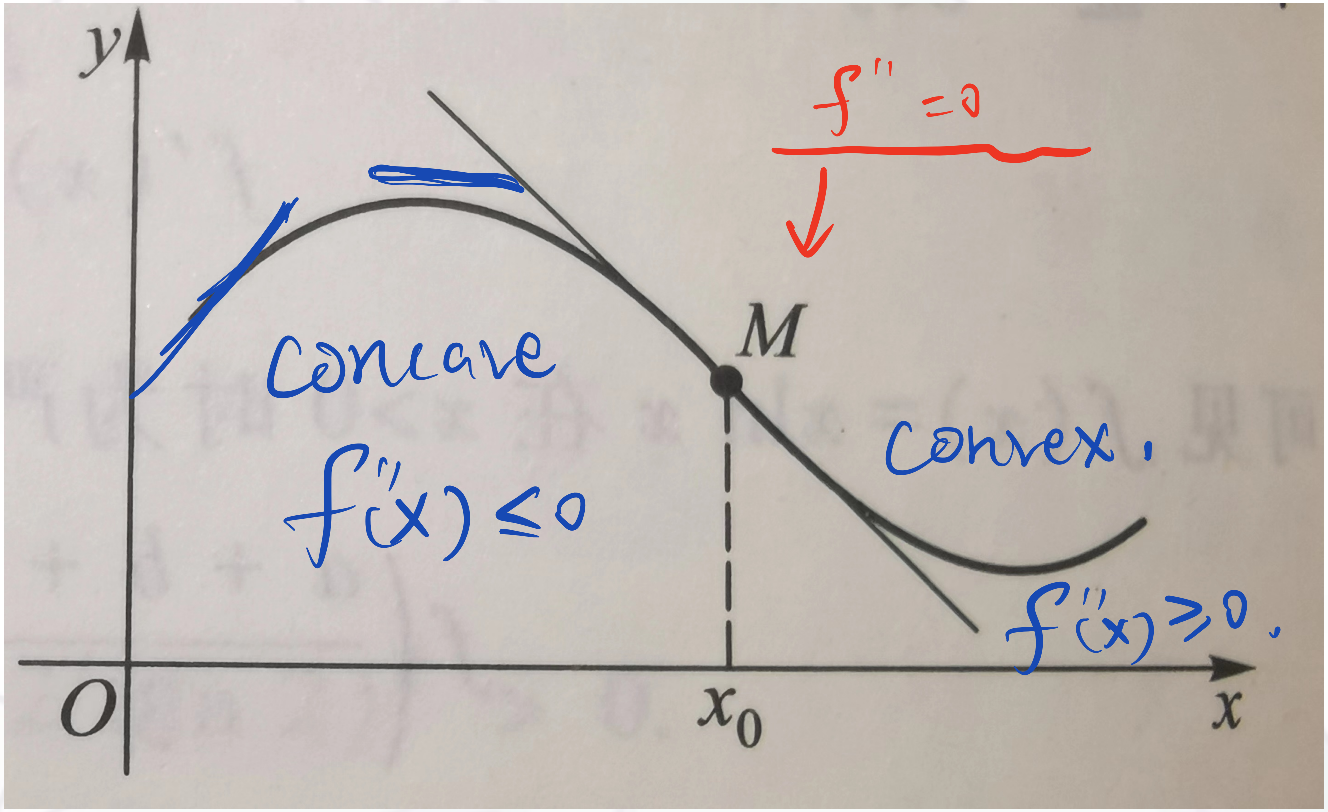
3. Let I be an interval, $f\colon I \to \mathbb{R}$ differentiable and f' strictly increasing. If $a, b \in I$, $a<b$ and $f(a)=f(b)$, then

$$f(x) < f(a) = f(b) \text{ for all } x \in (a,b)$$



Inflection Point

Definition: inflection point is a point on a smooth plane curve at which the curvature changes sign. In particular, in the case of the graph of a function, it is a point where the function changes from being concave (concave downward) to convex (concave upward), or vice versa.



Exercise

1. This exercise will show why convexity is useful.
   i Let f be a convex function on $[a, b]$. Prove that

$$f(\sum_{i=1}^{n}\lambda_i x_i) \leq \sum_{i=1}^{n}\lambda_i f(x_i),\ \ x_i \in [a, b],\ \sum_{i=1}^{n}\lambda_i = 1,\ \ \lambda_i > 0$$

   This inequality is known as **Jensen's Inequality**(for discrete measure.) Mathematical Induction.

   ii Show that

$$\prod_{i=1}^{n} a_i^{\lambda_i} \leq \sum_{i=1}^{n}\lambda_i a_i,\ \ a_i \geq 0,\ \sum_{i=1}^{n}\lambda_i = 1,\ \ \lambda_i > 0.$$

   This is the inequality you will encounter in your assignment.

1° λ_i $i=1$ / $i=2$.

$$\begin{cases} \lambda_1+\lambda_2=1 \\ x=\lambda_1 x_1+\lambda_2 x_2 \end{cases} \qquad \lambda_1=\frac{x_2-x}{x_2-x_1} \qquad \lambda_2=\frac{x-x_1}{x_2-x_1}$$

$$(\lambda_1+\lambda_2)f(x) \le \lambda_1 f(x_1)+\lambda_2 f(x_2).$$

$$\lambda_1(f(x)-f(x_1)) \le \lambda_2(f(x_2)-f(x)).$$

$$\frac{f(x)-f(x_1)}{x-x_1} \le \frac{f(x_2)-f(x)}{x_2-x}.$$

2°

$$f\Big(\sum_{i=1}^{k+1}\lambda_i x_i\Big)=f\left(\frac{\lambda_1 x_1}{\underbrace{\lambda_1+\cdots+\lambda_{k+1}}_{=1}}+\frac{\lambda_2+\cdots+\lambda_{k+1}}{\underbrace{\lambda_1+\cdots+\lambda_{k+1}}_{=1}}\cdot\frac{\lambda_2 x_2+\cdots+\lambda_{k+1}x_{k+1}}{\lambda_2+\cdots+\lambda_{k+1}}\right)$$

k

$$\le \frac{\lambda_1}{\Sigma\lambda}f(x_1)+\frac{\lambda_2+\cdots\lambda_{k+1}}{\Sigma\lambda}f\left(\frac{\lambda_2 x_2+\cdots+\lambda_{k+1}x_{k+1}}{\lambda_2+\cdots+\lambda_{k+1}}\right)$$

$$\le \frac{\cancel{\lambda_2+\cdots\lambda_{k+1}}}{\underbrace{\Sigma\lambda}_{=1}}\cdot\frac{\lambda_2 f(x_2)+\lambda_3 f(x_3)+\cdots+\lambda_{k+1}f(x_{k+1})}{\cancel{\lambda_2+\cdots\lambda_{k+1}}}$$

$$\prod_{i=1}^{n} a_i^{\lambda_i} \leq \sum_{i=1}^{n} \lambda_i a_i, \ \ a_i \geq 0, \ \sum_{i=1}^{n} \lambda_i = 1, \ \ \lambda_i > 0.$$

$$\prod_{i=1}^{n} a_i^{\lambda_i} = e^{\sum x_i \lambda_i} \quad f(\lambda_1 x_1 + \cdots \lambda_n x_n).$$

$$e^{x_i} = a_i. \quad f(x) = e^x.$$

$$f(x_i) = a_i.$$

$$f(\lambda_1 x_1 + \cdots \lambda_n x_n) \leq \lambda_1 f(x_1) + \cdots \lambda_n f(x_n).$$

$f(x)$ convex.

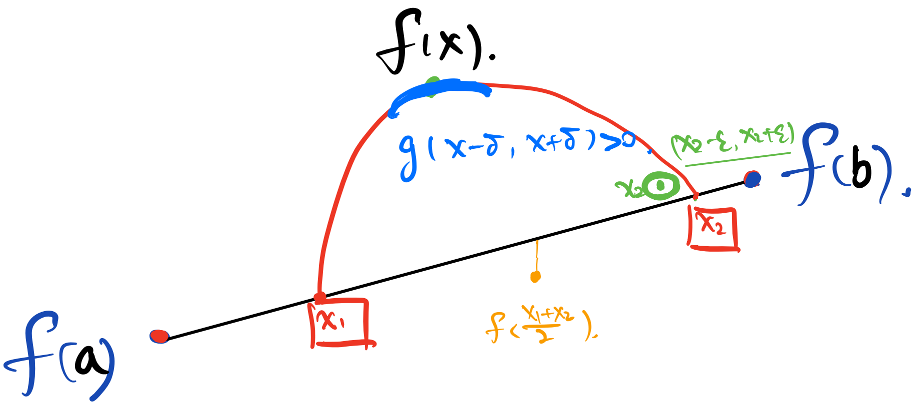
$$f''(x) = e^x > 0.$$

Exercise

2. Let $f\colon [0,1] \to \mathbb{R}$ be a continuous function. Prove that if f satisfies

$$f(\frac{x_1+x_2}{2}) \leq \frac{1}{2}(f(x_1)+f(x_2))$$

, where $x_1, x_2 \leq [0,1]$, then f is convex.



$\mathcal{L}$.

$$h(x) = f(a) + (x-a)\cdot\frac{f(b)-f(a)}{b-a}.$$

$g(x) = f(x) - h(x).$ $\quad \exists x.\ g(x) > 0.$

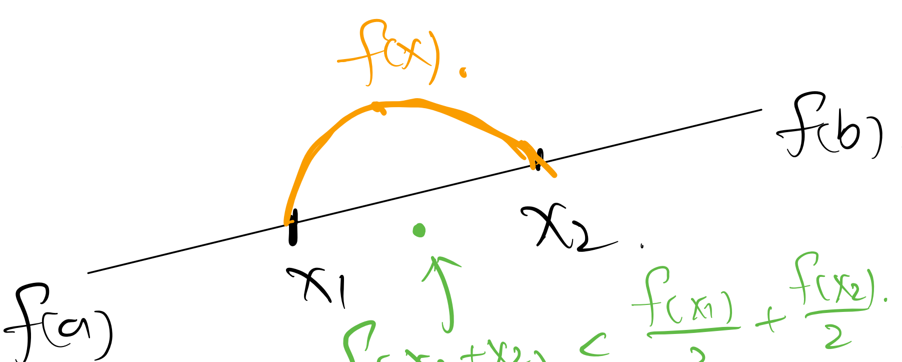
$x_2 > x,\ g(x_2) = 0,$ $\quad \forall t \in (x_1, x_2),$

$x_1 < x,\ g(x_1) = 0.$ $\quad h(t) > 0.$

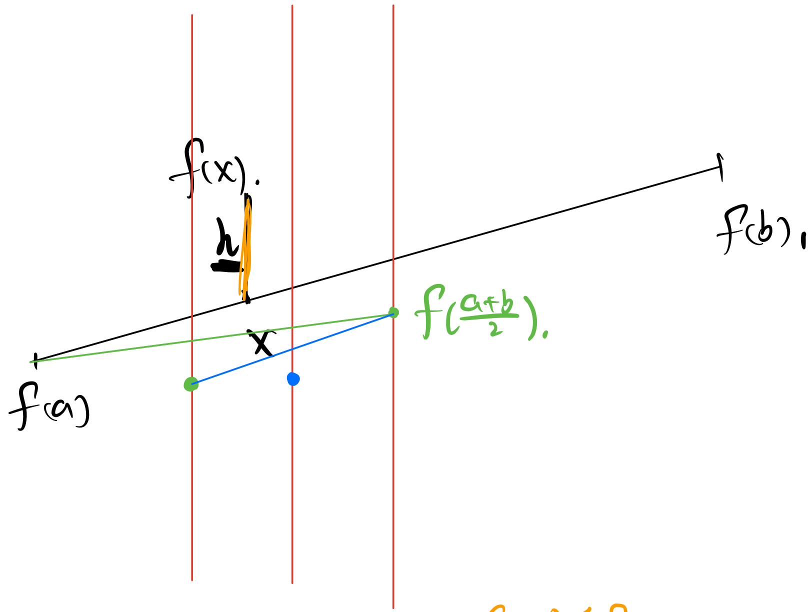
$g(x) = 0.$ $\quad \boxed{A \cap [x+\delta, +\infty)},$

$\mathbb{R}$ $\quad$ infimum, x_2

$(x_2, x_2+\varepsilon)$ No zeros.



$$f\left(\frac{x_1+x_2}{2}\right) \le \frac{f(x_1)}{2} + \frac{f(x_2)}{2}.$$



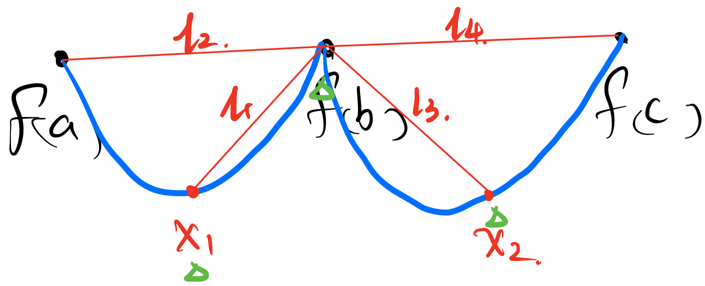
$x \in (a-\varepsilon, a+\varepsilon)$ $\quad$ $g(a) \leq 0.$

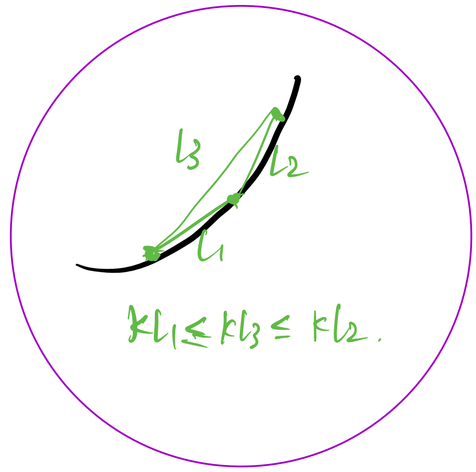
$$\forall \varepsilon.\ \exists \delta.\ |x_1 - x_2| < \delta \ \ldots < \varepsilon$$

$$|\underbrace{g(x)}_{=h} - \underbrace{g(a)}_{\leq 0}| < \varepsilon.$$

Exercise

3. Left f be a convex function, f:[a,b]$\to \mathbb{R}$. If there exists c$\in$(a,b), f(a)=f(b)=f(c), prove that f is a constant function.





$k_{l_1} \geq k_{l_2} = 0$ $\quad$ $k_{l_1} \leq k_{l_3}$.

$k_{l_3} \leq k_{l_4} = 0$.

$k_{l_3} \geq k_{l_1} \geq k_{l_2} = 0 \geq k_{l_3} = 0$.

Exercise

4. Let f be a continuous convex real function on $[a, b]$. Show that f either has one local minimum or infinitely many local minimums on $[a, b]$.

Exercise

5. $f(x)$ is a concave function on (a,b), and it is not constant. Prove that $f(x)$ can't attain its maximum on (a,b).

Exercise

6. (Darboux Theorem) If f is differentiable on [a,b], prove that f' can fetch all values between $f'(a)$ and $f'(b)$.

The Cauchy Mean Value Theorem

3.2.17. Theorem. Let f, g be real functions and $[a, b] \subset \operatorname{dom} f \cap \operatorname{dom} g$. If f and g are continuous on $[a, b]$ and differentiable on (a, b), then there exists an $x \in (a, b)$ such that

$$(f(b) - f(a))g'(x) = (g(b) - g(a))f'(x).$$

$$\frac{f(b)-f(a)}{\underset{b-a}{g(b)-g(a)}} = \frac{f'(x)}{\underset{1}{g'(x)}}$$

Proof.
We apply Rolle's Theorem to

$$h(x) = f(x)(g(b) - g(a)) - g(x)(f(b) - f(a)).$$

□

$g(x) = x.$

Exercise

7. Function f is continuous on [a,b], and differentiable on (a,b), prove that there exists $\xi \in (a, b)$, such that:

$(\ln x)' = \frac{1}{x}$

$$f(b) - f(a) = \xi \underbrace{\ln \frac{b}{a}}_{\ln b - \ln a} f'(\xi)$$

$$\frac{f(b) - f(a)}{\ln b - \ln a} = \frac{f'(\xi)}{\frac{1}{\xi}} = \xi f'(\xi).$$

Exercise

8. Prove that if f is differentiable on interval [a,b], and ab>0, then there exists a $\xi \in (a, b)$,such that:

$$\frac{af(b) - \overset{bf(a).}{\cancel{b(a)}}}{a-b} = f(\xi) - \xi f'(\xi)$$

$$\frac{g(a)-g(b)}{h(a)-h(b)} = \frac{g'(\xi)}{h'(\xi)}.$$

$$\frac{f(b)}{b} - \frac{f(a)}{a}.$$

$$g(x) = \frac{f(x)}{x}. \qquad \boxed{h(x).}$$

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)} < 0.$$

9. Function f and g are differentiable on interval (a,b).

$$F(x)=f(x)g'(x)-f'(x)g(x) > 0,$$

Prove that if F(x)>0 on (a,b), there must exist a solution for g(x)=0 between two different solutions for f(x)=0.

$f(x_1) = f(x_2) = 0, \quad x_1 < x_2.$

$\frac{f(x)}{g(x)} \downarrow$

$\frac{f(x_1)}{g(x_1)} = 0 \quad \frac{f(x_2)}{g(x_2)} = 0$

$g(t) = 0. \quad t \in (x_1, x_2).$

$g(t) \neq 0$

$(e^x \cdot f(x))' = e^x \cdot f(x) + e^x \cdot f'(x).$

10. f(x) is differentiable on (a,b), prove that there is a solution for f(x)+f'(x)=0 between two solutions for f(x)=0 on (a,b).

existence. $\boxed{} = 0.$

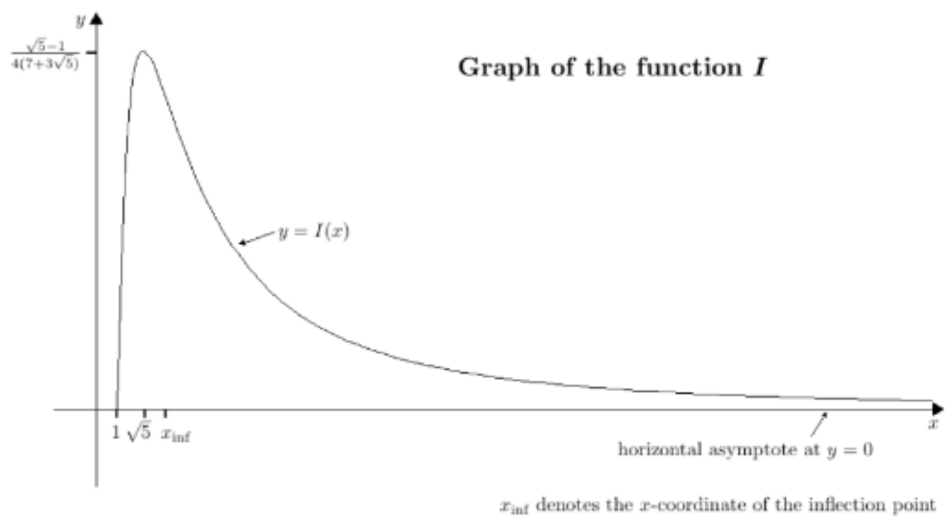
Rolle Theorem.

$f(a) = f(b)$ $\quad f'(t) = 0.$

11. f(x) is continuous on [0,1] and differentiable on(a,b), and f(0)=f(1)=0. Assume there exists $t_0 \in (0,1)$, f(t_0)=α. Prove that there exists $\xi \in (0,1)$, such that $f'(\xi) = \alpha$.

Curve sketching

- 99% possibility to appear in your midterm 2.
- Follow the guidelines provided by Horst. (The rubric will be generally the same as the guidelines.)[2]



Two advice:

1. Do not forget to mark the *asymptote line*.
2. Do not add any **redundant** marks.

Reference

- Exercises from 2021VV186-Niyinchen.
- Graph from 2021VV186-Huangyue

End

Thanks!