Exercises for Mid 1

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VV186 - Honors Mathmatics II

Exercise: Points

Professor Ding has already discussed.

Please identify the interior, exterior, boundary and accumalation points of the set

$$\{\frac{1}{z} : z \in \mathbb{Z}\backslash\{0\}\} \cup (\bigcap_{j=1}^{\infty}(-2-\frac{1}{j}, -1+\frac{1}{j}))$$

Just a reminder:

① Discrete points (like $\frac{1}{n}$, $n \in \mathbb{Z}\backslash\{0\}$) are $\begin{cases} \text{boundary points } \checkmark \\ \text{accumulation points } \times \end{cases}$

② if x is an accumulation point for Set S. RC2

$\forall \varepsilon > 0$. There are infinite points in $(x-\varepsilon, x+\varepsilon) \cap S$.

Imagine that infinite points "accumulate" near x

Exercise: Bounds

Step 1: Graph! Visualize!

When doing this exercise: recall their definition, and whey they exist?

(2.5min)

Consider the set $U \subset \mathbb{R}$, where $U = A \cup B \cup C$ with

$$A = \{x \in \mathbb{R} : 0 < x \leq 1\},$$
$$B = \{x \in \mathbb{R} : x = 2 - 1/n,\ n \in \mathbb{N} \setminus \{0\}\},$$
$$C = \{x \in \mathbb{R} : x = -1/n,\ n \in \mathbb{N} \setminus \{0\}\}.$$

State (without proof) $\min U$, $\max U$, $\inf U$, $\sup U$, $\underline{\lim}\, U$ and $\overline{\lim}\, U$ (if one or more of these do not exist, simply state this).

Consider the set $U \subset \mathbb{R}$, where $U = A \cup B \cup C$ with

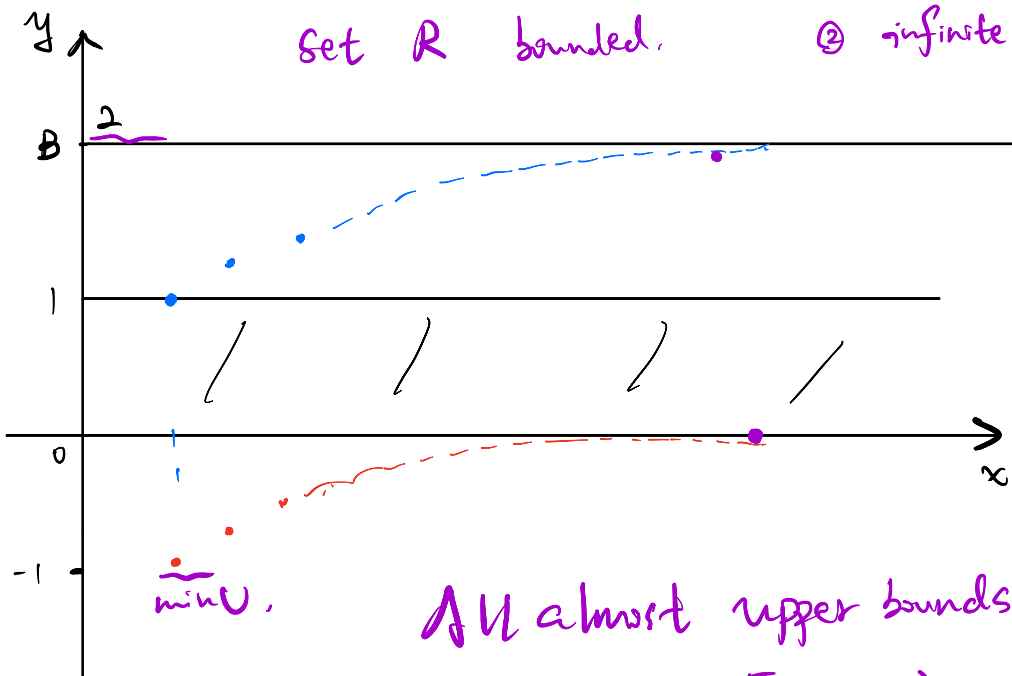
$$A = \{x \in \mathbb{R} : 0 < x \leq 1\},$$
$$B = \{x \in \mathbb{R} : x = 2 - 1/n,\ n \in \mathbb{N} \setminus \{0\}\},$$
$$C = \{x \in \mathbb{R} : x = -1/n,\ n \in \mathbb{N} \setminus \{0\}\}.$$

State (without proof) $\min U$, $\max U$, $\inf U$, $\sup U$, $\underline{\lim}\, U$ and $\overline{\lim}\, U$ (if one or more of these do not exist, simply state this).

-1 × -1 2   0   2

Set R bounded.

① bounded above/below
② infinite.



All almost upper bounds:
$[2, +\infty)$,
$\forall x < 2$.
infinite t in U
$t > 2$.

All almost lower bounds:
$(-\infty, 0)$.

Exercise: Limit and Accumulation Point

Easy, but check whether you should only USE WORDS !

Exercise 2.
Describe in words the difference between **an accumulation point of a sequence** and **a limit of a sequence**. How are these two concepts different? If you can, state some properties that they have in common or that serve to differentiate them from each other. Is one of them also always an example of the other? Give examples. **(5 Marks)**

Solution. A limit of a sequence of numbers is a number such that for any given distance, all terms of the sequence eventually are within that distance of the limit. **(1 Mark)** An accumulation point is a number such that for any given distance and any given sequence term, there will always be some other succeeding sequence term within that distance of the accumulation point. **(1 Mark)**

After these two marks, a maximum of **(3 Marks)** can be obtained as follows:

$\forall, \varepsilon, \exists$ ~~X~~

Give **(1 Mark)** for any of the following statements:

- Any limit is also an accumulation point.
- A sequence can have at most a single limit, but may have zero or more accumulation points.
- An accumulation point is always a limit of a subsequence and vice-versa.
- If a sequence has a limit, it must be bounded, but the same statement is not if the wrd "limit" is replaced with "accumulation point".
- If a Cauchy sequence has an accumulation point, that point is also the (unique) limit of the Cauchy sequence.

Give **(1/2 Mark)** for any coherent example to illustrate any of the above statements or the definitions above.

Exercises: Find limits for a recursively defined sequence

Just briefly discuss the methods. Check details by yourself later.

Show that the sequence defined by

$c = \frac{1}{3-c}$, c.

$$a_1 = 2, \qquad a_{n+1} = \frac{1}{3 - a_n}, \qquad (n \geq 1)$$

satisfies $0 < a_n \leq 2$ and is decreasing. Deduce that the sequence is convergent and find its limit.

1° calculate limit (in draft),

2° Use M I ① monotonic. ② bounded.

3°

Exercises (Important) Very similar Exercise in Mid 1 (Last year)

(2 min)

Step 1: First calculate the limit (in your draft)

Step 2: Mathematical Induction.

Show that the sequence defined by $x^2-3x+1=0$ $x=\frac{3-\sqrt{5}}{2}$ or $\frac{3+\sqrt{5}}{2}$.

$$a_1 = 2, \qquad a_{n+1} = \frac{1}{3-a_n}, \qquad (n \geq 1)$$

→ Even no hints, set the target to prove monotonic and bounded.

satisfies $0 < a_n \leq 2$ and is decreasing. Deduce that the sequence is convergent and find its limit.

We use Mathematical Induction to show that

$\forall n$, $a_{n+1} \leq a_n$ and $\frac{3-\sqrt{5}}{2} \leq a_{n+1} \leq 2$.

1° Base: $a_1 = 2 \in [\frac{3-\sqrt{5}}{2}, 2]$

2° $a_{n+1} - a_n = \frac{1}{3-a_n} - a_n = \frac{a_n^2-3a_n+1}{3-a_n} \leq 0$, So $a_{n+1} \leq a_n$ (a_n) is ↓

$a_{n+1} \geq \frac{3-\sqrt{5}}{2} \Leftrightarrow \frac{1}{3-a_n} \geq \frac{3-\sqrt{5}}{2} \Leftrightarrow 3-a_n \leq \frac{2}{3-\sqrt{5}} = \frac{3+\sqrt{5}}{2} = a_n \geq \frac{3-\sqrt{5}}{2}$

So (a_n) is ↓ and bounded ⇒ converge.

3° $a_{n+1} = \frac{1}{3-a_n}$. $\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{1}{3-a_n} \Rightarrow \lim_{n\to\infty} a_n = \frac{3-\sqrt{5}}{2}$. (since $a_n \leq 2$).

Exercise: Upper Limits and Lower Limits

(2min)

First recall their definitions, their properties...

Consider the sequence (a_n) given by

$$a_n = \frac{1}{2} + (-1)^n \frac{2+n}{2n}.$$

Calculate $\overline{\lim}\, a_n$ and $\underline{\lim}\, a_n$.

Take $\overline{\lim}\, a_n$ as example.

Definition 1: define: $x_n = \sup\{a_k : k \geq n\}$.

$(x_n) \downarrow$ and bounded $\longrightarrow$ converge!

$\lim\limits_{n\to\infty} x_n = \overline{\lim}\, a_n$.

Definition 2: The largest accumulation point for (a_n).

Consider the sequence (a_n) given by

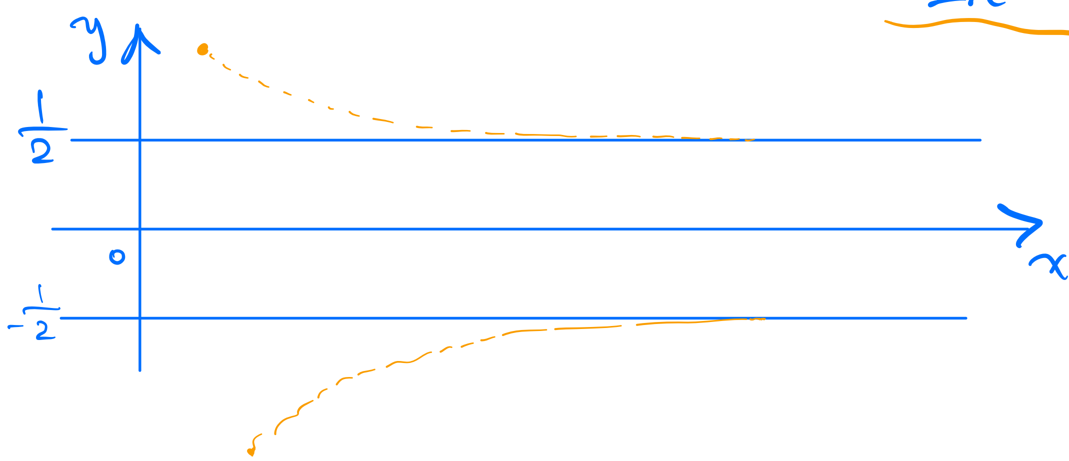
trivial

$$a_n = \frac{1}{2} + (-1)^n \frac{2+n}{2n}.$$

Calculate $\overline{\lim}\, a_n$ and $\underline{\lim}\, a_n$.

Step 1: Graph! Visualize.

$$\frac{2+n}{2n} = \frac{1}{2} + \frac{1}{n}.$$



To prove that x is an accumulation point.

Method 1: Definition. $\forall \varepsilon > 0$, $\forall N \in \mathbb{Z}^+$, $\exists n > N$

Method 2: Show that a subsequence converges to x

$$a_{2n} = \frac{1}{2} + (-1)^{2n} \cdot \frac{2+2n}{4n} = \frac{1}{2} + \frac{1}{2n} + \frac{1}{2} \xrightarrow{+\infty} 1.$$

$$a_{2n+1} = \frac{1}{2} + (-1)^{2n+1} \cdot \frac{2+(2n+1)}{4n+2} = \frac{1}{2} - \frac{1}{2n+1} - \frac{1}{2} \xrightarrow{\infty} 0.$$

Since there are only two accumulation point.

and, $\overline{\lim}$ is largest -----

$\underline{\lim}$ is smallest ----

(Horst's proof).

Exercise: Upper Limits and Lower Limits

(4min)

Given (x_n) a real bounded sequence, prove that:

(1) $\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n > N, x_n < \overline{lim}(x_n) + \epsilon.$

(2) $\forall \epsilon > 0, \forall k, \exists n_k > k, x_{n_k} > \overline{lim}(x_n) - \epsilon.$

Given (x_n) a real bounded sequence, prove that:

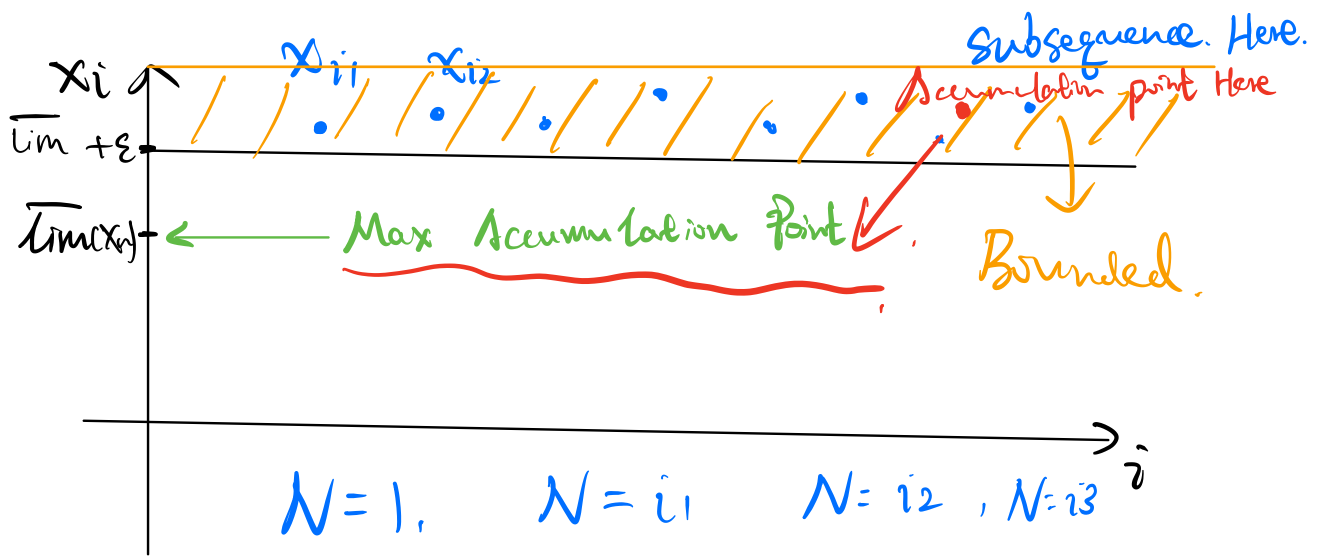
(1) $\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n > N,\ x_n < \overline{lim}(x_n) + \epsilon.$

(1) (2) $\forall \epsilon > 0, \forall k, \exists n_k > k,\ x_{n_k} > \overline{lim}(x_n) - \epsilon.$

☆ Most important: Visualize.

Prove by contradiction. (why? more condition!)

Suppose:
$\exists \varepsilon > 0, \forall N \in \mathbb{N}, \exists n > N, x_n \geq \overline{\lim}(x_n) + \varepsilon$



Contradiction!

Given (x_n) a real bounded sequence, prove that:

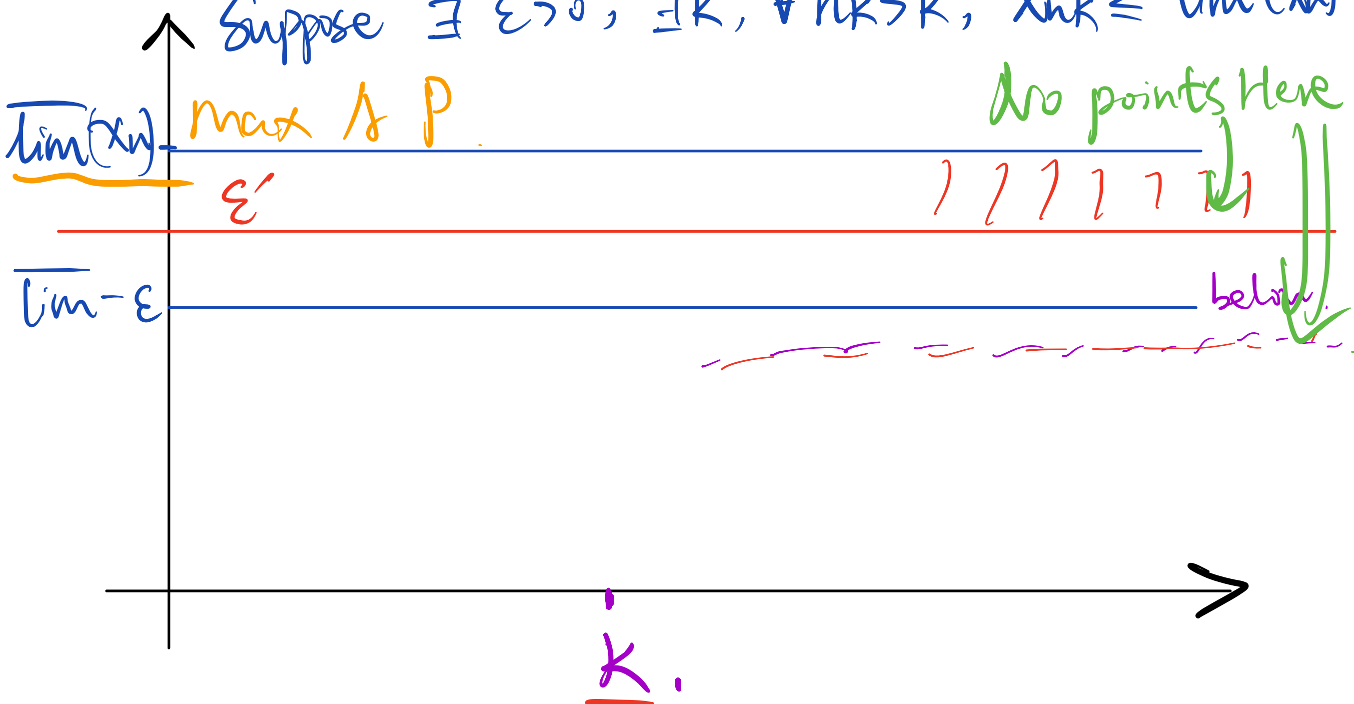
(1) $\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n > N, x_n < \overline{lim}(x_n) + \epsilon.$

(2) $\forall \epsilon > 0, \forall k, \exists n_k > k, x_{n_k} > \overline{lim}(x_n) - \epsilon.$

(2) ☆ Most important: Visualize.

Prove by contradiction. (why ? more condition!)

Suppose $\exists\, \varepsilon > 0, \exists K, \forall n_k > K, x_{n_k} \leq \overline{\lim}(x_n) - \varepsilon$



$\forall \varepsilon' > 0, \forall N, \exists n > N, |x_n - \overline{\lim}(x_n)| < \varepsilon'$.

$\varepsilon' < \varepsilon$

Exercise: Upper Limits and Lower Limits

(2min)

A valuable question asked in piazza.

For a real bounded sequence (a_n), prove that if $\mathrm{ran}(a_n)$ doesn't have a maximum, then $\sup \mathrm{ran}(a_n) = \overline{lim} a_n$.

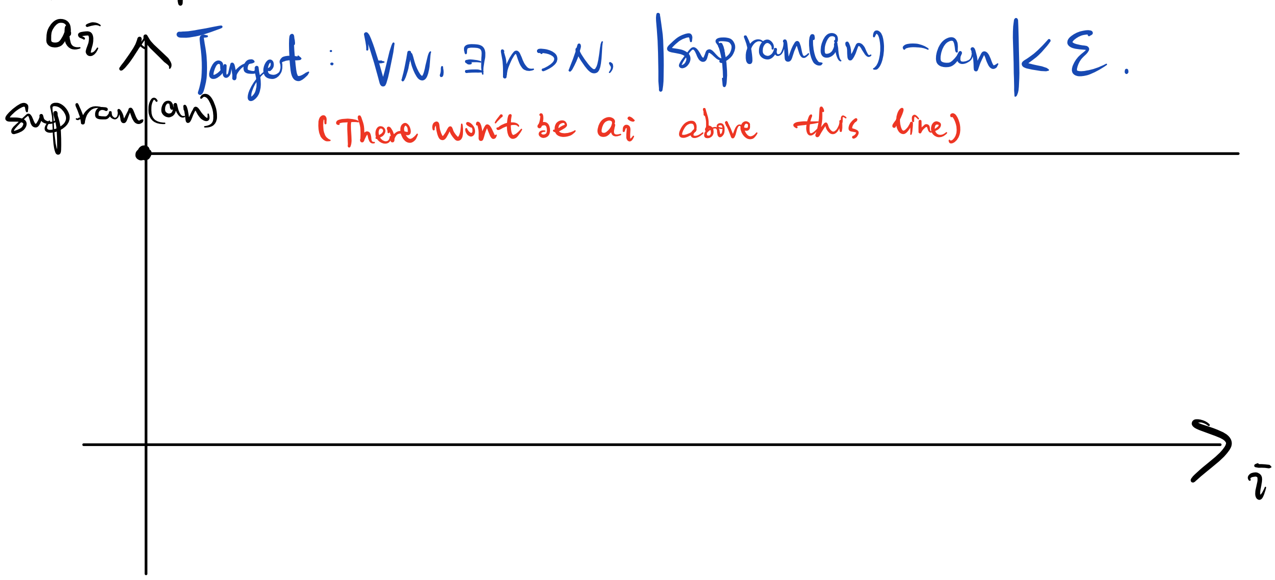
First recall the definition of $\mathrm{ran}(a_n)$. Divide the procedure of the exercise into several steps, set up your goal!

For a real bounded sequence (a_n), prove that if $\mathrm{ran}(a_n)$ doesn't have a maximum, then $\sup \mathrm{ran}(a_n) = \overline{\lim} a_n$.

Target: $\sup \mathrm{ran}(a_n)$ is the max accumulation point.

1° $\sup \mathrm{ran}(a_n)$ is an accumulation point.

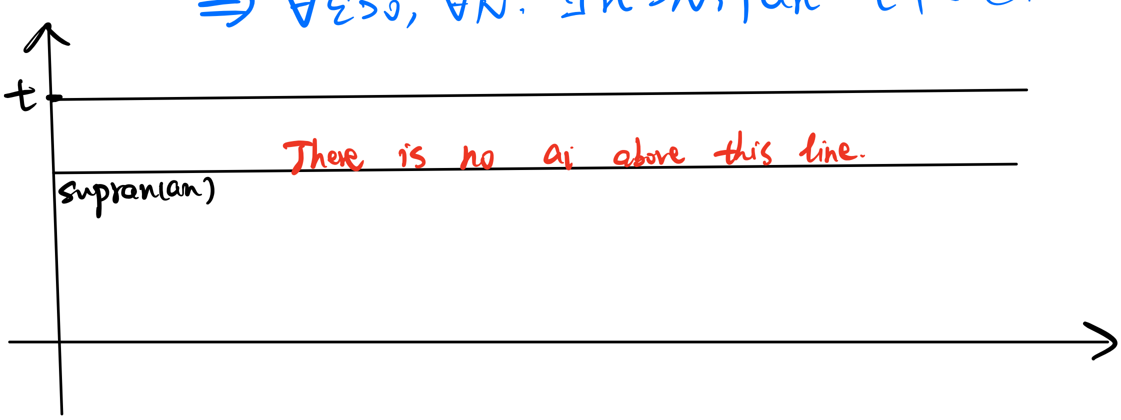
Target: $\forall N, \exists n > N, |\sup \mathrm{ran}(a_n) - a_n| < \varepsilon$.



2° $\forall t > \sup \mathrm{ran}(a_n)$, t is not an accumulation point.

Suppose t is accumulation point.

$\Rightarrow \forall \varepsilon > 0, \forall N. \exists n > N, |a_n - t| < \varepsilon$.



Exercise: Upper Limits and Lower Limits

$\underline{\lim}\, a_n$.

① Smallest accumulation point.

② $x_n = \inf\{x_k : k \geq n\}$.

$x_n \uparrow$. Bounded $\longrightarrow \underline{\lim}\, a_n$.

Exercise 3.6

Let (a_n) and (b_n) be two bounded real sequences. Prove that:

$$-\overline{\lim}(-b_n)$$

$$\underline{lim} a_n + \underline{lim} b_n \leq \underline{lim}(a_n + b_n) \leq \overline{lim} a_n + \underline{lim} b_n \leq \overline{lim} a_n + \overline{lim} b_n.$$

$n \to +\infty$.

$$\inf_{k \geq n} a_k + \inf_{k \geq n} b_k \leq a_k + b_k \quad \text{for all } k \geq n.$$

$$\underline{\lim}\, a_n \xleftarrow{n\to+\infty} \inf_{k\ge n} a_k, \qquad \underline{\lim}\, b_n \xleftarrow{n\to+\infty} \inf_{k\ge n} b_k$$

$$\leq \inf_{k \geq n}(a_k + b_k) \xrightarrow{} \underline{\lim}(a_n + b_n).$$

Exercise: Metric Space

Just briefly discuss the methods. Check details by yourself later.
A metric space is complete when all cauchy sequence in this metric space converges.

> **2.2.46. Example.** Consider the metric $\varrho\colon \mathbb{R}\times\mathbb{R}\to\mathbb{R}$ defined by
> $$\varrho(x,y)=\left|\frac{x}{1+|x|}-\frac{y}{1+|y|}\right|.$$

Prove that the metric space $(\mathbb{R},\rho)$ is incomplete by analyzing the sequence: $a_n = n$.

For convergence, accumulation points and so on
we used to use abs ($|\cdot|$) to describe "distance", now replace all distance with metrics.

$a_n = n.$

$\downarrow$

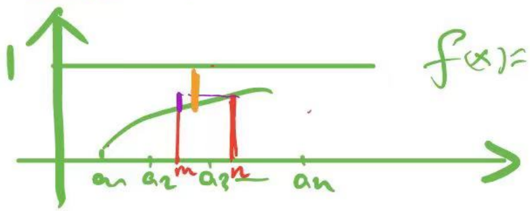
Cauchy Sequence.

1° $\forall \varepsilon>0, \exists N \in \mathbb{N}, \forall m,n>N, \ \rho(m,n)<\varepsilon.$

$$\left|\frac{m}{1+m} - \frac{n}{1+n}\right| < \varepsilon. \checkmark$$

$$\frac{x}{1+x} = \frac{1}{1+\frac{1}{x}} \uparrow \to 1$$

$a_n = n.$



$f(x) = \frac{x}{1+x}$

$$\left|\frac{m}{1+m} - \frac{n}{1+n}\right| < \left|\frac{m}{1+m} - 1\right|.$$

$$= \frac{1}{1+m} < \varepsilon$$

Select N such that $\frac{1}{1+N} < \varepsilon$!

$a_n = n.$

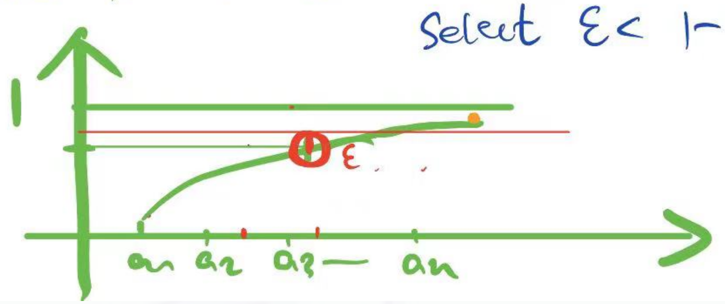
doesn't converge.

2° Proof by contradiction

if $a_n \to a$.

$\forall \varepsilon>0, \exists N \in \mathbb{N}, \forall n>N, \ \rho(n,a)<\varepsilon.$

Select $\varepsilon < 1 - \frac{a}{1+a}$!



Exercise Important! (A Former Midterm Question)

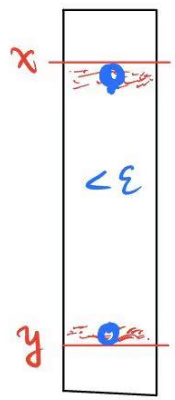
TAs have discussed this exercise in their RC3.
Just briefly discuss the methods. Check details by yourself later.

Prove that every Cauchy sequence has at most one accumulation point.

ρ abstract.

Tips:

- You should work on an abstract metric space, using ρ instead of $|\cdot|$.
- Visualize to help you think !



Suppose there are two different Accumulation Points. x, y,

Cauchy. So

$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall m, n > N,$

$\rho(a_m, a_n) < \varepsilon.$

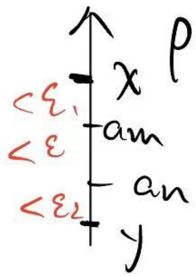
P. 1.1

Because x, y are Accumulation Points.

So $\forall \varepsilon_1 > 0, \forall N_1 \in \mathbb{N}, \exists m, \rho(a_m, x) < \varepsilon_1$

So $\forall \varepsilon_2 > 0, \forall N_2 \in \mathbb{N}, \exists n, \rho(a_n, y) < \varepsilon_2$

$\begin{cases} N_1 > N \\ N_2 > N \end{cases}$ $\rho(a_m, a_n) < \varepsilon.$



$\rho(x, y) \le \rho(x, a_m) + \rho(a_m, a_n) + \rho(a_n, y)$

$< \varepsilon_1 + \varepsilon_2 + \varepsilon.$

Let $\varepsilon_1, \varepsilon_2, \varepsilon < \frac{1}{3}\rho(x, y).$

Contradiction!

Exercise: Big O and Small o

(3 min)

First recall the definition...

Please prove, or disprove by giving counterexamples of the following statements:

i) $\left(1 + O(x)\right)^2 = 1 + O(x^2)$ as $x \to 0$

ii) $o(x)^n = o(x^n), n \in \mathbb{N}^*$, as $x \to 0$

2.4.15. Definition. Let f, ϕ be real- or complex-valued functions defined on a subset $\Omega \subset \mathbb{R}$ containing the interval (L, ∞) for some $L \in \mathbb{R}$. We say that

$$f(x) = O(\phi(x)) \qquad \text{as } x \to \infty$$

if and only if

$$\underset{C>0}{\exists}\ \underset{M>L}{\exists} \quad x > M \quad \Rightarrow \quad |f(x)| \le C|\phi(x)|$$

2.4.17. Definition. Let f, ϕ be real- or complex-valued functions defined on a subset $\Omega \subset \mathbb{R}$ and let x_0 be an accumulation point of Ω. We say that

$$f(x) = o(\phi(x)) \qquad \text{as } x \to x_0$$

if and only if

$$\underset{C>0}{\forall}\ \underset{\varepsilon>0}{\exists}\ \underset{x\in\Omega\setminus\{x_0\}}{\forall} \quad |x - x_0| < \varepsilon \quad \Rightarrow \quad |f(x)| < C|\phi(x)| \qquad (2.4.3)$$

Please prove, or disprove by giving counterexamples of the following statements:

i) $(1 + O(x))^2 = 1 + O(x^2)$ as $x \to 0$ ✗

ii) $o(x)^n = o(x^n), n \in \mathbb{N}^*$, as $x \to 0$ ✓

Set our target. $\exists C . \exists \varepsilon , \forall |x| < |\varepsilon| , |f(x)| \le C|x|$

(i) $f(x) = O(x)$

$$(1 + f(x))^2 = \cancel{1} + 2f(x) + \overset{O(x^2)}{f^2(x)} = \cancel{1} + O(x^2),$$

Setting $f(x) = x$, $\quad 2x$ not $O(x^2)$.

Suppose $\exists C, \exists \varepsilon . \forall |x| < |\varepsilon|$.

$|2x| \le C|x^2|$

$\Leftrightarrow \quad 2 \le C|x|$. ↯

$|x| \ge \frac{2}{C}$ ↯.

(ii) $f(x) = o(x)$

Method 1 :

Use definition.

$f(x) = o(x)$, $\forall c, \forall \varepsilon.\ \forall\, 0 < |x| < |\varepsilon|.\ |f(x)| \le c|x|$

power $1/n$ $\qquad |f(x)|^n \le c^n |x|^n$

$\forall c^{\frac{1}{n}}, \forall \varepsilon$ ---- $|f(x)|^n \le c\,|x|^n$.

Method 2: (Important conclusion !)

$$\lim_{x \to x_0} \frac{|f(x)|}{|\phi(x)|} = 0 \quad \Leftrightarrow \quad f(x) = o(\phi(x)) \text{ as } x \to x_0.$$

$$\lim_{x \to 0} \frac{f(x)}{x} = 0,$$

$$\lim_{x \to 0} \frac{(f(x))^n}{x^n} = 0,$$

Exercise: Small o and limits

(3min)

A function f(x) satisfies that: $\lim\limits_{x \to 0} f(x) = 0$. And f(x)-f($\frac{x}{2}$)=o(x) (x→0).

Prove that:

$$f(x) = o(x) \ (x \to 0)$$

A function f(x) satisfies that: $\lim_{x\to 0} f(x)=0$. And $f(x)-f(\frac{x}{2})=o(x)$ $(x\to 0)$. Prove that:

$$f(x) = o(x)\ (x\to 0)$$

Step 1: Translate !!

What you have?

① $\lim_{x\to 0} f(x)=0$: $\forall \varepsilon>0, \exists \delta_1>0. \forall |x|<\delta_1, |f(x)|<\varepsilon$.

② $f(x)-f(\frac{x}{2})=o(x)$, $x\to 0$: $\forall c>0, \exists \delta_2>0. \forall 0<|x|<\delta_2$, $|f(x)-f(x/2)|<c|x|$.

What's your target?

bridge:

$f(x)=o(x)$ $(x\to 0)$:

$\forall c>0, \exists \delta_3>0. \forall 0<|x|<\delta_3, |f(x)|<c|x|$.

$\frac{x}{2}$ half and half.

$\longrightarrow 0$,

$$\left|f(x)-f(\cancel{\tfrac{x}{2}})\right| < c|x|$$

$$\left|f(\cancel{\tfrac{x}{2}})-f(\cancel{\tfrac{x}{4}})\right| < c\cdot\left|\frac{x}{2}\right|$$

$$\left|f(\cancel{\tfrac{x}{4}})-f(\tfrac{x}{8})\right| < c\cdot\left|\frac{x}{4}\right|$$

Sum. ; Triangle Inequality

$$\left|f(x)-f(\frac{x}{2^k})\right| < c\cdot|x|\cdot\left(1+\frac{1}{2}+\frac{1}{4}\cdots\right)$$

$< 2c|x|$

$$f(x) < 2c|x| + f(\frac{x}{2^k}) \to 0.$$

Exercise: Continuous and Uniformly Continuous

Professor Sun has already discussed it.

(2min)

A function f(x) is continuous on [a,+∞), and $\lim_{x\to\infty} f(x)$ exists. Prove that:

f(x) is uniformly continuous on [a,+∞)

Important Conclusion

A continuous function on closed interval is uniformly continuous!

A function f(x) is continuous on [a,$+\infty$), and $\lim_{x\to\infty} f(x)$ exists. Prove that:

$$f(x) \text{ is uniformly continuous on } [a,+\infty)$$

Step 1: Translate!

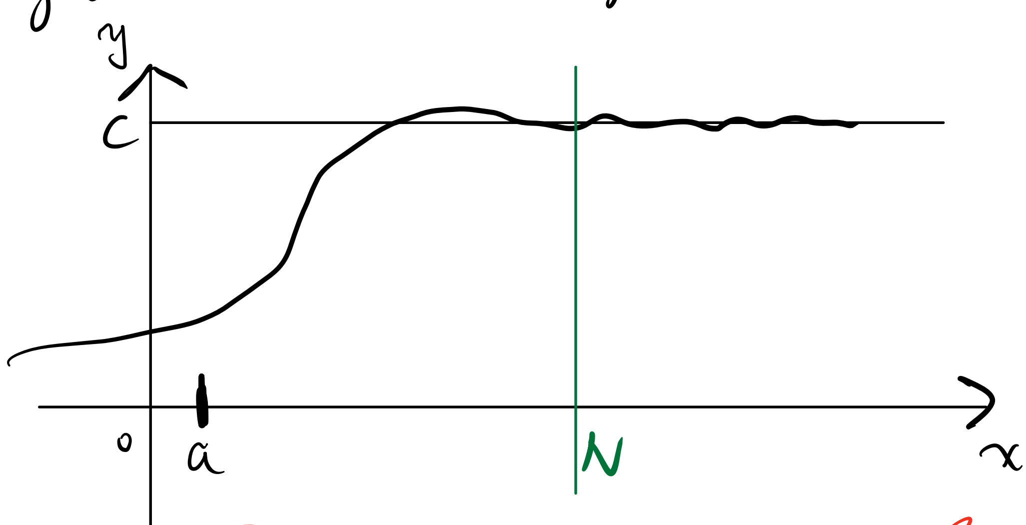
$\lim_{x\to\infty} f(x)$ exists. $\forall \varepsilon>0, \exists C>0, N>0, \forall x>N, |f(x)-C|<\varepsilon$

$\Leftrightarrow \forall \varepsilon>0, \exists C>0, \exists N, \forall x,y>N, |f(x)-f(y)|<\varepsilon. (*)$

(Similar to Cauchy Sequence. Actually it's the cauchy for functions.)

Step 2: Graph!

Target: $\forall \varepsilon>0, \exists \delta, \forall |x-y|<\delta, |f(x)-f(y)|<\varepsilon.$



By $(*)$, $[N+1, +\infty)$ is already uniformly continuous.

By conclusion in last page. $[a, N+1]$ is uniformly continuous.

So $[a, +\infty)$ is uniformly continuous.!!

Exercise: Continuous and Uniformly Continuous

Skip

It has been proved in lecture that: if f and g are two functions continuous at x. Then f+g and f · g are also continous at x.

What about uniformly continous properties ?
f(x) and g(x) are two uniformly continuous real functions defined in $[0,+\infty)$.
(1) Prove that: Function $h(x) = f(x) + g(x)$, $h(x)$ is also uniformly continuous.
T
(2) Given that g(x) is bounded. Function $h(x) = f(x)g(x)$. Is h(x) also uniformly continuous ? F

$f(x) = x \quad g(x) = \sin x,$

$2k\pi$

$h(x) = \underline{x \sin x}$ not uniformly continuous

(1)

$\forall \varepsilon_1, \exists \delta_1, \forall |x-y| < \delta_1, |f(x)-f(y)| < \varepsilon_1$

$\forall \varepsilon_2, \exists \delta_2, \forall |x-y| < \delta_2, |g(x)-g(y)| < \varepsilon_2$

Let $\delta < \min\{\delta_1, \delta_2\}$.

$\forall |x-y| < \delta, |h(x)-h(y)|$

(2) $f(x) = x$ uniformly

$g(x) = \sin x$ uniformly & bounded.

But $h(x) = x \sin x$ is not uniformly continuous

$\forall \varepsilon > 0, \exists \delta > 0, \forall |x_1 - x_2| < \delta, \underline{|h(x_1) - h(x_2)| < \varepsilon}$.

Counter: $x_1 = 2k\pi$, $k \in \mathbb{Z}^+$, $h(x_1) = 0$.

$x_2 = 2k\pi + \delta$. $|h(x_2)| < \varepsilon$.

$\underbrace{(2k\pi+\delta)}_{} \underbrace{\sin(2k\pi+\delta)}_{\text{not } 0} > \varepsilon$. let $k \to +\infty$.

$+\infty$ >0

Exercise: Continuous and Uniformly Continuous

(3min)

Let $f : \Omega \to \mathbb{R}$ be a real function that satisfies Lipschitz condition, that is, there is a constant $M > 0$ such that for all x and y in the domain of f, $|f(x) - f(y)| \leq M \cdot |x - y|$ (Peter A. Loeb *Real Analysis*, P92)

i) Show that f is uniformly continuous

ii) Now Let $\Omega =: [a, +\infty)$, where $a > 0$. Show that $\frac{f(x)}{x}$ is uniformly continuous

Let $f: \Omega \to \mathbb{R}$ be a real function that satisfies Lipschitz condition, that is, there is a constant $M > 0$ such that for all x and y in the domain of f, $|f(x) - f(y)| \le M \cdot |x - y|$ (Peter A. Loeb *Real Analysis*, P92)

i) Show that f is uniformly continuous

ii) Now Let $\Omega =: [a, +\infty)$, where $a > 0$. Show that $\frac{f(x)}{x}$ is uniformly continuous

$y > a$

(i) Set up target.

$$\forall \varepsilon > 0,\ \exists \delta > 0.\ \forall |x-y| < \delta,\ |f(x) - f(y)| < \varepsilon$$

$$|f(x) - f(y)| \le \underbrace{M \cdot \overset{\delta}{\underset{\text{bounded}}{|x-y|}}} < \varepsilon.$$

wish.

Select $\delta < \frac{\varepsilon}{M}$.

(ii) $\forall \varepsilon > 0,\ \exists \delta > 0,\ \forall |x-y| < \delta,\ \left|\frac{f(x)}{x} - \frac{f(y)}{y}\right| < \varepsilon.$

Target. $\left|\frac{f(x)}{x} - \frac{f(y)}{y}\right| <$ Bounded $\cdot\ |x-y|$

$a_n \to a$ $\qquad$ $a_n \cdot b_n \to ab$.

$b_n \to b$ $\qquad$ $a_n b_n - ab = \underline{a_n b_n - a_n b + a_n b - ab}$

$a_n(b_n - b) + \cancel{b(a_n - a)}$,

δ

$|y-x|$ bounded, $< \varepsilon$.

$$\underbrace{\left|\frac{f(x)}{x}-\frac{f(y)}{y}\right|} = \left|\frac{yf(x)-xf(y)}{xy}\right|$$

$$= \left|\frac{\underline{yf(x)-xf(x)}+\underline{xf(x)-xf(y)}}{xy}\right|$$

bounded

$$\le \frac{|y-x|\,|f(x)|}{xy} + \frac{\cancel{x}\,|f(x)-f(y)|}{\cancel{x}y}$$

bounded

$$\le \frac{|x-y|\,M}{y} \;?$$

target: $\dfrac{f(x)}{xy}$ bounded ?

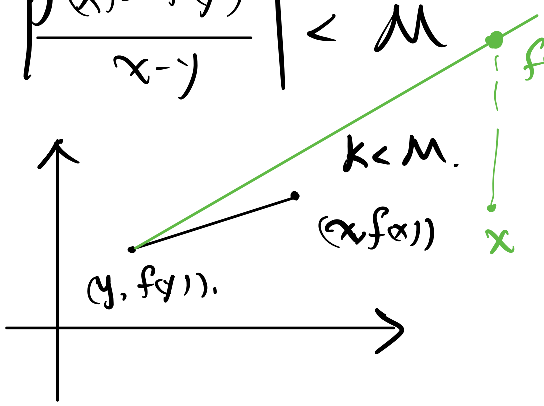
$xy \longrightarrow a$

$$\le \left(\frac{f(a)}{a}+M\right)\frac{1}{a}.$$

$$\le \frac{|x-y|\,M}{a}$$

under control

$$\left|\frac{f(x)-f(y)}{x-y}\right| < M$$



$$|f(x)| \le f(a) + (x-a)\cdot M \le f(a) + xM.$$

$$\left|\frac{f(x)}{x}\right| \le \frac{f(a)}{x} + M \le \frac{f(a)}{a}.$$

Exercise: Continuous and Uniformly Continuous

Function f(x) is defined on $(-\infty, +\infty)$. Satisfying that $\forall x, y \in \mathbb{R}$,

$$|f(x) - f(y)| \leq k\,|x - y|\,.$$

Here k is a constant satisfying that $0 < k < 1$. Prove that:
(1) $kx - f(x)$ is increasing.
(2) $\exists c \in \mathbb{R}$, f(c)=c.
(Hint: Bolzano Intermediate Value Theorem)

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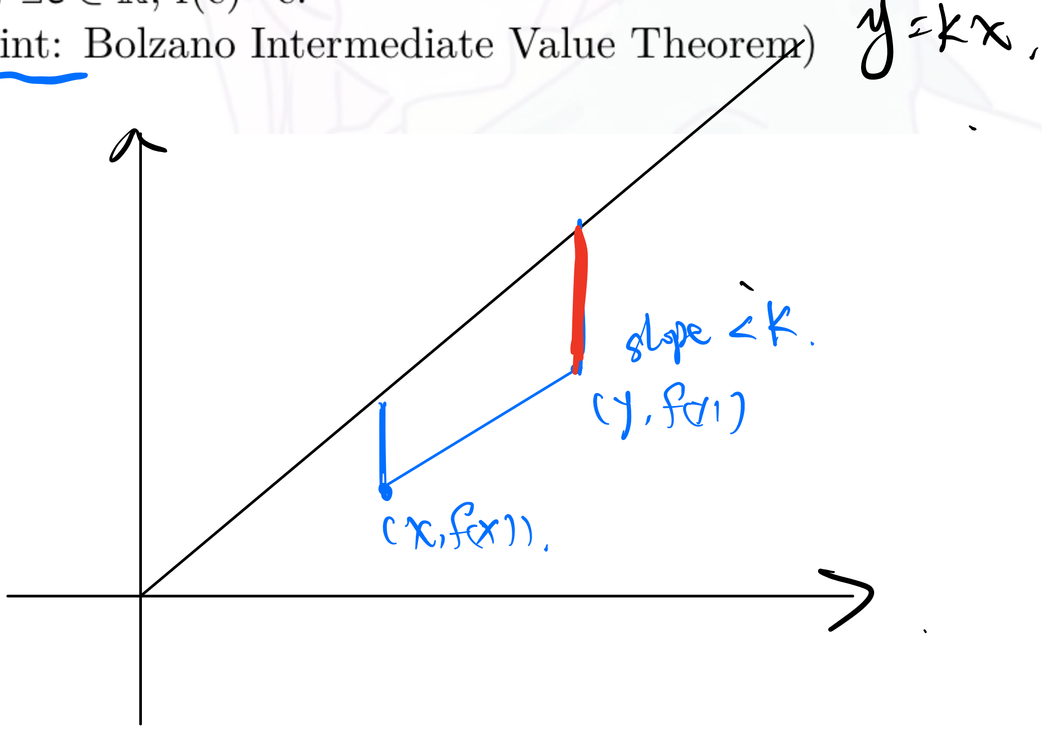

$g(x) = kx - f(x)$,

$\forall x_1 < x_2$, $f(x_2) - f(x_1) \geq 0$.

$kx_2 - f(x_2) \geq kx_1 - f(x_1)$.

$k(x_2 - x_1) \geq f(x_2) - f(x_1)$

$$k \geq \left|\frac{f(x_2)-f(x_1)}{x_2 - x_1}\right| \geq \frac{f(x_2)-f(x_1)}{x_2 - x_1}$$

Function f(x) is defined on $(-\infty, +\infty)$. Satisfying that $\forall x, y \in \mathbb{R}$,

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Her
(1)
(2)
(Hi

Here k is a constant satisfying that $0 < k < 1$. Prove that:

(1) $kx - f(x)$ is increasing.

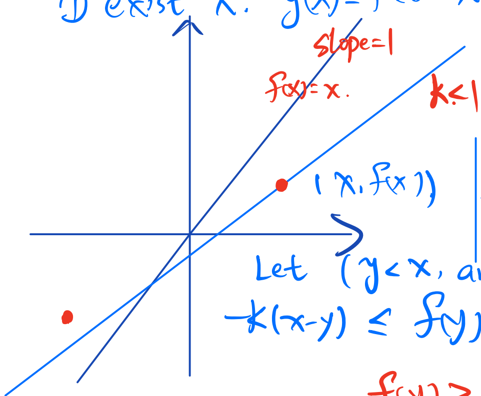
(2) $\exists c \in \mathbb{R}$, f(c)=c.

(Hint: Bolzano Intermediate Value Theorem) $y = kx$,

If there doesn't exist any $c \in \mathbb{R}$, $f(c) = c$.

$g(x) = f(x) - x$.

① exist x. $g(x) = f(x) - x < 0$.



Hope: Find y. $g(y) = f(y) - y > 0$

$$\left|\frac{f(y) - f(x)}{y - x}\right| \leq k.$$

Let $(y < x$, and $y \to -\infty)$

$-k(x-y) \leq f(y) - f(x) \leq k(x-y)$

$f(y) \geq f(x) - kx + ky$.

$f(y) - y \geq f(x) - kx + \underbrace{(k-1)}_{<0}\underbrace{y}_{\to -\infty}$.

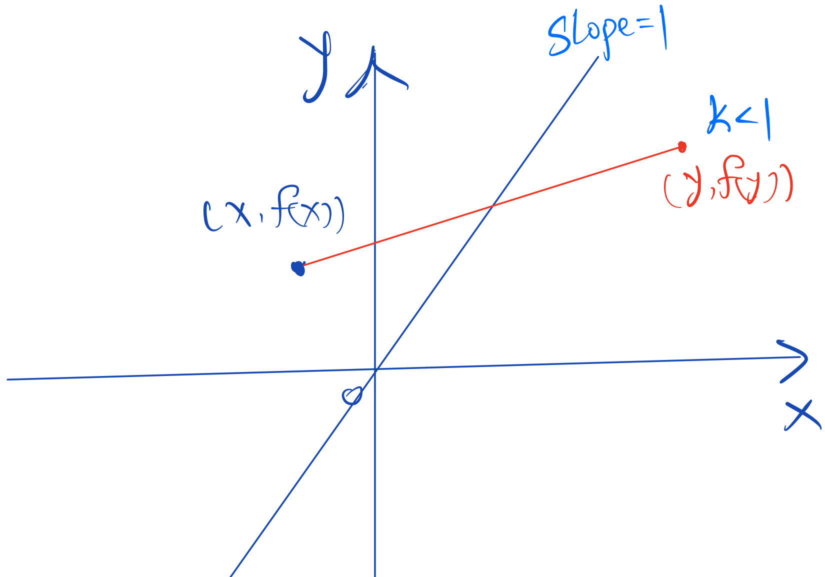
$\longrightarrow +\infty$.

So $y \to -\infty$. $f(y) - y > 0$.

$(f(y) - y)(f(x) - x) < 0$.

$\exists t \in (y, x)$. $g(t) = f(t) - t = 0$.

② exist x, $g(x) = f(x) - x > 0$,



let $y \to +\infty$.

$\left|\frac{f(x)-f(y)}{x-y}\right| \le k \qquad -k(y-x) \le f(y) - f(x) \le k(y-x)$

$f(y) \le k(y-x) + f(x)$

$g(y) = f(y) - y \le \underbrace{(k-1)}_{<0}\underbrace{y}_{\to +\infty} - kx + f(x) \longrightarrow -\infty$.

So $y \to +\infty$. $g(y) < 0$,

$g(x) > 0$. $g(y) < 0$. $\exists\, t \in (x, y)$. $g(t) = 0$.

Reference

- Vv186 Lecture Slides Professor Horst
- Vv186 Sample Exam from Professor Horst
- 2020 Vv186 TA-Huangqiyue

Say at the end of our RC

There are also many exercises in our regular RCs. And the most important and helpful exercises are all in your sample exam. Some of them are easy and many are hard. If you can't tackle the exercises from today's RC and sample exam, that's totally normal ! Just try to understand the structure of the proof and get some thought from the solutions.

Don't waste too much time on out-of-class materials. The most valuable materials will always be Professor's slide and homework. Get familiar with them !

Hope you all relax and achieve good grade !

THANKS

Announcement.

Reschedule Regular RC.

Thursday 20:30 — 22:00.