Review II(Slides 71 - 118)

Sets, Points, Rational and Real Numbers, Functions

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VV186 - Honors Mathmatics II

Assignment 1

- The first homework is graded rigorously.
- Please check the rubric and comments on common mistakes on Piazza.
- If you have questions about grading, please contact the related TA.
    - 1.1&1.2 Ma Tianyi
    - 1.3 Heyinong
    - 1.4 Ding Zizhao
    - 1.5 Sun Meng

Interval

Remind: $(a,b) \cup (c,d]$ is not an interval
$\{1\}$ is not an interval

$a=2$ $b=1$

Remark: Difference between (a,b) and the set $\{x \in \mathbb{R} : a < x < b\}$?

$a<b$ always! can be empty. ϕ

1.5.8. Definition. Let $a, b \in \mathbb{R}$ with $a < b$. Then we define the following special subsets of $\mathbb{R}$, which we call ***intervals***:

$$[a,b] := \{x \in \mathbb{R} \colon (a \le x) \wedge (x \le b)\} = \{x \in \mathbb{R} \colon a \le x \le b\},$$
$$[a,b) := \{x \in \mathbb{R} \colon (a \le x) \wedge (x < b)\} = \{x \in \mathbb{R} \colon a \le x < b\},$$
$$(a,b] := \{x \in \mathbb{R} \colon (a < x) \wedge (x \le b)\} = \{x \in \mathbb{R} \colon a < x \le b\},$$
$$(a,b) := \{x \in \mathbb{R} \colon (a < x) \wedge (x < b)\} = \{x \in \mathbb{R} \colon a < x < b\}.$$

Furthermore, for any $a \in \mathbb{R}$ we set

$$[a,\infty) := \{x \in \mathbb{R} \colon x \ge a\}, \qquad (a,\infty) := \{x \in \mathbb{R} \colon x > a\},$$
$$(-\infty,a] := \{x \in \mathbb{R} \colon x \le a)\}, \qquad (-\infty,a) := \{x \in \mathbb{R} \colon x < a\}$$

Finally, we set $(-\infty,\infty) := \mathbb{R}$.

Sets & Points

Recall the following definition and notation, it is very likely to appear in your exam ! let $x \in S$.

neighborhood.

- in Interior point $\exists \varepsilon > 0$, $(x-\varepsilon, x+\varepsilon) \subset S$
- out Exterior point $\exists \varepsilon > 0$, $(x-\varepsilon, x+\varepsilon)$ not in S.
- in/out Boundary point $\forall \varepsilon > 0$ $(x-\varepsilon, x+\varepsilon) \cap S \neq \phi$ $(x-\varepsilon, x+\varepsilon) \cap S^c \neq \phi$
- in/out Accumalation point $\forall \varepsilon > 0$ $(x-\varepsilon, x+\varepsilon) \cap S \setminus \{x\} \neq \phi$.

For each kind of points, discuss whether the point should be in the set or not ?

Whether a boundary point for a set must be an accumalation point for a set ? X

Tips:

- **Remember the definitions!**
- **Draw the pictures!**
- **Pay attention to some speacial cases!**

Better understand accumulation point

Helpful when you want to decide whether a point is accumulation point.

- Try to prove that : if x is an accumulation point of set A, for every $\epsilon > 0$, the set $(x-\epsilon, x+\epsilon) \cap A \backslash \{x\}$ contains infinite elements.
- Consequently, you can take elements from the set to construct a sequence that converges to x.

Try to prove that : if x is an accumulation point of set A, for every $\epsilon > 0$, the set $(x-\epsilon, x+\epsilon) \cap A \backslash \{x\}$ contains infinite elements.

Two Methods. (continuously find x_{i+1} after finding x_i.)

Method I : x is an accumulation point

So $\forall \varepsilon > 0$, $(x-\varepsilon, x+\varepsilon) \cap A \backslash \{x\} \neq \phi$

$x_1 \subset (x-\varepsilon, x+\varepsilon) \cap A \backslash \{x\}$

$|x - x_i| < |x - x_{i-1}|$

$\varepsilon_1 = x - x_1$, $(x-\varepsilon_1, x+\varepsilon_1) \cap A \backslash \{x\} \neq \phi$.

x_2

$x_2 \subset (x-\varepsilon_1, x+\varepsilon_1)$.

$|x - x_2| < |x - x_1|$.

Mathmetical Induction.

ε_i. x_{i+1}.

Method II: proof by contradiction.

Suppose $(x-\varepsilon, x+\varepsilon) \cap A \backslash \{x\}$ contains finite elements.

$x_1, x_2 \cdots x_n$ $\quad |x - x_i|_{min} = \varepsilon'$

Consequently, you can take elements from the set to construct a sequence that converges to x.

Rewind: finite means you can find min and max.

two more conditions to play with!

How to construct the sequence ?

$\varepsilon_1 = 1$

$\varepsilon_2 = \frac{1}{2}$

$\vdots$

$\varepsilon_k = \frac{1}{k}$

$\exists x_k \in (x - \frac{1}{k}, x + \frac{1}{k}) \cap A \backslash \{x\}$.

(x_k) is a sequence in S that converges to x.

Examples in class

Try to draw pictures, special cases.

Those two examples can give us some thought.

Two cases { ① interval boundary. ② discrete points.

Consider them additionally.

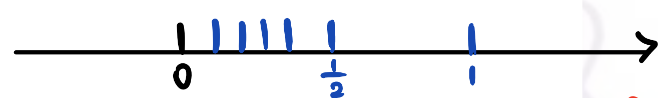
1.5.11. Example. For the interval $A = [0, 1)$,

- $\operatorname{int} A = (0, 1)$,
- Any $x \in \mathbb{R} \setminus [0, 1]$ is an exterior point,
- $\partial A = \{0, 1\}$,
- Any $x \in [0, 1]$ is an accumulation point.

discrete.

1.5.12. Example. For the set $A = \{x \in \mathbb{R} : x = \frac{1}{n}, n \in \mathbb{N} \setminus \{0\}\}$,

- $\operatorname{int} A = \emptyset$,
- Any $x \in \mathbb{R} \setminus (A \cup \{0\})$ is an exterior point,
- $\partial A = A \cup \{0\}$,
- Only $x = 0$ is an accumulation point.



discrete points are boundary points but not accumulation points.

1° $[a,b)$ Try to draw pictures. special cases.

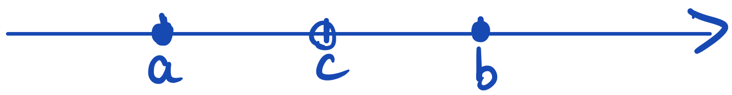
Interior $x\in(a,b)$ $\exists\varepsilon.$ $(x-\varepsilon.\ x+\varepsilon)$ Two cases $\begin{cases} ① \text{ interval boundary.} \\ ② \text{ discrete points.} \end{cases}$

Exterior $(-\infty, a)$ $(b,+\infty)$ $(x-\varepsilon, x+\varepsilon)\cap S=\phi.$

Boundary: $\{a, b\}.$

Accumulation: $[a,b].$

2° $[a, c)\cup(c, b)$



Interior : $(a.c)\cup(c,b).$

Exterior : $(-\infty, a)\cup(b,+\infty)$

Boundary : $\{a, c, b\}$

Accumulation: $[a,b].$

not that $c\notin S.$
but c can be boundary and accumulation point.

Exercise about points

Accumulation Point: $[-2,-1]$.

Answer: Int: $(-2,-1)$

Ext: $\bigcup_{i=1}^{+\infty}(\frac{1}{i+1},\frac{1}{i}) \cup \bigcup_{i=1}^{+\infty}(-\frac{1}{i},-\frac{1}{i+1}) \cup (-\infty,-2) \cup (1,+\infty)$

Boundary: $\{\frac{1}{z}: z\in\mathbb{Z}\setminus\{0\}\} \cup \{-2\}$

Please identify the interior, exterior, boundary and accumalation points of the set

what ??

$$\{\frac{1}{z} : z \in \mathbb{Z}\setminus\{0\}\} \cup (\bigcap_{j=1}^{\infty}(-2-\frac{1}{j}, -1+\frac{1}{j}))$$

$\Updownarrow$

$[-2,-1]$.

prove that

Sets & Points

Remark: Set A is closed $\Leftrightarrow$ $\partial A \subset A$

S

Recall the following definition:

- Open Set $\forall x \in S$, x interior point. for S^c
- Closed set S^c is open set. $\Leftrightarrow \forall x \in S^c$, x is interior point.
- Closure In lecture: $\bar{A}: A \cup \partial A$; In other textbooks, $\bar{A}$ is also defined as $A \cup A'$.

where A' is the set of all accumulation points

Remark: Remember that a set does **NOT** have to be either open or closed.

The set

$$\overline{A} := A \cup \partial A$$

is called the ***closure*** of A. It can be shown that $\overline{A}$ is the smallest closed set that contains A.

Conceptual Exercises

complement of set.

Remark! S closed $\Leftrightarrow$ S^c open.

Please judge true or false:

- The set $\mathbb{R}$ is an open set ? T. $\forall x \in R$ $(x-\varepsilon, x+\varepsilon) \subset R$
- The set $\mathbb{R}$ is a closed set ? T
- An empty set is an open set ? T
- An empty set is a closed set ? T
- The set $(a, b]$ is an open set or a closed set ? F
- The set $\{x \in \mathbb{R} : x = \frac{1}{n}, n \in \mathbb{N}\backslash\{0\}\}$ is closed ? F.

$S \Rightarrow$

$$S^c = \underbrace{(-\infty, 0]}_{\text{not open.}} \cup \bigcup_{i=1}^{+\infty} \left(\frac{1}{i+1}, \frac{1}{i}\right) \cup (1, +\infty).$$

Boundness

S (defined in $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$) discuss $\mathbb{C}$ later.

How we define those concepts for a set ?

- bounded/unbounded lower / upper bound
  x
  $\forall t \in S,\ t \leq x.$
  $\forall t \in S,\ t \geq x.$
- max/min $\in S$.
- sup/inf supremum: The least upper bound.
  x ↓ $\forall\ t < x$, t is not an upper bound.

Quick check:

- 1. What's the relationship between max/min and sup/inf.
- 2. Does max/min or sup/inf always exists for bounded sets? When ?

Important Conclusion: $\inf S = \xi \in S \Leftrightarrow \xi = \min S$ DIY.

Judge: A finite set must have maximum/minimum ? F

Get familiar with this!

$\emptyset$

Boundness

Check the scope ! Q or R ...... $\mathbb{Q}^+$

Example:

1. The set $A = (-\infty, a)$ is bounded above in $\mathbb{R}$ with $\sup A = a$. It isn't in A. open interval case
2. The set $B = [b, +\infty)$ is bounded below in $\mathbb{R}$ with $\inf B = b$. It's in B since b is the minimum of B. closed interval case
3. The set $C = [c, d) \cup (e, f)$ is bounded above and below in $\mathbb{R}$, so it's bounded with $\sup C = f$, $\inf C = c$. d<e.
4. The set $D = \{x \in \mathbb{Q}^+ : x = \frac{1}{n}, n \in \mathbb{N}^*\}$ is bounded above in $\mathbb{Q}^+$, but not bounded below in $\mathbb{Q}^+$. check the scope. The set is defined in $\mathbb{Q}^+$.

$\nearrow 1$

$0 \notin \mathbb{Q}^+$

$1\ \frac{1}{2}\ \frac{1}{3}\ \frac{1}{4} \cdots \frac{1}{n} \to 0.$

Exercise

The conclusion is really useful, we will frequently use it !

Let A be a bounded set in $\mathbb{R}$. Prove that for any $\epsilon > 0$, there is an element x in A such that $|x-\sup A| < \epsilon$.

☆ This conclusion tells us that we can always find an element in S that's sufficiently close to inf / sup.

We proof by contradiction.

Suppose that $\exists \varepsilon > 0, \forall x \in A, |x - \sup A| \geq \varepsilon$

Since $x \leq \sup A$ (definition of $\sup A$)

$\forall x \in A, |x - \sup A| \geq \varepsilon \Rightarrow \forall x \in A, x \leq \sup A - \varepsilon.$

Then let $t = \sup A - \frac{\varepsilon}{2}$. and $t < \sup A$ is also an upper bound

$\forall x \in A, x \leq \sup A - \varepsilon < \sup A - \frac{\varepsilon}{2} = t.$

Essence: we can always find an element from the set close enough to sup/inf

Tips on proving a supremum or infimum

least upper bound.

Generally, proving η is a supremum of a set S has two steps:

1. Firstly, show that η is a upper bound for S. i.e. $\forall x \in S, x \leq \eta$.
2. Secondly, show that $\forall \alpha < \eta, \exists x_0 \in S, x_0 > \alpha$

Sometimes an inequality is useful (directly come from the definition):

- For a set S, if $\forall x \in S, x \leq y$, then sup S $\leq$ y.

The steps for proof and properties for infimum is quite similar to supremum.

y is upper bound, so $\sup S \leq y$

Exercise

Suppose A and B are two nonempty sets of numbers such that $x \leq y$ for $\forall x \in A$ and $\forall y \in B$. Prove that :

1. sup A $\leq$ y , for $\forall y \in B$.

1. $\forall x \in A$, $x \leq y$. So $\forall y \in B$, y is an upper bound for set A. $\Rightarrow$ $\sup A \leq y$. Q.E.D.

2. sup A $\leq$ inf B

2. Prove by contradiction

Suppose $\sup A > \inf B$.

For an $\varepsilon > 0$. (we later construct ε)

There exists $a \in A$, such that $\sup A > a > \sup A - \varepsilon$.

There exits $b \in B$, such that $\inf B < b < \inf B + \varepsilon$

We want that: $a > \sup A - \varepsilon > \inf B + \varepsilon > b$.

(let $\varepsilon < \dfrac{\sup A - \inf B}{2}$)

$\Rightarrow$ contradiction.

It's hard to solve exercise just by staring at sup and inf.

Get familiar with this strategy!

Draw graph to help understanding.

Exercise

Let A, B be bounded and non-empty sets. S=$A \cup B$, please prove that:

(i) sup S = max {sup A, sup B}

(ii) inf S = min {inf A, inf B}

Just prove (i), (ii) is similar.

1° $\forall x \in S$, $x \in A$ or $x \in B$, so $x \le \sup A$ or $x \le \sup B$

So $\forall x \in S, x \le \underline{\max\{\sup A, \sup B\}}$ an upper bound for S

$\Rightarrow \sup S \le \max\{\sup A, \sup B\}$ recall the strategy introduced before

2° Further prove that $\sup S \ge \max\{\sup A, \sup B\}$,

Proof by contradiction suppose $\sup S < \max\{\sup A, \sup B\}$ why?

given $\varepsilon > 0$. $\exists\, t \in S$, $\max\{\sup A, \sup B\} > t > \max\{\sup A, \sup B\} - \varepsilon$.

$\sup S$ $\quad$ t $\quad$ ε $\quad$ $\max\{\sup A, \sup B\}$ let $\varepsilon < \max\{\sup A, \sup B\} - \sup S$.

Exercise

$t > \max\{\sup A, \sup B\} - \varepsilon > \sup S$. Contradiction! Q.E.D.

Let A and B be two bounded and non-empty sets in $\mathbb{R}$.

Define A+B $:= \{z \mid z = x + y,\ x \in A,\ y \in B\}$.

Prove that:

$$\sup(A+B) = \sup A + \sup B$$

1° By definition of $\sup A$, $\sup B$.

$\forall x \in A, y \in B$, $x \leq \sup A$, $y \leq \sup B$.

So $x + y \leq \sup A + \sup B$.

$\sup A + \sup B$ is an upper bound for $\underline{A+B}$

So $\sup(A+B) \leq \sup A + \sup B$.

2° We further prove that $\sup(A+B) \geq \sup A + \sup B$.

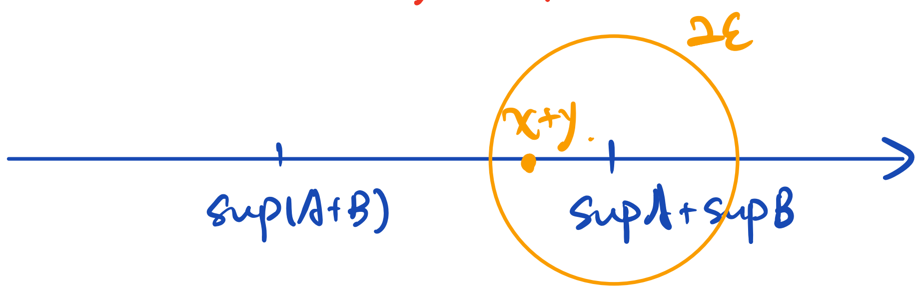
proof by contradiction:

Suppose that $\sup(A+B) < \sup A + \sup B$.

Same strategy as before. Get familiar with it!

$\varepsilon > 0$,

$$\begin{cases} \exists\ x \in A, & \sup A > x > \sup A - \varepsilon \\ \exists\ y \in B, & \sup B > y > \sup B - \varepsilon \end{cases}$$



$x+y > \sup A + \sup B - 2\varepsilon$

We wish that $x+y > \sup A + \sup B - 2\varepsilon > \sup(A+B)$

Just let $\varepsilon < \dfrac{\sup A + \sup B - \sup(A+B)}{2}$

$\Rightarrow x+y > \sup(A+B)$ contradiction!

Rational Numbers

$\mathbb{Z}$

We define that the set of rational numbers is

$$\mathbb{Q} = \{\frac{p}{q} : p, q \in \mathbb{Z} \land q \neq 0\}$$

together with the following properties(P1-P9).

Porperties	Addition	Multiplication
Associativity	$a + (b + c) = (a + b) + c$	$a \cdot (b \cdot c) = (a \cdot b) \cdot c$
unique. *NeutralElement*	$a + 0 = 0 + a = a$	$a \cdot 1 = 1 \cdot a = a$
Commutativity	$a + b = b + a$	$a \cdot b = b \cdot a$
unique. *InverseElement*	$(-a) + a = a + (-a) = 0$	$a \cdot a^{-1} = a^{-1} \cdot a = 1$
Distributivity	$a \cdot (b + c) = a \cdot b + a \cdot c$	

Rational Numbers

Property 10: Trichotomony Law (Using a set P to divide Q into three parts)

set for positive rational numbers.

Property 11 and 12: Feature the set P, such that positive numbers are closed under addition and multiplication

$\forall a, b \in P.$

$a+b \in P$

$a \cdot b \in P.$

$(-1) \cdot (-1) = 1.$

Important Inequality

$\triangle$

Triangle Inequality.

For all rational numbers $a, b \in \mathbb{Q}$, we have $||a| - |b|| \leq |a + b| \leq |a| + |b|$

You don't have to turn every a_i into $|a_i|$. Eg: You can turn 3 a into $|a|$ and save others.

Prove it using Mathematical Induction !

Corollary: $|\sum_{i=1}^{n} a_i| \leq \sum_{i=1}^{n} |a_i|$

1° Base Case. $i = 1$. $|a_1| \le |a_1|$ ✓

$i = 2$.

2° Suppose for $i = k$. $|\sum_{i=1}^{k} a_i| \le \sum_{i=1}^{k} |a_i|$ holds true.

Then consider the case where $i = k+1$.

$|\sum_{i=1}^{k+1} a_i| = |\sum_{i=1}^{k} a_i + a_{k+1}| \le |\sum_{i=1}^{k} a_i| + |a_{k+1}|$

$\sum_{i=1}^{k+1} |a_i|$.

The Square Root Problem

defined in $\mathbb{R}$ here!

Let $M = \{t \in \mathbb{R} : t > 0 \wedge t^2 > x\}$, $y = \inf M$. We want to prove that $y^2 = x$ by showing that $y^2 > x$ and $y^2 < x$ lead to contradictions.

$M: \{t \in \mathbb{R},\ t > 0 \wedge t^2 > x\}$. Find infimum y.

Part 1: Prove that $y^2 > x$ leads to contradiction.

Suppose that $y^2 > x$. Since y is an infimum, $\forall\, 0 < t < y,\ t \notin M \Rightarrow t^2 \le x$

(By definition of inf).

We want to find a $t < y$, but $t^2 > x$ to find a contradiction

let: $t = y - \varepsilon$, $\varepsilon > 0$. 看 ε 取多小时 $t^2 > x$

$t^2 > x \iff (y-\varepsilon)^2 > x \iff y^2 - 2y\varepsilon + \varepsilon^2 > x$ ← we want an ε to satisfy this inequality.

$\underbrace{(y^2 - x)}_{>0} + \varepsilon^2 > 2y\varepsilon$.

more ε

So we just need to let. $0 < \varepsilon \le \dfrac{y^2 - x}{2y} = \dfrac{1}{2}\left(y - \dfrac{x}{y}\right)$

$t = y - \varepsilon$. $t^2 > x$.

$\left(y + \dfrac{x}{y}\right)$

Part 2 for contradiction

Part 2 prove that $y^2<x$ leads to contradiction.

Suppose that $y^2<x$, since y is an infimum. $\forall 0<t<y$, $t^2\le x$.

This time we search for $t>y$, such that t is a greater lower bound.

let $t=y+\varepsilon$. $\varepsilon>0$, and find ε such that $t^2<x$

Then $t^2<x \Leftrightarrow (y+\varepsilon)^2<x \Leftrightarrow y^2+2y\varepsilon+\varepsilon^2<x$.

Remember that, you still can't use "not" in this exercise

This exercise aims to define "not" in $\mathbb{R}$.

Arbitrarily find ε such that $\begin{cases} 2y\varepsilon<\dfrac{x-y^2}{2} & ① \\ \varepsilon^2<\dfrac{x-y^2}{2} & ② \end{cases}$

(Advantage: also easy to use when ε has higher dimensions)

① $\Rightarrow \varepsilon < \frac{x-y^2}{4y}$

② let $\varepsilon = \frac{1}{k}$, $(k \in \mathbb{Z}^+)$. $(0, +\infty)$.

$\frac{1}{k^2} < \frac{x-y^2}{2} \Leftrightarrow k^2 > \frac{2}{x-y^2}$

fixed and limited.

So $\exists k_0 \in \mathbb{Z}^+$, $\varepsilon^2 = \frac{1}{k_0^2} < \frac{x-y^2}{2}$.

Finally, let $0 < \varepsilon < \min\left\{\frac{x-y^2}{4y}, \frac{1}{k_0}\right\}$

So the ε is found

$\Rightarrow t = y + \varepsilon$, $t^2 < x$.

$\Rightarrow$ t is a greater lower bound

~~then~~ inf M $\Rightarrow$ contradiction!

Important Conclusion

Infimum and Supremum don't necessarily exist in a bounded set defined in Q.

Eg: $\{x \in \mathbb{Q} : x^2 > 2\}$ don't have a infimum.

Real Numbers and Important Conclusion

The square root problem tells us that: Bounded sets may not have infimum or supremum.
The definition of real numbers guarantees that for a set in $\mathbb{R}$, **if it is bounded above, then it has an supremum; if it is bounded below, then it has a infimum.**

The Real Numbers

Similary. let x in A becomes $-x$,
$\Rightarrow$ conclusions: bounded below, then exists infimum

We define the set of real numbers $\mathbb{R}$ as the smallest extension of the rational numbers $\mathbb{Q}$ such that the following property holds:

(P13) *If $A \subset \mathbb{R}$, $A \neq \emptyset$ is bounded above, then there exists a least upper bound for A in $\mathbb{R}$.*

We call all real numbers that are not rational ***irrational numbers***.

Complex Numbers

In Vv186, you just need to know how to perform basic complex numbers' computation and some basic properties. Here, we just list some basic computation rules and formulas.

Given $z_1 = (a_1, b_1)$ and $z_2 = (a_2, b_2)$,

- $z_1 + z_2 = (a_1, b_1) + (a_2, b_2) = (a_1 + a_2, b_1, b_2)$
- $z_1 \cdot z_2 = (a_1, b_1) \cdot (a_2, b_2) = (a_1 a_2 - b_1 b_2, a_1 b_2 - a_2 b_1)$
- $c \cdot z_1 = c(a_1, b_2) = (ca_1, cb_1), c \in \mathbb{R}$
- $\bar{z_1} = (a_1, -b_1)$
- $|z_1|^2 = a_1^2 + b_1^2 = z_1 \bar{z_1}$
- $\mathrm{Re}\ z_1 = \frac{z_1 + \bar{z_1}}{2}$
- $(\mathrm{Im}\ z_1) i = \frac{z_1 - \bar{z_1}}{2}$

Conclusions for existence of max/min and sup/inf for a set in $\mathbb{Q}$ and in $\mathbb{R}$.

① Max and min may not exist in set defined in $\mathbb{Q}/\mathbb{R}$ even if the set is bounded above / below.

Eg: $(1,2)$.

② sup and inf may not exist in a bounded set defined in $\mathbb{Q}$.

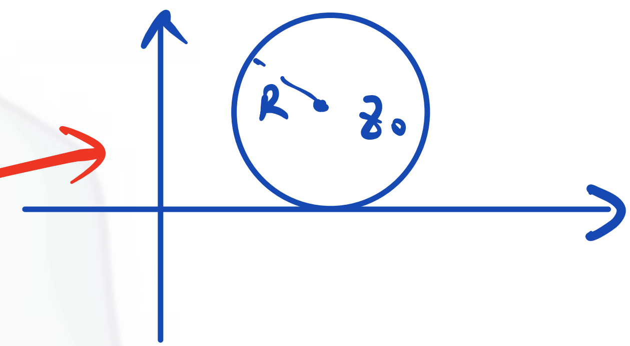
Eg: $\{x \in \mathbb{Q} : x^2 > 2\}$ don't have a infimum.

③ For a set defined in $\mathbb{R}$.

if the set has lower bound, then it has an infimum.

if the set has upper bound, then it has a supremum.

Open Ball



Let $z_0 \in \mathbb{C}$. Then we define the **open ball** of radius $R > 0$ centered at z_0 by

$$B_R(z_0) := \{z \in \mathbb{C} : |z - z_0| < R\}$$

- Geometric interpretation? $|z - z_0| < R$.
- Higher dimensions? NORM learn later.

How to define the boundness of a set in $\mathbb{C}$? $\rightarrow S \subset B_R(0)$.

Are there lower bound or upper bound for a bounded set in $\mathbb{C}$?

No. You can't compare $1+i$ and $2+i$ which one is larger.

Important Definition for Function

different!!!

Recall the definition of domain, codomain and range.

Let's start from the notation for functions.

range: $\{y : \exists x \in \text{domain}. \ y = f(x)\}$.

domain. codomain.

① $f : \Omega \to Y, \qquad x \mapsto f(x).$

or alternatively

② $f : \Omega \to Y, \qquad \mathrm{y} = \mathrm{f(x)}.$

Example: Point out the domain, codomain and range for the function:

domain codomain

$f : \mathbb{R}^2 \to \mathbb{R}, \qquad \mathrm{f(x,y)} = x^2 + y^2$

$\mathbb{R} \times \mathbb{R}$. $\{(x,y)\}$ range. $[0, +\infty)$.

cartesian product.

Exercise

Will detailedly discuss next time in RC!

We will spend some time discussing about $\overline{lim}$ and $\underline{lim}$, and their relation with sup/inf for sets in next RC.

Consider the set $U \subset \mathbb{R}$, where $U = A \cup B \cup C$ with

$$A = \{x \in \mathbb{R} \colon 0 < x \leq 1\},$$
$$B = \{x \in \mathbb{R} \colon x = 2 - 1/n,\ n \in \mathbb{N} \setminus \{0\}\},$$
$$C = \{x \in \mathbb{R} \colon x = -1/n,\ n \in \mathbb{N} \setminus \{0\}\}.$$

State (without proof) $\min U$, $\max U$, $\inf U$, $\sup U$, $\underline{\lim}\, U$ and $\overline{\lim}\, U$ (if one or more of these do not exist, simply state this).

Exercise

Let $A \subset \mathbb{R}$ be a non-empty set.

- ☐ If $\inf A$ exists, then $\underline{\lim}\, A$ exists.
- ☐ If $\underline{\lim}\, A$ exists, then $\inf A$ exists.
- ☐ $\underline{\lim}\, A$ exists if and only if A is bounded below.
- ☐ $\inf A$ exists if and only if A is bounded below.

Reference

- Exercises from 2021-Vv186 TA-Ni Yinchen.
- Exercises from 2021-Vv186 TA-Tu Yiwen.
- Exercises from 2022-Vv186 TA-Sun Meng