Review III(Slides 119 - 167)

Sequence & Convergence

Kulu

University of Michigan-Shanghai Jiao Tong University Joint Institute

October 12, 2022

VV186 - Honors Mathmatics II

Reschedule

THURSDAY : RC : 20:30 - 22:10

Midterm is Coming

Mid-term Exam - Schedule

Week	Lecture Subject	Date
1	Introduction Elements of Logic Set Theory, Natural Numbers, Induction	13-9-2022 15-9-2022 16-9-2022
2	Rational and Real Numbers Complex Numbers; Functions	20-9-2022 22-9-2022
3	Sequences Metric Spaces, Cauchy Sequences Real Functions	27-9-2022 29-9-2022 30-9-2022
4	Chinese National Day Holiday Chinese National Day Holiday	
5	Asymptotic Behavior of Functions Continuous Functions Continuous Functions	11-10-2022 13-10-2022 14-10-2022
6	Differentiation of Real Functions First Midterm Exam	18-10-2022 20-10-2022

Scope:

Slides 1 - 252
Topics 1 - 11

Most important:
Topic 5, 8, 10, 11

(Others are also important!)

Midterm Preparation

Suggestions:

1, Lecture Slides - Definition & Lemma & Theory

2, Homework - Do it youself! Especially Assignment 2, 3, 4 (Last questions)

3, Sample Exam - Better than doing exercises from elsewhere

4, Sleep more & Relax

Kulu's Personal Tips:

- Understanding instead of reciting ! Explain concepts and theories in your own intuitive words.
- Visualize is most important. For set, for function, for sequence.

$\overline{lim}$ and $\underline{lim}$ for set

First, let's recall the definition.

Exercise 2.4

A number x is called an *almost upper bound* for a set $A \subset \mathbb{R}$ if there are only finitely many numbers $y \in A$ with $y \geq x$.[3] An *almost lower bound* is defined similarly.

i) State (without proof) all amost upper and almost lower bounds for the sets

a) $\{1 + 2^{-n} : n \in \mathbb{N}^*\}$, b) $\left\{(-1)^n + \frac{1}{n^2} : n \in \mathbb{N}^*\right\}$

c) $\left\{\frac{1}{n} : n \in \mathbb{Z} \setminus \{0\}\right\}$, d) $\left\{x \in \mathbb{Q} : 0 \leq x \leq \sqrt{2}\right\}$

(4 Marks)

ii) Suppose that X is a bounded infinite set. Prove that the set Y of all almost upper bounds of X is nonempty, and bounded below.
(2 Marks)

iii) By (P13), the infimum $\inf Y$ exists; this number is called the *limit superior* of X and denoted by $\limsup X$ or $\overline{\lim}\, X$. Find the limit superior for the sets given in i).
(2 Marks)

iv) Formulate a definition for the *limit inferior* $\underline{\lim}\, X$ and find the limit inferior for the sets given in i).
(2 Marks)

$\overline{lim}$ and $\underline{lim}$ properties

Prove them "in a second". Be familiar enough with every concept and conclusion in your assignment !
Warning: A finite set doesn't necessarily have a maximum/minimum ! It can be empty !

almost upper bound. Y_1
lower Y_2.

v) Let A be an infinite bounded set.[4] Prove that

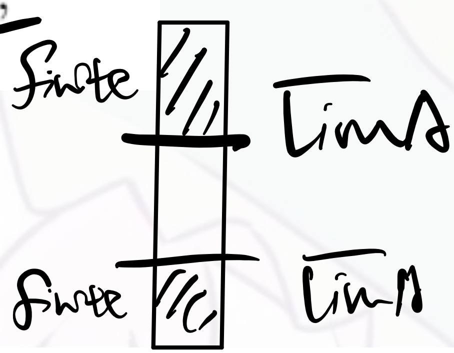
(a) $\underline{\lim}\, A \le \overline{\lim}\, A$,
(2 Marks)

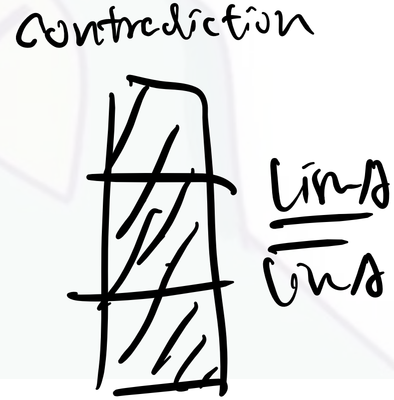
(b) $\overline{\lim}\, A \le \sup A$, $\underline{\lim}\, A \ge \inf A$.
(2 Marks)

(c) If $\overline{\lim}\, A < \sup A$, then $\max A$ exists. If $\underline{\lim}\, A > \inf A$, then $\min A$ exists.
(2 Marks)

Proof for $\overline{lim}$ and $\underline{lim}$ properties

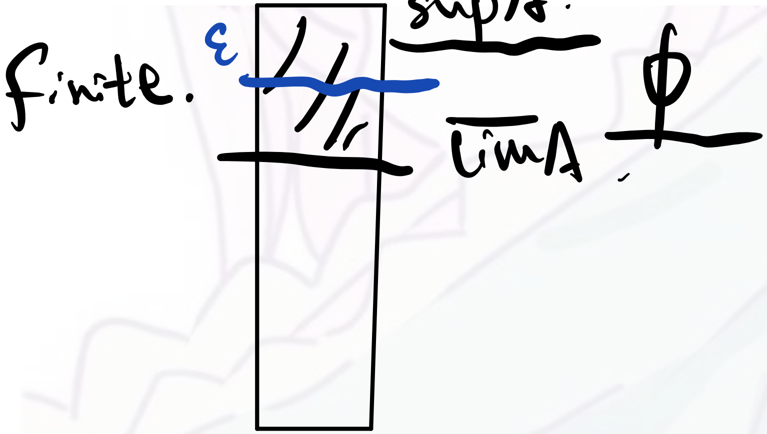
(a) $\underline{\lim}\, A \leq \overline{\lim}\, A$,
(2 Marks)





(b) $\overline{\lim}\, A \leq \sup A$, $\underline{\lim}\, A \geq \inf A$.
(2 Marks) $\in \tilde{I}$

(c) If $\overline{\lim}\, A < \sup A$, then $\max A$ exists. If $\underline{\lim}\, A > \inf A$, then $\min A$ exists.
(2 Marks)



$(\sup A - \varepsilon \,,\, \sup A)$

$\underline{\sup A - \frac{\varepsilon}{2}}$

Exercise (Left in RC2)

Consider the set $U \subset \mathbb{R}$, where $U = A \cup B \cup C$ with

$$\begin{aligned} A &= \{x \in \mathbb{R} : 0 < x \leq 1\}, \\ B &= \{x \in \mathbb{R} : x = 2 - 1/n,\ \ n \in \mathbb{N} \setminus \{0\}\}, \\ C &= \{x \in \mathbb{R} : x = -1/n,\ \ n \in \mathbb{N} \setminus \{0\}\}. \end{aligned}$$

State (without proof) $\min U$, $\max U$, $\inf U$, $\sup U$, $\underline{\lim}\, U$ and $\overline{\lim}\, U$ (if one or more of these do not exist, simply state this).

Exercise Answer (Left in RC2)

$$\min U = -1, \qquad \max U \text{ does not exist},$$

$$\inf U = -1, \qquad \sup U = 2,$$

$$\underline{\lim}\, U = 0, \qquad \overline{\lim}\, U = 2.$$

Exercise (Left in RC2) (1 min),

⟺ lower/upper bound exists

Question: When does inf/sup exist? When does $\overline{lim}$ and $\underline{lim}$ exist?

⇕ bounded + ~~infinite~~ infinite

Let $A \subset \mathbb{R}$ be a non-empty set.

lower bound infinite!

F ☐ If $\inf A$ exists, then $\underline{\lim} A$ exists. $\{1\}$ x

T ☐ If $\underline{\lim} A$ exists, then $\inf A$ exists. ✓

infinite!

F ☐ $\underline{\lim} A$ exists if and only if A is bounded below. $\{1\}$ x

T ☐ $\inf A$ exists if and only if A is bounded below. ✓

Sequence

Last year, a lot of students asked Prof. and TAs:

- Why is a sequence always have infinite items?
- What if a sequence only have finite items?
- ...

!!! When we say "sequence" we usually assume that it is infinite.
If it is finite, i.e., it contains only finite items , we usually say it is a "n-tuple". Similarly, a subsequence of a sequence is infinite.

Convergence & Divergence

Recall definition for converge and diverge.

Converge. Limit c.

negation. $\forall \varepsilon > 0,\ \exists N \in \mathbb{Z}^+,\ \forall n > N,\ |a_n - c| < \varepsilon.$

Quick check:

Diverge. $\exists \varepsilon > 0,\ \forall N \in \mathbb{Z}^+,\ \exists n > N,\ |a_n - c| > \varepsilon.$

- F A sequence is either convergent or divergent to (minus) infinity.
- T A sequence is either convergent or divergent. $A \vee (\neg A)$.
- F If a sequence diverges, then it will go to (minus) infinity.

Relationship between limit and accumulation ~~set~~ point.

First, recall the definition:

- Limit:
  - How to impretate it ?
  - Relationship with boundness ? if (a_n) has limit   bounded
  - How many limits a sequence can have ?   Limit.
- Accumulation Point: c (a_n).
  - How to impretate it ? $\forall \varepsilon > 0, \forall N \in \mathbb{Z}^+, \exists n > N, |a_n - c| < \varepsilon.$
  - How many accumulation points can a sequence have ? any number.
  - limit of the subsequence ? c
  - Existence ? Bounded sequence must have $|a_{i_1} - c| < 1$
  - Relationship with boundness ? $|a_{i_2} - c| < \frac{1}{2}$ (a_{i_k}).
  - $|a_{i_k} - c| < \frac{1}{k}$.
  - Difference between accumulation points for set and sequence ? Accumulation point of ran(a_n) must be accumulation point for (a_n) ? Vice versa ?

BW

A sequence have accumulation point, it doesn't need to be bounded.

a set S has accumulation point c.

$\forall \varepsilon > 0$. $\underline{(c-\varepsilon, c+\varepsilon)}$ S

infinite.

Helpful when you want to decide whether a point is accumulation point.

- Try to prove that : if x is an accumulation point of set A, for every $\epsilon > 0$, the set $(x-\epsilon, x+\epsilon) \cap A \backslash \{x\}$ contains infinite elements.
- Consequently, you can take elements from the set to construct a sequence that converges to x.

From RC2

(a_n) accumulation point c.

$\forall \varepsilon > 0$. $\underline{(c-\varepsilon, c+\varepsilon)}$

Accumulation point of $\boxed{\text{ran}(a_n)}$ must be accumulation point for (a_n)

p

$\text{ran}(a_n) = \{x : \exists i, x = a_i\}$ $\underline{\{1\}}$

$a_n = (-1)^n$ $\text{ran}(a_n) = \{-1, 1\}$.

$\forall \varepsilon > 0$, $(p-\varepsilon, p+\varepsilon)$

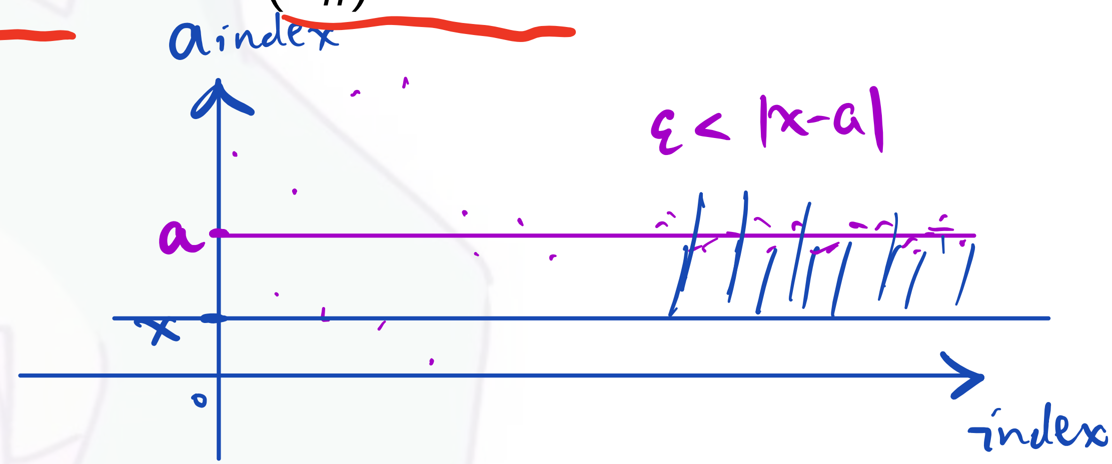
Converse. False. $\underline{a_n = 1}$ $\underline{1}$

Limit

Some results for limit.

Suppose $(a_n) \to a \in \mathbb{R}$ and $(b_n) \to b \in \mathbb{R}$

1. $\lim(a_n + b_n) = a + b$
2. $\lim(a_n \cdot b_n) = a \cdot b$
3. $\lim \frac{a_n}{b_n} = \frac{a}{b}, b \neq 0$



We will prove property 3 later.

Useful conclusion: if (a_n) converges to a > 0, $\forall x \in (0, a)$, there exists N>0, such that $\forall n > N$, $a_n > x > 0$. Visualize it to understand

Notice:

$a_n = n$

- $\lim(a_n + b_n) = \lim a_n + \lim b_n$? F.

$b_n = -n$

$\left(\frac{1}{2}+\frac{1}{2}\right)\left(\frac{1}{4}+\cdots+\frac{1}{4}\right)\ \frac{1}{8}+\frac{1}{8}$

$1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n} \to +\infty$

$\lim\limits_{n\to\infty}(|a_{n+1} - a_n|) = 0$, then (a_n) converges?

$a_1 = 1$, $\forall n \geq 2$, $a_{n+1} = a_n + \frac{1}{n}$.

Limit

$$\lim \frac{a_n}{b_n} = \frac{a}{b}, b \neq 0$$

Look at the definition and set your goal!

Goal:

$$\forall_{\varepsilon > 0} \ \exists_{N > 0} \ \forall_{n > N} \ \left|\frac{a_n}{b_n} - \frac{a}{b}\right| < \varepsilon$$

Another approach:

Using the second result and try to prove $\lim \frac{1}{b_n} = \frac{1}{b}$

Proof

Condition: $\forall \varepsilon>0, \exists N \in \mathbb{Z}^+, \forall n>N, |b_n - b| < \varepsilon$

Need to prove. $\forall \varepsilon>0, \exists N \in \mathbb{Z}^+, \forall n>N, \left|\frac{1}{b_n} - \frac{1}{b}\right| < \varepsilon.$

$$\left|\frac{b_n - b}{b_n \cdot b}\right| < \varepsilon.$$

try to construct.

$$\left|\frac{b_n - b}{b_n \cdot b}\right| < \frac{\varepsilon'}{|\boxed{b_n} \cdot b|} \leq \frac{\varepsilon'}{|\frac{b}{2}| \cdot |b|} \leq \frac{2\varepsilon'}{b^2} < \varepsilon.$$

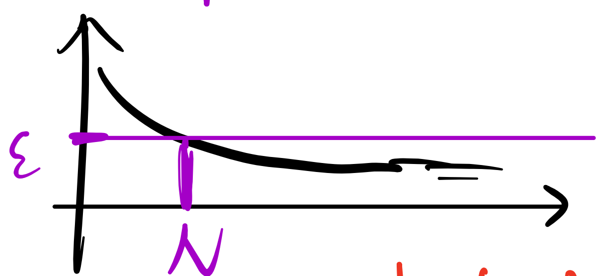
$$\varepsilon' < \frac{b^2 \varepsilon}{2}.$$

Exercises : Important limits

Prove by definition

Prove that, $\lim_{n\to\infty} \frac{1}{n^\alpha} = 0, \alpha \in (0, +\infty)$

Step 1. Generally analyze the trend of sequence



$n^\alpha \uparrow +\infty$

$\forall \varepsilon > 0$, Let $N > \log_\alpha \frac{1}{\varepsilon}$, then $0 < \frac{1}{n^\alpha} < \frac{1}{N^\alpha} < \varepsilon$.

Try to find N, such that $\forall n > N$,

$\frac{1}{n^\alpha} < \frac{1}{N^\alpha} < \varepsilon$. $N^\alpha > \frac{1}{\varepsilon}$

Let $N > \log_\alpha \frac{1}{\varepsilon}$

Exercises : Important limits (30s)

Squeeze Theorem.

Prove that, $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$

$$1 \le \lim_{n\to\infty} n^{\frac{1}{n}} \le \lim_{n\to\infty} \frac{2\sqrt{n}+n-2}{n} = 1.$$

$$1 \le n^{\frac{1}{n}}$$

$$(\underbrace{1+1+1+\cdots+1}_{n-2}+\sqrt{n}+\sqrt{n}) \ge n\sqrt[n]{n}.$$

$$\underbrace{\frac{2\sqrt{n}+n-2}{n}}_{\to 1} \ge n^{\frac{1}{n}} \ge 1.$$

Exercises : Important limits (30s),

squeeze theorem.

Prove that, $\lim_{n\to\infty} \sqrt[n]{a} = 1, a > 0$

Method 1:

① $\alpha > 1$, $1 \le \alpha^{\frac{1}{n}} \le n^{\frac{1}{n}}$, $n^{\frac{1}{n}} \to 1$

$\lim_{n\to\infty} \alpha^{\frac{1}{n}} = 1$.

② $\alpha = 1$ ✓

③ $\alpha < 1$.

$\underbrace{\alpha^{\frac{1}{n}}}_{\downarrow\, 1}$ $\quad (\frac{1}{\alpha})^{\frac{1}{n}} \to 1$. $\quad \frac{1}{\alpha} > 1$

Method 2:

$$\alpha^{\frac{1}{n}} = e^{\ln \alpha^{\frac{1}{n}}} = e^{\underbrace{\frac{1}{n}\ln\alpha}_{\downarrow\, 0}} = 1.$$

Exercises

(1 min)

Squeeze Theorem

Let (a_n) be a sequence that $a_n = \frac{1}{\sqrt{n^2+1}} + \cdots + \frac{1}{\sqrt{n^2+n}}$. Calculate the limit of (a_n).

$$\frac{n}{\sqrt{n^2+n}} \le a_n \le \frac{n}{\sqrt{n^2+1}}$$

$$\lim_{n\to\infty} \frac{n}{\sqrt{n^2+n}} \le \lim_{n\to\infty} a_n \le \lim_{n\to\infty} \frac{n}{\sqrt{n^2+1}}$$

$$\Downarrow \qquad \downarrow \qquad \Downarrow$$

$$1 \qquad 1 \qquad 1$$

$$\frac{1}{\sqrt{1+\frac{1}{n}}} \to 1 \qquad \frac{1}{\sqrt{1+\frac{1}{n^2}}} \to 1$$

Exercises ( )min

First describe relation between Sk and Sk+1 Using math language.

Analyze the trend of the sequence

A sequence is defined as

$$(S_n)_{n\in\mathbb{N}}, S_1 = \sqrt{2}, S_2 = \sqrt{2+\sqrt{2}}, S_3 = \sqrt{2+\sqrt{2+\sqrt{2}}}, \ldots$$

Please prove that it is convergent and calculate the limit of (S_n) as $n \to \infty$.

Exercises (Important) Very similar Exercise in Mid 1 (Last year)

(2 min)

Step 1: First calculate the limit (in heart)

Step 2: Mathematical Induction.

Show that the sequence defined by

$x^2-3x+1=0 \quad x=\frac{3-\sqrt{5}}{2}$ or $\frac{3+\sqrt{5}}{2}$.

$$a_1 = 2, \qquad a_{n+1} = \frac{1}{3 - a_n}, \qquad (n \geq 1)$$

Even no hints, set the target to prove monotonic and bounded.

satisfies $0 < a_n \leq 2$ and is decreasing. Deduce that the sequence is convergent and find its limit.

We use Mathematical Induction to show that

$\forall n, \quad a_{n+1} \leq a_n$ and $\frac{3-\sqrt{5}}{2} \leq a_{n+1} \leq 2$.

1° Base: $a_1 = 2 \in [\frac{3-\sqrt{5}}{2}, 2]$

2° $a_{n+1} - a_n = \frac{1}{3-a_n} - a_n = \frac{a_n^2-3a_n+1}{3-a_n} \leq 0$, So $a_{n+1} \leq a_n$ (a_n) is $\downarrow$

$a_{n+1} \geq \frac{3-\sqrt{5}}{2} \iff \frac{1}{3-a_n} \geq \frac{3-\sqrt{5}}{2} \iff 3-a_n \leq \frac{2}{3-\sqrt{5}} = \frac{3+\sqrt{5}}{2} = a_n \geq \frac{3-\sqrt{5}}{2}$

So (a_n) is $\downarrow$ and bounded $\Rightarrow$ converge.

3° $a_{n+1} = \frac{1}{3-a_n}$. $\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{1}{3-a_n} \Rightarrow \lim_{n\to\infty} a_n = \frac{3-\sqrt{5}}{2}$. (since $a_n \leq 2$).

Exercises

Let $(a_n), (b_n)$ be two real sequences. Furthermore, assume that $a_n < b_n$ for all n, $[a_{n+1}, b_{n+1}] \subseteq [a_n, b_n]$, $\lim(a_n - b_n) = 0$. Prove that there is an unique $m \in [a_n, b_n]$ for all n, such that

$$\lim a_n = \lim b_n = m$$

Exercises

1° Write down the definition of converge. That's your only tool.

2° Divide and Conquer.

Let (a_n) be a real sequence that converges to $a \in \mathbb{R}$. Prove that the sequence $(\frac{\sum_{i=1}^{n} a_i}{n})$ is convergent. Furthermore $\lim_{n\to\infty} (\frac{\sum_{i=1}^{n} a_i}{n}) = a$.

Condition: $\forall \varepsilon > 0, \exists N \in \mathbb{Z}^+, \forall n > N, |a_n - a| < \varepsilon$.

Exercises (Optional)

Let (a_n) be a real sequence that converges to $a \in \mathbb{R}$. Let (b_n) be a real sequence that converges to $b \in \mathbb{R}$. Prove that the sequence $(\frac{\sum_{i=1}^{n} a_i b_{n-i+1}}{n})$ is convergent. Furthermore $\lim_{n\to\infty}(\frac{\sum_{i=1}^{n} a_i b_{n-i+1}}{n}) = ab$.

$\overline{lim}$ and $\underline{lim}$ for sequence

3.6.

How to interpret $\overline{lim}$ and $\underline{lim}$ of a sequence ?

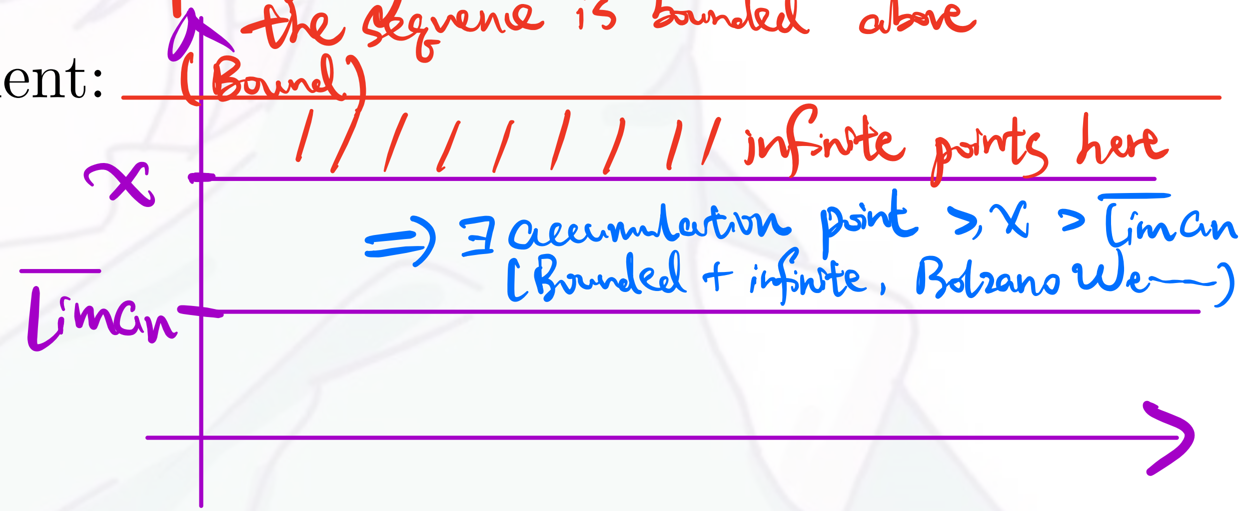
- Way 1: Let $\alpha_n = \underset{k\geq n}{\overset{\sup}{\cancel{\inf}}} a_k$, $\overline{lim} a_n = \lim_{n\to\infty} \alpha_n$
- Way 2: Largest and smallest accumulation point.

Properties:

- Important property: $\forall x > \overline{lim} a_n$, $\exists N \in \mathbb{N}$, such that $\forall n \geq N, a_n < x$.

Proof by contradiction, Since $\overline{\lim}$ an exists, the sequence is bounded above

- Property in Assignment:



$\overline{lim} a_n \geq \underline{lim} a_n$

$\overline{lim} a_n = \overline{lim} ran(a_n)$

Exercise (1.5min)

Prove that : $\liminf\limits_{n\to\infty} a_n + \liminf\limits_{n\to\infty} b_n \le \liminf\limits_{n\to\infty}(a_n + b_n) \le \liminf\limits_{n\to\infty} a_n + \limsup\limits_{n\to\infty} b_n$

$\alpha_n = \inf\limits_{k\ge n} a_k$ $\beta_n = \inf\limits_{k\ge n} b_k$

$\lim \alpha_n$ $\lim \beta_n$.

$-\liminf\limits_{n\to\infty}(-b_n)$.

Δ

$\inf\limits_{k\ge n} a_k + \inf\limits_{k\ge n} b_k \le a_p + b_p$ for all $p \ge n$.

$\le \inf\limits_{k\ge n}(a_k + b_k)$.

Important Results & Theorem

Should Be Very Familiar !

- A convergent sequence is bounded. (Slides 128)
- A convergent sequence has precisely one limit. (Slides 130)
- (**Squeeze Theorem**)
  Let $(a_n), (b_n)$ and (c_n) be real sequences with $a_n < c_n < b_n$ for sufficiently large $n \in \mathbb{N}$. Suppose that $\lim\ a_n = \lim\ b_n =: a$. Then (c_n) converges and $\lim\ c_n = a$. (Slide 133)
  Comment: It is extremely useful for examining the convergence of a sequence that is bounded.
- Let (a_n) be a convergent sequence with limit a. Then any subsequence of (a_n) is convergent with the same limit. (Slide 145)
- Every real sequence has a monotonic subsequence. (Slide 146)

Refer to slides, check the proofs on the weekend!

Important Results & Theorem

- If a sequence has an accumulation point x, then there is a subsequence that converge to this point x. (Slides 149)
- (**Bolzano–Weierstraß**)
  Every bounded real sequence has an accumulation point. (Slide 150)
  Comment. There are at least two proofs, which we will discuss later.
- Every monotonic and bounded (real) sequence is convergent. (Slide 141)
  Comment. This result holds for sequence in any space with an ordering (otherwise it's strange to even define "monotonic").

Refer to slides, check the proofs on the weekend!

Bolzano–Weierstraß (Read it yourself)

Optional

Bolzano–Weierstraß Every bounded real sequence has an accumulation point.

1. Proof–1: On Horst's Slides.
2. Proof–2: Since(a_n) is bounded, assume $-M \leq a_n \leq M$ for all n. Divide the interval $[-M, M]$ into 2 sections:$[-M, 0], [0, M]$. One of the interval, denoted by $I^{(1)}$, must contain infinitely many "a_n"s(otherwise $(a_n)$is finite). Choose an $a_{(n,1)}$ in $I^{(1)}$. We bisect $I^{(1)}$ into two intervals, one of which, denoted by $I^{(2)}$ must contain infinitely many "a_n "s. Choose an $a_{(n,2)}$ in $I^{(2)}$ that is different from $a_{(n,1)}$. By repeatedly doing this procedure, we find a subsequence $(a_{n,k})_{k\in\mathbb{N}}$ that converges.

Metric Space How do we describe "distance" before ?

- What is the definition of a metric? $\underline{|x-y|}$
- Why we want to introduce the idea of Metric Space?
- What new results can we explore from this new idea?

We want to generalize the idea of convergence, or close to some point. The most important thing is to define the **Length Function**. Metric is just a nice way of describing the distance.

What properties a usual length function should have?

1. Always positive.(distance) $\underline{\geq 0.}$
2. Symmetric.(distance)
3. Followed Triangle Inequality.(nice)

The remaining task is just transform these into mathematical language...

Metric Space

a pair (M, ρ) (set) (metric)

A two variables functions $\rho(\cdot,\cdot) : M \times M \to \mathbb{R}$ is called a metric if it satisfies:

1. $\forall x, y \in M,\ \rho(x, y) \geq 0$ and $\rho(x, y) = 0$ if and only if $x = y$.
2. $\forall x, y \in M,\ \rho(x, y) = \rho(y, x)$.
3. $\forall x, y, z \in M,\ \rho(x, z) \leq \rho(x, y) + \rho(y, z)$. Nice.

Examples: Examine yourself they are metrics!

- $M = \mathbb{R}^n$, the usual metric is given by

$$\rho((x_1, x_2, \dots, x_n), (y_1, y_2, \dots, y_n)) = \sqrt{\sum_{i=k}^{n} (x_i - y_i)^2}$$

and this is so-called Euclidean distance.

- $M = \mathbb{N}, \rho(x, y) = \#\{a : a \in [min\{x, y\}, max\{x, y\}]\ \}$
- $M = \mathbb{R}, \rho(x, y) = 1$ if $x \neq y$; $\rho(x, y) = 0$ if $x = y$

#OH. Prove Triangle Inequality for Euclidean distance

$$\sqrt{(x_1-z_1)^2+\cdots+(x_n-z_n)^2}$$
$$+\sqrt{(y_1-z_1)^2+\cdots+(y_n-z_n)^2}$$
$$\geq \sqrt{(x_1-y_1)^2+\cdots+(x_n-y_n)^2}$$

$\alpha = (x_1, x_2 \cdots x_n)$

$\gamma = (z_1, z_2 \cdots z_n)$ vector.

$\beta = (y_1, y_2 \cdots y_n)$

3 variables $\Downarrow$ 2 variables.

$$|\underbrace{\alpha-\gamma}_{A}| + |\underbrace{\beta-\gamma}_{B}| \geq |\underbrace{\alpha-\beta}_{A-B}|$$

$$|A| + |B| \geq |A-B|.$$

$$|A|^2 + |B|^2 + 2|A||B| \geq (A-B)(A-B)$$
$$= |A|^2 + |B|^2 - 2\vec{A}\cdot\vec{B}.$$

$$\Leftrightarrow 2|A||B| \geq -2\vec{A}\cdot\vec{B}.$$

Generalization of Convergence

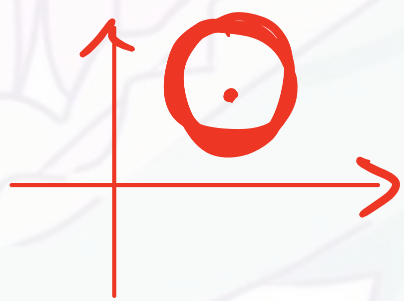
Then, by replacing the usual matric $\rho(x, y) = |x - y|$ and choosing our universal set M, we get the natural definition for generalize convergence in metric space (M, ρ) for a sequence $(a_n) : \mathbb{N} \to M$, which is given by:

$$\lim_{n\to\infty} a_n = a \quad :\Leftrightarrow \quad \forall_{\varepsilon>0} \; \exists_{N\in\mathbb{N}} \; \forall_{n>\mathbb{N}} \; a_n \in B_\varepsilon(a)$$

where

$|x-a| < \varepsilon.$

$$B_\varepsilon(a) = \{x \in M : \rho(x, a) < \varepsilon\}, \quad \varepsilon > 0, \quad a \in M.$$



(x_1, x_2) (y_1, y_2)

Cauchy Sequences

Meaning for metric space

$\longrightarrow$ labor saving

A Sequence (a_n) in a metric space (M, ρ) is called a **Cauchy Sequence** if

$$\underset{\varepsilon>0}{\forall}\ \underset{N\in\mathbb{N}}{\exists}\ \underset{m,n>N}{\forall}\ \rho(a_m, a_n) < \varepsilon$$

An intuitive description of a Cauchy sequence is that the elements are getting closer together. $|a_n - 0|$

Some properties: How is bounded defined now?

1. Every Cauchy sequence is bounded. $\forall n, \rho(a_n, x) < +\infty$.
2. Every convergent sequence is a Cauchy sequence.
3. But not every Cauchy sequence converges.
4. If all Cauchy sequences in a metric space converges, then the space is called **complete**.

Complete and Incomplete Metric Spaces

set metric

(M, ρ)

$\mathbb{Z}$ $\mathbb{Q}$, $\mathbb{R}$

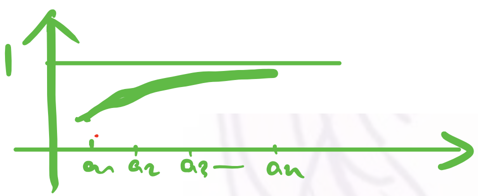
A metric space is complete when all cauchy sequence in this metric space converges.

Prove yourself that it's a metric. How ?

2.2.46. Example. Consider the metric $\rho \colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by

$a_n = n.$

$\frac{x}{1+x} = \frac{1}{1+\frac{1}{x}} \to 1$



$$\rho(x, y) = \left| \frac{x}{1+|x|} - \frac{y}{1+|y|} \right|.$$

How to prove.?

The metric space $(\mathbb{R}, \rho)$ is incomplete. (Know that the metric space can be incomplete) 1.4, 1.41, 1.414 ---- $\boxed{\sqrt{2}}$

The metric space $(\mathbb{Q}, |\cdot|)$ is incomplete. (Should remember)

The metric space $(\mathbb{R}, |\cdot|)$ is complete. (Should remember)

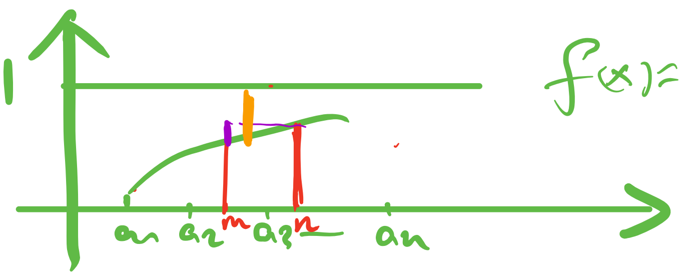
$\forall \varepsilon>0,\ \exists N\in\mathbb{N},\ \forall m,n>N,\ \rho(m,n)<\varepsilon.$

$\left|\frac{m}{1+m}-\frac{n}{1+n}\right|<\varepsilon.$ ✓

$\frac{x}{1+x}=\frac{1}{1+\frac{1}{x}}\uparrow \to 1$

$a_n=n.$

$f(x)=\frac{x}{1+x}$



$\left|\frac{m}{1+m}-\frac{n}{1+n}\right|<\left|\frac{m}{1+m}-1\right|$

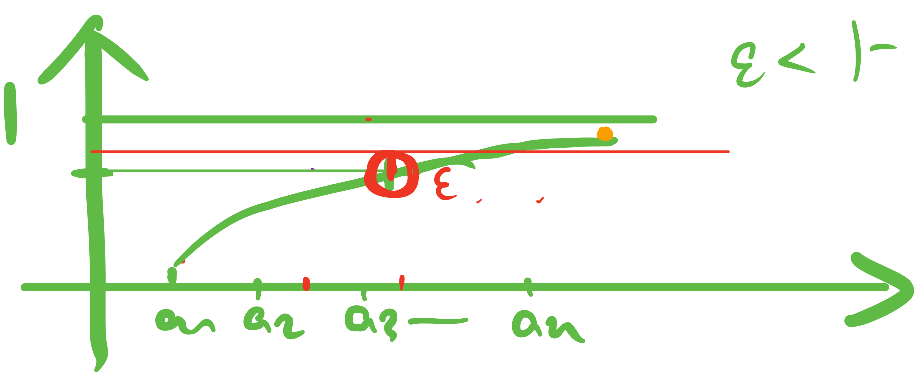
$=\frac{1}{1+m}<\varepsilon$

N JJ

$a_n\to a.$ $\frac{a}{1+a}$

$\forall \varepsilon>0,\ \exists N\in\mathbb{N},\ \forall n>N,\ \rho(n,a)<\varepsilon.$

$\varepsilon<1-\frac{a}{1+a}$



For $\rho(x,y)=|x-y|$, $(\mathbb{C}, \rho)$ is complete

Recall : A complex sequence converges $\Longleftrightarrow$ Real part image part.

In the lecture we have discussed that $(\mathbb{R}, \rho)$ is complete. For every Cauchy sequence (z_n) in $\mathbb{C}$, we can write it into 2 real sequences (x_n) and (y_n) by writing $z_n = x_n + i \cdot y_n$. Since

$$(x_m - x_n)^2 \le (x_m - x_n)^2 + (y_m - y_n)^2 = |z_m - z_n|^2 < \varepsilon$$

$$\Rightarrow |x_m - x_n| < \varepsilon$$

and similar for (y_n), both (x_n) and (y_n) are Cauchy and thus convergent. Since a complex sequence converges if the real and imaginary parts converge, (z_n) converges and $\mathbb{C}$ is complete.

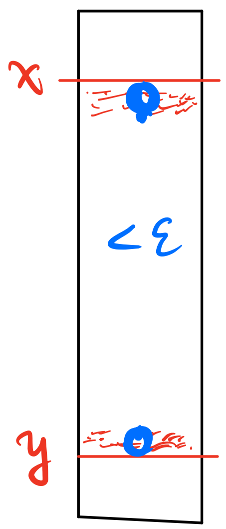
Exercise Important! (A Former Midterm Question)

(1 min)

Prove that every Cauchy sequence has at most one accumulation point.

Tips:

- You should work on an abstract metric space, using ρ instead of $|\cdot|$.
- Visualize to help you think !



Suppose there are two different Accumulation Points. x, y.

Cauchy. So

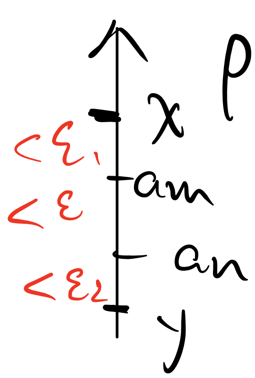
$$\forall \varepsilon > 0,\ \exists N \in \mathbb{N},\ \forall m, n > N,\ \rho(a_m, a_n) < \varepsilon.$$

Because x, y are A P.

So $\forall \varepsilon_1 > 0,\ \forall N_1 \in \mathbb{N},\ \exists m,\ \rho(a_m, x) < \varepsilon_1$

So $\forall \varepsilon_2 > 0,\ \forall N_2 \in \mathbb{N},\ \exists n,\ \rho(a_n, y) < \varepsilon_2$

$$\begin{cases} N_1 > N \\ N_2 > N \end{cases} \quad \rho(a_m, a_n) < \varepsilon.$$



$$\rho(x, y) \le \rho(x, a_m) + \rho(a_m, a_n) + \rho(a_n, y)$$

$$< \varepsilon_1 + \varepsilon_2 + \varepsilon.$$

Let $\varepsilon_1, \varepsilon_2, \varepsilon < \frac{1}{3}\rho(x, y)$.

Contradiction!

Get familiar with Cauchy !

(1 min) $(\mathbb{R}, |\cdot|)$ complete.

Given a sequence (a_n), And define the sequence (b_n) as:

$b_n = |a_2 - a_1| + |a_3 - a_2| + \cdots + |a_{n+1} - a_n|$

$$b_n = \sum_{i=1}^{n} |a_{n+1} - a_n|$$

$b_{n+1} \geq b_n$.

Prove that if (b_n) is bounded, (a_n) converges.

Tips: (b_n) is convergent $\Rightarrow$ is cauchy

- Cauchy is really useful for proving convergence !
- Think about how you can get the difference between two distant a_i and a_j.

Aim: $|a_m - a_n| < \varepsilon$.

$\forall \varepsilon > 0, \exists N, \forall m > n > N, |b_m - b_n| < \varepsilon$.

$\Rightarrow |a_{m+1} - a_m| + |a_m - a_{m-1}| + \cdots + |a_{n+2} - a_{n+1}|$.

$m > n, |a_m - a_n| = |(a_m - a_{m-1}) + (a_{m-1} - a_{m-2}) + \cdots + (a_{n+1} - a_n)| \leq |a_m - a_{m-1}| + \cdots + |a_{n+1} - a_n|$

Get familiar with Cauchy !

Metric Space $(\mathbb{R}, |\cdot|)$.

Let (a_n) be a real sequence that satisfies:
$|a_{n+2} - a_{n+1}| < c\,|a_{n+1} - a_n|, 0 < c < 1$
Does the sequence converge? Try to prove it.

$\forall \varepsilon > 0, \exists N, \forall m > n > N, |a_m - a_n| < \varepsilon.$

$|a_m - a_n| = |(a_m - a_{m-1}) + (a_{m-1} - a_{m-2}) + \cdots + (a_{n+1} - a_n)|$

$\leq |a_m - a_{m-1}| + |a_{m-1} - a_{m-2}| + \cdots + |a_{n+1} - a_n|.$

Tips:

- Cauchy is really useful for proving convergence !
- Think about how you can get the difference between two distant a_i and a_j.

$\leq |a_{n+1} - a_n| \cdot (1 + c + c^2 + \cdots\cdots)$ $n \uparrow\uparrow$.

$\leq \frac{1}{1-c} |a_{n+1} - a_n| \leq \frac{1}{1-c} \cdot c^{n-1} |a_2 - a_1| < \varepsilon$

Reference

Warning:

Student's proof

$$|a_{n+2} - a_{n+1}| \le c^n |a_2 - a_1|$$

$\lim_{n\to\infty} |a_{n+1} - a_n| = 0$. So (a_n) converges ?

X

- VV186 Lecture Slides Horst Hohberger
- 2021 Vv186 TA-Niyinchen
- 2022 Vv186 TA-Dingzizhao
- 2022 Vv186 TA-Sunmeng
- 2022 Vv186 TA-Matianyi

$a_n = \ln n$

or $a_{n+1} = a_n + \frac{1}{n}$. $\Rightarrow$ $|a_{n+1} - a_n| \to 0$

But diverge