1. (A) $U \cap W \neq \phi$.

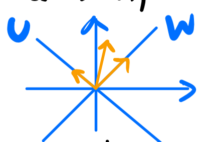
Yes. a unique neutral element

(B) $V \backslash U$ is a subspace of V.

No. $V \backslash U$ don't have neutral element.

(C) $U \cup W$. is a subspace of V.

No.



(D) $U \cap W$ is a subspace of V. Simply check that neutral element 0 is in $U \cap W$, $U \cap W \neq \phi$.

Yes. $\forall\, v_1, v_2 \in U \cap W$ $\begin{cases} v_1+v_2 \in U \\ v_1+v_2 \in W \end{cases} \Rightarrow v_1+v_2 \in U \cap W$. $\forall\, v \in U \cap W$. $\begin{cases} kv \in U \\ kv \in W \end{cases} \Rightarrow kv \in U \cap W$.

2. ACD.

3.3.8. Definition. Let V be a real (complex) vector space. Then a map $\|\cdot\| : V \to \mathbb{R}$ is said to be a ***norm*** if for all $u, v \in V$ and all $\lambda \in \mathbb{R}$ ($\mathbb{C}$),

1. $\|v\| \geq 0$ for all $v \in V$ and $\|v\| = 0$ if and only if $v = 0$,
2. $\|\lambda \cdot v\| = |\lambda| \cdot \|v\|$,
3. $\|u + v\| \leq \|u\| + \|v\|$.

A: $\|f\| = \|f\|_\infty$ (slide 347. Example 6).

B: $\|f\| = 0 \Longleftrightarrow f = 0$. Remember to check this property.

However, $f(x) = c \neq 0$, $f'(x) = 0$ and $\sup\limits_{x \in (a,b)} |f'(x)| = 0$.

C. D. add $|f|$ and $|f'|$, can ensure $\|f\| = 0 \Longleftrightarrow f = 0$.

Check the three properties by yourself.

For triangle inequality, can first easily prove $\sup\limits_{x\in(a,b)} |f(x)+g(x)| \leq \sup\limits_{x\in(a,b)} |f(x)| + \sup\limits_{x\in(a,b)} |g(x)|$. (analogous for others)

3. $f(x) = \sum\limits_{n=0}^{\infty} a_n x^n$ converges at least for $x \in [0,1]$.

Abel Theory.

① if power series $f(x) = \sum\limits_{n=0}^{\infty} a_n x^n$ converges at $x = x_0$.

Then $f(x)$ is absolutely convergent for $|x|<|x_0|$

proof: $\exists M>0, \forall n, |a_n x_0^n| \le M$ (prove it by Cauchy).

$$\sum_{n=0}^{\infty}|a_n x^n| = \sum_{n=0}^{\infty}|a_n x_0^n|\cdot\left|\frac{x}{x_0}\right|^n$$
$$\le M\cdot\sum_{n=0}^{\infty}\left|\frac{x}{x_0}\right|^n \text{ converge.}$$

② if power series $f(x)=\sum_{n=0}^{\infty}a_n x^n$ diverge at $x=x_0$

Then $f(x)$ diverge for $\forall |x|>|x_0|$

proof by contradiction. Otherwise, by ①. $x=x_0$ converge

So A ✓, C ✓.

4. $$f(x)=\begin{cases} x^4\sin\frac{1}{x} & x\neq 0. \text{ it is differentiable!} \\ 0 & x=0. \end{cases}$$

1° $f'(x)$:

① for $x\neq 0$. $f'(x)=4x^3\sin\frac{1}{x}+x^4\cos\frac{1}{x}\cdot\left(-\frac{1}{x^2}\right)$
$$=4x^3\sin\frac{1}{x}-x^2\cos\frac{1}{x}$$

② when $x=0$. $f'(x)=\lim_{h\to 0}\frac{f(h)-f(0)}{h}=\lim_{h\to 0}\left|h^3\sin\frac{1}{h}\right| \longrightarrow 0.$
$$\le |h^3| \to 0$$

$$f'(x)=\begin{cases} 4x^3\sin\frac{1}{x}-x^2\cos\frac{1}{x} & (x\neq 0) \\ 0 & (x=0) \end{cases}$$

2° $f''(x)$:

① $x\neq 0$ $f''(x)=12x^2\sin\frac{1}{x}+4x^3\cos\frac{1}{x}\left(-\frac{1}{x^2}\right)-2x\cos\frac{1}{x}+x^2\sin\left(\frac{1}{x}\right)\left(-\frac{1}{x^2}\right)$
$$=12x^2\sin\frac{1}{x}-6x\cos\frac{1}{x}-\sin\frac{1}{x}.$$

② $x=0$ $f''(x)=\lim_{h\to 0}\frac{f'(h)-f'(0)}{h}=\lim_{h\to 0}4h^2\sin\frac{1}{h}-h\cos\frac{1}{h}=0.$

$$f''(x)=\begin{cases} 12x^2\sin\frac{1}{x}-6x\cos\frac{1}{x}-\sin\frac{1}{x} \\ 0. \end{cases}$$

5. (A)

3.2.13. Definition. Let $\Omega \subset \mathbb{R}$ be any set and $I \subset \Omega$ an interval. A function $f\colon \Omega \to \mathbb{R}$ is called ***strictly convex*** on I if for all $a, x, b \in I$ with $a < x < b$,

$$\frac{f(x)-f(a)}{x-a} < \frac{f(b)-f(a)}{b-a}.$$

We say that f is ***strictly concave*** on I if $-f$ is strictly convex. If we replace "$<$" by "$\leq$" above, f is simply called convex and $-f$ concave.

$$f(x) = \begin{cases} x^2 \sin^2 \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

$$f'(x) = \begin{cases} 2x \sin^2 \frac{1}{x} - 2 \sin\frac{1}{x} \cos\frac{1}{x}. & x \neq 0 \\ 0 & x = 0. \end{cases}$$

$f'(x)$ is not continuous at $x=0$.
also not differentiable

$$f''(x) = \begin{cases} \frac{2\cos^2\frac{1}{x}}{x^2} - \frac{4\sin\frac{1}{x}\cos\frac{1}{x}}{x} + 2\sin^2\frac{1}{x} - \frac{2\sin^2\frac{1}{x}}{x^2} & x \neq 0 \\ \text{Not Differentiable at } x = 0. \end{cases}$$

Update the counterexample.

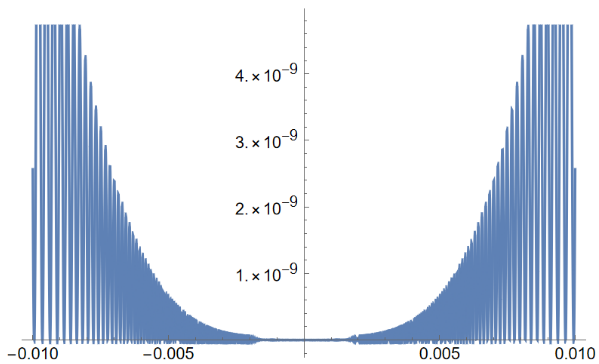
$$f(x) = \begin{cases} x^4 \sin^2 \frac{1}{x}, & x \neq 0 \\ 0 & x = 0. \end{cases}$$

$$f'(x) = \begin{cases} 4x^3 \sin^2 \frac{1}{x} - 2 \sin\frac{1}{x} \cos\frac{1}{x}\, x^2 & x \neq 0. \\ 0 & x = 0. \end{cases}$$

Then $f'(x)$ is continuous at $x=0$ and also differentiable. Refer to Exercise 4 to differentiate using limit.

$$f''(x) = \begin{cases} 2\cos\left(\frac{1}{x}\right)^2 - 12x\cos\left(\frac{1}{x}\right)\sin\left(\frac{1}{x}\right) - 2\sin\left(\frac{1}{x}\right)^2 + 12x^2\sin\left(\frac{1}{x}\right)^2 & x \neq 0. \\ 0 & x = 0 \end{cases}$$

```
Plot[x^4*(Sin[1/x]^2), {x, -0.01, 0.01}]
```
绘图 正弦



why is it always oscillating?

let $x = \frac{1}{k\pi}$ $(k \in \mathbb{Z})$. then get $f(x) = 0$.

let $x = \frac{1}{k\pi + \frac{\pi}{2}}$ $(k \in \mathbb{Z})$ then get $f(x) = x^4$.

The Function is always oscillating.

There is not a convex part.

Convex

(B) $f''(0) > 0$. No. It can be zero.

(C) Yes. Totally Right.

Local extrema Property! local minimum + $f''(x)$ exists $\Rightarrow f''(x) \geq 0$.

3.2.11. Theorem. Let f be a real function and $x \in \operatorname{dom} f$ such that $f'(x) = 0$. If $f''(x) > 0$, then f has a local minimum at x; if $f''(x) < 0$, then f has a local maximum at x.

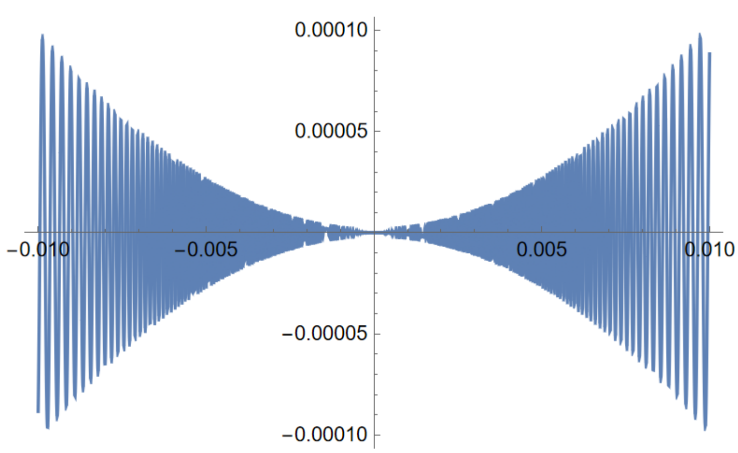
3.2.12. Theorem. Let f be a real function and $x \in \operatorname{dom} f$ such that $f''(x)$ exists. If f has a local minimum at x, then $f''(x) \geq 0$; if f has a local maximum at x, then $f''(x) \leq 0$.

D. No. $f'(x)$ may not be increasing at any neighborhood.

3.2.10. Corollary. Let f be a real function and $I \subset \operatorname{dom} f$. Assume that $f' > 0$ on I. Then f is strictly increasing on I. If $f' < 0$ on I, f is strictly decreasing on I.

We only know $f''(0) > 0$. it's not enough to decide a property of an interval.

```
Plot[-2 x^2 Cos[1/x] Sin[1/x] + 4 x^3 Sin[1/x]^2, {x, -0.01, 0.01}]
绘图        余弦        正弦
```



Graph for $f'(x)$

Also oscilating, not simply increasing.

6. A. ✗ $U_1 - U_2 \subset U_1 \Leftrightarrow U_2 \subseteq U_1$

B $\forall u \in U_1$, $u \in U_1 + U_2$ since $0 \in U_2$.

C $(U_1 - U_2) + U_2 = U_1$?

$\alpha \in U_1$, $\beta \in U_2$, $0 \in U_2$.

$\alpha - \beta \in (U_1 - U_2) + U_2$. not necessarily in U_1.

$U_1 \subset (U_1 - U_2) + U_2$.

D $U_1 - U_2 = U_1 + U_2$

① $U_1 - U_2 \subseteq U_1 + U_2$ $\forall t \in U_1 - U_2$ $t = \alpha - \beta$. $\begin{cases} \alpha \in U_1 \\ \beta \in U_2 \end{cases}$ $-\beta \in U_2$. $t \in U_1 + U_2$

② $U_1 + U_2 \subseteq U_1 - U_2$.

7. Does the following series converge?

$$\sum_{n=0}^{\infty} \frac{(2n)!(3n)!}{n!\,(4n)!}$$

$$\frac{(2n+2)!\,(3n+3)!}{(n+1)!\,(4n+4)!} \Big/ \frac{(2n)!\,(3n)!}{n!\,(4n)!}$$

$$= \frac{\cancel{(2n+2)}\cancel{(2n+1)}(3n+1)(3n+2)\cancel{(3n+3)}\,3}{\cancel{(n+1)}\cancel{(4n+4)}(4n+3)\cancel{(4n+2)}(4n+1)\,4}$$

$$= \frac{3(3n+1)(3n+2)}{4(4n+3)(4n+1)} \xrightarrow{n\to\infty} \frac{27}{64} < 1$$

By limit ratio test, the series converges

$\ast$ it's easy for $a \le 0$, since $\left(\frac{1}{n} - \sin(\frac{1}{n})\right)^a \to \infty$

Discuss simple cases may worth some points !

For which $a \in \mathbb{R}$ does the following series converge ?

$$\sum_{n=0}^{\infty} \left(\frac{1}{n} - \sin(\tfrac{1}{n})\right)^a \qquad \left(\frac{1}{6n^3} - o(\tfrac{1}{n^3})\right)^a .$$

$$\sin(\tfrac{1}{n}) = \frac{1}{n} - \frac{1}{6n^3} + o(\tfrac{1}{n^3})$$

$$\sin x \Leftarrow f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \qquad \sum \frac{1}{6n^{3a}} \underline{(1+o(1))^a}$$

$$= x - \frac{x^3}{6} + o(x^3).$$

$1 + a \cdot o(1) + o(1)$.

$= 1 + o(1)$.

$\le 1+\varepsilon$.

($\varepsilon > 0$ is fixed).

$$\left(\frac{1}{6n^3}\right)^a \quad a \le 0. \quad \text{No!!!}$$

Squeeze

$a > 0$.

$$\sum \left(\frac{1}{n^3}\right)^a \Rightarrow \sum \left(\frac{1}{n^{3a}}\right)$$

Find some larger

$(1+\varepsilon)$

when $a > \frac{1}{3}$, converge.

Find some smaller

$(1-\varepsilon)$

$0 < a \le \frac{1}{3}$?

$$\frac{1}{6n^{3\alpha}} (1 + o(1))^a$$

$$\ge \frac{1}{6n^{3\alpha}} (1-\varepsilon) \quad (0<\varepsilon<1 \text{ is fixed})$$

$$\ge (1-\varepsilon) \sum \frac{1}{6n^{3\alpha}}$$

diverge.

So $a \in (\frac{1}{3}, +\infty)$.

8. $f(x) = \dfrac{1}{1+e^{\pi-x}\sin(x)}$

$f'(x) = \dfrac{e^{\pi-x}(\sin x - \cos x)}{(1+e^{\pi-x}\sin x)^2}$

(i)

$f'(x) = 0 \iff \sin x - \cos x = 0 \iff \sin(x-\frac{\pi}{4}) = 0.$

$x = \frac{\pi}{4} + k\pi.\quad k\in\mathbb{Z},\ x\in[0,2\pi],\quad x=\frac{\pi}{4},\ \frac{5}{4}\pi.$

(ii) $[0,\frac{\pi}{4}],\ [\frac{5}{4}\pi, 2\pi]$ decreasing. $f'>0$

$[\frac{\pi}{4}, \frac{5}{4}\pi]$ increasing. $f'<0.$

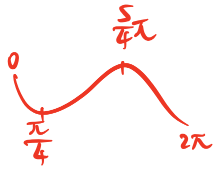
(iii) local extrema.

3.2.3. Definition. Let f be a real function and $\Omega \subset \operatorname{dom} f$. Then $x \in \Omega$ is called a ***local maximum (minimum) point for f on Ω*** if there exists some $\varepsilon > 0$ such that x is a maximum (minimum) point for f on $\Omega \cap B_\varepsilon(x)$.

or the trend.

where: $\begin{cases} ① \text{ differentiable and derivative } 0. \text{ (also check } f'') . \\ ② \text{ boundary} \end{cases}$

From (ii) a rough trend.



Local extrema:

$f=1$ at $x=0$ is a local maximum.

$f=\dfrac{1}{1+\frac{\sqrt{2}}{2}e^{\frac{3}{4}\pi}}$ at $x=\frac{\pi}{4}$ is a local minimum

$f = \dfrac{1}{1-\frac{\sqrt{2}}{2}e^{-\frac{1}{4}\pi}}$ at $x = \frac{5}{4}\pi$ is a local maximum.

$f = 1$ at $x = 2\pi$ is a local minimum.

Then get global max & min.

global maximum: $f(\frac{5}{4}\pi) = \dfrac{1}{1-\frac{\sqrt{2}}{2}e^{-\frac{1}{4}\pi}}$

global minimum: $f(\frac{\pi}{4}) = \dfrac{1}{1+\frac{\sqrt{2}}{2}e^{\frac{3}{4}\pi}}$

(iv) convexity and concavity of f:

Consider the trend of $f'(x)$ / analyze $f''(x)$.

$$f''(x) = \frac{2(e^{\pi-x})^2(\cos x-\sin x)^2+2e^{\pi-x}\cos x\,(1+e^{\pi-x}\sin x)}{(1+e^{\pi-x}\sin x)^3}$$

$1-\frac{\sin 2x}{2}$

$e^{\pi}(1-\sin x\cos x)+\cos x\, e^{x}$.

$$= \frac{2e^{\pi+x}\,\overbrace{[e^{\pi}+\cos x(e^{x}-e^{\pi}\sin x)]}^{A}}{(1+e^{\pi-x}\sin x)^3}$$

For Part A:

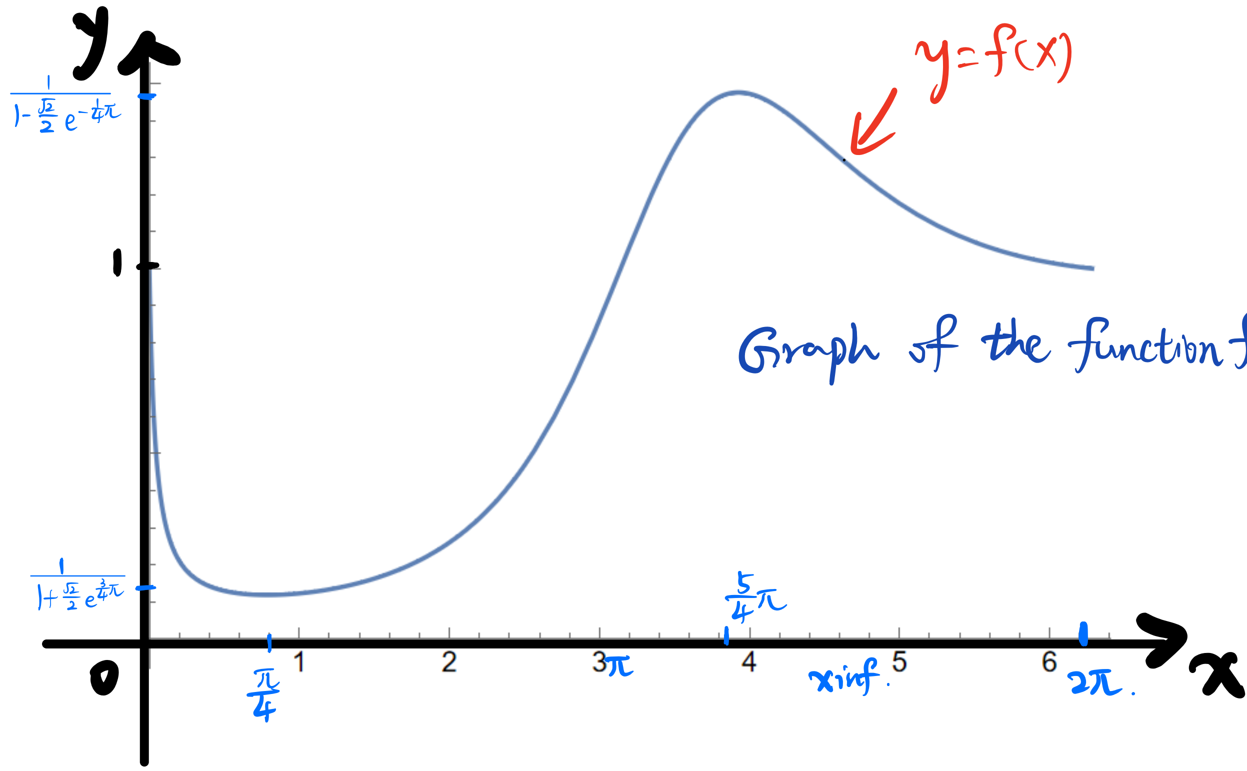
$$\begin{cases} f''(\pi) = 0. \\ f''(\frac{3}{2}\pi) = e^{\pi} > 0. \\ f''(2\pi) = e^{\pi} - e^{2\pi} < 0. \end{cases} \Rightarrow \text{So } \exists\, t \in (\tfrac{3}{2}\pi, 2\pi),\ f''(t) = 0.$$

$\Rightarrow t$ is an inflection point.

A Kind Notice: It's OK not to provide exact value if it's indeed hard to calculate.

f is convex at $[0, \pi]$, $[t, 2\pi]$

concave at $[\pi, t]$.



Graph of the function f.

π and xinf denotes the x-coordinate of the inflection point.

CHECK THE GUIDELINE!

Curve Sketching

We now have all the tools we need to sketch the graph of a given function. In orer to perform a sketch, we first need to perform a ***curve discussion***, where we explore the behavior of the curve at all points of its domain. In particular, we are interested in

1. the domain and range;
2. the continuity and behavior near points of discontinuity;
3. the behavior as $x \to \pm\infty$; in particular ***asymptotes*** (straight lines such that the values of the function approach the points on the line as $x \to \pm\infty$).;
4. local and global extrema; label asymptote.
5. intervals where the function is increasing, decreasing or constant;
6. ***inflection points***, where the second derivative changes sign;
7. other remarkable features of the curve. label inflection points.

Curve Sketching

The object of a sketch of a curve is to visualize the curve for the reader, and to convey all relevant information regarding the curve. Knowing what to include in a sketch is just as important as knowing what ***not*** to include.

First, here are some general guidelines:

- Decide what area of the coordinate system you will need (for the graph of a function, consider its domain and range). In most graphs, you should include the origin $(0, 0)$ of the coordinate system. decide range.
- Make sure your sketch is large enough; generally, it should be as wide as the page on which you are writing, and at least 8 cm high! Large!
- RULER Use a ruler to draw the axes. If the sketch includes the origin of the coordinate system, the axes should intersect there.
- Add an arrowhead pointing right to the right end of the horizontal axis (called the abscissa), and an arrowhead pointing upwards to the top end of the vertical axis (called the ordinate). Arrow!

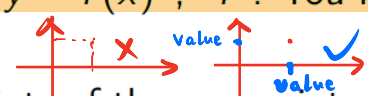
Curve Sketching

label x,y.

- Label the axes (generally, "x" for the abscissa and "y" for the ordinate, but you can use any other suitable variables), writing the labels under the horizontal arrowhead and on the left of the vertical arrowhead.
- You do not need to label the origin $(0, 0)$. It is understood that the axes intersect at $(0, 0)$. Only if in your sketch the axes do ***not*** intersect at the origin should you label the intersection.
- Sketch the curve so that all the space provided by the axes is used; for a graph of a function, the maximum should be near the top of the ordinate and the minimum should be near the bottom of the ordinate. The domain of the function should be clearly visible in the sketch. If the graph continues for $x \to \pm\infty$, then draw the curve until you are just above the ends of the abscissa - do not draw a curve further than the horizontal or vertical axes, and do not stop drawing before the ends are reached.

Label the curve

- Label the curve, e.g., "$y = f(x)$", "f". You may use little arrows to indicate the curve.



- Mark characteristic points of the curve: intersections with the axes, extrema, inflection points, other interesting points. Generally, it is sufficient to mark these points on the axes; only in rare cases should you resort to labeling a point (x, y) directly on the graph.
- If you do not have exact values for some characteristic points, indicate them through suitable symbols (e.g., $x_1, x_2, \ldots$) and explain these symbols briefly at the bottom of the graph.

It's OK to not have exact values for special points

just label and explain

9. Closed Intervals + Continuous

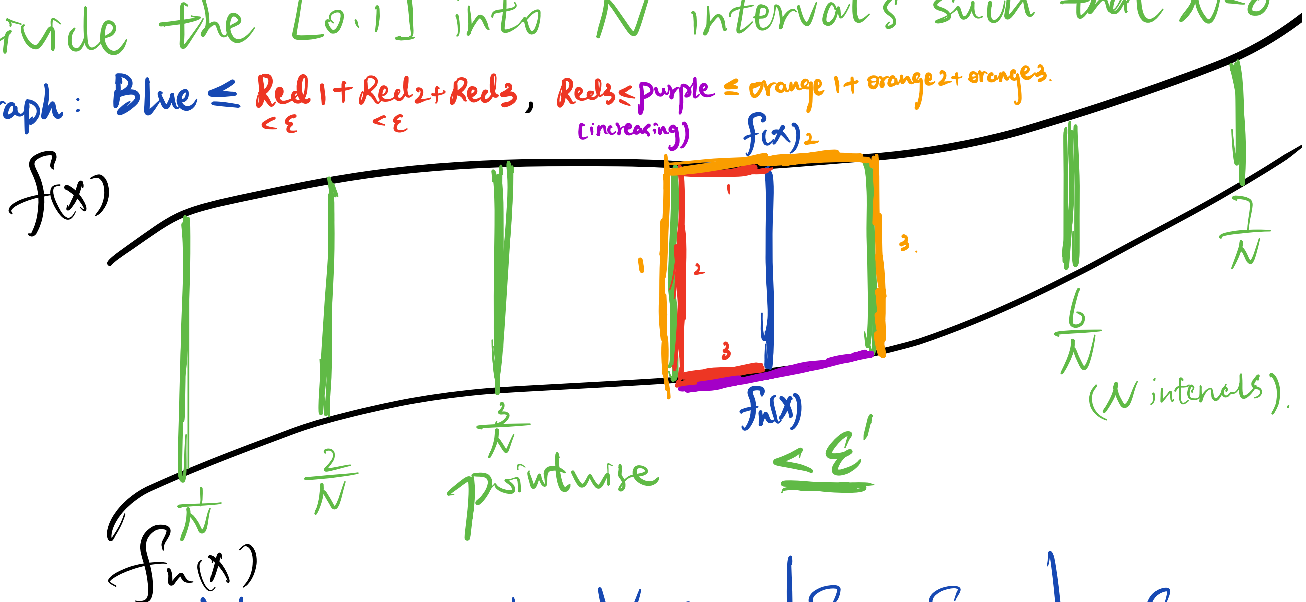
$\Rightarrow$ uniform continuous.

$\forall \varepsilon > 0,\ \exists \delta,\ \forall |x_1 - x_2| < \delta,\ |f(x_1) - f(x_2)| < \varepsilon.$

Divide the $[0,1]$ into N intervals such that $\frac{1}{N} < \delta$

Graph: Blue $\le$ $\underbrace{\text{Red}_1}_{<\varepsilon}$ + $\underbrace{\text{Red}_2}_{<\varepsilon}$ + Red$_3$, Red$_3$ $\le$ purple $\le$ orange 1 + orange 2 + orange 3.

(increasing)



Target: $\forall \varepsilon > 0.\ \exists \underline{N}\ \ \forall x.\ \ \forall n > N.\ \ |f(x) - f_n(x)| < \varepsilon.$

(Uniformly Continuous).

$\forall \varepsilon > 0,\ \exists \delta > 0,\ \forall |x_1 - x_2| < \delta,\ |f(x_1) - f(x_2)| < \varepsilon.$

Divide $[0,1]$ into N intervals with length $\frac{1}{N}$,

such that $\frac{1}{N} < \delta$.

$\Rightarrow [\xi_0, \xi_1], [\xi_1, \xi_2], \cdots [\xi_{n-1}, \xi_n].$

$$\sup |f_n(x) - f(x)| = \max_{1 \le k \le N} \left| \sup |f_n(x) - f(x)| \right| \le \underbrace{\max_{1 \le k \le N} \sup \left| f_n(x) - f_n(\xi_{k-1}) \right|}_{①}$$

$$+ \underbrace{\max_{1 \le k \le N} \sup \left| f_n(\xi_{k-1}) - f(\xi_{k-1}) \right|}_{②} + \underbrace{\max_{1 \le k \le N} \sup \left| f(\xi_{k-1}) - f(x) \right|}_{③}$$

$$① \le \max_{1\le k\le N} |f_n(\xi_k) - f_n(\xi_{k-1})| \quad \text{(increasing)}.$$

$$\le \max_{1\le k\le N} \underbrace{|f_n(\xi_k) - f(\xi_k)|}_{\le \varepsilon \text{ (pointwise)}} + \max_{1\le k\le N} \underbrace{|f(\xi_k) - f(\xi_{k-1})|}_{\le \varepsilon \text{ (continuous)}}$$

$$+ \max_{1\le k\le N} \underbrace{|f(\xi_{k-1}) - f_n(\xi_{k-1})|}_{\le \varepsilon \text{ (pointwise)}}.$$

②: $\le \varepsilon$ (pointwise)

③ $\le \varepsilon$ (continuous f). Since ①②③ are controlled.

Q.E.D.

10. Target: Since $f \ne 0$. Assume $f > 0$ for $\forall x$. without loss of generality.

Wish to prove that: $\forall \varepsilon > 0$, $\exists N > 0$, $\forall n > N$, for all $x \in [a,b]$,

$$\left| \frac{1}{f_n(x)} - \frac{1}{f(x)} \right| < \varepsilon.$$

Already have: $\forall \varepsilon' > 0$, $\exists N_1 > 0$, $\forall n > N_1$, $|f(x) - f_n(x)| < \varepsilon'$

$$\left| \frac{f(x) - f_n(x)}{f_n(x) \cdot f(x)} \right| < \left| \frac{\varepsilon'}{f_n(x) \cdot f(x)} \right| \quad ①$$

$f(x)$ is easy to control.

Closed Interval $[a,b]$ + continuous. So $f(x)$ has minimum.

$\Rightarrow f(x) \geq \min f$.

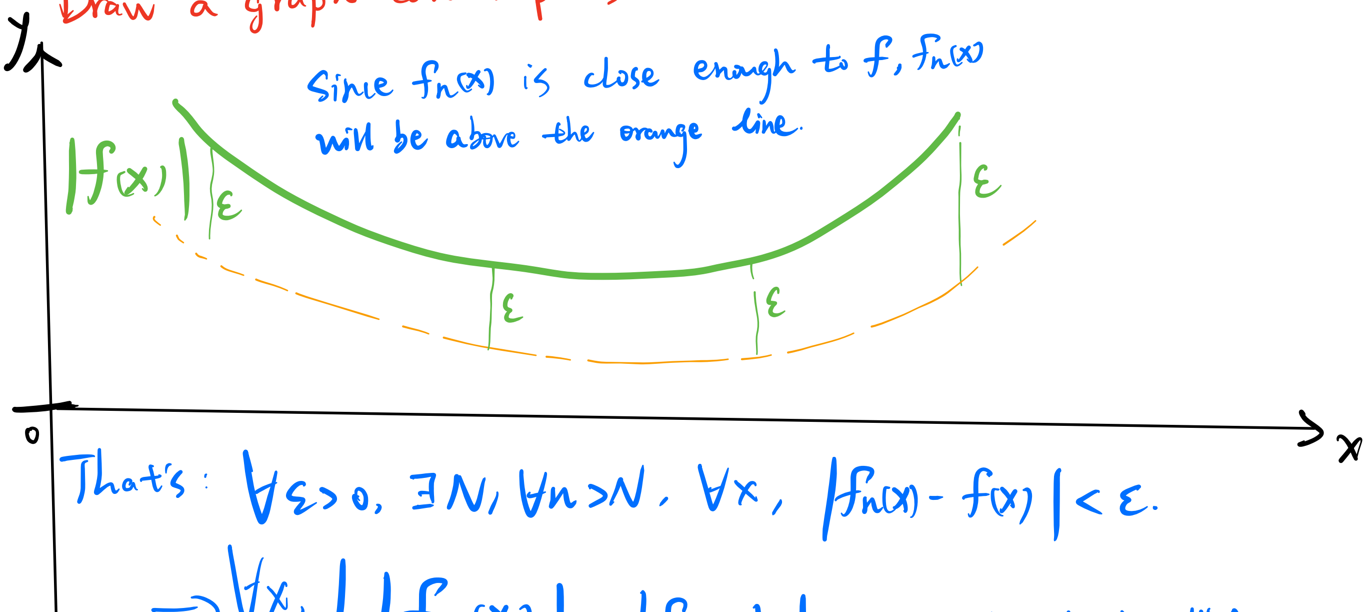
$$① \le \left| \frac{\varepsilon'}{f_n(x) \cdot \min f} \right|$$

② Then we want to control $|f_n(x)|$.

We know that $|f_n(x)|$ is getting closer and closer to $|f(x)|$

So it won't be too small, or $f_n(x)$ and f is not close enough.

Draw a graph can help us:



That's: $\forall \varepsilon > 0, \exists N, \forall n > N, \forall x, |f_n(x) - f(x)| < \varepsilon.$

$\Rightarrow \forall x, \big|\, |f_n(x)| - |f(x)| \,\big| < \varepsilon.$ (triangle inequality)

$\Rightarrow \forall x, |f_n(x)| > |f(x)| - \varepsilon \geqslant \min f - \varepsilon.$

That's $|f_n(x)| \geqslant \min f - \varepsilon$, let $\varepsilon = \frac{1}{2} \min f$.

Then $|f_n(x)| \geqslant \frac{1}{2} \min f$.

$$② \leq \left| \frac{\varepsilon'}{\frac{1}{2}\min f \cdot \min f} \right| = \frac{|2\varepsilon'|}{\min^2 f}$$

Now, simply let $\varepsilon' < \frac{1}{2} \min f$ and $② \leq \varepsilon$. Q.E.D.

11.

How to find the limit

$$\frac{[x]^n}{1+[x]^n}$$

Since function vector space is abstract, we can use pointwise convergence to help us find the limit of a function sequence:

1. Calculate the pointwise limit f of a given function sequence (f_n).
2. Find a formula or estimate of $||f_n - f||$ for any $n \in \mathbb{N}$.
3. If $||f_n - f|| \to 0$ as $n \to \infty$, then (f_n) converges uniformly to f. Otherwise the convergence is not uniform.

1. $f(x) = \begin{cases} 0 & 0 \leq |x| < 1 \\ \frac{1}{2} & |x| = 1 \\ 1 & |x| > 1 \end{cases}$

$f(x)$.

① fix x ② $n \to +\infty$.

2. Method 1.

Prove that $\lim\limits_{n\to\infty} \sup |f_n(x) - f(x)| \neq 0$.

Usually select special value to show it.

For $f_n(x)$ consider $x = 2^{\frac{1}{n}}$

$$f_n(2^{\frac{1}{n}}) = \frac{2}{1+2} = \frac{2}{3}.$$

$$\lim_{n\to\infty} \sup |f_n(x) - f(x)| \geq \lim_{n\to\infty} |f_n(2^{\frac{1}{n}}) - f(2^{\frac{1}{n}})|$$

$$= \left|\frac{2}{3} - 0\right| \neq 0. \quad \square$$

Method 2. $f_n(x)$ is continuous. if $f_n(x)$ converge uniformly to f. f should also be continuous. But f is not continuous at $x=1$. $\square$

3.4.3. Theorem. Let $[a, b] \subset \mathbb{R}$ be a closed interval. Let (f_n) be a sequence of continuous functions defined on $[a, b]$ such that $f_n(x)$ converges to some $f(x) \in \mathbb{R}$ as $n \to \infty$ for every $x \in [a, b]$. If the sequence (f_n) converges uniformly to the thereby defined function $f\colon [a, b] \to \mathbb{R}$, then f is continuous.

Just prove the cases for $|x| \geqslant q > 1$

$$\sup |f_n(x) - f(x)| = \sup \left| \frac{|x|^n}{1+|x|^n} - 1 \right| = \sup \left| \frac{1}{1+|x|^n} \right|$$

$$\text{So } \lim_{n\to\infty} \sup |f_n(x) - f(x)| = \lim_{n\to\infty} \sup \left| \frac{1}{1+|x|^n} \right| = 0.$$

The cases for $|x| \leqslant \frac{1}{q} < 1$ is quite similar.

Here is the end of Kulu's RC's Solution!

Good Luck for Mid 2 !!

Try to understand the thoughts flowing behind the solutions. They all come from our experience and step-by-step analysis. Draw graphs and visualize to help ~~thinking~~.