VV186 Honors Mathematics II

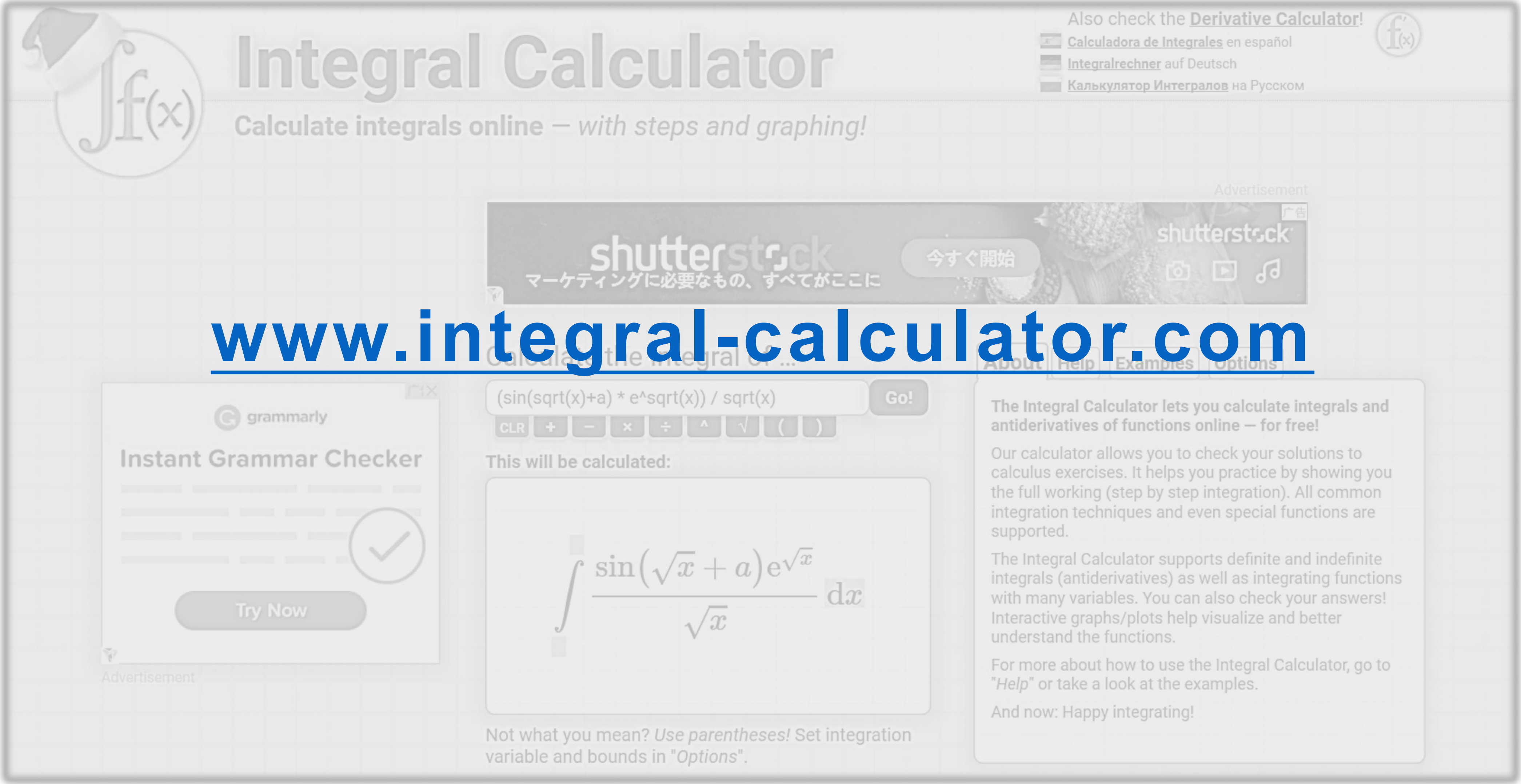
RC9 - The Last RC

KULU

Preview

1, Integration Calculation

2, Improper Integrals



www.integral-calculator.com

Tactic 1 - Integration by Inspection / Recitation

1. Write down the primitives of the following functions.

$e^{\alpha x}$, α is a constant

x^{α}, $\alpha \in \mathbb{Q} \setminus \{-1\}$ is a constant

lnx

$sinx$ & $cosx$

$\frac{1}{1+x^2}$ & $\frac{1}{\sqrt{1-x^2}}$

$\frac{x}{\sqrt{x^2-a^2}}$

DIY : $\frac{x}{\left(\sqrt{x^2+a^2}\right)^n}$

Tactic 2 - Substitution Rule

4.2.4. Substitution Rule. Let $f \in \operatorname{Reg}([\alpha, \beta])$ and $g\colon [a, b] \to [\alpha, \beta]$ continuously differentiable. Then

$$\int_a^b (f \circ g)(x) g'(x)\, dx = \int_{g(a)}^{g(b)} f(y)\, dy.$$

Tactic 2 - Substitution Rule

1, Integrate $\int \frac{1}{cosx}$

2, Show that $dx = \frac{2}{1+t^2} dt$, if we substitute $t = \tan\left(\frac{x}{2}\right)$ in the integral

3, $\int \left(\frac{1-\cos(2x)}{2}\right)^2 \sin(x)\cos(x)dx$

DIY : $\int \frac{1}{\sqrt{a^2 - x^2}}$, $a > 0$

$$\int \frac{1}{2 - \sin^2 x} dx$$

Tactic 3 - Integration by Parts

4.2.8. Theorem. Let $f, g \in C^1([a, b], \mathbb{C})$. Then

$$\int_a^b f'(x)g(x)\,dx = f(x)g(x)\Big|_a^b - \int_a^b f(x)g'(x)\,dx.$$

Tactic 3 - Integration by Parts

1, Calculate $\int e^{ax} sin(bx)$ and $\int e^{ax} cos(bx)$
2, Calculate $\int e^{ax} P_n(x)$ (e.g. $\int e^x \cdot x^4$)

Exercise 11.9

Let $a_n := \int_0^{\pi/2} \sin^n x \, dx$.

i) Show that $\{a_n\}_{n\in\mathbb{N}}$ is a convergent sequence by establishing that it is bounded below and decreasing.
(1 Mark)

ii) Prove the recursion formula

$$\int_0^{\pi/2} \sin^n x \, dx = \frac{n-1}{n} \int_0^{\pi/2} \sin^{n-2} x \, dx \qquad (*)$$

and calculate $\int_0^{\pi/2} \sin^2 x \, dx$, $\int_0^{\pi/2} \sin x \, dx$.
(2 Marks)

i) Prove (e.g., using integration by parts) that

$$\int_0^{\pi} \cos^{2m} \theta \sin^{2n} \theta \, d\theta = \frac{(2m)!}{2^{2m} m!} \frac{(2n)!}{2^{2n} n!} \frac{\pi}{(n+m)!}, \qquad n, m \in \mathbb{N}.$$

(2 Marks)

"$\int \frac{1}{ax^2+bx+c}$, $a > 0$": this case should be divided by three subcases

If $\Delta > 0$, use factorization of the denominator.

If $\Delta = 0$, use factorization and substitution rule and the primitive $\frac{1}{x}$.

If $\Delta < 0$, change the denominator into the form $a\left(x+\frac{b}{2a}\right)^2 + E$, $E > 0$, then use substitution rule and the primitive $\arctan x$

Tactic 4 - Comprehensive Methods

1, $sin\,(mx)\,cos(nx)$

2, $\int \frac{1}{x^2+4x-5}$

3, $\int \frac{x}{3x^2+6x+10}$

$$\int \frac{1}{x^4-16}, \int \frac{x}{x^2-1}$$

Integrate $\int \frac{1}{sin(x)\cos^3(x)}$ (SJTU Math textbook, P191)

Find: $\int \cos \ln x \, dx = \int x \cdot (\sin \ln x)' \, dx = x \sin \ln x - \int \sin \ln x \, dx$.

$\int \sin \ln x \, dx = -\int x \cdot (\cos \ln x)' \, dx = -x \cos \ln x + \int \cos \ln x \, dx$

$$\Rightarrow \begin{cases} \int \cos \ln x \cdot dx = \frac{x}{2}(\sin \ln x + \cos \ln x) \\ \int \sin \ln x \, dx = \frac{x}{2}(\sin \ln x - \cos \ln x) \end{cases}$$

Tactic 5 - Extra!

Multi-variable Method (VV285)

Complex Plane (VV286)

etc...

Improper Integrals - Definition

4.2.10. Definition. Assume that $b \le \infty$ and that $f\colon [a, b) \to \mathbb{C}$ is regulated on any closed subinterval $[a, x]$, $x < b$. Then

$$\int_a^b f(t)\,dt$$

is called an ***improper integral*** and is said to ***converge*** or ***exist*** if

$$\lim_{x \nearrow b} \int_a^x f(t)\,dt =: I$$

exists.

Improper Integrals - Practise

Whether $\int_1^\infty \frac{\ln x}{x^2+1}$ converge or not?

Whether $\int_1^\infty \frac{sinx}{x\sqrt[3]{x}}$ converge or not?

4.2.13. Cauchy Criterion. Let $a \in \mathbb{R}$ and $f : [a, \infty) \to \mathbb{R}$ be integrable on every interval $[a, x]$, $x \in \mathbb{R}$. The improper integral

$$\int_a^\infty f(x)\,dx$$

converges if and only if

$$\underset{\varepsilon>0}{\forall}\ \underset{R>0}{\exists}\ \underset{x,y>R}{\forall}\quad \left|\int_x^y f(t)\,dt\right| < \varepsilon.$$

4.2.16. Comparison Test. Let $I \subset \mathbb{R}$ and $f : I \to \mathbb{C}$, $g : I \to [0, \infty)$. Suppose that $|f(t)| \le g(t)$ for $t \in I$ and $\int_I g(t)\,dt$ converges. Then $\int_I f(t)\,dt$ also converges.

i) Show that $\int_1^\infty \frac{\ln x}{x^2+1}$ is integrable

ii) Show that $\int_0^1 \frac{\ln x}{x^2+1} = -\int_1^\infty \frac{\ln x}{x^2+1}$

iii) Calculate the value of the integral $\int_0^\infty \frac{\ln x}{x^2+1}$

What's left after integration.....

- Integral test for series.
- Function series.
- Taylor's theorem.
- Riemann Zeta Function
- Stirling's Formula

Important Detail 1:Taylor's Theorem

Taylor's Theorem

4.3.5. Taylor's Theorem. Let $I \subset \mathbb{R}$ an open interval and $f \in C^k(I)$. Let $x \in I$ and $y \in \mathbb{R}$ such that $x + y \in I$. Then for all $p \leq k$,

$$f(x+y) = f(x) + \frac{1}{1!}f'(x)y + \cdots + \frac{1}{(p-1)!}f^{(p-1)}(x)y^{p-1} + R_p \quad (4.3.5)$$

with the remainder term

$$R_p := \int_0^1 \frac{(1-t)^{p-1}}{(p-1)!} f^{(p)}(x+ty)y^p \, dt.$$

Very important to remember! 6 points in last years' final!

But how to use?

Just remember the formula and
Use it in approximation and inequality!

Important Detail 2:Stirling's Formula

4.3.15. Stirling's Formula. $n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$ as $n \to \infty$.

The proof is not important! Remember the conclusion!

When to use? Use it in approximation when n goes to infinity!

Important Detail 2:Stirling's Formula

Exercise 1. Comphrehensive Exercise

Prove the following:

- $\frac{e^n \cdot n!}{n^n} = \infty$ as n goes to infinity
- $\frac{n!}{n^n} = 0$ as n goes to infinity
- $\frac{n!}{(n/3)^n} = \infty$ as n goes to infinity
- $\frac{n!}{(n/2)^n} = 0$ as n goes to infinity
- Find a positive number $\alpha \in \mathbb{R}$, such that when $x > \alpha$, $\frac{n!}{(n/x)^n} = 0$

  when 0<x<α, $\frac{n!}{(n/x)^n} = \infty$

Note: remind yourself of stirling's formula whenever you see n factorial ! ! !

See you!

Thanks for your coming!
I am very glad to be your taa. Thanks for your support and companion!
Perhaps I am not going to apply for TA of any courses in the future...
But wish you the best and find your own beauty in your life!

VV186 is a start, which opens the huge door of Math for you...
Honors Mathematics: VV186 - VV285 - VV286
Discrete Mathematics: VE203
Linear Algebra: VV214 / VV417
Probability: VE401
Partial Differential Equations VV557
Want more?
Further explore in Analysis, Algebra, Geometry, Topology, Number Theory...

Reflection

Find the true beauty in your life.

Find your passion, and flourish.

Be brave, try something new in your youth.

Reference

2021 & 2022 VV186 Lecture Slides Horst Hohberger
2019 VV186 TA Zhang Leyang
2021 VV186 TA Ni Yinchen
2021 VV186 TA Huang Yue
2022 VV186 TA Sun Meng
2022 VV186 TA Ding zizhao
2022 VV186 TA Ma tianyi

Thanks and Have Fun!

This is not the end... See you in big RC!