Review VIII

Integral

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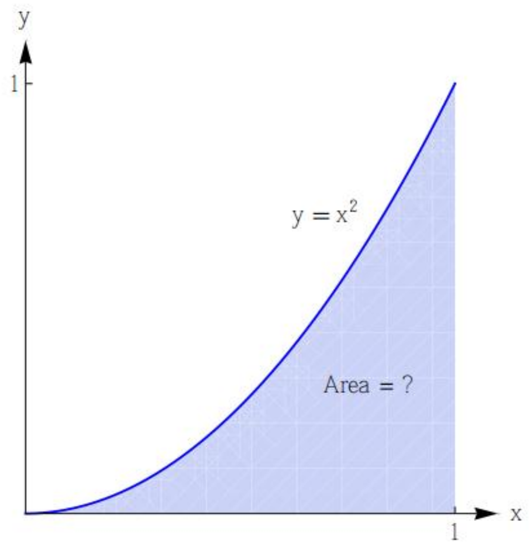
December 1, 2022

VV186 - Honors Mathmatics II


y
1
$y = x^2$
Area = ?
x
1

Our first step is to give a proper definition of a "piecewise constant" function and the area "under" its graph.

4.1.1. Definition. A ***partition*** P of an interval $[a, b] \subset \mathbb{R}$ is a tuple of numbers $P = (a_0, \ldots, a_n)$ such that

$$a = a_0 < a_1 < \ldots < a_n = b.$$

4.1.2. Definition. A function $\varphi\colon [a, b] \to \mathbb{R}$ is called a ***step function with respect to a partition $P = (a_0, \ldots, a_n)$*** if there exist numbers $y_i \in \mathbb{R}$, $i = 1, \ldots, n$, such that

$$\varphi(t) = y_i \qquad \text{whenever } a_{i-1} < t < a_i \tag{4.1.1}$$

for $i = 1, \ldots, n$. We denote the set of all step functions by $\mathrm{Step}([a, b])$

4.1.5. Definition. Let P be a partition of an interval $[a, b]$. We call a partition R of $[a, b]$ a ***sub-partition*** of P if $R \supset P$, where we naturally regard P and R as sets instead of tuples.

4.1.6. Remark. If $\varphi\colon [a, b] \to \mathbb{R}$ is a step function with respect to a partition P of $[a, b]$, then it as also a step function with respect to any sub-partition R of P.

4.1.7. Proposition. If φ, ψ are step functions on $[a, b]$ and $\lambda, \mu \in \mathbb{R}$, then $\lambda\varphi + \mu\psi$ is a step function on $[a, b]$.

4.1.8. Remark. This also shows that $\text{Step}([a, b])$ is a vector space.

4.1.9. Definition and Theorem. Let $\varphi\colon [a, b] \to \mathbb{R}$ be a step function of the form (4.1.1) with respect to some partition P. Then

$$I_P(\varphi) := (a_1 - a_0)y_1 + \cdots + (a_n - a_{n-1})y_n \tag{4.1.2}$$

is independent of the choice of the partition P. We define the ***integral*** of f as

$$\int_a^b \varphi := I_P(\varphi)$$

for any partition P with respect to which φ is a step function.

4.1.11. Definition. Let $I \subset \mathbb{R}$ be an interval. We say that a function $f\colon I \to \mathbb{R}$ is ***bounded*** if

$$\|f\|_\infty := \sup_{x \in I} |f(x)| < \infty. \tag{4.1.5}$$

4.1.13. Definition. A function $f \in L^\infty([a,b])$ is said to be ***regulated*** if for any $\varepsilon > 0$ there exists a step function φ such that

$$\sup_{x \in [a,b]} |f(x) - \varphi(x)| < \varepsilon. \tag{4.1.6}$$

4.1.15. Theorem. The continuous functions on the interval $[a,b]$ are regulated, i.e., $C([a,b]) \subset \operatorname{Reg}([a,b])$.

4.1.18. Definition and Theorem. Let $f \in \operatorname{Reg}([a,b])$ and (φ_n) a sequence in $\operatorname{Step}([a,b])$ converging uniformly to f. Then the ***regulated integral*** of f, defined by

$$\int_a^b f := \lim_{n\to\infty} \int_a^b \varphi_n \tag{4.1.8}$$

exists and does not depend on the choice of (φ_n).

4.1.16. Definition. Let $I \subset \mathbb{R}$ be an interval, $\{a_1, \dots, a_n\} \subset I$ a finite set of points and $f\colon I \to \mathbb{C}$ a function such that

(i) $f\colon I \setminus \{a_1, \dots, a_n\} \to \mathbb{C}$ is continuous and

(ii) For all a_i, $i = 1, \dots, n$, both

$$\lim_{x \nearrow a_i} f(x) \qquad \text{and} \qquad \lim_{x \searrow a_i} f(x)$$

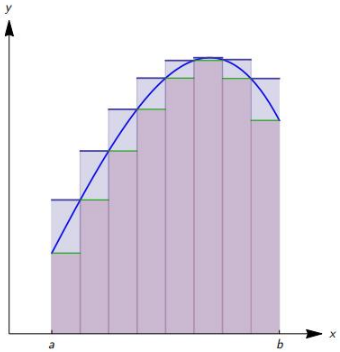
exist.

Then f is said to be ***piecewise continuous***. We denote the set of piecewise continuous functions on I by $\mathrm{PC}(I)$.

4.1.17. Theorem. The piecewise continuous functions on the interval $[a, b]$ are regulated, i.e., $\mathrm{PC}([a, b]) \subset \mathrm{Reg}([a, b])$.

In contrast, we will now try to approximate the area under f, ***without attempting to approximate f at all***.

We will do this by considering step functions that are larger than f (i.e., $\varphi(x) \geq f(x)$ for all x) and step functions that are smaller than f ($\psi(x) \leq f(x)$ for all x), as illustrated at right.



4.1.22. Definition. Let $[a, b] \subset \mathbb{R}$ be a closed interval and f a bounded real function on $[a, b]$. Let $\mathscr{U}_f$ denote the set of all step functions u on $[a, b]$ such that $u \geq f$ and $\mathscr{L}_f$ the set of all step functions v on $[a, b]$ such that $v \leq f$. The function f is then said to be ***Darboux-integrable*** if

$$\underline{I}(f) = \sup_{v \in \mathscr{L}_f} \int_a^b v = \inf_{u \in \mathscr{U}_f} \int_a^b u = \overline{I}(f).$$

In this case, the ***Darboux integral of f***, $\int_a^b f$, is defined to be this common value.

4.1.26. Theorem. If $f \in \text{Reg}([a,b])$, then f is Darboux-integrable and the Darboux integral coincides with the regulated integral.

A ***tagged partition*** (P, Ξ) on an interval $[a,b]$ consists of a partition $P = \{x_0, \dots, x_n\}$ together with numbers $\Xi = \{\xi_1, \dots, \xi_n\}$ such that each $\xi_k \in [x_{k-1}, x_k]$. The ***mesh size*** of P is defined as

$$m(P) := \max_{k=1,\dots,n} (x_k - x_{k-1})$$

A step function $\varphi \in \text{Step}([a,b])$ for a function $f \in L^\infty([a,b])$ with respect to (P, Ξ) is given by

$$\varphi(x) = f(\xi_k) \qquad \text{for } x_{k-1} < x < x_k, \qquad k = 1, \dots, n,$$

where as usual $\varphi(x_k)$ can be defined in an arbitrary manner. The sum

$$\int_a^b \varphi := \sum_{k=1}^{n} f(\xi_k)(x_k - x_{k-1}) \tag{4.1.12}$$

is then called a ***Riemann sum*** for f.

4.1.28. Definition. Let $[a, b] \subset \mathbb{R}$ be a closed interval and f a bounded real function on $[a, b]$. Then f is Riemann-integrable with integral

$$\int_a^b f \in \mathbb{R}$$

if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for any tagged partition (P, Ξ) on $[a, b]$ with mesh size $m(P) < \delta$

$$\left| \sum_{k=1}^{n} f(\xi_k)(x_k - x_{k-1}) - \int_a^b f \right| < \varepsilon.$$

4.1.29. Theorem. A bounded real function is Riemann-integrable on $[a, b]$ if and only if it is Darboux-integrable. Moreover, the values of the integrals coincide.

Integration Method

Method0: Symmetry

Suppose $f(x)$ is an odd integrable function, then

$$\int_{-a}^{a} f(x)dx = 0$$

Exercises: For $a > 0$, calculate

$$\int_{-a}^{a} \frac{\cos(x)}{1 + even(x)^{odd(x)}} dx$$

where $even(x)$ is a continuous strictly positive even function, and $odd(x)$ is an odd function

Integration Method

Method1: Recite!

Common indefinite integrals include:

- $\int x^\alpha dx = \frac{x^{\alpha+1}}{\alpha+1} + C$
- $\int e^x dx = e^x + C$
- $\int \frac{1}{x} dx = \ln(|x|) + C$
- $\int \cos x \, dx = \sin x + C$
- $\int \sin x \, dx = -\cos x + C$
- $\int \frac{1}{1+x^2} dx = arctan(x) + C$
- $\int \ln(x) = ?$

For more complex integrals, we need other theorems to help us evaluate them.

Exercise

Calculte the following integrals:

- $$\int \frac{2}{\sqrt{x^3}} dx$$
- $$\int \frac{1}{x^2 + 6x + 5} dx$$

Comment. Partial fraction is sometimes powerful!

Integration Method

Method2: Substitution!

- Let $u = g(x)$.
- Compute $du = g'(x)$.
- Substitute $g(x) = u$ and $g'(x) = du$. **At this moment, only u, no x !!!**
- Calculate the above integral of u, it should be easier.
- Replace u by $g(x)$ to get the result with x.
- For definite integral, pay attention to the range.

Demo

$$\int sin(x)cos(x)dx$$

Exercise

Calculte the following integrals:

- $$\int_{-1}^{3} \sqrt{9 - x^2} dx$$
- $$\int \tan(x) dx$$
- $$\int \frac{x}{3x^2 + 6x + 10} dx$$
- $$\int \frac{e^{4x}}{1 + e^{2x}} dx$$

Integral Method

Method3: Integration by part!

For definite integral:

$$\int_a^b f'(x)g(x)dx = f(x)g(x)\Big|_a^b - \int_a^b f(x)g'(x)dx$$

For indefinite integral:

$$\int f'(x)g(x)dx = f(x)g(x) - \int f(x)g'(x)dx$$

Demo:

$$\int x\sin(x)dx$$

Exercise

- $$\int x^2 e^{-x} dx$$
- $$\int (\ln x)^2 dx$$
- $$\int_0^{\frac{\pi}{2}} \sin^n x \, dx$$

Reference

- 2021-Vv186 TA-Niyinchen

End

Thanks!