RC 4(Slides 255 - 307)

Differentiation

Kulu

University of Michigan-Shanghai Jiao Tong University Joint Institute

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VV186 - Honors Mathmatics II

Differentiation – An Introduction

In order to investigate a function's derivative, we should first take a close look of **Linear map**.

Definition : A linear map on $\mathbb{R}$ is a function given by :

$$L : \mathbb{R} \to \mathbb{R}, \qquad L(x) = \alpha x, \alpha \in \mathbb{R}$$

Clearly, such a function has lots of good properties, which made our discussion becomes easier.

In this perspective, we would like to approximate any functions which we are interested in by a linear map. And if such linear map exists, we say this function is differentiable.

Differentiation – An Introduction

Translating into mathematical language...

Definition : Let $\Omega \subseteq \mathbb{R}$ be a set and $x \in \text{int}\Omega$. Moreover, Let $f : \Omega \to \mathbb{R}$ be a real function. Then we say f is **differentiable** if there exists a linear map L_x such that for all sufficiently small $h \in \mathbb{R}$,

$$f(x+h) = f(x) + L_x(h) + o(h) \quad as\ h \to 0$$

This linear map is **unique**, if it exists.

We call L_x "the derivative of f at x". If f is differentiable at all points of some open set $U \subseteq \Omega$, we say f is differentiable on U.

Derivative

Common misunderstandings:

$$L_x \text{ is a number for a fixed } x \in \Omega, \text{ because } L_x = \alpha.$$

L_x is **not a number**, but a **linear map**, or one can say "linear function", so it essentially is a function. $L_x \cdot h = \alpha \cdot h$ (for some α) doesn't mean $L_x = \alpha$.

To see this, one can consider a function given by

$$f(x) = 2x$$

,which doesn't mean $f = 2$.

Linear Map

- A more general case.

Derivative

Common misunderstandings:

$$\text{For } f(x) = x^4,\ f'(x) = 4x^3, \text{ so } L_x \text{ may not be linear}$$

You are confusing "derivative at a point" with "function that gives derivative". At certain point x, $4x^3$ is just a number in $\mathbb{R}$. Using our notation for L_x(or $f'(x)$), we can express L_x as

$$L_x(\cdot) = 4x^3(\cdot)$$

, the variable of L_x is not x, so L_x is **linear** for its input $(\cdot)$

Given a differentiable function $f\colon \Omega \to \mathbb{R}$, the function that gives a derivative can be denoted by $L : (\Omega \to \mathbb{R}) \to (\Omega \to \mathbb{R}),\ L(\cdot)(x) = L_x(\cdot)$. **It is a function that maps function to function.**

Derivative

Common misunderstandings:

The derivative of f at x is a line passing through $(x, f(x))$

Although it is usually a good idea to sketch something to help you to understand some mathematical concepts, but you always need to aware of the essential reason why such a graph make sense.

The derivative of f at x is a function, not a graph. We simply use the graph to illustrate our function sometimes, in this case($\mathbb{R}$), it will be a straight line, but in other case, it can be more complicated.

Rules of Differentiation

We now assume both f and g are differentiable functions, then:

- $(f+g)'(x) = f'(x) + g'(x)$
- $(f \cdot g)'(x) = f'(x)g(x) + f(x)g'(x)$
- $(f \circ g)'(x) = f'(g(x))g'(x)$
- $(\frac{f}{g})'(x) = \frac{f'(x)g(x) - f(x)g'(x))}{g^2(x)}$

Exercise

1. Practical calculation is really important! Please calculate the derivatives of the following functions.

- $(2x + 5x^2)^6$
- $\frac{\sqrt{x}}{x+1}$
- $\sqrt[3]{\frac{3x^2+1}{x^2+1}}$

2. More general Cases! Please calculate following functions' derivative. (Suppose g' always exists and doesn't vanish)

i. $f(x) = g(x \cdot g(a))$
ii. $f(x) = g(x + g(x)) + \frac{1}{g(x)}$
iii. $f(x) = g(x)(x - a)$

Inverse Function Theorem

Let I be an open interval and let $f : I \to \mathbb{R}$ be differentiable and strictly monotonic. Then the inverse map $f^{-1} : f(I) \to I$ exists and is differentiable at all points $y \in f(I)$ for which $f'(f^{-1}(y)) \neq 0$.

$$(f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))}$$

Demo

- Calculate $(\arctan x)'$
- Calculate $(\arcsin x)'$
- Calculate $(\arccos x)'$

L'Hopital's Rule

$$\lim_{x \searrow b} \frac{f(x)}{g(x)} = \lim_{x \searrow b} \frac{f'(x)}{g'(x)}, \text{ if } \lim_{x \searrow b} \frac{f(x)}{g(x)} = \frac{0}{0} \text{ or } \frac{\infty}{\infty} \text{ and } \lim_{x \searrow b} \frac{f'(x)}{g'(x)} \text{ exists.}$$

What is wrong?

$$\lim_{x \to 1} \frac{x^3 - x - 2}{x^2 - 3x + 2} = \lim_{x \to 1} \frac{3x^2 - 1}{2x - 3} = \lim_{x \to 1} \frac{6x}{2} = 3$$

Put a gun on your head: do write down the word "L'Hopital" !

L'Hopital's Rule Exercise

Calculate: $\lim\limits_{x \to 0} (\sin x)^x$

Application of Differentiation

We list some useful Results and Theorems.

1. If a real function is differentiable at x, then it is continuous at x.
2. Hierarchy of local smoothness.
   1. Arbitrary function
   2. Function continuous at x
   3. Function differentiable at x
   4. Function continuously differentiable at x
   5. Function twice differentiable at x
   6. ...

Application of Differentiation

Result and Theorems.

3. Let f be a function and $(a, b) \subseteq \text{dom } f$ and open interval. If $x \in (a, b)$ is a maximum(or minimum) point of $f \subseteq (a, b)$ and if f is differentiable at x, then $f'(x) = 0$.

4. Let f be a function and $[a, b] \subseteq \text{dom } f$. Assume that f is differentiable on (a, b) and $f(a) = f(b)$. Then there is a number $x \in (a, b)$ such that $f'(x) = 0$.

   Comment. We need the requirement that f is **differentiable everywhere** on (a, b). Otherwise, a counterexample can be:

$$[a, b] = [0, 2], \quad \begin{cases} f(x) = x & x \in [0, 1] \\ f(x) = 2 - x & x \in (1, 2] \end{cases}$$

Application of Differentiation

Result and Theorems.

5. Let $[a, b] \subseteq \operatorname{dom} f$ be a function that is continuous on $[a, b]$ and differentiable on (a, b). Then there exists a number $x \in (a, b)$ such that $f'(x) = \frac{f(b)-f(a)}{b-a}$.
6. Let f be a real function and $x \in \operatorname{dom} f$ such that $f'(x) = 0$. If $f''(x) > 0$, then f has a local minimum at x, if $f''(x) < 0$, then f has a local maximum at x.

Comment

The case in which $f''(x) = 0$ is more complicated, different conditions may occur.

Example 1: $f'(x) = x^2$. Example 2: $f'(x) = x^3$.

As you can see from example 2, f may not even have a local extremum if $f''(x) = 0$.

Application of Differentiation

Result and Theorems.

7. Let f be a twice differentiable function on an open set $\Omega \subseteq \mathbb{R}$. If f has a local minimum at some point $a \in \Omega$, then $f''(a) \geq 0$.

Proof :

Suppose f has a local minimum at a. If $f''(a) < 0$, then f would also have a local maximum at a. Thus, f would be constant in some interval containing a. So $f''(a) = 0$. But this contradicts to our assumption.

Comment. An analogous statement is : If f has a local maximum at some point $a \in \Omega$, then $f''(a) \leq 0$.

Exercise

3. This exercise aims to show that differentiation can also be used to prove sequential results. Recall the inequality (see also review 2)

$$|a+b|^n \leq 2^{n-1}(|a|^n + |b|^n)$$

Now try to use differentiable function to prove it.

Exercise

$$\text{Prove that}: \ \arctan x \text{ is uniformly continuous in } (-\infty, \infty)$$

Exercise

Prove that if

$$\frac{a_0}{1} + \frac{a_1}{2} + \cdots + \frac{a_n}{n+1} = 0$$

Then $a_0 + a_1 x + \cdots + a_n x^n = 0$ for some $x \in [0, 1]$

Exercise

Suppose that f satisfies $f'' + f'g - f = 0$ for some function g. Prove that if f is 0 at two distinct points, then f is 0 on the interval between them.

Exercise

Suppose $f\colon [0, n]$, $n \in \mathbb{N}$ is a continuous function, and is differentiable on $(0, n)$. Furthermore, assume that

$$f(0) + f(1) + \cdots + f(n-1) = n, \;\; f(n) = 1$$

Show that there must exist $c \in (0, n)$ such that $f'(c) = 0$.

Exercise

In this exercise, we would like to give a deeper investigation of Lipschitz condition. If a real function $T : \Omega \to \mathbb{R}$ satisfies

$$|T(x) - T(y)| \leq k \cdot |x - y|^\alpha$$

for any $x, y \in \Omega$, we say T satisfies "Lipschitz condition of order α".

1. Show that if $\alpha > 0$, then T is continuous.
2. Show that if $\alpha > 1$, then T is a constant function, i.e.,

$$\exists_{C \in \mathbb{R}} \; T(x) = C$$

Reference

- Exercises from 2021-Vv186 TA-Niyinchen.

End

Thanks