Review VI(Slides 333 - 365)

Vector Space & Sequence of Functions

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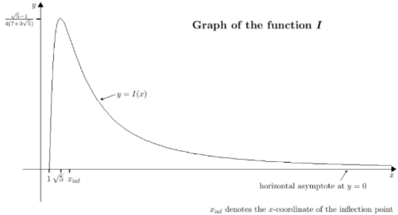
VV186 - Honors Mathmatics II

Curve sketching

- 99% possibility to appear in your midterm 2.
- Follow the guidelines provided by Horst. (The rubric will be generally the same as the guidelines.)[2]



Two advice:

1. Do not forget to mark the *asymptote line*.
2. Do not add any **redundant** marks.

Vector Space

By introducing vector space, one can treat a specific group of functions which all have some shared properties to form such a set, then we can find out some convenient operation that will make us easier to deal with them.

We then make a specific definition to which **set** can be called a **Vector Space**.

Vector Space

We have eight axioms of vector space V (in $\mathbb{C}$ or $\mathbb{R}$)

$$+ : V \times V \to V$$

i $(u + v) + w = u + (v + w)$

ii $u + v = v + u$

iii $\exists e \in V$ such that $v + e = e + v = v$

iv $\underset{v \in V}{\forall} \underset{(-v) \in V}{\exists}$ such that $v + (-v) = (-v) + v = e$

$$\cdot : \mathbb{F} \times V \to V$$

i $1 \cdot u = u \cdot 1 = u$

ii $\lambda \cdot (u + v) = \lambda \cdot u + \lambda \cdot v$

iii $(\lambda + \mu) \cdot u = \lambda \cdot u + \mu \cdot u$

iv $(\lambda\mu) \cdot u = \lambda(\mu \cdot u)$

Vector Space

Common Misunderstanding

A real vector space is a subset of $\mathbb{R}^n$; a complex vector space is a subset of $\mathbb{C}^n$.

When we say a vector space is real, or complex, we just refer to the scalar multiplication – the scalar is real, or complex. We don't set extra limitation on the element of the vector space.

3. The set $\mathbb{C}^n$ is a real vector space if we define addition as in (3.3.1) and scalar multiplication

$$\lambda z := (\lambda z_1, \ldots, \lambda z_n), \qquad \lambda \in \mathbb{R},\ z \in \mathbb{C}^n.$$

Subspace

Let $(V, +, \cdot)$ be a real (complex) vector space and $U \subset V$. If $u_1 + u_2 \in U$ for $u_1, u_2 \in U$, and $\lambda u \in U$ for all $u \in U$ and $\lambda \in \mathbb{F}$, then $(U, +, \cdot)$ is a subspace of $(V, +, \cdot)$.

- This lemma actually states that, when the maps "+" and "·" makes sense in the subset U, then U will inherit the eight axioms of V.
- for a subspace, we don't need to check the 8 axioms as they are inherited from the original vector space. Only check the addition and product is closed is enough.

Recap: Metric

The definition of metric is as follows.

- $\forall x, y \in M,\ \rho(x, y) \geq 0$ and $\rho(x, y) = 0$ if and only if $x = y$.
- $\forall x, y \in M,\ \rho(x, y) = \rho(y, x)$.
- $\forall x, y, z \in M,\ \rho(x, z) \leq \rho(x, y) + \rho(y, z)$

Since the metric measure the distance between two points(elements) in the set, now we want to directly measure the length of one single element.

We modified our length function as follows.

- Still positive, and equal to zero if and only if this vector is 0 (e).
- Triangle inequality still holds(with some modification).
- The symmetric property makes no sense in this case, so...

"Metric" in Vector Space

The symmetric property makes no sense in this case, so...

Replace it by another important property in vector space. As you might notice, we care about scalar multiplication, so the new property is

$$\text{The length of } \alpha u?$$

Equal to α times the length of u ,where u is a vector and α is a scalar.

Just as metric, we would like to give this function a name since it is so important, we will call it a **norm**. We then give the explicit definition of a norm.

Norm

Let V be a real(complex) vector space. Then a map

$$||\cdot|| : V \to \mathbb{R}$$

is called a norm if for all $u, v \in V$ and all $\lambda \in \mathbb{F}$, we have the following:

- $||\cdot|| \geq 0$ for all $v \in V$ and $||v|| = 0$ if and only if $v = 0$
- $||\lambda \cdot v|| = |\lambda| \cdot ||v||$
- $||u + v|| \leq ||u|| + ||v||$

Comment. Obviously, any normed can be considered as a metric of that space. However, not all the distance function generating metrics can be considered as a norm. A counterexample is:$\rho(x, y) = 0$ if $x = y$ and $\rho(x, y) = 1$ if $x \neq y$. The reason is when defining metric we don't assume the second property.

Norm

Examples

- $V = \mathbb{R}^n, ||(a_1, a_2, \dots, a_n)|| = \sum_{k=1}^{n} |a_k|$
- $V = C[a, b], ||f|| = \max_{x \in [a,b]} |f(x)|$
- $f : U \to V, ||f|| = \sup_{x \in U \setminus \{0\}} \frac{||f(x)||_2}{||x||_1}$, where $|| \cdot ||_1$ is a norm defined on U and $|| \cdot ||_2$ is a norm defined on V.
- $V = C[a, b], ||u|| = \sup\{|u(z)| \cdot p(z) : z \in [a, b]\}$, where p is a real-valued function on $[a, b]$ and $0 < \alpha \leq p(z) \leq \beta$ for some $\alpha, \beta > 0$.

Comment. The third example is often called "operator norm"; while the last example is often called "weighted norm", a modification of which is useful in complex analysis.

More Examples in the Slides

3.3.9. Examples.

1. $\mathbb{R}^n$ with $\|x\|_2 = \left(\sum_{j=1}^{n} x_i^2\right)^{1/2}$,
2. $\mathbb{R}^n$ with $\|x\|_p = \left(\sum_{j=1}^{n} |x_i|^p\right)^{1/p}$ for any $p \in \mathbb{N} \setminus \{0\}$,
3. $\mathbb{R}^n$ with $\|x\|_\infty = \max_{1 \le k \le n} |x_k|$,
4. ℓ^∞ with $\|(a_n)\|_\infty = \sup_{n \in \mathbb{N}} |a_n|$,
5. c_0 with $\|(a_n)\|_\infty = \sup_{n \in \mathbb{N}} |a_n|$,
6. $C([a, b])$, $[a, b] \subset \mathbb{R}$, with $\|f\|_\infty = \sup_{x \in [a,b]} |f(x)|$.

Exercise

1. Prove the (reverse) triangle inequality for a norm

$$||\cdot|| : V \to \mathbb{R}.$$

That is, to prove

$$|\ ||x|| - ||y||\ | \leq ||x \pm y||, \text{ where } x, y \in V$$

Exercise

2*. Prove that a weighted norm is a norm on $C([a,b])$

Exercise

3. Check whether the following sentences are true or false:
   - Given a vector space V, and its two non-empty subspaces V_1, V_2, then $V_1 \cup V_2$ is a subspace of V.
   - Given a vector space V, and its two subspaces V_1, V_2, then $V_1 \cap V_2$ is a subspace of V.
   - The set of all linear maps on $\mathbb{R}$ is a subspace of $C(\mathbb{R})$.
   - Given a vector space $\mathbb{R}^n$, for any two distince norms $||\cdot||_1, ||\cdot||_2$ of $\mathbb{R}^n$, $||\cdot|| := \sqrt{||\cdot||_1 \cdot ||\cdot||_2}$ is also a norm of $\mathbb{R}^n$.
   - Given a vector space V, given two norms $||\cdot||_1 : V \to \mathbb{R}, ||\cdot||_2 : \mathbb{R} \to \mathbb{R}$, then the $||\cdot|| := ||\cdot||_2 \circ ||\cdot||_1$ is a norm of V.

Convergence in Vector Space

Now we have our length function in the vector space, namely a norm. Then we can talk about the convergence and continuity in vector space. We start with convergence.

Let $(V, ||\cdot||)$ be a normed vector space. A sequence in V is a map $(a_n) : \mathbb{N} \to V$. We say that (a_n) converges to $a \in V$ if

$$\underset{\varepsilon>0}{\forall} \ \underset{N\in\mathbb{N}}{\exists} \ \underset{n>N}{\forall} \ ||a_n - a|| < \varepsilon$$

Continuity in Vector Space

Now let us introduce an interesting theorem.

Let $(V, ||\cdot||)$ be a normed vector space. The norm

$$||\cdot|| : V \to \mathbb{R}$$

is a continuous function on V.

Proof. By (reverse) triangle inequality,

$$|\ ||x|| - ||y||\ | \leq ||x - y||.$$

Fix arbitrary $\varepsilon > 0$, choose $\delta = \frac{1}{2}\varepsilon$ and we are done.

Exercise

5. Given a vector space $(V, ||\cdot||)$.
Let $a \in V$ be fixed; let $\lambda \neq 0 \in \mathbb{F}(\mathbb{C} \text{ or } \mathbb{R})$ be fixed. Prove the following. The scalar multiplication function $g : V \to V, g(x) = \lambda x$ is a continuous function and has a continuous inverse function.

Inner Product

Let $\mathbb{F}$ denote $\mathbb{R}$ or $\mathbb{C}$. An inner product in a real or complex vector space V is a map $(x, y) : V \times V \to \mathbb{F}$, such that the following holds:

- The inner product is linear in the first variable, i.e., for all $x, y, z \in V$ and all $\alpha, \beta \in \mathbb{F}$, $(\alpha x + \beta y, z) = \alpha(x, z) + \beta(y, z)$
- For all $x, y \in V$, $(x, y) = \overline{(y, x)}$ if V is complex, and $(x, y) = (y, x)$ if V is real.
- The inner product is positive definite, i.e., $(x, x) \geq 0$ for all x and $(x, x) = 0$ if and only if $x = 0$

If a vector space V is endowed with an inner product, we call it real (or complex) inner product space. If V is a real inner product space, then $(x, \alpha y + \beta z) = (\alpha y + \beta z, x) = \alpha(y, x) + \beta(z, x) = \alpha(x, y) + \beta(x, z)$

Examples

- $\mathbb{R}^n$ forms a real vector space with the inner product

$$(\cdot\ ,\ \cdot) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R},\ (x, y) = \sum_{i=1}^{n} x_i y_i$$

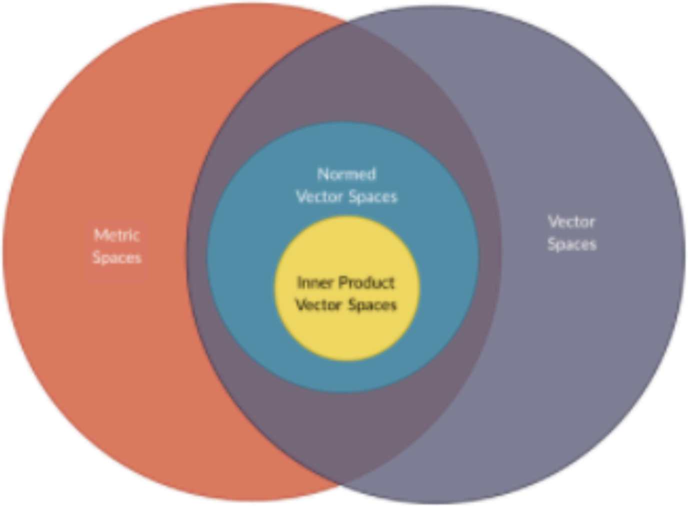
- $\mathbb{R}^2$ forms a real vector space with the inner product

$$(\cdot\ ,\ \cdot) : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R},\ (x, y) = 2x_1y_1 + x_1y_2 + x_2y_1 + 2x_2y_2$$

- $\mathbb{C}$ forms a complex vector space with the inner product

$$(\cdot\ ,\ \cdot) : \mathbb{C} \times \mathbb{C} \to \mathbb{C},\ (x, y) = x\bar{y}$$

Diagram


Metric Spaces
Normed Vector Spaces
Inner Product Vector Spaces
Vector Spaces

Convergence of Function Sequences

Let (f_n) be a sequence of real functions on $\Omega \subset \mathbb{R}$, then

1. Pointwise convergence. For every $x \in \Omega$,

$$f_n(x) \xrightarrow{(n\to\infty)} f(x) \quad :\Leftrightarrow \quad |f_n(x) - f(x)| \to 0$$

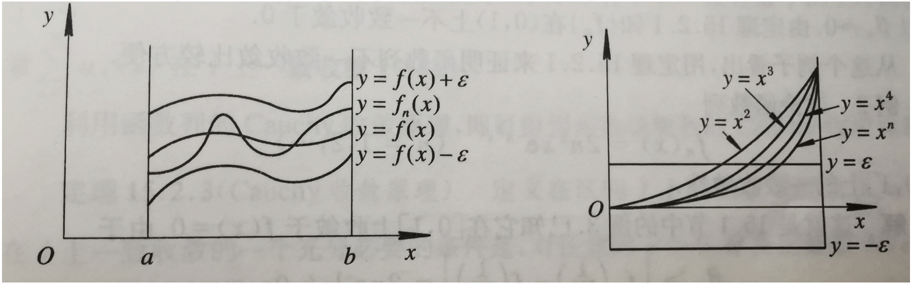
2. Uniform convergence.

$$f_n \xrightarrow{(n\to\infty)} f \quad :\Leftrightarrow \quad \sup_{x\in\Omega} |f_n(x) - f(x)| \to 0$$

Comment. For uniform convergence, we deal with the functions f_n as a whole, instead of each $f_n(x)$; for pointwise convergence, we deal with function values. Uniform convergence automatically implies pointwise convergence

An example to illustrate Uniform Convergence & Pointwise Convergence

whether the N(ϵ) is related to x ? Or only related to ϵ ?



Example

3.4.2. Example. The sequence (f_n),

$$f_n \colon [0,1] \to \mathbb{R}, \qquad f_n(x) = \begin{cases} 1 - nx, & 0 \le x \le 1/n, \\ 0, & \text{otherwise}, \end{cases}$$

converges to

$$f \colon [0,1] \to \mathbb{R}, \qquad f(x) = \begin{cases} 1, & x = 0, \\ 0, & \text{otherwise}, \end{cases}$$

pointwise, but not uniformly, as we now show.

How to find the limit

Since function vector space is abstract, we can use pointwise convergence to help us find the limit of a function sequence:

1. Calculate the pointwise limit f of a given function sequence (f_n).
2. Find a formula or estimate of $||f_n - f||$ for any $n \in \mathbb{N}$.
3. If $||f_n - f|| \to 0$ as $n \to \infty$, then (f_n) converges uniformly to f. Otherwise the convergence is not uniform.

Exercise

7. Calculate the limit of (f_n), sketch their graph, and determine whether the convergence is uniform or not.

- $f_n : \mathbb{R} \to \mathbb{R}, f_n(x) = \frac{(|x|^n)}{(1+|x|^n)}, n \in \mathbb{N}$
- $f_n : \mathbb{R} \to \mathbb{R}, f_n(x) = \sqrt{x^2 + \frac{1}{n^2}}, n \in \mathbb{N}$
- $f_n : \mathbb{R} \to \mathbb{R}, f_n(x) = \frac{x^2+nx}{n}, n \in \mathbb{N}^*$

Exercises

8. Let (f_n) be a sequence of functions in $C([a, b])$, and (f_n) converges to some function f uniformly. Prove that if $f \neq 0$ on $[a, b]$, then $(\frac{1}{f_n})$ converges to $\frac{1}{f}$ uniformly.

Important Theorem & Learn The Proof

Sequences of Functions

In the previous example, the sequence of continuous functions f_n converged to the discontinuous function pointwise, but not uniformly. This no accident. In fact, a uniformly convergent sequence of continuous functions will always converge to a continuous function:

3.4.3. Theorem. Let $[a, b] \subset \mathbb{R}$ be a closed interval. Let (f_n) be a sequence of continuous functions defined on $[a, b]$ such that $f_n(x)$ converges to some $f(x) \in \mathbb{R}$ as $n \to \infty$ for every $x \in [a, b]$. If the sequence (f_n) converges uniformly to the thereby defined function $f : [a, b] \to \mathbb{R}$, then f is continuous.

Sequences of Functions

Proof.
We need to show that f is continuous for all $x \in [a, b]$. We will here deal only with $x \in (a, b)$; the cases $x = a$ and $x = b$ are left to you.

Let $x \in (a, b)$. We will show that for any $\varepsilon > 0$ there exists a $\delta > 0$ such that $|h| < \delta$ implies $|f(x+h) - f(x)| < \varepsilon$ (for h so small that $x + h \in (a, b)$). Fix $\varepsilon > 0$. Then there exists some $N \in \mathbb{N}$ such that

$$\|f_n - f\|_\infty = \sup_{x \in [a,b]} |f_n(x) - f(x)| < \frac{\varepsilon}{3}.$$

for all $n > N$. Choose some such $n \in \mathbb{N}$. Since each f_n is continuous on $[a, b]$, there exists some $\delta > 0$ such that $|h| < \delta$ implies

$$|f_n(x) - f_n(x+h)| < \frac{\varepsilon}{3}.$$

Sequences of Functions

Proof (continued).
Then for $|h| < \delta$ we have

$$\begin{aligned} |f(x+h) - f(x)| &\le |f(x+h) - f_n(x+h)| + |f_n(x+h) - f_n(x)| \\ &\quad + |f_n(x) - f(x)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

□

Challenging

$9^*.(f_n)$ is a sequence of increasing functions, f_n:[0,1]$\to$[0,1] is pointwise convergent. Suppose f is continuous, show uniform convergence.

Exercises

10*.Let (f_n) be a sequence of functions such that for each $n \in \mathbb{N}$, $f_n \in C([a,b])$, $\forall x \in [a,b]$, $(f_n(x))$ is a bounded monotonic sequence.

- Please show that (f_n) converges point-wisely to some function f
- Suppose $f \in C([a,b])$, prove that (f_n) converges uniformly to f

Reference

- Many Contents from 2021-Vv186TA Niyinchen

End

Thanks!