Final Big RC

Exercises

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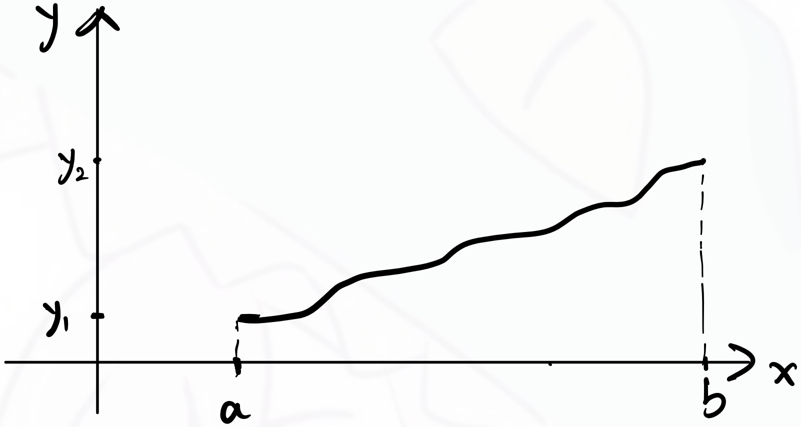
VV186 - Honors Mathmatics II

Integral Type	Basis	Value Construction	Comment
Regulated Integral	A step function sequence (ϕ_n) converging uniformly to f	Limit of $\left(\int_a^b \phi_n\right)$	Arbitrary choice of step function value, one desired step function sequence is enough
Darboux Integral	Upper & lower step functions "u"s and "v"s	Compare $\sup \int_a^b v$ and $\inf \int_a^b u$	Less freedom of choice of step function value. Can be used to treat Riemann integrals
Riemann Integral	Step functions with value taken from f	Limit of $\int_a^b \phi$	Very limited choice of step function value

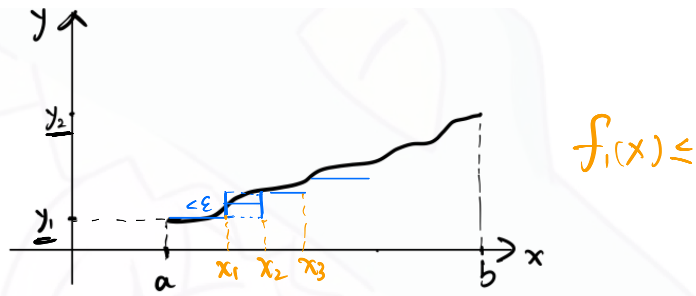
Regulated & Step Function Exam Problem (last year)

3min



Let f:[a,b]$\to \mathbb{R}$, f is monotonic on [a,b], prove that f is regulated.

3min



Let f:[a,b]→ ℝ, f is monotonic on [a,b], prove that f is regulated.

Target: $\forall \varepsilon > 0$, there exists a step function $\varphi(x): [a,b] \to \mathbb{R}$, Such that: $\sup\limits_{x \in [a,b]} |\varphi(x) - f(x)| < \varepsilon$.

By Symmetry, let f be increasing (without loss of generality).

For any ε', N sufficiently large, $\varepsilon' < \frac{y_2 - y_1}{N}$.

$$x_n =: \inf\{x \in [a,b], f(x) \geq y_1 + n\varepsilon'\}, \quad n = 0, 1, 2, \cdots N+1.$$

We obtain a partition: $P = (x_0, x_1, \cdots x_{N+1})$ for $[a,b]$.

$$\begin{cases} x_0 = a \\ x_{N+1} = b \end{cases}$$

$$\varphi_N(x) = \begin{cases} y_1 + n\varepsilon', & x_n < x < x_{n+1}. \\ f(x), & \text{otherwise}. \end{cases}$$

Then $\sup\limits_{x \in [a,b]} |f - \varphi_N(x)| \leq \varepsilon' < \varepsilon$.

∴ step function $\varphi_N(x)$ approximates $f(x)$.

Symmetry

MTY has covered it.

1. $\int_0^{+\infty} \frac{\ln x}{1+x^2} dx$ consider $\int_0^1 \frac{\ln x}{1+x^2} dx$ and $\int_1^{\infty} \frac{\ln x}{1+x^2} dx$
2. $\int_0^{\frac{\pi}{2}} \ln(\sin x) dx$
3. $\int_0^{\pi} x \ln(\sin x) dx$

consider $\sin\theta = \sin(\pi - \theta)$

$\sin\theta = \cos(\frac{\pi}{2} - \theta)$

Properties:

① $\int_0^{\frac{\pi}{2}} \ln\sin x dx = \int_0^{\frac{\pi}{2}} \ln\cos x dx$

② $\int_0^{\frac{\pi}{2}} \ln\sin x dx = \int_{\frac{\pi}{2}}^{\pi} \ln\sin x dx = \frac{1}{2}\int_0^{\pi} \ln\sin x dx$

$\Rightarrow \int_0^{\frac{\pi}{2}} \ln\sin x dx = \int_0^{\frac{\pi}{2}} \ln\sin 2x dx$

③ $\ln\sin x + \ln\cos x = \ln(\sin x \cdot \cos x) = \ln\sin 2x - \ln 2.$

$(t = \pi - x)$

$$\int_0^{\pi} x \ln\sin x dx = \int_0^{\pi} (\pi - t)\ln\sin t dt$$

$$= \pi\int_0^{\pi} \ln\sin t dt - \int_0^{\pi} t\ln\sin t dt.$$

$$\Rightarrow \int_0^{\pi} x\ln\sin x = \frac{\pi}{2}\int_0^{\pi} \ln\sin x dx$$

$(\int_0^{\pi} \ln\sin x dx = 2\int_0^{\frac{\pi}{2}} \ln\sin x dx)$ symmetry.

$$= \pi \times \int_0^{\frac{\pi}{2}} \ln\sin x dx$$

Improper Integral —— Useful Conclusions

Comparison Test.

f is defined on $[a, +\infty]$ (a>0), K>0, then

(1) if $0 \le f(x) \le \frac{K}{x^p}$ for $x \in [a, +\infty)$, and $p > 1$, then $\int_a^{+\infty} f(x)dx$ converges.

(2) if $f(x) \ge \frac{K}{x^p}$ for $x \in [a, +\infty)$, and $p \le 1$, then $\int_a^{+\infty} f(x)dx$ diverges.

$$\int_a^b \frac{1}{x^p} = \frac{1}{-p+1} \cdot x^{-p+1} \Big|_a^b = \frac{b^{1-p} - a^{1-p}}{1-p}$$

So. $\lim_{b \to \infty} \int_a^b \frac{1}{x^p}$ converge when $p > 1$

Improper Integral

$$\left|\frac{\sin x}{x^p}\right| \le \left|\frac{1}{x^p}\right| \qquad \int_1^{+\infty} \left|\frac{1}{x^p}\right| \qquad \text{convergent.}$$

1. Is $\int_1^{+\infty} \frac{\sin(x)}{x^p} dx (p > 1)$ convergent or divergent ?
2. Is $\int_0^{+\infty} [(1 - \frac{\sin x}{x})^{-\frac{1}{3}} - 1] dx$ convergent? Is it absolutely convergent?

$$\int_0^{\infty}\left[\left(1-\frac{\sin x}{x}\right)^{-\frac{1}{3}}-1\right]dx \quad (5\,\text{min})$$

Some Facts you should know:

Hint 1: $\sin x = x - \frac{1}{3!}x^3 + o(x^3)$

So $1-\frac{\sin x}{x} = -\frac{1}{3!}x^2 + o(x^2) = O(x^2)$

Hint 2: $(1+x)^{\alpha} = \sum_{n=0}^{+\infty}\binom{\alpha}{n}x^n$ (Assignment 9.2 Binomial Series) Taylor Series.

for $|x|<1$ $\quad \binom{-\frac{1}{3}}{0}=1 \quad \binom{-\frac{1}{3}}{1}=-\frac{1}{3}$

$x>1. \quad \left|\frac{\sin x}{x}\right|<1$

Dirichlet slide. absolutely convergent.

$$-1+\left(1-\frac{\sin x}{x}\right)^{-\frac{1}{3}} = \frac{1}{3}\cdot\frac{\sin x}{x} + O\left(\frac{1}{x^2}\right)$$

$\cancel{-1}+\cancel{1}+\frac{1}{3}\frac{\sin x}{x}+O$ Hint 2. Convergent.

But not absolutely convergent.

$$\int_1^{+\infty}\left[\left(1-\frac{\sin x}{x}\right)^{-1/3}-1\right]dx$$

$O(x^{-2})$

$\rightarrow$ convergent, but not absolutely convergent

$$\int_0^1\left[\left(1-\frac{\sin x}{x}\right)^{-1/3}-1\right]dx = \int_0^1\left(1-\frac{\sin x}{x}\right)^{-1/3}dx - \int_0^1 1\,dx$$

$\int_0^1 1\,dx$: 1

$\sin x = x-\frac{1}{3!}x^3+o(x^3)$

$1-\frac{\sin x}{x} = -\frac{1}{3!}x^2+o(x^2) = O(x^2)$

$\left(1-\frac{\sin x}{x}\right)^{-1/3} = o\left(\frac{1}{x^{2/3}}\right)$.

$\Rightarrow \int_0^1 o\left(\frac{1}{x^{2/3}}\right)dx.$

$x^{-\frac{2}{3}}$

$\int_0^1 \frac{1}{x^{2/3}}\,dx$

$\Rightarrow 3x^{\frac{1}{3}}\Big|_0^1$

$$\int_0^{\infty}\left[\left(1-\frac{\sin x}{x}\right)^{-1/3}-1\right]dx = \int_0^1\left[\left(1-\frac{\sin x}{x}\right)^{-1/3}dx - \int_0^1 1\,dx\right.$$
$$+\int_1^{\infty}\left[\left(1-\frac{\sin x}{x}\right)^{-1/3}-1\right]dx.$$

Taylor Series

(5 min)

Let $f(x) = \ln(1+x)$.
(1) Find the Taylor Series of f.
(2) Find its radius of convergence.
(3) Find points of which the series converge, and prove it.

Let $f(x) = \ln(1+x)$. $\quad \rho(f(x)) = \rho(f'(x))$.

(1) Find the Taylor Series of f. $\quad \rho\ \frac{1}{1+x} = 1 \quad |x|<1$

(2) Find its radius of convergence. 1

(3) Find points of which the series converge, and prove it.

4.3.5. Taylor's Theorem. Let $I \subset \mathbb{R}$ an open interval and $f \in C^k(I)$. Let $x \in I$ and $y \in \mathbb{R}$ such that $x + y \in I$. Then for all $p \leq k$,

$$f(x+y) = \underbrace{f(x) + \frac{1}{1!}f'(x)y + \cdots + \frac{1}{(p-1)!}f^{(p-1)}(x)y^{p-1}}_{T_p} + \underbrace{R_p}_{R_p} \quad (4.3.5)$$

$x \to 0$, $y \to x$; $\quad R_p \to 0$, $p \to +\infty$.

with the remainder term

$$R_p := \int_0^1 \frac{(1-t)^{p-1}}{(p-1)!} f^{(p)}(x+ty)y^p\, dt.$$

$$f(x) = f(0) + \frac{1}{1!}f'(0)x + \cdots + \frac{1}{(p-1)!}f^{(p-1)}(0)\,x^{p-1} + R_p.$$

$$\ln'(1+x) = \frac{1}{1+x} \qquad f^{(n)}(x).$$

$$f(x) = \int f'(x)\,dx \qquad \text{linear property ensures it.}$$

$$= \int \frac{1}{1+x}\,dx = \int \boxed{\sum_{n=0}^{\infty} (-1)^n}\ x^n\,dx = \sum_{n=0}^{\infty} \int (-1)^n x^n\,dx.$$

4.3.8. Remark. The Taylor series for the previous example could have been obtained in an easier way: using the geometric series expansion, we have

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty}(-x)^n$$

whenever $|x| < 1$. Since the Taylor series of a function f at x_0 is unique (the coefficients are determined by (4.3.4)) it ***doesn't matter how we obtain a series expansion*** - it is always the Taylor series.

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}.$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \cdots$$

① $\overline{\lim_{n\to\infty}} \sqrt[n]{\left|\frac{(-1)^{n-1}}{n}\right|} = 1$

$\lim_{n\to\infty} \sqrt[n]{n} = 1.$

② $\overline{\lim_{n\to\infty}} \left|\frac{\frac{(-1)^n}{n+1}}{\frac{(-1)^{n-1}}{n}}\right| = 1.$

3. $\rho = 1.$ ① $(-1, 1)$ convergent

② check boundaries. $|x| > 1$ diverge.

(i) $x = 1$ $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ (Leibniz) converge $|x| = 1$

$x = -1$ $\sum_{n=1}^{\infty} -\frac{1}{n}$ diverge

$T_n \longrightarrow f.$

$f = T_n + \underline{R_n}.$

$$R_p := \int_0^1 \frac{(1-t)^{p-1}}{(p-1)!} f^{(p)}(x+ty) y^p \, dt. \longrightarrow 0.$$

$$f^{(n)}(x) = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n}$$

① $\lim_{p\to\infty} \int_0^1 \frac{(1-t)^{p-1}}{(p-1)!} f^{(p)}(tx)x^p\,dt = 0.$ $x \in (-1,1)$

$$\int_0^1 \frac{(t-1)^{p-1}\, x^p}{(1+tx)^p} \to 0.$$

$$\le \sup_{t\in[0,1]} \left| \frac{(t-1)^{p-1} \cdot x^p}{(1+tx)^p} \right|$$

$$\sup \frac{1}{1+tx} = \begin{cases} 1 & 0 \le x \le 1 \\ \frac{1}{1+x} & -1<x<0 \end{cases}$$

$$\le |x^p| \cdot \underbrace{\sup_{t\in[0,1]} \Big(\overbrace{\frac{1-t}{1+tx}}^{\le 1}\Big)^{p-1}} \cdot \underbrace{\overbrace{\sup \left| \frac{1}{1+tx} \right|}^{\boxed{C}}}_{x\in(-1,1)}.$$

$g(t) = \frac{1-t}{1+tx}$. $g(t)$ decreasing.

$g'(t) \le 0.$

$\le |x^p| \cdot 1 \cdot C = C \cdot |x^p| \longrightarrow 0$ when $|x|<1$.

② $x = 1$

$$|R_n(1)| = \int_0^1 \frac{(1-t)^{n-1}}{(1+t)^n}\,dt.$$

$$\le \int_0^1 (1-t)^{n-1}\,dt$$

$$\Rightarrow \frac{1}{n} \longrightarrow 0.$$

Reference

- 2021-VV186 Final Big RC Exercises.

The End

This is our last RC! Thanks for all the support and companion throughout the all semester! We are very glad to be your VV186 Taa! Maybe you are sill indulged in integration and Taylor expansions, but here comes a full stop for VV186.
VV186 is the first stop to manifest the beauty and rigidity of college mathematics. You've been formally accepted to such vast universe of mathematics, roaming about those rigid logics and mind-blowing proofs. You will learn more theories and applications in the future, and here is the start.
Wish you all best grades. Believe us, no matter what the final grade is, your efforts will never tell lies. The true happiness hides itself in the comprehension and reflection along the journey.
Wish you good luck, wish you a lifetime keen on math. You are always welcome to contact us! Bye.