Exercises for Mid 1

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VV186 - Honors Mathmatics II

Exercise: Points

Please identify the interior, exterior, boundary and accumalation points of the set

$$\{\frac{1}{z} : z \in \mathbb{Z}\backslash\{0\}\} \cup (\bigcap_{j=1}^{\infty}(-2-\frac{1}{j}, -1+\frac{1}{j}))$$

Exercise: Bounds

Consider the set $U \subset \mathbb{R}$, where $U = A \cup B \cup C$ with

$$\begin{aligned} A &= \{x \in \mathbb{R} \colon 0 < x \leq 1\}, \\ B &= \{x \in \mathbb{R} \colon x = 2 - 1/n,\ n \in \mathbb{N} \setminus \{0\}\}, \\ C &= \{x \in \mathbb{R} \colon x = -1/n,\ n \in \mathbb{N} \setminus \{0\}\}. \end{aligned}$$

State (without proof) $\min U$, $\max U$, $\inf U$, $\sup U$, $\underline{\lim}\, U$ and $\overline{\lim}\, U$ (if one or more of these do not exist, simply state this).

Exercise: Limit and Accumulation Point

Easy, but check whether you should only USE WORDS !

Exercise 2.
Describe in words the difference between **an accumulation point of a sequence** and **a limit of a sequence**. How are these two concepts different? If you can, state some properties that they have in common or that serve to differentiate them from each other. Is one of them also always an example of the other? Give examples.
(5 Marks)

Solution. A limit of a sequence of numbers is a number such that for any given distance, all terms of the sequence eventually are within that distance of the limit. **(1 Mark)** An accumulation point is a number such that for any given distance and any given sequence term, there will always be some other succeeding sequence term within that distance of the accumulation point. **(1 Mark)**

After these two marks, a maximum of **(3 Marks)** can be obtained as follows:

Give **(1 Mark)**for any of the following statements:

- Any limit is also an accumulation point.
- A sequence can have at most a single limit, but may have zero or more accumulation points.
- An accumulation point is always a limit of a subsequence and vice-versa.
- If a sequence has a limit, it must be bounded, but the same statement is not if the wrd "limit" is replaced with "accumulation point".
- If a Cauchy sequence has an accumulation point, that point is also the (unique) limit of the Cauchy sequence.

Give **(1/2 Mark)** for any coherent example to illustrate any of the above statements or the definitions above.

Exercises: Find limits for a recursively defined sequence

Just briefly discuss the methods. Check details by yourself later.

Show that the sequence defined by

$$a_1 = 2, \qquad a_{n+1} = \frac{1}{3 - a_n}, \qquad (n \geq 1)$$

satisfies $0 < a_n \leq 2$ and is decreasing. Deduce that the sequence is convergent and find its limit.

Exercises (Important) Very similar Exercise in Mid 1 (Last year)

(2 min)

Step 1: First calculate the limit (in your draft)

Step 2: Mathematical Induction.

Show that the sequence defined by $x^2-3x+1=0$ $x=\frac{3-\sqrt{5}}{2}$ or $\frac{3+\sqrt{5}}{2}$.

$$a_1 = 2, \qquad a_{n+1} = \frac{1}{3-a_n}, \qquad (n \geq 1)$$

→ Even no hints, set the target to prove monotonic and bounded.

satisfies $0 < a_n \leq 2$ and is decreasing. Deduce that the sequence is convergent and find its limit.

We use Mathematical Induction to show that

$\forall n$, $a_{n+1} \leq a_n$ and $\frac{3-\sqrt{5}}{2} \leq a_{n+1} \leq 2$.

1° Base: $a_1 = 2 \in [\frac{3-\sqrt{5}}{2}, 2]$

2° $a_{n+1} - a_n = \frac{1}{3-a_n} - a_n = \frac{a_n^2-3a_n+1}{3-a_n} \leq 0$, So $a_{n+1} \leq a_n$ (a_n) is ↓

$a_{n+1} \geq \frac{3-\sqrt{5}}{2} \Leftrightarrow \frac{1}{3-a_n} \geq \frac{3-\sqrt{5}}{2} \Leftrightarrow 3-a_n \leq \frac{2}{3-\sqrt{5}} = \frac{3+\sqrt{5}}{2} = a_n \geq \frac{3-\sqrt{5}}{2}$

So (a_n) is ↓ and bounded ⇒ converge.

3° $a_{n+1} = \frac{1}{3-a_n}$. $\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{1}{3-a_n} \Rightarrow \lim_{n\to\infty} a_n = \frac{3-\sqrt{5}}{2}$. (since $a_n \leq 2$).

Exercise: Upper Limits and Lower Limits

First recall their definitions, their properties...

Consider the sequence (a_n) given by

$$a_n = \frac{1}{2} + (-1)^n \frac{2+n}{2n}.$$

Calculate $\overline{\lim}\, a_n$ and $\underline{\lim}\, a_n$.

Exercise: Upper Limits and Lower Limits

Given (x_n) a real bounded sequence, prove that:

(1) $\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n > N, x_n < \overline{lim}(x_n) + \epsilon.$

(2) $\forall \epsilon > 0, \forall k, \exists n_k > k, x_{n_k} > \overline{lim}(x_n) - \epsilon.$

Exercise: Upper Limits and Lower Limits

A valuable question asked in piazza.
For a real bounded sequence (a_n), prove that if $\operatorname{ran}(a_n)$ doesn't have a maximum, then $\sup \operatorname{ran}(a_n) = \overline{lim} a_n$.

First recall the definition of $\operatorname{ran}(a_n)$. Divide the procedure of the exercise into several steps, set up your goal!

Exercise: Upper Limits and Lower Limits

Let (a_n) and (b_n) be two bounded real sequences. Prove that:

$$\underline{lim} a_n + \underline{lim} b_n \leq \underline{lim}(a_n + b_n) \leq \overline{lim} a_n + \underline{lim} b_n \leq \overline{lim} a_n + \overline{lim} b_n.$$

Exercise: Metric Space

Just briefly discuss the methods. Check details by yourself later.
A metric space is complete when all cauchy sequence in this metric space converges.

> **2.2.46. Example.** Consider the metric $\varrho\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by
> $$\varrho(x, y) = \left| \frac{x}{1 + |x|} - \frac{y}{1 + |y|} \right|.$$

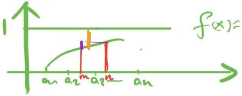
Prove that the metric space $(\mathbb{R}, \rho)$ is incomplete by analyzing the sequence: $a_n = n$.

1° $\forall \varepsilon>0, \exists N\in \mathbb{N}, \forall m,n>N, \rho(m,n)<\varepsilon.$

$\left|\frac{m}{1+m}-\frac{n}{1+n}\right|<\varepsilon.$ ✓

$\frac{x}{1+x}=\frac{1}{1+\frac{1}{x}}\uparrow \to 1$

$a_n=n.$



$f(x)=\frac{x}{1+x}$

$\left|\frac{m}{1+m}-\frac{n}{1+n}\right|<\left|\frac{m}{1+m}-1\right|.$

$=\frac{1}{1+m}<\varepsilon$

Select N such that $\frac{1}{1+N}<\varepsilon$!

2° Proof by contradiction

if $a_n\to a$.

$\forall \varepsilon>0, \exists N\in \mathbb{N}, \forall n>N, \rho(n,a)<\varepsilon.$

Select $\varepsilon<1-\frac{a}{1+a}$!

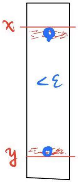
Exercise Important! (A Former Midterm Question)

TAs have discussed this exercise in their RC3.
Just briefly discuss the methods. Check details by yourself later.

Prove that every Cauchy sequence has at most one accumulation point.

Tips:

- You should work on an abstract metric space, using ρ instead of $|\cdot|$.
- Visualize to help you think !



Suppose there are two different Accumulation Points. x, y,

Cauchy. So

$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall m, n > N,$

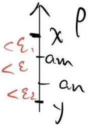
$\rho(a_m, a_n) < \varepsilon.$

Because x, y are Accumulation Points.

So $\forall \varepsilon_1 > 0, \forall N_1 \in \mathbb{N}, \exists m, \ \rho(a_m, x) < \varepsilon_1$

So $\forall \varepsilon_2 > 0, \forall N_2 \in \mathbb{N}, \exists n, \ \rho(a_n, y) < \varepsilon_2$

$\begin{cases} N_1 > N \\ N_2 > N \end{cases} \quad \rho(a_m, a_n) < \varepsilon.$



$\rho(x, y) \le \rho(x, a_m) + \rho(a_m, a_n) + \rho(a_n, y)$

$< \varepsilon_1 + \varepsilon_2 + \varepsilon.$

Let $\varepsilon_1, \varepsilon_2, \varepsilon < \frac{1}{3}\rho(x, y).$

Contradiction!

Exercise: Big O and Small o

First recall the definition...

Please prove, or disprove by giving counterexamples of the following statements:

i) $(1+O(x))^2 = 1+O(x^2)$ as $x \to 0$

ii) $o(x)^n = o(x^n), n \in \mathbb{N}^*$, as $x \to 0$

Exercise: Small o and limits

A function f(x) satisfies that: $\lim_{x \to 0} f(x) = 0$. And $f(x) - f(\frac{x}{2}) = o(x)$ $(x \to 0)$.
Prove that:

$$f(x) = o(x) \ (x \to 0)$$

Exercise: Continuous and Uniformly Continuous

A function f(x) is continuous on $[a,+\infty)$, and $\lim_{x\to\infty} f(x)$ exists. Prove that:

$$f(x) \text{ is uniformly continuous on } [a,+\infty)$$

Exercise: Continuous and Uniformly Continuous

It has been proved in lecture that: if f and g are two functions continuous at x. Then f+g and $f \cdot g$ are also continous at x.

What about uniformly continous properties ?
f(x) and g(x) are two uniformly continuous real functions defined in $[0,+\infty)$.
(1) Prove that: Function $h(x) = f(x) + g(x)$, h(x) is also uniformly continuous.
(2) Given that g(x) is bounded. Function $h(x) = f(x)g(x)$. Is h(x) also uniformly continuous ?

Exercise: Continuous and Uniformly Continuous

Let $f\colon \Omega \to \mathbb{R}$ be a real function that satisfies Lipschitz condition, that is, there is a constant $M > 0$ such that for all x and y in the domain of f, $|f(x) - f(y)| \leq M \cdot |x - y|$ (Peter A. Loeb *Real Analysis*, P92)

i) Show that f is uniformly continuous

ii) Now Let $\Omega =: [a, +\infty)$, where $a > 0$. Show that $\frac{f(x)}{x}$ is uniformly continuous

Exercise: Continuous and Uniformly Continuous

Function f(x) is defined on $(-\infty, +\infty)$. Satisfying that $\forall x, y \in \mathbb{R}$,

$$|f(x) - f(y)| \leq k\,|x - y|\,.$$

Here k is a constant satisfying that $0 < k < 1$. Prove that:
(1) $kx - f(x)$ is increasing.
(2) $\exists c \in \mathbb{R}$, f(c)=c.
(Hint: Bolzano Intermediate Value Theorem)

Reference

- Vv186 Lecture Slides Professor Horst
- Vv186 Sample Exam from Professor Horst
- 2020 Vv186 TA-Huangqiyue

Say at the end of our RC

There are also many exercises in our regular RCs. And the most important and helpful exercises are all in your sample exam. Some of them are easy and many are hard. If you can't tackle the exercises from today's RC and sample exam, that's totally normal ! Just try to understand the structure of the proof and get some thought from the solutions.

Don't waste too much time on out-of-class materials. The most valuable materials will always be Professor's slide and homework. Get familiar with them !

Hope you all relax and achieve good grade !