Review V(Slides 308-330)

Differentiation

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VV186 - Honors Mathmatics II

Exercise (Left in RC4)

Suppose $f\colon [0, n]$, $n \in \mathbb{N}$ is a continuous function, and is differentiable on $(0, n)$. Furthermore, assume that

$$f(0) + f(1) + \cdots + f(n-1) = n, \;\; f(n) = 1$$

Show that there must exist $c \in (0, n)$ such that $f'(c) = 0$.

Exercise (Left in RC4)

In this exercise, we would like to give a deeper investigation of Lipschitz condition. If a real function $T : \Omega \to \mathbb{R}$ satisfies

$$|T(x) - T(y)| \leq k \cdot |x - y|^{\alpha}$$

for any $x, y \in \Omega$, we say T satisfies "Lipschitz condition of order α".

1. Show that if $\alpha > 0$, then T is continuous.
2. Show that if $\alpha > 1$, then T is a constant function, i.e.,

$$\underset{C \in \mathbb{R}}{\exists} \; T(x) = C$$

Differentiation and Uniformly Continuity

(1) If the derivative for f(x) is bounded for $x \in (a, b)$, then f(x) is uniformly continuous on f(x). (2) Show that f(x)=sin(x) is uniformly continous. (3) Show that f(x)=arctan(x) is uniformly continuous.

Convexity and Concavity

For further analysis of functions, we would introduce the concept of **Convexity** and **Concavity**.
The definition of these two concepts are as follows.

Let $\Omega \subseteq \mathbb{R}$ be any set and $I \subseteq \Omega$ an interval. A function $f : \Omega \to \mathbb{R}$ is called convex on I if for all

$$x, a, b \in I \text{ with } a < x < b, \frac{f(x) - f(a)}{x - a} \leq \frac{f(b) - f(a)}{b - a}$$

A strictly convex function is a function that satisfies

$$\frac{f(x) - f(a)}{x - a} < \frac{f(b) - f(a)}{b - a}. \tag{1}$$

We say a function f is concave if $-f$ is convex. We say a function f is strictly concave if $-f$ is strictly convex.

Convexity and Concavity

Comment 1.

We often use "−"(minus sign) to define a new definition from an existing one. The benefit is that these two definitions can be strongly related with each other.

Comment 2.

There is a quick way to memorize it... Con**cave**...

Convexity and Concavity

Results/Theorem & Comment

1. Let $f: I \to \mathbb{R}$ be strictly convex on I and differentiable at $a, b \in I$. Then:

   i For any $h > 0 (h < 0)$ such that $a + h \in I$, the graph of f over the interval $(a, a + h)$ lies below the secant line through the points $(a, f(a))$ and $(a + h, f(a + h))$

   ii The graph of f over all I lies above the tangent line through the point $(a, f(a))$

   iii If $a < b$, then $f'(a) < f'(b)$

Draw some pictures to visualize these results!

Convexity and Concavity

Results/Theorem & Comment

2. A function $f: I \to \mathbb{R}$(I is an interval) is convex if and only if

$$\underset{t\in(0,1)}{\forall}\ \underset{x,y\in I}{\forall} \text{ with } x<y, f(tx+(1-t)y) \leq tf(x)+(1-t)f(y)$$

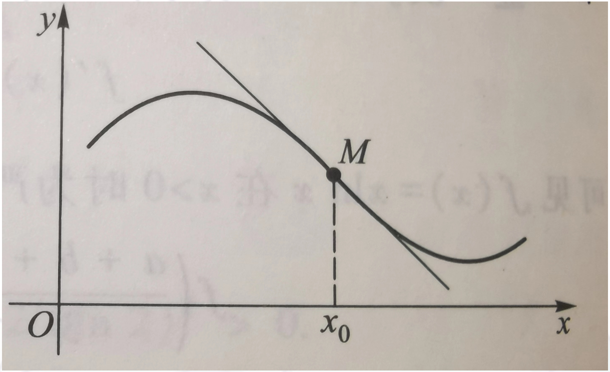
Draw some pictures to visualize these results!

3. Let I be an interval, $f: I \to \mathbb{R}$ differentiable and f' strictly increasing. If $a, b \in I$, $a<b$ and $f(a)=f(b)$, then

$$f(x) < f(a) = f(b) \text{ for all } x \in (a, b)$$

Inflection Point

Definition: inflection point is a point on a smooth plane curve at which the curvature changes sign. In particular, in the case of the graph of a function, it is a point where the function changes from being concave (concave downward) to convex (concave upward), or vice versa.



Exercise

1. This exercise will show why convexity is useful.
   i Let f be a convex function on $[a, b]$. Prove that

$$f(\sum_{i=1}^{n} \lambda_i x_i) \leq \sum_{i=1}^{n} \lambda_i f(x_i), \ \ x_i \in [a, b], \ \sum_{i=1}^{n} \lambda_i = 1, \ \ \lambda_i > 0$$

   This inequality is known as **Jensen's Inequality**(for discrete measure.)

   ii Show that

$$\prod_{i=1}^{n} a_i^{\lambda_i} \leq \sum_{i=1}^{n} \lambda_i a_i, \ \ a_i \geq 0, \ \sum_{i=1}^{n} \lambda_i = 1, \ \ \lambda_i > 0.$$

   This is the inequality you will encounter in your assignment.

Exercise

2. Let $f : [0,1] \to \mathbb{R}$ be a continuous function. Prove that if f satisfies

$$f(\frac{x_1 + x_2}{2}) \leq \frac{1}{2}(f(x_1) + f(x_2))$$

, where $x_1, x_2 \leq [0,1]$, then f is convex.

Exercise

3. Left f be a convex function, $f:[a,b] \to \mathbb{R}$. If there exists $c \in (a,b)$, $f(a)=f(b)=f(c)$, prove that f is a constant function.

Exercise

4. Let f be a continuous convex real function on $[a, b]$. Show that f either has one local minimum or infinitely many local minimums on $[a, b]$.

Exercise

5. $f(x)$ is a concave function on (a,b), and it is not constant. Prove that $f(x)$ can't attain its maximum on (a,b).

Exercise

6. (Darboux Theorem) If f is differentiable on [a,b], prove that f' can fetch all values between $f'(a)$ and $f'(b)$.

The Cauchy Mean Value Theorem

3.2.17. Theorem. Let f, g be real functions and $[a, b] \subset \operatorname{dom} f \cap \operatorname{dom} g$. If f and g are continuous on $[a, b]$ and differentiable on (a, b), then there exists an $x \in (a, b)$ such that

$$(f(b) - f(a))g'(x) = (g(b) - g(a))f'(x).$$

Proof.
We apply Rolle's Theorem to

$$h(x) = f(x)(g(b) - g(a)) - g(x)(f(b) - f(a)). \qquad \square$$

Exercise

7. Function f is continuous on [a,b], and differentiable on (a,b), prove that there exists $\xi \in (a, b)$, such that:

$$f(b) - f(a) = \xi \ln \frac{b}{a} f'(\xi)$$

Exercise

8. Prove that if f is differentiable on interval [a,b], and ab>0, then there exists a $\xi \in (a, b)$,such that:

$$\frac{af(b) - b(a)}{a - b} = f(\xi) - \xi f'(\xi)$$

9. Function f and g are differentiable on interval (a,b).

$$F(x)=f(x)g'(x)-f'(x)g(x)$$

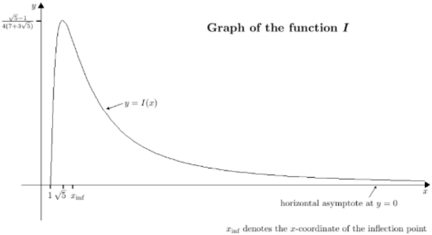
Prove that if $F(x)>0$ on (a,b), there must exist a solution for $g(x)=0$ between two different solutions for $f(x)=0$.

10. $f(x)$ is differentiable on (a,b), prove that there is a solution for $f(x)+f'(x)=0$ between two solutions for $f(x)=0$ on (a,b).

11. f(x) is continuous on [0,1] and differentiable on(a,b), and f(0)=f(1)=0. Assume there exists $t_0 \in (0,1)$, f(t_0)=α. Prove that there exists $\xi \in (0,1)$, such that $f'(\xi) = \alpha$.

Curve sketching

- 99% possibility to appear in your midterm 2.
- Follow the guidelines provided by Horst. (The rubric will be generally the same as the guidelines.)[2]



Two advice:

1. Do not forget to mark the *asymptote line*.
2. Do not add any **redundant** marks.

Reference

- Exercises from 2021VV186-Niyinchen.
- Graph from 2021VV186-Huangyue

End

Thanks!