Review II(Slides 62 - 108)

Numbers

We define everything as they now defined for some good reasons...

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VV186 - Honors Mathmatics II

Natural Numbers

The natural numbers can be constructed using the **Peano Axioms** and set theory, which will be one of the main topics in **Ve203 Discrete Mathematics**. The Peano Axioms is also the theoretical basis of **mathematical induction**. But here in vv186 we don't care much and just simply denote the set of natural numbers by $\mathbb{N}$ and define it as:

$$\mathbb{N} = \{0, 1, 2, 3, 4, \dots\}$$

We will have following **properties** we simply take it as axioms.

Porperties	Addition	Multiplication
Associativity	$a + (b + c) = (a + b) + c$	$a \cdot (b \cdot c) = (a \cdot b) \cdot c$
Existence	$a + 0 = 0 + a = a$	$a \cdot 1 = 1 \cdot a = a$
Commutativity	$a + b = b + a$	$a \cdot b = b \cdot a$
Distributivity	$a \cdot (b + c) = a \cdot b + a \cdot c$	

Mathematical Induction

Let A be a statement that has to do with **natural numbers**. We denote the statement with respect to a specific number n as $A(n)$.

There are two types of Mathematical Induction.

Mathematical Induction I

1. $A(n_0)$ is true
2. $A(n+1)$ is true whenever $A(n)$ is true for $n \geq n_0$

Mathematical Induction II

1. $A(n_0)$ is true
2. $A(k+1)$ is true whenever $A(n)$ is true for all $n_0 \leq n \leq k$

Proof of Mathematical Induction II

Suppose that a statement A holds for mathematical induction II, but there is a set $F \subseteq \mathbb{N}$ such that $\forall m \in F$, $A(m)$ doesn't hold. i.e., A does not hold for all $n \in \mathbb{N}$.

$\Rightarrow$ There is a smallest number $m_0 \in F$ such that $A(m_0)$ doesn't hold (because the set $\mathbb{N}$ is well-ordered).

However, $A(m_0 - 1)$ holds, according to our induction, $A(m_0)$ holds. This leads to a contradiction.

Thus mathematicl induction II holds.

Application of Mathematical Induction¿

A strange example shared by Horst. What's going wrong?

1.3.5. Example. Let us use mathematical induction to argue that every set of $n \geq 2$ lines in the plane, no two of which are parallel, meet in a common point.

The statement is true for $n = 2$, since two lines are not parallel if and only if they meet at some point. Since these are the only lines under considerations, this is the common meeting point of the lines.

We next assume that the statement is true for n lines, i.e., any n non-parallel lines meet in a common point. Let us now consider $n+1$ lines, which we number 1 through $n+1$. Take the set of lines 1 through n; by the induction hypothesis, they meet in a common point. The same is true of the lines $2, \ldots, n+1$. We will now show that these points must be identical.

Application of Mathematical Induction¿

A strange example shared by Horst. What's going wrong?

Assume that the points are distinct. Then all lines $2, \dots, n$ must be the same line, because any two points determine a line completely. Since we can choose our original lines in such a way that we consider distinct lines, we arrive at a contradiction. Therefore, the points must be identical, so all $n + 1$ lines meet in a common point. This completes the induction proof.

Where is the mistake in the above "proof" of our (obviously false) supposition?

How to use Induction

- Examine the complexity of the problem, because using induction is sometimes much more complicated than using a direct method to prove a statement.
- Determine the initial condition (i.e. n_0) for your induction proof.
- Decide the part of the proof that you use induction and which induction you want to use.
- Make a short test of your method on draft paper to see whether it works and is easy to write down.

! One of the solid example is Slides 67-70, the proof of Binomial Formula, please check it!

Rational Numbers

We define that the set of rational numbers is

$$\mathbb{Q} = \{\frac{p}{q} : p, q \in \mathbb{Z} \wedge q \neq 0\}$$

together with the following properties(P1-P9). You will learn this in **Ve203 Discrete Mathematics**.

Porperties	Addition	Multiplication
Associativity	$a + (b + c) = (a + b) + c$	$a \cdot (b \cdot c) = (a \cdot b) \cdot c$
NeutralElement	$a + 0 = 0 + a = a$	$a \cdot 1 = 1 \cdot a = a$
Commutativity	$a + b = b + a$	$a \cdot b = b \cdot a$
InverseElement	$(-a) + a = a + (-a) = 0$	$a \cdot a^{-1} = a^{-1} \cdot a = 1$
Distributivity	$a \cdot (b + c) = a \cdot b + a \cdot c$	

Trichotomy law

There are still three axioms we will define, and together with the nine axioms above, we build our rational numbers $\mathbb{Q}$.

We assume that we know what a strictly positive rational number is, then we know we can find such a set P with the property that :

1. $a = 0$
2. $a \in P$
3. $-a \in P$

which is so-called **trichotomy law**.

Further more, we assume that the set of positive number P is *closed under addition and multiplication.*

The Square Root Problem

Let $M = \{t \in \mathbb{R} : t > 0 \wedge t^2 > x\}$, $y = \inf M$. We want to prove that $y^2 = x$ by showing that $y^2 > x$ and $y^2 < x$ lead to contradictions.

Assume that $y^2 > x$, so $y > \frac{x}{y}$. Then

$$0 < \left(y - \frac{x}{y}\right)^2 = y^2 - 2y\frac{x}{y} + \left(\frac{x}{y}\right)^2,$$

so that

$$y^2 + \left(\frac{x}{y}\right)^2 > 2y\frac{x}{y} = 2x.$$

since $y > \frac{x}{y}$, $\Rightarrow$ $y^2 + y^2 > y^2 + \left(\frac{x}{y}\right)^2 > 2x$

Roots, Bounds & Real Numbers Slide 84

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The Square Root Problem

$y^2 > \boxed{\left(\frac{y^2 + \frac{x}{y}}{2}\right)^2} > x$ $\rightarrow$ exist another greater lower bound

Proof (continued).

Set $s := \frac{1}{2}(y + \frac{x}{y})$. Then

Complex Numbers

In Vv186, you just need to know how to perform basic complex numbers' computation and some basic properties. Here, we just list some basic computation rules and formulas.

Given $z_1 = (a_1, b_1)$ and $z_2 = (a_2, b_2)$,

- $z_1 + z_2 = (a_1, b_1) + (a_2, b_2) = (a_1 + a_2, b_1, b_2)$
- $z_1 \cdot z_2 = (a_1, b_1) \cdot (a_2, b_2) = (a_1 a_2 - b_1 b_2, a_1 b_2 - a_2 b_1)$
- $c \cdot z_1 = c(a_1, b_2) = (ca_1, cb_1), c \in \mathbb{R}$
- $\bar{z_1} = (a_1, -b_1)$
- $|z_1|^2 = a_1^2 + b_1^2 = z_1 \bar{z_1}$
- $\mathrm{Re}\ z_1 = \frac{z_1 + \bar{z_1}}{2}$
- $(\mathrm{Im}\ z_1) i = \frac{z_1 - \bar{z_1}}{2}$

Interval

Recall...

- Open interval
- Closed interval
- Half-open (or half-closed) interval

"(" and ")" = open
"[" and "]" = closed

Sets & Points

Recall the following definition and try to write down their **notation**...

- Interior point
- Exterior point
- Boundary point
- Accumalation point
- Open Set
- Closed set
- Closure

Tips:

- **Remember the definitions!**
- **Draw the pictures!**
- **Pay attention to some speacial cases!**

Boundness

How we define...

- bounded/unbounded
- max/min
- sup/inf

Quick check:

1. What's the relationship between 1. and 2.?
2. Does a set in $\mathbb{Q}$ necessarily has a max or sup in $\mathbb{Q}$?

Get familiar with this!

Boundness

Example:

1. The set $A = (-\infty, a)$ is bounded above in $\mathbb{R}$ with $\sup A = a$. It isn't in A.
2. The set $B = [b, +\infty)$ is bounded below in $\mathbb{R}$ with $\inf B = b$. It's in B since b is the minimum of B.
3. The set $C = [c, d) \cup (e, f)$ is bounded above and below in $\mathbb{R}$, so it's bounded with $\sup C = f$, $\inf C = c$.
4. The set $D = \{x \in \mathbb{Q}^+ : x = \frac{1}{n}, n \in \mathbb{N}^*\}$ is bounded above in $\mathbb{Q}^+$, but not bounded below in $\mathbb{Q}^+$.

Open Ball

Let $z_0 \in \mathbb{C}$. Then we define the **open ball** of radius $R > 0$ centered at z_0 by

$$B_R(z_0) := \{z \in \mathbb{C} : |z - z_0| < R\}$$

- Geometric interpretation?
- Higher dimensions?

Exercises

Recall Triangle Inequality:

$$||a| - |b|| \leq |a \pm b| \leq |a| + |b|$$

1. Prove that $|a - c| \leq |a - b| + |c - b|$
2. Prove that for $a_i \in \mathbb{Q}, i \in \{1, 2, \dots n\}$, and n is a natural number,

$$|\sum_{i=1}^{n} a_i| \leq \sum_{i=1}^{n} |a_i|$$

Why is $a_i \in \mathbb{Q}$?

Exercises

3. When learning the axioms of rational number, one student found that the operation of subsets of a non-empty set X is somewhat similar to that of rational number:

If we regard $\cup$ as $+$, $\cap$ as $\cdot$, then the equation

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

is just the distributivity law. Help him check whether P1 – P9 also hold for such operations.

Exercises

4. Please identify the interior, exterior, and boundary points of the set

$$\{\frac{1}{z} : z \in \mathbb{Z}\backslash\{0\}\} \cup (\bigcap_{j=1}^{\infty}(-2-\frac{1}{j}, -1+\frac{1}{j}))$$

Exercise

5. Let A be bounded set in $\mathbb{R}$ (which means that the total set is $\mathbb{R}$), for any $\epsilon > 0$, there is an element x in A such that $|x - \sup A| < \epsilon$.

Then what about $\inf$?

Exercises

6. The Fibonacci sequence is defined as follows:

$$a_1 = 1$$

$$a_2 = 1$$

$$a_n = a_{n-1} + a_{n-2};\ n > 2$$

Use Mathematical induction II to prove that

$$a_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}$$

Exercises

7*. Let $a, b \in \mathbb{R}$, prove that $|a + b|^n \leq 2^{n-1}(|a|^n + |b|^n)$, $n \in \mathbb{N}$

Reference

- Exercises from 2019–Vv186 TA-Zhang Leyang.
- Exercises from 2020–Vv186 TA-Hu Pingbang.
- Exercises from 2020–Vv186 TA-Zhang Xingjian.
- Lecture notes from Yang Shaoze.