Review 3

Exercise 1

1. When learning the axioms of rational number, student KWAELEN found that the operation of subsets of a non-empty set X is somewhat similar to that of rational number: if we regard "∪" as "+", and "∩" as " ·", then the equation $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ is just the distributivity law. Help him check whether P1 – P9 also hold for such operations

Proof: To solve such kind of problem, what we need to do is simply check the axioms one by one:

P1: (associativity) $\mathrm{A} \cup (\mathrm{B} \cup \mathrm{C}) = (\mathrm{A} \cup \mathrm{B}) \cup \mathrm{C}$ holds

P2: (existence of neutral element) $\mathrm{A} \cup \emptyset = \emptyset \cup \mathrm{A}$, so the empty set is our neutral element

P3: (inverse element) For any $\mathrm{A} \neq \emptyset, \forall \mathrm{B} \subseteq \mathrm{X},\ \mathrm{A} \cup \mathrm{B} \neq \emptyset$. Therefore, the inverse element doesn't exist. P3 fails

P4: (commutativity) $\mathrm{A} \cup \mathrm{B} = \mathrm{B} \cup \mathrm{A}$ holds

P5: (associativity) $\mathrm{A} \cap (\mathrm{B} \cap \mathrm{C}) = (\mathrm{A} \cap \mathrm{B}) \cap \mathrm{C}$ holds

P6: (existence of neutral element) $\mathrm{A} \cap \mathrm{X} = \mathrm{X} \cap \mathrm{A} = \mathrm{A}$, so the neutral element is the total set X

P7: (existence of inverse element) $\mathrm{A} \cap \mathrm{B} \subseteq \mathrm{A} \subseteq \mathrm{X}$, so $\mathrm{A} \cap \mathrm{B} \neq \mathrm{X}$ is A or B is proper set of X. This means: given an A, A^{-1} doesn't necessarily exists

P8: (commutativity) $\mathrm{A} \cap \mathrm{B} = \mathrm{B} \cap \mathrm{A}$ holds

Exercise 6

6. Please identify the interior, exterior, and boundary point of the set

$$\left\{\frac{1}{z} : z \in \mathbb{Z} \backslash \{0\}\right\} \cup \left(\bigcap_{i=1}^{\infty} \left(-2 - \frac{1}{i}, -1 + \frac{1}{i}\right)\right)$$

For problems like this one, we first need to simplify the expression of our set. For this exercise, the set is $\{\frac{1}{\mathrm{z}} : z \in \mathbb{Z}\backslash\{0\}\} \cup [-2,-1]$. The reason is $lim_{n\to\infty}\left(\frac{1}{n}\right) = 0$. Any $\mathrm{x} \in (-2,-1)$ is an interior point. Any point $\mathrm{x} \notin \{0\} \cup \{\frac{1}{\mathrm{z}} : z \in \mathbb{Z}\backslash\{0\}\} \cup [-2,-1]$ is an exterior point. Any $\mathrm{x} \in \{0\} \cup \{-1\} \cup \{-2\} \cup \{\frac{1}{\mathrm{z}} : z \in \mathbb{Z}\backslash\{0\}\}$ is a boundary point.

Exercise* 9

9. Let A be bounded set in $\mathbb{R}$ (which means that the total set is $\mathbb{R}$), for any $\varepsilon > 0$ there is an element x in A such that $|\mathrm{x} - \sup\mathrm{A}| < \varepsilon$

Proof: Suppose this is not true, i.e., there is an $\varepsilon > 0$ such that for all x in A we have $|\mathrm{x} - \sup\mathrm{A}| \geq \varepsilon$. Then $\sup\mathrm{A} - \frac{1}{2}\varepsilon$ is an upper bound of A, leading to a contradiction.