Review II(Slides 71 - 118)

Sets, Points, Rational and Real Numbers, Functions

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September 28, 2022

VV186 - Honors Mathmatics II

Assignment 1

- The first homework is graded rigorously.
- Please check the rubric and comments on common mistakes on Piazza.
- If you have questions about grading, please contact the related TA.
  - 1.1&1.2 Ma Tianyi
  - 1.3 Heyinong
  - 1.4 Ding Zizhao
  - 1.5 Sun Meng

Interval

Remark: Difference between (a,b) and the set $\{x \in \mathbb{R} : a < x < b\}$?

1.5.8. Definition. Let $a, b \in \mathbb{R}$ with $a < b$. Then we define the following special subsets of $\mathbb{R}$, which we call ***intervals***:

$$[a, b] := \{x \in \mathbb{R} \colon (a \le x) \wedge (x \le b)\} = \{x \in \mathbb{R} \colon a \le x \le b\},$$
$$[a, b) := \{x \in \mathbb{R} \colon (a \le x) \wedge (x < b)\} = \{x \in \mathbb{R} \colon a \le x < b\},$$
$$(a, b] := \{x \in \mathbb{R} \colon (a < x) \wedge (x \le b)\} = \{x \in \mathbb{R} \colon a < x \le b\},$$
$$(a, b) := \{x \in \mathbb{R} \colon (a < x) \wedge (x < b)\} = \{x \in \mathbb{R} \colon a < x < b\}.$$

Furthermore, for any $a \in \mathbb{R}$ we set

$$[a, \infty) := \{x \in \mathbb{R} \colon x \ge a\}, \qquad (a, \infty) := \{x \in \mathbb{R} \colon x > a\},$$
$$(-\infty, a] := \{x \in \mathbb{R} \colon x \le a)\}, \qquad (-\infty, a) := \{x \in \mathbb{R} \colon x < a\}$$

Finally, we set $(-\infty, \infty) := \mathbb{R}$.

Sets & Points

Recall the following definition and notation, it is very likely to appear in your exam !

- Interior point
- Exterior point
- Boundary point
- Accumalation point

For each kind of points, discuss whether the point should be in the set or not ?
Whether a boundary point for a set must be an accumalation point for a set ?

Tips:

- **Remember the definitions!**
- **Draw the pictures!**
- **Pay attention to some speacial cases!**

Better understand accumulation point

- Try to prove that : if x is an accumulation point of set A, for every $\epsilon > 0$, the set $(x-\epsilon, x+\epsilon) \cap A \backslash \{x\}$ contains infinite elements.
- Consequently, you can take elements from the set to construct a sequence that converges to x.

Examples in class

Those two examples can give us some thought.

1.5.11. Example. For the interval $A = [0, 1)$,

- $\operatorname{int} A = (0, 1)$,
- Any $x \in \mathbb{R} \setminus [0, 1]$ is an exterior point,
- $\partial A = \{0, 1\}$,
- Any $x \in [0, 1]$ is an accumulation point.

1.5.12. Example. For the set $A = \{x \in \mathbb{R} : x = \frac{1}{n}, n \in \mathbb{N} \setminus \{0\}\}$,

- $\operatorname{int} A = \emptyset$,
- Any $x \in \mathbb{R} \setminus (A \cup \{0\})$ is an exterior point,
- $\partial A = A \cup \{0\}$,
- Only $x = 0$ is an accumulation point.

Exercise about points

Please identify the interior, exterior, boundary and accumalation points of the set

$$\{\frac{1}{z} : z \in \mathbb{Z}\backslash\{0\}\} \cup (\bigcap_{j=1}^{\infty}(-2-\frac{1}{j}, -1+\frac{1}{j}))$$

Sets & Points

Recall the following definition:

- Open Set
- Closed set
- Closure

Remark: Remember that a set does **NOT** have to be either open or closed.

Conceptual Exercises

Please judge true or false:

- The set $\mathbb{R}$ is an open set ?
- The set $\mathbb{R}$ is a closed set ?
- An empty set is an open set ?
- An empty set is a closed set ?
- The set $(a, b]$ is an open set or a closed set ?
- The set $\{x \in \mathbb{R} : x = \frac{1}{n}, n \in \mathbb{N} \backslash \{0\}\}$ is closed ?

Boundness

How we define those concepts for a set ?

- bounded/unbounded
- max/min
- sup/inf

Quick check:

- 1. What's the relationship between max/min and sup/inf.
- 2. Does max/min or sup/inf always exists for bounded sets? When ?

Important Conclusion: $\inf S = \xi \in S \Leftrightarrow \xi = \min S$

Get familiar with this!

Boundness

Check the scope ! Q or R ......

Example:

1. The set $A = (-\infty, a)$ is bounded above in $\mathbb{R}$ with $\sup A = a$. It isn't in A.
2. The set $B = [b, +\infty)$ is bounded below in $\mathbb{R}$ with $\inf B = b$. It's in B since b is the minimum of B.
3. The set $C = [c, d) \cup (e, f)$ is bounded above and below in $\mathbb{R}$, so it's bounded with $\sup C = f$, $\inf C = c$.
4. The set $D = \{x \in \mathbb{Q}^+ : x = \frac{1}{n}, n \in \mathbb{N}^*\}$ is bounded above in $\mathbb{Q}^+$, but not bounded below in $\mathbb{Q}^+$.

Exercise

The conclusion is really useful, we will frequently use it !

Let A be a bounded set in $\mathbb{R}$. Prove that for any $\epsilon > 0$, there is an element x in A such that $|x - \sup A| < \epsilon$.

Tips on proving a supremum or infimum

Generally, proving η is a supremum of a set S has two steps:

1. Firstly, show that η is a upper bound for S. i.e. $\forall x \in S, x \leq \eta$.
2. Secondly, show that $\forall \alpha < \eta, \exists x_0 \in S, x_0 > \eta$.

Sometimes an inequality is useful (directly come from the definition):

- For a set S, if $\forall x \in S, x \leq y$, then sup S $\leq$ y.

The steps for proof and properties for infimum is quite similar to supremum.

Exercise

Suppose A and B are two nonempty sets of numbers, such that $x \leq y$ for $\forall x \in A$ and $\forall y \in B$. Prove that :

1. sup A $\leq$ y , for $\forall y \in B$.
2. sup A $\leq$ inf B

Exercise

Let A, B be bounded and non-empty sets. S=$A \cup B$, please prove that:

(i) sup S = max {sup A, sup B}

(ii) inf S = min {inf A, inf B}

Exercise

Let A and B be two bounded and non-empty sets in $\mathbb{R}$.

Define $A+B := \{z \mid z = x + y,\ x \in A,\ y \in B\}$.

Prove that:

$$\sup(A+B) = \sup A + \sup B$$

Rational Numbers

We define that the set of rational numbers is

$$\mathbb{Q} = \{\frac{p}{q} : p, q \in \mathbb{Z} \wedge q \neq 0\}$$

together with the following properties(P1-P9).

Porperties	Addition	Multiplication
Associativity	$a + (b + c) = (a + b) + c$	$a \cdot (b \cdot c) = (a \cdot b) \cdot c$
NeutralElement	$a + 0 = 0 + a = a$	$a \cdot 1 = 1 \cdot a = a$
Commutativity	$a + b = b + a$	$a \cdot b = b \cdot a$
InverseElement	$(-a) + a = a + (-a) = 0$	$a \cdot a^{-1} = a^{-1} \cdot a = 1$
Distributivity	$a \cdot (b + c) = a \cdot b + a \cdot c$	

Rational Numbers

Property 10: Trichotomy Law (Using a set P to divide Q into three parts)
Property 11 and 12: Feature the set P, such that positive numbers are closed under addition and multiplication

Important Inequality

For all rational numbers $a, b \in \mathbb{Q}$, we have $||a| - |b|| \leq |a + b| \leq |a| + |b|$

Prove it using Mathematical Induction !
Corollary: $|\sum_{i=1}^{n} a_i| \leq \sum_{i=1}^{n} |a_i|$

The Square Root Problem

Let $M = \{t \in \mathbb{R} : t > 0 \wedge t^2 > x\}, y = \inf M$. We want to prove that $y^2 = x$ by showing that $y^2 > x$ and $y^2 < x$ lead to contradictions.

$M: \{t \in \mathbb{R}, t > 0 \wedge t^2 > x\}$. Find infimum y.

Part 1: Prove that $y^2 > x$ leads to contradiction.

Suppose that $y^2 > x$. Since y is an infimum. $\forall 0 < t < y, t \notin M \Rightarrow t^2 \le x$. (By definition of inf).

We want to find a $t < y$, but $t^2 > x$ to find a contradiction

let: $t = y - \varepsilon$. $\varepsilon > 0$. 看 ε 取多小时 $t^2 > x$

$t^2 > x \Leftrightarrow (y-\varepsilon)^2 > x \Leftrightarrow y^2 \underline{- 2y\varepsilon} + \varepsilon^2 > x \leftarrow$ we want an ε to satisfy this inequality.

$y^2 - x + \varepsilon^2 > 2y\varepsilon$.

So we just need to let. $0 < \varepsilon \le \frac{y^2 - x}{2y} = \frac{1}{2}\left(y - \frac{x}{y}\right)$

Part 2 for contradiction

Important Conclusion

Infimum and Supremum don't necessarily exist in a bounded set defined in $\mathbb{Q}$.

Real Numbers and Important Conclusion

The square root problem tells us that: Bounded sets may not have infimum or supremum.
The definition of real numbers guarantees that for a set in $\mathbb{R}$, **if it is bounded above, then it has an supremum; if it is bounded below, then it has a infimum.**

The Real Numbers

We define the set of real numbers $\mathbb{R}$ as the smallest extension of the rational numbers $\mathbb{Q}$ such that the following property holds:

> **(P13)** *If $A \subset \mathbb{R}$, $A \neq \emptyset$ is bounded above, then there exists a least upper bound for A in $\mathbb{R}$.*

We call all real numbers that are not rational ***irrational numbers***.

Complex Numbers

In Vv186, you just need to know how to perform basic complex numbers' computation and some basic properties. Here, we just list some basic computation rules and formulas.

Given $z_1 = (a_1, b_1)$ and $z_2 = (a_2, b_2)$,

- $z_1 + z_2 = (a_1, b_1) + (a_2, b_2) = (a_1 + a_2, b_1, b_2)$
- $z_1 \cdot z_2 = (a_1, b_1) \cdot (a_2, b_2) = (a_1 a_2 - b_1 b_2, a_1 b_2 - a_2 b_1)$
- $c \cdot z_1 = c(a_1, b_2) = (ca_1, cb_1), c \in \mathbb{R}$
- $\bar{z_1} = (a_1, -b_1)$
- $|z_1|^2 = a_1^2 + b_1^2 = z_1 \bar{z_1}$
- $\mathrm{Re}\ z_1 = \frac{z_1 + \bar{z_1}}{2}$
- $(\mathrm{Im}\ z_1) i = \frac{z_1 - \bar{z_1}}{2}$

Open Ball

Let $z_0 \in \mathbb{C}$. Then we define the **open ball** of radius $R > 0$ centered at z_0 by

$$B_R(z_0) := \{z \in \mathbb{C} : |z - z_0| < R\}$$

- Geometric interpretation?
- Higher dimensions?

How to define the boundness of a set in $\mathbb{C}$?
Are there lower bound or upper bound for a bounded set in $\mathbb{C}$?

Important Definition for Function

Recall the definition of domain, codomain and range.

Let's start from the notation for functions.

$$f : \Omega \to Y, \qquad x \mapsto f(x).$$

or alternatively

$$f : \Omega \to Y, \qquad \text{y} = \text{f(x)}.$$

Example: Point out the domain, codomain and range for the function:

$$f : \mathbb{R}^2 \to \mathbb{R}, \qquad \text{f(x,y)}{=}x^2 + y^2$$

Exercise

We will spend some time discussing about $\overline{\lim}$ and $\underline{\lim}$, and their relation with sup/inf for sets in next RC.

Consider the set $U \subset \mathbb{R}$, where $U = A \cup B \cup C$ with

$$\begin{aligned} A &= \{x \in \mathbb{R} \colon 0 < x \leq 1\}, \\ B &= \{x \in \mathbb{R} \colon x = 2 - 1/n,\ n \in \mathbb{N} \setminus \{0\}\}, \\ C &= \{x \in \mathbb{R} \colon x = -1/n,\ n \in \mathbb{N} \setminus \{0\}\}. \end{aligned}$$

State (without proof) $\min U$, $\max U$, $\inf U$, $\sup U$, $\underline{\lim}\, U$ and $\overline{\lim}\, U$ (if one or more of these do not exist, simply state this).

Exercise

Let $A \subset \mathbb{R}$ be a non-empty set.

- ☐ If $\inf A$ exists, then $\underline{\lim}\, A$ exists.
- ☐ If $\underline{\lim}\, A$ exists, then $\inf A$ exists.
- ☐ $\underline{\lim}\, A$ exists if and only if A is bounded below.
- ☐ $\inf A$ exists if and only if A is bounded below.

Reference

- Exercises from 2021-Vv186 TA-Ni Yinchen.
- Exercises from 2021-Vv186 TA-Tu Yiwen.
- Exercises from 2022-Vv186 TA-Sun Meng