Review VII(Slides 362 - 443)

Series

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VV186 - Honors Mathmatics II

Series

Let (a_n) be a sequence in a normed vector space $(V, ||\cdot||)$. We define $s_n := \sum_{k=0}^{n} a_k$ as the **n-th partial sum** of (a_n). We say that (a_n) is **summable** with sum $s \in V$ if $\lim\limits_{n\to\infty} s_n = s$. We use $\sum_{k=0}^{\infty} a_k$, or a_k to denote s as well as the "procedure of summing the sequence (a_n)", and call this notation **infinite series**.

Comment. While the definition of series is in general vector space, we will focus on real series and real function series later on.

Cauchy Criterion

Generally, a closed form sum of a sequence is hard to find. Instead, we will mostly focus on whether the series converges. The starting point will be the **Cauchy Criterion**(Slides 380):

Let $\sum a_k$ be a sequence in a complete vector space $(V, ||\cdot||)$ Then

$$\begin{aligned}\sum a_k \text{ converges} &\Leftrightarrow (s_n) \text{ converges}\\ &\Leftrightarrow (s_n) \text{ is Cauchy}\\ &\Leftrightarrow \forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ such that } \forall m, n > N, ||s_m - s_n|| < \varepsilon\\ &\Leftrightarrow \forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ such that } \forall m > n > N, ||\sum_{k=n+1}^{m} a_k|| < \varepsilon\end{aligned}$$

Two important colloaries are:

- If (a_n) is summable, $a_n \to 0$ as $n \to \infty$
- If (a_n) is summable, $\sum_{k=n}^{\infty} a_k \to 0$ as $n \to \infty$

Tests for Convergence

A number of tests (for series of positive real numbers) are given throughout the slides:

- 1. The Comparison Test (P392, 3.5.15)
- 2. The Root Test (P401, 3.5.22)
- 3. The Root Test in Limits Form (P405, 3.5.26)
- 4. The Ratio Test (P407, 3.5.28)
- 5. The Ratio Test in Limits Form (P410, 3.5.30)
- 6. The Ratio Comparison Test (P411, 3.5.31)
- 7. Raabe's Test (P413, 3.5.32)
- 8. Leibniz Theorem (P413, 3.5.38)

The Comparison Test

In the rest of this section we will use the phrase "for sufficiently large k" to mean "for $k > k_0$ for some $k_0 \in \mathbb{N}$". For example, the statement "$1/k < 0.001$ for sufficiently large k" should be interpreted in this way.

Many of the following criteria apply to series of positive real numbers. They are therefore suitable for establishing the ***absolute convergence*** of a series.

3.5.15. Comparison Test. Let (a_k) and (b_k) be real-valued sequences with $0 \le a_k \le b_k$ for sufficiently large k. Then

$$\sum b_k \text{ converges} \qquad \Rightarrow \qquad \sum a_k \text{ converges.}$$

3.5.16. Remark. In applications one also often uses the contrapositive: if $0 \le a_k \le b_k$ for sufficiently large k, then

$$\sum a_k \text{ diverges} \qquad \Rightarrow \qquad \sum b_k \text{ diverges.}$$

The Root Test

A further consequence of the Comparison Test 3.5.15 is the following, quite fundamental criterion for convergence:

3.5.22. Root Test. Let $\sum a_k$ be a series of positive real numbers $a_k \geq 0$.

(i) Suppose that there exists a $q < 1$ such that

$$\sqrt[k]{a_k} \leq q \qquad \text{for all sufficiently large } k.$$

Then $\sum a_k$ converges.

(ii) Suppose that

$$\sqrt[k]{a_k} \geq 1 \qquad \text{for all sufficiently large } k.$$

Then $\sum a_k$ diverges.

The Root Test using Limits

3.5.26. Root Test. Let a_k be a sequence of positive real numbers $a_k \geq 0$. Then

$$\overline{\lim_{k\to\infty}} \sqrt[k]{a_k} < 1 \qquad \Rightarrow \qquad \sum_{k=0}^{\infty} a_k \qquad \text{converges,}$$

$$\overline{\lim_{k\to\infty}} \sqrt[k]{a_k} > 1 \qquad \Rightarrow \qquad \sum_{k=0}^{\infty} a_k \qquad \text{diverges.}$$

3.5.27. Remarks.

(i) No statement is possible if $\overline{\lim\limits_{k\to\infty}} \sqrt[k]{a_k} = 1$.

(ii) If $\lim\limits_{k\to\infty} \sqrt[k]{a_k}$ exists, it equals $\overline{\lim\limits_{k\to\infty}} \sqrt[k]{a_k}$. This will be the case in many applications.

Prove that the series $\sum a_m = \frac{1}{2} + \frac{1}{3} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{2^3} + \frac{1}{3^3} + \dots$ is convergent.

3.5.28. Ratio Test. Let $\sum a_k$ be a series of strictly positive real numbers $a_k > 0$.

(i) Suppose that there exists a $q < 1$ such that

$$\frac{a_{k+1}}{a_k} \le q \qquad \text{for all sufficiently large } k.$$

Then $\sum a_k$ converges.

(ii) Suppose that

$$\frac{a_{k+1}}{a_k} \ge 1 \qquad \text{for all sufficiently large } k.$$

Then $\sum a_k$ diverges.

3.5.30. Ratio Test. Let (a_k) be a sequence of strictly positive real numbers $a_k > 0$. Then

$$\varlimsup_{k\to\infty} \frac{a_{k+1}}{a_k} < 1 \qquad \Rightarrow \qquad \sum_{k=0}^{\infty} a_k \qquad \text{converges,}$$

$$\varliminf_{k\to\infty} \frac{a_{k+1}}{a_k} > 1 \qquad \Rightarrow \qquad \sum_{k=0}^{\infty} a_k \qquad \text{diverges.}$$

The proof is left to you; note the ***inferior limit*** in the condition for divergence!

3.5.31. Ratio Comparison Test. Let (a_k) and (b_k) be sequences of strictly positive real numbers $a_k, b_k > 0$. Suppose that $\sum b_k$ converges. If

$$\frac{a_{k+1}}{a_k} \le \frac{b_{k+1}}{b_k} \qquad \text{for sufficiently large } k,$$

then $\sum a_k$ converges.

3.5.32. Raabe's Test. Let $\sum a_k$ be a series of positive real numbers $a_k \geq 0$. Suppose that there exists a number $p > 1$ such that

$$\frac{a_{k+1}}{a_k} \leq 1 - \frac{p}{k} \qquad \text{for sufficiently large } k.$$

Then the series $\sum a_k$ converges.

The Weierstraß M-test

3.5.19. Weierstraß M-test. Let $\Omega \subset \mathbb{R}$ and (f_k) be a sequence of functions defined on Ω, $f_k \colon \Omega \to \mathbb{C}$, satisfying

$$\sup_{x\in\Omega} |f_k(x)| \le M_k, \qquad k \in \mathbb{N} \tag{3.5.9}$$

for a sequence of real numbers (M_k). Suppose that $\sum M_k$ converges. Then the limit

$$f(x) := \sum_{k=0}^{\infty} f_k(x) \qquad \text{exists for every } x \in \Omega.$$

Furthermore, the sequence (F_n) of partial sums

$$F_n(x) = \sum_{k=0}^{n} f_k(x)$$

converges uniformly to f.

3.5.38. Leibniz Theorem. Let $\sum \alpha_k$ be a complex series whose partial sums are bounded but need not converge. Let (a_k) be a decreasing convergent sequence with limit zero, $a_k \searrow 0$. Then the series

$$\sum \alpha_k a_k \qquad \text{converges.}$$

A typical application of the Leibniz Theorem 3.5.38 are alternating series, for which $\alpha_k = (-1)^k$. In particular, the Leibniz series

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$$

converges by the Leibniz Theorem.

Procedure of Determining Convergence

We rank the "usefulness" of all these tests as follows:

Cauchy Criteria
$>$ Comparison Test
$>$ Ratio Test (in Limits)
$>$ Root Test (in Limits)
$>$ Ratio Comparison Test/Raabe's Test...

When you are asked to determine whether a series converges, it's recommended to use the tests in this order. Thus if you have a hard time memorizing all the tests, do first memorize the more "important" tests.

Exercises

Suppose $\sigma > 0$, $a_n > 0$, prove that:
(1) If $\forall$n>N, $(\ln \frac{1}{a_n})(\ln n)^{-1} \geq 1 + \sigma$, then $\sum_{n=1}^{\infty} a_n$ converges
(2) If $\forall$n>N, $(\ln \frac{1}{a_n})(\ln n)^{-1} \leq 1$, then $\sum_{n=1}^{\infty} a_n$ diverges

Exercises

1. Please determine whether the following series converge or not!

- $$\sum_{n=0}^{\infty} \frac{4n(n+2)!}{(2n)!}$$

- $$\sum_{n=1}^{\infty} \frac{\sin(n\theta)}{n^2}, \text{ where } \theta \text{ is fixed}$$

- $$\sum_{n=0}^{\infty} \frac{(2n)!}{4^n (n+1)! n!} \text{ (Hint: This appears in the slides!)}$$

- $$\sum_{n=2}^{\infty} \frac{1}{(\ln n)^{(\ln n)}}$$

Exercises

2. Prove the **limit comparison test**:

For two positive series $\sum a_n$ and $\sum b_n$, if

$$0 < \lim_{n\to\infty} \frac{a_n}{b_n} < \infty$$

then a_n and b_n both converges or diverges.

Exercises

3. Prove that if a positive series a_n diverges, then
(1) $\sum \frac{a_n}{1+a_n}$ diverges.
(2) $\sum \frac{a_n}{1+n^2 a_n}$ converges.

Absolute and Conditionally Convergence

- A series $\sum a_n$ is called **absolutely convergent** if $\sum ||a_n||$ converges.
- If $\sum a_n$ converges while $\sum ||a_n||$ doesn't, than it's called **conditionally convergent**.
- In a complete vector space (which is the case in our cases), absolutely convergent implies convergent.

To test for conditionally convergence, we have the following theorem: Let $\sum \alpha_k$ be a complex series whose partial sum are bounded but need not converge. Let (a_k) be a decreasing convergent sequence with limit zero, then the series $\sum \alpha_k a_k$ converges (Slide 418)

Comment. With this result, it is easy to see that $\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$ converges

3.5.36. Theorem. Assume that $\sum a_k$ is an absolutely convergent series in a complete normed vector space. If the summands of the series are rearranged, the new series $\sum b_j$, $b_j = a_{k(j)}$, $k\colon \mathbb{N} \to \mathbb{N}$ bijective, converges absolutely with the same sum as $\sum a_k$.

In particular, if $\sum a_k$ and $\sum b_k$ are absolutely convergent, then $\sum(a_k + b_k)$ is absolutely convergent and equal to $\sum a_k + \sum b_k$.

Contrast this with the following result:

3.5.37. Theorem. Let $\sum a_k$ be a conditionally convergent series of real numbers. Then for any $\alpha \in \mathbb{R}$ there exists a rearrangement $b_j = a_{k(j)}$, $k\colon \mathbb{N} \to \mathbb{N}$ bijective, of $\sum a_k$ such that $\sum b_j = \alpha$.

3.5.40. Theorem. Let $\sum a_k$ and $\sum b_k$ be absolutely convergent series. Then the ***Cauchy product*** $\sum c_k$ given by

$$c_k := \sum_{i+j=k} a_i b_j$$

converges absolutely and $\sum c_k = \left(\sum a_k\right)\left(\sum b_k\right)$.

3.5.41. Remark. If $a = (a_k)$ and $b = (b_k)$ are two absolutely summable sequences, the sequence

$$a * b := (c_k), \qquad c_k := \sum_{i+j=k} a_i b_j,$$

is called the ***convolution*** of a and b.

Power Series

Of all function series, one useful kind is the **power series**, which is the infinite sum of monomials.

$$\sum_{k=0}^{\infty} a_k z^k \text{ or simply } \sum a_k z_k$$

We call this *formal* as we are yet to find whether the series converge or not for given z.

We can add and multiply two power series:

- $\sum a_n z^n + \sum b_n z^n = \sum (a_n + b_n) z^n$
- $\sum a_n z^n \cdot \sum b_n z^n = \sum (a * b)_n z^n$

Why convolution?

Raduis of Convergence

Let $\sum a_k z^k$ be a complex power series. Then there exists a unique number $\rho \in (0, +\infty)$ such that

i) the power series $\sum a_k z^k$ is absolutely convergent at $z_0 \in \mathbb{C}$ if $|z_0| < \rho$;

ii) the power series diverges at $z_0 \in \mathbb{C}$ if $|z_0| > \rho$

Hadamard's formula:

$$\rho = \frac{1}{\overline{\lim_{k\to\infty}} \sqrt[k]{|a_k|}}$$

where ρ is called the radius of convergence, if we informally write $1/\infty = 0, 1/0 = \infty$.

Remarks

- For a complex power series, the set of z which the series converge will always be a circle. For a real power series, the set will be a line segment and the radius of convergence is one half of the length.
- We can't say much if we have $|z| = \rho$. The series may converge or diverge or conditionally converge.

Do check for the boundary!

3.6.6. Example. The formal power series $\sum_{k=0}^{\infty} \frac{(-1)^k}{k} z^k$ has radius of convergence $\varrho = 1$. The series converges for $z_0 = 1$ and diverges for $z_0 = -1$. Other values of z_0 with $|z_0| = 1$ can be checked individually.

Exercise

5. Decide for the following real power series, on which interval would it converge?

- $$\sum_{n=1}^{\infty} \frac{(x-1)^n}{2^n n}$$
- $$\sum_{n=1}^{\infty} \frac{x^{2n}}{n-3^{2n}}$$

Abel Theorem

(1) If a power series is convergent at x=x_0, then it is absolutely convergent for $|x| < |x_0|$;
(2) If a power series is divergent at x=x_0, then it is divergent for $|x| > |x_0|$.

Uniform Convergence of Series of Functions

3.5.20. Definition. Let $\Omega \subset \mathbb{R}$ and (f_k) be a sequence of functions defined on Ω, $f_k : \Omega \to \mathbb{C}$. We say that the series

$$\sum_{k=0}^{\infty} f_k$$

converges uniformly (to a function $f : \Omega \to \mathbb{C}$) if the sequence of partial sums $F_n = \sum_{k=0}^{n} f_k$ converges uniformly to f.

Uniform Convergence of Power Series

3.6.8. Lemma. If $\sum a_k z^k$ is a complex power series with radius of convergence ϱ, then for any $R < \varrho$ the series converges uniformly on $B_R(0) = \{z : |z| < R\}$.

Proof.
Let $0 < R < \varrho$ be fixed. Then

$$\sup_{z \in B_R(0)} |a_k z^k| < |a_k| R^k =: M_k$$

Now

$$\sum_{k=0}^{\infty} M_k = \sum_{k=0}^{\infty} |a_k| R^k$$

converges, because the point $R < \varrho$ lies within the radius of convergence of the series and the series convergences absolutely within its radius of convergence. By the Weierstraß M-test 3.5.19, the series $\sum a_k z^k$ of functions $a_k z^k : B_R(0) \to \mathbb{C}$ converges uniformly on $B_R(0)$. $\square$

Continuity of Power Series

3.6.9. Corollary. A power series $\sum_{k=0}^{\infty} a_k z^k$ with radius of convergence ϱ defines a continuous function

$$f : B_\varrho(0) \to \mathbb{C}, \qquad f(z) = \sum_{k=0}^{\infty} a_k z^k.$$

Differentiability of Power Series

The power series $\sum a_k z^k$ with radius of convergence ρ defines a differentiable function $f\colon B_\rho(0) \to \mathbb{C}$. Furthermore,

$$f'(z_0) = \sum k a_k z_0^{k-1}$$

Remarks:

1. This means that we can differentiate a power series "term by term" inside the radius of convergence.
2. Recursively apply this theorem to see that any power series is infinitely differentiable inside its radius of convergence. In fact, for a function to be expressable as a power series (which we call it **analytic**) is stronger than being infinitely differentiable. (You will learn more about this in Vv286!)

Reference

- 2021-Vv186 TA-Niyinchen

End

Thanks!