1.

Solution: i) $(2x+5x^2)^6$: $6(2x+5x^2)^5 \cdot (2+10x)$

ii) $\frac{\sqrt{x}}{x+1}$: $\frac{1}{2\sqrt{x}(x+1)} - \frac{\sqrt{x}}{(x+1)^2}$

iii) $\sqrt[3]{\frac{3x^2+1}{x^2+1}}$: $\frac{1}{3}\left(\frac{3x^2+1}{x^2+1}\right)^{-2/3} \cdot \frac{4x}{(x^2+1)^2}$

3.

Proof: The key is to construct a differentiable function, denoted by f. Furthermore, instead of proving $|a+b|^n \le 2^{n-1}(|a|^n+|b|^n)$, we would like to prove $(|a|+|b|)^n \le 2^{n-1}(|a|^n+|b|^n)$ first.

If $|a|+|b|=0$, then the inequality holds trivially. Otherwise $(|a|+|b| \neq 0)$, without losing generality let's suppose that $|b| \neq 0, |a| = t|b|$ for some $t \ge 0$. Fix arbitrary $p \ge 1$. Construct our f be a function $f: [0,+\infty) \to \mathbb{R}, f(t) = (t+1)^p - 2^{p-1}(t^p+1)$; so $f(1)=0$. Since $f'(t) = p(t+1)^{p-1} - 2^{p-1}\cdot(pt^{p-1}) = p(t+1)^{p-1} - p(2t)^{p-1}$, we know

$$\begin{cases} f' > 0 \text{ when } t \in [0,1) \\ f' = 0 \text{ when } t = 1 \\ f' < 0 \text{ when } t > 1 \end{cases}$$

It follows that f is increasing on $[0,1)$ and decreasing on $[1,+\infty)$, which means that $f(1)$ is the maximum of our f. Therefore, $f \le 0$. It follows that for any n (which is just a specific number p), $|b|^p \cdot f = (|a|+|b|)^n - 2^{n-1}(|a|^n+|b|^n) \le 0$. The rest follows from $|a+b|^n \le (|a|+|b|)^n \le 2^{n-1}(|a|^n+|b|^n)$.

5.

Proof: Let the two points where f attains zero be a and b, namely, $f(a)=f(b)=0$. Let $I=(a,b), \bar{I}=[a,b]$. Suppose the maximum of f on $\bar{I}$, denoted by $f(x_0)$ is strictly greater than zero. Then $f'(x_0)=0$ Since $f(x_0)>0$, $f''(x_0) = -f'(x_0)g(x_0)+f(x_0) = f(x_0) > 0$. But this means there exists some $\varepsilon > 0$ such that on the neighborhood $(x_0-\varepsilon, x_0+\varepsilon)$, $f(x_0)$ is a strict local minimum. It follows that $f(x_0)$ is not the maximum point of f, leading to a contradiction. Suppose the maximum of f on $\bar{I}$, denoted by $f(x_0)$ is strictly smaller than zero. A similar proof shows that this assumption fails. $\max_{x\in\bar{I}} f(x) = \min_{x\in\bar{I}} f(x) = 0$. It follows that f is 0 on the interval between $f(a)$ and $f(b)$.

6.

Proof: i) First we would like to show that $\forall \lambda_1, \lambda_2 > 0, \lambda_1 + \lambda_2 = 1, \{x_1, x_2\} \subseteq [a, b]$, $f(\lambda_1 x_1 + \lambda_2 x_2) \leq \lambda_1 f(x_1) + \lambda_2 f(x_2)$: Without losing generality suppose $x_1 < x_2$. Now, since $\lambda_1 x_1 + \lambda_2 x_2 \in (x_1, x_2)$, by convexity we have $\frac{f(\lambda_1 x_1 + \lambda_2 x_2) - f(x_1)}{(\lambda_1 x_1 + \lambda_2 x_2) - x_1} \leq \frac{f(x_2) - f(x_1)}{x_2 - x_1}$. This inequality is equivalent to

$$(x_2 - x_1)\big(f(\lambda_1 x_1 + \lambda_2 x_2) - f(x_1)\big) \leq ((\lambda_1 x_1 + \lambda_2 x_2) - x_1)(f(x_2) - f(x_1))$$

$$\Leftrightarrow\ f(\lambda_1 x_1 + \lambda_2 x_2) - f(x_1) \leq \lambda_2\big(f(x_2) - f(x_1)\big)$$

$$\Leftrightarrow\ f(\lambda_1 x_1 + \lambda_2 x_2) \leq \lambda_1 f(x_1) + \lambda_2 f(x_2)$$

Therefore, the case of Jensen's inequality for $n = 1, 2$ is trivial. Suppose Jensen's inequality holds for $n = k$, we would like to show that it holds for $n = k + 1$. Let $\{x_1, x_2, \dots, x_k, x_{k+1}\} \subseteq [a, b]$ be a set of k+1 points, and fix arbitrary positive "λ_i"s, $i \in \{1, 2, \dots, k + 1\}$, such that $\sum_{i=1}^{k+1} \lambda_i = 1$. Moreover, let $y := \sum_{i=2}^{k+1} \frac{\lambda_i}{\sum_{i=2}^{k+1} \lambda_i} x_i$.

Of course, $\sum_{i=2}^{n} \frac{\lambda_i}{\sum_{i=2}^{n} \lambda_i} = 1$. Then we have

$$f\left(\sum_{i=1}^{k+1} \lambda_i x_i\right) = f\left(\frac{\sum_{i=1}^{k+1} \lambda_i x_i}{\sum_{i=1}^{k+1} \lambda_i}\right)$$

$$= f\left(\frac{\lambda_1 x_1}{\sum_{i=1}^{k+1} \lambda_i} + \frac{\sum_{i=2}^{k+1} \lambda_i x_i}{\sum_{i=1}^{n} \lambda_i}\right)$$

$$= f\left(\frac{\lambda_1}{\sum_{i=1}^{k+1} \lambda_i} x_1 + \frac{\sum_{i=2}^{k+1} \lambda_i}{\sum_{i=1}^{n} \lambda_i} y\right)$$

$$\leq \frac{\lambda_1}{\sum_{i=1}^{k+1} \lambda_i} f(x_1) + \frac{\sum_{i=2}^{k+1} \lambda_i}{\sum_{i=1}^{k+1} \lambda_i} f(y) \text{ .... by convexity}$$

$$\leq \frac{\lambda_1}{\sum_{i=1}^{k+1} \lambda_i} f(x_1) + \frac{\sum_{i=2}^{k+1} \lambda_i}{\sum_{i=1}^{k+1} \lambda_i} \cdot \left(\sum_{i=2}^{k+1} \frac{\lambda_i}{\sum_{i=2}^{k+1} \lambda_i} f(x_i)\right) \text{ .... by induction}$$

$$=\frac{\lambda_1}{\sum_{i=1}^{k+1}\lambda_i}f(x_1)+\frac{\lambda_2}{\sum_{i=1}^{k+1}\lambda_i}f(x_2)+\cdots+\frac{\lambda_{k+1}}{\sum_{i=1}^{k+1}\lambda_i}f(x_{k+1})$$

$$=\sum_{i=1}^{k+1}\lambda_i f(x_i)\ \ldots\ldots\ldots\ldots \sum_{i=1}^{k+1}\lambda_i=1$$

This proves Jensen's inequality.

ii) (If you don't know that the function $f:\mathbb{R}\to\mathbb{R}, f(x)=e^x$ is convex, you can skip this exercise) The case for $a_i=0$ for some i is trivial, since the left-hand side equals to zero and the right-hand side is always no less than zero. Now, suppose $a_i>0$ for all i. For each a_i we can assign some $x_i \in \mathbb{R}$ such that $e^{x_i}=a_i$. Since e^x is a convex function, by i) we have $e^{\sum_{i=1}^{n}\lambda_i x_i}\le\sum_{i=1}^{n}\lambda_i e^{x_i}$. So,

$$\prod_{i=1}^{n} a_i^{\lambda_i}=e^{\sum_{i=1}^{n}\lambda_i x_i}\le\sum_{i=1}^{n}\lambda_i e^{x_i}=\sum_{i=1}^{n}\lambda_i a_i$$

This proves the inequality.

8.

Proof: Since f is continuous on a closed interval $[a,b]$, f attains its minimum on it, so f has at least one local minimum. Then we would like to show that f cannot have finitely many local minimums: suppose $\{f(x_1),f(x_2),f(x_3),\ldots,f(x_n)\}$ are all of f's local minimums, and $n\ge 2$. Between two adjacent points $f(x_i),f(x_{i+1})$, since f is continuous, f attains a maximum, denoted by $f(y_i)$, on $[x_i,x_{i+1}]$. Furthermore, $f(y)>max\,\{f(x_i),f(x_{i+1})\}$. It follows that

$$\frac{f(x_i)-f(y)}{x_i-y}>0>\frac{f(x_{i+1})-f(y)}{x_{i+1}-y}$$

So f is not convex, leading to a contradiction. Third we need to show that f can have infinitely many local minimums: the example is $f:[-2,2], f(x)=\begin{cases}-x, & -2\le x<-1\\ 1, & -1\le x\le 1\\ x, & 1<x\le 2\end{cases}$.

9.

Proof: If $f(0)=f(1)=\cdots=f(n-1)$, then $f(i)=1, i\in\{0,1,2,\ldots n\}$. This means for any interval $[i,i+1], i\in\{0,1,2,\ldots,n-1\}$, there is some $c_i\in(i,i+1)$ with $f'(c_i)=0$. If "f(i)"s are not all equal, then there is some $f(i)>1$ and some $gf(j)<1$. Without losing generality we assume that $i<j$. By Mean Value Theorem of continuous function, there is some point $x\in(i,j)$ such that $f(x)=1$. Thus, on the interval (x,n), there is some $c\in(x,n)$ such that $f'(c)=0$.

10.

Proof: i) Given $\varepsilon>0$, choose $\delta<\left(\frac{1}{k}\cdot\varepsilon\right)^{\frac{1}{\alpha}}$. Fix $x\in\Omega$, $\forall y\in\Omega$ with $|x-y|<\delta$, we have the following:

$$|T(x) - T(y)| \leq k \cdot |x - y|^{\alpha} < k \cdot \left(\frac{1}{k} \cdot \varepsilon\right) = \varepsilon$$

Therefore, T is continuous.

ii) Fix $x \in \Omega$, $\forall y \neq x$, $\left|\frac{T(x)-T(y)}{x-y}\right| \leq k \cdot |x - y|^{\alpha-1}$. Thus, when $y \to x$, $|x - y| \to 0$, $\left|\frac{T(x)-T(y)}{x-y}\right| \to 0$. But this means $T'(x) = 0$. Since this is true for all $x \in \Omega$, $T' = 0$ on Ω. It follows that $T = C$ for some $C \in \mathbb{R}$.