Exercises for Midterm 2

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November 21, 2022

VV186 - Honors Mathmatics II

Frequently Asked Question in Sample Exam

Exercise 2.
Find the radius of convergence of the following series

$$\sum_{k=1}^{\infty} \frac{k(x-4)^k}{k^3+1}, \qquad \sum_{k=1}^{\infty} \frac{k^2 x^k}{2 \cdot 4 \cdot 6 \cdots (2k)}.$$

(4 Marks)

Solution. We calculate

$$\boxed{\varlimsup_{k\to\infty} \left| \frac{\frac{k+1}{(k+1)^3+1}}{\frac{k}{k^3+1}} \right| = \varlimsup_{k\to\infty} \left| \frac{(k+1)(k^3+1)}{k(k+1)^3+k} \right| = 1,}$$

since the numerator and denominator both are polynomials of the type $k^4 + c_3 k^3 + c_2 k^2 + c_1 k + c_0$ for suitable constants c_0, c_1, c_2, c_3.**($1\frac{1}{2}$ Mark)** Hence the radius of convergence is $\rho = 1/1 = 1$. **(1/2 Mark)**
Note that

$$2 \cdot 4 \cdot 6 \cdots (2k) = 2^k \cdot 1 \cdot 2 \cdot 3 \cdots k = 2^k k!.$$

Then

$$\boxed{\varlimsup_{k\to\infty} \left| \frac{\frac{(k+1)^2}{2^{k+1}(k+1)!}}{\frac{k^2}{2^k k!}} \right| = \varlimsup_{k\to\infty} \frac{\frac{(k+1)}{2k!}}{\frac{k^2}{k!}} = \varlimsup_{k\to\infty} \frac{(k+1)}{2k^2} = 0}$$

($1\frac{1}{2}$ Marks) Hence the radius of convergence is $\rho = \infty$. **(1/2 Mark)**

Exercises

1. Suppose that $(V, +, \cdot)$ is a vector space and let $U, W \subset V$ be two subspaces. Then:

(A) $U \cap W \neq \emptyset$

(B) $V \setminus U$ is also a subspace of V

(C) $U \cup W$ is a subspace of V

(D) $U \cap W$ is a subspace of V

Exercises

2. Let $(a, b) \subset \mathbb{R}$ be an open interval and denote $C^1(a, b)$ the vector space of continuously differentiable functions on (a, b). On this space a norm is defined by

(A) $||f|| := \sup_{x \in (a,b)} |f(x)|$

(B) $||f|| := \sup_{x \in (a,b)} |f'(x)|$

(C) $||f|| := \sup_{x \in (a,b)} |f(x)| + \sup_{x \in (a,b)} |f'(x)|$

(D) $||f|| := \sup_{x \in (a,b)} (|f(x)| + |f'(x)|)$

Exercises

3. Let (a_n) be a sequence of real numbers such that $f(x) = \sum_{n=0}^{\infty} a_n x^n$ converges at least for $x \in [0,1]$. Then $f(x)$

(A) $f(x)$ must converge for $x = -1/2$

(B) $f(x)$ must converge for $x = -1$

(C) $f(x)$ may or may not converge for $x = -1$

(D) $f(x)$ never converges for $x = 2$

Exercises

4.

$$f(x) = \begin{cases} x^4 \sin \dfrac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Calculate $f''(x)$

Exercises

5. Suppose that $f\colon \mathbb{R} \to \mathbb{R}$ is twice differentiable and has a local minimum at $x = 0$. Then

(A) f is convex in a neighborhood of $x = 0$

(B) $f''(0) > 0$

(C) $f''(0) \geq 0$ and $f''(0) = 0$ is possible

(D) $f'(x)$ is increasing in a neighborhood of $x = 0$

Exercises

6. Let V be a vector space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. If one were to define

$$U_1 + U_2 := \{z \in V : \underset{x \in U_1}{\exists} \underset{y \in U_2}{\exists} z = x + y\}$$

$$U_1 - U_2 := \{z \in V : \underset{x \in U_1}{\exists} \underset{y \in U_2}{\exists} z = x - y\}$$

for subspaces U_1, U_2 of V, then one would have:

(A) $U_1 - U_2 \subset U_1$
(B) $U_1 \subset U_1 + U_2$
(C) $(U_1 - U_2) + U_2 = U_1$
(D) $U_1 - U_2 = U_1 + U_2$

Exercises

7. (i) Does the following series converge?

$$\sum_{n=0}^{\infty} \frac{(2n)!(3n)!}{n!(4n)!}$$

(ii) For which $a \in \mathbb{R}$ does the following series converge?

$$\sum_{n=0}^{\infty} \left(\frac{1}{n} - \sin\left(\frac{1}{n}\right)\right)^{a}$$

Exercises Very likely to appear in mid2

8. Let $f\colon [0, 2\pi] \to \mathbb{R}$ be given by

$$f(x) = \frac{1}{1 + e^{\pi - x}\sin(x)}$$

(i) For which x is $f'(x) = 0$? Derive the solution to the equation $\sin(x) = \cos(x)$, $x \in [0, 2\pi]$

(ii) Where is f increasing? Where is f decreasing?

(iii) Find the local extrema of f

(iv) What can you say about the convexity and concavity of f?

(v) Sketch the graph of f, clearly indicating any siginicant features of the graph

Exercises

9*. Suppose that (f_n) is a sequence of increasing functions $f_n : [0,1] \to [0,1]$, $n \in \mathbb{N}$, such that

$$\lim_{n\to\infty} f_n(x) = f(x) \qquad \text{for all } x \in [0,1],$$

Suppose that f is a continuous function. Show that the convergence is uniform.

(Note that the functions f_n are not assumed to be continuous.)

Comment. This is also called *Dini's theorem.*

Exercises

10. Let (f_n) be a sequence of functions in $C([a, b])$, and (f_n) converges to some function f uniformly. Prove that if $f \neq 0$ on $[a, b]$, then $(\frac{1}{f_n})$ converges to $\frac{1}{f}$ uniformly.

Exercises Very likely to appear in mid2

11. From Sample Test, but professor didn't give answers for this exercise.

Define the functions

$$f_n : \mathbb{R} \to \mathbb{R}, \qquad f_n(x) = \frac{|x|^n}{1 + |x|^n}$$

for $n \in \mathbb{N}$.

i) Find the pointwise limit of (f_n).

ii) Show that the convergence is not uniform on $\mathbb{R}$.

iii) Show that for any $q > 1$ the convergence on $I_q = \{x \in \mathbb{R} : |x| \le 1/q\} \cup \{x \in \mathbb{R} : |x| \ge q\}$ is uniform.

(1 + 1 + 2 Marks)

End

Good Luck!

Reference

- Exercises from 2020 Vv186 Midterm 2.