$x \in \partial U \Leftrightarrow$ every open ball $B(x,r)$ meets both U and U^c

$$\begin{cases} -\Delta u = f & \text{in } U \\ u = g & \text{on } \partial U \end{cases}$$

boundary condition

Poisson's equation

Dirichlet problem

$U \subset \mathbb{R}^n$ is open, bounded, and ∂U is C^1

The boundary is locally the graph of a continuously differentiable function.

ensure the validity of tools like the divergence theorem and the smoothness of solutions up to the boundary.

Φ (2) fundamental solution

$$\Phi(x) = \begin{cases} -\frac{1}{2\pi} \log|x| & (n=2) \\ \frac{1}{n(n-2)\alpha(n)} \frac{1}{|x|^{n-2}} & (n \geq 3). \end{cases}$$

Suppose $u \in C^2(\bar{U})$ (arbitrary). Fix $x \in U$, choose $\varepsilon > 0$ so small that $B(x,\varepsilon) \subset U$.

THEOREM 3 (Green's formulas). *Let $u, v \in C^2(\bar{U})$. Then*

(i) $\int_U \Delta u \, dx = \int_{\partial U} \frac{\partial u}{\partial \nu} dS$,

(ii) $\int_U Dv \cdot Du \, dx = -\int_U u \Delta v \, dx + \int_{\partial U} \frac{\partial v}{\partial \nu} u \, dS$,

(iii) $\int_U u\Delta v - v\Delta u \, dx = \int_{\partial U} u \frac{\partial v}{\partial \nu} - v \frac{\partial u}{\partial \nu} dS$.

$\Phi(y-x)$ singular at $y=x$, avoid singularity by removing the ball

apply on the region $V_\varepsilon := U - B(x,\varepsilon)$ to $u(y)$ and $\Phi(y-x)$

(*)
$$\int_{V_\varepsilon} u(y)\Delta\Phi(y-x) - \Phi(y-x)\Delta u(y) \, dy = \underbrace{\int_{\partial V_\varepsilon} u(y)\frac{\partial \Phi}{\partial \nu}(y-x) - \Phi(y-x)\frac{\partial u}{\partial \nu}(y) \, dS(y)}_{(\mathrm{I})},$$
ν denoting outer normal vector on ∂V_ε.

$\Delta\Phi(y-x) = 0$ away from x, term on the left is just $-\int_{V_\varepsilon} \Phi \Delta u$.

$$\int_{\partial V_\varepsilon} = \int_{\partial U} + \int_{\partial B(x,\varepsilon)}$$

ν the outward normal for V_ε points into the ball

$$(\mathrm{I}) = \int_{\partial U} u(y)\frac{\partial \Phi}{\partial \nu}(y-x) - \Phi(y-x)\frac{\partial u}{\partial \nu}(y) \, dS(y) + \int_{\partial B(x,\varepsilon)} u(y)\frac{\partial \Phi}{\partial \nu}(y-x) - \Phi(y-x)\frac{\partial u}{\partial \nu}(y) \, dS(y).$$

Φ grows like ε^{2-n} (or $|\log\varepsilon|$ in $n=2$)

$$\left| \int_{\partial B(x,\varepsilon)} \Phi(y-x)\frac{\partial u}{\partial \nu}(y) \, dS(y) \right| \leq C\varepsilon^{n-1} \max_{\partial B(0,\varepsilon)} |\Phi| = o(1) \quad \text{as } \varepsilon \to 0.$$

surface measure is $O(\varepsilon^{n-1})$

$u \in C^2(\bar{U})$, bounded on $\partial B(x,\varepsilon)$

$$(\mathrm{II}) = \int_{\partial B(x,\varepsilon)} u(y)\frac{\partial \Phi}{\partial \nu}(y-x) \, dS(y)$$ have shown in Thm 1.

$(n \geq 3)$
$$D\Phi(y-x) = \frac{-1}{n\alpha(n)} \frac{y-x}{|y-x|^n}, \quad \nu = \frac{-(y-x)}{|y-x|} = -\frac{y-x}{\varepsilon} \text{ on } \partial B(x,\varepsilon) \Rightarrow \frac{\partial \Phi}{\partial \nu}(y-x) = \frac{1}{n\alpha(n)\varepsilon^{n-1}} \text{ on } \partial B(x,\varepsilon).$$

$n\alpha(n)\varepsilon^{n-1}$ is the surface area of the sphere $\partial B(x,\varepsilon)$

$$(\mathrm{II}) = \frac{1}{n\alpha(n)\varepsilon^{n-1}} \int_{\partial B(x,\varepsilon)} u(y) \, dS(y) = ⨍_{\partial B(x,\varepsilon)} u(y) \, dS(y) \to u(x) \quad \text{as } \varepsilon \to 0.$$

(*) gives
$$-\int \Phi(y-x)\Delta u(y) \, dy = \int_{\partial U} u(y)\frac{\partial \Phi}{\partial \nu}(y-x) - \Phi(y-x)\frac{\partial u}{\partial \nu}(y) dS(y) + u(x), \quad \varepsilon \to 0$$

$$u(x) = \int_{\partial U} \Phi(y-x)\frac{\partial u}{\partial \nu}(y) - u(y)\frac{\partial \Phi}{\partial \nu}(y-x) \, dS(y) - \int_U \Phi(y-x)\Delta u(y) \, dy. \quad (**)$$

Need to know the values of Δu within U and the values of u, $\frac{\partial u}{\partial \nu}$ along ∂U.

unknown

Build a corrector $\phi^x(y)$ that cancels the offending boundary terms.

Fix x, $\phi^x = \phi^x(y)$. $\begin{cases} \Delta\phi^x = 0 & \text{in } U \\ \phi^x = \Phi(y-x) & \text{on } \partial U \end{cases}$.

Apply Green's formula once more,

$$-\int_U \phi^x(y)\Delta u(y)\,dy = \int_U \Delta\phi^x(y)u(y)\,dy - \phi^x(y)\Delta u(y)\,dy = \int_{\partial U} u(y)\frac{\partial\phi^x}{\partial\nu}(y) - \phi^x(y)\frac{\partial u}{\partial\nu}(y)\,dS(y)$$

$$= \int_{\partial U} u(y)\frac{\partial\phi^x}{\partial\nu}(y) - \Phi(y-x)\frac{\partial u}{\partial\nu}(y)\,dS(y) \qquad (**)$$

Definition. Green's function for the region U is $G(x,y) := \Phi(y-x) - \phi^x(y)$ $(x,y\in U,\ x\neq y)$.

$$u(x) = \int_{\partial U} \Phi(y-x)\frac{\partial u}{\partial\nu}(y) - u(y)\frac{\partial\Phi}{\partial\nu}(y-x)\,dS(y) - \int_U \Phi(y-x)\Delta u(y)\,dy.$$

$$-\int_U \phi^x(y)\Delta u(y)\,dy = \int_{\partial U} u(y)\frac{\partial\phi^x}{\partial\nu}(y) - \Phi(y-x)\frac{\partial u}{\partial\nu}(y)\,dS(y).$$ adding them,

$$u(x) = \int_{\partial U}\Big[\Phi(y-x)\frac{\partial u}{\partial\nu}(y) - u(y)\frac{\partial\Phi}{\partial\nu}(y-x) + u\,\partial_\nu\phi^x(y) - \Phi(y-x)\frac{\partial u}{\partial\nu}(y)\Big]dS(y)$$
$$- \int_U (\Phi(y-x) - \phi^x(y))\Delta u(y)\,dy$$
$$= \int_{\partial U}\Big[\underbrace{(\Phi-\phi^x)}_{0}\partial_\nu u - u(\partial_\nu\Phi - \partial_\nu\phi^x)\Big]dS - \int_U G(x,y)\Delta u(y)\,dy$$
$$= -\int_{\partial U} u(y)\,\partial_\nu G(x,y)\,dS(y) - \int_U G(x,y)\Delta u(y)\,dy \qquad (*)$$

$\frac{\partial G}{\partial\nu}(x,y) = D_y G(x,y)\cdot\nu(y)$ is the outer normal derivative of G with respect to y. $\partial u/\partial\nu$ disappear

Suppose $u\in C^2(\bar U)$,
$\begin{cases} -\Delta u = f & \text{in } U \\ u = g & \text{on } \partial U \end{cases}$, Plugging into $(*)$, we get

Theorem (Representation formula using Green's function). If $u\in C^2(\bar U)$ solve $\begin{cases} -\Delta u = f & \text{in } U \\ u = g & \text{on } \partial U \end{cases}$, then
$$u(x) = -\int_{\partial U} g(y)\frac{\partial G}{\partial\nu}(x,y)\,dS(y) + \int_U f(y)G(x,y)\,dy \quad (x\in U).$$

Construct G for a given domain U. difficult. can be done only when U is simple.

Fix $x\in U$. Then regarding G as a function of y, we may symbolically write
$\begin{cases} -\Delta G = \delta_x & \text{in } U \\ G = 0 & \text{on } \partial U, \end{cases}$

δ_x denoting the Dirac measure giving unit mass to the point x.

G is symmetric in the variables x and y.

Theorem (Symmetry of Green function). For all $x, y \in U$, $x \neq y$, we have $G(y,x) = G(x,y)$.

Pf. Fix $x, y \in U$, $x \neq y$. $v(z) := G(x,z)$, $w(z) := G(y,z)$ $(z \in U)$.

Then $\Delta v(z) = 0$ $(z \neq x)$, $\Delta w(z) = 0$ $(z \neq y)$ and $w = v = 0$ on ∂U.

★ $\begin{cases} v \text{ harmonic on } U \setminus \{x\} \\ w \text{ harmonic on } U \setminus \{y\} \end{cases}$

$v = w = 0$ on ∂U

Let $V := U - [B(x,\varepsilon) \cup B(y,\varepsilon)]$, $\varepsilon > 0$ sufficiently small.

$$0 = \int_V (v \Delta w - w \Delta v)\, dz \underset{\text{Green's formula}}{=} \int_{\partial V} (v \partial_\nu w - w \partial_\nu v)\, dS$$

$$\partial V = \underbrace{\partial U}_{\text{on } \partial U,\ v = w = 0} \sqcup \partial B(x,\varepsilon) \sqcup \partial B(y,\varepsilon).$$

$$\int_{\partial B(x,\varepsilon)} (v \partial_\nu w - w \partial_\nu v)\, dS + \int_{\partial B(y,\varepsilon)} (v \partial_\nu w - w \partial_\nu v)\, dS = 0$$

$\int_{\partial B(x,\varepsilon)} \frac{\partial v}{\partial \nu} w - \frac{\partial w}{\partial \nu} v\, dS(z) = \int_{\partial B(y,\varepsilon)} \frac{\partial w}{\partial \nu} v - \frac{\partial v}{\partial \nu} w\, dS(z)$, ν denoting the inward pointing unit vector field on $\partial B(x,\varepsilon) \cup \partial B(y,\varepsilon)$

$w(z) = G(y,z)$ is harmonic in the neighborhood of $x \Rightarrow w$ is smooth near x.

THEOREM 6 (Smoothness). *If $u \in C(U)$ satisfies the mean-value property (16) for each ball $B(x,r) \subset U$, then*

$$u \in C^\infty(U).$$

$$\left| \int_{\partial B(x,\varepsilon)} \frac{\partial w}{\partial \nu} v\, dS \right| \le \underbrace{\sup_{\partial B(x,\varepsilon)} \left| \frac{\partial w}{\partial \nu} \right|}_{\le C} \cdot \sup_{\partial B(x,\varepsilon)} |v| \cdot \underbrace{O(\varepsilon^{n-1})}_{\text{surface measure}} = o(1) \quad \text{as } \varepsilon \to 0.$$

$v(z) = \Phi(z-x) - \phi^x(z)$ — harmonic, smooth in U

$$\sup_{\partial B(x,\varepsilon)} |v| \lesssim \begin{cases} \varepsilon^{2-n}, & n \ge 3 \\ |\log \varepsilon|, & n = 2. \end{cases}$$

$$\lim_{\varepsilon \to 0} \int_{\partial B(x,\varepsilon)} \frac{\partial v}{\partial \nu} w\, dS = \lim_{\varepsilon \to 0} \int_{\partial B(x,\varepsilon)} \underbrace{-\frac{\partial \phi^x}{\partial \nu}}_{\text{smooth near } x \text{, thus vanish}} w\, dS + \lim_{\varepsilon \to 0} \int_{\partial B(x,\varepsilon)} \frac{\partial \Phi}{\partial \nu}(x-z) w(z)\, dS \quad (★★)$$

Same in Thm 1

$D\Phi(x-z) = \frac{-1}{n\alpha(n)} \frac{z-x}{|x-z|^n}$, $\nu = \frac{x-z}{|x-z|} = \frac{x-z}{\varepsilon}$ on $\partial B(0,\varepsilon)$

$\frac{\partial \Phi}{\partial \nu}(x-z) = \frac{1}{n\alpha(n)\varepsilon^{n-1}}$ on $\partial B(x,\varepsilon)$

$$(★★) = ⨍_{\partial B(x,\varepsilon)} w(z)\, dS(z) \to w(x) \quad \text{as } \varepsilon \to 0.$$

Likewise $\int_{\partial B(y,\varepsilon)} \frac{\partial w}{\partial \nu} v - \frac{\partial v}{\partial \nu} w\, dS(z)$ converges to $v(y)$.

Thus $G(y,x) = w(x) = v(y) = G(x,y)$.

Green's function for a half-space

$\mathbb{R}^n_+ = \{ x = (x_1, \cdots, x_n) \in \mathbb{R}^n \mid x_n > 0 \}$. The region is unbounded

Definition. If $x = (x_1, \ldots, x_{n-1}, x_n) \in \mathbb{R}^n_+$, its reflection in the plane $\partial \mathbb{R}^n_+$ is the point $\tilde{x} = (x_1, \cdots, x_{n-1}, -x_n)$.

Set $\phi^x(y) := \Phi(y - \tilde{x}) = \Phi(y_1 - x_1, \cdots, y_{n-1} - x_{n-1}, y_n + x_n)$ $\quad (x, y \in \mathbb{R}^n_+)$

The idea is that the corrector ϕ^x is built from Φ by "reflecting the singularity" from $x \in \mathbb{R}^n_+$ to $\tilde{x} \notin \mathbb{R}^n_+$.

$\phi^x(y) = \Phi(y - x)$ if $y \in \partial \mathbb{R}^n_+$.

Thus
$$\begin{cases} \Delta \phi^x = 0 & \text{in } \mathbb{R}^n_+ \\ \phi^x = \Phi(y - x) & \text{on } \partial \mathbb{R}^n_+. \end{cases}$$

Definition. Green's function for the half-space $\mathbb{R}^n_+$ is $G(x, y) := \Phi(y - x) - \Phi(y - \tilde{x})$ $\quad (x, y \in \mathbb{R}^n_+, x \neq y)$.

Then $G_{y_n}(x, y) = \Phi_{y_n}(x - y) - \Phi_{y_n}(y - \tilde{x}) = \frac{-1}{n\alpha(n)} \left[\frac{y_n - x_n}{|y - x|^n} - \frac{y_n + x_n}{|y - \tilde{x}|^n} \right]$

$y_n = 0 \quad |y - x| = |y - \tilde{x}|$

If $y \in \partial \mathbb{R}^n_+$, $\frac{\partial G}{\partial \nu}(x, y) = -G_{y_n}(x, y) = -\frac{2x_n}{n\alpha(n)} \frac{1}{|x - y|^n}$, the outward normal on $\partial \mathbb{R}^n_+$ is $\nu = -e_n$

Suppose u solve
$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^n_+ \\ u = g & \text{on } \partial \mathbb{R}^n_+. \end{cases}$$

From Representation formula. $u(x) = -\int_{\partial U} g(y) \frac{\partial G}{\partial \nu}(x, y)\, dS(y) + \int_U f(y) G(x, y)\, dy \quad (x \in U)$.

we expect
$$u(x) = \frac{2x_n}{n\alpha(n)} \int_{\partial \mathbb{R}^n_+} \frac{g(y)}{|x - y|^n}\, dy \quad (x \in \mathbb{R}^n_+)$$ Poisson's formula

$K(x, y) := \frac{2x_n}{n\alpha(n)} \frac{1}{|x - y|^n}$ $\quad (x \in \mathbb{R}^n_+, y \in \partial \mathbb{R}^n_+)$ Poisson's kernel for $\mathbb{R}^n_+$.

smooth function of x

Theorem (Poisson's formula for half-space). Assume $g \in C(\mathbb{R}^{n-1}) \cap L^\infty(\mathbb{R}^{n-1})$, and define $u(x) = \frac{2x_n}{n\alpha(n)} \int_{\partial\mathbb{R}^n_+} \frac{g(y)}{|x-y|^n} dy$ $(x \in \mathbb{R}^n_+)$. Then

$x_n > 0$

(i) $u \in C^\infty(\mathbb{R}^n_+) \cap L^\infty(\mathbb{R}^n_+)$,

(ii) $\Delta u = 0$ in $\mathbb{R}^n_+$,

(iii) $\lim_{\substack{x \to x^0 \\ x \in \mathbb{R}^n_+}} u(x) = g(x^0)$ for each point $x^0 \in \partial\mathbb{R}^n_+$

$g \in L^\infty(\mathbb{R}^{n-1})$

Pf. 1) $u \in L^\infty(\mathbb{R}^n_+)$: $|u(x)| \le \|g\|_\infty \underbrace{\int_{\partial\mathbb{R}^n_+} K(x,y)dy}_{\shortparallel} = \|g\|_\infty$

$\times \int_{\partial\mathbb{R}^n_+} K(x,y)\,dy = \int_{\partial\mathbb{R}^n_+} G_{y_n}(x,y)\,dy \overset{?}{=} \int_{\mathbb{R}^n_+} \Delta_y G(x,y)\,dy = \int_{\mathbb{R}^n_+} \delta_x = 1$

$x = (x', x_n)$

$$\int_{\partial\mathbb{R}^n_+} K(x,y)\,dy = \frac{2x_n}{n\alpha(n)} \int_{\mathbb{R}^{n-1}} \frac{d\eta}{(|\eta-x'|^2+x_n^2)^{n/2}} = \frac{2x_n}{n\alpha(n)} \int_{\mathbb{R}^{n-1}} \frac{d\eta}{(|\eta|^2+x_n^2)^{n/2}} = \frac{2x_n}{n\alpha(n)} \int d\theta \int \frac{r^{n-2}\,dr}{(r^2+x_n^2)^{n/2}} = \frac{2(n-1)\alpha(n-1)}{n\alpha(n)} \int_0^\infty \frac{s^{n-2}}{(1+s^2)^{n/2}} ds$$

write $y = (\eta, 0)$, $\eta \in \mathbb{R}^{n-1}$ translate $\eta \mapsto \eta - x'$ switch to polar coordinate $r = x_n s$

$$\shortparallel$$

$$\frac{2\Gamma(\frac{n}{2})}{\Gamma(\frac{n-1}{2})\sqrt{\pi}} \int_0^\infty \frac{s^{n-2}}{(1+s^2)^{n/2}} ds = 1$$

$$n\alpha(n) = n \cdot \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)} = n \frac{\pi^{\frac{n}{2}}}{\frac{n}{2}\Gamma(\frac{n}{2})} = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}$$

Claim: $\int_0^\infty \frac{s^{n-2}}{(1+s^2)^{\frac{n}{2}}} ds = \frac{\Gamma(\frac{n-1}{2})\sqrt{\pi}}{2\Gamma(\frac{n}{2})}$

$n=2$: $\int_0^\infty \frac{1}{1+s^2} ds \underset{s=\tan\theta}{=} \int_0^{\frac{\pi}{2}} \frac{1}{1+\tan^2\theta} \sec^2\theta\, d\theta = \int_0^{\frac{\pi}{2}} \frac{1}{\cos^2\theta(1+\frac{\sin^2\theta}{\cos^2\theta})} d\theta = \frac{\pi}{2}$

$I := \int_0^\infty \frac{s^{n-2}}{(1+s^2)^{\frac{n}{2}}} ds$ Integration by part

$u = s^{n-3}$, $dv = \frac{s}{(1+s^2)^{\frac{n}{2}}} ds$, then $du = (n-3)s^{n-4} ds$, $v = \int \frac{s}{(1+s^2)^{\frac{n}{2}}} ds = \frac{1}{2}\int (1+s^2)^{-\frac{n}{2}} d(1+s^2) = \frac{(1+s^2)^{1-\frac{n}{2}}}{2-n} = -\frac{1}{n-2}\frac{1}{(1+s^2)^{\frac{n-2}{2}}}$

$$I = [uv]_0^\infty - \int_0^\infty v\,du = \underbrace{\left[-\frac{1}{n-2}\frac{s^{n-3}}{(1+s^2)^{\frac{n-2}{2}}}\right]_0^\infty}_{0} + \frac{n-3}{n-2}\int_0^\infty \frac{s^{n-4}}{(1+s^2)^{\frac{n-2}{2}}} ds$$

By induction hypothesis: $\int_0^\infty \frac{s^{n-4}}{(1+s^2)^{\frac{n-2}{2}}} ds = \frac{\Gamma(\frac{n-3}{2})\sqrt{\pi}}{2\Gamma(\frac{n-2}{2})}$

Thus $\int_0^\infty \frac{s^{n-2}}{(1+s^2)^{\frac{n}{2}}} ds = \frac{\frac{n-3}{2}}{\frac{n-2}{2}} \frac{\Gamma(\frac{n-3}{2})\sqrt{\pi}}{2\Gamma(\frac{n-2}{2})} = \frac{\Gamma(\frac{n-1}{2})\sqrt{\pi}}{2\Gamma(\frac{n}{2})}$

$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt$. $\Gamma(n) = (n-1)!$ $\Gamma(1/2) = \sqrt{\pi}$, $\Gamma(z+1) = z\Gamma(z)$

$$\partial_i u(x) = \lim_{h\to 0} \frac{u(x+e_i h)-u(x)}{h} = \lim_{h\to 0} \frac{\int_{\partial\mathbb{R}_+^n} K(x+e_i h,y)g(y)dy - \int_{\partial\mathbb{R}_+^n} K(x,y)g(y)dy}{h} = \lim_{h\to 0}\int_{\partial\mathbb{R}_+^n} \frac{K(x+e_i h,y)-K(x,y)}{h} g(y)dy = \int_{\partial\mathbb{R}_+^n} \partial_{i,x}K(x,y)g(y)dy.$$

??

$$\int_{\partial\mathbb{R}_+^n} \left|\frac{K(x+he_i,y)-K(x,y)}{h} g(y)\right| dy \le \int_{\partial\mathbb{R}_+^n} \left|\frac{K(x+he_i,y)-K(x,y)}{h}\right| dy \cdot \|g\|_\infty.$$

2.24 The Dominated Convergence Theorem. *Let $\{f_n\}$ be a sequence in L^1 such that (a) $f_n \to f$ a.e., and (b) there exists a nonnegative $g \in L^1$ such that $|f_n| \le g$ a.e. for all n. Then $f \in L^1$ and $\int f = \lim_{n\to\infty}\int f_n$.*

For $i \in \{1,\dots,n-1\}$.

It suffice to argue $\exists\, \varepsilon > 0$ sufficiently small and $\psi(x,y)$ s.t. for all $0<h<\varepsilon$

$\left|\frac{K(x+he_i,y)-K(x,y)}{h}\right| \le \psi(x,y)$ and $\int_{\partial\mathbb{R}_+^n} \psi(x,y)\,dy$ is finite for each fixed $x\in\mathbb{R}_+^n$

$$K_{x_i}(x,y) = \frac{2x_n}{n\alpha(n)}\frac{\partial}{\partial x_i}|x-y|^{-n} = \frac{2x_n}{n\alpha(n)}(-n)\frac{x_i-y_i}{|x-y|^{n+2}} = -\frac{2x_n}{\alpha(n)}\frac{x_i-y_i}{|x-y|^{n+2}}$$ smooth, no singularity.

$$\left|\frac{K(x+he_i,y)-K(x,y)}{h}\right| = \frac{2x_n}{\alpha(n)}\frac{|x_i+\theta_{h,x_i}-y_i|}{|x+\theta_{h,x_i}e_i-y|^{n+2}} \le \frac{2x_n}{\alpha(n)}\frac{1}{|x+\theta_{h,x_i}e_i-y|^{n+1}} \le C(x)\frac{1}{|x-y|^{n+1}}$$

mean value theorem

for $|y|$ sufficiently large

where $\theta_{h,x_i}\in(0,h)$ depends on h and h and x, $C(x)$ constant depends on x.

$$\int_{\partial\mathbb{R}_+^n}\frac{1}{|x-y|^{n+1}}dy < +\infty$$

The partial derivative of u with respect to x_n is computed in a similar manner. For every $0<|h|<\frac{x_n}{2}$ we have

$$\frac{u(x+he_n)-u(x)}{h} = \frac{2}{n\alpha(n)}\int_{\mathbb{R}^{n-1}}\frac{g(y)}{h}\left(\frac{x_n+h}{|x+he_n-y|^n} - \frac{x_n}{|x-y|^n}\right)dy$$

$$\frac{\partial}{\partial x_n}\left(\frac{x_n}{|x-y|^n}\right) = \frac{1}{|x-y|^n} - \frac{nx_n^2}{|x-y|^{n+2}}$$

$$\left|\frac{K(x+he_n,y)-K(x,y)}{h}\right| = \frac{2}{n\alpha(n)}\left|\frac{1}{|x+\theta_{h,x_n}e_n-y|^n} - \frac{n(x_n+\theta_{h,x_n})^2}{|x+\theta_{h,x_n}e_n-y|^{n+2}}\right| \le C'(x)\frac{1}{|x-y|^n}$$

y sufficiently large

$\theta_{h,x_n}\in(0,h)$

Thus, ?? is true.

u inherit the smoothness of K.

To prove the smoothness of u, we can always repeat the process for higher order derivatives. $D^\alpha K$

Just need to prove For each α, $\exists\ C_{\alpha,x}$ s.t. $|\partial_x^\alpha K(x,y)| \le C_{\alpha,x}|x-y|^{-n}$ $(x\in \mathbb{R}^n_+, y\in\partial\mathbb{R}^n_+)$

for large $|y|$

Pf: $K(x,y) = \frac{2x_n}{n\alpha(n)}\frac{1}{|x-y|^n} = C\,x_n \frac{1}{|x-y|^n}$

Leibniz rule

$$\partial_x^\alpha\left[x_n|x-y|^{-n}\right] = x_n\partial_x^\alpha|x-y|^{-n} + \alpha_n\partial_x^{\alpha-e_n}|x-y|^{-n}$$

$$\partial_x^\alpha(f\cdot g) = \sum_{\beta\le\alpha}\binom{\alpha}{\beta}(\partial_x^\beta f)(\partial_x^{\alpha-\beta}g).$$?

where e_n is the unit vector in the x_n-direction, the second term appears only if $\alpha_n\ge 1$.

$\left|\partial_x^\alpha|x-y|^{-n}\right| \le C_\alpha|x-y|^{-n-\alpha}$ (Induction)

$$|\partial_x^\alpha K(x,y)| \le C\left[|x_n|\,|\partial_x^\alpha|x-y|^{-n}| + |\alpha_n|\,|\partial_x^{\alpha-e_n}|x-y|^{-n}|\right] \le C\left[C_\alpha|x_n|\,|x-y|^{-n-|\alpha|} + |\alpha|C_{\alpha-e_n}|x-y|^{-n-|\alpha|+1}\right]$$

$$= |x-y|^{-n}\left[C\,C_\alpha|x_n|\,|x-y|^{-|\alpha|} + |\alpha|C_{\alpha-e_n}|x-y|^{-|\alpha|+1}\right] \qquad |\alpha|\ge 1$$

$\le C_{\alpha,x}|x-y|^{-n}$ for large y.

Thus $u\in C^\infty(\mathbb{R}^n_+)$. (i) is proved.

$(G(x,y) = \Phi(y-x) - \Phi(y-\tilde{x}))$

For each fixed x, the mapping $y\mapsto G(x,y)$ is harmonic, except for $y=x$. As $G(x,y)=G(y,x)$, $x\mapsto G(x,y)$ is harmonic, except for $x=y$.

$$\Delta_x K(x,y) = \Delta_x(-\partial_{y_n}G(x,y)) = -\partial_{y_n}(\Delta_x G(x,y)) = -\partial_{y_n}0 = 0$$

Thus $\Delta_x u(x) = \int_{\partial\mathbb{R}^n_+}\Delta_x K(x,y)g(y)\,dy = 0.$ $(x\in\mathbb{R}^n_+)$ (ii) is proved.

Now fix $x^0\in\partial\mathbb{R}^n_+$, $\varepsilon>0$. Choose $\delta>0$ so small that

$|g(y)-g(x^0)|<\varepsilon$ if $|y-x^0|<\delta$, $y\in\partial\mathbb{R}^n_+$

Then if $|x-x^0|<\frac{\delta}{2}$, $x\in\mathbb{R}^n_+$,

$$|u(x)-g(x^0)| = \left|\int_{\partial\mathbb{R}^n_+}K(x,y)[g(y)-g(x^0)]dy\right| \le \underbrace{\int_{\partial\mathbb{R}^n_+\cap B(x^0,\delta)}K(x,y)|g(y)-g(x^0)|dy}_{I}$$

$$+ \underbrace{\int_{\partial\mathbb{R}^n_+-B(x^0,\delta)}K(x,y)|g(y)-g(x^0)|dy}_{J}$$

$I\le\varepsilon\int_{\partial\mathbb{R}^n_+}K(x,y)\,dy = \varepsilon$

If $|x-x^0|\le\frac{\delta}{2}$ and $|y-x^0|\ge\delta$, we have $|y-x^0|\le|y-x|+\frac{\delta}{2}\le|y-x|+\frac{1}{2}|y-x^0|$; so $|y-x|\ge\frac{1}{2}|y-x^0|$

$$J\le 2\|g\|_\infty\int_{\partial\mathbb{R}^n_+-B(x^0,\delta)}K(x,y)dy = 2\|g\|_\infty\int_{\partial\mathbb{R}^n_+-B(x^0,\delta)}\frac{2x_n}{n\alpha(n)}\frac{1}{|x-y|^n}dy \le \frac{2^{n+2}\|g\|_\infty x_n}{n\alpha(n)}\int_{\partial\mathbb{R}^n_+-B(x^0,\delta)}|y-x^0|^{-n}dy$$

$\to 0$ as $x_n\to 0^+$

Thus $|u(x)-g(x^0)| \le 2\varepsilon$ for $|x-x^0|$ sufficiently small.

Green function for a ball $B(0,1)$

Definition. If $x \in \mathbb{R}^n - \{0\}$, the point $\tilde{x} = \frac{x}{|x|^2}$ is called the point dual to x with respect to $\partial B(0,1)$. The mapping $x \mapsto \tilde{x}$ is inversion through the unit sphere $\partial B(0,1)$

Fix $x \in B^\circ(0,1)$, find corrector $\phi^x = \phi^x(y)$ solving

$$\begin{cases} \Delta \phi^x = 0 & \text{in } B^\circ(0,1) \\ \phi^x = \Phi(y-x) & \text{on } \partial B(0,1) \end{cases}$$

$$G(x,y) = \Phi(y-x) - \phi^x(y)$$

Invert the singularity from $x \in B^\circ(0,1)$ to $\tilde{x} \notin B(0,1)$. Assume for the moment $n \ge 3$. Now the mapping $y \mapsto \Phi(x-\tilde{y})$ is harmonic for $y \neq \tilde{x}$. Thus $y \mapsto |x|^{2-n}\Phi(y-\tilde{x})$ is harmonic for $y \neq \tilde{x}$, and so ~~$\phi^x(y) := \Phi(|x|(y-\tilde{x}))$ is harmonic in U.~~

$\phi^x(y) := \Phi(|x|(y-\tilde{x}))$ is harmonic in U.

If $y \in \partial B(0,1)$ and $x \neq 0$,

$$|x|^2|y-\tilde{x}|^2 = |x|^2\left(|y|^2 - \frac{2y\cdot x}{|x|^2} + \frac{1}{|x|^2}\right) = |x|^2 - 2y\cdot x + 1 = |x-y|^2.$$

Thus $(|x||y-\tilde{x}|)^{-(n-2)} = |x-y|^{-(n-2)}$.

$$\phi^x(y) = \Phi(y-x) \qquad (y \in \partial B(0,1)),$$

as required.

Definition. Green's function for the unit ball is $G(x,y) := \Phi(y-x) - \Phi(|x|(y-\tilde{x}))$ $\quad (x,y \in B(0,1),\ x \neq y)$

Same for $n=2$

Assume now u solves the boundary-value problem $\begin{cases} \Delta u = 0 & \text{in } B^\circ(0,1) \\ u = g & \text{on } \partial B(0,1). \end{cases}$

From Representation formula. $u(x) = -\int_{\partial U} g(y)\frac{\partial G}{\partial \nu}(x,y)\,dS(y) + \int_U f(y)G(x,y)\,dy \quad (x \in U).$

$$u(x) = -\int_{\partial B(0,1)} g(y)\frac{\partial G}{\partial \nu}(x,y)\,dS(y)$$

$G_{y_i}(x,y) = \Phi_{y_i}(y-x) - \Phi(|x|(y-\tilde{x}))_{y_i}$. $\qquad \Phi_{y_i}(y-x) = \frac{1}{n\alpha(n)}\frac{x_i-y_i}{|x-y|^n}$,

If $y \in \partial B(0,1)$, $\quad \Phi(|x|(y-\tilde{x}))_{y_i} = \frac{-1}{n\alpha(n)}\frac{y_i|x|^2-x_i}{(|x||y-\tilde{x}|)^n} = -\frac{1}{n\alpha(n)}\frac{y_i|x|^2-x_i}{|x-y|^n}$

$$\frac{\partial G}{\partial \nu}(x,y) = \sum_{i=1}^{n} y_i G_{y_i}(x,y)$$

$$= \frac{-1}{n\alpha(n)} \frac{1}{|x-y|^n} \sum_{i=1}^{n} y_i\left((y_i - x_i) - y_i|x|^2 + x_i\right)$$ Because $y \in \partial B(0,1)$

$$= \frac{-1}{n\alpha(n)} \frac{1-|x|^2}{|x-y|^n}$$

Thus $u(x) = \frac{1-|x|^2}{n\alpha(n)} \int_{\partial B(0,1)} \frac{g(y)}{|x-y|^n}\, dS(y)$.

$$\begin{cases} \Delta u = 0 & \text{in } B^\circ(0,r) \\ u = g & \text{on } \partial B(0,r) \end{cases} \qquad r>0.$$

Then $\tilde{u}(x) = u(rx)$ in $B(0,1)$ with boundary data $\hat{g}(x) = g(rx)$ solve for case $B(0,1)$

$$u(x) = \tilde{u}\left(\frac{x}{r}\right) = \frac{1-\left|\frac{x}{r}\right|^2}{n\alpha(n)} \int_{\partial B(0,1)} \frac{g(ry)}{\left|\frac{x}{r}-y\right|^n}\, dS(y)$$

$$\overset{y'=ry}{=} \frac{1-\left|\frac{x}{r}\right|^2}{n\alpha(n)} \int_{\partial B(0,r)} \frac{g(y')}{\left|\frac{x}{r}-\frac{y'}{r}\right|^n}\, dS\left(\tfrac{1}{r}y'\right)$$

$$= \frac{1-\left|\frac{x}{r}\right|^2}{n\alpha(n)} \int_{\partial B(0,r)} \frac{g(y')}{|x-y'|^n}\, r\, dS(y')$$

$$= \frac{r^2-|x|^2}{n\alpha(n)r} \int_{\partial B(0,r)} \frac{g(y)}{|x-y|^n}\, dS(y) \quad . \quad x \in B(0,r) \qquad ✡$$

$$K(x,y) := \frac{r^2-|x|^2}{n\alpha(n)r} \frac{1}{|x-y|^n} \qquad (x \in B^\circ(0,r),\ y \in \partial B(0,r))$$ Poisson's kernel for $B(0,r)$

Theorem. Assume $g \in C(\partial B(0,r))$ and define u by ✡. Then

(i) $u \in C^\infty(B^\circ(0,r))$

(ii) $\Delta u = 0$ in $B^\circ(0,r)$

(iii) $\lim_{\substack{x \to x^0 \\ x \in B^\circ(0,r)}} u(x) = g(x^0)$ for each point $x^0 \in \partial B(0,r)$.

$$\binom{\beta}{\gamma} = \prod_{i=1}^{n} \binom{\beta_i}{\gamma_i} = \frac{\beta!}{\gamma!(\beta-\gamma)!}$$

Pf: $u(x) = \frac{r^2-|x|^2}{n\alpha(n)r} \int_{\partial B(0,r)} \frac{g(y)}{|x-y|^n}\, dS(y)$

$$\partial_{x_i} u(x) \overset{?}{=} \int_{\partial B(0,r)} \partial_{x_i} K(x,y)\, g(y)\, dS(y)$$

$$\partial^\beta K(x,y) = \frac{1}{n\alpha(n)r}\, \partial^\beta\left((r^2-|x|^2)\frac{1}{|x-y|^n}\right) = \frac{1}{n\alpha(n)r} \sum_{\gamma \le \beta} \binom{\beta}{\gamma} \partial_x^{\beta-\gamma}(r^2-|x|^2)\, \partial_x^\gamma |x-y|^{-n}$$

What is the definition of $\gamma \le \beta$ $\binom{\beta}{\gamma}$ here?

The derivative of $r^2-|x|^2$ are bounded and vanish for order ≥ 3 $\quad |\partial_x^{\beta-\gamma}(r^2-|x|^2)| \le C_\beta\, \mathbb{I}_{|\beta-\gamma| \le 2}$.

$\underline{|\partial_x^\gamma |x-y|^{-n}| \le C_\gamma |x-y|^{-n-|\gamma|}}$? — By induction

Hence $|\partial^\beta K(x,y)| \le C_\beta |x-y|^{-n}$

$K(x,y) = -\frac{\partial G}{\partial \nu}(x,y)$.

Fix x, $y \to G(x,y)$ harmonic. $G(x,y) = G(y,x)$

$x \to G(x,y)$ harmonic. $K(x,y)$ harmonic.

$\Delta u(x) = \int_{\partial B(0,r)} \Delta_x K(x,y)\, g(y)\, dS(y) = 0$.

$\int_{\partial B(0,r)} K(x,y)\, dS(y) = 1$

$$\begin{cases} \Delta u = 0 & \text{in } B^0(0,r) \\ u = g & \text{on } \partial B(0,r) \end{cases}$$

Imply.

Here Let $g = 1$. By maximum and minimum principle, $u = 1$ in the ball.

Thus $\int_{\partial B(0,r)} K(x,y)\, dS(y) = 1$ for $x \in B^0(0,r)$

Fix $x^0 \in \partial B(0,r)$

Fix $\varepsilon > 0$, choose $\delta > 0$ s.t. $|g(y) - g(x^0)| < \varepsilon$ when $|y - x^0| < \delta$, $y \in \partial B(0,r)$

$$u(x) - g(x^0) = \underbrace{\int_{|y-x^0|<\delta} K(x,y)[g(y) - g(x^0)]\, dS}_{I} + \underbrace{\int_{|y-x^0|\geq\delta} K(x,y)[g(y) - g(x^0)]\, dS}_{J}$$

$|I| \leq \varepsilon \int K = \varepsilon$

If $|x - x^0| \leq \frac{\delta}{2}$ and $|y - x^0| \geq \delta$, we have $|y - x| \geq \frac{1}{2}|y - x^0|$.

$$|J| \leq 2\|g\|_\infty \int_{|y-x^0|\geq\delta} K(x,y)\, dS(y) = 2\|g\|_\infty \int_{|y-x^0|\geq\delta} \frac{r^2 - x^2}{n\alpha(n) r} \cdot \frac{1}{|x-y|^n}$$

$$\leq \frac{2\|g\|_\infty (r^2 - x^2)}{n\alpha(n)\, r} 2^n \int_{|y-x^0|\geq\delta} |y - x^0|^{-n}\, dS(y) \to 0 \quad \text{a.s } x^2 \to r^2$$

$$\Phi(x) = \begin{cases} -\frac{1}{2\pi} \log|x| & (n=2) \\ \frac{1}{n(n-2)\alpha(n)} \frac{1}{|x|^{n-2}} & (n \geq 3) \end{cases} \quad . \quad |x| = \sqrt{x_1^2 + \cdots + x_n^2}$$

$$D\Phi(x) = ?$$

$n=2$, $-\frac{1}{2\pi} \cdot \frac{1}{|x|} \left(\frac{\frac{1}{2}(2x_1+2x_2)}{\sqrt{x_1^2+x_2^2}} \right) = -\frac{1}{2\pi} \cdot \frac{x}{|x|^2}$

$n \geq 3$

$$\frac{1}{n(n-2)\alpha(n)} \cdot (2-n) \frac{1}{|x|^{n-1}} \cdot \frac{x}{\sqrt{x_1^2 + \cdots x_n^2}}$$

$$= -\frac{1}{n\alpha(n)} \frac{x}{|x|^n}$$

$$\alpha(n) = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}-1)}$$

$2-x \qquad \partial B(x,\varepsilon)$

$\Phi\ |2-x|$