Proposition 3.3.7 (Modulus and length of geodesics) *The modulus of a cylinder A of finite modulus is a non-Euclidean invariant: on a cylinder of modulus M, there is a unique simple closed geodesic for its hyperbolic geometry, and the hyperbolic length of this geodesic is π/M.*

a geodesic $\gamma : I \to M$ is simple if it is an embedding on I (injective and without self-crossings).

A closed geodesic is a geodesic: a curve $\gamma : \mathbb{R} \to M$ with some $T>0$ s.t. $\gamma(t+T) = \gamma(t)$ and $\dot\gamma(t+T) = \dot\gamma(t)$ for all t.

Equivalently, it's a smooth map $\gamma : S^1 \to M$ satisfying the geodesic equation everywhere.

A (unit speed) geodesic $\gamma(t)$ on a Riemannian manifold (M, g) satisfies $D_{\dot\gamma}\dot\gamma = 0$.

Definition 3.8.2 (Collar) Let γ be a simple closed geodesic of length l on a hyperbolic surface X. If the δ-neighborhood

$$A_\delta(\gamma) := \{ x \in X \mid d(x, \gamma) < \delta \} \qquad 3.8.2$$

is isometric to the δ-neighborhood of the unique simple closed geodesic on the cylinder of modulus π/l, we say that γ admits a *δ-collar*, or equivalently, that $A_\delta(\gamma)$ is the δ-collar around γ.

The modulus is the conformal invariant that measures how "thin" an annulus (or cylinder) is.

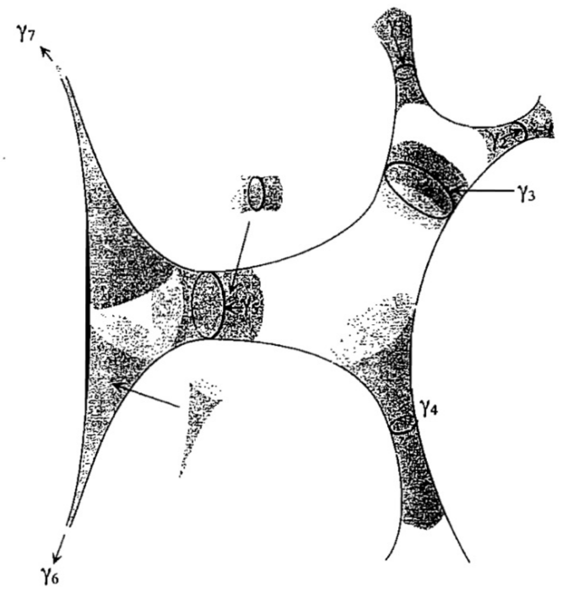


FIGURE 3.8.2. Seven collars (shaded), five around geodesics $\gamma_1, \dots \gamma_5$, the other two around punctures.

We have reproduced, outside the main figure, in miniature, the collars corresponding to γ_5 and γ_6.

In this context **bounded geometry** means: there is a **uniform, positive lower bound on injectivity radius** (and, more generally, uniform C^k bounds on the metric in local charts).

For hyperbolic surfaces the curvature is already fixed $K \equiv -1$, so "bounded geometry" reduces to:

- There exists $\rho_0 > 0$ such that for **every** point x the geodesic ball $B(x, \rho_0)$ is embedded and is uniformly bi-Lipschitz (indeed, isometric) to a standard hyperbolic disk of radius ρ_0.

How it's used here (Mumford/"thin–thick"): remove the explicit **collars** around short simple closed geodesics (the "thin part," whose geometry is completely understood by the Collar Lemma). The **remainder** is the "thick part," on which $\mathrm{inj}(x) \geq \rho_0 = \frac{c}{2}$ for some fixed $c > 0$. Hence that remainder has bounded geometry.

Theorem 3.8.3 (The collaring theorem) *Let X be a complete hyperbolic surface, and let $\Gamma := \{\gamma_1, \gamma_2, \dots\}$ be a (finite or infinite) collection of disjoint simple closed geodesics, each γ_i of length ℓ_i. Then the $A_{\eta(l_i)}(\gamma_i)$ are collars around the γ_i, and they are disjoint.*

Definition 6.4.13 (Teichmüller modular group) The group $\mathbf{QC}^0(S)$ is a normal subgroup of $\mathbf{QC}(S)$. The *mapping class group*, also known as the *Teichmüller modular group*, is the quotient group

$$\mathrm{MCG}(S) = \mathbf{QC}(S)/\mathbf{QC}^0(S). \qquad 6.4.20$$

Clearly $\mathrm{MCG}(S)$ acts on $\mathcal{T}_S$ as in the discussion following Definition 6.4.7.

?

REMARK If S is of finite type, the group $\mathrm{MCG}(S)$ is simply the set of homotopy classes of orientation-preserving homeomorphisms of S that fix the punctures, if any. △

Definition 6.4.14 (Moduli space) The *moduli space* of S, denoted $\mathrm{Moduli}(S)$, is the quotient $\mathcal{T}_S/\mathrm{MCG}(S)$.

When the ideal boundary of X is nonempty, the quasisymmetric homeomorphisms of the ideal boundary form a quotient of $\mathrm{MCG}(X)$, which is then not discrete. But when X has no ideal boundary, and in particular when X is compact or of finite type, the Teichmüller modular group is a discrete group. We will see in Chapter 7 that when X is of finite type and $\mathcal{T}_X$ has dimension > 1, then $\mathrm{MCG}(X)$ is the full group of isomorphisms of $\mathcal{T}_X$ as a complex manifold.

If a radius r geodesic ball $B_X(x,r)$ is not embedded, then there exists a simple closed geodesic of length $<2r$.

" Not embedded " means: $\exists$ two different points of the geodesic disc that map to the same point of X. Equivalently, there are two distinct minimizing geodesic segments

$\sigma_1, \sigma_2 : [0, \ell_i] \to X$, $\sigma_i(0) = x$, $\sigma_i(\ell_i) = y$, $\ell_i < r$.

Lift to the universal cover H^2. in H^2 geodesic between two points ?? are unique, so the lifts of σ_1, σ_2 cannot form a " bigon "; hence on X the interiors of σ_1, σ_2 are disjoint. Therefore the concatenation $\gamma = \sigma_1 \cdot \overline{\sigma_2}$ is a simple closed curve of length $L(\gamma) = \ell_1 + \ell_2 < 2r$.

Point of T_1X: a pain (x,v) where $x \in X$ and $v \in T_xX$ is a unit tangent vector $(\|v\|_g = 1)$

$\pi : R_c(S) \to \mathrm{Moduli}_c(S)$ $\quad (X,P) \mapsto X$ proper ? compact preimage of compact sets.

The fiber over X means

$\pi^{-1}(X) = \{ (X,P) \in R_c(S) : P \text{ is a maximal } c/4\text{-packing on } X \}$

Write $P = \{ \gamma_i : D_{c/4} \hookrightarrow X \}_{i=1}^{k}$ with $k \le 4$ and disjoint images.

Each γ_i is completely determined by:

- its center $x_i = \gamma_i(0) \in X$, and
- a unit tangent direction $v_i = d\gamma_i(0)(\partial x) \in T^1_{x_i}X$.

Because in the hyperbolic disc model, isometries fixing the center are just rotations; prescribing x_i and the derivative at 0 (a unit vector) picks a unique isometry onto the embedded $c/4$-ball in X.

Hence we can encode the fiber by ordered k-tuples $(x_i, v_i)_{i=1}^{k}$ with the property corresponding embedded discs are disjoint and maximal. This gives the inclusion

$\pi^{-1}(X) \subset \bigsqcup_{k \le N} (T_1X)^k$

S compact $\overset{?}{\Rightarrow} X$ compact $\overset{?}{\Rightarrow} T_1X$ compact

X compact, choose a finite trivializing cover $\{U_i\}_{i=1}^{m}$ with bundle charts

$\Phi_i : \pi^{-1}(U_i) \xrightarrow{\cong} U_i \times S^1$.

$T_1X = \bigcup_{i=1}^{m} \pi^{-1}(\overline{U_i})$ is a finite union of compact sets.

K is closed in Moduli$_c$(S) because Moduli$_c$(S) is Hausdorff. (It is the quotient of Teichmüller space — a proper metric space — by a properly discontinuous action of the mapping class group, hence a Hausdorff orbifold)

$K \subset$ Moduli$_c$(S) compact

$\pi^{-1}(K) \subset \bigsqcup_{X\in K} \bigsqcup_{k\in\mathbb{N}} (T_1X)^k$ is a closed subset of a compact space.

π continuous — projection

Disjoint interiors $\Leftrightarrow$ all center distances $\geq \frac{\varepsilon}{\Sigma}$ (continuous inequality) $\Rightarrow$ closed. ?

Embeddedness of each disc is stable under limits $\Rightarrow$ closed

Maximality is closed.

Lemma: compact base + compact fiber $\Rightarrow$ compact total space.

Let $p : E \to B$ be a **locally trivial fiber bundle** with fiber F.
If B is **compact** and F is **compact**, then E is compact.

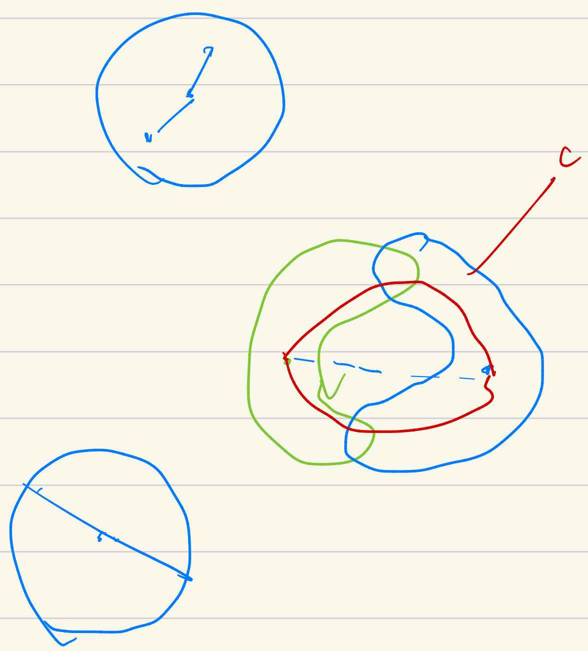
Proof (one paragraph).
Because p is locally trivial, there is an open cover $\{U_i\}$ of B with $p^{-1}(U_i) \cong U_i \times F$. Since B is compact, choose a **finite** subcover $U_1, \ldots, U_m$. Then

$$p^{-1}(\overline{U_i}) \cong \overline{U_i} \times F$$

is compact for each i (product of compact sets). Hence

$$E = p^{-1}(B) = \bigcup_{i=1}^{m} p^{-1}(\overline{U_i})$$

is a **finite union of compact sets**, so it's compact. ∎



?

Fix a nontrivial free homotopy class; it corresponds to a conjugacy $[g] \subset \pi_1(X)$ deck transformation

Lift the loop $c: S^1 \to X$ to a path $\tilde{c}: [0,1] \to \mathbb{H}^2$ with $\tilde{c}(1) = g\tilde{c}(0)$.

in $\mathbb{H}^2$ $\exists!$ geodesic line from $\tilde{c}(0)$ to $g\tilde{c}(0)$.

Project this line to X, get the unique closed geodesic γ in the class

Let $\pi: \mathbb{H}^2 \to X$ be the universal cover.

Fix one lift $\tilde{U}_i \subset \mathbb{H}^2$ of $U_i = B_X(p_i, \frac{c}{2})$. Both U_i and U_j are embedded hyperbolic discs, hence their lifts are embedded hyperbolic discs in $\mathbb{H}^2$, which are geodesic convex

The full preimage of U_j is a disjoint union of its lifts:

$\pi^{-1}(U_j) = \bigsqcup_{g \in \mathrm{Deck}(\pi)} g\tilde{U}_j^{\circ}$;

where $\tilde{U}_j^{\circ}$ is one chosen lift and the deck group acts by isometries.

Suppose $U_i \cap U_j$ has two distinct components, call them C_1 and C_2.

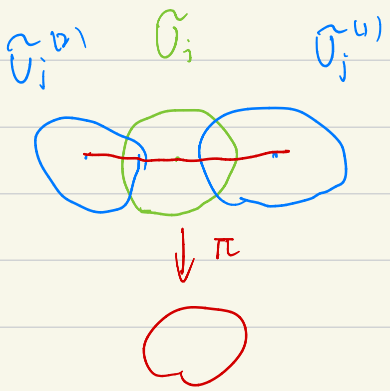
Pick points $x_k \in C_k$ and lift each to $\tilde{x}_k \in \tilde{U}_i$. Each $\tilde{x}_k$ must lie in some lift of U_j; say

$\tilde{x}_1 \in \tilde{U}_j^{(1)}$, $\tilde{x}_2 \in \tilde{U}_j^{(2)}$.

If $\tilde{U}_j^{(1)} = \tilde{U}_j^{(2)}$, then

$\tilde{U}_i \cap \tilde{U}_j^{(1)}$ would contain both $\tilde{x}_1$ and $\tilde{x}_2$. But $\tilde{U}_i$ and $\tilde{U}_j^{(1)}$ are convex discs in $\mathbb{H}^2$, so their intersection is convex, hence connected. Its projection $\pi(\tilde{U}_i \cap \tilde{U}_j^{(1)})$ would then be a single connected component of $U_i \cap U_j$ — contradicting that x_1 and x_2 lie in different components $C_1 \neq C_2$.

Therefore $\tilde{U}_j^{(1)} \neq \tilde{U}_j^{(2)}$



Let $G = \mathrm{Isom}(\mathbb{H}^2)$ and fix a basepoint $o \in \mathbb{H}^2$ (your 0).
Consider the evaluation map

$$\mathrm{ev}_o : G \to \mathbb{H}^2, \qquad g \mapsto g \cdot o.$$

- ev_o is continuous and surjective.
- The stabilizer $K := \{g \in G : g \cdot o = o\} \cong O(2)$ is **compact**.
- ev_o is a (locally trivial) fiber bundle with **compact fiber** K.

Let $A := \{x \in \mathbb{H}^2 : \ d(o, x) \in [c/2, c]\}$ be the closed metric annulus; it is **compact** in $\mathbb{H}^2$. Then

$$M_c \ = \ \{g \in G : \ d(o, g \cdot o) \in [c/2, c]\} \ = \ \mathrm{ev}_o^{-1}(A).$$

Since $\mathrm{ev}_o^{-1}(A) \to A$ is a bundle with compact base A and compact fiber K, the total space $\mathrm{ev}_o^{-1}(A)$ is compact. Hence M_c is compact.

Each (X, Γ) determines:

- an integer $k \in \{0, 1, \ldots, N\}$,
- an incidence set $I \subset \{1, \ldots, k\} \times \{1, \ldots, k\}$ telling which $U_i \cap U_j$ are nonempty (at most one component each), and
- a matrix $(A_{i,j})_{(i,j) \in I} \in \prod_{(i,j) \in I} M_c$.

Thus we get a map $R_c(S) \to Z := \bigsqcup_{k \leq N} \bigsqcup_I \prod_{(i,j) \in I} M_c$. Since M_c is compact and only **finitely many** k, I occur, Z is compact.