We will first define a "quasisymmetric mapping with modulus h" and then show that such mappings and quasiconformal mappings coincide.

Let $\{a_1, a_2, a_3\}$ be a triple of points in $\mathbb{C}$, and define its *skew*:

$$\operatorname{skew}(a_1, a_2, a_3) := \sup \frac{|a_i - a_j|}{|a_i - a_k|}, \qquad 4.5.1$$

where the supremum is taken over the six permutations of the three points. Thus the skew of a triangle is "long side/short side". The skew is smallest, and equal to 1, for an equilateral triangle; it becomes large when two vertices come together. It measures how far from equilateral a triangle is.

Definition 4.5.1 (Quasisymmetric mapping with modulus h) Let $h\colon [1,\infty) \to [1,\infty)$ be a monotone increasing continuous function. A mapping $f: U \to V$ will be called *quasisymmetric with modulus* h if it is a homeomorphism and if

1. any point $u \in U$ has a neighborhood $D_u \subset U$ such that for all triples $\{a, b, c\} \subset D_u$,
$$\operatorname{skew}\big(f(a), f(b), f(c)\big) \le h\big(\operatorname{skew}(a, b, c)\big), \quad \text{and} \qquad 4.5.2$$
2. any point $v \in V$ has a neighborhood $D_v \subset V$ such that for all triples $\{a', b', c'\} \subset D_v$,
$$\operatorname{skew}\big(f^{-1}(a'), f^{-1}(b'), f^{-1}(c')\big) \le h\big(\operatorname{skew}(a', b', c')\big). \qquad 4.5.3$$

Definition 4.5.13 (Labeled quasisymmetry) Let X, Y be metric spaces, and let $\eta\colon [0,\infty) \to [0,\infty)$ be a homeomorphism. A mapping $f: X \to Y$ is *L-quasisymmetric of modulus* η if for any three distinct points $x, y, z \in \cancel{W}$ X we have

metric space version

$$\left|\frac{f(x) - f(y)}{f(x) - f(z)}\right| \le \eta\left(\left|\frac{x - y}{x - z}\right|\right). \qquad 4.5.35$$

f distorts relative distance in a controlled way

$$\frac{d_X(x,y)}{d_X(x,z)} = t$$

只控相对比例变形，不看位置与尺寸

η 只依赖比例 t

图形的"相对形状"在各处、各尺度上一样受控.

The "L" in L-quasisymmetric stands for "labeled". Exchanging the roles of y and z, we see that the inequality is automatically "symmetric":

$$1\Big/\eta\left(\left|\frac{x - y}{x - z}\right|^{-1}\right) \le \left|\frac{f(x) - f(y)}{f(x) - f(z)}\right|. \qquad 4.5.36$$

Definition 4.9.3 ($\mathbb{R}$-quasisymmetry) Let $h\colon \mathbb{R} \to \mathbb{R}$ be a homeomorphism. Then h is $\mathbb{R}$-*quasisymmetric with modulus* M if for all $x \in \mathbb{R}$ and all $t > 0$ it satisfies

$$\frac{1}{M} \le \frac{h(x+t) - h(x)}{h(x) - h(x-t)} \le M. \qquad 4.9.3$$

Proposition 4.5.3 (Easy properties of h-quasisymmetric maps)

1. *Let $f : U \to V$ be quasisymmetric with modulus h_1, and let $g : V \to W$ be quasisymmetric with modulus h_2. Then the composition $(g \circ f) : U \to W$ is quasisymmetric with modulus $(h_2 \circ h_1)$.*
2. *If $f : U \to V$ is quasisymmetric with modulus h, then so is its inverse $f^{-1} : V \to U$.*

Let f be quasisymmetric with modulus η. We want to show $f^{-1}: Y \to X$ is also quasisymmetric.

Take any three distinct points $a, b, c \in Y$. Let:

$x = f^{-1}(a)$, $y = f^{-1}(b)$, $z = f^{-1}(c)$

Then $\dfrac{|f^{-1}(a) - f^{-1}(b)|}{|f^{-1}(a) - f^{-1}(c)|} = \dfrac{|x-y|}{|x-z|}$.

$$\frac{|a-b|}{|a-c|} = \frac{|f(x)-f(y)|}{|f(x)-f(z)|} \leq \eta\left(\frac{|x-y|}{|x-z|}\right)$$

$$\frac{|x-y|}{|x-z|} \geq \eta^{-1}\left(\frac{|a-b|}{|a-c|}\right)$$

since η strictly increasing

$$\frac{|f^{-1}(a) - f^{-1}(b)|}{|f^{-1}(a) - f^{-1}(c)|} \geq \eta^{-1}\left(\frac{|a-b|}{|a-c|}\right)$$

$$\frac{|f^{-1}(a) - f^{-1}(c)|}{|f^{-1}(a) - f^{-1}(b)|} \leq \frac{1}{\eta^{-1}\left(\frac{|a-b|}{|a-c|}\right)}$$

$$\eta'(t) := \frac{1}{\eta^{-1}\left(\frac{1}{t}\right)}$$

$$\eta'\left(\frac{|a-c|}{|a-b|}\right)$$

Exercise

$\gamma_2, \gamma_1 : D \to H$. boundary extension

$\overline{\gamma}_1 : \overline{D} \to \overline{H}$

$\partial D = S^1$ $\quad$ $\partial \overline{H} = \mathbb{R}$

such that for each boundary point $\zeta \in \partial\Omega$, $\lim_{\Omega \ni z \to \zeta} F(z) = \overline{F}(\zeta)$

Ex. Cayley transform. $C(z) = i\frac{1+z}{1-z}$ maps $D \to H$. On the boundary,

for $\zeta = e^{it} \neq 1$

$C(\zeta) = i\frac{1+e^{it}}{1-e^{it}}$

as $\zeta \to 1$ (i.e. $t \to 0$) $C(\zeta) \to \infty$

So the boundary extension is $C: S^1 \setminus \{1\} \to \mathbb{R}$, $C(1) = \infty$.

An analytic isomorphism (biholomorphic) between domains $U, V \subset \mathbb{C}$ is a map

$$F: U \to V$$

that is holomorphic, bijective, and whose inverse is holomorphic.

$F: \mathbb{D} \to H$

$F = \psi \circ C \circ \phi$, $\phi \in \mathrm{Aut}(D)$, $\psi \in \mathrm{Aut}(H)$

Cayley $\quad$ Möbius $\quad$ ⇓ Exercise 2.1.4 Möbius

The boundary extension of γ_1, γ_2 are Möbius maps $S^1 \setminus \{a\} \to \mathbb{R}$ and $S^1 \setminus \{f(a)\} \to \mathbb{R}$

Quasisymmetry is stable under composition.

If h is η-quasisymmetric and ϕ, ψ are quasisymmetric with moduli η_ϕ, η_ψ, then

$\psi \circ h \circ \phi^{-1}$

is quasisymmetric with a modulus that can be chosen as $\eta_{new}(t) = \eta_\psi(\eta(\eta_{\phi^{-1}}(t)))$

$\gamma : S^1 \to \mathbb{R}$ Möbius

we can prove $\gamma = L \circ C \circ A$ where A is a Möbius self-map of S^1.
C is the Cayley map, L is a Möbius self-map of $\overline{\mathbb{R}}$

A has the form:

$$A_{a,\theta}(z) = e^{i\theta} \frac{z-a}{1-\bar{a}z}, \quad |a|<1$$

$$\frac{1-|a|}{1+|a|} \le |A'_{a,\theta}(e^{it})|^{1/2} \le \frac{1+|a|}{1-|a|}$$

A is bi-Lipschits on S^1 $\Rightarrow$ A quasisymmetric

Let $L(x) = \dfrac{ax+b}{cx+d}$ with real a,b,c,d and $ad-bc \ne 0$

Its spherical derivative is continuous and never zero on $\overline{\mathbb{R}}$,
hence bounded above and below.

$$0 < m \le |L'|_{\sigma}(x) \le M < \infty \qquad L'_{\sigma}(x) = \frac{1+x^2}{1+L(x)^2} |L'(x)|$$

Thus biLipschitz.

For C, $C'_{\sigma}(z) = \frac{1}{2}$ for all $z \in S^1$

Setup

Let (X, d_X), (Y, d_Y) be metric spaces and $f : X \to Y$.

- f is **bi-Lipschitz** with constants (ℓ, L) if for all $x, y \in X$

$$\ell\, d_X(x,y) \le d_Y\big(f(x), f(y)\big) \le L\, d_X(x,y), \qquad 0 < \ell \le L < \infty.$$

(When people say "L-bi-Lipschitz" they usually mean $\ell = L^{-1}$.)

- f is **quasisymmetric** if there exists an increasing homeomorphism $\eta : [0,\infty) \to [0,\infty)$ such that for all distinct $x, a, b \in X$,

$$\frac{d_Y(f(x), f(a))}{d_Y(f(x), f(b))} \le \eta\left(\frac{d_X(x,a)}{d_X(x,b)}\right).$$

Proof (one line)

Assume $d_X(x,a) \le t\, d_X(x,b)$. Then

$$\frac{d_Y(f(x), f(a))}{d_Y(f(x), f(b))} \le \frac{L\, d_X(x,a)}{\ell\, d_X(x,b)} \le \frac{L}{\ell} t.$$

So f is quasisymmetric with the **linear control**

$$\boxed{\eta(t) = \frac{L}{\ell} t}.$$

In the common symmetric case $\ell = L^{-1}$, this is $\eta(t) = L^2 t$.

(1) $\Rightarrow$ (2)

$f: S^1 \to S^1$ is quasisymmetric with modulus η.

$g = r_2 \circ f \circ r_1^{-1}$ is also quasisymmetric on $\mathbb{R}$ with some modulus $\eta_{\mathbb{R}}$ depending only on η. and the fixed C.

Thus

$$\frac{g(x+t)-g(x)}{g(x)-g(x-t)} \leq \eta_{\mathbb{R}}(1) \quad \text{and} \quad \frac{g(x)-g(x-t)}{g(x+t)-g(x)} \leq \eta_{\mathbb{R}}(1)$$

$$\Downarrow$$

$$\frac{g(x+t)-g(x)}{g(x)-g(x-t)} \geq \frac{1}{\eta_{\mathbb{R}}(1)}$$

Take $M := \eta_{\mathbb{R}}(1)$

(2) $\Rightarrow$ (1)

$$\frac{1}{M} \leq \frac{g(x+t)-g(x)}{g(x)-g(x+t)} \leq M \quad \text{for all } x \in \mathbb{R},\ t > 0$$

This implies g is quasisymmetric on $\mathbb{R}$ with modulus $\eta_{\mathbb{R}}$ depending only on M.

g increasing or decreasing because g is homeomorphism between $\mathbb{R}$

$\Downarrow$

Beurling - Ahlfors condition

From Beurling–Ahlfors to Quasisymmetry

We work on $\mathbb{R}$ with an increasing homeomorphism h. Set the pushforward measure $\mu := h_{\#}\mathcal{L}^1$; for any interval I, $\mu(I) = |h(I)|$. The Beurling–Ahlfors condition says that for all $x \in \mathbb{R}$, $t > 0$,

$$\frac{1}{M} \le \frac{\mu([x, x+t])}{\mu([x-t, x])} \le M. \tag{BA}$$

We need to show: for all $x \in \mathbb{R}$, $s, t > 0$,

$$\frac{\mu([x, x+s])}{\mu([x-t, x])} \le \eta\left(\frac{s}{t}\right)$$

for some increasing homeomorphism $\eta : [0, \infty) \to [0, \infty)$ depending only on M.

Proof

Write $B(x, r) := [x - r, x + r]$.

1) Doubling (grow radii by 2).

From (BA),

$$\mu(B(x, 2r)) \le (M+1)\,\mu(B(x, r)) \quad \text{for all } x, r > 0.$$

Reason: Decompose $B(x, 2r)$ into four adjacent length-r pieces and compare the two outer ones to the inner ones using (BA).

Iterating,

$$\mu(B(x, 2^k r)) \le (M+1)^k\,\mu(B(x, r)).$$

Equivalently, for $\lambda \ge 1$,

$$\mu(B(x, \lambda r)) \lesssim_M \lambda^{\beta(M)}\,\mu(B(x, r)), \quad \beta(M) := \log_2(M+1).$$

2) Halving (shrink by 2).

Again from (BA),

$$\frac{\mu([x, x+r])}{\mu(B(x, r))} \le \frac{M}{M+1} =: \theta < 1.$$

Applying this at scales $r, r/2, r/4, \ldots$ gives, for all integers $k \ge 0$,

$$\mu([x, x + 2^{-k} r]) \le \theta^k\,\mu(B(x, r)).$$

If $0 < \lambda \le 1$, choose k with $2^{-(k+1)} < \lambda \le 2^{-k}$. Then

$$\mu([x, x + \lambda r]) \lesssim_M \lambda^{\alpha(M)}\,\mu(B(x, r)), \quad \alpha(M) := -\log_2 \theta = \log_2\left(1 + \frac{1}{M}\right).$$

Finally, compare symmetric vs one-sided using (BA):

$$\mu(B(x, r)) = \mu([x - r, x]) + \mu([x, x + r]) \le (M+1)\,\mu([x - r, x]).$$

Assemble the quasisymmetry bound

Putting the pieces together, with $r = t$ and $s = \lambda t$:

If $0 < \lambda \le 1$,

$$\frac{\mu([x, x+s])}{\mu([x-t, x])} \le (M+1) \cdot \frac{\mu([x, x+s])}{\mu(B(x,t))} \lesssim_M \lambda^{\alpha(M)}.$$

If $\lambda \ge 1$,

$$\frac{\mu([x, x+s])}{\mu([x-t, x])} \le \frac{\mu(B(x,s))}{\mu([x-t, x])} \le (M+1)\frac{\mu(B(x,s))}{\mu(B(x,t))} \lesssim_M \lambda^{\beta(M)}.$$

Translating back to h,

$$\frac{h(x+s) - h(x)}{h(x) - h(x-t)} \le \eta_M\left(\frac{s}{t}\right),$$

with the explicit control function

$$\eta_M(\lambda) = C(M) \times \begin{cases} \lambda^{\alpha(M)}, & 0 \le \lambda \le 1, \\ \lambda^{\beta(M)}, & \lambda \ge 1, \end{cases} \qquad \alpha(M) = \log_2\left(1 + \frac{1}{M}\right), \quad \beta(M) = \log_2(M+1)$$

where one can take $C(M) = (M+1)^2$. This η_M is increasing, continuous, $\eta_M(0) = 0$, and depends only on M. Hence h is η_M-quasisymmetric.

Definition 4.1.1 (Quasiconformal map: first analytic definition) Let U, V be open subsets of $\mathbb{C}$, take $K \geq 1$, and set $k := (K-1)/(K+1)$, so that $0 \leq k < 1$. A mapping $f: U \to V$ is *K-quasiconformal* if it is a homeomorphism whose distributional partial derivatives are in L^2_{loc} (locally in L^2) and satisfy

$$\left|\frac{\partial f}{\partial \bar{z}}\right| \leq k \left|\frac{\partial f}{\partial z}\right| \qquad 4.1.1$$

in L^2_{loc}, i.e., almost everywhere.

Definition 4.7.3 ($\mathbf{QC}(U, V)$ We denote by $\mathbf{QC}(U, V)$ the set of quasiconformal maps from U to V, with the topology of uniform convergence on compact subsets. The subset $\mathbf{QC}_K(U, V)$ consists of those mappings $f \in \mathbf{QC}(U, v)$ that are K-quasiconformal.

Proposition 4.5.14 *Let U, V be open subsets of $\mathbb{C}$. A homeomorphism $f: U \to V$ is K-quasiconformal if and only if f is L-quasisymmetric with some modulus η depending only on K.*

Theorem 4.5.4 (Quasisymmetric maps and quasiconformal maps coincide) *Let U, $V \subset \mathbb{C}$ be open. If a homeomorphism $f: U \to V$ is quasisymmetric with modulus h, then it is K-quasiconformal, where K depends only on h. Conversely, if it is K-quasiconformal, then it is quasisymmetric with modulus h, where h depends on K.*

Barycenter

(1)

$\exists!$ $\mu \mapsto \tilde{\xi}_\mu$ with

① $\xi_\mu(0) = \int_{S^1} \zeta \, \mu(d\zeta)$ " average unit vector " on the boundary

② $\tilde{\xi}_{\gamma_* \mu} = \gamma_* \tilde{\xi}_\mu$ for all $\gamma \in \mathrm{Aut}\, D$

Why 5.1.4 holds: for rotations $\gamma(z) = e^{i\theta} z$

$\gamma_* \tilde{\xi}_\mu(0) = D\gamma(0)\, \tilde{\xi}_\mu(0) = e^{i\theta} \int \zeta \, \mu(d\zeta)$.

$\tilde{\xi}_{\gamma_* \mu}(0) = \int_{S^1} \zeta (\gamma_* \mu)(d\zeta) = \underbrace{\int_{S^1} (\zeta \circ \gamma)(\eta)\, \mu(d\eta) = \int_{S^1} \gamma(\eta)\, \mu(d\eta)}$

$= \int_{S^1} e^{i\theta} \eta \, \mu(d\eta)$

Think of ζ as the identy / inclusion map $\zeta : S^1 \to \mathbb{C}$. $\zeta(x) = x$

Using definition $\tilde{\xi}_\mu(a) := [D\gamma(a)]^{-1} \tilde{\xi}_{\gamma_* \mu}(0)$. compute with γ_1

$D(\delta \circ \gamma)(a) = D\delta(\gamma(a))\, D\gamma(a) = D\delta(0)\, D\gamma(a)$

$\tilde{\xi}_{\gamma_{1*}\mu}(0) \underset{|}{=} \tilde{\xi}_{\delta_*(\gamma_* \mu)}(0) = \delta_* \tilde{\xi}_{(\gamma_* \mu)}(0) = [D\delta(0)]\, \tilde{\xi}_{(\gamma_* \mu)}(0)$

$(\delta \circ \gamma)_* \mu(A) = \mu((\delta \circ \gamma)^{-1}(A)) = \mu(\gamma^{-1}(\delta^{-1}(A))) = (\gamma_* \mu)(\delta^{-1}(A)) = \delta_*(\gamma_* \mu)(A)$

Thus $(\delta \circ \gamma)_* \mu = \delta_*(\gamma_* \mu)$

Thus $[D\gamma_1(a)]^{-1} \tilde{\xi}_{\gamma_{1*}\mu}(0) = [D\gamma(a)]^{-1} \tilde{\xi}_{\gamma_* \mu}(0)$

Well - defined .

$D\phi_a(a) = \frac{1}{1-|a|^2}$

$\tilde{\xi}_\mu(a) = (1-|a|^2) \underbrace{\int_{S^1} \zeta \, d(\phi_{a*}\mu)(\zeta)}$

$\quad ''$

$\int_{S^1} \frac{\zeta - a}{1 - \bar{a}\zeta} \mu(d\zeta)$

Uniqueness argument:

Fix $a \in D$ and choose any $\gamma \in Aut(D)$ with $\gamma(a) = 0$.

$\vec{\xi}_{\gamma_*\mu}(0) = (\gamma_* \vec{\xi}_{\mu})(0) = D\gamma(a)\vec{\xi}_{\mu}(a)$.

so

$$\vec{\xi}_{\mu}(a) = [D\gamma(a)]^{-1} \vec{\xi}_{\gamma_*\mu}(0).$$

Note that if $U, V \subset \mathbb{C}$ are open and $f: U \to V$ is a continuous map whose distributional derivatives are locally in L^2, then

$$\operatorname{Jac} f = \left|\frac{\partial f}{\partial z}\right|^2 - \left|\frac{\partial f}{\partial \bar{z}}\right|^2 \quad \text{and} \quad \|[Df]\|^2 = \left(\left|\frac{\partial f}{\partial z}\right| + \left|\frac{\partial f}{\partial \bar{z}}\right|\right)^2 \qquad 4.1.10$$

are locally in L^1.

Thus Definition 4.1.1 can be restated as follows:

Definition 4.1.5 (Quasiconformal map: 2nd analytic definition) Let U, V be open subsets of $\mathbb{C}$ and take $K \geq 1$. A map $f: U \to V$ is *K-quasiconformal* if

1. it is a homeomorphism,
2. its distributional partial derivatives are locally in L^2, and
3. its distributional partial derivatives satisfy

$$\operatorname{Jac} f \geq \frac{1}{K}\|[Df]\|^2 \quad \text{locally in } L^1. \qquad 4.1.11$$

(2)

$$\tilde{\xi}_\mu(z) = (1-|z|^2)\int_{S^1} \frac{\zeta - z}{1-\bar{z}\zeta}\,\mu(d\zeta)$$

Expand the fraction at $z=0$:

$$\frac{\zeta - z}{1-\bar{z}\zeta} = (\zeta - z)(1+\bar{z}\zeta + (\bar{z}\zeta)^2 + \cdots) = \zeta - z + \bar{z}\zeta^2 + o(|z|)$$

$$\tilde{\xi}_\mu(z) = \tilde{\xi}_\mu(0) - \bar{z} + z\int_{S^1} \zeta^2\,\mu(d\zeta) + o(|z|)$$

$$\frac{\partial \tilde{\xi}_\mu}{\partial \bar{z}}(0) = -1, \quad \frac{\partial \tilde{\xi}_\mu}{\partial z}(0) = \int_{S^1} \zeta^2\,\mu(d\zeta)$$

For a C^1 complex vector field F, the Jacobian is $J_F = |F_z|^2 - |F_{\bar{z}}|^2$

Thus $J_{\tilde{\xi}_\mu}(0) = 1 - \left|\int_{S^1} \zeta^2\mu(d\zeta)\right|^2 = 1 - \int \zeta_1^2\,\mu(d\zeta_1)\overline{\int \zeta_2^2\,\mu(d\zeta_2)}$

Fubini

$$= 1 - \iint \zeta_1^2\overline{\zeta_2^2}\,\mu(d\zeta_1)\,\mu(d\zeta_2) = \iint (1-\operatorname{Re}(\zeta_1^2\overline{\zeta_2^2}))\,\mu(d\zeta_1)\,\mu(d\zeta_2)$$

Use the elementary identity: $|a-b|^2 = |a|^2 + |b|^2 - 2\operatorname{Re}(a\bar{b})$

$$= \frac{1}{2}\int_{S^1\times S^1} |\zeta_1^2 - \zeta_2^2|^2$$

μ has no atoms, $J > 0$.

$\vec{\xi}_\mu : D \to \mathbb{R}^2$

If $\vec{\xi}_\mu(a) = 0$ and $J(a) = \det D\vec{\xi}_\mu(a) \neq 0$, then $\exists\ U \ni a$ s.t. $\xi_\mu|_U : U \to \xi_\mu(U)$ is a diffeomorphism.

$\xi_\mu^{-1}(\{0\}) \cap U = \{a\}$. the zero is isolated.

Let $V : \mathbb{R}^2 \to \mathbb{R}^2$ be a C^1 vector field with an isolated zero at a.

Take a tiny circle $C_\varepsilon = \{x : |x - a| = \varepsilon\}$.

$$n_\varepsilon : C_\varepsilon \longrightarrow S^1, \quad x \mapsto \frac{V(x)}{\|V(x)\|}.$$

The **index** $\mathrm{ind}_a(V)$ is the **degree** of n_ε (how many times the vector $V(x)$ turns as you go once around the circle). It's independent of small ε.

Equivalent characterizations for a continuous $f : S^1 \to S^1$: Fundamental group: $f_* : \pi_1(S^1) \cong \mathbb{Z} \to \mathbb{Z}$ is multiplication by an integer k; $k = \deg f$.

The Jacobian determinant $J(a) = \det DV(a) \neq 0$. On a tiny neighborhood,

$$V(x) = DV(a)(x - a) + o(|x - a|),$$

so V is homotopic (through nonvanishing fields on C_ε) to the **linear** field $x \mapsto DV(a)(x - a)$. Hence

$$\mathrm{ind}_a(V) = \deg\left(x \in S^1 \mapsto \frac{DV(a)\,x}{\|DV(a)\,x\|}\right).$$

For a linear map $A \in GL(2, \mathbb{R})$,

$$\deg\left(x \mapsto \frac{Ax}{\|Ax\|}\right) = \mathrm{sgn}(\det A) = \begin{cases} +1, & \det A > 0 \\ -1, & \det A < 0 \end{cases}$$

.

Geometrically: if $\det A > 0$, the arrow rotates once counterclockwise as you go once around $C_\varepsilon \Rightarrow$ index $+1$; if $\det A < 0$, it rotates once clockwise $\Rightarrow$ index -1.

In our complex notation, the "Jacobian"

$$J = |V_z|^2 - |V_{\bar{z}}|^2$$

equals $\det DV$. So $J > 0$ means the linearization preserves orientation, and the isolated (nondegenerate) zero has **index** $+1$.

Let M be a differentiable manifold, of dimension n, and v a vector field on M. Suppose that x is an isolated zero of v, and fix some local coordinates near x. Pick a closed ball D centered at x, so that x is the only zero of v in D. Then the index of v at x, $\mathrm{index}_x(v)$, can be defined as the degree of the map $u : \partial D \to \mathbb{S}^{n-1}$ from the boundary of D to the $(n-1)$-sphere given by $u(z) = v(z)/\|v(z)\|$.

Theorem Let M be a compact differentiable manifold. Let v be a vector field on M with isolated zeroes. If M has boundary, then we insist that v be pointing in the outward normal direction along the boundary. Then we have the formula

$$\sum_i \mathrm{index}_{x_i}(v) = \chi(M)$$

where the sum of the indices is over all the isolated zeroes of v and $\chi(M)$ is the Euler characteristic of M.

Prop 5.1.7

Lemma 5.1.10

Set $F := u + iv : \overline{D} \to \mathbb{C}$ $\qquad F|_{S^1} = f$

$$J_F = \det \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = |F_z|^2 - |F_{\bar{z}}|^2 = |c_1|^2 - |c_{-1}|^2$$

Fix real number a, b (not both 0),

$\phi_{a,b} := au + bv = \mathrm{Re}((a - ib)F)$.

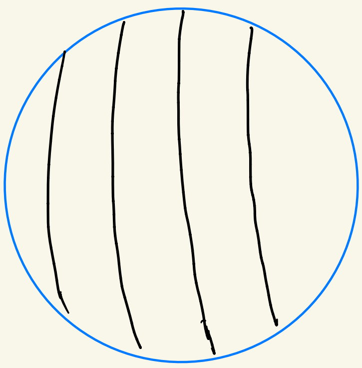
The level set $\{\phi_{a,b} = t\}$ can't contain a closed loop inside D by maximum principle.

$f : S^1 \to S^1$ is a degree $+1$ homeomorphism. Then there is a strictly increasing, 2π-periodic lift

$f(e^{i\theta}) = e^{i\psi(\theta)}$ with $\psi(\theta + 2\pi) = \psi(\theta) + 2\pi$.

Fix $a, b \in \mathbb{R}$ and set $\alpha = \arg(a - ib)$. $|a - ib| = \rho > 0$. On the boundary, $\phi_{a,b}(e^{i\theta}) = \rho\cos(\psi(\theta) - \alpha)$

For any regular value $|t| < |(a - ib)|$, the equation $\phi_{a,b} = t$ on S^1 has exactly two solutions.



Thus there is no trivial linear combination $aDu + bDv$ vanishes anywhere in $D \Rightarrow$ Du, Dv linearly independent everywhere $\Rightarrow$ $J_F = \det(Du, Dv) \neq 0$

Given $a \in D$, choose $\gamma_2 \in \mathrm{Aut}(D)$ with $\gamma_2(0) = a$.

Then $\gamma_2^{-1}(a) = 0$. Next choose $\gamma_1 \in \mathrm{Aut}(D)$ with $\gamma_1(\hat{f}(a)) = 0$.

Let $\tilde{F} := \widehat{\gamma_1 \circ f \circ \gamma_2}$. $\tilde{F} = \gamma_1 \circ \hat{f} \circ \gamma_2$

$$D\tilde{F}(0) = D\gamma_1(\hat{f}(a))\, D\hat{f}(a)\, D\gamma_2(0)$$

For a holomorphic map $\phi = u + iv : \mathbb{C} \to \mathbb{C}$ (e.g., a disk automorphism)

$$D\phi = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}.$$

Holomorphicity gives $u_x = v_y$, $u_y = -v_x$.

$$\det D\phi = u_x^2 + u_y^2 = (u_x + iv_x)(u_x - iv_x) = |\phi'(z)|^2$$

Thus $\det D\tilde{F}(0) > 0 \iff \det D\hat{f}(a) > 0$.

By the argument in the book, we know $\hat{f}$ is a local diffeomorphism.

We still need to prove $\hat{f}$ is bijective.

$\hat{f}$ local diffeomorphism $\Rightarrow$ $\hat{f}$ open map

$\hat{f}$ extends continuously on $\bar{D}$ with $\hat{f}|_{S^1} = f$. For any compact $K \subset D$, the preimage $(\hat{f}|_D)^{-1}(K) = \hat{f}^{-1}(K)$ is closed in the compact set $\bar{D}$ and disjoint from S^1, hence compact in D.

Thus $\hat{f}|_D$ proper; proper maps into Hausdorff space have closed images.

The image $\hat{f}(D)$ is open and closed in D. (nonempty) $\Rightarrow$ $\hat{f}(D) = D$

A proper local homeomorphism $\hat{f} : D \to D$ is a covering map. Any covering of the simply connected disk by the disk must be trivial, so a proper local homeomorphism $D \to D$ is a homeomorphism $\Rightarrow$ injective.

Prop 5.1.11

Corollary 4.9.7 *The space of homeomorphisms* $h:\mathbb{R} \to \mathbb{R}$ *that are* $\mathbb{R}$*-quasisymmetric with modulus* M *and satisfy* $h(0) = 0$, $h(1) = 1$ *is compact for the topology of uniform convergence on* $\overline{\mathbb{R}}$.

Given any $z \in \mathbb{D}$, pick γ_2 with $\gamma_2(0) = z$ and γ_1 with $\gamma_1\Big(\hat{f}(z)\Big) = 0$. Differentiating, and using that $D\gamma_j$ are complex scalars, we get

$$\left|\frac{\hat{f}_{\bar{z}}}{\hat{f}_z}(z)\right| = \left|\frac{\big(\widehat{\gamma_1 \circ f \circ \gamma_2}\big)_{\bar{z}}(0)}{\big(\widehat{\gamma_1 \circ f \circ \gamma_2}\big)_z(0)}\right|.$$

Thus a uniform bound at 0 for normalized maps yields a uniform bound at every point for arbitrary f.

By the normal-family theorem for quasisymmetric homeomorphisms (your Cor. 4.9.7), the set

$$QS^0_M(S^1) = \{f \in QS_M(S^1) : f(1) = 1,\ f(i) = i,\ f(-1) = -1\}$$

is compact in the uniform topology on S^1.

From Prop. 5.1.6: $f \mapsto \hat{f}$ is continuous (uniformly on $\mathbb{D}$), and on $\hat{f}^{-1}(\mathbb{D}) = \mathbb{D}$ the map is real-analytic. Moreover, the partial derivatives $\hat{f}_z(0)$ and $\hat{f}_{\bar{z}}(0)$ depend continuously on f (differentiate the implicit integral formula under the integral; we used this earlier to compute the Jacobian at 0).

Hence the function

$$G(f) := \left|\frac{\hat{f}_{\bar{z}}(0)}{\hat{f}_z(0)}\right|$$

is a continuous function on the compact set $QS^0_M(S^1)$. Therefore G attains a maximum

$$\beta(M) := \max_{f \in QS^0_M(S^1)} G(f).$$

For every boundary homeomorphism f, we previously showed (via the Jacobian computation)

$$|\hat{f}_{\bar{z}}(0)| < |\hat{f}_z(0)|,$$

equivalently $J_f(0) = |\hat{f}_z(0)|^2 - |\hat{f}_{\bar{z}}(0)|^2 > 0$.

Thus $G(f) < 1$ for each f. Since G is continuous and strictly < 1 pointwise on a compact set, the maximum satisfies

$$\beta(M) = \max_{QS^0_M} G(f) < 1.$$

$$|\mu_f(z)| := \left|\frac{\hat{f}_{\bar{z}}}{\hat{f}_z}(z)\right| \le \beta(M) \quad \text{for all } z \in \mathbb{D}.$$

Hence $\hat{f}$ is K-quasiconformal with

$$K = \frac{1 + \|\mu_f\|_\infty}{1 - \|\mu_f\|_\infty} \le \frac{1 + \beta(M)}{1 - \beta(M)} =: \alpha(M).$$