Trousers are building blocks for general hyperbolic surface

Define a *hyperbolic surface with geodesic boundary* to be an orientable surface with boundary such that every interior point has a neighborhood isometric to an open subset of $\mathbf{H}$ (with its hyperbolic metric), and every boundary point has a neighborhood isometric to a neighborhood of a purely imaginary number in the part of $\mathbf{H}$ where $\operatorname{Re} z \geq 0$.

Definition 3.5.1 (Trouser) A *trouser* is a complete hyperbolic surface with geodesic boundary, whose interior is homeomorphic to the complement of three points in the 2-sphere.

simple closed geodesic

loops has no self-intersection. curves come back to where it started

Proposition 3.5.2 *Let X be a compact connected hyperbolic surface with geodesic boundary. If all simple closed geodesics of X are boundary components, then X is homeomorphic to a trouser.*

Hopf - Rinow Thm.

Let (M, g) be a connected Riemannian manifold, and let d be the distance function induced by g. The following conditions are equivalent:

1. (M, d) is a complete metric space. (Cauchy sequence converges)
2. (M, g) is geodesically complete. (every geodesic can be extended to be defined on all of $\mathbb{R}$)
3. Every closed and bounded subset of M is compact.
4. For any two points $p, q \in M$, $\exists$ a minimizing geodesic from p to q; that is, a geodesic whose length equals the length $d(p, q)$.

In particular, any compact Riemannian manifold is complete (and hence geodesic complete).

在一个双曲曲面里，如果没有"内部的"本质简单测地线可以再切了，那它的拓扑只能是裤子.

PROOF Suppose first that X has at least two distinct boundary components, A and B. These boundary components are homeomorphic to circles, since they are compact 1-dimensional manifolds, and since X is connected, there is a simple arc C joining A to B. Let U be a small neighborhood of $A \cup B \cup C$ with smooth boundary, and let $D := \partial U$. Then D is a simple closed curve in X, and it is not homotopic to A or B. Indeed, C is an element of $H_1(X, A \cup B)$, and any curve in $\mathring{X}$ homotopic to A or to B must have algebraic intersection number 1 with this class, whereas D has intersection number 0. So D is homotopic to some third boundary component E. The component of $X - D$ containing E is homeomorphic to an annulus, so X is homeomorphic to a 3-times punctured sphere.

compact 1-dimensional manifold is a finite union of circles.

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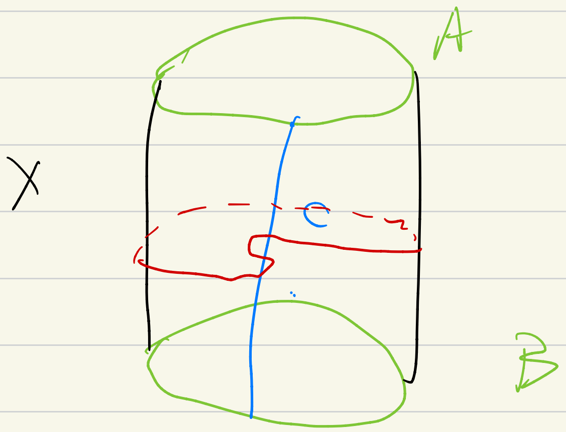


(thickening)

U: neighborhood of $A \cup B \cup C$.

$D = \partial U$.

D is not homotopic to A or to B

Let S be an oriented and α, β two closed curves in general position. For each intersection point $p \in \alpha \cap \beta$, define $\text{sign}_p(\alpha, \beta) = +1$ if the ordered pair of tangent vectors (v_α, v_β) at p agrees with the orientation of S, and -1 otherwise

看有序对 (v_α, v_β) 是否给出了曲面在p处的正向基.

The algebraic intersection number of α and β is

$$I(\alpha, \beta) = \sum_{p \in \alpha \cap \beta} \text{sign}_p(\alpha, \beta).$$

This depends only on the homology classes $[\alpha], [\beta] \in H_1(S; \mathbb{Z})$, and thus defines a bilinear pairing

$$I: H_1(S; \mathbb{Z}) \times H_1(S; \mathbb{Z}) \to \mathbb{Z}.$$?

成对生灭

If a closed curve is homotopic to A or to B, it wraps once around that boundary circle. Geometrically, any such loop must cross the arc C exactly once. Its algebraic intersection number with $[C]$ is 1 ??

D: 0

Intersection number is invariant under homotopy. D is not homotopic to A or B

The curve D is not null-homotopic in X.

Proof. Suppose for contradiction that D is null-homotopic in X. Since X is a surface, this means that D bounds an embedded disc $\Delta \subset X$ with $\partial\Delta = D$. Consider the subsurface

$$Y := U \cup \Delta \subset X.$$

Y has two boundary components, namely A and B. Moreover, the Euler characteristic of Y is

$$\chi(Y) = \chi(U) + \chi(\Delta) - \chi(D) = (-1) + 1 - 0 = 0,$$

so Y is an annulus.

Now restrict the hyperbolic metric on X to Y. The Gaussian curvature is still $K \equiv -1$ on the interior of Y, and the boundary curves A and B are geodesics in X, hence also in Y, so the geodesic curvature along ∂Y is $k_g \equiv 0$. By the Gauss–Bonnet theorem, we obtain

$$\int_Y K\, dA + \int_{\partial Y} k_g\, ds = 2\pi\, \chi(Y) = 0.$$

However, since $K = -1$ and $\mathrm{area}(Y) > 0$, we have $\int_Y K\, dA < 0$, while $\int_{\partial Y} k_g\, ds = 0$. Thus the left-hand side is strictly negative, which contradicts the equality above. Therefore our assumption that D is null-homotopic must be false, and D is nontrivial. □

P in U is non-null-homotopic
trouser (P boundary)

$i: U \hookrightarrow X \Rightarrow i_*: \pi_1(U) \to \pi_1(X)$ injective (facts of 2-manifold)

X has no cusps

D essential simple closed curve

Thus we need only worry about the cases where X has either no boundary, or only one boundary component. If X has no boundary, it is a compact Riemann surface with trivial fundamental group, hence it is homeomorphic to the sphere, hence not hyperbolic.

$\Rightarrow$ Don't have essential simple closed curve $\Rightarrow \pi_1(X)$ trivial

It is more or less obvious that the case where X has exactly one boundary component A cannot occur either. Indeed, the one boundary component A must then bound X, and so X is simply connected, so its interior is isomorphic to a disc by the uniformization theorem (but the hyperbolic structure is not the hyperbolic structure of the disc, since the boundary is geodesic, not at infinity). Apply the Gauss-Bonnet formula

$$\int_X K\,dS + \int_A k\,ds = 2\pi\chi(X). \qquad 3.5.1$$

The first integral gives the negative of the area of X, the second gives 0 since A is geodesic. Since $2\pi\chi(X) = 2\pi$, this is a contradiction, so X cannot have a single boundary component either. □

$K \equiv -1 \quad \int_X K\,dS = \int_X (-1)\,dS = -\mathrm{Area}(X)$

boundary geodesic $\Rightarrow k_g = 0 \Rightarrow \int_A k\,ds = 0.$

$2\pi\chi(X) = 2\pi$
disk

Theorem 3.6.2 *Let X be a connected hyperbolic Riemann surface that is not simply connected, with its hyperbolic metric. Then there exists a multicurve Y on X such that if $\overline{Z}$ denotes the closure of*

$$Z := \{ x \in \gamma \mid \gamma \in Y \}, \qquad 3.6.1$$

then the closure of each component of $X - \overline{Z}$ is isometric to either

1. *a trouser, with anywhere from zero to three cusps,*
2. *a half-annulus $|z| \geq 1$ in $\{1/R < |z| < R\}$ for some $0 < R < \infty$, with its hyperbolic metric, or*
3. *a halfplane $\operatorname{Im} z \geq 0$ in $\mathbf{D}$, with its hyperbolic metric.*

Moreover, each component of $\overline{Z} - Z$ is a simple infinite geodesic bounding a halfplane (i.e., case 3 above).

PROOF OF THEOREM 3.6.2 If X is compact, the theorem is easy: simply choose on X a maximal multicurve Y. If we replace each curve by the geodesic in its homotopy class, Proposition 3.5.2 gives us a decomposition of X into trousers.

非紧不能随便取最大曲线.

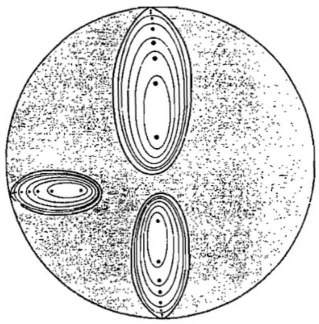


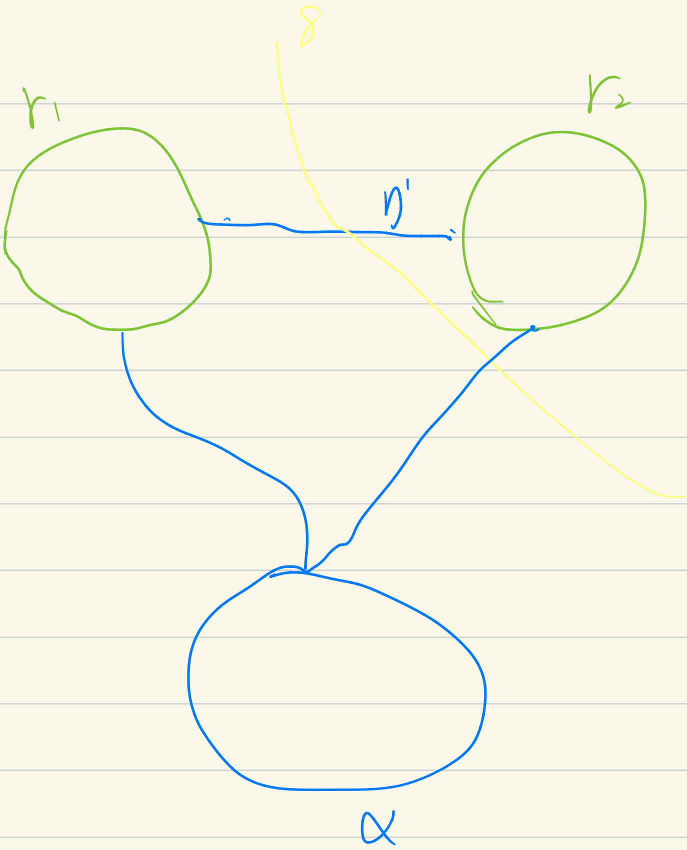
FIGURE 3.6.2. The Riemann surface X at left consists of the open disc minus three sequences of points tending to the boundary. A maximal multicurve Y is sketched in. Denote by Z the set of points of elements of Y. The set $\overline{Z} - Z$ consists of three geodesics. The shaded component of $X - \overline{Z}$ is not a halfplane, an annulus, or a trouser.

Proposition 1.4.1 (A nice exhaustion of X) *Let X be a Riemann surface satisfying the conditions of Theorem 1.1.2: i.e., connected, non-compact, and satisfying $H^1(X, \mathbb{R}) = 0$. Let $x_0 \in X$ be a "base point" of X. Then there exists an increasing sequence*

$$X_0 \subset X_1 \subset \cdots \subset X \qquad 1.4.1$$

of connected compact C^∞ pieces of X such that

1. *$x_0 \in X_0$,*
2. *each X_n is contained in the interior of X_{n+1},*
3. *$\bigcup X_n = X$,*
4. *each X_n satisfies $H^1(X_n, \mathbb{R}) = 0$.*



$\eta' \subset \eta$