A Teichmüller space is the set of Riemann surfaces of a given quasiconformal type. There is one Teichmüller space for every quasiconformal surface: we speak of the "Teichmüller space modeled on S", where S is a quasiconformal surface. This requires knowing what a quasiconformal surface is.

A quasiconformal surface S is a topological surface with a Riemann-surface structure; two Riemann surface structures on S define the same quasiconformal structure if the identity map between them is quasiconformal. If S_1, S_2 are two quasiconformal surfaces, a map $f: S_1 \to S_2$ is quasiconformal if it is a quasiconformal homeomorphism for one, hence all, analytic structures on each of S_1 and S_2. In particular, by definition all quasiconformal maps are isomorphisms.

If X is a Riemann surface, we denote by $\mathrm{qc}(X)$ its equivalence class. By Rado's theorem, all connected quasiconformal surfaces are σ-compact.

4 Teichmüller spaces

是否确定S结构.

The definitions of *Teichmüller equivalence* and *Teichmüller space* are among the most important definitions of this book, and you should not expect to come to terms with them easily.

该α?

未确定S结构，何来φ_1, φ_2.

?

Definition 6.4.1 (Teichmüller equivalence) Let X_1 and X_2 be Riemann surfaces, S a hyperbolic quasiconformal surface, and $\varphi_1: S \to X_1$ and $\varphi_2: S \to X_2$ quasiconformal mappings. The pairs (X_1, φ_1) and (X_2, φ_2) are *Teichmüller equivalent* if there exists an analytic isomorphism $\alpha: X_1 \to X_2$ such that $\varphi_2 = \alpha \circ \varphi_1$ on $I(S)$ and φ_2 is homotopic to $\alpha \circ \varphi_1$ rel the ideal boundary $I(S)$.

$\varphi_i: S \to \mathrm{qc}(X_i)$

S is a QC-equivalence class

Proposition and Definition 6.4.4 (Teichmüller metric) *Define*

$$d\big((X_1, \varphi_1),\ (X_2, \varphi_2)\big) := \inf_f\ \ln K(f), \qquad 6.4.2$$

where $K(f)$ is the quasiconformal constant of f (Definition 4.1.2) and the infimum is taken over all quasiconformal homeomorphisms f such that $\varphi_2 = f \circ \varphi_1$ on $I(S)$, and φ_2 and $f \circ \varphi_1$ are homotopic rel $I(S)$. Then d defines a metric on $\mathcal{T}_S$. With this metric, $\mathcal{T}_S$ is a complete metric space.

Recall (Definition 6.1.3) that a Beltrami form on a quasiconformal surface S is represented by a pair $((\varphi: S \to X), \mu)$ where X is a Riemann surface, φ is an isomorphism $S \to \mathrm{qc}(X)$, and μ is a Beltrami form on X. Recall also the definition of X_μ given in Proposition and Definition 4.8.12.

Definition 6.4.6 (The map from Beltrami forms to Teichmüller space) Let $m \in \mathcal{M}(S)$ be represented by $((\varphi: S \to X), \mu)$. Then $\Phi_S(m) \in \mathcal{T}_S$ is the element represented by

$\mu \in \mathcal{M}(X)$

quasi conformal

关系?

$$\Phi_S(m) = (\varphi: S \to X_\mu). \qquad 6.4.6$$

new Riemann surface structure on the same set

Definition 1.8.12 (Finite type) A Riemann surface is of *finite type* if it is isomorphic to a compact surface from which at most finitely many points have been removed.

Definition 1.8.9 (Hyperbolic Riemann surface) A Riemann surface is *hyperbolic* if its universal covering space is isomorphic to $\mathbf{D}$.

6.5 Analytic structure of Teichmüller space

Goal: give T_S the structure of a Banach analytic manifold.

charts land in Banach space

transition maps are holomorphic in Banach space.

A map $f: U \to F$ is holomorphic at $x \in U$ if there exists a bounded complex linear operator $Df(x): E \to F$ such that $\lim_{h \to 0} \frac{\| f(x+h) - f(x) - Df(x)h \|_F}{\| h \|_E} = 0$

Answer

Theorem 6.5.1 (Analytic structure on Teichmuller space)

1. There exists a unique structure of a complex analytic manifold on T_S such that the mapping $\Phi_S: M(S) \to T_S$ is analytic.
2. With this structure, Φ_S is a split submersion.

Q: dF_p bounded? ① finite dimensional, ✓ ② Banach: By definition, ✓

F is a split submersion at p if its differential $dF_p: T_pM \to T_{F(p)}N$ is surjective and its kernel is complemented: there exists a closed subspace $H_p \subset T_pM$ with $T_pM = \ker dF_p \oplus H_p$.

If this holds at every point, F is a split submersion.

$\Updownarrow$

Equivalent, there is a bounded right inverse $R_p: T_{F(p)}N \to T_pM$ with $dF_p \circ R_p = id$.

$\Rightarrow$) $dF_p|_{H_p}: H_p \to T_{F(p)}N$ is a bijection since dF_p surjective. Define $R_p = (dF_p|_{H_p})^{-1}$. Then R_p is a bounded right inverse because $dF_p(R_p(w)) = w$ for all $w \in T_{F(p)}N$. (Bounded automatic in finite dimensions by linear)

Open mapping theorem: Let X and Y be Banach spaces. If $T: X \to Y$ is a bounded linear operator and surjective, then T is an open map.

Corollary: If T is a bijective bounded linear operator between Banach spaces, then its inverse T^{-1} is also bounded.

$\Leftarrow$) Assume R_p exists. Define $H_p = im(R_p)$.

For any $v \in T_pM$, write $v = \underbrace{v - R_p(dF_p(v))}_{\in \ker dF_p} + \underbrace{R_p(dF_p(v))}_{\in H_p}$.

If $k \in \ker(dF_p) \cap H_p$, write $k = R_p y$. Then $0 = dF_p(R_p y) = y$, hence $k = 0$.

Hence $T_pM = \ker dF_p \oplus H_p$

H_p is closed: The restriction $dF_p|_{H_p}: H_p \to T_{F(p)}N$ is bijective and its inverse is R_p. Since R_p is bounded, $dF_p|_{H_p}$ is a bounded isomorphism of Banach space; therefore H_p must be closed in X.

Proof. As usual, when defining a manifold structure, we need to find an atlas. Choose a universal cover $\pi: \hat{S} \to S$, with covering group Γ.

Why the universal cover exists?

If a topological space X is connected, locally path-connected and semilocally connected, then X has a universal covering space $\pi: \tilde{X} \to X$, and it is unique up to unique isomorphism.

S: a topological surface is a 2-manifold: every point has a neighborhood homeomorphism to an open disk.

Disks are path-connected $\Rightarrow$ S is lpc.

Small disks are simply connected $\Rightarrow$ S is slsc.

connected? Q: Is Riemann surface connected? Is S connected?
hyperbolic?

The deck group is $T = \{g: \tilde{S} \to \tilde{S} \text{ homeo} \mid \pi \circ g = \pi\}$.

If g_1, g_2 are deck maps, then $\pi \circ (g_1 \circ g_2) = (\pi \circ g_1) \circ g_2 = \pi \circ g_2 = \pi$.

$id_{\tilde{S}}$ is a deck map. If $\pi \circ g = p$, then $\pi \circ g^{-1} = p$. $T = \mathrm{Deck}(\pi)$ is a group under composition.

$T \cong \pi_1(S, x_0)$

We will construct coordinates $\varphi_{\tilde{\varphi}}$ labeled by quasiconformal maps $\tilde{\varphi}: \tilde{S} \to H$ such that $T_{\tilde{\varphi}} := \tilde{\varphi} \circ T \circ (\tilde{\varphi})^{-1}$ is a Fuchsian group, i.e., such that for every $\gamma \in T$, the homeomorphism $\tilde{\varphi} \circ \gamma \circ (\tilde{\varphi})^{-1}$ is an analytic automorphism of H.

Q: why $\tilde{\varphi}$ exists? How to construct such $\tilde{\varphi}$?
Why $T_{\tilde{\varphi}}$ is a Fuchsian group? $\quad \tilde{S} \to \tilde{X} \to H$

Start from a base Riemann surface X on S. Let $p: \tilde{X} \to X$ be its universal cover and $u: \tilde{X} \xrightarrow{\cong} H$ a conformal uniformization (hyperbolic case).

Let $h: \tilde{S} \to \tilde{X}$ be the lift covering the identity on S. Give $\tilde{S}$ the pulled back complex structure so that h is biholomorphic. Set $\tilde{\varphi} := u \circ h : \tilde{S} \to H$. ?

$\tilde{\varphi} \circ \gamma \circ \tilde{\varphi}^{-1} = u \circ (h \gamma h^{-1}) \circ u^{-1}$ is a composition of biholomorphisms of $\tilde{X}$ and H, hence a biholomorphic automorphism of H. ?

But $\mathrm{Aut}(H)$ equals $PSL_2(\mathbb{R})$ acting by Möbius maps $z \mapsto \frac{az+b}{cz+d}$ with $ad-bc=1$. Therefore $T_{\tilde{\varphi}} := \tilde{\varphi} T \tilde{\varphi}^{-1} \subset PSL_2(\mathbb{R})$. ?

Remark Such a map $\tilde{\varphi}$ carries more information than a representative $\varphi: S \to X$ of a point of T_S. Indeed, to $\tilde{\varphi}$ we can associate the representative

$S = \tilde{S}/T \to X = H/T_{\tilde{\varphi}}$. ??? $\quad S = H/T \xrightarrow{[\tilde{\varphi}]} H/T_{\tilde{\varphi}} = X$

but the map $\tilde{\varphi}$ also contains an identification of $\tilde{X}$ with H and a lifting of $\varphi: S \to X$ to a homeomorphism $\tilde{S} \to H$. $\triangle$ $\quad D \underset{?}{=} H$

S is hyperbolic quasiconformal surface.

it exists for all finite type.

?

AI: Assume S has $\chi(S) > 0$. Then every conformal structure on S carries the unique complete hyperbolic metric of curvature -1.

Lift that metric to the universal cover $\tilde{S}$. The lifted surface is simply connected, complete, and have curvature -1. By the Cartan-Hadamard / Uniformization classification, any such surface is isometric and biholomorphic to the upper half-plane H (equivalently the unit disk D)

D — H homeomorphism, isometry?

Choose an isometry / biholomorphism $I : \tilde{S} \xrightarrow{\cong} H$

$T_I = I \circ T \circ I^{-1} \subset PSL(2, \mathbb{R})$ is a Fushsian group acting by Möbius transformation.

$T \overset{?}{=} \pi_1(S)$

?

If later we change the complex structure via a quasiconformal map, the new deck group is the $T_{\tilde{\varphi}} = \tilde{\varphi} \cdot T \circ \tilde{\varphi}^{-1}$, still a Fuchsian group acting on H.

Definition 1.8.13 (Fuchsian group, Kleinian group) A *Fuchsian group* is a discrete subgroup of Aut $\mathbf{D}$. A *Kleinian group* is a discrete subgroup of $\mathrm{PSL}_2\,\mathbb{C}$.

Aut H.

Exercise 2.1.3 Show that $\mathbf{H}$ and $\mathbf{B}$ are isometric to $\mathbf{D}$. ♢

Exercise 2.1.4 1. Show that the complex analytic automorphisms of $\mathbf{H}$ are the maps $z \mapsto \frac{az+b}{cz+d}$ with $a, b, c, d \in \mathbb{R}$ and $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} > 0$.

2. Show that this identifies Aut $\mathbf{H}$ with $\mathrm{PSL}_2\,\mathbb{R} := \mathrm{SL}_2\,\mathbb{R}/\pm I$.

3. Show that $\mathrm{PSL}_2\,\mathbb{R}$ is precisely the set of orientation-preserving isometries of $\mathbf{H}$ for the hyperbolic metric. ♢

The Cayley transform $w = \frac{z-i}{z+i}$ is a homeomorphism between H and D. When H and D are equipped with their standard hyperbolic metrics.

$Q^\infty(\Omega)$: Banach space of bounded holomorphic quadratic differentials on Ω.

Each candidate local coordinate $\psi_{\tilde\varphi}$ is a map

$$T_S \to (Q^\infty)^{\Gamma_{\tilde\varphi}}(H^*), \qquad q(\gamma z)(\gamma'(z))^2 = q(z) \quad (\gamma \in \Gamma_{\tilde\varphi})$$

whose image is a subset of ~~the~~ Banach space of bounded holomorphic quadratic differentials on H^* invariant under $\Gamma_{\tilde\varphi}$. The coordinates are defined using maps $\tilde\Psi_{\tilde\varphi}: M(S) \to (Q^\infty)^{\Gamma_{\tilde\varphi}}(H^*)$ such that if $m_1, m_2 \in M(S)$ satisfy $\Phi_S(m_1) = \Phi_S(m_2)$, then $\tilde\Psi_{\tilde\varphi}(m_1) = \tilde\Psi_{\tilde\varphi}(m_2)$. ?

To carry out the construction, we use the canonical identification of $M(S)$ with the space $M^{\Gamma_{\tilde\varphi}}(H)$ of $\Gamma_{\tilde\varphi}$-invariant Beltrami forms on H. ?

合理: $M(S) \longrightarrow T_S$ quotient

Definition 6.5.2 Given $\mu \in \mathcal{M}^{\Gamma_{\tilde\varphi}}(\mathbf{H})$, we define

$$\widetilde{\Psi}_{\tilde\varphi}(\mu) := \mathcal{S}\{f^{\widehat{\mu}}|_{\mathbf{H}^*}, z\}, \qquad 6.5.4$$

where $\widehat{\mu} \in \mathcal{M}^{\Gamma_{\tilde\varphi}}(\mathbb{P}^1)$ is μ extended by 0 to $\mathbb{P}^1 - \mathbf{H}$.

$M^{\Gamma_{\tilde\varphi}}(H)$: The unit ball of $\Gamma_{\tilde\varphi}$-invariant Beltrami forms on H: $\mu \in L^\infty(H)$, $\|\mu\|_\infty < 1$, $\mu \circ \gamma \frac{\overline{\gamma'}}{\gamma'} = \mu \quad (\forall \gamma \in \Gamma_{\tilde\varphi})$.

Define $\hat\mu = \mathbb{P}^1 \overset{=}{} \overline{\mathbb{C}}$ by

$$\hat\mu(z) = \begin{cases} \mu(z), & z \in H \\ 0, & z \in \overline{\mathbb{C}} \setminus H \end{cases}$$

In $\overline{\mathbb{C}}$:

$H = \{z \in \overline{\mathbb{C}} : \operatorname{Im}(z) > 0\}$

$\overline{H} = \{z \in \overline{\mathbb{C}} : \operatorname{Im}(z) \geq 0\} \cup \{\infty\}$

Definition 6.3.2 (Schwarzian derivative) Let U be a Riemann surface. Let $f, g: U \to \mathbb{P}^1$ be maps with nonvanishing derivatives and let A be the unique Möbius transformation such that the Taylor series of f and of $A \circ g$ coincide to second order at a point $z \in U$. The *Schwarzian derivative* $\mathcal{S}\{f, g\}$ at z is given by

$$\mathcal{S}\{f,g\}(z) = 6\Big(\big(Df(z)\big)^{-1} \circ D^3(f - A \circ g)(z)\Big). \qquad 6.3.5$$

The fundamental example is the following, which is the standard definition.

Proposition 6.3.3 (Computing the Schwarzian derivative) *Let $U \subset \mathbb{C}$. If $f: U \to \mathbb{C}$ is an analytic function with nonvanishing derivative, and $g(z) := z$, then*

$$\mathcal{S}\{f, z\} = \left(\frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2\right) dz^2. \qquad 6.3.6$$

What is $f^{\hat\mu}$?

Choose a universal covering map $\pi: \mathbf{H} \to X$, with covering group Γ. Then $\mu \mapsto \pi^*\mu$ maps $\mathcal{M}(X)$ to $\mathcal{M}^\Gamma(\mathbf{H})$, where $\mathcal{M}^\Gamma$ denotes the Γ-invariant Beltrami forms. Bers had the brilliant idea of extending $\pi^*\mu$ to all of $\mathbb{C}$ by setting it to be 0 in the lower halfplane $\mathbf{H}^*$, i.e., by defining $\widehat{\mu} \in \mathcal{M}^\Gamma(\mathbb{C})$ to be $\pi^*\mu$ in $\mathbf{H}$, extended by 0 to the remainder of $\mathbb{C}$.

We can then consider the quasiconformal homeomorphism $f^{\widehat{\mu}}: \mathbb{C} \to \mathbb{C}$ that fixes 0 and 1 and solves the Beltrami equation $\bar\partial f = \partial f \circ \widehat{\mu}$.

$f^{\hat\mu}(\infty) = \infty$

Fix the identification coming from your lift $\tilde{\varphi}$:

$$S \;\cong\; \mathbb{H}/\Gamma, \qquad \pi:\mathbb{H}\to S \text{ the quotient map}, \Gamma=\Gamma_{\tilde{\varphi}}\subset \mathrm{PSL}(2,\mathbb{R}).$$

Up: lift Beltrami forms on S to Γ-invariant ones on $\mathbb{H}$

A Beltrami form on S is (in a chart w) $\mu(w)\,\frac{d\bar{w}}{dw}$ with $\|\mu\|_\infty<1$. Define

$$L:\mathcal{M}(S)\longrightarrow \mathcal{M}^\Gamma(\mathbb{H}), \qquad (L\mu)(z):=\mu(\pi(z))\,\frac{\overline{\pi_z(z)}}{\pi_z(z)}.$$

- $L\mu$ is measurable and $\|L\mu\|_\infty=\|\mu\|_\infty$.
- Γ-invariance: for any $\gamma\in\Gamma$,

$$(L\mu)\big(\gamma z\big)\,\frac{\overline{\gamma'(z)}}{\gamma'(z)}=\mu\big(\pi(\gamma z)\big)\,\frac{\overline{(\pi\circ\gamma)_z}}{(\pi\circ\gamma)_z}=\mu\big(\pi(z)\big)\,\frac{\overline{\pi_z}}{\pi_z}=(L\mu)(z),$$

since $\pi\circ\gamma=\pi$ and γ is conformal.

So $L\mu\in\mathcal{M}^\Gamma(\mathbb{H})$.

Down: descend Γ-invariant Beltrami forms to S

Given $\nu\in\mathcal{M}^\Gamma(\mathbb{H})$, define μ on S by

$$\mu(\pi(z)):=\nu(z)\,\frac{\pi_z(z)}{\overline{\pi_z(z)}}.$$

This is **well-defined**: if $z'=\gamma z$ lies in the same fiber,

$$\nu(z')\,\frac{\pi_z(z')}{\overline{\pi_z(z')}}=\Big(\nu(\gamma z)\,\frac{\overline{\gamma'(z)}}{\gamma'(z)}\Big)\,\frac{\pi_z(z)\gamma'(z)}{\overline{\pi_z(z)}\,\overline{\gamma'(z)}}=\nu(z)\,\frac{\pi_z(z)}{\overline{\pi_z(z)}}.$$

Again $\|\mu\|_\infty=\|\nu\|_\infty$.

Canonical identification

The two constructions are inverse linear isometries:

$$\mathcal{M}(S)\;\overset{L}{\underset{D}{\longleftrightarrow}}\;\mathcal{M}^\Gamma(\mathbb{H}),$$

so once you have fixed the identification $S\simeq\mathbb{H}/\Gamma$ (i.e., fixed $\tilde{\varphi}$), there is a **canonical** 1–1 correspondence between Beltrami forms on S and Γ-invariant Beltrami forms on $\mathbb{H}$.

Proposition 6.5.3 (Candidate coordinates $\Psi_{\tilde{\varphi}}$)

1. *The mapping*

$$\widetilde{\Psi}_{\tilde{\varphi}} : \mathcal{M}^{\Gamma_{\tilde{\varphi}}}(\mathbf{H}) \to (Q^{\infty})^{\Gamma_{\tilde{\varphi}}}(\mathbf{H}^*) \qquad 6.5.5$$

of equation 6.5.4 is analytic.

2. *If $\mu_1, \mu_2 \in \mathcal{M}^{\Gamma_{\tilde{\varphi}}}(\mathbf{H})$ correspond to $m_1, m_2 \in \mathcal{M}(S)$, and if $\Phi_S(m_1) = \Phi_S(m_2)$, then $\widetilde{\Psi}_{\tilde{\varphi}}(\mu_1) = \widetilde{\Psi}_{\tilde{\varphi}}(\mu_2)$. Thus $\widetilde{\Psi}_{\tilde{\varphi}}$ induces a continuous map*

$$\Psi_{\tilde{\varphi}} : \mathcal{T}_S \to (Q^{\infty})^{\Gamma_{\tilde{\varphi}}}(\mathbf{H}^*), \qquad 6.5.6$$

so $\Psi_{\tilde{\varphi}}$ is well defined on $\mathcal{T}_S$.

3. *The map $\Psi_{\tilde{\varphi}}$ is injective.*

linear and bounded

$L^\infty(H) \to L^\infty(\bar{\mathbb{C}})$. Holomorphic

$\mu \to \hat{\mu} \to f^{\hat{\mu}} \to \cdots$

PROOF 1. The map $\mu \mapsto \widehat{\mu}$ is evidently analytic. The map $\widehat{\mu} \mapsto f^{\widehat{\mu}}$ is analytic by Theorem 4.7.4. Clearly taking the Schwarzian derivative is analytic.

2. This follows immediately from the "only if" part of Proposition 6.4.12.

3. This follows immediately from the "if" part of Proposition 6.4.12. □

(2) $\widehat{\Psi}_{\tilde{\varphi}}$ descends to a continuous map $\Psi_{\tilde{\varphi}} : \mathcal{T}_S \to (Q^\infty)^\Gamma(\mathbb{H}^*)$

Let $\Phi_S : \mathcal{M}(S) \to \mathcal{T}_S$ be the projection. If $\mu_1, \mu_2 \in \mathcal{M}^\Gamma(\mathbb{H})$ represent the *same* Teichmüller point, then by your **Prop. 6.4.12** the normalized solutions agree on the lower half-plane:

$$f^{\widehat{\mu_1}}\big|_{\mathbb{H}^*} = f^{\widehat{\mu_2}}\big|_{\mathbb{H}^*}.$$

Applying the Schwarzian, we get $\widehat{\Psi}_{\tilde{\varphi}}(\mu_1) = \widehat{\Psi}_{\tilde{\varphi}}(\mu_2)$. Thus $\widehat{\Psi}_{\tilde{\varphi}}$ is **constant on the fibers** of Φ_S, so by the universal property of the quotient topology there exists a unique **continuous** map

$$\Psi_{\tilde{\varphi}} : \mathcal{T}_S \to (Q^\infty)^\Gamma(\mathbb{H}^*), \qquad \widehat{\Psi}_{\tilde{\varphi}} = \Psi_{\tilde{\varphi}} \circ \Phi_S.$$

---

(3) $\Psi_{\tilde{\varphi}}$ is injective

Suppose $\Psi_{\tilde{\varphi}}([m_1]) = \Psi_{\tilde{\varphi}}([m_2])$. Pick representatives μ_1, μ_2 in $\mathcal{M}^\Gamma(\mathbb{H})$. Then

$$S\big(f^{\widehat{\mu_1}}\big|_{\mathbb{H}^*}\big) = S\big(f^{\widehat{\mu_2}}\big|_{\mathbb{H}^*}\big).$$

Hence $f^{\widehat{\mu_2}}\big|_{\mathbb{H}^*} = M \circ f^{\widehat{\mu_1}}\big|_{\mathbb{H}^*}$ for some Möbius map M (Schwarzian vanishes precisely on Möbius differences). With the standard normalization $0, 1, \infty$ fixed for both global solutions, $M = \mathrm{id}$. Therefore

$$f^{\widehat{\mu_1}}\big|_{\mathbb{H}^*} = f^{\widehat{\mu_2}}\big|_{\mathbb{H}^*}.$$

By **Prop. 6.4.12** again, $\Phi_S(m_1) = \Phi_S(m_2)$, i.e. $[m_1] = [m_2]$. Thus $\Psi_{\tilde{\varphi}}$ is injective.

Theorem 4.7.4 *Let $U \subset \mathbb{C}$ be a compact set. Let $B_U(\mathbb{C}) \subset L^\infty(\mathbb{C})$ be the subset of the unit ball consisting of functions with support in U. Then the mapping $B_U(\mathbb{C}) \to \mathbf{QC}(\mathbb{C}, \mathbb{C})$ given by $\mu \mapsto w^\mu$ is analytic, in the sense that it is continuous, and for each $z \in \mathbb{C}$ the map $\mu \mapsto w^\mu(z)$ is analytic.*

Proposition 6.4.12 (Criterion for Teichmüller equivalence) *We have $\Phi_S(m_1) = \Phi_S(m_2)$ if and only if the restrictions of $f^{\widehat{\mu}_1}$ and $f^{\widehat{\mu}_2}$ to the lower halfplane $\mathbf{H}^*$ coincide.*

We will now study how solutions of the Beltrami equation

$$\frac{\partial f}{\partial \bar{z}} = \mu \frac{\partial f}{\partial z} \qquad 4.7.1$$

depend on the Beltrami coefficient μ.

Since μ lives in L^∞, it may seem that the most obvious question is whether solutions of the Beltrami equation depend continuously on μ in the L^∞ topology. The answer is that they do; we have essentially proved it above. But that statement is not interesting: the L^∞ topology is very strong and the hypothesis is too restrictive to be useful.

We will instead discuss both the continuity of solutions when μ varies continuously in L^1, and the analytic dependence of solutions on $\mu \in L^\infty$. The first is essentially obvious, at least with the proof of the mapping theorem we have given. The second is more elaborate.

In order to address these issues, we need to normalize the solutions of equation 4.7.1. There are several ways to do this; we will use the following.

$\mathbb{C}$ ☆

Notation 4.7.1 (w^μ) Suppose μ has compact support. We denote by w^μ the unique solution $w^\mu : \mathbb{C} \to \mathbb{C}$ of equation 4.7.1 that is a homeomorphism, and which at infinity can be written $w^\mu(z) = z + O(1/|z|)$.

Because μ is $\Gamma_{\tilde\varphi}$-invariant on H, $f^{\hat\mu}$ satisfies

$$f^{\hat\mu}\circ\gamma = A_\gamma\circ f^{\hat\mu} \quad (\gamma\in\Gamma_{\tilde\varphi}).$$ for some Möbius map A_γ.

For $\gamma\in\Gamma_{\tilde\varphi}$ (a Möbius automorphism of H), compute the Beltrami coefficient of $f\circ\gamma$ on H: (Here we write $f = f^{\hat\mu}$)

$$\mu_{f\circ\gamma}(z) = \hat\mu(\gamma z)\frac{\overline{\gamma'(z)}}{\gamma'(z)} = \mu(z) \quad (\text{by } \Gamma_{\tilde\varphi}\text{-invariance of } \mu).$$

On H^*, $\hat\mu = 0$ and γ preserves the real line, so still $\mu_{f\circ\gamma}=0$.

Hence $f\circ\gamma$ solves the same Beltrami equation as f on $\overline{\mathbb{C}}$.

Normalized solutions of a given Beltrami coefficient are unique up to postcomposition by a Möbius map A that preserves the normalization set. Therefore, for each γ $\exists$ a Möbius A_γ with $f\circ\gamma = A_\gamma\circ f$.

Take Schwartzians and use the chain rules

$S(f\circ\gamma) = (Sf\circ\gamma)(\gamma')^2 + S\gamma$, $\quad S(A\circ f) = Sf$, and $S\gamma = 0$ for Möbius γ. From

$S(f^{\hat\mu}\circ\gamma) = S(A_\gamma\circ f^{\hat\mu}) = S(f^{\hat\mu})$

we get on H^*:

$S(f^{\hat\mu})\circ\gamma\cdot(\gamma')^2 = S(f^{\hat\mu})$ which is exactly the automorphy condition for a $\Gamma_{\tilde\varphi}$-invariant quadratic differential.

(2) $\Phi_S(m_1) = \Phi_S(m_2) \Rightarrow f^{\hat\mu_1} = f^{\hat\mu_2}$ on $H^* \Rightarrow \hat\Psi_{\tilde\varphi}(\mu_1) = \hat\Psi_{\tilde\varphi}(\mu_2)$. The map depends only on $[m]\in T_S$, giving a well-defined continuous map

$\Psi_{\tilde\varphi}: T_S \to (Q^\infty)^{\Gamma_{\tilde\varphi}}(H^*)$ Universal property of quotient map. ?

Why not analytic ?

Definition 6.5.2 Given $\mu\in\mathcal{M}^{\Gamma_{\tilde\varphi}}(\mathbf{H})$, we define

$$\widetilde\Psi_{\tilde\varphi}(\mu) := \mathcal{S}\{f^{\hat\mu}|_{\mathbf{H}^*}, z\}, \qquad (6.5.4)$$

where $\hat\mu\in\mathcal{M}^{\Gamma_{\tilde\varphi}}(\mathbb{P}^1)$ is μ extended by 0 to $\mathbb{P}^1 - \mathbf{H}$.

(3) $\Psi_{\tilde\varphi}([m_1]) = \Psi_{\tilde\varphi}([m_2])$

$\quad\mu_1 \qquad\qquad \mu_2$ — representation

$$S\{f^{\hat\mu_1}|_{H^*}, z\} = S\{f^{\hat\mu_2}|_{H^*}, z\}.$$

Two locally univalent maps with the same Schwarzian on a simply connected domain differed by a Möbius transformation there: $f^{\hat\mu_1} = A\circ f^{\hat\mu_2}$ on H^* ? for some Möbius A.

But our normalization fix $0,1,\infty$ on the boundary, so $A = \mathrm{id}$. Therefore

$f^{\hat\mu_1} = f^{\hat\mu_2}$ on H^*

Thus $\Phi_S(m_1) = \Phi_S(m_2)$. $[m_1] = [m_2]$

$X \cong \mathbf{H}/\Gamma_{\tilde\varphi}$

$X^* \cong \mathbf{H}^*/\Gamma_{\tilde\varphi}$

Definition 6.5.4 (Bers embedding) If $\tau \in \mathcal{T}_S$ is represented by $\varphi : S \to X$, then the map $\Psi_{\tilde\varphi} : \mathcal{T}_S \to (Q^\infty)^{\Gamma_{\tilde\varphi}}(\mathbf{H}^*)$ of Proposition 6.5.3 induces a natural map

conjugate surface

$$\Psi_\tau : \mathcal{T}_S \to Q^\infty(X^*), \qquad 6.5.7$$

which does not depend on the choice of $\tilde\varphi$. This map Ψ_τ is called the *Bers embedding.*

回到证明

(*)
- That each $\Psi_{\tilde\varphi}$ is an open map near its base point τ, i.e., that there is a neighborhood of τ in $\mathcal{T}_S$ that is mapped homeomorphically to a neighborhood of 0 in $(Q^\infty)^\Gamma(\mathbf{H}^*)$. we have already know $\Psi_{\tilde\varphi}$ injective and continuous
- That the changes of coordinates are analytic.

In finite dimensions, proving that a map is open is almost never difficult and follows from the implicit function theorem. When S is of finite type, then $\mathcal{T}_S$ is finite dimensional, and proving that the $\Psi_{\tilde\varphi}$ are open is not difficult either. But in the infinite-dimensional case, proving that the $\Psi_{\tilde\varphi}$ are open is not obvious at all. The proof depends on the Ahlfors-Weill theorem, Theorem 6.3.10, which constructs a local section of $\Psi_{\tilde\varphi}$.

? For a marked Riemann surface X on the underlying finite-type surface S, the complex cotangent space is $T^*_{[X]}\mathcal{T}_S \cong Q(X)$, the vector space of holomorphic quadratic differentials on X.

By Riemann-Roch, $Q(X)$ is finite-dimensional. Cotangent space finite-dimensional $\Rightarrow$ tangent space is too.

Consider a continuous function $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ such that $f(x_0, y_0) = 0$. If there exist open neighbourhoods $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^m$ of x_0 and y_0, respectively, such that, for all y in B, $f(\cdot, y) : A \to \mathbb{R}^n$ is locally one-to-one, then there exist open neighbourhoods $A_0 \subset \mathbb{R}^n$ and $B_0 \subset \mathbb{R}^m$ of x_0 and y_0, such that, for all $y \in B_0$, the equation $f(x, y) = 0$ has a unique solution

$$x = g(y) \in A_0,$$

where g is a continuous function from B_0 into A_0.

The Ahlfors-Weill construction

The Ahlfors-Weill construction can be seen as a converse of Nehari's theorem. It asserts that if the Schwarzian is small, then the map is injective.

Let $q \in Q^\infty(\mathbf{H}^*)$ be a bounded holomorphic quadratic differential, with $\|q\|_\infty < 1/2$. We can then define a Beltrami form μ_q on $\mathbb{P}^1$ as follows: write $q = q(z)\,dz^2$, and set

What is y ? $\longrightarrow$ $z = x + iy$

$$\mu_q(z) := \begin{cases} 2y^2 q(\bar z)\frac{d\bar z}{dz} & \text{if } z \in \mathbf{H} \\ 0 & \text{if } z \in \mathbf{H}^*, \end{cases} \qquad 6.3.24$$

where y is the imaginary part of z. It is then clear that $\|\mu_q\| = \|q\|_\infty < 1$, so that μ_q is indeed a Beltrami form and can be integrated. The map f^{μ_q} is injective and analytic in $\mathbf{H}^*$, and has a Schwarzian derivative. In view of how f^{μ_q} is defined (Notation 4.7.5), it might seem unlikely that this can be computed, but Ahlfors and Weill [10] found that the answer is amazingly simple.

Theorem 6.3.10 (Ahlfors-Weill) *We have* $S\{f^{\mu_q}|_{\mathbf{H}^*}, z\} = q$.

(*): The Banach spaces that model Teichmuller space in the Bers set-up are typically non-separable. For infinite-type surfaces and especially the universal Teichmüller space, the resulting Teichmuller spaces are likewise not separable in their natural topologies.

Therefore you cannot expect a countable atlas covering all of $\mathcal{T}_S$ in those cases.

The definition of Banach manifold does not require second countability; we only need charts with Banach targets and holomorphic transition maps.

Let $V_{\tilde\varphi} \subset (Q^\infty)^{\Gamma_{\tilde\varphi}}(\mathbf{H}^*)$ be the ball of radius $1/2$, set $U_{\tilde\varphi} := \Psi_{\tilde\varphi}^{-1}(V_{\tilde\varphi})$, and let

$$\sigma_{\tilde\varphi} : V_{\tilde\varphi} \to \mathcal{M}^{\Gamma_{\tilde\varphi}}(\mathbf{H}) \qquad 6.5.8$$

be the map given by

$$(\sigma_{\tilde\varphi}(q))(z) := 2y^2 q(\bar z)\frac{d\bar z}{dz}. \qquad 6.5.9$$

(We saw the expression in the right side in equation 6.3.24. The letter σ is supposed to suggest "section".) The diagram below, where Φ_S is the map defined in Theorem 6.5.1, should help keep notation straight; the notation $\nearrow\subset$ means inclusion map:

$$\begin{tikzcd}
\mathcal{M}(S) \arrow[d, "\Phi_S"] & = & \mathcal{M}^{\Gamma_{\tilde\varphi}}(\mathbf{H}) \arrow[d, "\tilde\Psi_{\tilde\varphi}"] \\
\mathcal{T}(S) \arrow[rr, "\Psi_{\tilde\varphi}"'] & & (Q^\infty)^{\Gamma_{\tilde\varphi}}(\mathbf{H}^*) \\
U_{\tilde\varphi} \arrow[u, hook] \arrow[rr, "\Psi_{\tilde\varphi}"'] & & V_{\tilde\varphi} \arrow[u, hook] \arrow[uu, bend right, "\sigma_{\tilde\varphi}"']
\end{tikzcd} \qquad 6.5.10$$

$(\Psi_{\tilde\varphi})^{-1} = \Phi_S \circ \sigma_{\tilde\varphi}$.

$V_{\tilde\varphi} := \{ q \in (Q^\infty)^{\Gamma_{\tilde\varphi}}(H^*) : \|q\|_\infty < \frac{1}{2} \}$

$U_{\tilde\varphi} := \Psi_{\tilde\varphi}^{-1}(V_{\tilde\varphi})$.

与 Ahlfors-Weill 相符.

$(\sigma_{\tilde\varphi}(q))(z) \in M^{\Gamma_{\tilde\varphi}}(H)$.:

Need to check

$(\sigma_{\tilde\varphi}(q)\circ\gamma)(z)\dfrac{\overline{\gamma'(z)}}{\gamma'(z)} = \sigma_{\tilde\varphi}(q)(z) \quad (\gamma \in \Gamma_{\tilde\varphi})$

Direct coordinate check (if you like the algebra)

Let $\gamma(z) = \dfrac{az+b}{cz+d} \in \mathrm{PSL}_2(\mathbb{R})$. Use:

- $\Im(\gamma z) = \dfrac{\Im z}{|cz+d|^2}$,
- $\gamma'(z) = \dfrac{1}{(cz+d)^2}$,
- $\gamma(\bar z) = \overline{\gamma(z)}$,
- and automorphy of q on $\mathbb{H}^*$: $q(\gamma(\bar z))\,(\gamma'(\bar z))^2 = q(\bar z)$, with $\gamma'(\bar z) = \overline{\gamma'(z)}$.

Then

$$\begin{aligned}(\sigma(q)\circ\gamma)(z) &= 2\,(\Im\gamma z)^2\,q(\overline{\gamma z})\,\frac{d\bar z}{dz} \\ &= 2\,\frac{(\Im z)^2}{|cz+d|^4}\,\frac{q(\bar z)}{\gamma'(\bar z)^2}\,\frac{d\bar z}{dz} \\ &= 2\,(\Im z)^2\,q(\bar z)\,\frac{(cz+d)^4}{|cz+d|^4}\,\frac{d\bar z}{dz}.\end{aligned}$$

Multiplying by $\dfrac{\overline{\gamma'(z)}}{\gamma'(z)} = \dfrac{(cz+d)^2}{\overline{(cz+d)}^2}$ gives exactly

$$2(\Im z)^2\,q(\bar z)\,\frac{d\bar z}{dz} = \sigma(q)(z),$$

which is $(*)$.

Lemma 6.5.5 *The mapping $\Psi_{\tilde\varphi}$ induces a homeomorphism $U_{\tilde\varphi} \to V_{\tilde\varphi}$.*

Proof Indeed, the composition $\Phi_S \circ \sigma_{\tilde\varphi}$ is a right inverse of $\Psi_{\tilde\varphi}$:

$$\Psi_{\tilde\varphi} \circ \Phi_S \circ \sigma_{\tilde\varphi} = \tilde\Psi_{\tilde\varphi} \circ \sigma_{\tilde\varphi} \underbrace{=}_{\text{Thm. 6.3.10}} \mathrm{id}, \qquad 6.5.11$$

and since $\Psi_{\tilde\varphi}$ is injective, this also implies $\Phi_S \circ \sigma_{\tilde\varphi} \circ \Psi_{\tilde\varphi} = \mathrm{id}$. □

$\tilde\Phi_{\tilde\varphi}($

$$V_{\tilde\varphi} \xrightarrow{\sigma_{\tilde\varphi}} M^{\Gamma_{\tilde\varphi}}(H) = \underbrace{M(S) \xrightarrow{\Phi_S} T(S) \xrightarrow{\Psi_{\tilde\varphi}} V_{\tilde\varphi}}_{\tilde\Psi_{\tilde\varphi}}.$$

$q \in V_{\tilde\varphi}$, $(\sigma_{\tilde\varphi}(q))(z) := 2y^2 q(\bar z)\dfrac{d\bar z}{dz}$.

Injectivity of $\Psi_{\tilde\varphi}$ gives the left inverse.

Take any $x \in U_{\tilde\varphi}$. Then $\Psi_{\tilde\varphi}(x) \in V_{\tilde\varphi}$, so applying the identity above at $q = \Psi_{\tilde\varphi}(x)$ gives $\Psi_{\tilde\varphi}(\Phi_S \circ \sigma_{\tilde\varphi} \circ \Psi_{\tilde\varphi}(x)) = \Psi_{\tilde\varphi}(x)$.

Because $\Psi_{\tilde\varphi}$ is injective, $\Phi_S \circ \sigma_{\tilde\varphi} \circ \Psi_{\tilde\varphi}(x) = x \quad (x \in U_{\tilde\varphi})$.

Hence $\Phi_S \circ \sigma_{\tilde\varphi}$ is also a left inverse of $\Psi_{\tilde\varphi}$ on $U_{\tilde\varphi}$.

$\Psi_{\tilde\varphi}$ continuous, $\sigma_{\tilde\varphi}$ analytic, Φ_S continuous ?

quotient

To prove Theorem 6.5.1, part 1, the only thing left to check is that the changes of coordinates are analytic, but that follows also. If $\Psi_{\tilde{\varphi}}$ and $\Psi_{\tilde{\varphi}_1}$ are two coordinates, then

$$\underline{\Psi_{\tilde{\varphi}_1} \circ \Psi_{\tilde{\varphi}}^{-1}} = \Psi_{\tilde{\varphi}_1} \circ \Phi_S \circ \sigma_{\tilde{\varphi}} = \widetilde{\Psi}_{\tilde{\varphi}_1} \circ \sigma_{\tilde{\varphi}} \qquad 6.5.12$$

is an analytic map !??

$\Psi_{\tilde{\varphi}_1}$ analytic

$$V_{\tilde{\varphi}} \longrightarrow (Q^{\infty})^{\Gamma_{\tilde{\varphi}_1}}. \qquad 6.5.13$$

This gives $\mathcal{T}_S$ the structure of a Banach analytic manifold. It is clear that with this structure the map Φ_S is analytic, since composing it with the coordinate $\Psi_{\tilde{\varphi}}$ gives $\widetilde{\Psi}_{\tilde{\varphi}}$, which is analytic. This proves part 1 of Theorem 6.5.1.

For part 2, observe that Φ_S is a submersion, because the composition $\Psi_{\tilde{\varphi}} \circ \Phi_S$ is a submersion. Indeed, $\Psi_{\tilde{\varphi}}$ has a right inverse. In fact, locally having a right inverse is the definition of being a split submersion. □

$$\underbrace{\Psi_{\tilde{\varphi}}}_{\text{analytic?}} \circ \Phi_S = \underbrace{\widetilde{\Psi}_{\tilde{\varphi}}}_{\text{analytic}}.$$

On a small ball $V_{\tilde{\varphi}}$ we have a **right inverse**

$$(\Psi_{\tilde{\varphi}} \circ \Phi_S) \circ \sigma_{\tilde{\varphi}} = \mathrm{id}_{V_{\tilde{\varphi}}}.$$

Any map that admits a local right inverse is a **split submersion**: differentiating at a point $q \in V_{\tilde{\varphi}}$ gives

$$D(\Psi_{\tilde{\varphi}} \circ \Phi_S)_p \circ D\sigma_{\tilde{\varphi}}(q) = I,$$

so $D(\Psi_{\tilde{\varphi}} \circ \Phi_S)_p$ is onto and has the bounded right inverse $D\sigma_{\tilde{\varphi}}(q)$.

$\Psi_{\tilde{\varphi}}$ is a chart, hence $D\Psi_{\tilde{\varphi}}$ is a Banach-space isomorphism.

$D(\Psi_{\tilde{\varphi}} \circ \Phi_S)_p = D\Psi_{\tilde{\varphi}}|_{\Phi_S(p)} \circ D\Phi_S|_p$ is surjective with the splitting $\Leftrightarrow D\Phi_S|_p$ is.

Therefore $D\Phi_{S_p}$ is surjective and split.