INDIAN INSTITUTE OF SCIENCE EDUCATION AND RESEARCH PUNE

Mid-Sem Exam, 2019 August Semester

IISER PUNE

Course Name: Introductory Mechanics
Date: 27 September 2019
Instructor(s): Sudarshan Ananth & Mukul Kabir

Course Code: PH1113
Duration: 2 hrs
Total Marks: 50

Instructions:

1. Printouts, photocopies, notes and textbooks are not allowed.
2. There should be 3 pages in this question paper. Please check that all the pages are there in your copy of the question paper.
3. There are 5 questions, 10 marks each. Answer all questions.

Questions:

1. **2 + 4 + (2 + 2) marks**

   (a) The vectors $\vec{A} = 2\hat{i} - 3\hat{j}$ and $\vec{B} = -\hat{i} + a\hat{j} - 5\hat{k}$ are perpendicular to each other. What is the value of a?

   (b) Find two unit vectors which are perpendicular to both $\vec{C} = 2\hat{i} - 3\hat{j}$ and $\vec{D} = -\hat{i} + 4\hat{j} - 5\hat{k}$

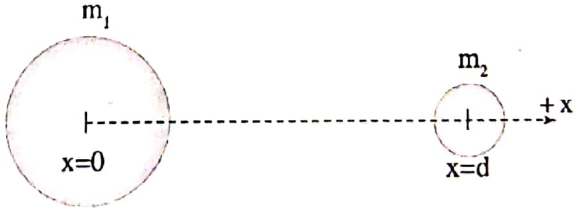
   (c) A rocket has launched straight up into the air. As time $t = 0$ the rocket is at rest, and about to be launched. The position of the rocket as a function of time is given by,

   $$y(t) = \frac{1}{2}(a_0 - g)t^2 - \frac{1}{30}\frac{a_0}{t_0^4}t^6, \text{ for } 0 < t < t_0,$$

   where a_0 is a positive constant, g is the acceleration of gravity, and $a_0 > g$. The constant t_0 is the time that the rocket takes to burn out the entire fuel.
   Find out the y-component of the acceleration as a function of time t. Argue that your result is correct.

2. **3 + (1 + 1 + 1) + (3 +1) marks**

   (a) The center of two spherical planets of mass m_1 and m_2 ($m_1 \neq m_2$) are separated by a distance d. Consider the origin of the coordinate system to be at the center of planet with mass m_1.



   At what location x measured from the center of planet m_1 will a third planet of mass m experience zero gravitational force?

   (b) A person on a spherical asteroid of mass m_1 and radius R, sees a small satellite of mass m_2 orbiting the asteroid in a circular orbit of period T. For the following questions, express your results in terms of m_1, m_2, π, T and universal gravitational constant G.

(I) Derive an expression for the radius r_{sat} of the satellite's orbit.

(II) What is the magnitude of the velocity of the satellite, v_{sat}?

(III) The asteroid rotates about its axis with a period T_a such that the satellite appears stationary to the person on the asteroid. Find an expression for T_a.

(c) A particle of mass m is subjected to two forces, a central force $\vec{f_1}$ and a frictional force $\vec{f_2}$, with

$$\begin{aligned}\vec{f_1} &= f(r)\hat{r} \\ \vec{f_2} &= -\lambda\vec{v} \quad (\lambda > 0),\end{aligned}$$

where v is the velocity of the particle.

(I) If the particle initially has angular momentum L_0 about $r = 0$, find its angular momentum for all subsequent times.

(II) Using the result in (I), comment on the situation $\lambda = 0$.

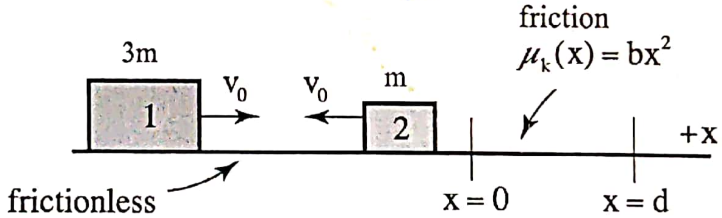
> **Help:** In polar coordinates,
>
> $$\begin{aligned}\dot{\vec{r}} &= \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} \\ \ddot{\vec{r}} &= (\ddot{r} - r\dot{\theta}^2)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta}\end{aligned}$$

3. **4 + (2 + 2 + 2) marks**

(a) Block 1 of mass $3m$ is sliding along a frictionless horizontal table to the right with speed v_0, and collides with block 2 of mass m that was moving to the left with speed v_0. After collision, the two blocks stick together and the blocks enter a rough surface at $x = 0$ with a coefficient of kinetic friction that increases with distance as $\mu(x) = bx^2$ for $0 \leq x \leq d$, where b is a positive constant. The blocks come to rest at $x = d$. The downward gravitational acceleration is g. Derive an expression for the initial speed v_0 of the blocks.

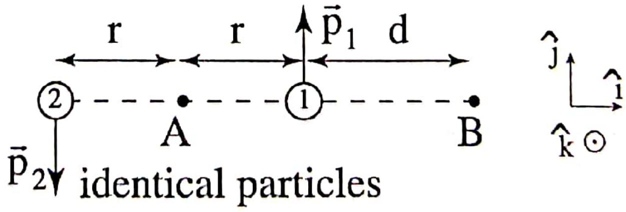
3mv 2mv 4m $\frac{v_0}{2}$

μm



F(x)

(b) Two identical particles 1 and 2 form a system. At the instant shown in the picture, the particles have equal and opposite momentums, $p_2 = -p_1 = p$.



(I) Determine a vector expression for the angular momentum of the system about the point A.

(II) Determine a vector expression for the angular momentum of the system about the point B.

(III) How do your results for angular momentum about A and B compare, and discuss your result.

4. (2 + 1) + (3 + 1) + 3 marks

(a) The displacement of a simple harmonic oscillator is given by $x = a \sin \omega t$. If the values of the displacement x and the velocity $\dot{x}$ are plotted on perpendicular axes, eliminate t to show that the equation of the points $(x, \dot{x})$ is an ellipse. Show that this ellipse represents a path of constant energy.

(b) A particle oscillates with simple harmonic motion along the x axis with a displacement amplitude a. While in motion, it moves from x to $x + dx$ in time dt. Show that the probability of finding it between x to $x + dx$ is given by,

$$\frac{dx}{\pi\sqrt{(a^2 - x^2)}}$$

$\frac{\omega^2}{c^2} = k^2 m^2 c^2$ $\qquad$ $\frac{2\omega}{c^2} \frac{d\omega}{dk} = 2k$

Discuss the reasons that you think for the correctness of the result.

(c) Verify that the solution, $x = (A + Ct)e^{-bt/2m}$ satisfies the equation $m\ddot{x} + b\dot{x} + kx = 0$,

$$\text{when } \frac{b^2}{4m^2} = \frac{k}{m}.$$

5. 2 + (2 + 1) + 3 + 2 marks

(a) Show that $y = f(x + ct)$ is a solution of the wave equation,

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}$$

$\sqrt{\frac{k}{m}}$

(b) Show that the wave profile, that is $y_1 = f_1(x - ct)$ remains unchanged with time when c is the wave velocity. Show that the same is true for the wave profile $y_2 = f_2(x + ct)$.

(c) The phase velocity v of transverse waves in a crystal of atomic separation a is given by,

$$v = c\left[\frac{\sin(ka/2)}{(ka/2)}\right],$$

where k is the wave number and c is the constant. Show that the value of the group velocity is $c\cos(ka/2)$. Determine the limiting value of the group velocity for long wavelengths?

(d) In relativistic wave mechanics the dispersion relation for an electron of velocity $v = \hbar k/m$ is given by $\omega^2/c^2 = k^2 + m^2c^2/\hbar^2$, where c is the velocity of light, m is the electron mass (considered constant at a given velocity), $\hbar = h/2\pi$, h is Planck's constant. Show that the product of group velocity and particle (phase) velocities is c^2.

$2y \sin(kx + \omega t)$ $\quad$ $\cos\left(\frac{dk}{2}x + \frac{d\omega}{2}\right)$

$\int a \sin(kx - \omega t)$

$\int_0^T a \sin\left(kx - \frac{2\pi}{T}t\right)dt$