Estimating Dynamic Value Effects with High Dimensional States

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INTRODUCTION

Weighted averages of the value function are important objects of interest for dynamic choice models.

Gives the value of an optimal policy by groups and can be used to quantify effects of changes in distribution of time invariant state variables, like initial wealth .

This paper is about estimation of these objects via neural nets and other machine leaners that allow for high dimensional state variables.

We give debiased machine learners of the weighted average, average derivative, and other objects that depend linearly on the value function.

These estimators are doubly robust and allow for a wide variety of estimators of per period returns.

Neural nets and other machine learners give good predictions in high dimensional settings but are inherently biased.

The source of the bias is regularization and model selection that controls the variance in order to get good out of sample predictions.

For example, neural nets often approximately minimize an objective function that includes a penalty term proportional to the sum of squares of the coefficients.

Neural nets also often regularize by averaging over estimates obtained by running stochastic gradient starting from different random starting values while stopping iteration short of convergence.

These and other regularization methods make the neural nets and other machine learning methods biased.

This bias is a problem for estimating objects of interest like weighted averages of value functions.

The bias passes through to the weighted average while taking the weighted average reduces variance.

Consequently weighted averages of value function estimators may have biases much larger than standard deviations, making standard inference methods are poor.

A solution is Neyman orthogonal estimating equations, where value function estimator has no first-order effect.

Cross-fitting can also help reduce bias (and variance in some cases).

Neyman orthogonal estimating equations generally depend on a debiasing function that we denote by α_0.

Orthogonality can be used to estimate α_0.

Orthogonality implies estimating equations for α_0 and in some cases objective functions for estimation of α_0, that are automatic in only using estimating equation for parameter of interest.

Describe here automatic estimation of α_0 for weighted average value function.

MODEL AND OBJECTS OF INTEREST

We a consider a setting where X is a vector of state variables and $\{X_t\}_{(t=0,1,...)}$ is a stationary, first-order time-homogeneous Markov process.

Our objects will depend on the expected value function $V_0(X)$ that is the expected discounted utility of optimal choices given a current state of X.

The expected value function will depend on a per period expected reward function $\zeta(X)$ as

$$V_0(X) = \sum_{t=0}^{\infty} \beta^t E[\zeta(X_t)|X_0 = X],$$

where $\beta \in [0, 1)$ is a known discount factor.

For example $\zeta(X)$ could be the expectation of per period utility assuming dynamically optimal choices.

We shall work with the recursive characterization of $V_0(X)$ as

$$V_0(X) = \zeta(X) + \beta E[V_0(X_+)|X],$$

where X is the current and X_+ is the next period.

We are interested in the effect of time invariant state variables on average value function.

We consider a state variable vector of the form

$$X = (S, K)$$

where K is time invariant and S can vary over time.

We will be interested in the effect of the time invariant variables K on weighted averages of the value function.

Some examples of parameters of this form are weighted averages of the value function and expected derivatives of the value function.

Example 1: A weighted average of value function

$$\delta_0 = E[w(X)V(X)].$$

May want a weight that depends on state variables.

A special case is when R is discrete valued and $w(X) = 1(R = r)/\Pr(R = r)$.

Here $P(R = r)$ unknown but ignore that for now.

Example 2: The average derivative of the value function with respect to a time invariant state variable K is

$$\delta_0 = E[\frac{\partial V(X)}{\partial K}] = E[\frac{\partial V(S,K)}{\partial K}].$$

This parameter quantifies average change in the value function with a change in a time invariant, continuous state variable, like initial wealth.

This parameter quantifies the effect a change in initial wealth on the value of following an optimal policy.

Examples 1 and 2 are both special cases of a parameter of the form

$$\delta_0 = E[m(W, V_0)],$$

where $m(W,V)$ is a known function of a data observation W and that is linear in a possible value function V.

In Example 1 $m(W,V) = w(X)V(X)$ and in Example 2 $m(W,V) = \partial V(X)/\partial R$.

Later we will generalize this set up to cover other examples.

We will view the true $V_0(X)$ as the solution to a conditional moment restriction of the form

$$E[\lambda(X_+, X, V)|X] = 0,\ \lambda(X_+, X, V) = \beta V(X_+) - V(X) + \zeta(X),$$

which is another version of the recursive representation of $V_0(X)$.

Here we can see that V_0 solves a nonparametric instrumental variable (NPIV) condition with residual given by $\lambda(X_+, X, V)$ and instrumental variable that is X.

Such NPIV problems were considered by Newey and Powell (1989, 1991, 2003).

Ai and Chen (2003, 2007, 2012) developed and analyzed estimators of parameters like $\delta_0 = E[m(W, V_0)]$ that depend on the solution to a conditional moment restriction.

In his paper we give and analyze debiased machine learners of such parameters as in Chernozhukov et al. (2022).

For now we assume that the per period utility $\zeta(X)$ is known; we return later in the paper to consider the common case where $\zeta(X)$ is estimated.

DEBIASED MACHINE LEARNER

Neyman orthogonal moment functions identify the parameter of interest and have the property that there is zero first order effect of the unknown nonparametric function, which is $V(X)$ here.

As in Chernozhukov et al. (2022, "Locally Robust Semiparametric Estimation," LR) a Neyman orthogonal moment function can be formed as the sum of an identifying moment function and the first step influence function.

Here the identifying moment function is

$$g(W,V,\delta) = m(W,V) - \delta.$$

From Ichimura and Newey (2022) the FSIF is the product of an unknown function of the instrument $\alpha(X)$ and the residual $\lambda(X_+, X, V)$ given by

$$\phi(W,V,\alpha) = \alpha(X)\lambda(X_+, X, V).$$

The orthogonal moment function is then

$$\begin{aligned}\psi(W,V,\alpha,\delta) &= g(W,V,\delta) + \phi(W,V,\alpha) \\ &= m(W,V) - \delta + \alpha(X)\lambda(X_+, X, V).\end{aligned}$$

$$\psi(W, V, \alpha, \delta) = m(W, V) - \delta + \alpha(X)\lambda(X_+, X, V).$$

Using this orthogonal moment function we can construct a debiased machine learner of δ_0 from estimators $\hat{V}$ of V_0 and $\hat{\alpha}$ of α_0.

We do this using cross-fitting, where the orthogonal moment function is averaged over different observations than used to form $\hat{V}$ and $\hat{\alpha}$.

Partition the observation indices $(i = 1, ..., n)$ into L groups I_ℓ, $(\ell = 1, ..., L)$.

Let $\hat{V}_\ell$ and $\hat{\alpha}_\ell$, be estimators that are constructed using all observations *not* in I_ℓ.

A debiased machine learner is

$$\hat{\theta} = \frac{1}{n} \sum_{\ell=1}^{L} \sum_{i \in I_\ell} \{m(W_i, \hat{V}_\ell) + \hat{\alpha}_\ell(X_i)\lambda(X_{+i}, X_i, \hat{V}_\ell)\}.$$

An estimator of the asymptotic variance is

$$\hat{\Omega} = \frac{1}{n} \sum_{\ell=1}^{L} \sum_{i \in I_\ell} \hat{\psi}_{i\ell},\ \ \hat{\psi}_{i\ell} = m(W_i, \hat{V}_\ell) + \hat{\alpha}_\ell(X_i)\lambda(X_{+i}, X_i, \hat{V}_\ell) - \hat{\theta}.$$

$$\hat{\theta} = \frac{1}{n}\sum_{\ell=1}^{L}\sum_{i\in I_\ell}\{m(W_i,\hat{V}_\ell) + \hat{\alpha}_\ell(X_i)\lambda(X_{+i},X_i,\hat{V}_\ell)\}.$$

$L = 5$ works well based on a variety of other empirical examples and in simulations, for medium sized data sets; see Chernozhukov et al. (2018).

$L = 10$ works well for small data sets.

This cross-fitting

– eliminates "own observation" bias, like jackknife instrumental variables;

– helps remainders converge faster to zero, e.g. Newey and Robins (2017);

– eliminates need for Donsker conditions not satisfied by many machine learners

Example 1:

$$\hat{\theta} = \frac{1}{n} \sum_{\ell=1}^{L} \sum_{i \in I_\ell} \{w(X_i)\hat{V}_\ell(X_i) + \hat{\alpha}_\ell(X_i)\lambda(X_{+i}, X_i, \hat{V}_\ell)\}$$

Example 2:

$$\hat{\theta} = \frac{1}{n} \sum_{\ell=1}^{L} \sum_{i \in I_\ell} \{\frac{\partial \hat{V}_\ell(X_i)}{\partial R} + \hat{\alpha}_\ell(X_i)\lambda(X_{+i}, X_i, \hat{V}_\ell)\}.$$

Examples have different $\hat{\alpha}_\ell(X_i)$.

The Neyman orthogonal moment function is

$$\begin{aligned}\psi(W,V,\alpha,\delta) &= m(W,V)-\delta+\alpha(X)\lambda(X_+,X,V),\\ \lambda(X_+,X,V) &= \beta V(X_+)-V(X)+\zeta(X).\end{aligned}$$

It is linear in V and so by LR we know that it is doubly robust in V and α, meaning that for all V and α,

$$0=E[\psi(W,V,\alpha_0,\delta_0)]=E[\psi(W,V_0,\alpha,\delta_0)].$$

CHARACTERIZING AND ESTIMATING α_0

Can use Neyman orthogonality in γ, which here is robustness in γ, to characterize and estimate α_0.

To characterize α_0 suppose that $E[m(W,V)]$ is linear and mean square continuous in V.

Then by Riesz representation theorem there is $w(X)$ such that

$$E[m(W,V)] = E[w(X)V(X)].$$

Example 1 has $w(X) = w(X)$.

Example 2 we integrate by parts to get

$$w(X) = -\frac{\partial \ln[f(R|S)]}{\partial R}.$$

For $\Delta(X) = V(X) - V_0(X)$, global robustness in $V(X)$ and stationarity give

$$\begin{aligned} 0 &= E[w(X)\Delta(X) + \alpha_0(X)\{\beta\Delta(X_+) - \Delta(X)\}] \\ &= E[w(X)\Delta(X) + \beta\alpha_0(X_-)\Delta(X) - \alpha_0(X)\Delta(X)\}] \\ &= E[\{w(X) + \beta\alpha_0(X_-) - \alpha_0(X)\}\Delta(X)], \end{aligned}$$

for all $\Delta(X)$.

That is, for all $\Delta(X)$,

$$0 = E[\{w(X) + \beta\alpha_0(X_-) - \alpha_0(X)\}\Delta(X)].$$

This implies

$$0 = w(X) + \beta E[\alpha_0(X_-)|X] - \alpha_0(X).$$

The $\alpha_0(X)$ that solves this is

$$\alpha_0(X) = \sum_{t \geq 0} \beta^t E[w(X_{-t})|X].$$

Thus $\alpha_0(X)$ is the net present backward discounted value of the Riesz representer for the $E[m(W, V)]$.

$$\alpha_0(X) = \sum_{t \geq 0} \beta^t E[w(X_{-t})|X].$$

There is an important dual representation of $\delta_0 = E[w(X)V_0(X)]$ in terms of $\alpha_0(X)$.

By the value function characterization $V(X) = \sum_{t=0}^{\infty} \beta^t E[\zeta(X_t)|X]$, iterated expectations, and stationarity we have

$$\begin{aligned}
\delta_0 &= E[w(X) \sum_{t=0}^{\infty} \beta^t E[\zeta(X_t)|X]] = \sum_{t=0}^{\infty} \beta^t E[w(X)\zeta(X_t)] \\
&= \sum_{t=0}^{\infty} \beta^t E[w(X_{-t})\zeta(X)] = \sum_{t=0}^{\infty} \beta^t E[E[w(X_{-t})|X]\zeta(X)] \\
&= E[\sum_{t=0}^{\infty} \beta^t E[w(X_{-t})|X]\zeta(X)] = E[\alpha_0(X)\zeta(X)].
\end{aligned}$$

This is a dual representation of δ_0 in terms of $\alpha_0(X)$.

$$\delta_0 = E[\alpha_0(X)\zeta(X)]$$

This dual representation is very useful when $w(X)$ depends only on the time invariant R.

One such important special case is $w(X) = w(R) = 1(R = r)/\Pr(R = r)$.

When $w(X) = w(R)$,

$$\begin{aligned} \alpha_0(X) &= \sum_{t \geq 0} \beta^t E[w(R_{-t})|X] = \sum_{t \geq 0} \beta^t E[w(R)|X] = \sum_{t \geq 0} \beta^t w(R) \\ &= \frac{1}{1-\beta} w(R). \end{aligned}$$

Then

$$\delta_0 = E[\alpha_0(X)\zeta(X)] = \frac{1}{1-\beta} E[w(R)\zeta(X)].$$

Simple to estimate by

$$\hat{\delta} = \frac{1}{1-\beta} \frac{1}{n} \sum_{i=1}^{n} w(R_i)\delta(X_i).$$

More generally, need to estimate $\alpha_0(X)$.

Could solve sample version of

$$0 = w(X) + \beta E[\alpha_0(X_-)|X] - \alpha_0(X),$$

but this requires estimation of $w(X)$, which is not convenient for average derivative where $w(X)$ is log derivative of conditional pdf.

Instead can use automatic method.

Let $b(X) = (b_1(X), ..., b_p(X))'$ be dictionary of functions and suppose

$$\alpha_0(X) \approx b(X)'\gamma_0.$$

Double robustness implies that approximately,

$$\begin{aligned} 0 &= E[m(W,b) + b(X)'\gamma_0\{\beta b(X_+) - b(X)\}] \\ &= E[m(W,b) + \{\beta b(X_+) - b(X)\}b(X)'\gamma_0] \\ &= E[m(W,b)] + E[\{\beta b(X_+) - b(X)\}b(X)']\gamma_0. \end{aligned}$$

Plug-in sample moments and solve for $\hat{\gamma}$ as solution to this equation to get $\hat{\alpha}(X) = b(X)'\hat{\gamma}$.

Regularize to allow for large p.

DML FOR ESTIMATED $\zeta(X)$

First step influence function is sum of influence functions for unknown components.

Need to include first step influence function for $\zeta(X)$ to debias for estimated $\zeta(X)$.

Use dual characterization that $\delta_0 = E[\alpha_0(X)\zeta(X)]$.

Let $\hat{\phi}_\zeta(W)$ be estimated influence function for $\int \alpha_0(X)\hat{\zeta}(X)F_0(dx)$.

Will in general depend on $\alpha_0(X)$ so construction of $\hat{\phi}_\zeta(W)$ may require $\hat{\alpha}(X)$.

Debiased machine learner is then

$$\hat{\theta} = \frac{1}{n}\sum_{\ell=1}^{L}\sum_{i\in I_\ell}\{m(W_i,\hat{V}_\ell) + \hat{\alpha}_\ell(X_i)\lambda(X_{+i},X_i,\hat{V}_\ell) + \hat{\phi}_{\zeta\ell}(W_i)\}.$$

TO BE COMPLETED

–Estimation of value of policy effects.

–Mean square convergence rate for $\hat{\alpha}$.

–Large sample theory that includes debiasing for estimation of $\zeta(X)$.

–Application.