# Adaptive Estimation of Intersection Bounds – a Classification Approach

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## Introduction

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Covariate set $\mathcal{X}$ maps to a class of bounds

- any covariate subset $\mathcal{X}' \subseteq \mathcal{X}$ maps to a pair of valid bounds
- the full set $\mathcal{X}$ maps to sharp (the tightest possible) bounds

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- the full set $\mathcal{X}$ maps to sharp (the tightest possible) bounds

Sharp bounds are difficult to estimate, non-sharp bounds may not be very useful.

# Outline

1. Example
2. Debiased Inference
3. Linear Programming
4. Envelope Theorem

## Literature Review

1. **Envelope theorems. Stochastic programming** Shapiro (Annals of Statistics, 1989), Milgrom and Segal (2002)
2. **Bounds/partial identification**: (**identification**) Heckman (1976), Heckman (1979), Manski (1989), Manski (1990), Manski (1997), Heckman et al. (1997), Fan et al. (2017), Tetenov (2012), Kamat (2019) (**inference**) Fan and Park (2010, 2012), Chernozhukov et al. (2013), Kaido and Santos (2014), Kaido and White (2012), Kaido (2017) Kaido (2016), Kline and Tartari (2016), Abdulkadiroglu et al. (2020), Kaido et al. (2019), Hsieh et al. (2021), Fang et al. (2020)
3. **Policy Learning and Classification**: Tsybakov (2004), Qian and Murphy (2011), Kitagawa and Tetenov (2018), Athey and Wager (2021), Mbakop and Tabord-Meehan (2021), Sun (2021)
4. **Directional Differentiability** Fang and Santos (2018), Ponomarev (2022).
5. **Orthogonal/debiased machine learning**: Newey (1994), Belloni and Chernozhukov (2011), Chernozhukov et al. (2022), Belloni et al. (2017), Chernozhukov et al. (2018), Chiang et al. (2019), Sasaki and Ura (2020), Sasaki et al. (2020), Cha et al. (2022)

# (1): Example

## Example: setup

Notation

- $D = 1$ if subject wins a lottery
- $S(1) = 1$ employed if $D = 1$
- $S(0) = 1$ employed if $D = 0$
- $X$ pre-treatment (baseline) characteristics

Observed data: $(X, D, S)$ where $S = D \cdot S(1) + (1 - D)S(0)$.

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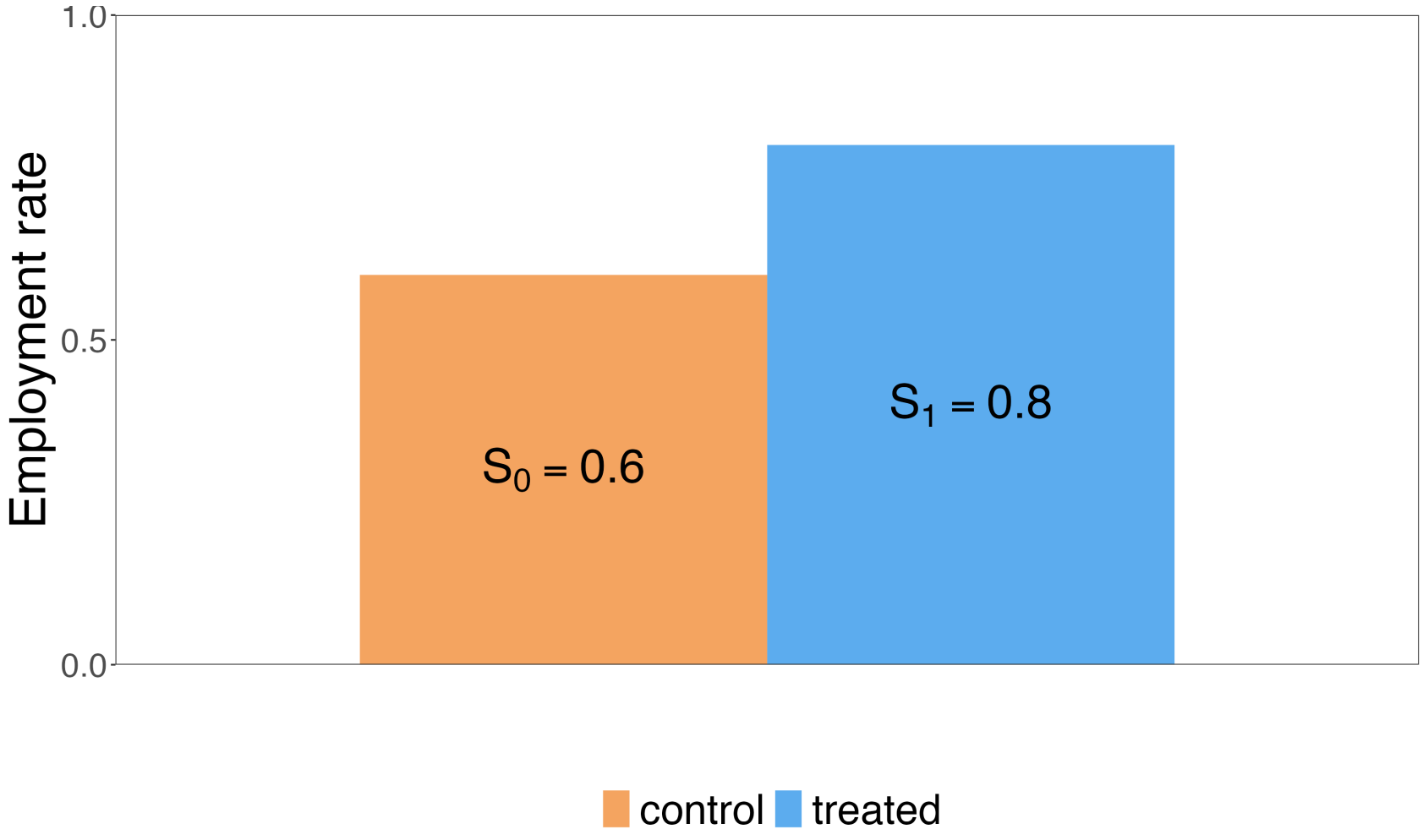
Observed data: $(X, D, S)$ where $S = D \cdot S(1) + (1 - D)S(0)$.

| | | Control ($D = 0$) | |
|---|---|---|---|
| | | $S(0) = 1$ | $S(0) = 0$ |
| Treated ($D = 1$) | $S(1) = 1$ | always-takers ($\pi_{AT}$) | compliers ($\pi_{comp}$) |
| | $S(1) = 0$ | defiers ($\pi_{defier}$) | never-takers ($\pi_{NT}$) |

Target parameter is

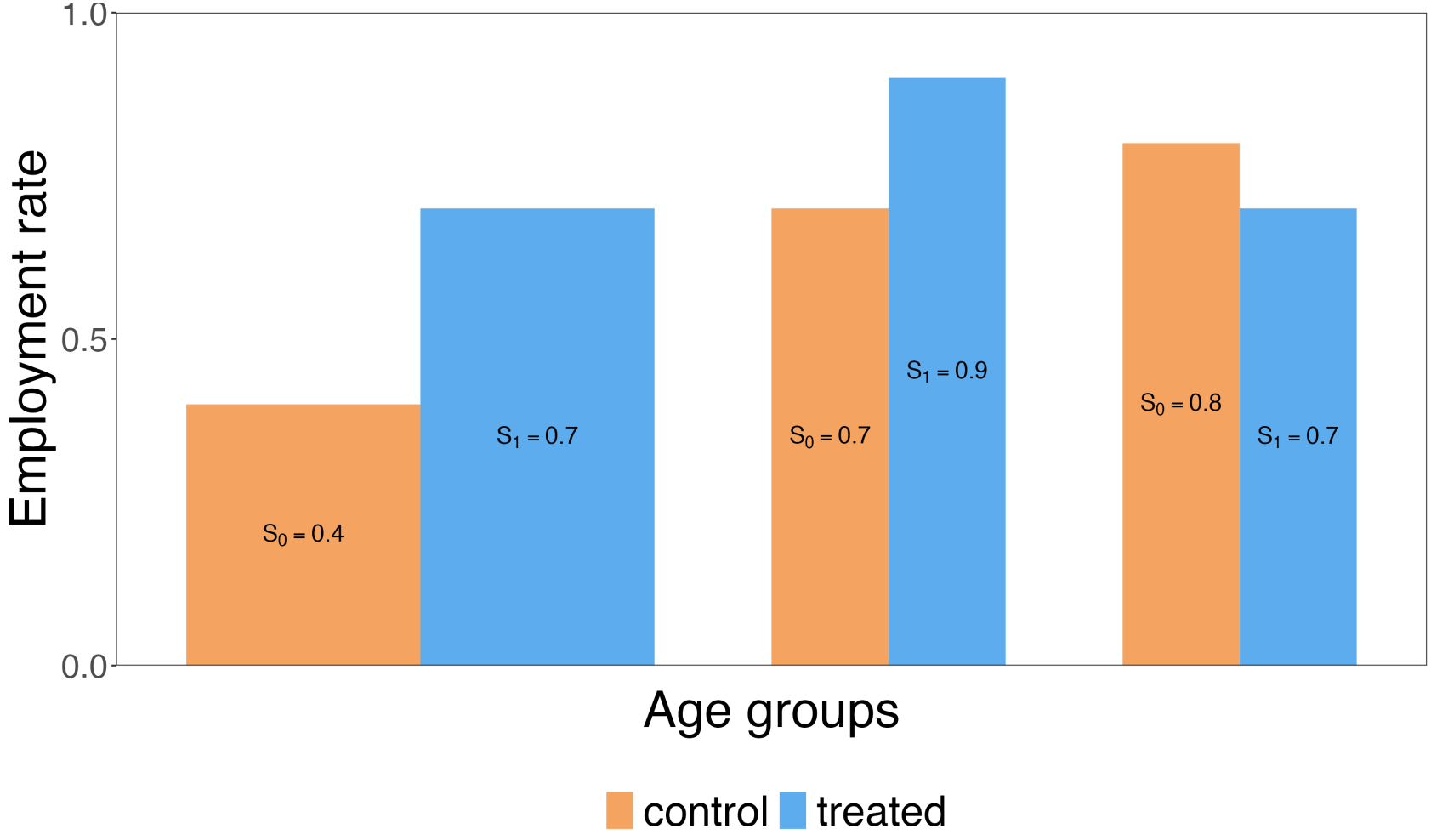
$$\bar{\pi} = (\pi_{AT}, \pi_C, \pi_D) \cdot (1, 0, 0) = \pi_{AT}$$

## Example: basic bound on $\pi_{AT}$




$$\pi_{AT} \leq \min(S_1, S_0) = \min(0.8, 0.6) = 0.6$$

## Example: tighter bound on $\pi_{AT}$




$$\mathbb{E}\min(s(0,X), s(1,X)) =$$

$$1/2\min(0.4, 0.7) + 1/4\min(0.7, 0.9) + 1/4\min(0.8, 0.7) = 0.55$$

Jensen: $0.55 < 0.6 = \min(S_1, S_0)$

# (2): Debiased Inference

## Example: age as a continuous variable

employment probability

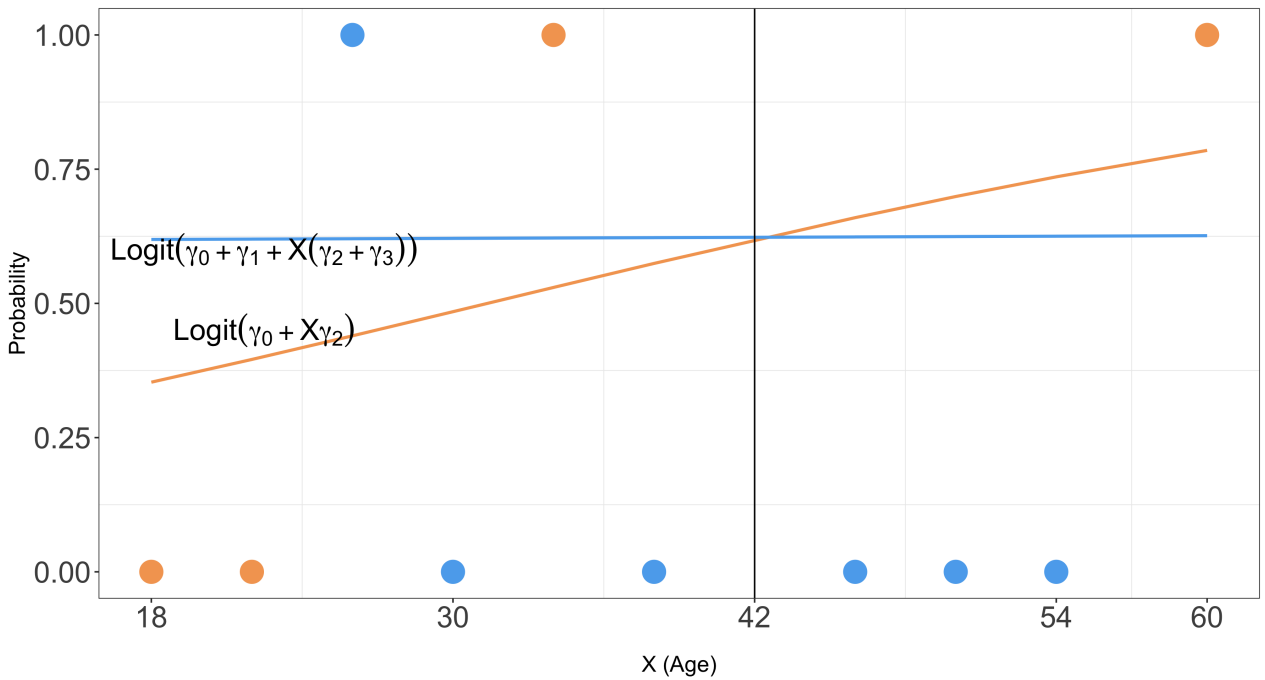
$$s(D, X) = \text{Logit}(\gamma_0 + D \cdot \gamma_1 + X \cdot \gamma_2 + D \cdot X \cdot \gamma_3)$$

regions of positive conditional ATE $s(1, X) - s(0, X)$

$$G := \{X : \quad s(1, X) - s(0, X) \geq 0\} = \{X : X \leq 42\}.$$

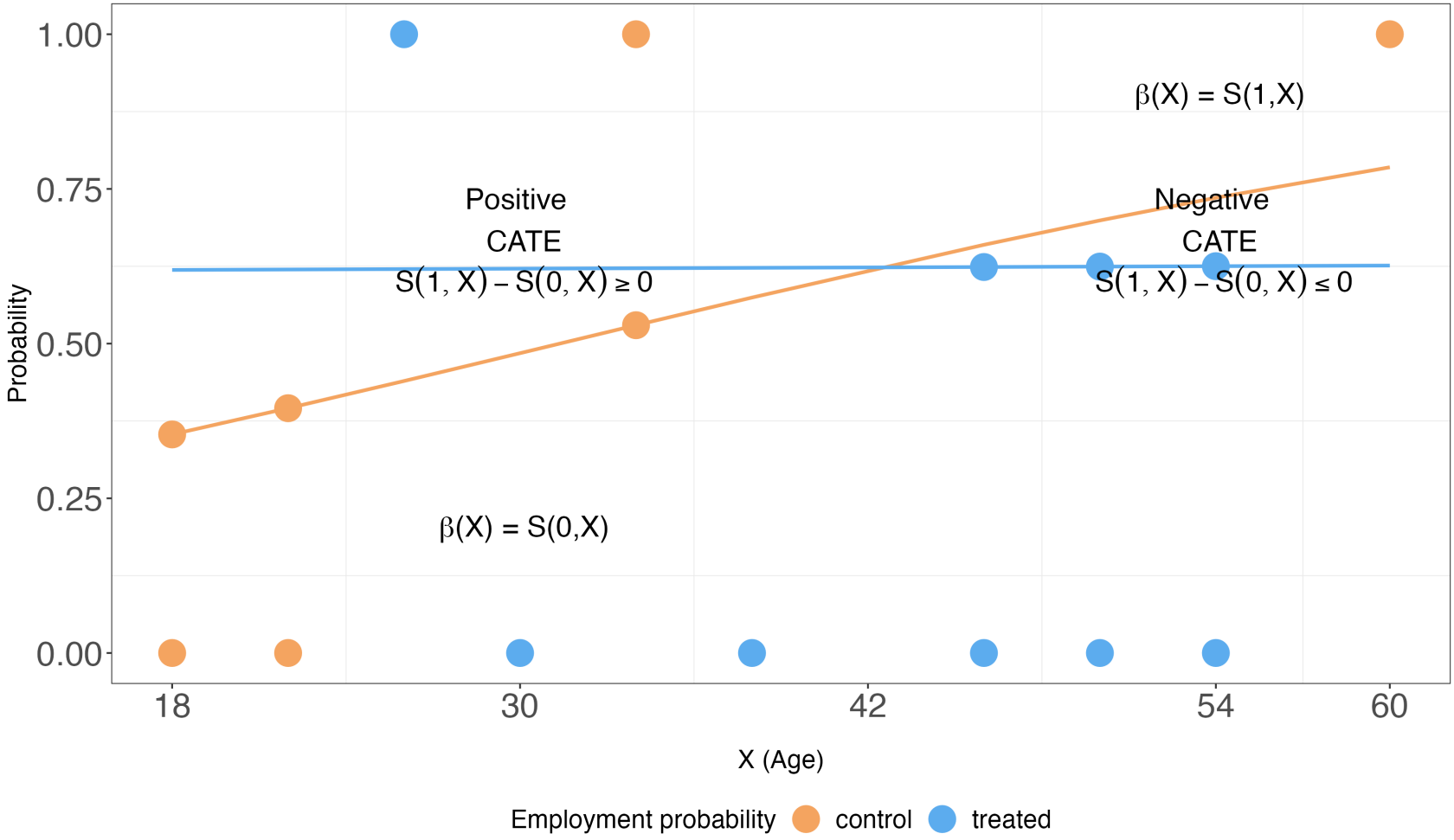
sharp bound is

$$\phi_0 = \mathbb{E} \min(s(0, X), s(1, X)) = 0.579$$



# Example: envelope regression

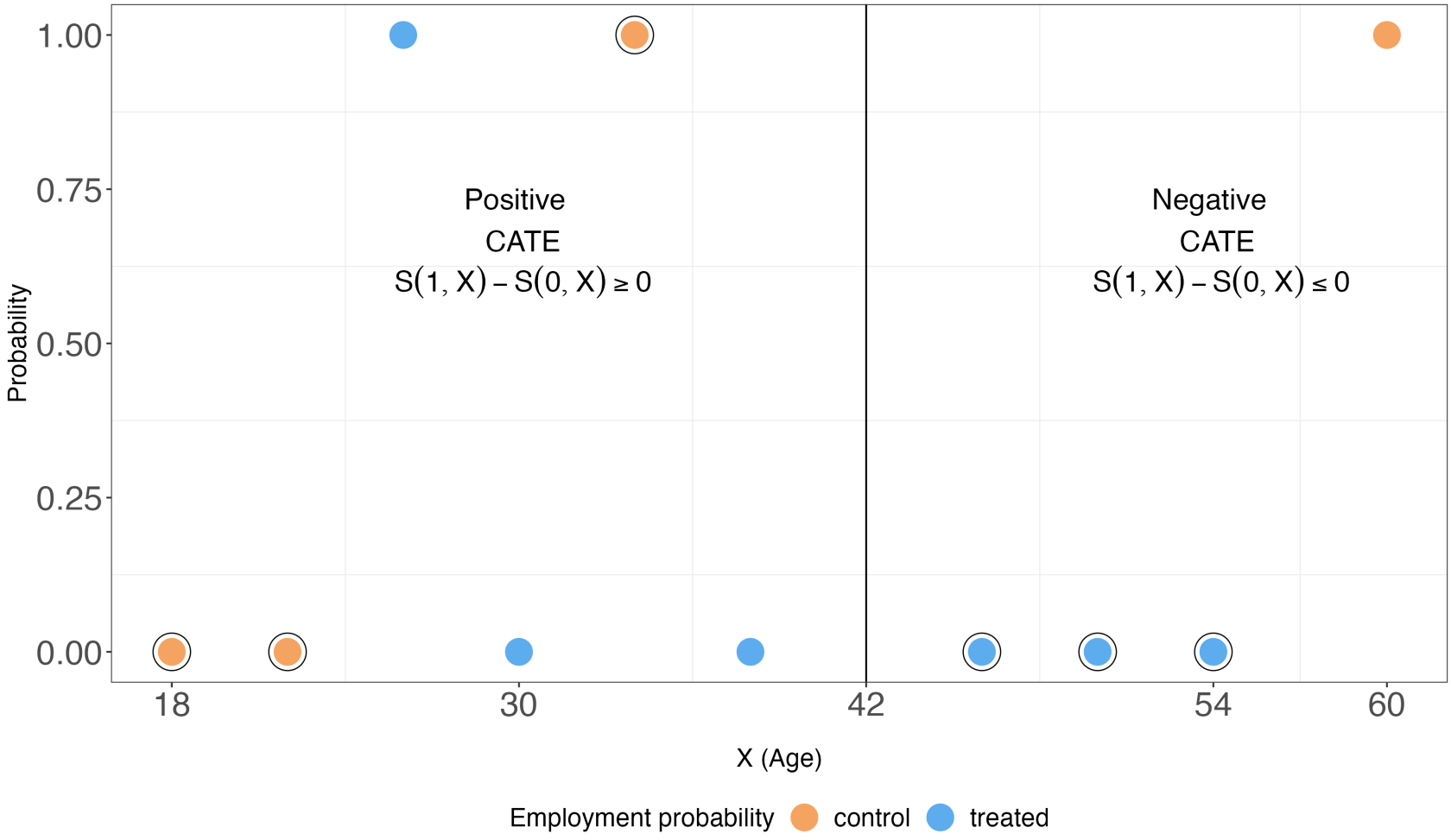
$$\widehat{\bar{\pi}}(\widehat{S}) := N^{-1} \sum_{i=1}^{N} \min(\widehat{S}(1, X_i), \widehat{S}(0, X_i))$$



# Example: envelope moment

region of positive CATE: $\widehat{G} := \{X : \widehat{s}(1, X) - \widehat{s}(0, X) \geq 0\}$

$$\widehat{\bar{\pi}}(\widehat{G}) := (N^{-1} \sum_{i=1}^{N} 2S_i(1 - D_i)\{X_i \in \widehat{G}\} + 2S_i D_i\{X_i \in \widehat{G}^c\})$$



## Oracle property: result

**Assumptions.**

1. the covariate $X$ has bounded density (margin condition, Tsybakov (2004))
2. $\sup_{x\in\mathcal{X}} \sup_{t\in T} |\widehat{S}(t,x) - S_0(t,x)| = o_P(n^{-1/4})$

**Result.** The envelope moment $\widehat{\pi}(\widehat{G}) = \pi(\widehat{S})$ based on the plug-in estimator

$$\widehat{G} := \{X : \widehat{s}(1,X) - \widehat{s}(0,X) \geq 0\}$$

obeys oracle property

$$\sqrt{N}(\widehat{\pi}(\widehat{S}) - \widehat{\pi}(S_0)) \Rightarrow^P 0.$$

As a result, $\widehat{\pi}(\widehat{S})$ is asymptotically Gaussian with oracle variance

$$\sqrt{N}(\widehat{\pi}(\widehat{S}) - \pi_0) \Rightarrow N(0, V_\pi), \quad V_\pi = (2-\pi_0)\pi_0.$$

## Oracle property: discussion

- The result generalizes to infimum over infinite set $T$

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$$\phi_0 = \mathbb{E}_X \inf_{t \in T} s(t, X)$$

  - treatment effect CDF $\Pr(S(1) - S(0) \leq t)$ with $T = \mathrm{R}$
  - positive treatment effects $\mathbb{E}(S(1) - S(0))_+$ with $\cup_{t \in \mathrm{R}} T_t = \{1, 0\}$
  - best linear predictor of intersection bounds $\inf_{t \in T} s(t, X)$

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- Regularity of $\mathbb{E}_X \inf_{t \in T} s(t, X)$
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  - this paper: $\mathbb{E}_X \inf_{t \in T} s(t, X)$ bootstrap applies

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- Applications beyond bounds
  - welfare in statistical treatment choice (Kitagawa and Tetenov (2018))
  - Bayes (optimal) risk in classification literature

# (3): Linear Programming (LP)

## LP: setup

1. the constraint set is

$$A\pi = \mathbf{S}(x),$$

   where the RHS is a conditional expectation

$$\mathbf{S}(x) = \mathbb{E}[\mathcal{S} \mid X = x]$$

2. the conditional bound is

$$\bar{\pi}(x) := \min_{\pi \in \mathrm{R}^+} -\pi_1 \text{ s. t. } A\pi = \mathbf{S}(x)$$

3. the target is the average conditional bound

$$\mathbb{E}_X \bar{\pi}(X)$$

4. by Jensen's inequality,

$$\mathbb{E}_X \bar{\pi}(X) \leq \bar{\pi}.$$

## LP: upper bound as an envelope of regression

1. Dual feasible set is data-free

$$\nu \in \mathrm{R}^r : A'\nu \geq (1, 0, \ldots, 0)'.$$

2. Dual set reduces to its vertices $\mathcal{T} = \{\nu_t\}$

$$\underbrace{\min_{\nu \,:\, A'\nu \geq e_1}}_{\text{infinite}} \nu'\mathbf{S}(x) = \underbrace{\min_{\nu_t \in \mathcal{T}}}_{\textit{finite}} \nu_t'\mathbf{S}(x)$$

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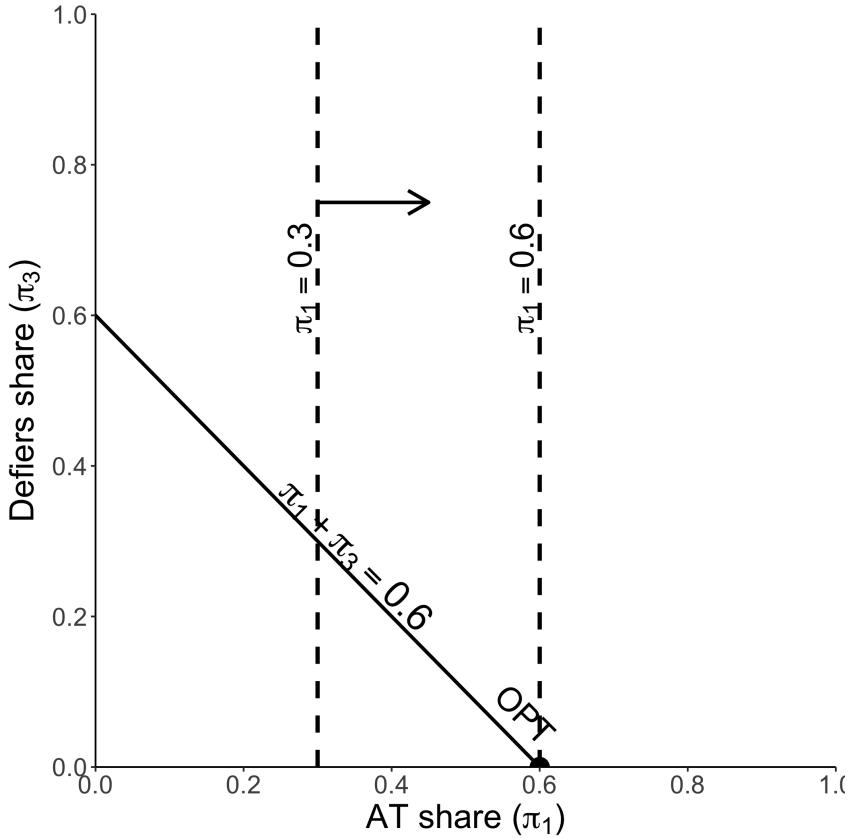
3. At the optimum, primal LP $=$ dual LP

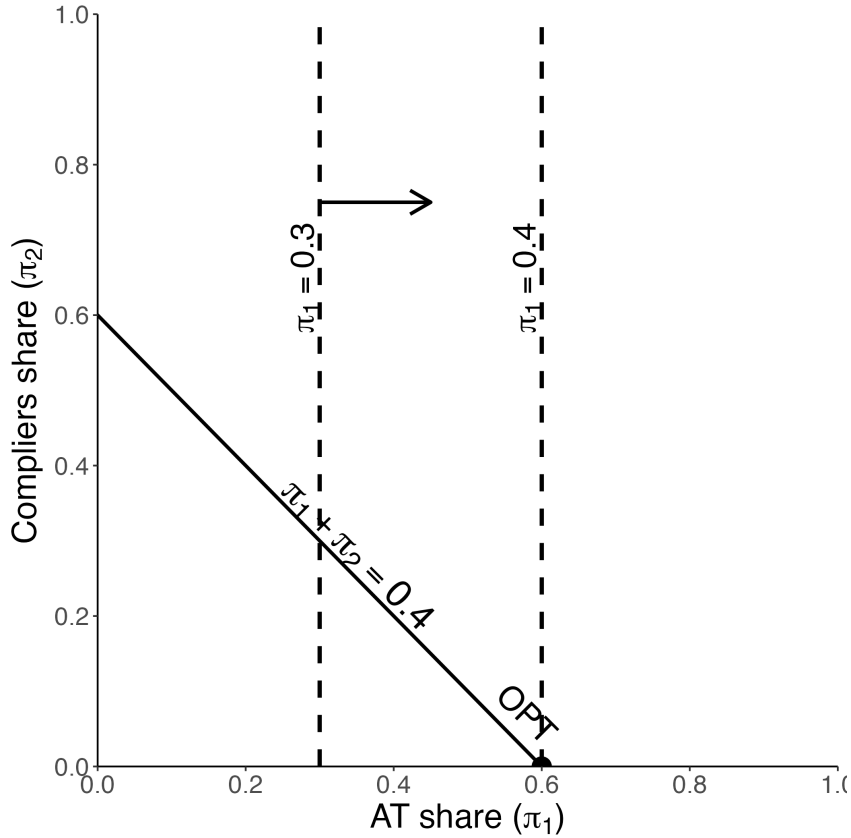
$$\bar{\pi}(x) = \inf_{\nu_t \in \mathcal{T}} \underbrace{\mathbf{S}(x)'\nu_t}_{} =: s(t, x) = \inf_{t \in T} s(t, x).$$

Duality has been used in Kaido (2017), Fang et al. (2020), Hsieh et al. (2021) (JoE, 2021)

# Example: primal LP is a regression problem

Example. $A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$

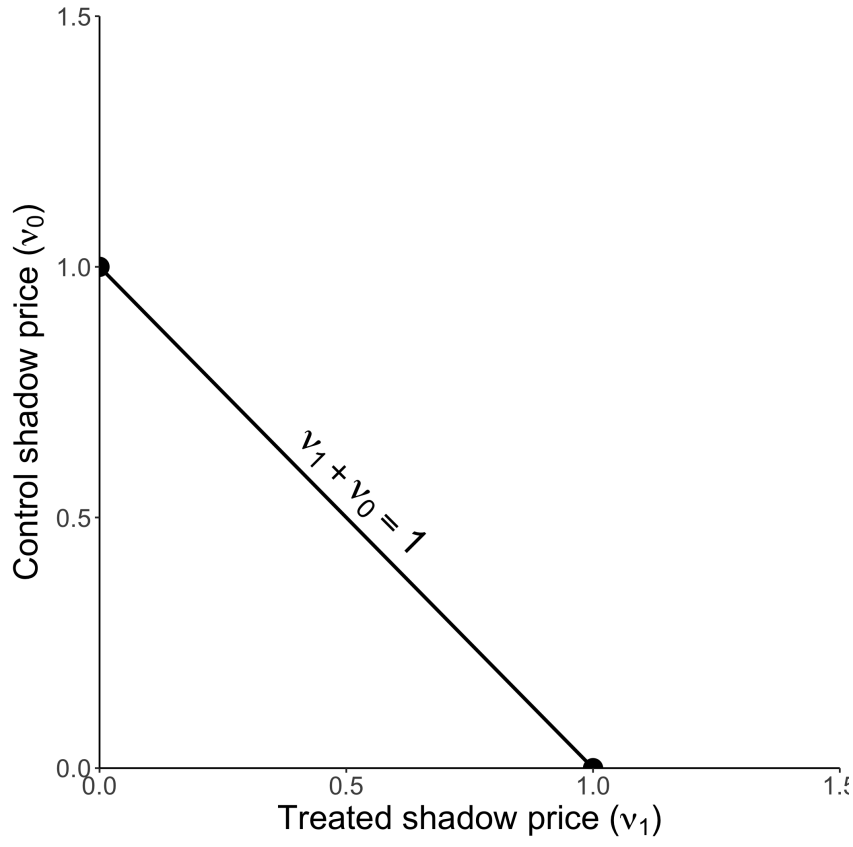



$$\mathbf{S}(x_1) = \begin{pmatrix} 0.8 \\ 0.6 \end{pmatrix}$$

$$\mathbf{S}(x_2) = \begin{pmatrix} 0.4 \\ 0.5 \end{pmatrix}$$

# Example: dual feasible set

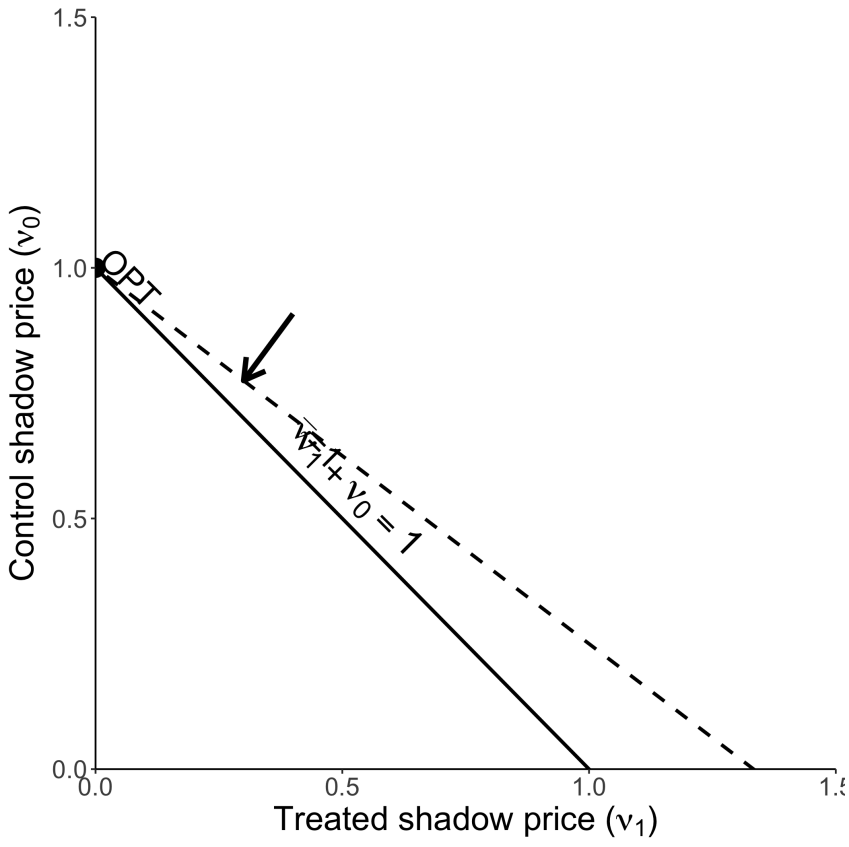



$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \nu, \quad A'\nu \geq (1,0,0)'$$

The vertex set $\mathcal{T} = \{(1,0),(0,1)\}$

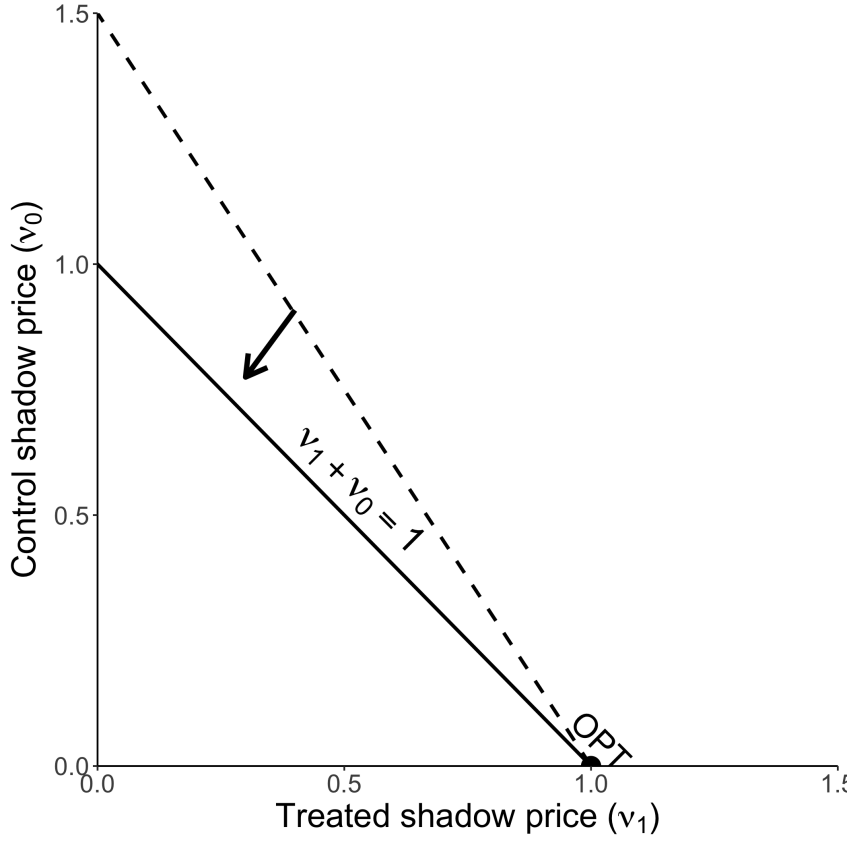
# Example: dual LP reduces to classification problem

$0.8 > 0.6$




$\min 0.8\nu_1 + 0.6\nu_2$ s. to $A'\nu \geq (1, 0, 0)'$

$0.4 < 0.5$



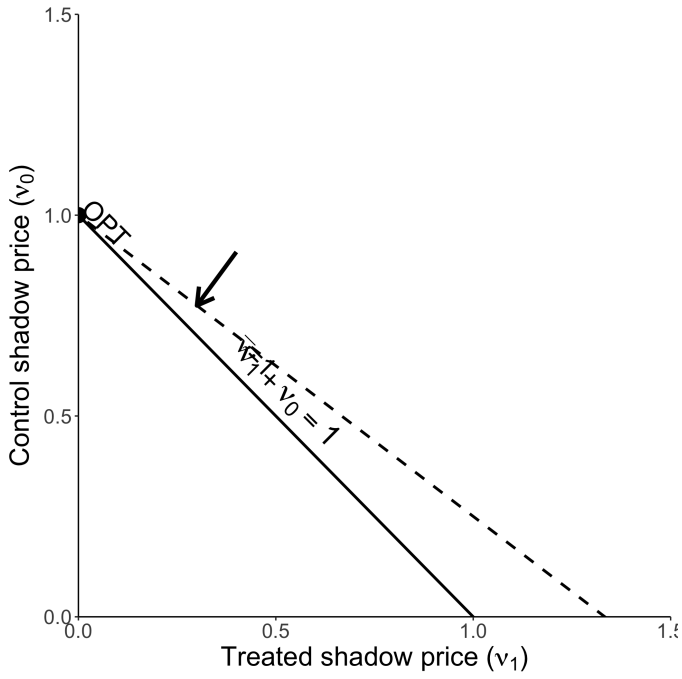

$\min 0.4\nu_1 + 0.5\nu_2$ s. to $A'\nu \geq (1, 0, 0)'$

## Example: estimate reduces to weighted sample average

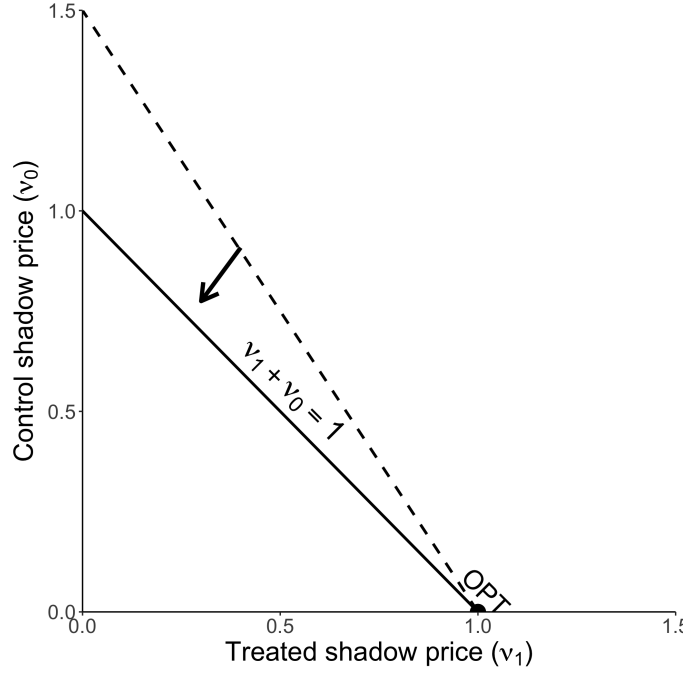
region of positive CATE:

$$\widehat{G} := \{x : \widehat{s}(1,x) - \widehat{s}(0,x) \geq 0\}$$

$x_1 \in \widehat{G}$




$x_2 \notin \widehat{G}$




The estimator is the sample average:

$$\widehat{\bar{\pi}}(\widehat{G}) := (N^{-1} \sum_{i=1}^{N} 2S_i(1 - D_i)\{X_i \in \widehat{G}\} + 2S_i D_i \{X_i \in \widehat{G}^c\}).$$

## Example: main take-aways

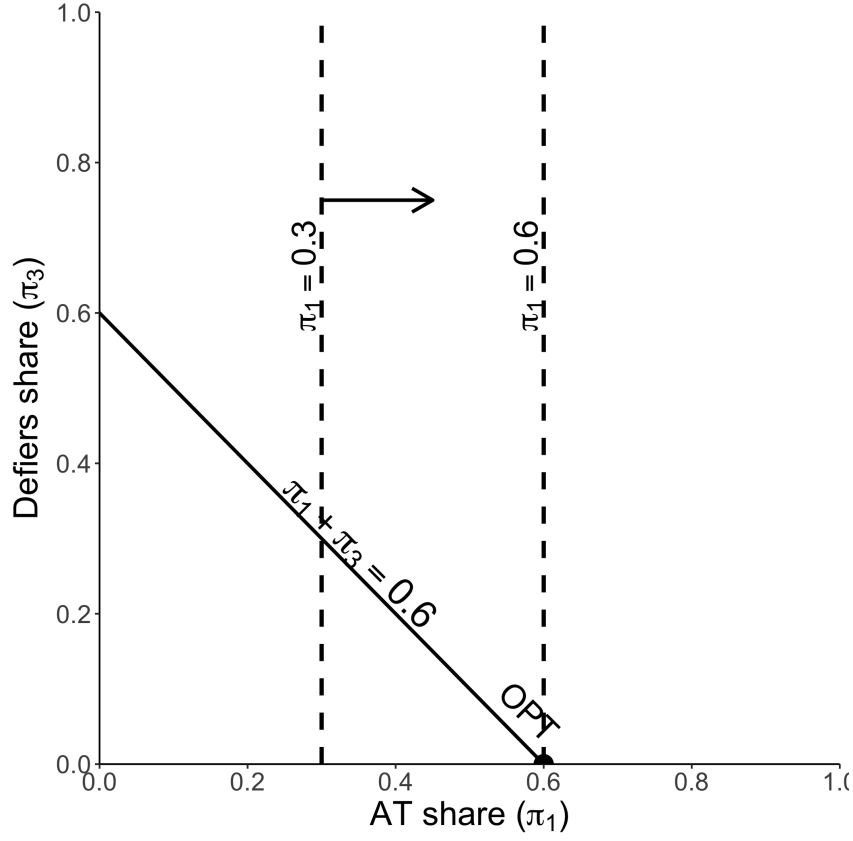
1. covariates tighten bounds
2. dual LP is a classification problem
3. dual LP is first-order insensitive to misclassification mistake
   - The dual vector $\bar{\nu}(X)$ is the Riesz representer function for the RHS function

# (3.b): Linear Programming (LP)

$A(x) \neq A$ depends on $x$

## General case: primal LP

$$\bar{\pi}(x) = \min_{\pi} -\pi_1 \quad \text{subject to} \quad A(x) \cdot \pi = \mathbf{S}(x)$$




Example. $$A(x) = \begin{pmatrix} 1/0.8 & 1/0.8 & 0 \\ 1/0.6 & 0 & 1/0.6 \end{pmatrix}, \qquad \mathbf{S} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

## General case: from envelopes to saddle values

1. Define Lagrangian function

$$L(\pi, \nu, x) := -\pi_1 + \nu^\top (A(x)\pi - \mathbf{S}(x)).$$

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1. Define Lagrangian function

$$L(\pi, \nu, x) := -\pi_1 + \nu^\top (A(x)\pi - \mathbf{S}(x)).$$

2. The objective function is the *saddle value* of regression

$$\bar{\pi}(x) = \max_{\nu} \min_{\pi} L(\pi, \nu, x) = L(\pi^*(x), \nu^*(x), x)$$

3. $(\pi^*(x), \nu^*(x))$ is a saddle value of $L(\pi, \nu, x)$

4. Envelope moment is

$$g(W, \pi, \nu) := \sum_{\nu,\pi} g_{\nu,\pi}(W) 1\{\pi = \pi^*(X), \nu = \nu^*(X)\},$$

where $g_{\nu,\pi}(W)$ is an unbiased signal for Lagrangian

$$\mathbb{E}[g_{\nu,\pi}(W) \mid X = x] = L(\pi, \nu, x).$$

## LP: oracle property for saddle moments

**Assumptions.**

1. the covariate $X$ has bounded density
2. $\sup_x \|\widehat{S}(x) - S_0(x)\| + \|\widehat{A}(x) - A_0(x)\| = o_P(n^{-1/4})$
3. new condition!: $(\widehat{\pi}, \widehat{\nu})$ must be a saddle-value, that is

$$L(x, \widehat{\pi}, \widehat{\nu}) = \max_\nu \min_\pi L(x, \widehat{\pi}, \widehat{\nu}) = \min_\pi \max_\nu L(x, \widehat{\nu}, \widehat{\pi})$$

**Result.** The saddle moment

$$\widehat{\phi}(\widehat{\nu}, \widehat{\pi}) := N^{-1} \sum_{i=1}^{N} \sum_{d=0}^{d=1} g_{\nu,\pi}(W_i) 1\{\nu = \widehat{\nu}(X_i), \pi = \widehat{\pi}(X_i)\}.$$

obeys oracle property

$$\sqrt{N}(\widehat{\phi}(\widehat{\nu}, \widehat{\pi}) - \widehat{\phi}(\nu_0, \pi_0)) \Rightarrow^P 0.$$

As a result, $\widehat{\phi}(\widehat{\nu}, \widehat{\pi})$ is asymptotically Gaussian with oracle variance

$$\sqrt{N}(\widehat{\phi}(\widehat{\nu}, \widehat{\pi}) - \phi_0) \Rightarrow N(0, V_\phi).$$

# (4): Envelope Theorem

## Key definitions and facts

(Envelope Theorem, Milgrom and Segal (2002)) The envelope function $V(\tau) := \inf_{t \in T} s(t, \tau)$ is differentiable. Its derivative

$$V'(\tau) = s_\tau(t^*(\tau), \tau), \qquad \tau \in [0, 1]$$

is calculated as if $t^*(\tau)$ was known.

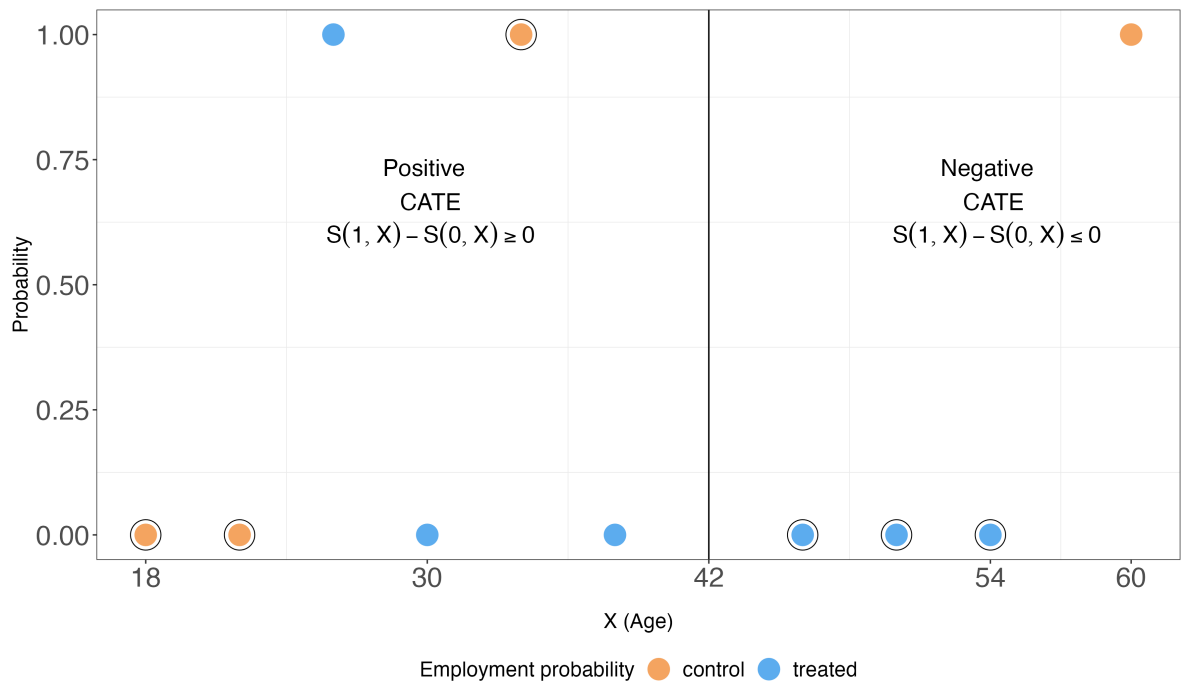
Let $\{P_\tau\}$ is a parametric submodel containing true distribution $P_0$. The target parameter $\phi(P_\tau), \tau \in [0, 1]$,

$$\frac{\partial \phi(P_\tau)}{\partial \tau} = \mathbb{E}\psi(W) S_\tau(W), \qquad \tau \in [0, 1]$$

where $\psi(W)$ is the **influence** function and $S_\tau(W)$ is the score. van der Vaart (1991)

# Influence function: $\phi(W) = \psi_1(W) + \psi_2(W)$

$$\phi(W) = \sum_{d=0}^{d=1} 2S\{D = d\}1\{s(d, X) = t^*(X)\} - \phi_0.$$



## Conclusion and Future Work

Proposed asymptotic theory for

- envelopes $\mathbb{E} \inf_{t \in T} s(t, X)$
- saddle-values $\mathbb{E} \max_{\nu} \inf_{\pi} L(\nu, \pi, X)$

Future Work

- quantify sharpness-complexity tradeoff

$$\arg\max_{\phi_0} \underbrace{\phi_0}_{\text{parameter}} + N^{-1/2} \underbrace{\sqrt{\phi_0(1-\phi_0)}}_{\text{std. error}} \text{ not sharp bound!}$$

- incorporate semi-supervised classifiers (use pre-treatment covariates to draw boundaries)

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