DM-Homework-2

April 8, 2023

2.1

25.
a) $\mathcal{P}(\{a, b, \{a, b\}\})$
$\mathcal{P}(\{a, b, \{a, b\}\})$ is the set of all subsets of $\{a, b, \{a, b\}\}$, so it has $2^3 = 8$ elements.
b)$\mathcal{P}(\{\emptyset, a, \{a\}, \{\{a\}\}\})$ has 16 elements.
c)$\mathcal{P}(\{\mathcal{P}(\emptyset)\})$ has 4 elements.

43.
Let $|A| = a, |B| = b$, then $|A \times B| = ab$. So we know that $|\mathcal{P}(A \times B)| = 2^{ab}$ but $|\mathcal{P}(A) \times \mathcal{P}(B)| = 2^{a+b}$, which are not always equal to each other. Thus, we can disprove that if A and B are sets, then $\mathcal{P}(A \times B) = \mathcal{P}(A) \times \mathcal{P}(B)$.

2.2

35.

$$
\begin{aligned}
\overline{(A \cup B)} \cap \overline{(B \cup C)} \cap \overline{(A \cup C)} &= \overline{A} \cap \overline{B} \cap \overline{B} \cap \overline{C} \cap \overline{A} \cap \overline{C} \qquad (by \quad De \quad Morgan's \quad laws) \\
&= (\overline{A} \cap \overline{A}) \cap (\overline{B} \cap \overline{B}) \cap (\overline{C} \cap \overline{C}) \qquad (by \quad Associative \quad laws) \\
&= \overline{A} \cap \overline{B} \cap \overline{C} \qquad (by \quad Idempotent \quad laws)
\end{aligned}
$$

37.
a)

$$\begin{aligned} A \times (B - C) &= \{(x, y) | x \in A \wedge y \in (B - C)\} \\ &= \{(x, y) | x \in A \wedge y \in B \wedge y \notin C\} \end{aligned}$$

$(A \times B) = \{(x, y)|x \in A \wedge y \in B\}$ and $(A \times C) = \{(x, y)|x \in A \wedge y \in C\}$, so

$$\begin{aligned} (A \times B) - (A \times C) &= \{(x, y)|x \in A \wedge y \in B \wedge \neg(x \in A \wedge y \in C)\} \\ &= \{(x, y)|x \in A \wedge y \in B \wedge (x \notin A \vee y \notin C)\} \\ &= \{(x, y)|(x \in A \wedge y \in B \wedge x \notin A) \vee (x \in A \wedge y \in B \wedge y \notin C)\} \\ &= \{(x, y)|F \vee (x \in A \wedge y \in B \wedge y \notin C)\} \\ &= \{(x, y)|x \in A \wedge y \in B \wedge y \notin C\} \end{aligned}$$

Thus, we can **prove** that for all sets A, B and C, we have $A \times (B - C) = (A \times B) - (A \times C)$

b)
$\overline{A} \times \overline{(B \cup C)} = \{(x, y)|x \notin A \wedge y \notin B \wedge y \notin C\}$
and $A \times (B \cup C) = \{(x, y)|x \in A \wedge (y \in B \vee y \in C)\}$
so $\overline{A \times (B \cup C)} = \{(x, y)|x \notin A \vee (y \notin B \wedge y \notin C)\}$
Thus, we can **disprove** that for all sets of A, B and C, we have $\overline{A} \times \overline{(B \cup C)} = \overline{A \times (B \cup C)}$

53.
a. $\bigcup_{i=1}^{n} A_i = \{1, 2, 3, ..., n\}$
b. $\bigcap_{i=1}^{n} A_i = \{1\}$

2.3

23.
a)
injection: For arbitrary x,$y \in R$, if $f(x) = f(y)$, that says $2x + 1 = 2y + 1$,

so we have $x = y$. Thus, the function is an injection from **R** to **R**.
surjection: For an arbitrary element $y \in R$, we have an element $x = \frac{y-1}{2} \in R$, Thus, the function is a surjection from **R** to **R**.
So, this function a bijection from **R** to **R**.

b)
not injection: Let $x = 1$ and $y = -1$, we have $f(x) = 2 = f(y)$, while $x \neq y$, so the function is not an injection. Thus, the function is **not** a bijection from **R** to **R**.

c)
injection: For arbitrary x,$y \in R$, if $f(x) = f(y)$, that says $x^3 = y^3$, so we have $x = y$. Thus, the function is an injection from **R** to **R**.
surjection: For an arbitrary element $y \in R$, we have an element $x = y^{\frac{1}{3}} \in R$, Thus, the function is a surjection from **R** to **R**.
So, this function a bijection from **R** to **R**.

d)
the domain of x is $\{x|x \neq \sqrt{2} \wedge x \neq -\sqrt{2}\} \neq R$, so the function is **not** a bijection from **R** to **R**.

47.
$f^{-1}(S) = \{a \in A|f(a) \in S\}$
So $\overline{f^{-1}(S)} = \{a \in A|a \notin f^{-1}(S)\} = \{a \in A|f(a) \notin S\}$
$f^{-1}(\overline{S}) = \{a \in A|f(a) \notin S\}$
Thus, $f^{-1}(\overline{S}) = \overline{f^{-1}(S)}$

81.
a) The set S can be denoted as $S = \{s_1, s_2, ..., s_m\}$, then we have a function f whose domain is $\{1, 2, 3, ..., m\}$ where $f(i) = s_i$. It's a bijection. So we can say that if there is a one-to-one correspondence between S and the set $\{1, 2, 3, ..., m\}$.

b) The set S can be denoted as $S = \{s_1, s_2, ..., s_m\}$, the set T can be denoted as $T = \{t_1, t_2, ..., t_m\}$. then we have a function f whose domain is $T = \{t_1, t_2, ..., t_m\}$ where $f(t_i) = s_i$. It's a bijection. So we can say that if there is a one-to-one correspondence between S and T.

2.4

31.

a)

$$\sum_{j=0}^{8} 3 \cdot 2^j = \frac{3 \cdot 2^{8+1} - 3}{2-1} = 1533$$

b)

$$\sum_{j=1}^{8} 2^j = \sum_{j=0}^{8} 2^j - 2^0 = 510$$

c)

$$\sum_{j=2}^{8} (-3)^j = \sum_{j=0}^{8} (-3)^j - \sum_{j=0}^{2} (-3)^j = 4914$$

d)

$$\sum_{j=0}^{8} 2 \cdot (-3)^j = 9842$$

41.

$$\begin{aligned}
\sum_{k=10}^{20} k^2(k-3) &= \sum_{k=10}^{20} (k^3 - 3k^2) \\
&= \sum_{k=10}^{20} k^3 - \sum_{k=10}^{20} 3k^2 \\
&= \sum_{k=1}^{20} k^3 - \sum_{k=1}^{10} k^3 - \sum_{k=1}^{20} 3k^2 + \sum_{k=1}^{10} 3k^2 \\
&= \frac{20^2 \times (20+1)^2}{4} - \frac{10^2 \times (10+1)^2}{4} - 3 \times (\frac{20 \times 21 \times 41}{6} - \frac{10 \times 11 \times 21}{6}) \\
&= 33620
\end{aligned}$$

2.5

Page 156. Example 4:

$$\begin{array}{cccccc}
\frac{1}{1} & \frac{2}{1} & \frac{3}{1} & \frac{4}{1} & \frac{5}{1} & \cdots \\
\frac{1}{2} & \cancel{\frac{2}{2}} & \frac{3}{2} & \cancel{\frac{4}{2}} & \cancel{\frac{5}{2}} & \cdots \\
\frac{1}{3} & \frac{2}{3} & \cancel{\frac{3}{3}} & \cancel{\frac{4}{3}} & \cancel{\frac{5}{3}} & \cdots \\
\frac{1}{4} & \cancel{\frac{2}{4}} & \cancel{\frac{3}{4}} & \cancel{\frac{4}{4}} & \cancel{\frac{5}{4}} & \cdots \\
\frac{1}{5} & \cancel{\frac{2}{5}} & \cancel{\frac{3}{5}} & \cancel{\frac{4}{5}} & \cancel{\frac{5}{5}} & \cdots \\
\ddots & \ddots & \ddots & \ddots & \ddots &
\end{array}$$

So the set of positive rational numbers is countable.

5.1

5.
a) reflexive, not symmetric, transitive.
b) not reflexive, symmetric, not transitive.
c) reflexive, symmetric, not transitive.
d) reflexive, symmetric, not transitive.

29. $R^{-1} = \{(f(a), a) | a \in A\}$

49.
a) If the relation is to be symmetric, each of the n ordered pairs (a,b) for $a, b \in A$ and $a \neq b$, must occur with (b,a). In the meantime, (a,a) for $a \in A$ may or may not in R. So, there are $2^{\frac{n(n-1)}{2}+n} = 2^{\frac{n(n+1)}{2}}$ symmetric relations.
b) $3^{\frac{n(n-1)}{2}} \cdot 2^n$

c) $3^{\frac{n(n-1)}{2}}$
d) $2^{n(n-1)}$
e) $2^{\frac{n(n-1)}{2}}$
f) $2^{n^2} - 2^{n^2-n+1}$

5.2

21. $s_{C_1}(s_{C_2}(R)) = s_{C_1}(R) \cap s_{C_2}(R) = s_{C_2}(s_{C_1}(R))$

5.3

11. $M_{\overline{R}} = M_U - M_R$, where M_U is a matrix whose entries are all 1.

17. There will have $n^2 - k$ nonzero entries in $M_{\overline{R}}$.