Derivation of the equations

1 Ch.1

2 Ch.2

2.1 2.2 Quantization of field

$$\vec{\nabla} \cdot \vec{A} = 0 \tag{1}$$

$$\nabla^2 \vec{A} - \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} \tag{2}$$

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t}, \vec{B} = \vec{\nabla} \times \vec{A} \tag{3}$$

fourier series expansion of A:

$$\vec{A}(\vec{r}, t) = \sum_k (\vec{A_k}(t) e^{i\vec{k} \cdot \vec{r}} + \vec{A_k}^*(t) e^{-i\vec{k} \cdot \vec{r}}) \tag{4}$$

$$k_x = \frac{2\pi n_x}{L}, k_y = \frac{2\pi n_y}{L}, k_z = \frac{2\pi n_z}{L} \tag{5}$$

$$\vec{k} \cdot \vec{A_k} = \vec{k} \cdot \vec{A_k}^* = 0 \tag{6}$$

substitute eq(2.4) with (2.2)

$$\nabla^2 = (\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2})$$

$$\sum_k (\vec{A_k}(t)(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}) e^{i(k_x x + k_y y + k_z z)} + c.c)$$

$$\sum_k (-k^2 \vec{A_k}(t) e^{i\vec{k} \cdot \vec{r}} + -k^2 \vec{A_k}^*(t) e^{-i\vec{k} \cdot \vec{r}})$$

for single k,

$$k^2 \vec{A_k}(t) + \frac{1}{c^2} \frac{\partial^2 \vec{A_k}(t)}{\partial t^2} = 0 \tag{7}$$

$$\vec{A_k} = \vec{A_k} e^{-i\omega_k t}, \omega_k = ck \tag{8}$$

$$\vec{A}(\vec{r}, t) = \sum_k (\vec{A_k} e^{-i\omega_k t + i\vec{k}\cdot\vec{r}} + \vec{A_k}^* e^{-i\vec{k}\cdot\vec{r} + i\omega_k t}) \tag{9}$$

$$\vec{E_k}(\vec{r}, t) = \sum_k \vec{E_k} = i \sum_k \omega_k (\vec{A_k} e^{-i\omega_k t + i\vec{k}\cdot\vec{r}} + \vec{A_k}^* e^{-i\vec{k}\cdot\vec{r} + i\omega_k t}) \tag{10}$$

$$\vec{B_k}(\vec{r}, t) = \sum_k \vec{B_k} = i \sum_k \vec{k} \times (\vec{A_k} e^{-i\omega_k t + i\vec{k}\cdot\vec{r}} + \vec{A_k}^* e^{-i\vec{k}\cdot\vec{r} + i\omega_k t}) \tag{11}$$

calculation process for eq(11):

$$\nabla \times \vec{A} = \nabla \times (\vec{A_k} e^{-i\omega_k t + i\vec{k}\cdot\vec{r}} + \vec{A_k}^* e^{-i\vec{k}\cdot\vec{r} + i\omega_k t})$$

$$(\frac{\partial}{\partial y} A_{k_z} e^{i\omega_k t + i\vec{k}\cdot\vec{r}} - \frac{\partial}{\partial z} A_{k_y} e^{i\omega_k t + i\vec{k}\cdot\vec{r}})\hat{i}$$

$$+(\frac{\partial}{\partial z} A_{k_x} e^{i\omega_k t + i\vec{k}\cdot\vec{r}} - \frac{\partial}{\partial x} A_{k_z} e^{i\omega_k t + i\vec{k}\cdot\vec{r}})\hat{j}$$

$$+(\frac{\partial}{\partial x} A_{k_y} e^{i\omega_k t + i\vec{k}\cdot\vec{r}} - \frac{\partial}{\partial y} A_{k_x} e^{i\omega_k t + i\vec{k}\cdot\vec{r}})\hat{z}$$

$$\overline{\eta_k} = \frac{1}{2} \int (\epsilon_0 \overline{E_k^2} + \frac{\overline{B_k^2}}{\mu_0}) dV \tag{12}$$

$$\overline{\vec{E_k^2}} = \frac{1}{T} \int_0^T dt E_k^2 = 2\omega_k^2 \left|\vec{A_k}\right|^2 \tag{13}$$

$$\overline{\vec{B_k^2}} = \frac{1}{T} \int_0^T dt E_k^2 = 2k_k^2 \left|\vec{A_k}\right|^2 \tag{14}$$

$$\overline{\eta_k} = (\epsilon_0 \omega_k^2 \left|\vec{A_k}\right|^2 + \frac{k_k^2}{\mu_0} \left|\vec{A_k}\right|^2) = 2\epsilon_0 \omega_k^2 V \left|\vec{A_k}\right|^2 \tag{15}$$

$$\vec{A_k} = \frac{\vec{\epsilon_k}}{\sqrt{4\epsilon_0 V \omega_k^2}} (\omega_k q_k + i p_k) \tag{16}$$

$$\overline{\eta_k} = \frac{1}{2}(\omega_k^2 q_k^2 + p_k^2) \tag{17}$$

$$\overline{\eta} = \sum_k \overline{\eta_k} = 2\epsilon_0 V \sum_k \omega_k^2 \left|\vec{A_k}\right|^2 = \frac{1}{2} \sum_k (\omega_k^2 q_k^2 + p_k^2) \tag{18}$$

$$\hat{H} = \frac{1}{2} \sum_k (\omega_k^2 \hat{q}_k^2 + \hat{p}_k^2) \tag{19}$$

since

$$[\hat{q}_k, \hat{p}_k] = i\hbar$$

$$\hat{a}_k = \frac{1}{\sqrt{2\hbar\omega_k}} (\omega_k \hat{q}_k + i\hat{p}_k) \tag{20}$$

$$\hat{a^\dagger}_k = \frac{1}{\sqrt{2\hbar\omega_k}} (\omega_k \hat{q}_k - i\hat{p}_k) \tag{21}$$

$$[\hat{a}_k, \hat{a}^\dagger_k] = \hat{a}_k \hat{a}^\dagger_k - \hat{a}^\dagger_k \hat{a}_k$$

$$\hat{a}_k \hat{a}^\dagger_k = \frac{1}{2\hbar\omega_k}(\omega_k \hat{q}_k + i\hat{p}_k)(\omega_k \hat{q}_k - i\hat{p}_k)$$

$$= \frac{1}{2\hbar\omega_k}(\omega_k^2 \hat{q}_k^2 + \hat{p}_k^2 - i\omega_k \hat{q}\hat{p} + i\omega_k \hat{p}\hat{q})$$

$$= \frac{1}{2\hbar\omega_k}(\omega_k^2 \hat{q}_k^2 + \hat{p}_k^2 - i\omega_k[\hat{q}_k, \hat{p}_k])$$

$$\hat{a}^\dagger_k \hat{a}_k = \frac{1}{2\hbar\omega_k}(\omega_k^2 \hat{q}_k^2 + \hat{p}_k^2 + i\omega_k[\hat{q}_k, \hat{p}_k])$$

$$[\hat{a}_k, \hat{a}^\dagger_k] = \frac{1}{2\hbar\omega_k}(-2i\omega_k[\hat{q}_k, \hat{p}_k]) = -\frac{i}{\hbar}[\hat{q}_k, \hat{p}_k] = -\frac{i}{\hbar}i\hbar = 1$$

$$[\hat{a}_k, \hat{a}^\dagger_k] = \delta_{kk^*} \tag{22}$$

$$\hat{\vec{A}}_k = \vec{\epsilon_k}\sqrt{\frac{\hbar}{2\epsilon_0 \omega_k V}}\hat{A}_k \tag{23}$$

$$\vec{A}(\vec{r}, t) = \sum_k \vec{\epsilon_k}\sqrt{\frac{\hbar}{2\epsilon_0 \omega_k V}}(\hat{a}_k e^{-i\omega_k t + i\vec{k}\cdot\vec{r}} + \hat{a}_k^\dagger e^{-i\vec{k}\cdot\vec{r} + i\omega_k t}) \tag{24}$$

$$\vec{E}(\vec{r}, t) = i\sum_k \vec{\epsilon_k}\sqrt{\frac{\hbar\omega_k}{2\epsilon_0 V}}(\hat{a}_k e^{-i\omega_k t + i\vec{k}\cdot\vec{r}} + \hat{a}_k^\dagger e^{-i\vec{k}\cdot\vec{r} + i\omega_k t}) \tag{25}$$

$$\vec{B}(\vec{r}, t) = i\sum_k \vec{k} \times \vec{\epsilon_k}\sqrt{\frac{\hbar}{2\epsilon_0 \omega_k V}}(\hat{a}_k e^{-i\omega_k t + i\vec{k}\cdot\vec{r}} + \hat{a}_k^\dagger e^{-i\vec{k}\cdot\vec{r} + i\omega_k t}) \tag{26}$$

2.2 Quantized single mode field

single mode, wave vector k, angular frequency $\omega = ck$

$$\hat{H} = \frac{1}{2}(\omega^2 \hat{q}^2 + \hat{p}^2) \tag{27}$$

$\hat{q}$: position operator , $\hat{p}$: momentum operator

$$\hat{q} = \sqrt{\frac{\hbar}{2\omega}}(\hat{a} + \hat{a}^\dagger) \tag{28}$$

$$\hat{p} = i\sqrt{\frac{\hbar\omega}{2}}(\hat{a}^\dagger - \hat{a}) \tag{29}$$

$$\hat{H} = \frac{\hbar\omega}{2}(\hat{a}^\dagger \hat{a} + \hat{a}\hat{a}^\dagger + \frac{1}{2}) \tag{30}$$

$$\hat{H}|n\rangle = \hbar\omega(\hat{a}^\dagger \hat{a} + \frac{1}{2})|n\rangle = \eta_n |n\rangle \tag{31}$$

$$\hbar\omega\hat{a}^\dagger \hat{a} \hat{a}^\dagger \hat{a} + \frac{1}{2}\hat{a}^\dagger |n\rangle = \hbar\omega(\hat{a}^\dagger \hat{a} - \frac{1}{2})|n\rangle \tag{32}$$

$$= \eta_n \hat{a}\hat{a}^\dagger |n\rangle$$