Derivation of the equations

1 Ch.1

$$m = 0$$
$$\mathscr{E} = h\nu = \frac{hc}{\lambda}$$
$$p = \frac{h\nu}{c} = \frac{h}{\lambda}$$
$$\Phi = \frac{AI}{h\nu_0} = \frac{P}{h\nu_0}$$

2 Ch.2

2.1 2.2 Quantization of field

$$\vec{\nabla} \cdot \vec{A} = 0 \tag{1}$$

$$\nabla^2 \vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} \tag{2}$$

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t}, \vec{B} = \vec{\nabla} \times \vec{A} \tag{3}$$

fourier series expansion of A:

$$\vec{A}(\vec{r}, t) = \sum_k (\vec{A_k}(t) e^{i\vec{k}\cdot\vec{r}} + \vec{A_k}^*(t) e^{-i\vec{k}\cdot\vec{r}}) \tag{4}$$

$$k_x = \frac{2\pi n_x}{L}, k_y = \frac{2\pi n_y}{L}, k_z = \frac{2\pi n_z}{L} \tag{5}$$

$$\vec{k} \cdot \vec{A_k} = \vec{k} \cdot \vec{A_k}^* = 0 \tag{6}$$

substitute eq(2.4) with (2.2)

$$\nabla^2 = (\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2})$$

$$\sum_k (\vec{A_k}(t) (\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}) e^{i(k_x x + k_y y + k_z z)} + c.c)$$

$$\sum_k(-k^2\vec{A_k}(t)e^{i\vec{k}\cdot\vec{r}}+-k^2\vec{A_k}^*(t)e^{-i\vec{k}\cdot\vec{r}})$$

for single k,

$$k^2\vec{A_k}(t)+\frac{1}{c^2}\frac{\partial^2\vec{A_k}(t)}{\partial t^2}=0 \tag{7}$$

$$\vec{A_k}=\vec{A_k}e^{-i\omega_k t},\omega_k=ck \tag{8}$$

$$\vec{A}(\vec{r},t)=\sum_k(\vec{A_k}e^{-i\omega_k t+i\vec{k}\cdot\vec{r}}+\vec{A_k}^*e^{-i\vec{k}\cdot\vec{r}+i\omega_k t}) \tag{9}$$

$$\vec{E_k}(\vec{r},t)=\sum_k\vec{E_k}=i\sum_k\omega_k(\vec{A_k}e^{-i\omega_k t+i\vec{k}\cdot\vec{r}}+\vec{A_k}^*e^{-i\vec{k}\cdot\vec{r}+i\omega_k t}) \tag{10}$$

$$\vec{B_k}(\vec{r},t)=\sum_k\vec{B_k}=i\sum_k\vec{k}\times(\vec{A_k}e^{-i\omega_k t+i\vec{k}\cdot\vec{r}}+\vec{A_k}^*e^{-i\vec{k}\cdot\vec{r}+i\omega_k t}) \tag{11}$$

calculation process for eq(11):

$$\nabla\times\vec{A}=\nabla\times(\vec{A_k}e^{-i\omega_k t+i\vec{k}\cdot\vec{r}}+\vec{A_k}^*e^{-i\vec{k}\cdot\vec{r}+i\omega_k t})$$

$$(\frac{\partial}{\partial y}A_{k_z}e^{i\omega_k t+i\vec{k}\cdot\vec{r}}-\frac{\partial}{\partial z}A_{k_y}e^{i\omega_k t+i\vec{k}\cdot\vec{r}})\hat{i}$$

$$+(\frac{\partial}{\partial z}A_{k_x}e^{i\omega_k t+i\vec{k}\cdot\vec{r}}-\frac{\partial}{\partial x}A_{k_z}e^{i\omega_k t+i\vec{k}\cdot\vec{r}})\hat{j}$$

$$+(\frac{\partial}{\partial x}A_{k_y}e^{i\omega_k t+i\vec{k}\cdot\vec{r}}-\frac{\partial}{\partial y}A_{k_x}e^{i\omega_k t+i\vec{k}\cdot\vec{r}})\hat{z}$$

$$\overline{\mathscr{E}_k}=\frac{1}{2}\int(\epsilon_0\overline{E_k^2}+\frac{\overline{B_k^2}}{\mu_0})dV \tag{12}$$

$$\overline{\vec{E}_k^2}=\frac{1}{T}\int_0^T dtE_k^2=2\omega_k^2\left|\vec{A_k}\right|^2 \tag{13}$$

$$\overline{\vec{B}_k^2}=\frac{1}{T}\int_0^T dtE_k^2=2k_k^2\left|\vec{A_k}\right|^2 \tag{14}$$

$$\overline{\mathscr{E}_k}=(\epsilon_0\omega_k^2\left|\vec{A_k}\right|^2+\frac{k_k^2}{\mu_0}\left|\vec{A_k}\right|^2)=2\epsilon_0\omega_k^2V\left|\vec{A_k}\right|^2 \tag{15}$$

$$\vec{A_k}=\frac{\vec{\epsilon_k}}{\sqrt{4\epsilon_0V\omega_k^2}}(\omega_kq_k+ip_k) \tag{16}$$

$$\overline{\mathscr{E}_k}=\frac{1}{2}(\omega_k^2q_k^2+p_k^2) \tag{17}$$

$$\overline{\mathscr{E}}=\sum_k\overline{\mathscr{E}_k}=2\epsilon_0V\sum_k\omega_k^2\left|\vec{A_k}\right|^2=\frac{1}{2}\sum_k(\omega_k^2q_k^2+p_k^2) \tag{18}$$

$$\hat{H}=\frac{1}{2}\sum_k(\omega_k^2\hat{q}_k^2+\hat{p}_k^2) \tag{19}$$

since

$$[\hat{q}_k, \hat{p}_k] = i\hbar$$

$$\hat{a}_k = \frac{1}{\sqrt{2\hbar\omega_k}}(\omega_k \hat{q}_k + i\hat{p}_k) \tag{20}$$

$$\hat{a^\dagger}_k = \frac{1}{\sqrt{2\hbar\omega_k}}(\omega_k \hat{q}_k - i\hat{p}_k) \tag{21}$$

$$[\hat{a}_k, \hat{a^\dagger}_k] = \hat{a}_k \hat{a^\dagger}_k - \hat{a^\dagger}_k \hat{a}_k$$

$$\hat{a}_k \hat{a^\dagger}_k = \frac{1}{2\hbar\omega_k}(\omega_k \hat{q}_k + i\hat{p}_k)(\omega_k \hat{q}_k - i\hat{p}_k)$$

$$= \frac{1}{2\hbar\omega_k}(\omega_k^2 \hat{q}_k^2 + \hat{p}_k^2 - i\omega_k \hat{q}\hat{p} + i\omega_k \hat{p}\hat{q})$$

$$= \frac{1}{2\hbar\omega_k}(\omega_k^2 \hat{q}_k^2 + \hat{p}_k^2 - i\omega_k [\hat{q}_k, \hat{p}_k])$$

$$\hat{a^\dagger}_k \hat{a}_k = \frac{1}{2\hbar\omega_k}(\omega_k^2 \hat{q}_k^2 + \hat{p}_k^2 + i\omega_k [\hat{q}_k, \hat{p}_k])$$

$$[\hat{a}_k, \hat{a^\dagger}_k] = \frac{1}{2\hbar\omega_k}(-2i\omega_k [\hat{q}_k, \hat{p}_k]) = -\frac{i}{\hbar}[\hat{q}_k, \hat{p}_k] = -\frac{i}{\hbar} i\hbar = 1$$

$$[\hat{a}_k, \hat{a^\dagger}_k] = \delta_{kk^*} \tag{22}$$

$$\hat{\vec{A}}_k = \vec{\epsilon_k}\sqrt{\frac{\hbar}{2\epsilon_0 \omega_k V}}\hat{A}_k \tag{23}$$

$$\vec{A}(\vec{r}, t) = \sum_k \vec{\epsilon_k}\sqrt{\frac{\hbar}{2\epsilon_0 \omega_k V}}(\hat{a}_k e^{-i\omega_k t + i\vec{k}\cdot\vec{r}} + \hat{a}_k^\dagger e^{-i\vec{k}\cdot\vec{r} + i\omega_k t}) \tag{24}$$

$$\vec{E}(\vec{r}, t) = i\sum_k \vec{\epsilon_k}\sqrt{\frac{\hbar\omega_k}{2\epsilon_0 V}}(\hat{a}_k e^{-i\omega_k t + i\vec{k}\cdot\vec{r}} + \hat{a}_k^\dagger e^{-i\vec{k}\cdot\vec{r} + i\omega_k t}) \tag{25}$$

$$\vec{B}(\vec{r}, t) = i\sum_k \vec{k} \times \vec{\epsilon_k}\sqrt{\frac{\hbar}{2\epsilon_0 \omega_k V}}(\hat{a}_k e^{-i\omega_k t + i\vec{k}\cdot\vec{r}} + \hat{a}_k^\dagger e^{-i\vec{k}\cdot\vec{r} + i\omega_k t}) \tag{26}$$

2.2 Quantized single mode field

single mode, wave vector k, angular frequency $\omega = ck$

$$\hat{H} = \frac{1}{2}(\omega^2 \hat{q}^2 + \hat{p}^2) \tag{27}$$

$\hat{q}$: position operator , $\hat{p}$: momentum operator

$$\hat{q} = \sqrt{\frac{\hbar}{2\omega}}(\hat{a} + \hat{a}^\dagger) \tag{28}$$

$$\hat{p} = i\sqrt{\frac{\hbar\omega}{2}}(\hat{a}^\dagger - \hat{a}) \tag{29}$$

$$\hat{H} = \frac{\hbar\omega}{2}(\hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger + \frac{1}{2}) \tag{30}$$

$$\hat{H}\left|n\right\rangle = \hbar\omega(\hat{a}^\dagger\hat{a} + \frac{1}{2})\left|n\right\rangle = \mathscr{E}_n\left|n\right\rangle \tag{31}$$

$$\begin{aligned} \hbar\omega\hat{a}^\dagger\hat{a}a^\dagger\hat{a} + \frac{1}{2}\hat{a}^\dagger\left|n\right\rangle &= \hbar\omega(\hat{a}^\dagger\hat{a} - \frac{1}{2})\left|n\right\rangle \\ &= \mathscr{E}_n\hat{a}^\dagger\left|n\right\rangle \end{aligned} \tag{32}$$

$$\hat{H}\hat{a}^\dagger\left|n\right\rangle = \hbar\omega(\hat{a}^\dagger\hat{a} + \frac{1}{2})\hat{a}^\dagger\left|n\right\rangle = (\mathscr{E}_n + \hbar\omega)\hat{a}^\dagger\left|n\right\rangle \tag{33}$$

$$\hat{H}\hat{a}\left|n\right\rangle = (\mathscr{E}_n - \hbar\omega)(\hat{a}\left|n\right\rangle) \tag{34}$$

$$\hat{a}^\dagger\left|n\right\rangle \propto \left|n+1\right\rangle, \mathscr{E}_{n+1} = \mathscr{E}_n + \hbar\omega \tag{35}$$

$$\hat{a}\left|n\right\rangle \propto \left|n-1\right\rangle, \mathscr{E}_{n-1} = \mathscr{E}_n - \hbar\omega \tag{36}$$

$$\hat{H}\hat{a}\left|0\right\rangle = (\mathscr{E}_0 - \hbar\omega)(\hat{a}\left|0\right\rangle) \tag{37}$$

$$\hat{a}\left|0\right\rangle = 0 \tag{38}$$

$$\hat{H}\left|0\right\rangle = \frac{1}{2}\hbar\omega\left|0\right\rangle = \mathscr{E}\left|0\right\rangle \tag{39}$$

$$\hat{a}^\dagger\hat{a}\left|n\right\rangle = n\left|n\right\rangle \tag{40}$$