

Derivation of the equations

1 Ch.1

2 Ch.2

2.1 2.2 Quantization of field

$$\vec{\nabla} \cdot \vec{A} = 0 \quad (1)$$

$$\nabla^2 \vec{A} - \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} \quad (2)$$

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t}, \vec{B} = \vec{\nabla} \times \vec{A} \quad (3)$$

fourier series expansion of A:

$$\vec{A}(\vec{r}, t) = \sum_k (\vec{A}_k(t) e^{i\vec{k} \cdot \vec{r}} + \vec{A}_k^*(t) e^{-i\vec{k} \cdot \vec{r}}) \quad (4)$$

$$k_x = \frac{2\pi n_x}{L}, k_y = \frac{2\pi n_y}{L}, k_z = \frac{2\pi n_z}{L} \quad (5)$$

$$\vec{k} \cdot \vec{A}_k = \vec{k} \cdot \vec{A}_k^* = 0 \quad (6)$$

substitute eq(2.4) with (2.2)

$$\begin{aligned} \nabla^2 &= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \\ \sum_k &(\vec{A}_k(t) \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) e^{i(k_x x + k_y y + k_z z)} + c.c) \\ &\sum_k (-k^2 \vec{A}_k(t) e^{i\vec{k} \cdot \vec{r}} - k^2 \vec{A}_k^*(t) e^{-i\vec{k} \cdot \vec{r}}) \end{aligned}$$

for single k,

$$k^2 \vec{A}_k(t) + \frac{1}{c^2} \frac{\partial^2 \vec{A}_k(t)}{\partial t^2} = 0 \quad (7)$$

$$\vec{A}_k = \vec{A}_k e^{-i\omega_k t}, \omega_k = ck \quad (8)$$

$$\vec{A}(\vec{r}, t) = \sum_k (\vec{A}_k e^{-i\omega_k t + i\vec{k} \cdot \vec{r}} + \vec{A}_k^* e^{-i\vec{k} \cdot \vec{r} + i\omega_k t}) \quad (9)$$

$$\vec{E}_k(\vec{r}, t) = \sum_k \vec{E}_k = i \sum_k \omega_k (\vec{A}_k e^{-i\omega_k t + i\vec{k} \cdot \vec{r}} + \vec{A}_k^* e^{-i\vec{k} \cdot \vec{r} + i\omega_k t}) \quad (10)$$

$$\vec{B}_k(\vec{r}, t) = \sum_k \vec{B}_k = i \sum_k \vec{k} \times (\vec{A}_k e^{-i\omega_k t + i\vec{k} \cdot \vec{r}} + \vec{A}_k^* e^{-i\vec{k} \cdot \vec{r} + i\omega_k t}) \quad (11)$$

calculation process for eq(11):

$$\begin{aligned} \nabla \times \vec{A} &= \nabla \times (\vec{A}_k e^{-i\omega_k t + i\vec{k} \cdot \vec{r}} + \vec{A}_k^* e^{-i\vec{k} \cdot \vec{r} + i\omega_k t}) \\ &= \left(\frac{\partial}{\partial y} A_{k_z} e^{i\omega_k t + i\vec{k} \cdot \vec{r}} - \frac{\partial}{\partial z} A_{k_y} e^{i\omega_k t + i\vec{k} \cdot \vec{r}} \right) \hat{i} \\ &+ \left(\frac{\partial}{\partial z} A_{k_x} e^{i\omega_k t + i\vec{k} \cdot \vec{r}} - \frac{\partial}{\partial x} A_{k_z} e^{i\omega_k t + i\vec{k} \cdot \vec{r}} \right) \hat{j} \\ &+ \left(\frac{\partial}{\partial x} A_{k_y} e^{i\omega_k t + i\vec{k} \cdot \vec{r}} - \frac{\partial}{\partial y} A_{k_x} e^{i\omega_k t + i\vec{k} \cdot \vec{r}} \right) \hat{z} \\ \overline{\eta_k} &= \frac{1}{2} \int (\epsilon_0 \overline{E_k^2} + \frac{\overline{B_k^2}}{\mu_0}) dV \end{aligned} \quad (12)$$

$$\overline{E_k^2} = \frac{1}{T} \int_0^T dt E_k^2 = 2\omega_k^2 |\vec{A}_k|^2 \quad (13)$$

$$\overline{B_k^2} = \frac{1}{T} \int_0^T dt E_k^2 = 2k_k^2 |\vec{A}_k|^2 \quad (14)$$

$$\overline{\eta_k} = (\epsilon_0 \omega_k^2 |\vec{A}_k|^2 + \frac{k_k^2}{\mu_0} |\vec{A}_k|^2) = 2\epsilon_0 \omega_k^2 V |\vec{A}_k|^2 \quad (15)$$

$$\vec{A}_k = \frac{\vec{\epsilon}_k}{\sqrt{4\epsilon_0 V \omega_k^2}} (\omega_k q_k + i p_k) \quad (16)$$

$$\overline{\eta_k} = \frac{1}{2} (\omega_k^2 q_k^2 + p_k^2) \quad (17)$$

$$\overline{\eta} = \sum_k \overline{\eta_k} = 2\epsilon_0 V \sum_k \omega_k^2 |\vec{A}_k|^2 = \frac{1}{2} \sum_k (\omega_k^2 q_k^2 + p_k^2) \quad (18)$$

$$\hat{H} = \frac{1}{2} \sum_k (\omega_k^2 \hat{q}_k^2 + \hat{p}_k^2) \quad (19)$$

since

$$\begin{aligned} [\hat{q}_k, \hat{p}_k] &= i\hbar \\ \hat{a}_k &= \frac{1}{\sqrt{2\hbar\omega_k}} (\omega_k \hat{q}_k + i\hat{p}_k) \end{aligned} \quad (20)$$

$$\hat{a}_k^\dagger = \frac{1}{\sqrt{2\hbar\omega_k}} (\omega_k \hat{q}_k - i\hat{p}_k) \quad (21)$$

$$\begin{aligned}
[\hat{a}_k, \hat{a}_k^\dagger] &= \hat{a}_k \hat{a}_k^\dagger - \hat{a}_k^\dagger \hat{a}_k \\
\hat{a}_k \hat{a}_k^\dagger &= \frac{1}{2\hbar\omega_k} (\omega_k \hat{q}_k + i\hat{p}_k)(\omega_k \hat{q}_k - i\hat{p}_k) \\
&= \frac{1}{2\hbar\omega_k} (\omega_k^2 \hat{q}_k^2 + \hat{p}_k^2 - i\omega_k \hat{q}_k \hat{p} + i\omega_k \hat{p} \hat{q}) \\
&= \frac{1}{2\hbar\omega_k} (\omega_k^2 \hat{q}_k^2 + \hat{p}_k^2 - i\omega_k [\hat{q}_k, \hat{p}_k]) \\
\hat{a}_k^\dagger \hat{a}_k &= \frac{1}{2\hbar\omega_k} (\omega_k^2 \hat{q}_k^2 + \hat{p}_k^2 + i\omega_k [\hat{q}_k, \hat{p}_k]) \\
[\hat{a}_k, \hat{a}_k^\dagger] &= \frac{1}{2\hbar\omega_k} (-2i\omega_k [\hat{q}_k, \hat{p}_k]) = -\frac{i}{\hbar} [\hat{q}_k, \hat{p}_k] = -\frac{i}{\hbar} i\hbar = 1 \\
[\hat{a}_k, \hat{a}_k^\dagger] &= \delta_{kk^*}
\end{aligned} \tag{22}$$

$$\hat{A}_k = \epsilon_k \sqrt{\frac{\hbar}{2\epsilon_0\omega_k V}} \hat{A}_k \tag{23}$$

$$\vec{A}(\vec{r}, t) = \sum_k \epsilon_k \sqrt{\frac{\hbar}{2\epsilon_0\omega_k V}} (\hat{a}_k e^{-i\omega_k t + i\vec{k} \cdot \vec{r}} + \hat{a}_k^\dagger e^{-i\vec{k} \cdot \vec{r} + i\omega_k t}) \tag{24}$$

$$\vec{E}(\vec{r}, t) = i \sum_k \epsilon_k \sqrt{\frac{\hbar\omega_k}{2\epsilon_0 V}} (\hat{a}_k e^{-i\omega_k t + i\vec{k} \cdot \vec{r}} + \hat{a}_k^\dagger e^{-i\vec{k} \cdot \vec{r} + i\omega_k t}) \tag{25}$$

$$\vec{B}(\vec{r}, t) = i \sum_k \vec{k} \times \epsilon_k \sqrt{\frac{\hbar}{2\epsilon_0\omega_k V}} (\hat{a}_k e^{-i\omega_k t + i\vec{k} \cdot \vec{r}} + \hat{a}_k^\dagger e^{-i\vec{k} \cdot \vec{r} + i\omega_k t}) \tag{26}$$

2.2 Quantized single mode field

single mode, wave vector k , angular frequency $\omega = ck$

$$\hat{H} = \frac{1}{2}(\omega^2 \hat{q}^2 + \hat{p}^2) \tag{27}$$

\hat{q} : position operator , \hat{p} : momentum operator

$$\hat{q} = \sqrt{\frac{\hbar}{2\omega}} (\hat{a} + \hat{a}^\dagger) \tag{28}$$

$$\hat{p} = i\sqrt{\frac{\hbar\omega}{2}} (\hat{a}^\dagger - \hat{a}) \tag{29}$$

$$\hat{H} = \frac{\hbar\omega}{2} (\hat{a}^\dagger \hat{a} + \hat{a} \hat{a}^\dagger + \frac{1}{2}) \tag{30}$$

$$\hat{H} |n\rangle = \hbar\omega (\hat{a}^\dagger \hat{a} + \frac{1}{2}) |n\rangle = \eta_n |n\rangle \tag{31}$$

$$\begin{aligned}
\hbar\omega \hat{a}^\dagger \hat{a} \hat{a}^\dagger \hat{a} + \frac{1}{2} \hat{a}^\dagger |n\rangle &= \hbar\omega (\hat{a}^\dagger \hat{a} - \frac{1}{2}) |n\rangle \\
&= \eta_n \hat{a}^\dagger |n\rangle
\end{aligned} \tag{32}$$

$$\hat{H} \hat{a}^\dagger |n\rangle = \hbar\omega (\hat{a}^\dagger \hat{a} + \frac{1}{2}) \hat{a}^\dagger |n\rangle = (\eta_n + \hbar\omega) \hat{a}^\dagger |n\rangle \tag{33}$$