

# Derivation of the equations

## 1 Ch.1

This is a section about properties of photon.

$$m = 0$$

$$\mathcal{E} = h\nu = \frac{hc}{\lambda}$$

$$p = \frac{h\nu}{c} = \frac{h}{\lambda}$$

Mean photon flux can be wrote as:

$$\Phi = \frac{AI}{h\nu_0} = \frac{P}{h\nu_0}$$

Notice that We can count the number of the photon per time using this equation.  
Unit of P is Watt, and  $h\nu$  is eV.

## 2 Ch.2

### 2.1 2.1 Quantization of field

$$\vec{\nabla} \cdot \vec{A} = 0 \quad (1)$$

$$\nabla^2 \vec{A} - \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} \quad (2)$$

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t}, \vec{B} = \vec{\nabla} \times \vec{A} \quad (3)$$

fourier series expansion of A:

$$\vec{A}(\vec{r}, t) = \sum_k (\vec{A}_k(t) e^{i\vec{k} \cdot \vec{r}} + \vec{A}_k^*(t) e^{-i\vec{k} \cdot \vec{r}}) \quad (4)$$

$$k_x = \frac{2\pi n_x}{L}, k_y = \frac{2\pi n_y}{L}, k_z = \frac{2\pi n_z}{L} \quad (5)$$

$$\vec{k} \cdot \vec{A}_k = \vec{k} \cdot \vec{A}_k^* = 0 \quad (6)$$

substitute eq(2.4) with (2.2)

$$\begin{aligned}\nabla^2 &= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) \\ \sum_k (\vec{A}_k(t) \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) e^{i(k_x x + k_y y + k_z z)} + c.c) \\ \sum_k (-k^2 \vec{A}_k(t) e^{i\vec{k} \cdot \vec{r}} + -k^2 \vec{A}_k^*(t) e^{-i\vec{k} \cdot \vec{r}})\end{aligned}$$

for single k,

$$k^2 \vec{A}_k(t) + \frac{1}{c^2} \frac{\partial^2 \vec{A}_k(t)}{\partial t^2} = 0 \quad (7)$$

$$\vec{A}_k = \vec{A}_k e^{-i\omega_k t}, \omega_k = ck \quad (8)$$

$$\vec{A}(\vec{r}, t) = \sum_k (\vec{A}_k e^{-i\omega_k t + i\vec{k} \cdot \vec{r}} + \vec{A}_k^* e^{-i\vec{k} \cdot \vec{r} + i\omega_k t}) \quad (9)$$

$$\vec{E}_k(\vec{r}, t) = \sum_k \vec{E}_k = i \sum_k \omega_k (\vec{A}_k e^{-i\omega_k t + i\vec{k} \cdot \vec{r}} + \vec{A}_k^* e^{-i\vec{k} \cdot \vec{r} + i\omega_k t}) \quad (10)$$

$$\vec{B}_k(\vec{r}, t) = \sum_k \vec{B}_k = i \sum_k \vec{k} \times (\vec{A}_k e^{-i\omega_k t + i\vec{k} \cdot \vec{r}} + \vec{A}_k^* e^{-i\vec{k} \cdot \vec{r} + i\omega_k t}) \quad (11)$$

calculation process for eq(11):

$$\begin{aligned}\nabla \times \vec{A} &= \nabla \times (\vec{A}_k e^{-i\omega_k t + i\vec{k} \cdot \vec{r}} + \vec{A}_k^* e^{-i\vec{k} \cdot \vec{r} + i\omega_k t}) \\ &= \left(\frac{\partial}{\partial y} A_{k_z} e^{i\omega_k t + i\vec{k} \cdot \vec{r}} - \frac{\partial}{\partial z} A_{k_y} e^{i\omega_k t + i\vec{k} \cdot \vec{r}}\right) \hat{i} \\ &+ \left(\frac{\partial}{\partial z} A_{k_x} e^{i\omega_k t + i\vec{k} \cdot \vec{r}} - \frac{\partial}{\partial x} A_{k_z} e^{i\omega_k t + i\vec{k} \cdot \vec{r}}\right) \hat{j} \\ &+ \left(\frac{\partial}{\partial x} A_{k_y} e^{i\omega_k t + i\vec{k} \cdot \vec{r}} - \frac{\partial}{\partial y} A_{k_x} e^{i\omega_k t + i\vec{k} \cdot \vec{r}}\right) \hat{z} \\ \overline{\mathcal{E}}_k &= \frac{1}{2} \int (\epsilon_0 \overline{E}_k^2 + \frac{\overline{B}_k^2}{\mu_0}) dV \quad (12)\end{aligned}$$

$$\overline{E}_k^2 = \frac{1}{T} \int_0^T dt E_k^2 = 2\omega_k^2 |\vec{A}_k|^2 \quad (13)$$

$$\overline{B}_k^2 = \frac{1}{T} \int_0^T dt E_k^2 = 2k_k^2 |\vec{A}_k|^2 \quad (14)$$

$$\overline{\mathcal{E}}_k = (\epsilon_0 \omega_k^2 |\vec{A}_k|^2 + \frac{k_k^2}{\mu_0} |\vec{A}_k|^2) = 2\epsilon_0 \omega_k^2 V |\vec{A}_k|^2 \quad (15)$$

$$\vec{A}_k = \frac{\vec{\epsilon}_k}{\sqrt{4\epsilon_0 V \omega_k^2}} (\omega_k q_k + i p_k) \quad (16)$$

$$\overline{\mathcal{E}}_k = \frac{1}{2}(\omega_k^2 q_k^2 + p_k^2) \quad (17)$$

$$\overline{\mathcal{E}} = \sum_k \overline{\mathcal{E}}_k = 2\epsilon_0 V \sum_k \omega_k^2 \left| \vec{A}_k \right|^2 = \frac{1}{2} \sum_k (\omega_k^2 q_k^2 + p_k^2) \quad (18)$$

$$\hat{H} = \frac{1}{2} \sum_k (\omega_k^2 \hat{q}_k^2 + \hat{p}_k^2) \quad (19)$$

since

$$[\hat{q}_k, \hat{p}_k] = i\hbar$$

$$\hat{a}_k = \frac{1}{\sqrt{2\hbar\omega_k}}(\omega_k \hat{q}_k + i\hat{p}_k) \quad (20)$$

$$\hat{a}_k^\dagger = \frac{1}{\sqrt{2\hbar\omega_k}}(\omega_k \hat{q}_k - i\hat{p}_k) \quad (21)$$

# calculation process

$$\begin{aligned} [\hat{a}_k, \hat{a}_k^\dagger] &= \hat{a}_k \hat{a}_k^\dagger - \hat{a}_k^\dagger \hat{a}_k \\ \hat{a}_k \hat{a}_k^\dagger &= \frac{1}{2\hbar\omega_k}(\omega_k \hat{q}_k + i\hat{p}_k)(\omega_k \hat{q}_k - i\hat{p}_k) \\ &= \frac{1}{2\hbar\omega_k}(\omega_k^2 \hat{q}_k^2 + \hat{p}_k^2 - i\omega_k \hat{q}_k \hat{p} + i\omega_k \hat{p} \hat{q}) \\ &= \frac{1}{2\hbar\omega_k}(\omega_k^2 \hat{q}_k^2 + \hat{p}_k^2 - i\omega_k [\hat{q}_k, \hat{p}_k]) \\ \hat{a}_k^\dagger \hat{a}_k &= \frac{1}{2\hbar\omega_k}(\omega_k^2 \hat{q}_k^2 + \hat{p}_k^2 + i\omega_k [\hat{q}_k, \hat{p}_k]) \\ [\hat{a}_k, \hat{a}_k^\dagger] &= \frac{1}{2\hbar\omega_k}(-2i\omega_k [\hat{q}_k, \hat{p}_k]) = -\frac{i}{\hbar} [\hat{q}_k, \hat{p}_k] = -\frac{i}{\hbar} i\hbar = 1 \end{aligned}$$

end.

$$[\hat{a}_k, \hat{a}_k^\dagger] = \delta_{kk^*} \quad (22)$$

$$\hat{A}_k = \vec{\epsilon}_k \sqrt{\frac{\hbar}{2\epsilon_0 \omega_k V}} \hat{A}_k \quad (23)$$

$$\vec{A}(\vec{r}, t) = \sum_k \vec{\epsilon}_k \sqrt{\frac{\hbar}{2\epsilon_0 \omega_k V}} (\hat{a}_k e^{-i\omega_k t + i\vec{k} \cdot \vec{r}} + \hat{a}_k^\dagger e^{-i\vec{k} \cdot \vec{r} + i\omega_k t}) \quad (24)$$

$$\vec{E}(\vec{r}, t) = i \sum_k \vec{\epsilon}_k \sqrt{\frac{\hbar \omega_k}{2\epsilon_0 V}} (\hat{a}_k e^{-i\omega_k t + i\vec{k} \cdot \vec{r}} + \hat{a}_k^\dagger e^{-i\vec{k} \cdot \vec{r} + i\omega_k t}) \quad (25)$$

$$\vec{B}(\vec{r}, t) = i \sum_k \vec{k} \times \vec{\epsilon}_k \sqrt{\frac{\hbar}{2\epsilon_0 \omega_k V}} (\hat{a}_k e^{-i\omega_k t + i\vec{k} \cdot \vec{r}} + \hat{a}_k^\dagger e^{-i\vec{k} \cdot \vec{r} + i\omega_k t}) \quad (26)$$

## 2.2 Quantized single mode field

In this section, the word "mode" is defined as value of wave vector  $k$  varies. single mode, wave vector  $k$ , angular frequency  $\omega = ck$

$$\hat{H} = \frac{1}{2}(\omega^2 \hat{q}^2 + \hat{p}^2) \quad (27)$$

$\hat{q}$  : position operator ,  $\hat{p}$  : momentum operator

$$\hat{q} = \sqrt{\frac{\hbar}{2\omega}}(\hat{a} + \hat{a}^\dagger) \quad (28)$$

$$\hat{p} = i\sqrt{\frac{\hbar\omega}{2}}(\hat{a}^\dagger - \hat{a}) \quad (29)$$

$$\hat{H} = \frac{\hbar\omega}{2}(\hat{a}^\dagger \hat{a} + \hat{a} \hat{a}^\dagger + \frac{1}{2}) \quad (30)$$

$$\hat{H} |n\rangle = \hbar\omega(\hat{a}^\dagger \hat{a} + \frac{1}{2}) |n\rangle = \mathcal{E}_n |n\rangle \quad (31)$$

$$\begin{aligned} \hbar\omega \hat{a}^\dagger \hat{a} \hat{a}^\dagger \hat{a} + \frac{1}{2} \hat{a}^\dagger |n\rangle &= \hbar\omega(\hat{a}^\dagger \hat{a} - \frac{1}{2}) |n\rangle \\ &= \mathcal{E}_n \hat{a}^\dagger |n\rangle \end{aligned} \quad (32)$$

$$\hat{H} \hat{a}^\dagger |n\rangle = \hbar\omega(\hat{a}^\dagger \hat{a} + \frac{1}{2}) \hat{a}^\dagger |n\rangle = (\mathcal{E}_n + \hbar\omega) \hat{a}^\dagger |n\rangle \quad (33)$$

$$\hat{H} \hat{a} |n\rangle = (\mathcal{E}_n - \hbar\omega) \hat{a} |n\rangle \quad (34)$$

$$\hat{a}^\dagger |n\rangle \propto |n+1\rangle, \mathcal{E}_{n+1} = \mathcal{E}_n + \hbar\omega \quad (35)$$

$$\hat{a} |n\rangle \propto |n-1\rangle, \mathcal{E}_{n-1} = \mathcal{E}_n - \hbar\omega \quad (36)$$

$$\hat{H} \hat{a} |0\rangle = (\mathcal{E}_0 - \hbar\omega) \hat{a} |0\rangle \quad (37)$$

$$\hat{a} |0\rangle = 0 \quad (38)$$

$$\hat{H} |0\rangle = \frac{1}{2} \hbar\omega |0\rangle = \mathcal{E} |0\rangle \quad (39)$$

$$\hat{a}^\dagger \hat{a} |n\rangle = n |n\rangle \quad (40)$$

$$\hat{a}^\dagger = c_n |n+1\rangle \hat{a}^\dagger = \sqrt{n+1} |n+1\rangle \quad (41)$$

$$\hat{a} = d_n |n-1\rangle \quad (42)$$

$$\hat{a}^\dagger = \sqrt{n+1} |n+1\rangle \quad (43)$$

$$\hat{a} = \sqrt{n} |n-1\rangle \quad (44)$$

$$|n\rangle = \frac{1}{\sqrt{n!}} (\hat{a}^\dagger)^n |0\rangle \quad (45)$$

### 2.3 2.3 quantized multi mode radiation field

$$\hat{H} = \sum_k \hbar \omega_k (\hat{a}^\dagger \hat{a} + \frac{1}{2}) \quad (46)$$

$$\mathcal{E}_{k_1} + \mathcal{E}_{k_2} + \mathcal{E}_{k_3} + \dots + \mathcal{E}_{k_n} = \sum_{i=1}^N (n_{k_i} + \frac{1}{2}) \hbar \omega_{k_i}$$

$$\hat{H} |n_k\rangle = (\sum_k \hbar \omega_k (\hat{a}^\dagger \hat{a} + \frac{1}{2})) |n_k\rangle = (\sum_k \hbar \omega_k (n_k + \frac{1}{2})) |n_k\rangle \quad (47)$$

### 2.4 2.4 General states of the quantized radiation field

If single mode light is in a pure state, Generally we can describe its state as linear superposition.

$$|\phi\rangle = \sum_n a_n |n\rangle \quad (48)$$

Probability, Expectation value, Uncertainty:

$$P_n = |a_n|^2, \langle n \rangle = \sum_n n P_n = \sum_n n |a_n|^2$$

$$\Delta n = \sqrt{(n - \langle n \rangle)^2} = \sqrt{(\langle n^2 \rangle - \langle n \rangle^2)} = \sqrt{\sum_n n^2 P_n - (\sum_n n P_n)^2} \quad (49)$$

$$a_n = e^{-\frac{|\alpha|^2}{2}} \frac{\alpha^n}{\sqrt{n!}} \quad (50)$$

If single mode light is in a mixed state, than we can describe its state by using density operator,  $\hat{\rho} = \sum_\phi P_\phi |\phi\rangle \langle \phi|$

$$\hat{\rho} = \sum_n P_n |n\rangle \langle n| \quad (51)$$

Expectation value of number of the photon n is:

$$\langle n \rangle = \langle \hat{a}^\dagger \hat{a} \rangle = Tr\{\hat{\rho} \hat{a}^\dagger \hat{a}\} \quad (52)$$

In case of the blackbody radiation, which is an general example of the mixed state light, probability of the state is:

$$P_n = e^{\frac{-n\hbar\omega}{kT}} (1 - e^{\frac{-\hbar\omega}{kT}}) \quad (53)$$

density operator is:

$$\hat{\rho} = (1 - e^{\frac{-\hbar\omega}{kT}}) \sum_n e^{\frac{-n\hbar\omega}{kT}} |n\rangle \langle n| \quad (54)$$

## 2.5 Vacuum fluctuation and zero point energy

$$\hat{A}(\vec{r}, t) = \vec{\epsilon}_k \sqrt{\frac{\hbar}{2\epsilon_0\omega_k V}} (\hat{a}_k e^{-i\omega_k t + i\vec{k} \cdot \vec{r}} + \hat{a}_k^\dagger e^{-i\vec{k} \cdot \vec{r} + i\omega_k t}) \quad (55)$$

$$\hat{E}(\vec{r}, t) = i\vec{\epsilon}_k \sqrt{\frac{\hbar\omega_k}{2\epsilon_0 V}} (\hat{a}_k e^{-i\omega_k t + i\vec{k} \cdot \vec{r}} + \hat{a}_k^\dagger e^{-i\vec{k} \cdot \vec{r} + i\omega_k t}) \quad (56)$$

$$\hat{B}(\vec{r}, t) = i\vec{k} \times \vec{\epsilon}_k \sqrt{\frac{\hbar}{2\epsilon_0\omega_k V}} (\hat{a}_k e^{-i\omega_k t + i\vec{k} \cdot \vec{r}} + \hat{a}_k^\dagger e^{-i\vec{k} \cdot \vec{r} + i\omega_k t}) \quad (57)$$

Eigenstate  $|n\rangle$  of the hamiltonian  $H$ , expectation value of operators turn out to be 0. For instance,

$$\begin{aligned} \langle n | \hat{E} | n \rangle &= i\vec{\epsilon}_k \sqrt{\frac{\hbar\omega_k}{2\epsilon_0 V}} (\langle n | \hat{a}_k | n \rangle e^{-i\omega_k t + i\vec{k} \cdot \vec{r}} + \langle n | \hat{a}_k^\dagger | n \rangle e^{-i\vec{k} \cdot \vec{r} + i\omega_k t}) \\ &= i\vec{\epsilon}_k \sqrt{\frac{\hbar\omega_k}{2\epsilon_0 V}} (\sqrt{n} \langle n | n-1 \rangle e^{-i\omega_k t + i\vec{k} \cdot \vec{r}} + \sqrt{n+1} \langle n | n+1 \rangle e^{-i\vec{k} \cdot \vec{r} + i\omega_k t}) \\ \langle n | \hat{E} | n \rangle &= 0 \end{aligned} \quad (58)$$

Expectation value of  $\hat{E}^2 = \hat{\vec{E}} \cdot \hat{\vec{E}}$  :

$$\langle n | \hat{\vec{E}} \cdot \hat{\vec{E}} | n \rangle = \frac{\hbar\omega}{2\epsilon_0 V} \langle n | \hat{a}\hat{a}^\dagger + \hat{a}\hat{a}^\dagger | n \rangle = \frac{\hbar\omega}{2\epsilon_0 V} (n + \frac{1}{2}) \quad (59)$$

What this equation states is that there are always fluctuation exists in a Electronic field. When light is in fock state, uncertainty of field is:

$$\Delta E = \sqrt{\langle \hat{E}^2 \rangle - \langle \hat{E} \rangle^2} = \sqrt{\frac{\hbar\omega}{\epsilon_0 V} (n + \frac{1}{2})} \quad (60)$$