$$\langle \text{even}_l | \cos\phi | \text{even}_k \rangle = \int_0^{2\pi} d\phi \left(\frac{a_0^{(l)}}{\sqrt{2\pi}} + \sum_{m=1} \frac{a^{(l)}}{\sqrt{\pi}} \cos m\phi \right) \left(\frac{a_0^{(k)}}{\sqrt{2\pi}} \cos\phi + \sum_{n=1} \frac{a^{(k)}}{\sqrt{\pi}} \cos\phi \cos n\phi \right)$$

$$= \int_0^{2\pi} d\phi \left(\frac{a_0^{(k)} a_0^{(l)}}{2\pi} + \sum_{n=1} \frac{a_n^{(k)} a_0^{(l)}}{\pi\sqrt{2}} \cos n\phi + \sum_{m=1} \frac{a_0^{(k)} a_m^{(l)}}{\pi\sqrt{2}} \cos m\phi + \sum_{n,m=1} \frac{a_n^{(k)} a_l^{(m)}}{\pi} \cos n\phi \cos m\phi \right) \cos\phi$$

$$= \begin{cases} \text{if k} = \text{l}: & \frac{2}{\sqrt{2}} a_0^{(k)} a_1^{(k)} + \sum_{n=1}^{2k} a_{2n-1}^{(k)} a_{2n}^{(k)} \\ \text{if k} \neq \text{l}: & \frac{1}{\sqrt{2}} (a_0^{(k)} a_1^{(l)} + a_1^{(k)} a_0^{(l)}) + \frac{1}{2} \sum_{n=1}^{2k} (a_n^{(k)} a_{n+1}^{(l)} + a_n^{(l)} a_{n+1}^{(k)}) \end{cases}$$

$$\langle \text{odd}_l | \cos\phi | \text{odd}_k \rangle = \sum_{n,m=1} \int_0^{2\pi} d\phi \left(\frac{b_n^{(k)} b_m^{(l)}}{\pi} \cos\phi \sin n\phi \sin m\phi \right)$$

$$= \begin{cases} \text{if k} > \text{l}: & \sum_{n=1}^{N} \left(\frac{\hat{b}_n^{(k)} \hat{b}_{n-1}^{(l)}}{2} + \frac{\hat{b}_n^{(k)} \hat{b}_{n+1}^{(l)}}{2} \right) \\ \text{if k} < \text{l}: & \sum_{n=1}^{N} \left(\frac{\hat{b}_{n-1}^{(k)} \hat{b}_n^{(l)}}{2} + \frac{\hat{b}_{n+1}^{(k)} \hat{b}_n^{(l)}}{2} \right) \end{cases}$$

$$\hat{\cos}\phi = \begin{pmatrix} \ddots & & \vdots & & \\ \langle \text{even}_k | \cos\phi | \text{even}_k \rangle & 0 & \langle \text{even}_k | \cos\phi | \text{even}_{k+1} \rangle & \cdots \\ 0 & \langle \text{odd}_k | \cos\phi | \text{odd}_k \rangle & 0 & \langle \text{odd}_k | \cos\phi | \text{odd}_{k+1} \rangle \\ \langle \text{even}_{k+1} | \cos\phi | \text{even}_k \rangle & 0 & \langle \text{even}_{k+1} | \cos\phi | \text{even}_{k+1} \rangle & \\ & \vdots & & \ddots \end{pmatrix}$$