Matsubara Summary(2) : About τ

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To answering the question : What is the reason that τ has its boundary condition as : '$0 < \tau < \beta$'? Following the derivation in the textbook, written by Bruus and K, Ch.11 section 2,
First, retarded single particle Green's function can be written as:

$$C^R_{AB}(\omega) = \frac{1}{Z}\sum_{nn'}\frac{\langle n|A|n'\rangle\langle n'|B|n\rangle}{\omega + E_n - E_{n'} + i\eta}\left(e^{\beta E_n} - (\pm)e^{-\beta E_{n'}}\right)$$

And by the definition of Matsubara frequency,

$$\begin{aligned}
C_{AB}(\tau) = -\frac{1}{Z}Tr[e^{\beta H}e^{\tau H}Ae^{-\tau H}B] \\
&= -\frac{1}{Z}\int \psi^{0*}_n(x)e^{\beta H}e^{\tau H}a^\dagger e^{-\tau H}a\psi^0_n(x)dx \\
&= -\frac{1}{Z}\sum_n e^{\beta E_n}\int \psi^{0*}_n(x)e^{\tau H}a^\dagger e^{-\tau H}a\psi^0_n(x)dx \\
&= -\frac{1}{Z}\sum_n e^{\beta E_n}\int \psi^{0*}_n(x)a^\dagger|\psi_{n'}\rangle\langle\psi_n|a\psi^0_n(x)dxe^{-\tau E_n - E_{n'}}
\end{aligned}$$

$$\begin{aligned}
C_{AB}(\tau) &= -\frac{1}{Z}Tr[e^{\beta H}e^{\tau H}Ae^{-\tau H}B] \\
&= -\frac{1}{Z}\sum_{nn'}e^{\beta H}\langle n|A|n'\rangle\langle n'|B|n\rangle\left(e^{\beta E_n} - (\pm)e^{-\beta E_{n'}}\right)
\end{aligned}$$

Using Fourier Transformation,

$$\begin{aligned}
C_{AB} &= \int_0^\beta d\tau e^{i\omega_n\tau}[-\frac{1}{Z}\sum_{nn'}e^{\beta H}e\langle n|A|n'\rangle\langle n'|B|n\rangle\left(e^{\beta E_n} - (\pm)e^{-\beta E_{n'}}\right)] \\
&= \frac{1}{(E_n - E_n' + \tau\omega_n i)}(-\frac{e^{\beta H}}{Z}c)e^{\tau(E_n - E_{n'} + i\omega + n)}|^\beta_0
\end{aligned}$$

Therefore,

$$C_{AB} = \frac{1}{Z}\sum_{nn'}\frac{\sum_{nn'}\langle n|A|n'\rangle\langle n'|B|n\rangle}{i\omega_n + E_n - En'}(e^{betaE_n} - \pm e^{\beta E_{n'}})$$

Consider the function in entire complex plane, where $z = x + iy$,

$$C_{AB} = \frac{1}{Z}\sum_{nn'}\frac{\sum_{nn'}\langle n|A|n'\rangle\langle n'|B|n\rangle}{z + E_n - En'}(e^{betaE_n} - \pm e^{\beta E_{n'}})$$

According to the theory of analytic functions : if two functions coincide in an a infinite set of points then they are fully identical functions within the entire domain where at least one of them is analyitc function.
That is, if $C_{AB}(i\omega)$ is known, then $C^R_{AB}(\omega)$ can be found :

$$C^R_{AB}(\omega) = C_{AB}(i\omega \to \omega + i\eta)$$

Where η is infinitesimal real value.