Matsubara Summary(2)

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Trial for Deriving Matsubara frequency

At given condition, $H_0 = \sum_k \omega_k a_k^\dagger a_k$.

$$\begin{aligned}
\partial_\tau C_{AB} &= \langle \partial_\tau e^{\tau H_0} a_k e^{-\tau H_0} a_k^\dagger - e^{\tau H_0} a_k \partial_\tau e^{-\tau H_0} a_k^\dagger \rangle \\
&= \langle e^{\tau H_0} H_0 a_k e^{-\tau H_0} a_k^\dagger - e^{\tau H_0} a_k H_0 e^{-\tau H_0} a_k^\dagger \rangle \\
&= \langle e^{\tau H_0} [H_0, a] e^{-\tau H_0} a_k^\dagger \rangle
\end{aligned}$$

$$\Leftrightarrow \quad \partial_\tau C_{AB} = \langle e^{\tau H_0} (\sum_{k'} a_{k'} a_{k'}) a_k e^{-\tau H_0} a_k^\dagger \rangle - \langle e^{\tau H_0} a_k (\sum_{k'} a_{k'} a_{k'}) e^{-\tau H_0} a_k^\dagger \rangle$$

Considering bosonic case, If $k \neq k'$,

$$\begin{aligned}
&\langle e^{\tau H_0} \sum_{k'} a_{k'}^\dagger a_{k'} a_k e^{\tau H_0} a_k^\dagger \rangle - \langle e^{\tau H_0} a_k \sum_{k'} a_{k'}^\dagger a_{k'} e^{\tau H_0} a_k^\dagger \rangle \\
&= 0 \qquad (*[a_{k'}, a_k] = 0 \quad , \quad \text{if} \quad k \neq k')
\end{aligned}$$

If $k = k'$,

$$\begin{aligned}
\Leftrightarrow \quad &= \langle e^{\tau H_0} a_k a_k a_k e^{-\tau H_0} a_k^\dagger \rangle - \langle e^{\tau H_0} a_k a_k a_k e^{-\tau H_0} a_k^\dagger \rangle \\
&= \langle e^{\tau H_0} a_k (1 + a_k a_k) e^{-\tau H_0} a_k^\dagger \rangle - \langle e^{\tau H_0} a_k a_k a_k e^{-\tau H_0} a_k^\dagger \rangle \\
&= \langle e^{\tau H_0} a_k e^{-\tau H_0} a_k^\dagger \rangle + \langle e^{\tau H_0} a_k a_k a_k e^{-\tau H_0} a_k^\dagger \rangle - \langle e^{\tau H_0} a_k a_k a_k e^{-\tau H_0} a_k^\dagger \rangle
\end{aligned}$$

$$\begin{aligned}
\Leftrightarrow \quad &= \langle e^{\tau H_0} a_k e^{-\tau H_0} a_k^\dagger \rangle \\
&= C_{AB}(\tau)
\end{aligned}$$

Using Diff.equation

$$\partial_\tau C_{AB}(\tau) = C_{AB}(\tau) \quad \Leftrightarrow \quad C_{AB}(\tau) = e^{\alpha\tau}$$

Where α comes from initial condition. To apply Fourier transform

$$\begin{aligned}
G(n) &= \frac{1}{2} \int_{-\beta}^{\beta} d\tau e^{\frac{i\pi n \tau}{\beta}} C_{AB}(\tau) \\
&= \frac{1}{2} \int_{-\beta}^{\beta} d\tau e^{\frac{i\pi n \tau}{\beta}} A_0 e^{\tau} \\
&= \frac{1}{2} \int_{-\beta}^{\beta} d\tau A_0 e^{\frac{i\pi n}{\beta} + 1} \tau \\
&= \frac{1}{2} \frac{A_0}{\frac{i\pi n}{\beta} + 1} (e^{(\frac{i\pi n}{\beta} + 1)\beta} - e^{-(\frac{i\pi n}{\beta} + 1)\beta}) \\
&= \frac{A_0 i}{\frac{i\pi n}{\beta} + 1} \left(\frac{e^{(\frac{i\pi n}{\beta} + 1)\beta} - e^{-(\frac{i\pi n}{\beta} + 1)\beta}}{2i} \right) \\
&= \frac{A_0 i}{\frac{i\pi n}{\beta} + 1} \sin(\pi n - i\beta) \\
&= \frac{A_0 i}{\frac{i\pi n}{\beta} + 1} \cos(i\beta)
\end{aligned}$$

Using distribution function

Started from $k = k'$ condition,

$$\partial_\tau C_{AB}(\tau) = 1 + n_B = \frac{e^{\beta E_k}}{e^{\beta E_k} - 1}$$

Where n_B is Bosonic distribution Function. The form of $1 + n_B$ comes from :

$$\begin{aligned} C_{AB}(\tau) &= \langle e^{\tau H_0} a_k e^- \tau H_0 a_k^\dagger \rangle \\ &= \langle e^{\tau H_0} e^- \tau H_0 a_k a_k^\dagger \rangle \\ &= \langle a_k a_k^\dagger \rangle \\ &= \langle 1 + a_k^\dagger a_k \rangle \quad = 1 + \langle a_k^\dagger a_k \rangle \end{aligned}$$

Since $\langle a_k^\dagger a_k \rangle$ represents particle density function, substituting it to n_B could be possible.

$$\Leftrightarrow \quad C_{AB}(\tau) = \int_0^\tau (1 + n_B) = (1 + n_B)\tau = \frac{e^{\beta E_k}\tau}{1 + n_B}$$

$$\begin{aligned} C_{AB}(n) &= \frac{e^{\beta E_k}}{2} \int_{-\beta}^{\beta} \tau e^{\frac{i\pi n\tau}{\beta}} (e^{\beta E - 1} - 1)^{-1} d\tau \\ &= \frac{e^{\beta E_k}}{2} [(\text{odd function})] - \frac{e^{\beta E_k}}{2} \int_{-\beta}^{\beta} \frac{\beta}{i\pi n} \big(e^{i\pi n} - e^{i\pi - n} \big) (e^{\beta E} - 1)^{-1} \\ &= -\frac{\beta e^{\beta E}}{\pi n} \int_{-\beta}^{\beta} \sin(\pi n)(e^{\beta E} - 1)^{-1} d\tau \\ &= -\frac{2\beta^2}{\pi n} \sin(\pi n) \frac{e^{\beta E}}{e^{\beta E} - 1} \\ &= -\frac{2\beta^2}{\pi n} \sin(\pi n)(1 + C_{AB}(\tau)) \end{aligned}$$