Derivation of the Second Quantization of the two-particle Operator

$$S^{-}\hat{V}_{tot}\left|\psi_{\nu_{\alpha_1}}(\mathbf{r}_1)\right\rangle\left|\psi_{\nu_{\alpha_2}}(\mathbf{r}_2)\right\rangle,\cdots,\left|\psi_{\nu_{\alpha_N}}(\mathbf{r}_N)\right\rangle$$
$$=\hat{V}_{tot}c^{\dagger}_{\nu_{\alpha_1}}c^{\dagger}_{\nu_{\alpha_2}}\cdots c^{\dagger}_{\nu_{\alpha_N}}\left|0\right\rangle$$

$$=\frac{1}{2}\sum_{j\neq k}^{N}\sum_{\nu_a\nu_b,\nu_c\nu_d}V_{\nu_c\nu_d,\nu_a\nu_b}\delta_{\nu_a,\nu_{\nu_j}}\delta_{\nu_b,\nu_{\nu_k}}\left|\psi_{\nu_{\alpha_1}}(\mathbf{r}_1)\right\rangle,\left|\psi_{\nu_{\alpha_2}}(\mathbf{r}_2)\right\rangle\cdots\left|\psi_{\nu_c}(\mathbf{r}_j)\right\rangle,\cdots\left|\psi_{\nu_d}(\mathbf{r}_k)\right\rangle,\cdots,\left|\psi_{\nu_{\alpha_N}}(\mathbf{r}_N)\right\rangle$$

$$=\frac{1}{2}\sum_{j\neq k}^{N}\sum_{\nu_a\nu_b,\nu_c\nu_d}V_{\nu_c\nu_d,\nu_a\nu_b}\delta_{\nu_a,\nu_{\nu_j}}\delta_{\nu_b,\nu_{\nu_k}}c^{\dagger}_{\nu_n 1}c^{\dagger}_{\nu_{\alpha_2}}\cdots c^{\dagger}_{\nu_c},\cdots c^{\dagger}_{\nu_d}\cdots c^{\dagger}_{\nu_N}\left|0\right\rangle$$

$$=\frac{1}{2}\sum_{j\neq k}^{N}\sum_{\nu_a\nu_b,\nu_c\nu_d}V_{\nu_c\nu_d,\nu_a\nu_b}\delta_{\nu_a,\nu_{\nu_j}}\delta_{\nu_b,\nu_{\nu_k}}c^{\dagger}_{\nu_n 1}c^{\dagger}_{\nu_{\alpha_2}}\cdots c^{\dagger}_{\nu_c},\cdots c^{\dagger}_{\nu_d}\cdots c^{\dagger}_{\nu_N}\left|0\right\rangle$$

$$=\frac{1}{2}\sum_{j\neq k}^{N}\sum_{\nu_a\nu_b,\nu_c\nu_d}V_{\nu_c\nu_d,\nu_a\nu_b}c^{\dagger}_{\nu_c}c^{\dagger}_{\nu_d}c^{\dagger}_{\nu_1}c^{\dagger}_{\nu_2}\cdots\delta_{\nu_c,\nu_{\nu_j}}c_{\nu_c}c^{\dagger}_{\nu_c},\cdots\delta_{\nu_d,\nu_{\nu_k}}c_{\nu_d}c^{\dagger}_{\nu_d}\cdots c^{\dagger}_{\nu_N}\left|0\right\rangle$$

$$=\frac{1}{2}\sum_{\nu_a\nu_b,\nu_c\nu_d}V_{\nu_c\nu_d,\nu_a\nu_b}\left(\delta_{\nu_a,\nu_{\nu_1}}c_{\nu_c}c^{\dagger}_{\nu_1}+\delta_{\nu_a,\nu_{\nu_2}}c^{\dagger}_{\nu_c}c^{\dagger}_{\nu_2}+\cdots\delta_{\nu_d,\nu_{\nu_k}}c_{\nu_d}c^{\dagger}_{\nu_k}+\cdots+\delta_{\nu_d,\nu_{\nu_N}}c_{\nu_d}c^{\dagger}_{\nu_N}\right)c^{\dagger}_{\nu_1}c^{\dagger}_{\nu_2}\cdots c^{\dagger}_{\nu_N}\left|0\right\rangle$$

if $\nu_c=\nu_j$, $\nu_d=\nu_k$,

$$=\frac{1}{2}\sum_{\nu_a\nu_b,\nu_c\nu_d}V_{\nu_c\nu_d,\nu_a\nu_b}c^{\dagger}_{\nu_c}c^{\dagger}_{\nu_d}c_{\nu_a}c_{\nu_b}\left(c^{\dagger}_{\nu_1}c^{\dagger}_{\nu_2}\cdots c^{\dagger}_{\nu_N}\right)\left|0\right\rangle$$

$$V_{tot}=\frac{1}{2}\sum_{\nu_a\nu_b,\nu_c\nu_d}V_{\nu_c\nu_d,\nu_a\nu_b}c^{\dagger}_{\nu_c}c^{\dagger}_{\nu_d}c_{\nu_a}c_{\nu_b}$$