$$
\begin{aligned}
\langle \mathrm{even}_l | \cos\phi \, | \mathrm{even}_k \rangle &= \int_0^{2\pi} d\phi \left(\frac{a_0^{(l)}}{\sqrt{2\pi}} + \sum_{m=1} \frac{a^{(l)}}{\sqrt{\pi}} \cos m\phi \right) \left(\frac{a_0^{(k)}}{\sqrt{2\pi}} \cos\phi + \sum_{n=1} \frac{a^{(k)}}{\sqrt{\pi}} \cos\phi \cos n\phi \right) \\
&= \int_0^{2\pi} d\phi \left(\frac{a_0^{(k)} a_0^{(l)}}{2\pi} + \sum_{n=1} \frac{a_n^{(k)} a_0^{(l)}}{\pi\sqrt{2}} \cos n\phi + \sum_{m=1} \frac{a_0^{(k)} a_m^{(l)}}{\pi\sqrt{2}} \cos m\phi + \sum_{n,m=1} \frac{a_n^{(k)} a_l^{(m)}}{\pi} \cos n\phi \cos m\phi \right) \cos\phi \\
&= \begin{cases} \text{if k = l :} & \frac{2}{\sqrt{2}} a_0^{(k)} a_1^{(k)} + \sum_{n=1}^{2k} a_{2n-1}^{(k)} a_{2n}^{(k)} \\ \text{if k} \neq \text{l :} & \frac{1}{\sqrt{2}} (a_0^{(k)} a_1^{(l)} + a_1^{(k)} a_0^{(l)}) + \frac{1}{2} \sum_{n=1}^{2k} (a_n^{(k)} a_{n+1}^{(l)} + a_n^{(l)} a_{n+1}^{(k)}) \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
\langle \mathrm{odd}_l | \cos\phi \, | \mathrm{odd}_k \rangle &= \sum_{n,m=1} \int_0^{2\pi} d\phi \left(\frac{b_n^{(k)} b_m^{(l)}}{\pi} \cos\phi \sin n\phi \sin m\phi \right) \\
&= \begin{cases} \text{if n = 1, m = 2 :} & \frac{b_1^{(k)} b_2^{(l)}}{2} \\ \text{if n = 2, m = 1 :} & \frac{b_2^{(k)} b_1^{(l)}}{2} \end{cases}
\end{aligned}
$$

$$
\hat{\cos}\,\phi = \begin{pmatrix} \ddots & & \vdots & & \\ & \langle \mathrm{even}_k | \cos\phi \, | \mathrm{even}_k \rangle & 0 & \langle \mathrm{even}_k | \cos\phi \, | \mathrm{even}_{k+1} \rangle & \cdots \\ & 0 & \langle \mathrm{odd}_k | \cos\phi \, | \mathrm{odd}_k \rangle & 0 & \langle \mathrm{odd}_k | \cos\phi \, | \mathrm{odd}_{k+1} \rangle \\ & \langle \mathrm{even}_{k+1} | \cos\phi \, | \mathrm{even}_k \rangle & 0 & \langle \mathrm{even}_{k+1} | \cos\phi \, | \mathrm{even}_{k+1} \rangle & \\ & & \vdots & & \ddots \end{pmatrix}
$$