Matsubara Summary(1)

May 8, 2023

1 Interaction picture

Begin from the interaction picture, The interacting Hamiltonian $H_{int} = H - H_0$ with time-dependence and corresponding wavefunction can be written as:

$$H_{int}(t) = e^{iH_0 t} H_{int} e^{iH_0 t}, \qquad \psi_{(i)}(t) = e^{iH_0 t}\psi_{(s)}$$

where $\psi_{(i)}$ represents a wavefuction in the interaction picture, $\psi_{(s)}$ represents a wavefunction in the schrodinger picture, $\psi_{(s)} = e^{iHt}$.

Assume that, ψ_i can be represented as a series expansion form. then the wavefuction becomes:

$$\psi_{(i)}(t) = \psi_{(i)}^{(0)}(t) + \psi_{(i)}^{(1)}(t) + \cdots + \psi_{(i)}^{(n)}(t)$$

Where, nth term of the series is:

$$\psi_{(i)}^{(n)}(t) = (-i)^n \int_{t_0}^{t} H_{int}(t_1)dt_1 \int_{t_0}^{t_1} H_{int}(t_2)dt_2 \cdots \int_{t_0}^{t_{n-1}} H_{int}(t_n)\psi_{(i)}(0)dt_n$$

Let's call the multiplication of integral,$(-i)^n \int_{t_0}^{t} H_{int}(t_1)dt_1 \int_{t_0}^{t_1} H_{int}(t_2)dt_2 \cdots \int_{t_0}^{t_{n-1}} H_{int}(t_n)$ as $U^{(n)}(t,t_0)$. then the wavefunction $\psi_{(i)}(t)$ can be written as : $\psi_{(i)}(t) = U(t,t_0)\psi_{(i)}(0)$. For the multiplication of H_{int} terms, It can have a permutation feature for t_i.

After counting all possible permutations and arranging the integrals using time-ordering operator T, U becomes:

$$U^{(n)}(t,t_0) = \frac{(-i)^n}{n!}\int_{t_0}^{t}\cdots\int_{t_0}^{t} T(H_{int}(t_1)\ldots H_{int}(t_n))dt_1\ldots dt_n$$

The time-ordering operator T, arranges the terms into time-decreasing order(from left to right). The total series of the wavefuction of interaction picture is:

$$\begin{aligned}\psi_{(i)}(t) &= (U^{(0)}(t,t_0) + U^{(1)}(t,t_0) + \cdots + U^{(n)}(t,t_0))\psi_{(i)}(0) \\ &= U(t,t_0)\end{aligned}$$

And,

$$\begin{aligned}U(t,t_0) &= T(1 + (-i\int_{t_0}^{t} H_{int}(t_1)dt_1) - (\frac{1}{2!}\int_{t_0}^{t}\int_{t_0}^{t} H_{int}(t_1)H_{int}(t_2)dt_1dt_2) \\ &\quad \cdots + \frac{(-i)^n}{n!}\int_{t_0}^{t}\cdots\int_{t_0}^{t} T(H_{int}(t_1)\ldots H_{int}(t_n))dt_1\ldots dt_n \\ &= Te^{\{-i\int_{t_0}^{t} H_{int}(t')dt'\}}\end{aligned}$$

Following the argument, $U(t,t_0)$ is an a matrix which satisfies $\psi_(i)(t) = U(t,t_0)\psi_(0)(t)$, and $U(t,t) = 1$. Finally, the Expectation value of the Arbitrary Operator A in the interaction picture can be written in the following form:

$$\begin{aligned}\langle A|A\rangle &= \int \psi_(^* i)(t)A\psi_(i)(t)dx \\ &= \int \psi_(^* i)(0)U^{-1}(t_0,t)AU(t,t_0)\psi_(i)(0)dx\end{aligned}$$

Similar to the Heisenberg picture (e.g operator $B(t) = e^{iHt}Be^{-iHt}$), the Operator in the interaction picture depends on time can be represented as the form of:

$$A(t) = U^{-1}(t_0,t)AU(t,t_0)$$

2 Matsubara function first summary

Let's set a Correlation function as C_{AB}. According to the statistical mechanics, given correlationn function can be represented as:

$$C_{AB} = -\langle A(t)B(t')\rangle$$
$$= -\frac{1}{Z}Tr[e^{\beta H}A(t)B(t)]$$

Using the interaction picture representation,

$$C_{AB} = -\frac{1}{Z}Tr[e^{\beta H}U(0,t)A(t)U(t,t')B(t')U(t',0)]$$

To calculate the Energy term $e^{-\beta H}$, the Imginary time τ introduced instead of real-time t.

$$\tau = it$$

Using τ instead of t, correlation function $C_{AB} = -\frac{1}{Z}Tr[e^{\beta H}U(0,\tau)A(\tau)U(\tau,\tau')B(\tau')U(\tau',0)]$. Assume that $\tau - \tau' = \beta$. Then from the relation $U(\tau,\tau') = e^{\tau H_0}e^{(\tau-\tau')H}e^{-\tau' H_0}$,

$$e^{-\beta H} = e^{\beta H_0}U(\beta,0) \qquad = e^{\beta H_0}Te^{-\int_0^\beta d\tau_1 H_{int}(\tau_1)}$$

And the correlation function with using time-ordering operator T_τ,

$$\langle T_\tau A(\tau)B(\tau)\rangle = \frac{1}{Z}Tr[e^{-\beta H}T_\tau A(\tau)B(\tau')]$$
$$= \frac{1}{Z}Tr[e^{-\beta H_0}T_\tau U(\beta,0)A(\tau)B(\tau')]$$
$$= \frac{\langle U(\beta,0)A(\tau)B(\tau')\rangle_0}{\langle U(\beta,0)\rangle_0}$$

The definition of Matsubara Green function is:

$$C_{AB}(\tau,\tau') \equiv -\langle T_\tau(A(\tau)B(\tau'))|T_\tau(A(\tau)B(\tau'))\rangle$$

Where:

$$T_\tau(A(\tau)B(\tau')) = \theta(\tau-\tau')A(\tau)B(\tau') \pm \theta(\tau-\tau')B(\tau')A(\tau)$$