Ch.9 Equation of motion (1)

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1 Bruus

1.1 Derivation of Green's function

The given schrodinger equation is:

$$[H_0(\mathbf{r}) + H_i(\mathbf{r})]\Psi_E = \Psi_E$$

Define the corresponding Green's function as :

$$[E - H_0(r)]G_0(\mathbf{r}, \mathbf{r}'E) = \delta(\mathbf{r} - \mathbf{r}')$$

Once more, Define the inverse $G_0^{-1}(\mathbf{r}, E) = E - H_0(\mathbf{r})$, Then

$$G_0^{-1}(\mathbf{r}, E)G_0(\mathbf{r}, \mathbf{r}', E) = \delta(\mathbf{r} - \mathbf{r}')$$

Then Schrodinger equation rewritten as:

$$\begin{aligned}
[G_0^{-1}(\mathbf{r}, E) - H_i((r))]\Psi_E = 0& \\
&= (G_0^{-1}(\mathbf{r}, E) - H_i(\mathbf{r}))\Psi_E = 0 \\
&= G_0^{-1}(\mathbf{r}, E)\Psi_E - H_i(\mathbf{r})\Psi_E = 0 \\
&= \delta(\mathbf{r} - \mathbf{r}')G_0^{-1}(\mathbf{r}, \mathbf{r}'E)\Psi_E - H_i(\mathbf{r})\Psi_E = 0 \\
&= \delta(\mathbf{r} - \mathbf{r}')\Psi_E = G_0(\mathbf{r}, \mathbf{r}'E)H_i(\mathbf{r})\Psi_E
\end{aligned}$$

Then solution may be written as an integral equation

$$\Psi_E(\mathbf{r}) = \Psi_E^0(\mathbf{r}) + \int \mathbf{r}'G_0(\mathbf{r}, \mathbf{r}', E)H_i(\mathbf{r}')\Psi_E(\mathbf{r}')$$

Solving integral equation by iteration, and up to first order:

$$\Psi_E(\mathbf{r}) = \Psi_E^0(\mathbf{r}) + \int \mathbf{r}'G_0(\mathbf{r}, \mathbf{r}', E)H_i(\mathbf{r}')\Psi_E^0(\mathbf{r}') + \mathcal{O}(V^2)$$

Now based upon above discussion, in the case of time-dependent Hamiltonian, $H = H_0 + H_i$,

$$[i\partial_t - H_0(\mathbf{r}) - H_i(\mathbf{r})]\Psi(\mathbf{r}, t) = 0$$

Define Green's function by:

$$[i\partial_t - H_0(\mathbf{r})]G_0(\mathbf{r}t, \mathbf{r}'t') = \delta(\mathbf{r} - \mathbf{r}')\delta(t - t')$$
$$[i\partial_t - H_0(\mathbf{r}) - H_i(\mathbf{r})]G(\mathbf{r}t, \mathbf{r}'t') = \delta(\mathbf{r} - \mathbf{r}')\delta(t - t')$$

Then inverse of Green's function is:

$$G_0^{-1}(\mathbf{r}, t) = i\partial_t - H_0(\mathbf{r})$$
$$G^{-1}((\mathbf{r}, t)) = i\partial_t - H_0(\mathbf{r}) - H_i(\mathbf{r})$$

Adjust as a integral equation form:

$$G_0^{-1}(\mathbf{r}, t)G_0(\mathbf{r}t, \mathbf{r}'t') = \delta(\mathbf{r} - \mathbf{r}')\delta(t - t')$$
$$G^{-1}((\mathbf{r}, t))G(\mathbf{r}t, \mathbf{r}'t') = \delta(\mathbf{r} - \mathbf{r}')\delta(t - t')$$

$$G_0^{-1}(\mathbf{r}, t)\Psi(\mathbf{r}, t) - H_i(\mathbf{r})\Psi(\mathbf{r}, t) = 0$$
$$G^{-1}(\mathbf{r}, t)\Psi(\mathbf{r}, t) - H_i(\mathbf{r})\Psi(\mathbf{r}, t) = 0$$

$$\delta(\mathbf{r} - \mathbf{r}')\delta(t - t')\Psi(\mathbf{r}, t) = G_0(\mathbf{r}t, \mathbf{r}'t')\Psi(\mathbf{r}, t)$$
$$\delta(\mathbf{r} - \mathbf{r}')\delta(t - t')\Psi(\mathbf{r}, t) = G(\mathbf{r}t, \mathbf{r}'t')H_i(\mathbf{r})\Psi(\mathbf{r}, t)$$

Solution to time-dependent Schrodinger equation,

$$\Psi(\mathbf{r},t) = \Psi^0(\mathbf{r},t) + \int d\mathbf{r}' \int dt' G_0(\mathbf{r},\mathbf{r}';t,t')H_i(\mathbf{r}')\Psi^0(\mathbf{r}',t')$$

1.2 Fourier transforms of advanced and retarded functions

Retarded : a physical observable due to the action of some force or interaction at times prior to the measurement This means that the observing(standard of calculation) time t' must be small than the measurment time t, General form of retarded function is :

$$\begin{aligned} C^R_{AB}(t,t') &= -i\theta(t-t')\Big\langle \big[A(t),B(t')\big]\Big\rangle \\ &= -i\theta(t-t')\Big\langle A(t)B(t') - B(t')A(t)\Big\rangle \end{aligned}$$

Thermal average of $A(t)B(t')$ is, because of the time-dependent operator of interaction picture can be written as:

$$\hat{O}(t) = e^{iH_0t}\hat{O}e^{-iH_0t'}$$

Thus,

$$\begin{aligned} \langle A(t)B(t')\rangle &= \frac{1}{Z}Tr\Big[e^{\beta H_0}e^{iH_0t}Ae^{-iH_0t'}e^{iH_0t}Be^{-iH_0t'}\Big] \\ &= \frac{1}{Z}Tr\Big[e^{-iH_0t'}e^{\beta H_0}e^{iH_0t}Ae^{-iH_0t'}e^{iH_0t}B\Big] \\ &= \frac{1}{Z}Tr\Big[e^{\beta H_0}e^{iH_0(t-t')}Ae^{iH_0(t-t')}B\Big] \end{aligned}$$

C_{AB} depends on time variable, Fourier transform of $C^R(t-t')$ is :

$$C^R_{AB}(\omega) = \int_{\infty}^{\infty} dte^{i\omega t}C^R_{AB}(t)$$

This is available when time difference $t-t' \to \infty$. Guess $t'=0$ and because of long time interval, there are no interaction between two time dependent operators,

$$\begin{aligned} C^R_{AB}(t,t') &= i\theta(t-0)\Big(\langle A(t)B(t')\rangle - \langle B(t')A(t)\rangle\Big) \\ &= \Big(\langle A(t)B(t')\rangle - \langle B(t')A(t)\rangle\Big) \\ &= \Big(\langle A(t)\rangle\langle B(t')\rangle - \langle B(t')\rangle\langle A(t)\rangle\Big) \\ &= C^R_{AB}(t) \end{aligned}$$

2 Abriskov

2.1 Derivation of Green's fucntion

From the change of basis, the Quantum field operator is written by using eigenfunction which notes 'some state ν', and 'position basis $\mathbf{r}$',

$$\Psi(\mathbf{r}) \equiv \sum_\nu \langle r|\psi_\nu\rangle\, a_\nu = \sum_\nu \psi_\nu(r)a_\nu \quad , \quad \Psi^\dagger(\mathbf{r}) \equiv \sum_\nu \langle r|\psi^*_\nu\rangle\, a^\dagger_\nu = \sum_\nu \psi^*_\nu(r)a^\dagger_\nu \tag{1}$$

and its Fourier transform for momentum basis k is :

$$\Psi(\mathbf{r}) \equiv \sum_k e^{i\mathbf{k}\cdot\mathbf{r}}a_\nu \quad , \quad \Psi^\dagger(\mathbf{r}) \equiv \sum_k e^{-i\mathbf{k}\cdot\mathbf{r}}a^\dagger_\nu \tag{2}$$

Now Using these features, lets begin the derivation for Green's function. The one-particle operator $F^{(1)}$can be written as:

$$F^{(1)} = \int \Psi^\dagger(\xi)f^{(1)}\Psi(\xi)d\xi \tag{3}$$

Now we are treating the free-particle system, and guess that the Hamiltoniam become a quadratic form,

$$H = \sum \epsilon_0(p) n_p = \sum \epsilon_0 a_p^\dagger a_p \tag{4}$$

according to equation (2),and guess that the ψ is the free-particles eigenfunction, Ψ in the Heisenberg picture can be described as

$$\hat{\Psi}(r,t) = \frac{1}{\sqrt{V}} \sum_p e^{i\sum_{p'} \epsilon_0(p') n_p t} a_p e^{-i\sum_{p"} \epsilon_0(p") n_p t} = \frac{1}{\sqrt{V}} \sum_p e^{iHt} a_p e^{-iHt} \tag{5}$$

$$= \frac{1}{\sqrt{V}} \sum_p e^{i[(p\cdot t) - \epsilon_0(p) t]} \tag{6}$$

Now, the Definition of Green's function is,

$$G_{\alpha\beta}(x,x') = -i\langle T(\hat{\Psi}_\alpha(x)\hat{\Psi}^\dagger_\beta(x'))\rangle \tag{7}$$

Using the Green's function, the average over the ground state of any one-particle operator is :

$$F^{(1)} = \pm i \int d^3\mathbf{r} \left[\lim_{x' \to x+0} f^{(1)}_{\alpha\beta}(x) G_{\alpha\beta}(x,x') \right] \tag{8}$$

here $x = \{\mathbf{r}, r\}$.Same method, expand G into a Fourier integral :

$$G(x - x') = \int \frac{d^4p}{(2\pi)^4} G(p,\omega) e^{i[(p\cdot r - r') - \omega(t-t')]} (d^4p = d^3\mathbf{p} d\omega) \tag{9}$$