Lehmann representation

$$G^{>}(\nu;t,t') = -i(c_\nu(t)c_\nu^\dagger(t')) = i\frac{1}{Z}\sum_n \langle n| e^{\beta H} c_\nu(t)c_\nu^\dagger(t')|n\rangle \tag{1}$$

$$= -i\frac{1}{Z}\sum_{nn'} e^{-\beta E_n}\langle n| c_\nu |n'\rangle \langle n'| c_\nu^\dagger |n\rangle e^{i-(E_n-E_{n'})(t-t')} \tag{2}$$

$$G^{>}(\nu;\omega) = \frac{-2\pi i}{Z}\sum_{nn'} e^{-\beta E_n}\langle n| c_\nu |n'\rangle \langle n'| c_\nu^\dagger |n\rangle \delta(E_n - E_{n'} + \omega) \tag{3}$$

$$G^{<}(\nu;\omega) = \frac{-2\pi i}{Z}\sum_{nn'} e^{(-\beta E_n)}\langle n| c_\nu |n'\rangle \langle n'| c_\nu^\dagger |n\rangle \delta(E_n - E_{n'} + \omega) \tag{4}$$

$$= \frac{-2\pi i}{Z}\sum_{nn'} e^{(-\beta E_n)}\langle n| c_\nu^\dagger |n'\rangle \langle n'| c_n u |n\rangle \delta(E_n - E_{n'} + \omega) \tag{5}$$

$$= \frac{-2\pi i}{Z}\sum_{nn'} e^{(-\beta E_n+\omega)}\langle n| c_\nu^\dagger |n'\rangle \langle n'| c_n u |n\rangle \delta(E_n - E_{n'} + \omega) \tag{6}$$

$$= -G^{>}(\nu;\omega)e^{-\beta\omega} \tag{7}$$

$$G^R(\nu,\omega) = -i\int_0^\infty dt e^{i(\omega+i\eta)t}\frac{1}{Z}\sum_{nn'} e^{\beta E_n}\left(\langle n| c_\nu |n'\rangle \langle n'| c_\nu^\dagger |n\rangle e^{i(E_n-E_{n'})t} + \langle n| c_\nu^\dagger |n'\rangle \langle n'| c_n u |n\rangle e^{-i(E_n-E_{n'})t}\right) \tag{8}$$

$$= \frac{1}{Z}\sum_{nn'} e^{\beta E_n}\left(\frac{\langle n| c_\nu |n'\rangle \langle n'| c_\nu^\dagger |n\rangle}{\omega + E_n + E_n' + i\eta} + \frac{\langle n| c_\nu^\dagger |n'\rangle \langle n'| c_\nu |n\rangle}{\omega - E_n + E_n' + i\eta}\right) \tag{9}$$

$$= \frac{1}{Z}\sum_{nn'} \langle n| c_\nu |n'\rangle \langle n'| c_\nu^\dagger |n\rangle (e^{\beta E_n} + e^{\beta E_{n'}}) \tag{10}$$

$$2\mathrm{Im}G^R(\nu,\omega) = -\frac{2\pi}{Z}\sum_{nn'} \langle n| c_\nu |n'\rangle \langle n'| c_\nu^\dagger |n\rangle (e^{\beta E_n} + e^{\beta E_{n'}})\delta(\omega + E_n - E_{n'}) \tag{11}$$

$$= -\frac{2\pi}{Z}\sum_{nn'} \langle n| c_\nu |n'\rangle \langle n'| c_\nu^\dagger |n\rangle e^{\beta - E_n}(1 + e^{\beta\omega})\delta(\omega + E_n - E_{n'}) \tag{12}$$

$$= -i(1 + e^{\beta\omega})G^{>}(\nu,\omega) \tag{13}$$

$$A(\nu,\omega) = -2\mathrm{Im}G^R(\nu,\omega) \tag{14}$$

$$iG^{>}(\nu,\omega) = A(\nu,\omega)[1 - n_F(\omega)] \tag{15}$$

$$iG^{>}(\nu,\omega) = A(\nu,\omega)[1 - n_F(\omega)] \tag{16}$$

$$-iG^{<}(\nu,\omega) = A(\nu,\omega)n_F(\omega) \tag{17}$$

$$G^R(\nu,\omega) = \int \frac{d\omega}{2\pi}\frac{A(\nu,\omega)}{\omega - \omega' + i\eta} \tag{18}$$

$$G^A(\nu,\omega) = \int \frac{d\omega}{2\pi}\frac{A(\nu,\omega)}{\omega - \omega' - i\eta} \tag{19}$$

$$G^R(\nu,\omega) = [G^A(\nu,\omega)]* \tag{20}$$

$$G^R(\nu,\nu',\omega) = [G^A(\nu,\nu,\omega)]^* \tag{21}$$

The spectral function

$$G_0^R(\mathbf{k}\sigma,\omega) = -i\int_\infty^\infty dt\theta(t-t')e^{i\omega(t-t')\eta(t-t')} \tag{22}$$
$$= \frac{1}{\omega-\xi_k+i\eta} \tag{23}$$

$$A_0(k\sigma,\omega) = -2\mathrm{Im}G_0^R(\mathbf{k}\sigma,\omega) = 2\pi\delta(\omega-\xi_k) \tag{24}$$

$$\int_\infty^\infty \frac{d\omega}{2\pi}A(\nu,\omega) = 1 \tag{25}$$

$$\int_\infty^\infty \frac{d\omega}{2\pi}A(\nu,\omega) = -\int_\infty^\infty \frac{d\omega}{2\pi}2ImG^R(\nu,\omega) \tag{26}$$
$$= \int_\infty^\infty d\omega\frac{1}{Z}\sum_{nn'}\langle n|\,c_\nu\,|n'\rangle\,\langle n'|\,c_\nu^\dagger\,|n\rangle\,(e^{\beta E_n}+e^{\beta E_{n'}})\delta(\omega+E_n-E_{n'}) \tag{27}$$
$$= \frac{1}{Z}\sum_{nn'}\langle n|\,c_\nu\,|n'\rangle\,\langle n'|\,c_\nu^\dagger\,|n\rangle\,(e^{\beta E_n}+e^{\beta E_{n'}}) = \langle c_\nu c_\nu^\dagger\rangle + \langle c_\nu^\dagger c_\nu\rangle = \langle c_\nu c_\nu^\dagger + c_\nu^\dagger c_\nu\rangle \tag{28}$$

$$\int_\infty^\infty \frac{d\omega}{2\pi}G^R(\nu,\omega) = G^R(\nu,t=0) = -i\theta(0)\langle\{c_\nu,c_\nu^\dagger\}\rangle = -i\frac{1}{2} \tag{29}$$

Since for fermions the occupation n_ν of a given state ν is given by:

$$\bar{n}_\nu = \langle c_\nu^\dagger c_\nu\rangle = -iG^<(\nu,t=0) \tag{30}$$
$$= -i\int_\infty^\infty \frac{d\omega}{2\pi}G^<(\nu,\omega) \tag{31}$$
$$= \int_\infty^\infty A(\nu,\omega)n_F(\omega) \tag{32}$$