.

$$V_{jk} = V_{kj}, \qquad \sum_{j<k}^{N} V_{jk} = \frac{1}{2}\sum_{j}^{N}\sum_{k}^{N} V_{jk} \quad (j \neq k)$$

Applying the symmetric operator in both side of V expressed in first quantized form.

$$S^{\pm}\hat{V}_{tot}\left|\psi_{\nu_{\alpha_1}}(\mathbf{r}_1)\right\rangle\left|\psi_{\nu_{\alpha_2}}(\mathbf{r}_2)\right\rangle, \cdots, \left|\psi_{\nu_{\alpha_N}}(\mathbf{r}_N)\right\rangle$$
$$= \hat{V}_{tot} a^{\dagger}_{\nu_{\alpha_1}} a^{\dagger}_{\nu_{\alpha_2}} \cdots a^{\dagger}_{\nu_{\alpha_N}} |0\rangle$$

$$= \frac{1}{2}\sum_{j\neq k}^{N}\sum_{\nu_a\nu_b,\nu_c\nu_d} V_{\nu_c\nu_d,\nu_a\nu_b}\delta_{\nu_a,\nu_{\alpha_j}}\delta_{\nu_b,\nu_{\alpha_k}}\left|\psi_{\nu_{\alpha_1}}(\mathbf{r}_1)\right\rangle, \cdots |\psi_{\nu_c}(\mathbf{r}_j)\rangle, \cdots |\psi_{\nu_d}(\mathbf{r}_k)\rangle, \cdots, \left|\psi_{\nu_{\alpha_N}}(\mathbf{r}_N)\right\rangle$$
$$= \frac{1}{2}\sum_{j\neq k}^{N}\sum_{\nu_a\nu_b,\nu_c\nu_d} V_{\nu_c\nu_d,\nu_a\nu_b}\delta_{\nu_a,\nu_{\alpha_j}}\delta_{\nu_b,\nu_{\alpha_k}}(a^{\dagger}_{\nu_{\alpha_1}} a^{\dagger}_{\nu_{\alpha_2}} \cdots \underset{\substack{\uparrow \\ \text{jth}}}{a^{\dagger}_{\nu_c}}, \cdots \underset{\substack{\uparrow \\ \text{kth}}}{a^{\dagger}_{\nu_d}} \cdots a^{\dagger}_{\nu\alpha_N})|0\rangle$$

To get the final result of summation, first, count each calculation process of equation term which stated as a for j and k indices,

$$= \frac{1}{2}\sum_{\nu_a\nu_b,\nu_c\nu_d} V_{\nu_c\nu_d,\nu_a\nu_b}\Bigg(\underbrace{\sum_{j}^{N}\sum_{k}^{N}\delta_{\nu_a,\nu_{\alpha_j}}\delta_{\nu_b,\nu_{\alpha_k}}(a^{\dagger}_{\nu_{\alpha_1}} a^{\dagger}_{\nu_{\alpha_2}} \cdots a^{\dagger}_{\nu_c}, \cdots a^{\dagger}_{\nu_d} \cdots a^{\dagger}_{\nu\alpha_N})|0\rangle}_{a}\Bigg)$$

If the given operator represents a Coulomb interaction between two electrons, the difference compared to the case of single particle operator $\hat{T}$ is a using fermionic creation operator $c^{\dagger}$ and annihilation operator c instead of bosonic operators $b^{\dagger}, b$, due to the fermion property of electrons.

In the procedure below, all creation and annihilation operators $a^{\dagger}$and a are considered fermionic operators during the calculation, so it can be deduced that : $n_{\alpha} = 1$, and the range of a value of α is the same as the range of j, $1 \leq \alpha,\ j \leq N$.

The following 3 conditions are used to analyze the a.

$$(1): \quad a^{\dagger}|0\rangle = a^{\dagger}_j a_i a^{\dagger}_i |0\rangle$$
$$(2): \quad \{c_i, c^{\dagger}_j\} = \delta_{ij}, \quad [b_i, b^{\dagger}_j] = \delta_{ij}$$
$$(3): \quad \text{if} \quad \lambda = -\lambda, \quad \lambda = 0, \qquad \text{if} \quad \lambda \neq 0, \quad \lambda \neq -\lambda \qquad (\lambda \text{ is a real number})$$

First, counts each summation result for k indices, k=1,2,...,k'...,N.

k=1 $\quad \delta_{\nu_b\nu_{\alpha_1}}(a^{\dagger}_{\nu_d} a^{\dagger}_{\nu_{\alpha_2}} \cdots a^{\dagger}_{\nu_{\alpha_N}})|0\rangle$

$$= \delta_{\nu_b\nu_{\alpha_1}}(\underbrace{a^{\dagger}_{\nu_d} a_{\nu_{\alpha_1}} a^{\dagger}_{\nu_{\alpha_1}}}_{\text{condition (1)}} a^{\dagger}_{\nu_{\alpha_2}} \cdots a^{\dagger}_{\nu_{\alpha_N}})|0\rangle$$
$$= \delta_{\nu_b\nu_{\alpha_1}} a^{\dagger}_{\nu_d} a_{\nu_{\alpha_1}}(a^{\dagger}_{\nu_{\alpha_1}} a^{\dagger}_{\nu_{\alpha_2}} \cdots a^{\dagger}_{\nu_{\alpha_N}})|0\rangle \qquad \rightarrow \qquad (1, \bigcirc)$$

$(1, \bigcirc)$ refers to the result of the calculation of the left side of the right-headed-arrow. A $\bigcirc$ means that it is blank, the value will be given in the calculation process for j indices.

k=2 $\quad \delta_{\nu_b\nu_{\alpha_2}}(a^{\dagger}_{\nu_{\alpha_1}} a^{\dagger}_{\nu_d} \cdots a^{\dagger}_{\nu_{\alpha_N}})|0\rangle$

$$= \delta_{\nu_b\nu_{\alpha_2}}(\underset{\uparrow}{a^{\dagger}_{\nu_{\alpha_1}}} \underset{\uparrow}{a^{\dagger}_{\nu_d}} a_{\nu_{\alpha_2}} a^{\dagger}_{\nu_{\alpha_2}} \cdots a^{\dagger}_{\nu_{\alpha_N}})|0\rangle$$

two operators are in ordered place, first one is in 1st place, and after 2nd place.

$$= \underset{\substack{\uparrow \\ \text{condition(2), by swapping the order}}}{(-1)}\delta_{\nu_b\nu_{\alpha_2}} a^{\dagger}_{\nu_d}(a^{\dagger}_{\nu_{\alpha_1}} a_{\nu_{\alpha_2}} a^{\dagger}_{\nu_{\alpha_2}} \cdots a^{\dagger}_{\nu_{\alpha_N}})|0\rangle$$

$$= (-1)^2\delta_{\nu_b\nu_{\alpha_2}} a^{\dagger}_{\nu_d} a_{\nu_{\alpha_2}}(a^{\dagger}_{\nu_{\alpha_1}} a^{\dagger}_{\nu_{\alpha_2}} \cdots a^{\dagger}_{\nu_{\alpha_N}})|0\rangle \qquad \rightarrow \qquad (2, \bigcirc)$$

Notice that swapping the order of ν_d and ν_c is necessary because the overall summation process is equal to $\hat{V}_{tot} a^\dagger_{\nu_{\alpha_1}} a^\dagger_{\nu_{\alpha_2}} \cdots a^\dagger_{\nu_{\alpha_N}} |0\rangle > 0$, which requires the condition (3).

$\vdots$

$$\begin{aligned} \text{k=k'} \quad & \delta_{\nu_b \nu_{\alpha_2}} (a^\dagger_{\nu_{\alpha_1}} a^\dagger_{\nu_{\alpha_2}} \cdots a^\dagger_{\nu_d} \cdots a^\dagger_{\nu_{\alpha_N}}) |0\rangle \\ & = \delta_{\nu_b \nu_{\alpha_2}} (\underbrace{a^\dagger_{\nu_{\alpha_1}} a^\dagger_{\nu_{\alpha_2}} \cdots}_{\text{k'-1 operators}} a^\dagger_{\nu_d} a_{\nu_{\alpha_{k'}}} a^\dagger_{\nu_{\alpha_{k'}}} \cdots a^\dagger_{\nu_{\alpha_N}}) |0\rangle \\ & = (-1)^{\text{k'-1}} \delta_{\nu_b \nu_{\alpha_2}} a^\dagger_{\nu_d} (a^\dagger_{\nu_{\alpha_1}} a^\dagger_{\nu_{\alpha_2}} \cdots a_{\nu_{\alpha_{k'}}} a^\dagger_{\nu_{\alpha_{k'}}} \cdots a^\dagger_{\nu_{\alpha_N}}) |0\rangle \\ & = (-1)^{2(\text{k'-1})} \delta_{\nu_b \nu_{\alpha_{k'}}} a^\dagger_{\nu_d} a_{\nu_{\alpha_{k'}}} (a^\dagger_{\nu_{\alpha_1}} a^\dagger_{\nu_{\alpha_2}} \cdots a^\dagger_{\nu_{\alpha_{k'}}} \cdots a^\dagger_{\nu_{\alpha_N}}) |0\rangle \quad \rightarrow \quad (k', \bigcirc) \end{aligned}$$

Successively, count the summation for the j index, where k=k' and j=1,2,...j',...,N

$$\begin{aligned} \text{k=k', j=1} \quad & (-1)^{2(\text{k'-1})} \delta_{\nu_a \nu_{\alpha_1}} \delta_{\nu_b \nu_{\alpha_{k'}}} a^\dagger_{\nu_d} a_{\nu_{\alpha_{k'}}} (a^\dagger_{\nu_c} \cdots a^\dagger_{\nu_{\alpha_{k'}}} \cdots a^\dagger_{\nu_{\alpha_N}}) |0\rangle \\ & = (-1)^{2(\text{k'-1})} \delta_{\nu_a \nu_{\alpha_1}} \delta_{\nu_b \nu_{\alpha_{k'}}} a^\dagger_{\nu_d} a_{\nu_{\alpha_{k'}}} (a^\dagger_{\nu_c} a_{\nu_{\alpha_1}} a^\dagger_{\nu_{\alpha_1}} \cdots a^\dagger_{\nu_{\alpha_{k'}}} \cdots a^\dagger_{\nu_{\alpha_N}}) |0\rangle \\ & = (-1)^{2(\text{k'-1})} \delta_{\nu_a \nu_{\alpha_1}} \delta_{\nu_b \nu_{\alpha_{k'}}} a^\dagger_{\nu_d} a_{\nu_{\alpha_{k'}}} a^\dagger_{\nu_c} a_{\nu_{\alpha_1}} (a^\dagger_{\nu_{\alpha_1}} \cdots a^\dagger_{\nu_{\alpha_{k'}}} \cdots a^\dagger_{\nu_{\alpha_N}}) |0\rangle \\ & = (-1)^{|2(\text{k'-1})+1|} \delta_{\nu_a \nu_{\alpha_1}} \delta_{\nu_b \nu_{\alpha_{k'}}} \underset{\uparrow}{a^\dagger_{\nu_d}} (\underset{\uparrow}{a^\dagger_{\nu_c}} a_{\nu_{\alpha_1}} - \delta_{\nu_c \nu_{\alpha_1}}) a_{\nu_{\alpha_{k'}}} (a^\dagger_{\nu_{\alpha_1}} \cdots a^\dagger_{\nu_{\alpha_{k'}}} \cdots a^\dagger_{\nu_{\alpha_N}}) |0\rangle \\ & \qquad \text{two operators are in ordered place} \\ & = (-1)^{|2(\text{k'-1})+2|} \delta_{\nu_a \nu_{\alpha_1}} \delta_{\nu_b \nu_{\alpha_{k'}}} a^\dagger_{\nu_c} a^\dagger_{\nu_d} a_{\nu_{\alpha_1}} a_{\nu_{\alpha_{k'}}} (a^\dagger_{\nu_{\alpha_1}} \cdots a^\dagger_{\nu_{\alpha_{k'}}} \cdots a^\dagger_{\nu_{\alpha_N}}) |0\rangle \\ & - \delta_{\nu_c \nu_{\alpha_1}} (\text{products of operators}) |0\rangle \quad \rightarrow \quad (k', 1) \end{aligned}$$

$\vdots$

$$\begin{aligned} \text{k=k', j=j'} \quad & (-1)^{2(\text{k'-1})} \delta_{\nu_a \nu_{\alpha_1}} \delta_{\nu_b \nu_{\alpha_{k'}}} a^\dagger_{\nu_d} a_{\nu_{\alpha_{k'}}} (a^\dagger_{\nu_{\alpha_1}} \cdots a^\dagger_{\nu_{\alpha_{k'}}} \cdots a^\dagger_{\nu_c} \cdots a^\dagger_{\nu_{\alpha_N}}) |0\rangle \\ & = (-1)^{2(\text{k'-1})} \delta_{\nu_a \nu_{\alpha_1}} \delta_{\nu_b \nu_{\alpha_{k'}}} a^\dagger_{\nu_d} a_{\nu_{\alpha_{k'}}} (\underbrace{a^\dagger_{\nu_{\alpha_1}} \cdots a^\dagger_{\nu_{\alpha_{k'}}} \cdots}_{\text{j'-1 operators}} a^\dagger_{\nu_c} a_{\nu_{\alpha_{j'}}} a^\dagger_{\nu_{\alpha_{j'}}} \cdots a^\dagger_{\nu_{\alpha_N}}) |0\rangle \\ & = (-1)^{|2(\text{k'-1})+2(\text{j'-1})|} \delta_{\nu_a \nu_{\alpha_1}} \delta_{\nu_b \nu_{\alpha_{k'}}} a^\dagger_{\nu_d} a_{\nu_{\alpha_{k'}}} a^\dagger_{\nu_c} a_{\nu_{\alpha_{j'}}} (a^\dagger_{\nu_{\alpha_1}} \cdots a^\dagger_{\nu_{\alpha_{k'}}} \cdots a^\dagger_{\nu_{\alpha_{j'}}} \cdots a^\dagger_{\nu_{\alpha_N}}) |0\rangle \\ & = (-1)^{|2(\text{k'-1})+2(\text{j'-1})+1|} \delta_{\nu_a \nu_{\alpha_1}} \delta_{\nu_b \nu_{\alpha_{k'}}} \underset{\uparrow}{a^\dagger_{\nu_d}} (\underset{\uparrow}{a^\dagger_{\nu_c}} a_{\nu_{\alpha_{k'}}} - \delta_{\nu_c \nu_{\alpha_{k'}}}) a_{\nu_{\alpha_{j'}}} (a^\dagger_{\nu_{\alpha_1}} \cdots a^\dagger_{\nu_{\alpha_{k'}}} \cdots a^\dagger_{\nu_{\alpha_{j'}}} \cdots a^\dagger_{\nu_{\alpha_N}}) |0\rangle \\ & \qquad \text{two operators are in ordered place} \\ & = (-1)^{|2(\text{k'-1})+2(\text{j'-1})+2|} \delta_{\nu_a \nu_{\alpha_1}} \delta_{\nu_b \nu_{\alpha_{k'}}} a^\dagger_{\nu_c} a^\dagger_{\nu_d} a_{\nu_{\alpha_{k'}}} a_{\nu_{\alpha_{j'}}} (a^\dagger_{\nu_{\alpha_1}} \cdots a^\dagger_{\nu_{\alpha_{k'}}} \cdots a^\dagger_{\nu_{\alpha_{j'}}} \cdots a^\dagger_{\nu_{\alpha_N}}) |0\rangle \\ & - \delta_{\nu_c \nu_{\alpha_{k'}}} (\text{product of operators}) |0\rangle \quad \rightarrow \quad (k', j') \end{aligned}$$

Now arrange each number set in a square formation, ordering (k,j) as kth row and jth column. The arranged form can be shown as a matrix, so let the corresponding matrix named A, then each number set (k,j) is an element of a matrix A_{kj}. The result of summation a corresponds to the sum of all elements in matrix A.

$$A = \begin{pmatrix} 0 & (1,2) & (1,3) & \cdots & (1,N) \\ (2,1) & 0 & (2,3) & \cdots & (2,N) \\ \vdots & & & (k'j') & \\ (N,1) & (N,2) & (N,3) & \cdots & 0 \end{pmatrix} \quad , \quad a = \sum_j^N \sum_k^N A_{jk}$$

If $\nu_a = \nu'_j$ and $\nu_b = \nu_{k'}$, result of summation (a) is the value of matrix element $A_{k'j'}$,

$$
\begin{aligned}
&\frac{1}{2}\sum_{\nu_a\nu_b,\nu_c\nu_d} V_{\nu_c\nu_d,\nu_a\nu_b}\Big(\text{(a)}\Big) \\
&= \frac{1}{2}\sum_{\nu_a\nu_b,\nu_c\nu_d} V_{\nu_c\nu_d,\nu_a\nu_b}\Big(A_{k'j'}\Big) \\
&= \frac{1}{2}\sum_{\nu_a\nu_b,\nu_c\nu_d} V_{\nu_c\nu_d,\nu_a\nu_b}(-1)^{2(|k'-1|+|j'-1|+1)} c^\dagger_{\nu_c} c^\dagger_{\nu_d} c_{\nu_b} c_{\nu_a}\left(c^\dagger_{\nu_1} c^\dagger_{\nu_2}\cdots c^\dagger_{\nu_N}\right)|0\rangle \\
&- \delta_{\nu_c\nu_{\alpha_{k'}}}(\text{product of operators})\,|0\rangle
\end{aligned}
$$

Since $\nu_c \neq \nu_b = \nu_{\alpha_{k'}}$, and $(-1)^{2(|k'-1|+|j'-1|+1)} = 1$,

$$
= \frac{1}{2}\sum_{\nu_a\nu_b,\nu_c\nu_d} V_{\nu_c\nu_d,\nu_a\nu_b} c^\dagger_{\nu_c} c^\dagger_{\nu_d} c_{\nu_b} c_{\nu_a}\left(c^\dagger_{\nu_1} c^\dagger_{\nu_2}\cdots c^\dagger_{\nu_N}\right)|0\rangle
$$

Therefore,

$$
V_{tot} = \frac{1}{2}\sum_{\nu_a\nu_b,\nu_c\nu_d} V_{\nu_c\nu_d,\nu_a\nu_b} c^\dagger_{\nu_c} c^\dagger_{\nu_d} c_{\nu_a} c_{\nu_b}
$$