*Derivation of the Second Quantization of the one-particle Operator From the First Quantization of one-particle operator,

$$\hat{T}_{tot}\left|\psi_{\nu_{\alpha_1}}(\mathbf{r}_1)\right\rangle\left|\psi_{\nu_{\alpha_2}}(\mathbf{r}_2)\right\rangle,\cdots,\left|\psi_{\nu_{\alpha_N}}(\mathbf{r}_N)\right\rangle = \sum_{j=1}^{N}\sum_{\nu_a,\nu_b} T_{\nu_b,\nu_a}\left|\psi_{\nu_b}(r_j)\right\rangle\left\langle\psi_{\nu_a}(r_j)\right|\left(\left|\psi_{\nu_{\alpha_1}}(\mathbf{r}_1)\right\rangle\left|\psi_{\nu_{\alpha_2}}(\mathbf{r}_2)\right\rangle,\cdots,\left|\psi_{\nu_{\alpha_N}}(\mathbf{r}_N)\right\rangle\right) =$$

Adjusting the Bosonic Operator on both sides of equation,

$$S^{+}\hat{T}_{tot}\left|\psi_{\nu_{\alpha_1}}(\mathbf{r}_1)\right\rangle\left|\psi_{\nu_{\alpha_2}}(\mathbf{r}_2)\right\rangle,\cdots,\left|\psi_{\nu_{\alpha_N}}(\mathbf{r}_N)\right\rangle = \hat{T}_{tot}b^{\dagger}_{\nu_{\alpha_1}}b^{\dagger}_{\nu_{\alpha_2}}\cdots b^{\dagger}_{\nu_{\alpha_N}}|0\rangle = \sum_{j=1}^{N}\sum_{\nu_a\nu_b}T_{\nu_b\nu_a}\delta_{\nu_a,\nu_{\nu_j}}b^{\dagger}_{\nu_n1}b^{\dagger}_{\nu_{\alpha_2}}\cdots b^{\dagger}_{\nu_b}\cdots b^{\dagger}_{\nu_{\alpha_N}}|0\rangle =$$

Assumme in the case of single-particle system, with N particles in M states

$$= \sum_{j=1}^{N}\sum_{\nu_a\nu_b}T_{\nu_b\nu_a}\delta_{\nu_a,\nu_{\nu_j}}\sqrt{n_1!}\,|\nu_1\rangle\cdots|\nu_1\rangle\sqrt{n_2!}\,|\nu_2\rangle\cdots|\nu_2\rangle\cdots|\nu_b\rangle\cdots\sqrt{n_M!}\,|\nu_M\rangle\cdots|\nu_M\rangle = B^{+}\sum_{j=1}^{N}\sum_{\nu_a\nu_b}T_{\nu_b\nu_a}\delta_{\nu_a,\nu_{\nu_j}}\underbrace{|\nu_1\rangle\cdots|\nu_1\rangle}_{n_1}$$

$$= B^{+}\sum_{\nu_a\nu_b}T_{\nu_b\nu_a}\begin{cases}\delta_{\nu_a,\nu_{\nu_1}}|\nu_b\rangle|\nu_1\rangle\cdots|\nu_1\rangle|\nu_2\rangle\cdots|\nu_M\rangle\\ +\delta_{\nu_a,\nu_{\nu_1}}|\nu_1\rangle|\nu_b\rangle\cdots|\nu_1\rangle|\nu_2\rangle\cdots|\nu_M\rangle\\ \vdots\\ +\delta_{\nu_a,\nu_{\nu_2}}|\nu_1\rangle\cdots|\nu_b\rangle\cdots|\nu_2\rangle\cdots|\nu_M\rangle\\ \vdots\\ +\delta_{\nu_a,\nu_{\nu_2}}|\nu_1\rangle\cdots|\nu_2\rangle|\nu_M\rangle\cdots|\nu_b\rangle\end{cases}$$

$$= \sum_{\nu_a\nu_b}T_{\nu_b\nu_a}\begin{cases}n_1\delta_{\nu_a,\nu_1}b^{\dagger}_{\nu_b}(b^{\dagger}_{\nu_1})^{n_1-1}(b^{\dagger}_{\nu_2})^{n_2}\cdots(b^{\dagger}_{\nu_M})^{n_M}|0\rangle\\ +n_2\delta_{\nu_a,\nu_2}(b^{\dagger}_{\nu_1})^{n_1}b^{\dagger}_{\nu_b}(b^{\dagger}_{\nu_2})^{n_2-1}\cdots(b^{\dagger}_{\nu_M})^{n_M}|0\rangle\\ \vdots\\ +n_M\delta_{\nu_a,\nu_M}(b^{\dagger}_{\nu_1})^{n_1-1}(b^{\dagger}_{\nu_2})^{n_2}\cdots b^{\dagger}_{\nu_b}(b^{\dagger}_{\nu_M})^{n_M-1}|0\rangle\end{cases}$$

Since $b^{\dagger}_{\nu_b}(b^{\dagger}_{\nu_j})^{p-1}|0\rangle = \frac{1}{p}b^{\dagger}_{\nu_b}b_{\nu_j}b^{\dagger}_{\nu_j}(b^{\dagger}_{\nu_j})^{p-1}|0\rangle$,

$$= \sum_{\nu_a\nu_b}T_{\nu_b\nu_a}\begin{cases}\frac{n_1}{n_1}\delta_{\nu_a,\nu_1}b^{\dagger}_{\nu_b}b_{\nu_1}b^{\dagger}_{\nu_1}(b^{\dagger}_{\nu_1})^{n_1-1}(b^{\dagger}_{\nu_2})^{n_2}\cdots(b^{\dagger}_{\nu_M})^{n_M}|0\rangle\\ +\frac{n_2}{n_2}\delta_{\nu_a,\nu_2}(b^{\dagger}_{\nu_1})^{n_1}b^{\dagger}_{\nu_b}b_{\nu_2}b^{\dagger}_{\nu_2}(b^{\dagger}_{\nu_2})^{n_2-1}\cdots(b^{\dagger}_{\nu_M})^{n_M}|0\rangle\\ \vdots\\ +\frac{n_M}{n_M}\delta_{\nu_a,\nu_M}(b^{\dagger}_{\nu_1})^{n_1}(b^{\dagger}_{\nu_2})^{n_2}\cdots b^{\dagger}_{\nu_b}b^{\dagger}_{\nu_M}b_{\nu_2}b^{\dagger}_{\nu_M}(b^{\dagger}_{\nu_M})^{n_M-1}|0\rangle\end{cases}$$

$$= \sum_{\nu_a\nu_b}T_{\nu_b\nu_a}(\delta_{\nu_a,\nu_1}b^{\dagger}_{\nu_b}b_{\nu_1} + \delta_{\nu_a,\nu_2}b^{\dagger}_{\nu_b}b_{\nu_2}\cdots\delta_{\nu_a,\nu_M}b^{\dagger}_{\nu_b}b^{\dagger}_{\nu_M})((b^{\dagger}_{\nu_1})^{\alpha_1}(b^{\dagger}_{\nu_2})^{\alpha_2}\cdots b^{\dagger}_{\nu_b}\cdots(b^{\dagger}_{\nu_M})^{\alpha_M})|0\rangle$$

if $\nu_i = \nu_a$,

$$= \sum_{\nu_a\nu_b}T_{\nu_b\nu_a}b^{\dagger}_{\nu_b}b_{\nu_a}b^{\dagger}_{\nu_n1}b^{\dagger}_{\nu_{\alpha_2}}\cdots b^{\dagger}_{\nu_{\alpha_N}}|0\rangle$$

$$= \hat{T}_{tot}b^{\dagger}_{\nu_n1}b^{\dagger}_{\nu_{\alpha_2}}\cdots b^{\dagger}_{\nu_{\alpha_N}}|0\rangle$$

$$\therefore \hat{T}_{tot} = \sum_{\nu_a\nu_b}T_{\nu_b\nu_a}b^{\dagger}_{\nu_b}b_{\nu_a}$$

Derivation of the Second Quantization of the two-particle Operator