$$\hat{H}_c = \frac{(\hat{P}-q\hat{A})^2}{2m} + V(\hat{Q}) + \sum_k \hbar\omega_k \hat{a}^\dagger \hat{a}$$
$$= \frac{\hat{P}^2}{2m} + \frac{q^2\hat{A}^2}{2m} - \underbrace{\frac{q}{2m}(\hat{P}\hat{A}+\hat{A}\hat{P})}_{\text{(a)}} + V(\hat{Q}) + \sum_k \hbar\omega_k \hat{a}^\dagger \hat{a} \tag{1}$$

$$\hat{H}_c = \frac{\hat{P}}{2m} + V(\hat{Q}) - \underbrace{\hat{P}\sum_n \zeta_n(\hat{b}_n + \hat{b}_n^\dagger)}_{\text{(b)}} + \sum_n \hbar\Omega_n \hat{b}_n \hat{b}_n^\dagger \tag{4}$$

$$\hat{a}_k = \sum_n (O)_{kn}[cosh(r_{nk})\hat{b}_n - sinh(r_{nk})\hat{b}_n^\dagger] \tag{5}$$

$$\begin{cases} \hat{X}_k = \sqrt{\frac{\hbar}{2\omega_k}}(\hat{a}_k + \hat{a}_k^\dagger) \\ \hat{P}_k = \sqrt{\frac{\hbar\omega_k}{2}}(-\hat{a}_k + \hat{a}_k^\dagger) \end{cases}$$

$$\begin{cases} \hat{a}_k = \sqrt{\frac{\omega_k}{2\hbar}}\hat{X}_k - i\sqrt{\frac{1}{2\hbar\omega_k}}\hat{P}_k \\ \hat{a}_k^\dagger = \sqrt{\frac{\omega_k}{2\hbar}}\hat{X}_k + i\sqrt{\frac{1}{2\hbar\omega_k}}\hat{P}_k \end{cases}$$

$$\hat{P}_k = \sum_n O_{kn}\hat{\tilde{P}}_n \quad , \quad \hat{X}_k = \sum_n O_{kn}\hat{\tilde{X}}_n \tag{A3}$$

$$\begin{cases} \hat{a}_k = \sqrt{\frac{\omega_k}{2\hbar}}\sum_n O_{kn}\hat{\tilde{X}}_n - i\sqrt{\frac{1}{2\hbar\omega_k}}\sum_n O_{kn}\hat{\tilde{P}}_n \quad , \quad \hat{b}_n = \sqrt{\frac{\Omega_n}{2\hbar}}\hat{\tilde{X}}_n + i\sqrt{\frac{1}{2\hbar\Omega_n}}\hat{\tilde{P}}_n \\ \hat{a}_k^\dagger = \sqrt{\frac{\omega_k}{2\hbar}}\sum_n O_{kn}\hat{\tilde{X}}_n + i\sqrt{\frac{1}{2\hbar\omega_k}}\sum_n O_{kn}\hat{\tilde{P}}_n \quad , \quad \hat{b}_n^\dagger = \sqrt{\frac{\Omega_n}{2\hbar}}\hat{\tilde{X}}_n - i\sqrt{\frac{1}{2\hbar\Omega_n}}\hat{\tilde{P}}_n \end{cases}$$

$$\begin{cases} \hat{b}_n = \sqrt{\frac{\Omega_n}{2\hbar}}\hat{\tilde{X}}_n + i\sqrt{\frac{1}{2\hbar\Omega_n}}\hat{\tilde{P}}_n \quad , \quad \hat{\tilde{X}}_n = \sqrt{\frac{\hbar}{2\Omega_n}}(\hat{b}_n + \hat{b}_n^\dagger) \\ \hat{b}_n^\dagger = \sqrt{\frac{\Omega_n}{2\hbar}}\hat{\tilde{X}}_n - i\sqrt{\frac{1}{2\hbar\Omega_n}}\hat{\tilde{P}}_n \quad , \quad \hat{\tilde{P}}_n = i\sqrt{\frac{\hbar\Omega_n}{2}}(\hat{b}_n - \hat{b}_n^\dagger) \end{cases}$$

$$\sum_n O_{kn}\big(\sqrt{\frac{\omega_k}{2\hbar}}\sqrt{\frac{\hbar}{2\Omega_n}}(\hat{b}_n + \hat{b}_n^\dagger) + \sqrt{\frac{1}{2\hbar\omega_k}}i\sqrt{\frac{\hbar\Omega_n}{2}}(\hat{b}_n - \hat{b}_n^\dagger)\big)$$
$$= \sum_n O_{kn}\big(\frac{1}{2}\sqrt{\frac{\omega_k}{\Omega_n}}(\hat{b}_n + \hat{b}_n^\dagger) + \frac{1}{2}i\sqrt{\frac{\Omega_n}{\omega_k}}(\hat{b}_n - \hat{b}_n^\dagger)\big)$$

$$e^{r_{nk}} \equiv \sqrt{\frac{\Omega_n}{\omega_k}} \quad , \quad g_k \equiv q f_k \sqrt{\frac{\omega_k}{m\hbar}} \quad , \quad \hat{A} \equiv f_k(\hat{a} + \hat{a}^\dagger)$$

$$\begin{aligned}\hat{a}_k &= \sum_n \frac{1}{2} O_{kn}\big(e^{r_{nk}}(\hat{b}_n + \hat{b}_n^\dagger) + e^{r_{nk}}(\hat{b}_n - \hat{b}_n^\dagger)\big) \\ &= \sum_n O_{kn}\big(cosh(r_{nk})\hat{b}_n + sinh(r_{nk})\hat{b}_n^\dagger\big)\end{aligned}$$

$$\begin{aligned}\hat{a}_k^\dagger &= \sum_n \frac{1}{2} O_{kn}\big(e^{r_{nk}}(\hat{b}_n + \hat{b}_n^\dagger) + e^{r_{nk}}(\hat{b}_n - \hat{b}_n^\dagger)\big) \\ &= \sum_n O_{kn}\big(cosh(r_{nk})\hat{b}_n - sinh(r_{nk})\hat{b}_n^\dagger\big)\end{aligned}$$

$$\hat{a}_k + \hat{a}_k^\dagger = \sum_n O_{kn} e^{-r_{nk}}\big(\hat{b}_n + \hat{b}_n^\dagger\big)$$

Operator A and P are commute, $\hat{A}\hat{P} + \hat{P}\hat{A} = 2\hat{P}\hat{A} \quad , \quad (\text{a}) = -\frac{q}{m}(\hat{P}\hat{A})$

$$\begin{aligned}\hat{A} &= \sum_k f_k(\hat{a}_k + \hat{a}_k^\dagger) \\ &= \sum_{kn} f_k O_{nk}[e^{-r_{nk}}(\hat{b}_n + \hat{b}_n^\dagger)] \\ &= \sum_{kn} f_k O_{nk} \sqrt{\frac{\Omega_n}{\omega_k}}(\hat{b}_n + \hat{b}_n^\dagger) \\ &= \sum_{kn} O_{nk} \frac{g_k}{q} \sqrt{\frac{m\hbar}{\Omega_n}}(\hat{b}_n + \hat{b}_n^\dagger) \quad , \quad \big(f_k \sqrt{\frac{\omega_k}{\Omega_n}} = \frac{g_k}{q}\sqrt{\frac{m\hbar}{\Omega_n}}\big)\end{aligned}$$

$$\zeta_n \equiv \sqrt{\frac{\hbar}{m\Omega_n}} \sum_k g_k O_{kn}$$

$$\hat{A} = \sum_n \zeta_n(\hat{b}_n + \hat{b}_n^\dagger)$$

$$a :\equiv b \tag{1}$$

Thus (b) can be derived as a result.