Derivation of the equations

1 Ch.1

This is a section about properties of photon.

$$m = 0$$
$$\mathscr{E} = h\nu = \frac{hc}{\lambda}$$
$$p = \frac{h\nu}{c} = \frac{h}{\lambda}$$

Mean photon flux can be wrote as:

$$\Phi = \frac{AI}{h\nu_0} = \frac{P}{h\nu_0}$$

Notice that the number of the photon can be count by using equation above. Unit of P is Watt, and $h\nu$ is eV.

2 Ch.2

2.1 2.1 Quantization of the radiation field

$$\vec{\nabla} \cdot \vec{A} = 0 \tag{1}$$
$$\nabla^2 \vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} \tag{2}$$
$$\vec{E} = -\frac{\partial \vec{A}}{\partial t}, \vec{B} = \vec{\nabla} \times \vec{A} \tag{3}$$

fourier series expansion of A:

$$\vec{A}(\vec{r},t) = \sum_k (\vec{A_k}(t)e^{i\vec{k}\cdot\vec{r}} + \vec{A_k}^*(t)e^{-i\vec{k}\cdot\vec{r}}) \tag{4}$$
$$k_x = \frac{2\pi n_x}{L}, k_y = \frac{2\pi n_y}{L}, k_z = \frac{2\pi n_z}{L} \tag{5}$$
$$\vec{k} \cdot \vec{A_k} = \vec{k} \cdot \vec{A_k}^* = 0 \tag{6}$$

substitute eq(2.4) with (2.2)

$$\nabla^2 = (\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2})$$

$$\sum_k (\vec{A_k}(t)(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2})e^{i(k_x x + k_y y + k_z z)} + c.c)$$

$$\sum_k (-k^2 \vec{A_k}(t)e^{i\vec{k}\cdot\vec{r}} + -k^2 \vec{A_k}^*(t)e^{-i\vec{k}\cdot\vec{r}})$$

for single k,

$$k^2 \vec{A_k}(t) + \frac{1}{c^2}\frac{\partial^2 \vec{A_k}(t)}{\partial t^2} = 0 \tag{7}$$

$$\vec{A_k} = \vec{A_k} e^{-i\omega_k t}, \omega_k = ck \tag{8}$$

$$\vec{A}(\vec{r}, t) = \sum_k (\vec{A_k} e^{-i\omega_k t + i\vec{k}\cdot\vec{r}} + \vec{A_k}^* e^{-i\vec{k}\cdot\vec{r} + i\omega_k t}) \tag{9}$$

$$\vec{E_k}(\vec{r}, t) = \sum_k \vec{E_k} = i \sum_k \omega_k (\vec{A_k} e^{-i\omega_k t + i\vec{k}\cdot\vec{r}} + \vec{A_k}^* e^{-i\vec{k}\cdot\vec{r} + i\omega_k t}) \tag{10}$$

$$\vec{B_k}(\vec{r}, t) = \sum_k \vec{B_k} = i \sum_k \vec{k} \times (\vec{A_k} e^{-i\omega_k t + i\vec{k}\cdot\vec{r}} + \vec{A_k}^* e^{-i\vec{k}\cdot\vec{r} + i\omega_k t}) \tag{11}$$

calculation process for eq(11):

$$\nabla \times \vec{A} = \nabla \times (\vec{A_k} e^{-i\omega_k t + i\vec{k}\cdot\vec{r}} + \vec{A_k}^* e^{-i\vec{k}\cdot\vec{r} + i\omega_k t})$$

$$(\frac{\partial}{\partial y} A_{k_z} e^{i\omega_k t + i\vec{k}\cdot\vec{r}} - \frac{\partial}{\partial z} A_{k_y} e^{i\omega_k t + i\vec{k}\cdot\vec{r}})\hat{i}$$

$$+(\frac{\partial}{\partial z} A_{k_x} e^{i\omega_k t + i\vec{k}\cdot\vec{r}} - \frac{\partial}{\partial x} A_{k_z} e^{i\omega_k t + i\vec{k}\cdot\vec{r}})\hat{j}$$

$$+(\frac{\partial}{\partial x} A_{k_y} e^{i\omega_k t + i\vec{k}\cdot\vec{r}} - \frac{\partial}{\partial y} A_{k_x} e^{i\omega_k t + i\vec{k}\cdot\vec{r}})\hat{z}$$

$$\overline{\mathscr{E}_k} = \frac{1}{2}\int (\epsilon_0 \overline{E_k^2} + \frac{\overline{B_k^2}}{\mu_0})dV \tag{12}$$

$$\overline{\vec{E}_k^2} = \frac{1}{T}\int_0^T dt E_k^2 = 2\omega_k^2 \left|\vec{A_k}\right|^2 \tag{13}$$

$$\overline{\vec{B}_k^2} = \frac{1}{T}\int_0^T dt E_k^2 = 2k_k^2 \left|\vec{A_k}\right|^2 \tag{14}$$

$$\overline{\mathscr{E}_k} = (\epsilon_0 \omega_k^2 \left|\vec{A_k}\right|^2 + \frac{k_k^2}{\mu_0}\left|\vec{A_k}\right|^2) = 2\epsilon_0 \omega_k^2 V \left|\vec{A_k}\right|^2 \tag{15}$$

$$\vec{A_k} = \frac{\vec{\epsilon_k}}{\sqrt{4\epsilon_0 V \omega_k^2}}(\omega_k q_k + i p_k) \tag{16}$$

$$\overline{\mathscr{E}_k} = \frac{1}{2}(\omega_k^2 q_k^2 + p_k^2) \tag{17}$$

$$\overline{\mathscr{E}} = \sum_k \overline{\mathscr{E}_k} = 2\epsilon_0 V \sum_k \omega_k^2 \left|\vec{A_k}\right|^2 = \frac{1}{2}\sum_k (\omega_k^2 q_k^2 + p_k^2) \tag{18}$$

$$\hat{H} = \frac{1}{2}\sum_k (\omega_k^2 \hat{q}_k^2 + \hat{p}_k^2) \tag{19}$$

since

$$[\hat{q}_k, \hat{p}_k] = i\hbar$$

$$\hat{a}_k = \frac{1}{\sqrt{2\hbar\omega_k}}(\omega_k \hat{q}_k + i\hat{p}_k) \tag{20}$$

$$\hat{a^\dagger}_k = \frac{1}{\sqrt{2\hbar\omega_k}}(\omega_k \hat{q}_k - i\hat{p}_k) \tag{21}$$

calculation process

$$[\hat{a}_k, \hat{a^\dagger}_k] = \hat{a}_k \hat{a^\dagger}_k - \hat{a^\dagger}_k \hat{a}_k$$

$$\hat{a}_k \hat{a^\dagger}_k = \frac{1}{2\hbar\omega_k}(\omega_k \hat{q}_k + i\hat{p}_k)(\omega_k \hat{q}_k - i\hat{p}_k)$$

$$= \frac{1}{2\hbar\omega_k}(\omega_k^2 \hat{q}_k^2 + \hat{p}_k^2 - i\omega_k \hat{q}\hat{p} + i\omega_k \hat{p}\hat{q})$$

$$= \frac{1}{2\hbar\omega_k}(\omega_k^2 \hat{q}_k^2 + \hat{p}_k^2 - i\omega_k [\hat{q}_k, \hat{p}_k])$$

$$\hat{a^\dagger}_k \hat{a}_k = \frac{1}{2\hbar\omega_k}(\omega_k^2 \hat{q}_k^2 + \hat{p}_k^2 + i\omega_k [\hat{q}_k, \hat{p}_k])$$

$$[\hat{a}_k, \hat{a^\dagger}_k] = \frac{1}{2\hbar\omega_k}(-2i\omega_k [\hat{q}_k, \hat{p}_k]) = -\frac{i}{\hbar}[\hat{q}_k, \hat{p}_k] = -\frac{i}{\hbar} i\hbar = 1$$

end.

$$[\hat{a}_k, \hat{a^\dagger}_k] = \delta_{kk^*} \tag{22}$$

$$\hat{\vec{A}_k} = \vec{\epsilon_k}\sqrt{\frac{\hbar}{2\epsilon_0 \omega_k V}}\hat{A}_k \tag{23}$$

$$\vec{A}(\vec{r}, t) = \sum_k \vec{\epsilon_k}\sqrt{\frac{\hbar}{2\epsilon_0 \omega_k V}}(\hat{a}_k e^{-i\omega_k t + i\vec{k}\cdot\vec{r}} + \hat{a}_k^\dagger e^{-i\vec{k}\cdot\vec{r} + i\omega_k t}) \tag{24}$$

$$\vec{E}(\vec{r}, t) = i\sum_k \vec{\epsilon_k}\sqrt{\frac{\hbar\omega_k}{2\epsilon_0 V}}(\hat{a}_k e^{-i\omega_k t + i\vec{k}\cdot\vec{r}} + \hat{a}_k^\dagger e^{-i\vec{k}\cdot\vec{r} + i\omega_k t}) \tag{25}$$

$$\vec{B}(\vec{r}, t) = i\sum_k \vec{k} \times \vec{\epsilon_k}\sqrt{\frac{\hbar}{2\epsilon_0 \omega_k V}}(\hat{a}_k e^{-i\omega_k t + i\vec{k}\cdot\vec{r}} + \hat{a}_k^\dagger e^{-i\vec{k}\cdot\vec{r} + i\omega_k t}) \tag{26}$$

2.2 2.2 Quantized single mode radiation field

In this section, the word "mode" is defined as value of wave vector k varies. single mode, wave vector k, angular frequency $\omega = ck$

$$\hat{H} = \frac{1}{2}(\omega^2 \hat{q}^2 + \hat{p}^2) \tag{27}$$

$\hat{q}$: position operator , $\hat{p}$: momentum operator

$$\hat{q} = \sqrt{\frac{\hbar}{2\omega}}(\hat{a} + \hat{a}^\dagger) \tag{28}$$

$$\hat{p} = i\sqrt{\frac{\hbar\omega}{2}}(\hat{a}^\dagger - \hat{a}) \tag{29}$$

$$\hat{H} = \frac{\hbar\omega}{2}(\hat{a}^\dagger \hat{a} + \hat{a}\hat{a}^\dagger + \frac{1}{2}) \tag{30}$$

$$\hat{H}\left|n\right\rangle = \hbar\omega(\hat{a}^\dagger \hat{a} + \frac{1}{2})\left|n\right\rangle = \mathscr{E}_n \left|n\right\rangle \tag{31}$$

$$\begin{aligned} \hbar\omega \hat{a}^\dagger \hat{a} a^\dagger \hat{a} + \frac{1}{2}\hat{a}^\dagger \left|n\right\rangle &= \hbar\omega(\hat{a}^\dagger \hat{a} - \frac{1}{2})\left|n\right\rangle \\ &= \mathscr{E}_n \hat{a}^\dagger \left|n\right\rangle \end{aligned} \tag{32}$$

$$\hat{H}\hat{a}^\dagger \left|n\right\rangle = \hbar\omega(\hat{a}^\dagger \hat{a} + \frac{1}{2})\hat{a}^\dagger \left|n\right\rangle = (\mathscr{E}_n + \hbar\omega)\hat{a}^\dagger \left|n\right\rangle \tag{33}$$

$$\hat{H}\hat{a}\left|n\right\rangle = (\mathscr{E}_n - \hbar\omega)(\hat{a}\left|n\right\rangle) \tag{34}$$

$$\hat{a}^\dagger \left|n\right\rangle \propto \left|n+1\right\rangle, \mathscr{E}_{n+1} = \mathscr{E}_n + \hbar\omega \tag{35}$$

$$\hat{a}\left|n\right\rangle \propto \left|n-1\right\rangle, \mathscr{E}_{n-1} = \mathscr{E}_n - \hbar\omega \tag{36}$$

$$\hat{H}\hat{a}\left|0\right\rangle = (\mathscr{E}_0 - \hbar\omega)(\hat{a}\left|0\right\rangle) \tag{37}$$

$$\hat{a}\left|0\right\rangle = 0 \tag{38}$$

$$\hat{H}\left|0\right\rangle = \frac{1}{2}\hbar\omega\left|0\right\rangle = \mathscr{E}\left|0\right\rangle \tag{39}$$

$$\hat{a}^\dagger \hat{a}\left|n\right\rangle = n\left|n\right\rangle \tag{40}$$

$$\hat{a}^\dagger = c_n \left|n+1\right\rangle \hat{a}^\dagger = \sqrt{n+1}\left|n+1\right\rangle \tag{41}$$

$$\hat{a} = d_n \left|n-1\right\rangle \tag{42}$$

$$\hat{a}^\dagger = \sqrt{n+1}\left|n+1\right\rangle \tag{43}$$

$$\hat{a} = \sqrt{n}\left|n-1\right\rangle \tag{44}$$

$$\left|n\right\rangle = \frac{1}{\sqrt{n!}}(\hat{a}^\dagger)^n \left|0\right\rangle \tag{45}$$

2.3 2.3 quantized multi mode radiation field

$$\hat{H} = \sum_k \hbar\omega_k(\hat{a}^\dagger\hat{a} + \frac{1}{2}) \tag{46}$$

$$\mathscr{E}_{k_1} + \mathscr{E}_{k_2} + \mathscr{E}_{k_3} + ... + \mathscr{E}_{k_n} = \sum_{i=1}^{N}(n_{k_1} + \frac{1}{2})\hbar\omega_k$$

$$\hat{H}\left|n_k\right\rangle = (\sum_k \hbar\omega_k(\hat{a}^\dagger\hat{a} + \frac{1}{2}))\left|n_k\right\rangle = (\sum_k \hbar\omega_k(n_k + \frac{1}{2}))\left|n_k\right\rangle \tag{47}$$

2.4 2.4 General states of the quantized radiation field

If single mode light is in a pure state, Generally we can describe its state as linear superposition.

$$\left|\phi\right\rangle = \sum_n a_n\left|n\right\rangle \tag{48}$$

Probability, Expectation value, Uncertainty:

$$P_n = |a_n|^2, < n >= \sum_n nP_n = \sum_n n|a_n|^2$$

$$\Delta n = \sqrt{(n- < n >)^2} = \sqrt{(< n^2 > - < n >^2)} = \sqrt{\sum_n n^2P_n - (\sum_n nP_n)^2} \tag{49}$$

$$a_n = e^{-\frac{|\alpha^2|}{2}}\frac{\alpha^n}{\sqrt{n!}} \tag{50}$$

If single mode light is in a mixed state, than we can describe its state by using density operator, $\hat{\rho} = \sum_\phi P_\phi\left|\phi\right\rangle\left\langle\phi\right|$

$$\hat{\rho} = \sum_n P_n\left|n\right\rangle\left\langle n\right| \tag{51}$$

Expectation value of number of the photon n is:

$$< n >=< \hat{a}^\dagger\hat{a} >= Tr\{\hat{\rho}\hat{a}^\dagger\hat{a}\} \tag{52}$$

calculation process Using Eq 40

$$< \hat{a}^\dagger\hat{a} >= \left\langle n\right|\hat{a}^\dagger\hat{a}\left|a\right\rangle = n\left\langle n|n\right\rangle$$

In case of the blackbody radiation, which is an general example of the mixed state light, probability of the state is:

$$P_n = e^{\frac{-n\hbar\omega}{kT}}(1 - e^{\frac{-\hbar\omega}{kT}}) \tag{53}$$

density operator is:

$$\hat{\rho} = (1 - e^{\hbar\omega}kT)\sum_n e^{\frac{-n\hbar\omega}{kT}}\left|n\right\rangle\left\langle n\right| \tag{54}$$

2.5 2.5 Vacuum fluctuation and zero point energy

$$\hat{\vec{A}}(\vec{r},t) = \vec{\epsilon_k}\sqrt{\frac{\hbar}{2\epsilon_0\omega_k V}}(\hat{a}_k e^{-i\omega_k t+i\vec{k}\cdot\vec{r}} + \hat{a}_k^\dagger e^{-i\vec{k}\cdot\vec{r}+i\omega_k t}) \tag{55}$$

$$\hat{\vec{E}}(\vec{r},t) = i\vec{\epsilon_k}\sqrt{\frac{\hbar\omega_k}{2\epsilon_0 V}}(\hat{a}_k e^{-i\omega_k t+i\vec{k}\cdot\vec{r}} + \hat{a}_k^\dagger e^{-i\vec{k}\cdot\vec{r}+i\omega_k t}) \tag{56}$$

$$\hat{\vec{B}}(\vec{r},t) = i\vec{k}\times\vec{\epsilon_k}\sqrt{\frac{\hbar}{2\epsilon_0\omega_k V}}(\hat{a}_k e^{-i\omega_k t+i\vec{k}\cdot\vec{r}} + \hat{a}_k^\dagger e^{-i\vec{k}\cdot\vec{r}+i\omega_k t}) \tag{57}$$

Eigenstate $|n\rangle$ of the hamiltonian H, expectation value of operators turn out to be 0. For instance,

$$\langle n|\,\hat{\vec{E}}\,|n\rangle = i\vec{\epsilon_k}\sqrt{\frac{\hbar\omega_k}{2\epsilon_0 V}}(\langle n|\,\hat{a}_k\,|n\rangle\, e^{-i\omega_k t+i\vec{k}\cdot\vec{r}} + \langle n|\,\hat{a}_k^\dagger\,|n\rangle\, e^{-i\vec{k}\cdot\vec{r}+i\omega_k t})$$

$$= i\vec{\epsilon_k}\sqrt{\frac{\hbar\omega_k}{2\epsilon_0 V}}(\sqrt{n}\,\langle n|n-1\rangle\, e^{-i\omega_k t+i\vec{k}\cdot\vec{r}} + \sqrt{n+1}\,\langle n|n+1\rangle\, e^{-i\vec{k}\cdot\vec{r}+i\omega_k t})$$

$$\langle n|\,\hat{\vec{E}}\,|n\rangle = 0 \tag{58}$$

Expectation value of $\hat{E}^2 = \hat{\vec{E}}\cdot\hat{\vec{E}}$:

$$\langle n|\,\hat{\vec{E}}\cdot\hat{\vec{E}}\,|n\rangle = \frac{\hbar\omega}{2\epsilon_0 V}\,\langle n|\,\hat{a}\hat{a}^\dagger + \hat{a}\hat{a}\dagger\,|n\rangle = \frac{\hbar\omega}{2\epsilon_0 V}(n+\frac{1}{2}) \tag{59}$$

Equation states that there are always fluctuation exists in a Electronic field. When light is in fock state, uncertainty of field is:

$$\Delta E = \sqrt{<\hat{E}^2> - <\hat{E}>^2} = \sqrt{\frac{\hbar\omega}{\epsilon_0 V}}(n+\frac{1}{2}) \tag{60}$$