$$
\begin{aligned}
Z &= \langle \psi_a(\tau)\psi_b^\dagger(\tau')\rangle \\
&= \mathrm{Tr}[e^{\beta H(\hat{\tau})}\psi_a(\tau)\psi_b^\dagger(\tau')]
\end{aligned}
$$

$$H = H_{loc} + H_{bath} + \mathbf{H_{hyb}} + \mathbf{H_{hyb}^\dagger}$$

$$\mathbf{H_{hyb}} = \sum_i c_\nu \mathbf{V}_i b_i^\dagger$$

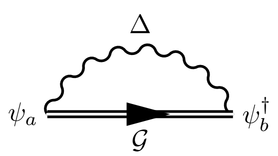
$$\mathcal{G} = Z_{\mathrm{bath}}\mathrm{Tr}_c\left[\frac{1}{Z_{\mathrm{bath}}}\mathrm{Tr}_{\mathrm{b}}\big[T_\tau e^{-\int_0^\beta d\tau H_{\mathrm{loc}}(\tau)+H_{\mathrm{bath}}(\tau)}\prod_\nu\sum_{k_\nu} Z_{k_\nu}\big]\right]$$

$$
\begin{aligned}
Z_{k_\nu} = &\sum_{i_1,\ldots,i_{k_\nu}}\sum_{i'_1,\ldots,i'_{k_\nu}} V^\nu_{i_1}V^{\nu*}_{i'_1}\cdots V^\nu_{i_{k_j}}V^{\nu*}_{i'_{k_j}}\int_0^\beta d\tau \\
&\times \int_{\tau_1}^\beta d\tau_2\cdots\int_{\tau_{k_\nu-1}}^\beta d\tau_{k_\nu}\int_{\tau_1^{'}}^\beta d\tau_2^{'}\cdots\int_{\tau^{'}_{k_\nu-1}}^\beta d\tau^{'}_{k_\nu} \\
&\times c_\nu(\tau_1)b^{\nu^\dagger}_{i_1}(\tau_1)b^\nu_{i'_1}(\tau_1^{'})c_\nu^\dagger(\tau_1^{'})c_\nu(\tau_2)b^{\nu^\dagger}_{i_2}(\tau_2)b^\nu_{i'_2}(\tau_2^{'})c_\nu^\dagger(\tau_2^{'}) \\
&\times \cdots c_\nu(\tau_{k_j})b^{\nu^\dagger}_{i_{k_j}}(\tau_{k_j})b^\nu_{i'_{k_j}}(\tau^{'}_{k_j})c_\nu^\dagger(\tau^{'}_{k_j})
\end{aligned}
$$

$$
M^{-1} = \begin{pmatrix}
\mathcal{G}'_{11} & \mathcal{G}'_{12} & \cdots & \mathcal{G}'_{1j} & \cdots \\
\mathcal{G}'_{21} & \mathcal{G}'_{22} & \cdots & \mathcal{G}'_{2j} & \cdots \\
\vdots & \vdots & \ddots & \vdots & \cdots \\
\mathcal{G}'_{i1} & \mathcal{G}'_{i2} & \cdots & \mathcal{G}'_{ij} & \cdots \\
\vdots & \vdots & \cdots & \cdots & \ddots
\end{pmatrix}
$$

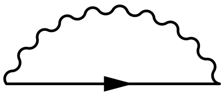
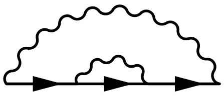
$$\mathrm{det}M^{-1} = \Delta$$

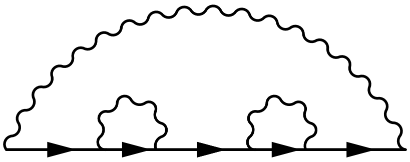
$$
\begin{aligned}
\mathcal{G}' &= \langle |V^\nu| b^{\dagger\nu}(\tau)b^\nu(\tau')\rangle \\
&= \begin{cases}
|V^a|^2 e^{-\epsilon(\tau-\tau')}\frac{1}{(1+e^{-\beta\epsilon_p})} & (\text{if} \quad \tau > \tau') \\
|V^a|^2 e^{-\epsilon(\tau'-\tau)}\frac{e^{-\beta\epsilon}}{(1+e^{-\beta\epsilon})} & (\text{if} \quad \tau < \tau')
\end{cases}
\end{aligned}
$$



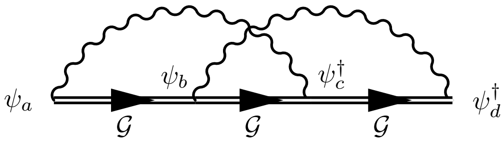
$$\mathcal{G} =$$

$$= \sum_{ab}\left[(\mathrm{sgn})\psi_a\mathcal{G}(\tau)\psi_b^\dagger\Delta_{ba}(\tau)\right]$$

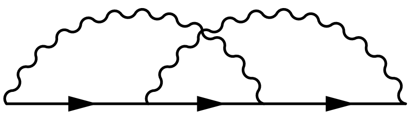
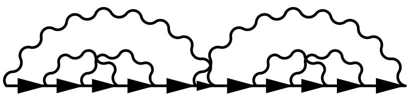
$=$  $+$ 

$+$  $+ \quad \cdots$

$\mathcal{G}_{\mathrm{OCA}} =$



$$= \sum_{\mathrm{abcd}} \int_0^{\tau} d\tau_1 \int_0^{\tau_1} d\tau_2 \Big[\mathrm{sgn}_1 \psi_d^\dagger \mathcal{G}(\tau-\tau_2)\psi_c^\dagger \mathcal{G}(\tau_2-\tau_1)\psi_b \mathcal{G}(\tau-\tau_1)\psi_a \Delta_{bd}(\tau-\tau_1)\Delta_{ca}(\tau_2) \Big]$$

$=$  $+$ 

$+ \quad \cdots$

$$Z = \mathrm{Tr}[\mathcal{G}(\tau)\lambda_1\mathcal{G}(0)\lambda_1\Delta(\tau)]$$

$$\begin{aligned} Z &= Z_{\mathrm{NCA}} + Z_{\mathrm{OCA}} \\ &= \mathrm{Tr}[\mathcal{G}(\tau)\lambda_1\mathcal{G}(0)\lambda_1\Delta(\tau)] \\ &\quad + \mathrm{Tr}[\mathcal{G}(\beta-\tau_2)\hat{N}\mathcal{G}(\tau_2-\tau)\lambda_1\mathcal{G}(\tau-\tau_1)\hat{N}\mathcal{G}(\tau_1)\lambda_1\Delta(\tau_2-\tau_1)] \end{aligned}$$

$$H_{\mathrm{tot}} = H_{\mathrm{loc}} + H_{\mathrm{int}} + H_{\mathrm{bath}}$$

$$H_{\mathrm{loc}} = E_c\hat{N} - E_J\cos\phi$$

$$H_{\mathrm{int}} = \sum_k g_k \hat{N}(b_k^\dagger + b_k)$$

$$H_{\mathrm{bath}} = \sum_k \omega_k b_k^\dagger b_k$$