

Derivation of the Second Quantization of Operators

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1 Second Quantization of the one-particle Operator

The N-single particle system, the state of the system can be written as $|\psi_{\nu_{\alpha_j}}(\mathbf{r}_j)\rangle$, which satisfies the following condition:

$$1 \leq j \leq N \quad 1 \leq \alpha \leq M, \quad M \leq N \quad (M, N = \text{integer})$$

Where α indicates the α^{th} quantum state $\nu_\alpha = \{\nu_1, \nu_2, \dots, \nu_M\}$, j indicates the j^{th} particle in the system, (e.g : 3-particle system ν , distinguishable particle 1 and particle 2 in 1st state, particle 3 in 2nd state, can be written as : $\psi_\nu(\mathbf{r}) = \psi_{\nu_1}(\mathbf{r}_1)\psi_{\nu_2}(\mathbf{r}_2)\psi_{\nu_3}(\mathbf{r}_3)$. here, the normalization factor was ignored.)

A N-single particle operator \hat{T} can be written the form of first quantization :

$$\begin{aligned} & \hat{T}_{tot} |\psi_{\nu_{\alpha_1}}(\mathbf{r}_1)\rangle |\psi_{\nu_{\alpha_2}}(\mathbf{r}_2)\rangle, \dots, |\psi_{\nu_{\alpha_N}}(\mathbf{r}_N)\rangle \\ &= \sum_{j=1}^N \sum_{\nu_a, \nu_b} T_{\nu_b, \nu_a} |\psi_{\nu_b}(r_j)\rangle \langle \psi_{\nu_a}(r_j)| (|\psi_{\nu_{\alpha_1}}(\mathbf{r}_1)\rangle |\psi_{\nu_{\alpha_2}}(\mathbf{r}_2)\rangle, \dots, |\psi_{\nu_{\alpha_N}}(\mathbf{r}_N)\rangle) \\ &= \sum_{j=1}^N \sum_{\nu_a, \nu_b} T_{\nu_b, \nu_a} \delta_{\nu_a, \nu_j} |\psi_{\nu_{\alpha_1}}(\mathbf{r}_1)\rangle |\psi_{\nu_{\alpha_2}}(\mathbf{r}_2)\rangle \dots |\psi_{\nu_b}(\mathbf{r}_j)\rangle, \dots, |\psi_{\nu_{\alpha_N}}(\mathbf{r}_N)\rangle \end{aligned}$$

Here, ν_a, ν_b indicates the arbitrary states, which is an element of set $\{\nu_\alpha\}$. Applying the Bosonic Operator on both sides of the equation,

$$\begin{aligned} & S^+ \hat{T}_{tot} |\psi_{\nu_{\alpha_1}}(\mathbf{r}_1)\rangle |\psi_{\nu_{\alpha_2}}(\mathbf{r}_2)\rangle, \dots, |\psi_{\nu_{\alpha_N}}(\mathbf{r}_N)\rangle \\ &= \hat{T}_{tot} b_{\nu_{\alpha_1}}^\dagger b_{\nu_{\alpha_2}}^\dagger \dots b_{\nu_{\alpha_N}}^\dagger |0\rangle \\ &= \sum_{j=1}^N \sum_{\nu_a, \nu_b} T_{\nu_b, \nu_a} \delta_{\nu_a, \nu_j} b_{\nu_{\alpha_1}}^\dagger b_{\nu_{\alpha_2}}^\dagger \dots b_{\nu_b}^\dagger \dots b_{\nu_{\alpha_N}}^\dagger |0\rangle \\ &= \sum_{j=1}^N \sum_{\nu_a, \nu_b} T_{\nu_b, \nu_a} \delta_{\nu_a, \nu_j} (b_{\nu_1}^\dagger)^{n_1} (b_{\nu_2}^\dagger)^{n_2} \dots b_{\nu_b}^\dagger \dots (b_{\nu_M}^\dagger)^{n_M} |0\rangle \\ &= \sum_{j=1}^N \sum_{\nu_a, \nu_b} T_{\nu_b, \nu_a} \delta_{\nu_a, \nu_j} \sqrt{n_1!} |n_1\rangle \sqrt{n_2!} |n_2\rangle \dots |\nu_b\rangle \dots \sqrt{n_M!} |n_M\rangle \end{aligned}$$

n_1, n_2, \dots, n_M are the number of times the $\nu_1, \nu_2, \dots, \nu_M$ state appears in the N-particle system, $\sum_{\alpha=1}^M n_\alpha = N$. Assume in the case of single-particle system, with N particles in ν_α states

$$\begin{aligned} &= \sum_{j=1}^N \sum_{\nu_a, \nu_b} T_{\nu_b, \nu_a} \delta_{\nu_a, \nu_j} \sqrt{n_1!} |\nu_1\rangle \dots |\nu_1\rangle \sqrt{n_2!} |\nu_2\rangle \dots |\nu_2\rangle \dots |\nu_b\rangle \dots \sqrt{n_M!} |\nu_M\rangle \dots |\nu_M\rangle \\ &= B^+ \sum_{j=1}^N \sum_{\nu_a, \nu_b} T_{\nu_b, \nu_a} \delta_{\nu_a, \nu_j} \underbrace{|\nu_1\rangle \dots |\nu_1\rangle}_{n_1} \underbrace{|\nu_2\rangle \dots |\nu_2\rangle}_{n_2} \dots |\nu_b\rangle \dots \underbrace{|\nu_M\rangle \dots |\nu_M\rangle}_{n_M} \end{aligned}$$

Here, $B^+ = \Pi_{n_\alpha} \sqrt{n_\alpha!}$.

$$\begin{aligned}
&= B^+ \sum_{\nu_a \nu_b} T_{\nu_b \nu_a} \left\{ \begin{array}{l} \delta_{\nu_a, \nu_1} |\nu_b\rangle \overbrace{|\nu_1\rangle \cdots |\nu_1\rangle}^{n_1-1} |\nu_2\rangle \cdots |\nu_M\rangle \\ + \delta_{\nu_a, \nu_1} |\nu_1\rangle |\nu_b\rangle \cdots |\nu_1\rangle |\nu_2\rangle \cdots |\nu_M\rangle \\ \vdots \\ + \delta_{\nu_a, \nu_2} |\nu_1\rangle \cdots |\nu_b\rangle \overbrace{|\nu_2\rangle \cdots |\nu_2\rangle}^{n_2-1} \cdots |\nu_M\rangle \\ \vdots \\ + \delta_{\nu_a, \nu_M} |\nu_1\rangle \cdots |\nu_2\rangle \cdots \overbrace{|\nu_M\rangle \cdots |\nu_M\rangle}^{n_M-1} |\nu_b\rangle \end{array} \right. \\
&= \sum_{\nu_a \nu_b} T_{\nu_b \nu_a} \left\{ \begin{array}{l} n_1 \delta_{\nu_a, \nu_1} b_{\nu_b}^\dagger (b_{\nu_1}^\dagger)^{n_1-1} (b_{\nu_2}^\dagger)^{n_2} \cdots (b_{\nu_M}^\dagger)^{n_M} |0\rangle \\ + n_2 \delta_{\nu_a, \nu_2} (b_{\nu_1}^\dagger)^{n_1} b_{\nu_b}^\dagger (b_{\nu_2}^\dagger)^{n_2-1} \cdots (b_{\nu_M}^\dagger)^{n_M} |0\rangle \\ \vdots \\ + n_M \delta_{\nu_a, \nu_M} (b_{\nu_1}^\dagger)^{n_1-1} (b_{\nu_2}^\dagger)^{n_2} \cdots b_{\nu_b}^\dagger (b_{\nu_M}^\dagger)^{n_M-1} |0\rangle \end{array} \right.
\end{aligned}$$

Use the calculation

$$b_{\nu_b}^\dagger (b_{\nu_\alpha}^\dagger)^{\alpha-1} |0\rangle = \frac{1}{\alpha} b_{\nu_b}^\dagger b_{\nu_\alpha} b_{\nu_\alpha}^\dagger (b_{\nu_\alpha}^\dagger)^{\alpha-1} |0\rangle, \quad (1)$$

$$\begin{aligned}
&= \sum_{\nu_a \nu_b} T_{\nu_b \nu_a} \left\{ \begin{array}{l} \frac{n_1}{n_1} \delta_{\nu_a, \nu_1} b_{\nu_b}^\dagger b_{\nu_1} b_{\nu_1}^\dagger (b_{\nu_1}^\dagger)^{n_1-1} (b_{\nu_2}^\dagger)^{n_2} \cdots (b_{\nu_M}^\dagger)^{n_M} |0\rangle \\ + \frac{n_2}{n_2} \delta_{\nu_a, \nu_2} (b_{\nu_1}^\dagger)^{n_1} b_{\nu_b}^\dagger b_{\nu_2} b_{\nu_2}^\dagger (b_{\nu_2}^\dagger)^{n_2-1} \cdots (b_{\nu_M}^\dagger)^{n_M} |0\rangle \\ \vdots \\ + \frac{n_M}{n_M} \delta_{\nu_a, \nu_M} (b_{\nu_1}^\dagger)^{n_1} (b_{\nu_2}^\dagger)^{n_2} \cdots b_{\nu_b}^\dagger b_{\nu_M} b_{\nu_M}^\dagger (b_{\nu_M}^\dagger)^{n_M-1} |0\rangle \end{array} \right. \\
&= \sum_{\nu_a \nu_b} T_{\nu_b \nu_a} (\delta_{\nu_a, \nu_1} b_{\nu_b}^\dagger b_{\nu_1} + \delta_{\nu_a, \nu_2} b_{\nu_b}^\dagger b_{\nu_2} \cdots \delta_{\nu_a, \nu_M} b_{\nu_b}^\dagger b_{\nu_M}^\dagger) ((b_{\nu_1}^\dagger)^{n_1} (b_{\nu_2}^\dagger)^{n_2} \cdots (b_{\nu_M}^\dagger)^{n_M}) |0\rangle
\end{aligned}$$

Considering only the case of $\nu_i = \nu_a$,

$$\begin{aligned}
&= \sum_{\nu_a \nu_b} T_{\nu_b \nu_a} (b_{\nu_b}^\dagger b_{\nu_a}) b_{\nu_{\alpha 1}}^\dagger b_{\nu_{\alpha 2}}^\dagger \cdots b_{\nu_{\alpha N}}^\dagger |0\rangle \\
&= \hat{T}_{tot} b_{\nu_{\alpha 1}}^\dagger b_{\nu_{\alpha 2}}^\dagger \cdots b_{\nu_{\alpha N}}^\dagger |0\rangle
\end{aligned}$$

Therefore,

$$\therefore \hat{T}_{tot} = \sum_{\nu_a \nu_b} T_{\nu_b \nu_a} b_{\nu_b}^\dagger b_{\nu_a}$$

2 Derivation of the Second Quantization of the two-particle Operator

2.1 $V \propto \frac{1}{r_j - r_k}$ in two-particle states

Suppose there are two different vectors representing the two-particle states, $|\psi_a(\mathbf{r}_j)\rangle |\psi_b(\mathbf{r}_k)\rangle$ and $|\psi_c(\mathbf{r}_j)\rangle |\psi_d(\mathbf{r}_k)\rangle$. Let V is a linear Operator which can be thought of as a function of $(\frac{1}{r_j - r_k})$. The index j and k represents the j^{th} and k^{th} coordinate of N -particle quantum system, where $1 \leq j, k \leq N$.

The expectation value of V_{jk} is :

$$V_{jk} = \int \langle \psi_c(\mathbf{r}_j) | \langle \psi_d(\mathbf{r}_k) | \frac{\gamma}{r_j - r_k} | \psi_a(\mathbf{r}_j) \rangle | \psi_b(\mathbf{r}_k) \rangle$$

and of V_{kj} :

$$V_{kj} = \int \langle \psi_c(\mathbf{r}_k) | \langle \psi_d(\mathbf{r}_j) | \frac{\gamma}{r_k - r_j} | \psi_a(\mathbf{r}_k) \rangle | \psi_b(\mathbf{r}_j) \rangle$$

Here, γ indicates the constant of proportionality.

Notice that the particles in N -particle quantum system are indistinguishable and thus result of the given integral will not change even if the given position of each particle was swapped. According to the symmetric properties of given system,

$$V_{jk} = V_{kj}, \quad \sum_{j < k}^N V_{jk} = \frac{1}{2} \sum_j^N \sum_k^N V_{jk} \quad (j \neq k)$$

2.2 2nd Quantization of two-body operator

A similar method with the second quantization of single-particle operator can be used. Here, Operator V considered as : $V = \frac{e^2}{4\pi\epsilon_0} \frac{1}{|r_j - r_k|}$.

Apply the symmetric operator S^\pm in both side of first quantized two-particle operator equation,

$$\begin{aligned} & S^\pm \hat{V}_{tot} |\psi_{\nu_{\alpha_1}}(\mathbf{r}_1)\rangle |\psi_{\nu_{\alpha_2}}(\mathbf{r}_2)\rangle, \dots, |\psi_{\nu_{\alpha_N}}(\mathbf{r}_N)\rangle \\ &= \hat{V}_{tot} a_{\nu_{\alpha_1}}^\dagger a_{\nu_{\alpha_2}}^\dagger \dots a_{\nu_{\alpha_N}}^\dagger |0\rangle \\ &= S^\pm \frac{1}{2} \sum_{j \neq k}^N \sum_{\nu_a \nu_b, \nu_c \nu_d} V_{\nu_c \nu_d, \nu_a \nu_b} \delta_{\nu_a, \nu_{\alpha_j}} \delta_{\nu_b, \nu_{\alpha_k}} |\psi_{\nu_{\alpha_1}}(\mathbf{r}_1)\rangle, \dots |\psi_{\nu_c}(\mathbf{r}_j)\rangle, \dots |\psi_{\nu_d}(\mathbf{r}_k)\rangle, \dots, |\psi_{\nu_{\alpha_N}}(\mathbf{r}_N)\rangle \\ &= \frac{1}{2} \sum_{j \neq k}^N \sum_{\nu_a \nu_b, \nu_c \nu_d} V_{\nu_c \nu_d, \nu_a \nu_b} \delta_{\nu_a, \nu_{\alpha_j}} \delta_{\nu_b, \nu_{\alpha_k}} (a_{\nu_{\alpha_1}}^\dagger a_{\nu_{\alpha_2}}^\dagger \dots \underset{jth}{a_{\nu_c}^\dagger} \dots \underset{kth}{a_{\nu_d}^\dagger} \dots a_{\nu_{\alpha_N}}^\dagger) |0\rangle \end{aligned}$$

To get the final result of summation, first, count each calculation process of equation term which stated as (a) for j and k indices,

$$= \frac{1}{2} \sum_{\nu_a \nu_b, \nu_c \nu_d} V_{\nu_c \nu_d, \nu_a \nu_b} \underbrace{\left(\sum_j^N \sum_k^N \delta_{\nu_a, \nu_{\alpha_j}} \delta_{\nu_b, \nu_{\alpha_k}} (a_{\nu_{\alpha_1}}^\dagger a_{\nu_{\alpha_2}}^\dagger \dots \underset{jth}{a_{\nu_c}^\dagger} \dots \underset{kth}{a_{\nu_d}^\dagger} \dots a_{\nu_{\alpha_N}}^\dagger) |0\rangle \right)}_{(a)}$$

If the given operator represents a Coulomb interaction between two electrons, the difference compared to the case of single particle operator \hat{T} is a using fermionic creation operator c^\dagger and annihilation operator c instead of bosonic operators b^\dagger, b , due to the fermion property of electrons.

In the procedure below, all creation and annihilation operators a^\dagger and a are considered fermionic operators during the calculation, so it can be deduced that : $n_\alpha = 1$, and the range of a value of α is the same as the range of

$j, 1 \leq \alpha, j \leq N$.

The following 3 conditions are used to analyze the (a).

- (1) : $a^\dagger |0\rangle = a_j^\dagger a_i a_i^\dagger |0\rangle$
- (2) : $\{c_i, c_j^\dagger\} = \delta_{ij}, \quad [b_i, b_j^\dagger] = \delta_{ij}$
- (3) : if $\lambda = -\lambda, \quad \lambda = 0, \quad \text{if } \lambda \neq 0, \quad \lambda \neq -\lambda \quad (\lambda \text{ is a real number})$

First, counts each summation result for k indices, $k=1,2,\dots,k',\dots,N$.

$$\begin{aligned}
k=1 \quad & \delta_{\nu_b \nu_{\alpha_1}} (a_{\nu_d}^\dagger a_{\nu_{\alpha_2}}^\dagger \cdots a_{\nu_{\alpha_N}}^\dagger) |0\rangle \\
& = \delta_{\nu_b \nu_{\alpha_1}} \underbrace{(a_{\nu_d}^\dagger a_{\nu_{\alpha_1}}^\dagger a_{\nu_{\alpha_1}}^\dagger a_{\nu_{\alpha_2}}^\dagger \cdots a_{\nu_{\alpha_N}}^\dagger)}_{\text{condition (1)}} |0\rangle \\
& = \delta_{\nu_b \nu_{\alpha_1}} a_{\nu_d}^\dagger a_{\nu_{\alpha_1}} (a_{\nu_{\alpha_1}}^\dagger a_{\nu_{\alpha_2}}^\dagger \cdots a_{\nu_{\alpha_N}}^\dagger) |0\rangle \quad \rightarrow \quad (1, \bigcirc)
\end{aligned}$$

(1, \bigcirc) refers the result of the calculation of the left side of the right-headed-arrow. A \bigcirc means that it is blank, the value will be given in the calculation process for j indices.

$$\begin{aligned}
k=2 \quad & \delta_{\nu_b \nu_{\alpha_2}} (a_{\nu_{\alpha_1}}^\dagger a_{\nu_d}^\dagger \cdots a_{\nu_{\alpha_N}}^\dagger) |0\rangle \\
& = \delta_{\nu_b \nu_{\alpha_2}} \underbrace{(a_{\nu_{\alpha_1}}^\dagger a_{\nu_d}^\dagger a_{\nu_{\alpha_2}}^\dagger a_{\nu_{\alpha_2}}^\dagger \cdots a_{\nu_{\alpha_N}}^\dagger)}_{\substack{\uparrow \quad \uparrow \\ \text{two operators are in ordered place, first one is in 1st place, and after 2nd place.}}} |0\rangle
\end{aligned}$$

two operators are in ordered place, first one is in 1st place, and after 2nd place.

$$\begin{aligned}
& = (-1) \delta_{\nu_b \nu_{\alpha_2}} a_{\nu_d}^\dagger (a_{\nu_{\alpha_1}}^\dagger a_{\nu_{\alpha_2}}^\dagger a_{\nu_{\alpha_2}}^\dagger \cdots a_{\nu_{\alpha_N}}^\dagger) |0\rangle \\
& \quad \uparrow \\
& \text{condition(2), by swapping the order} \\
& = (-1)^2 \delta_{\nu_b \nu_{\alpha_2}} a_{\nu_d}^\dagger a_{\nu_{\alpha_2}} (a_{\nu_{\alpha_1}}^\dagger a_{\nu_{\alpha_2}}^\dagger \cdots a_{\nu_{\alpha_N}}^\dagger) |0\rangle \quad \rightarrow \quad (2, \bigcirc)
\end{aligned}$$

Notice that swapping the order of ν_d and ν_c is necessary because a sign of the overall summation process should be equal to sign of $\hat{V}_{tot} a_{\nu_{\alpha_1}}^\dagger a_{\nu_{\alpha_2}}^\dagger \cdots a_{\nu_{\alpha_N}}^\dagger |0\rangle$, which requires the condition (3).

\vdots

$$\begin{aligned}
k=k' \quad & \delta_{\nu_b \nu_{\alpha_2}} (a_{\nu_{\alpha_1}}^\dagger a_{\nu_{\alpha_2}}^\dagger \cdots a_{\nu_d}^\dagger \cdots a_{\nu_{\alpha_N}}^\dagger) |0\rangle \\
& = \delta_{\nu_b \nu_{\alpha_2}} \underbrace{(a_{\nu_{\alpha_1}}^\dagger a_{\nu_{\alpha_2}}^\dagger \cdots a_{\nu_d}^\dagger a_{\nu_{\alpha_{k'}}}^\dagger a_{\nu_{\alpha_{k'}}}^\dagger \cdots a_{\nu_{\alpha_N}}^\dagger)}_{k'-1 \text{ operators}} |0\rangle \\
& = (-1)^{k'-1} \delta_{\nu_b \nu_{\alpha_2}} a_{\nu_d}^\dagger (a_{\nu_{\alpha_1}}^\dagger a_{\nu_{\alpha_2}}^\dagger \cdots a_{\nu_{\alpha_{k'}}}^\dagger a_{\nu_{\alpha_{k'}}}^\dagger \cdots a_{\nu_{\alpha_N}}^\dagger) |0\rangle \\
& = (-1)^{2(k'-1)} \delta_{\nu_b \nu_{\alpha_{k'}}} a_{\nu_d}^\dagger a_{\nu_{\alpha_{k'}}} (a_{\nu_{\alpha_1}}^\dagger a_{\nu_{\alpha_2}}^\dagger \cdots a_{\nu_{\alpha_{k'}}}^\dagger \cdots a_{\nu_{\alpha_N}}^\dagger) |0\rangle \quad \rightarrow \quad (k', \bigcirc)
\end{aligned}$$

Successively, count the summation for the j index, where $k=k'$ and $j=1,2,\dots,j',\dots,N$

$$\begin{aligned}
k=k', j=1 \quad & (-1)^{2(k'-1)} \delta_{\nu_a \nu_{\alpha_1}} \delta_{\nu_b \nu_{\alpha_{k'}}} a_{\nu_d}^\dagger a_{\nu_{\alpha_{k'}}} (a_{\nu_c}^\dagger \cdots a_{\nu_{\alpha_{k'}}}^\dagger \cdots a_{\nu_{\alpha_N}}^\dagger) |0\rangle \\
& = (-1)^{2(k'-1)} \delta_{\nu_a \nu_{\alpha_1}} \delta_{\nu_b \nu_{\alpha_{k'}}} a_{\nu_d}^\dagger a_{\nu_{\alpha_{k'}}} (a_{\nu_c}^\dagger a_{\nu_{\alpha_1}}^\dagger \cdots a_{\nu_{\alpha_{k'}}}^\dagger \cdots a_{\nu_{\alpha_N}}^\dagger) |0\rangle \\
& = (-1)^{2(k'-1)} \delta_{\nu_a \nu_{\alpha_1}} \delta_{\nu_b \nu_{\alpha_{k'}}} a_{\nu_d}^\dagger a_{\nu_{\alpha_{k'}}} a_{\nu_c}^\dagger a_{\nu_{\alpha_1}} (a_{\nu_{\alpha_1}}^\dagger \cdots a_{\nu_{\alpha_{k'}}}^\dagger \cdots a_{\nu_{\alpha_N}}^\dagger) |0\rangle \\
& = (-1)^{2(k'-1)+1} \delta_{\nu_a \nu_{\alpha_1}} \delta_{\nu_b \nu_{\alpha_{k'}}} \underbrace{a_{\nu_d}^\dagger (a_{\nu_c}^\dagger a_{\nu_{\alpha_1}}^\dagger - \delta_{\nu_c \nu_{\alpha_1}})}_{\substack{\uparrow \quad \uparrow \\ \text{two operators are in ordered place}}} a_{\nu_{\alpha_{k'}}} (a_{\nu_{\alpha_1}}^\dagger \cdots a_{\nu_{\alpha_{k'}}}^\dagger \cdots a_{\nu_{\alpha_N}}^\dagger) |0\rangle \\
& = (-1)^{2(k'-1)+2} \delta_{\nu_a \nu_{\alpha_1}} \delta_{\nu_b \nu_{\alpha_{k'}}} a_{\nu_c}^\dagger a_{\nu_d}^\dagger a_{\nu_{\alpha_1}} a_{\nu_{\alpha_{k'}}} (a_{\nu_{\alpha_1}}^\dagger \cdots a_{\nu_{\alpha_{k'}}}^\dagger \cdots a_{\nu_{\alpha_N}}^\dagger) |0\rangle \\
& - \delta_{\nu_c \nu_{\alpha_1}} (\text{products of operators}) |0\rangle \quad \rightarrow \quad (k', 1)
\end{aligned}$$

⋮

$$\begin{aligned}
k=j', j=j' & \quad (-1)^{2(k'-1)} \delta_{\nu_a \nu_{\alpha_1}} \delta_{\nu_b \nu_{\alpha_{k'}}} a_{\nu_d}^\dagger a_{\nu_{\alpha_{k'}}} (a_{\nu_{\alpha_1}}^\dagger \cdots a_{\nu_{\alpha_{k'}}}^\dagger \cdots a_{\nu_c}^\dagger \cdots a_{\nu_{\alpha_N}}^\dagger) |0\rangle \\
& = (-1)^{2(k'-1)} \delta_{\nu_a \nu_{\alpha_1}} \delta_{\nu_b \nu_{\alpha_{k'}}} a_{\nu_d}^\dagger a_{\nu_{\alpha_{k'}}} \underbrace{(a_{\nu_{\alpha_1}}^\dagger \cdots a_{\nu_{\alpha_{k'}}}^\dagger \cdots a_{\nu_c}^\dagger a_{\nu_{\alpha_{j'}}}^\dagger a_{\nu_{\alpha_{j'}}}^\dagger \cdots a_{\nu_{\alpha_N}}^\dagger)}_{j'-1 \text{ operators}} |0\rangle \\
& = (-1)^{|2(k'-1)+2(j'-1)|} \delta_{\nu_a \nu_{\alpha_1}} \delta_{\nu_b \nu_{\alpha_{k'}}} a_{\nu_d}^\dagger a_{\nu_{\alpha_{k'}}} a_{\nu_c}^\dagger a_{\nu_{\alpha_{j'}}} (a_{\nu_{\alpha_1}}^\dagger \cdots a_{\nu_{\alpha_{k'}}}^\dagger \cdots a_{\nu_{\alpha_{j'}}}^\dagger \cdots a_{\nu_{\alpha_N}}^\dagger) |0\rangle \\
& = (-1)^{|2(k'-1)+2(j'-1)+1|} \delta_{\nu_a \nu_{\alpha_1}} \delta_{\nu_b \nu_{\alpha_{k'}}} \underbrace{a_{\nu_d}^\dagger a_{\nu_c}^\dagger}_{\substack{\uparrow \quad \uparrow \\ \text{two operators are in ordered place}}} (a_{\nu_{\alpha_{k'}}}^\dagger - \delta_{\nu_c \nu_{\alpha_{k'}}}) a_{\nu_{\alpha_{j'}}} (a_{\nu_{\alpha_1}}^\dagger \cdots a_{\nu_{\alpha_{k'}}}^\dagger \cdots a_{\nu_{\alpha_{j'}}}^\dagger \cdots a_{\nu_{\alpha_N}}^\dagger) |0\rangle \\
& = (-1)^{|2(k'-1)+2(j'-1)+2|} \delta_{\nu_a \nu_{\alpha_1}} \delta_{\nu_b \nu_{\alpha_{k'}}} a_{\nu_c}^\dagger a_{\nu_d}^\dagger a_{\nu_{\alpha_{k'}}} a_{\nu_{\alpha_{j'}}} (a_{\nu_{\alpha_1}}^\dagger \cdots a_{\nu_{\alpha_{k'}}}^\dagger \cdots a_{\nu_{\alpha_{j'}}}^\dagger \cdots a_{\nu_{\alpha_N}}^\dagger) |0\rangle \\
& \quad - \delta_{\nu_c \nu_{\alpha_{k'}}} (\text{product of operators}) |0\rangle \quad \rightarrow \quad (k', j')
\end{aligned}$$

Now arrange each number set in a square formation, ordering (k,j) as kth row and jth column. The arranged form can be shown as a matrix, so let the corresponding matrix named A, then each number set (k,j) is an element of a matrix A_{kj} . The result of summation \textcircled{a} corresponds to the sum of all elements in matrix A.

$$A = \begin{pmatrix} 0 & (1,2) & (1,3) & \cdots & (1,N) \\ (2,1) & 0 & (2,3) & \cdots & (2,N) \\ \vdots & & & (k',j') & \\ (N,1) & (N,2) & (N,3) & \cdots & 0 \end{pmatrix}, \quad \textcircled{a} = \sum_j^N \sum_k^N A_{jk}$$

If $\nu_a = \nu_j'$ and $\nu_b = \nu_{k'}$, value of the \textcircled{a} corresponds to the matrix element $A_{k'j'}$,

$$\begin{aligned}
& \frac{1}{2} \sum_{\nu_a \nu_b, \nu_c \nu_d} V_{\nu_c \nu_d, \nu_a \nu_b} \left(\textcircled{a} \right) \\
& = \frac{1}{2} \sum_{\nu_a \nu_b, \nu_c \nu_d} V_{\nu_c \nu_d, \nu_a \nu_b} \left(A_{k'j'} \right) \\
& = \frac{1}{2} \sum_{\nu_a \nu_b, \nu_c \nu_d} V_{\nu_c \nu_d, \nu_a \nu_b} (-1)^{2(|k'-1|+|j'-1|+1)} c_{\nu_c}^\dagger c_{\nu_d}^\dagger c_{\nu_b} c_{\nu_a} (c_{\nu_1}^\dagger c_{\nu_2}^\dagger \cdots c_{\nu_N}^\dagger) |0\rangle \\
& \quad - \delta_{\nu_c \nu_{\alpha_{k'}}} (\text{product of operators}) |0\rangle
\end{aligned}$$

Since $\nu_c \neq \nu_b = \nu_{\alpha_{k'}}$, and $(-1)^{2(|k'-1|+|j'-1|+1)} = 1$,

$$= \frac{1}{2} \sum_{\nu_a \nu_b, \nu_c \nu_d} V_{\nu_c \nu_d, \nu_a \nu_b} c_{\nu_c}^\dagger c_{\nu_d}^\dagger c_{\nu_b} c_{\nu_a} (c_{\nu_1}^\dagger c_{\nu_2}^\dagger \cdots c_{\nu_N}^\dagger) |0\rangle$$

Therefore,

$$V_{tot} = \frac{1}{2} \sum_{\nu_a \nu_b, \nu_c \nu_d} V_{\nu_c \nu_d, \nu_a \nu_b} c_{\nu_c}^\dagger c_{\nu_d}^\dagger c_{\nu_a} c_{\nu_b}$$