# 1 First and second quantization

$A|\psi_\alpha\rangle = \alpha\psi_\alpha$
A : Hermitian Operator , $|\psi_\alpha\rangle$ : Eigenstate of A , $\alpha$ = Eigenvalue of A (real number)
$|\psi_\alpha\rangle$ , $|\phi_\beta\rangle$
$|\phi_\beta\rangle = \sum_\alpha |\psi_\alpha\rangle C_{\alpha\beta}$
$|C_{\alpha\beta}|^2$

## 1.1 First quantization,single-particle systems

$$i\hbar\partial_t |\psi(t)\rangle = H|\psi(t)\rangle \tag{1}$$

$$H = \frac{1}{2m}(\frac{\hbar}{i}\nabla_r + e\mathbf{A}(r,t))^2 - e\phi(r,t) \tag{2}$$

$$\psi_{\mathbf{K},\sigma}(r) = \langle\mathbf{r}|\mathbf{k},\sigma\rangle = \frac{1}{\sqrt{\nu}}e^{i\mathbf{k}\cdot\mathbf{r}}\chi_\sigma \tag{3}$$

$$\langle\mathbf{r}|n,l,m,\sigma\rangle = R_{nl}(r)Y_{l,m}(\theta,\phi)\chi_\sigma \tag{4}$$

$$\langle\mathbf{r}|n,k_y,k_z,\sigma\rangle = H_n(x/l - k_y l)e^{-\frac{1}{2}(x/l-k_y l)^2}\frac{1}{\sqrt{L_y L_z}}e^{ik_y y + k_z z}\chi_\sigma \tag{5}$$

$\psi_\nu(\mathbf{r}) = \langle\mathbf{r}|\nu\rangle$
$\int d\mathbf{r}|\psi_\nu(\mathbf{r})|^2$
$= \int d\mathbf{r}\,\langle\nu|\mathbf{r}\rangle\,\langle\mathbf{r}|\nu\rangle$
$= \langle\nu|\int d\mathbf{r}\,|\mathbf{r}\rangle\,\langle\mathbf{r}|\nu\rangle$
$\int d\mathbf{r}\,|\mathbf{r}\rangle\,\langle\mathbf{r}| = 1$
$\sum_\nu |\langle\nu|\psi\rangle|^2 = 1$
$\sum_\nu |\nu\rangle\,\langle\nu| = 1$

$$\psi(\mathbf{r}) = \sum_\nu \psi_\nu(\mathbf{r})\Big(\int d\mathbf{r}\psi^*_\nu(\mathbf{r})\psi(\mathbf{r})\Big) or\, \langle\mathbf{r}|\nu\rangle = \sum_\nu \langle\mathbf{r}|\nu\rangle\,\langle\nu|\psi\rangle$$

## 1.2 First quantization, many-particle systems

$\psi(\mathbf{r})$
$\psi(\mathbf{r}_1,\mathbf{r}_2,...,\mathbf{r}_N)$

$$|\psi(\mathbf{r}_1,\mathbf{r}_2,...,\mathbf{r}_N)|^2\prod_{j=1}^N d\mathbf{r}_j$$

$$\psi(\mathbf{r}_1,...\mathbf{r}_j,...,\mathbf{r}_k,...\mathbf{r}_N) = +\psi(\mathbf{r}_1,...\mathbf{r}_k,...,\mathbf{r}_j,...\mathbf{r}_N)$$

$$\psi(\mathbf{r}_1,...\mathbf{r}_j,...,\mathbf{r}_k,...\mathbf{r}_N) = -\psi(\mathbf{r}_1,...\mathbf{r}_k,...,\mathbf{r}_j,...\mathbf{r}_N)$$

$$\psi(\mathbf{r}_1,\mathbf{r}_2,...,\mathbf{r}_N) = \sum_{\nu 1,...\nu N} A_{\nu 1,\nu 2,...\nu N}\psi_{\nu 1}(\mathbf{r}_1)\psi_{\nu 2}(\mathbf{r}_2)...\psi_{\nu N}(\mathbf{r}_N)$$

$$\psi(\mathbf{r}_1,\mathbf{r}_2,...,\mathbf{r}_N)=\sum_{\nu 1,...\nu N} B_{\nu 1,\nu 2,...\nu N}\hat{S}_{\pm}\psi_{\nu 1}(\mathbf{r}_1)\psi_{\nu 2}(\mathbf{r}_2)...\psi_{\nu N}(\mathbf{r}_N)$$

**Operators in first quantization**

$$T_j=\sum_{\nu_a,\nu_b} T_{\nu_b,\nu_a}\left|\psi_{\nu_b}(\mathbf{r}_j)\right\rangle\left\langle\psi_{\nu_a}(\mathbf{r}_j)\right|$$

$$T_{\nu_b,\nu_a}=\int d\mathbf{r}_j\psi^*_{\nu_b}(\mathbf{r}_j)T(\mathbf{r}_j,\nabla_{\mathbf{r}_j})\psi^*_{\nu_a}(\mathbf{r}_j)$$

$$T_{tot}=\sum_{j=1}^{N}T_j$$

$$T_{tot}\psi(\mathbf{r}_1,\mathbf{r}_2,...,\mathbf{r}_N)=\sum_{j=1}^{N}\sum_{\nu_a,\nu_b}T_{\nu_b,\nu_a}\delta_{\nu_a,\nu_{n_j}}\left|\psi_{\nu_{n1}}(\mathbf{r}_1)\right\rangle,...\left|\psi_{\nu_b}(\mathbf{r}_j)\right\rangle,...,\left|\psi_{\nu_{n_N}}(\mathbf{r}_N)\right\rangle$$

$$V_{jk}=\sum_{\nu_a\nu_b\nu_c\nu_d}V_{\nu_c\nu_d,\nu_b\nu_a}\left|\psi_{\nu_c}(\mathbf{r}_j)\right\rangle\left|\psi_{\nu_d}(\mathbf{r}_k)\right\rangle\left\langle\psi_{\nu_a}(\mathbf{r}_j)\right|\left\langle\psi_{\nu_b}(\mathbf{r}_k)\right|$$

$$V_{\nu_c\nu_d,\nu_b\nu_a}=\int d\mathbf{r}_j\mathbf{r}_k\psi^*_{\nu_c}(\mathbf{r}_j)\psi^*_{\nu_d}(\mathbf{r}_k)V(\mathbf{r}_j-\mathbf{r}_k)\psi_{\nu_a}(\mathbf{r}_j)\psi_{\nu_b}(\mathbf{r}_k)$$

$$V_{tot}=\sum_{j>k}^{N}V_{jk}=\frac{1}{2}\sum_{j,k\neq j}^{N}V_{jk}$$

$$V_{tot}\psi(\mathbf{r}_1,\mathbf{r}_2,...,\mathbf{r}_N)=\sum_{j,k\neq j}^{N}\sum_{\nu_a\nu_b\nu_c\nu_d}V_{\nu_c\nu_d,\nu_b\nu_a}\delta_{\nu_a,\nu_{n_j}}\delta_{\nu_b,\nu_{n_k}}\left|\psi_{\nu_{n1}}(\mathbf{r}_1)\right\rangle,...\left|\psi_{\nu_c}(\mathbf{r}_j)\right\rangle,...\left|\psi_{\nu_d}(\mathbf{r}_k)\right\rangle,...\left|\psi_{\nu_{n_N}}(\mathbf{r}_N)\right\rangle$$

## 1.3 Second quantization, basic concepts

$$\left|n_{\nu 1},n_{\nu 2},n_{\nu 3},...\right\rangle,\sum_j n_{\nu j}=N$$

$$\hat{n}_{\nu j}\left|n_{\nu j}\right\rangle=n_{\nu j}\left|n_{\nu j}\right\rangle$$

$$n_{\nu j}=\begin{Bmatrix}0,1\\0,1,2,...\end{Bmatrix}$$

$\mathcal{F}=\mathcal{F}_0\bigoplus\mathcal{F}_1\bigoplus\mathcal{F}_2\bigoplus...,$
$\mathcal{F}_N=span\{\left|n_{\nu 1},n_{\nu 2},n_{\nu 3},...\right\rangle|\sum_j=N\}$

**boson creation and annihilation operators**

$$b^\dagger_{\nu j}\left|...,n_{\nu_{j-1}},n_{\nu_j},\nu_{j+1},...\right\rangle=B_{+(n_{\nu j})}\left|...,n_{\nu_{j-1}},n_{\nu_j}+1,\nu_{j+1},...\right\rangle$$

$\left\langle\nu_{j+1}\right|b^\dagger_{\nu_j}\left|\nu_j\right\rangle$

$$b_{\nu j}\left|...,n_{\nu_{j-1}},n_{\nu_j},\nu_{j+1},...\right\rangle=B_{-(n_{\nu j})}\left|...,n_{\nu_{j-1}},n_{\nu_j}-1,\nu_{j+1},...\right\rangle$$

$\langle \nu_j | \, (b^\dagger)^\dagger_{\nu_{j+1}} \, | \nu_j \rangle$

$$[b^\dagger_{\nu j}, b^\dagger_{\nu k}] = 0, \quad [b_{\nu j}, b_{\nu k}] = 0, \quad [b_{\nu j}, b^\dagger_{\nu k}] = \delta_{\nu_j, \nu_k}$$

$$[b^\dagger_\nu b_\nu, b_\nu] = 0, \quad [b^\dagger_\nu b_\nu, b_\nu] = 0$$

$$b^\dagger_\nu b_\nu = \hat{n}_\nu, \quad b^\dagger_\nu b_\nu \, |n_\nu\rangle = n_\nu \, |n_\nu\rangle \,, \quad n_\nu = 0, 1, 2...$$

$$b_\nu \, |n_\nu\rangle = \sqrt{n_\nu} \, |n_\nu - 1\rangle \,, \quad b^\dagger_\nu \, |n_\nu\rangle = \sqrt{n_\nu + 1} \, |n_\nu + 1\rangle \,, \quad (b^\dagger_\nu)^{n_\nu} \, |0\rangle = \sqrt{n!} \, |n_\nu\rangle$$

$$\hat{S}_+ \, |\psi_{\nu_{n1}}(\mathbf{r}_1)\rangle \,, |\psi_{\nu_{n2}}(\mathbf{r}_j)\rangle \,, ... \, |\psi_{\nu_{nN}}(\mathbf{r}_N)\rangle = b^\dagger_{\nu_{n1}} b^\dagger_{\nu_{n2}} ... b^\dagger_{\nu_{n_N}} \, |0\rangle$$

**fermion creation and annihilation operators**

$$\{c^\dagger_{\nu j}, c^\dagger_{\nu k}\} = 0, \quad \{c_{\nu j}, c_{\nu k}\} = 0, \quad \{c_{\nu j}, c^\dagger_{\nu k}\} = \delta_{\nu_j, \nu_k}$$

$$[c^\dagger_\nu c_\nu, c_\nu] = 0, \quad [c^\dagger_\nu c_\nu, c_\nu] = 0$$

$$c^\dagger_\nu c_\nu = \hat{n}_\nu, \quad c^\dagger_\nu c_\nu \, |n_\nu\rangle = n_\nu \, |n_\nu\rangle$$

$$c_\nu \, |0\rangle = 0 \quad c^\dagger_\nu \, |0\rangle = |1\rangle \,, \quad c_\nu \, |1\rangle = |0\rangle \,, \quad c^\dagger_\nu \, |1\rangle = 0, \quad n_\nu = 0, 1,$$

$$\hat{S}_- \, |\psi_{\nu_{n1}}(\mathbf{r}_1)\rangle \,, |\psi_{\nu_{n2}}(\mathbf{r}_j)\rangle \,, ... \, |\psi_{\nu_{nN}}(\mathbf{r}_N)\rangle = c^\dagger_{\nu_{n1}} c^\dagger_{\nu_{n2}} ... c^\dagger_{\nu_{n_N}} \, |0\rangle$$

**The general form for second quantization operators**

$$T_{tot} b^\dagger_{\nu n1} ... b^\dagger_{\nu n_N} \, |0\rangle = \sum_{\nu_a \nu_b} T_{\nu_b \nu_a} \sum_{j=1}^{N} \delta_{\nu_a, \nu_{n_j}} b^\dagger_{\nu_n 1} ... b^\dagger_{\nu_b} ... b^\dagger_{\nu_{n_N}} \, |0\rangle$$

$$T_{tot} = \sum_{\nu_i, \nu_j} T_{\nu_i \nu_j} a^\dagger_{\nu_i} a_{\nu_j}$$

$$V_{tot} = \frac{1}{2} \sum_{\nu_i \nu_j \nu_k \nu_l} V_{\nu_i \nu_j, \nu_k \nu_l} a^\dagger_{\nu_i} a^\dagger_{\nu_j} a_{\nu_l} a_{\nu_k}$$

**Change of basis in second quantization**

$\{|\psi_{\nu 1}\rangle \,, |\psi_{\nu 2}\rangle \, ...\} \; \{\left|\tilde{\psi}_{\mu 1}\right\rangle, \left|\tilde{\psi}_{\mu 2}\right\rangle ...\}$

$$\tilde{a}^\dagger_{\mu_{n1}} \tilde{a}^\dagger_{\mu_{n2}} ... \tilde{a}^\dagger_{\mu_{nN}} \, |0\rangle = \big(\sum_{\nu_{n1}} \langle \tilde{\psi}_{\mu_{n1}} | \psi \rangle^*_{\mu_{n1}} \, a^\dagger_{\nu_{n1}}\big) \ldots \big(\sum_{\nu_{n1}} \langle \tilde{\psi}_{\mu_{nN}} | \psi \rangle^*_{\mu_{nN}} \, a^\dagger_{\nu_{nN}}\big) \, |0\rangle$$

$$[\tilde{a}_{\mu_1}, \tilde{a}^\dagger_{\mu_2}]_\pm = \delta_{\mu_1, \mu_2}$$

$$\sum_\mu \tilde{a}^\dagger_\mu \tilde{a}_\mu = \sum_\mu \tilde{a}^\dagger_{\nu_j} \tilde{a}_{\nu_j}$$

**Second quatization and statstical mechanics**

$$Z = \sum_s e^(\beta E_s)$$

$$H\left|\nu\right\rangle = E_\nu \left|\nu\right\rangle$$

$$\rho \equiv e^{-\beta H} = \sum_\nu \left|\nu\right\rangle e^{-\beta E_\nu} \left\langle\nu\right|$$

$$Z = \sum_\nu \left\langle\nu\right| \rho \left|\nu\right\rangle = Tr[\rho]$$

$$\langle A\rangle = \frac{1}{Z} Z = \sum_\nu \left\langle\nu\right| A \left|\nu\right\rangle e^{-\beta E_\nu} = \frac{Tr[\rho A]}{Tr[\rho]}$$

$$n_F(\epsilon_{\mathbf{k}}) = \frac{Tr[\rho_G n_{\mathbf{k}}]}{Tr[\rho_G]} = \frac{1}{e^{\beta(\epsilon_{\mathbf{k}}-\mu)}+1}$$

$$n_B(\epsilon_{\mathbf{k}}) = \frac{Tr[\rho_G n_{\mathbf{k}}]}{Tr[\rho_G]} = \frac{1}{e^{\beta(\epsilon_{\mathbf{k}}-\mu)}-1}$$

**Quantum field operators and their Fourier transforms** $\{\left|\tilde{\psi}\right\rangle\}$ $\{\left|\mathbf{r}\right\rangle\}$

$$\Psi^\dagger(\mathbf{r}) \equiv \sum_\nu \left\langle\mathbf{r}|\psi_\nu\right\rangle^* a_\nu^\dagger = \sum_\nu \psi_\nu^*(\mathbf{r}) a_\nu^\dagger \qquad \Psi(\mathbf{r}) \equiv \sum_\nu \left\langle\mathbf{r}|\psi_\nu\right\rangle a_\nu = \sum_\nu \psi_\nu(\mathbf{r}) a_\nu$$

$$[\Psi(\mathbf{r}_1), \Psi^\dagger(\mathbf{r}_2)] = \delta(\mathbf{r}_1 - \mathbf{r}_2)$$

$$\{\Psi(\mathbf{r}_1), \Psi(\mathbf{r}_2), \Psi^\dagger(\mathbf{r}_2)\} = \delta(\mathbf{r}_1 - \mathbf{r}_2)$$

$$T = \sum_{\nu_i,\nu_j} \Big( \int d\mathbf{r} \psi_{\nu_i}^* Tr \psi_{\nu_j}(\mathbf{r})) a_{\nu_i}^\dagger a_{\nu_i}$$

$$\int d\mathbf{r} \big(\psi_{\nu_i}^* a_{\nu_i}^\dagger\big) Tr \big(\sum_{\nu_j} \psi_{\nu_j}(\mathbf{r}) a_{\nu_i}\big) = \int d\mathbf{r} \Psi^\dagger(\mathbf{r}) Tr \Psi(\mathbf{r})$$

$$\Psi^\dagger(\mathbf{r}) \frac{1}{\sqrt{\nu}} \sum_{\mathbf{k}} e^{-ik\cdot r} a_k^\dagger, \qquad \Psi(\mathbf{r}) \frac{1}{\sqrt{\nu}} \sum_{\mathbf{k}} e^{-ik\cdot r} a_k$$

$$a_q^\dagger(\mathbf{r}) \frac{1}{\sqrt{\nu}} \int d\mathbf{r} e^{-iq\cdot r} \Psi_k^\dagger, \qquad a_q(\mathbf{r}) \frac{1}{\sqrt{\nu}} \int d\mathbf{r} e^{-iq\cdot r} \Psi_k$$