大数据与机器智能

哈希表与哈希函数

许书畅

2019年10月

Outline

- Hash tables are another sort of data structure that allows fast INSERT/DELETE/SEARCH.
  - like self-balancing binary trees
  - The difference is we can get better performance in expectation by using randomness.

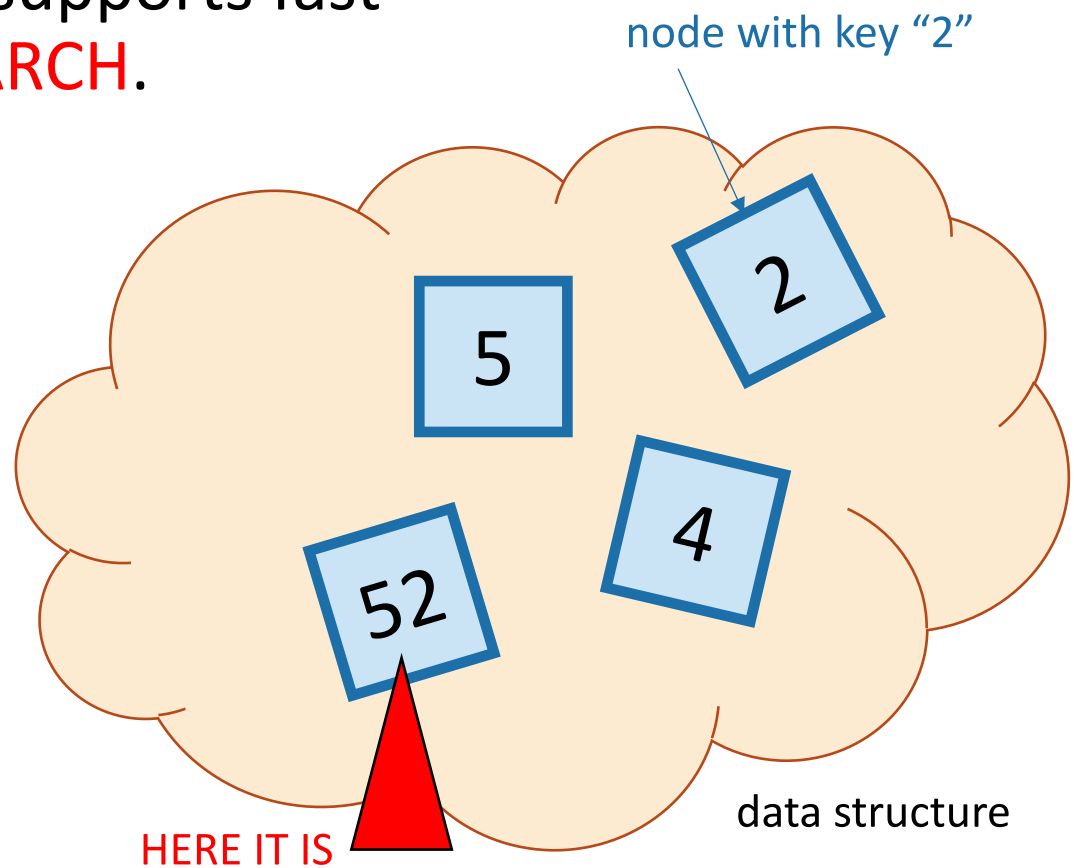
- **Hash families** are the magic behind hash tables.

- **Universal hash families** are even more magical.

Goal:
Just like last time

- We are interesting in putting nodes with keys into a data structure that supports fast INSERT/DELETE/SEARCH.

- INSERT 5
- DELETE 4
- SEARCH 52



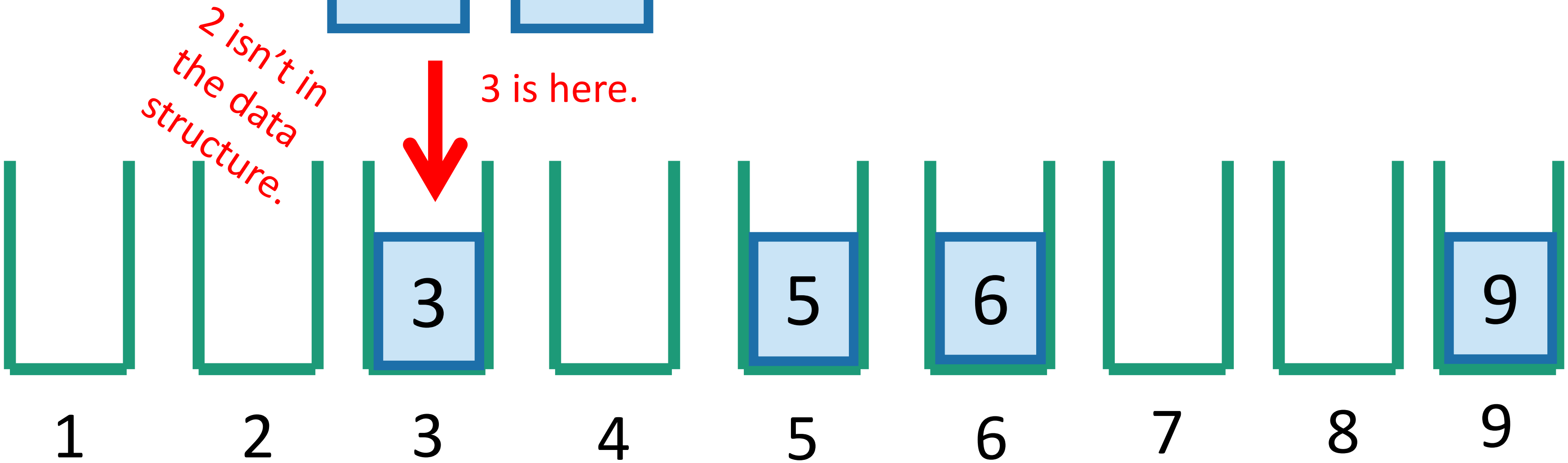
Today:

- Hash tables:
  - O(1) expected time INSERT/DELETE/SEARCH
- Worse worst-case performance, but often great in practice.

One way to get O(1) time

This is called "direct addressing"

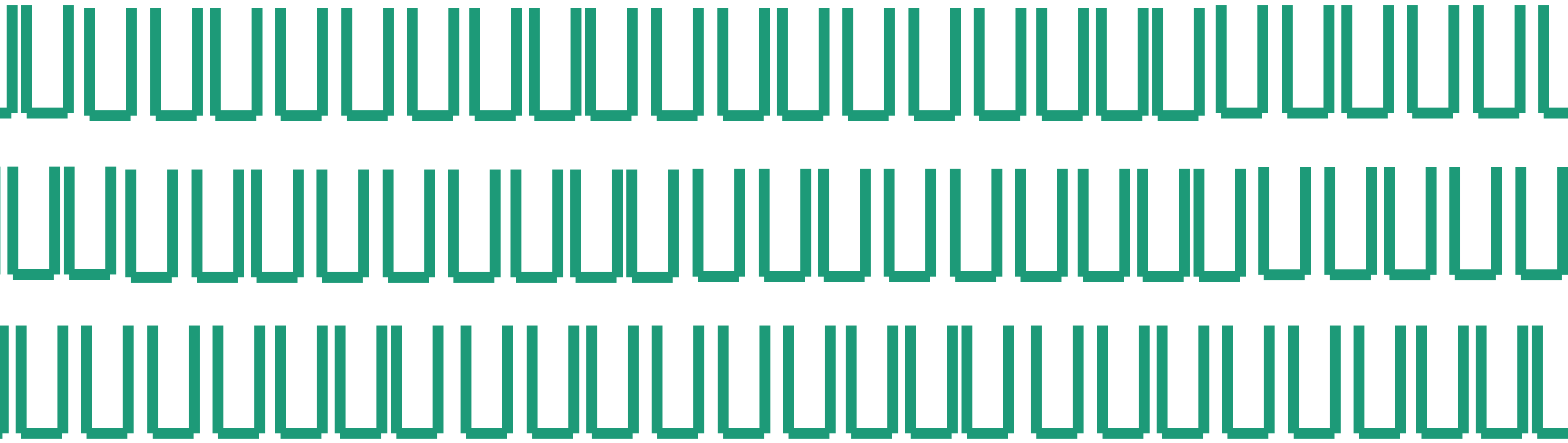
- Say all keys are in the set {1,2,3,4,5,6,7,8,9}.
- INSERT: 9 6 3 5
- DELETE: 6
- SEARCH: 3 2



That should look familiar

The universe is really big!

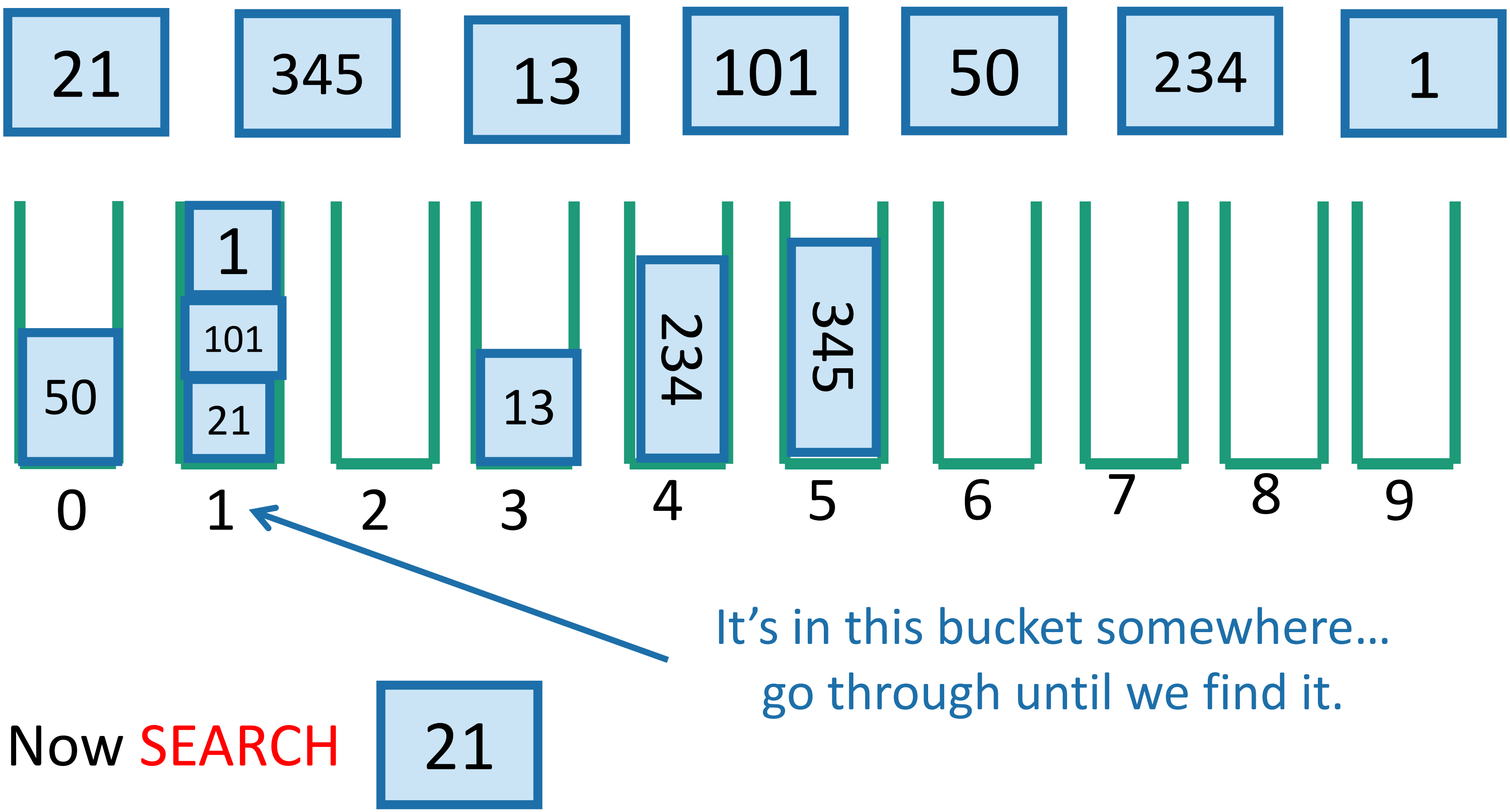
- Kind of like BUCKETSORT from Lecture 6.
- Same problem: if the keys may come from a "universe" U = {1,2, …., 10000000000}….



Solution?

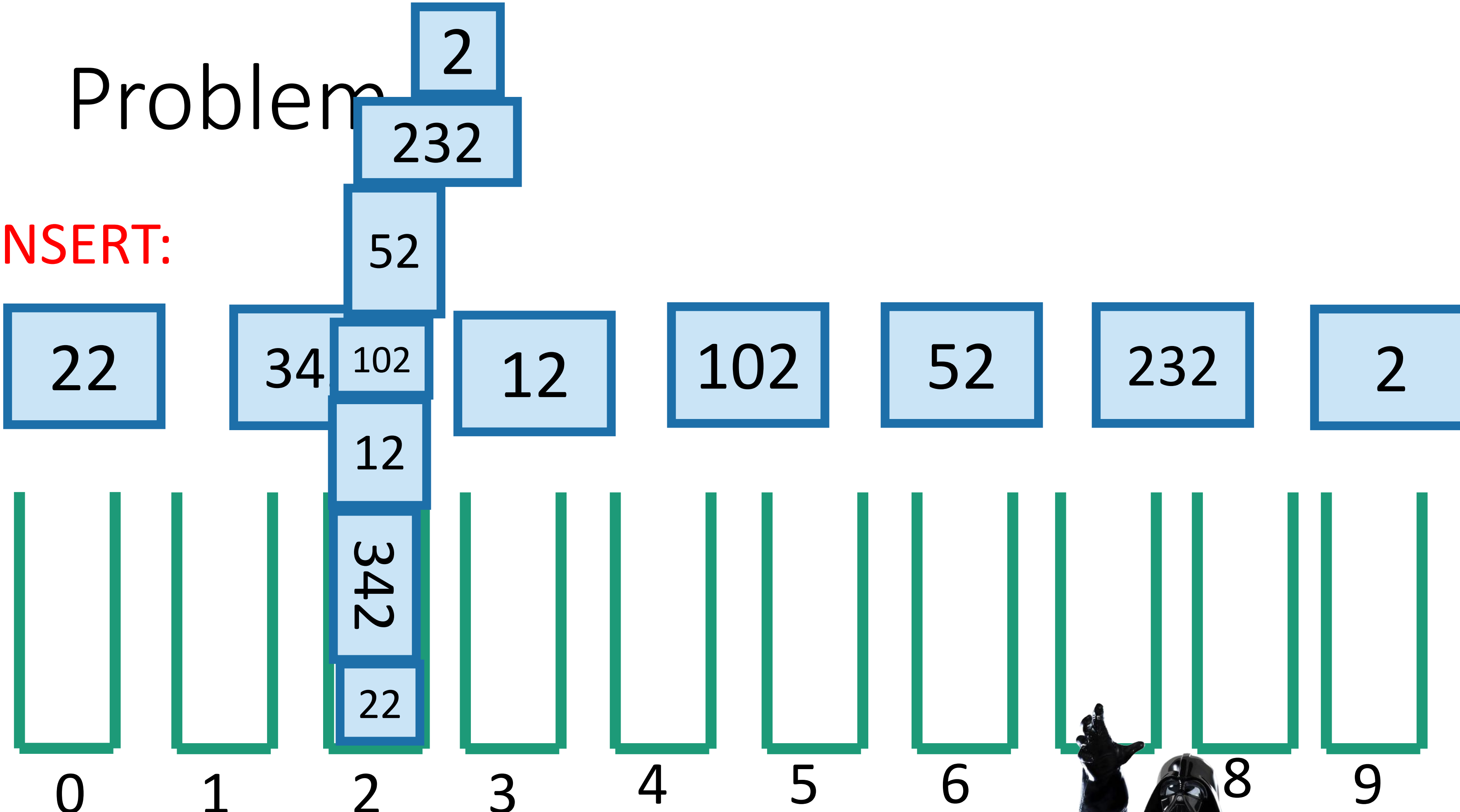
Put things in buckets based on one digit

INSERT:



Problem

INSERT:



Now SEARCH 22

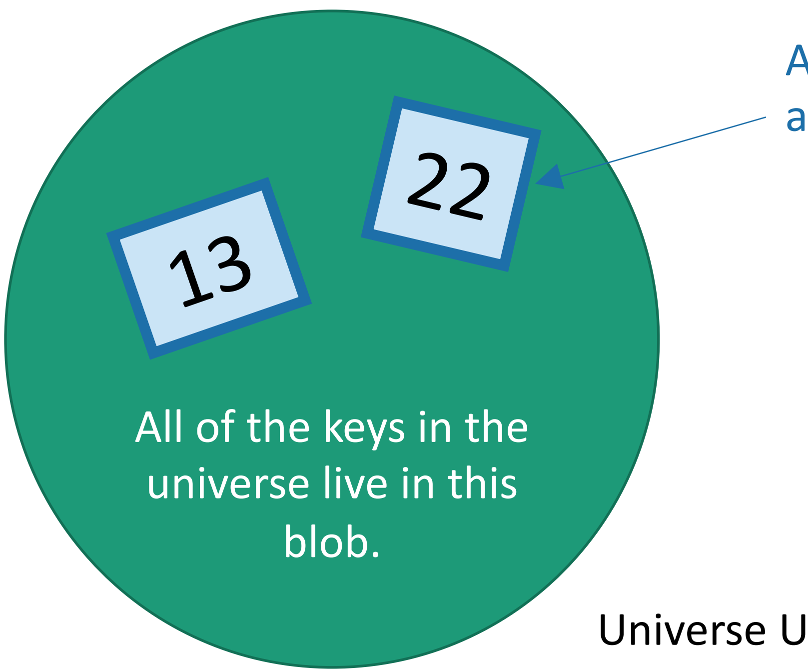
....this hasn't made our lives easier...

Hash tables

- That was an example of a hash table.
  - not a very good one, though.
- We will be **more clever** (and less deterministic) about our bucketing.
- This will result in fast (expected time) INSERT/DELETE/SEARCH.

But first! Terminology.

- We have a universe U, of size M.
  - M is really big.
- But only a few (say at most n for today's lecture) elements of M are ever going to show up.
  - M is waaaayyyyyyy bigger than n.
- But we don't know which ones will show up in advance.



Example: U is the set of all strings of at most 140 ascii characters. (128^{140} of them).

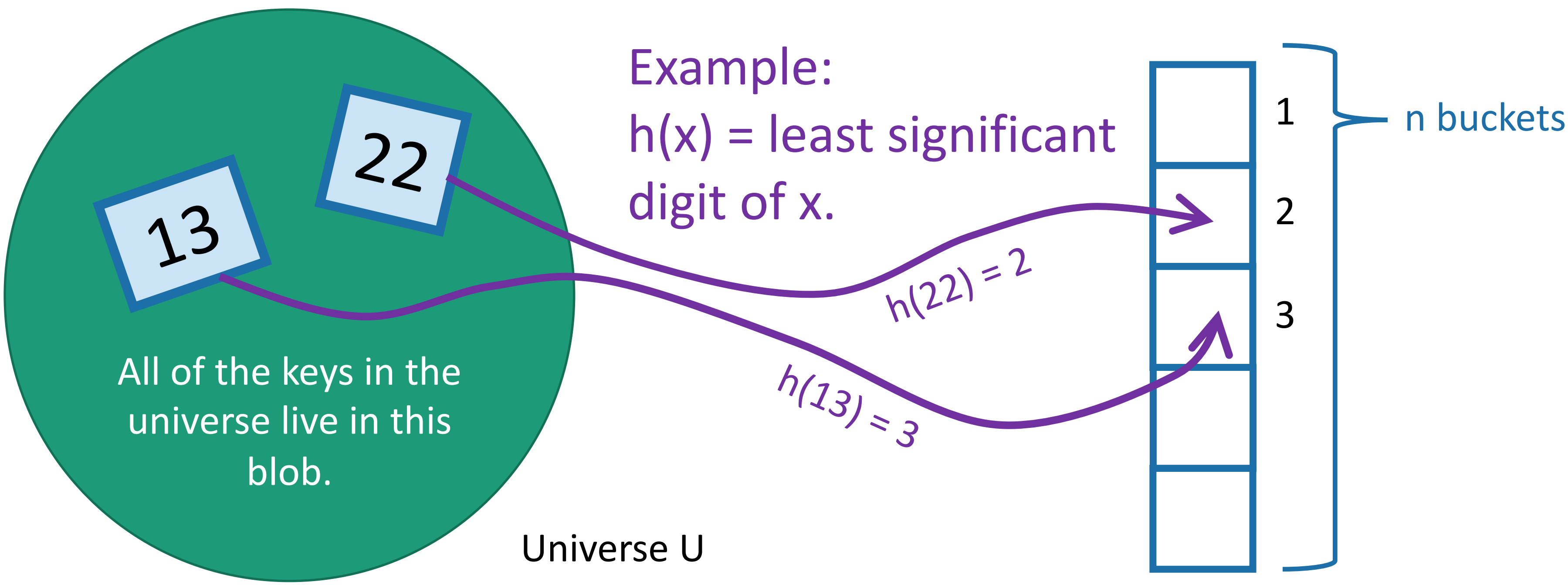
The only ones which I care about are those which appear as trending hashtags on twitter. #hashinghashtags
There are way fewer than 128^{140} of these.

Examples aside, I'm going to draw elements like I always do, as blue boxes with integers in them…

The previous example
with this terminology

For this lecture, I'm assuming that the number of things is the same as the number of buckets, both are n.
This doesn't have to be the case, although we do want:
#buckets = O(#things which show up)

- We have a universe U, of size M.
  - at most n of which will show up.
- M is waaaayyyyyy bigger than n.
- We will put items of U into n buckets.
- There is a *hash function* $h: U \to \{1, \dots, n\}$ which says what element goes in what bucket.



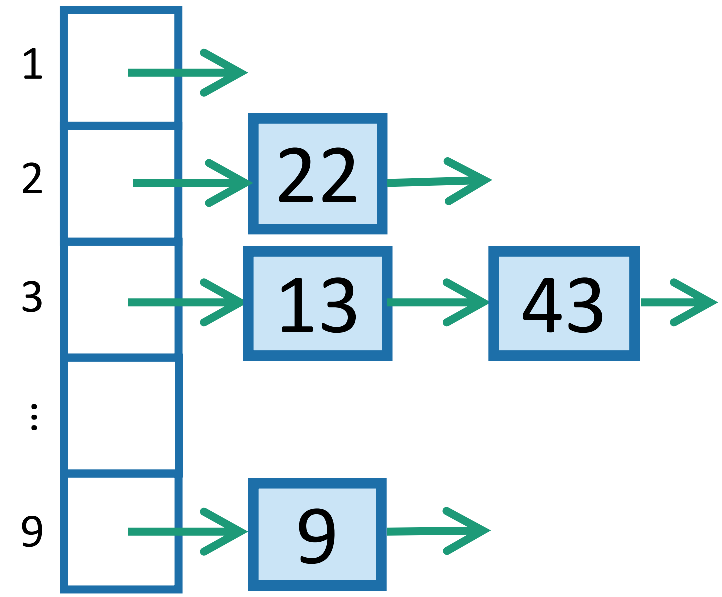
This is a **hash table** (with chaining)

For demonstration purposes only!
This is a terrible hash function! Don't use this!

- Array of n buckets.
- Each bucket stores a linked list.
  - We can insert into a linked list in time O(1)
  - To find something in the linked list takes time O(length(list)).
- $h: U \to \{1, \ldots, n\}$ can be any function:
  - but for concreteness let's stick with h(x) = least significant digit of x.

INSERT:

13 22 43 9



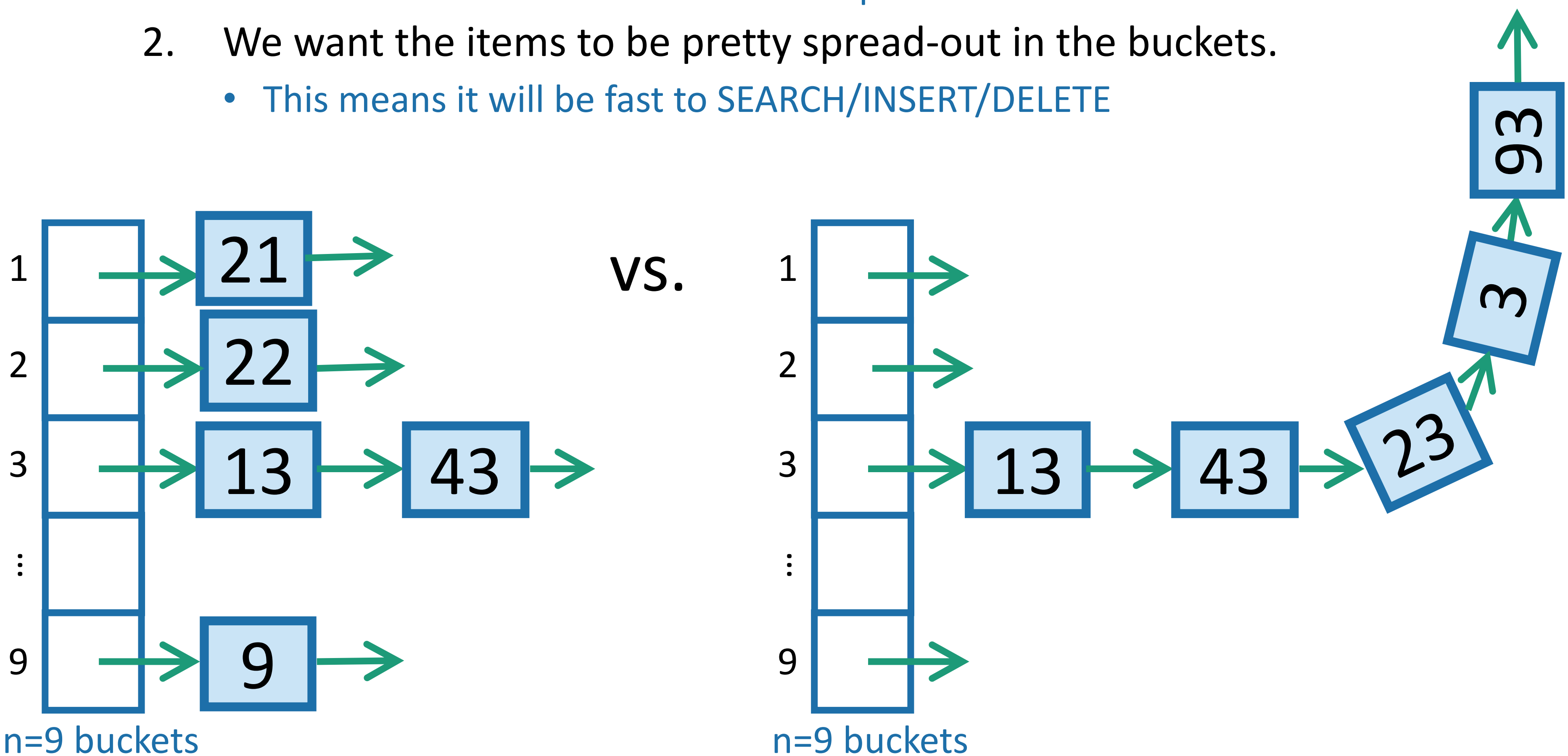
n buckets (say n=9)

SEARCH 43:

Scan through all the elements in bucket h(43) = 3.

Sometimes this a good idea
Sometimes this is a bad idea

- How do we pick that function so that this is a good idea?
  1. We want there to be not many buckets (say, n).
     - This means we don't use too much space
  2. We want the items to be pretty spread-out in the buckets.
     - This means it will be fast to SEARCH/INSERT/DELETE



Worst-case analysis

- Goal: Design a function $h: U \rightarrow \{1, \ldots, n\}$ so that:
  - No matter what input (fewer than n items of U) a bad guy chooses, the buckets will be balanced.
  - Here, balanced means O(1) entries per bucket.

- If we had this, then we'd achieve our dream of O(1) INSERT/DELETE/SEARCH

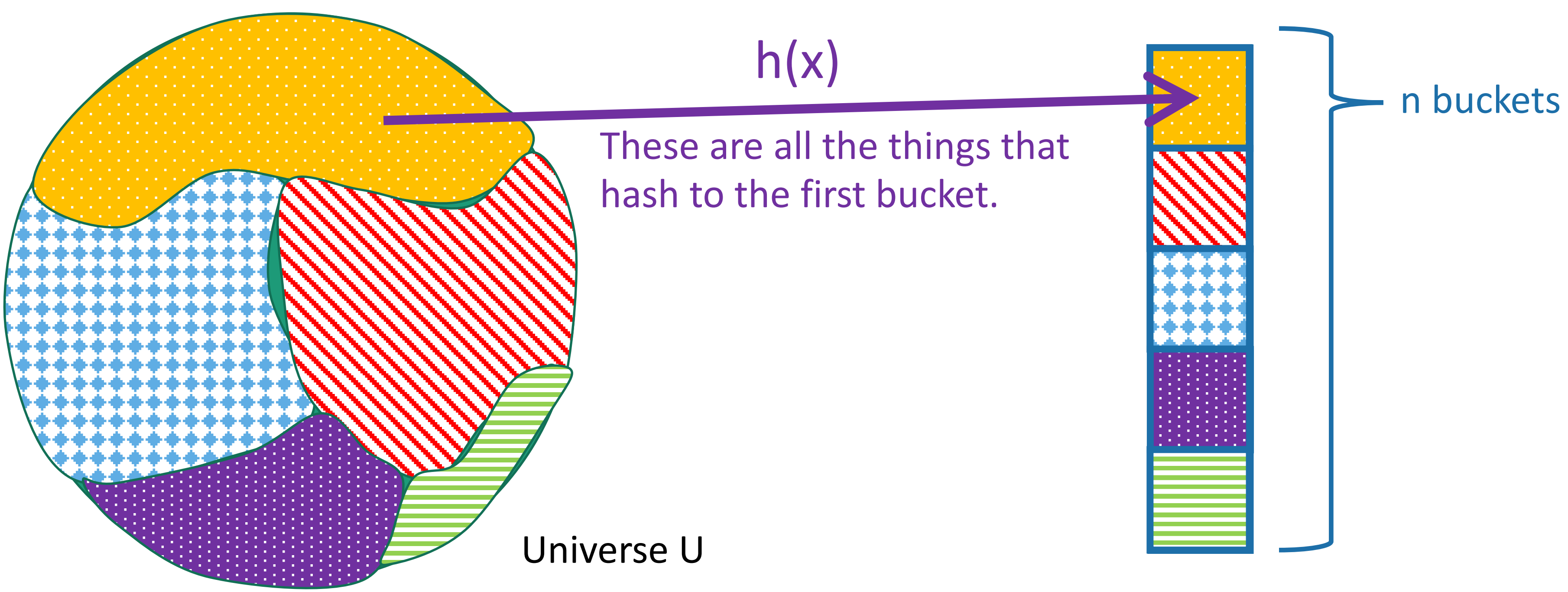
Can you come up with such a function?

This is impossible!

No deterministic hash function can defeat worst-case input!

We really can’t beat the bad guy here.

- The universe U has M items
- They get hashed into n buckets
- At least one bucket has at least M/n items hashed to it.
- M is waayyyy bigger then n, so M/n is bigger than n.
- **Bad guy chooses n of the items that landed in this very full bucket.**



Solution:

Randomness

The game

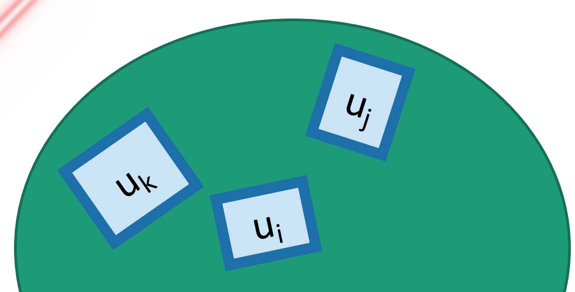
What does **random** mean here? Uniformly random?

Plucky the pedantic penguin

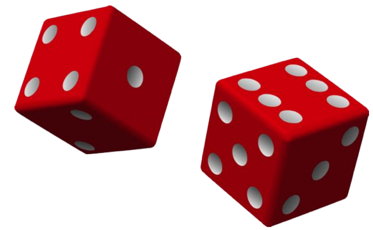
1. An adversary chooses any n items $u_1, u_2, \ldots, u_n \in U$, and any sequence of INSERT/DELETE/SEARCH operations on those items.

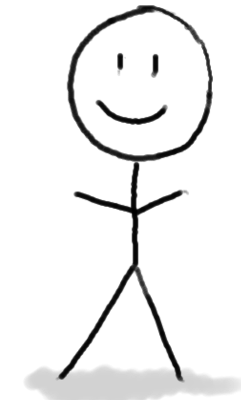
13 22 43 92 7

INSERT 13, INSERT 22, INSERT 43, INSERT 92, INSERT 7, SEARCH 43, DELETE 92, SEARCH 7, INSERT 92

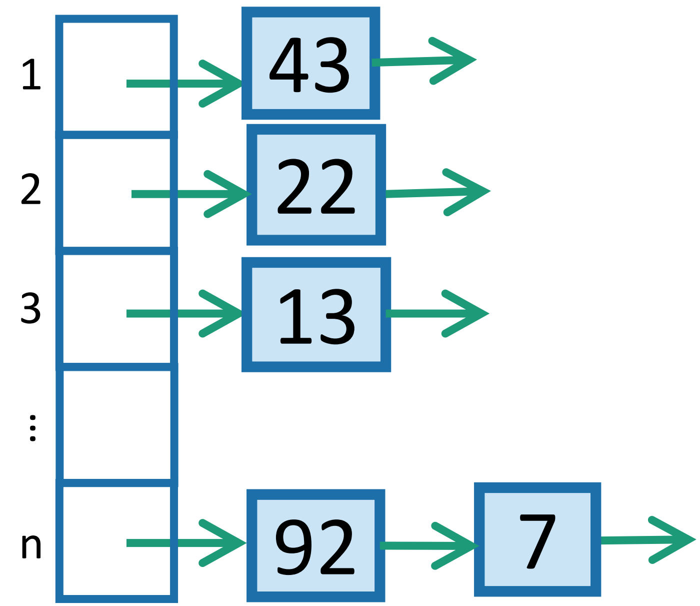


2. You, the algorithm, chooses a **random** hash function $h: U \to \{1, \ldots, n\}$.

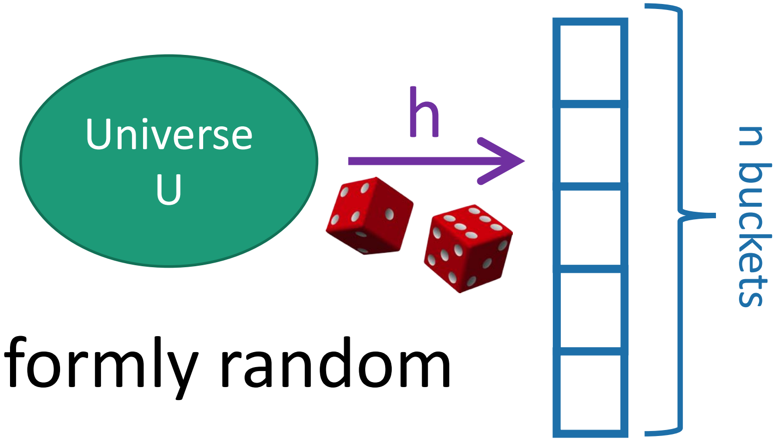




3. **HASH IT OUT** #hashpuns



Example



- Say that $h: U \to \{1, \ldots, n\}$ is a uniformly random function.
  - That means that **h(1)** is a **uniformly random** number between 1 and n.
  - **h(2)** is also a **uniformly random** number between 1 and n, independent of h(1).
  - **h(3)** is also a **uniformly random** number between 1 and n, independent of h(1), h(2).
  - …
  - **h(M)** is also a **uniformly random** number between 1 and n, independent of h(1), h(2), …, h(M-1).

Randomness helps

Intuitively: The bad guy can't foil a hash function that he doesn't yet know.

Lucky the
Lackadaisical Lemur

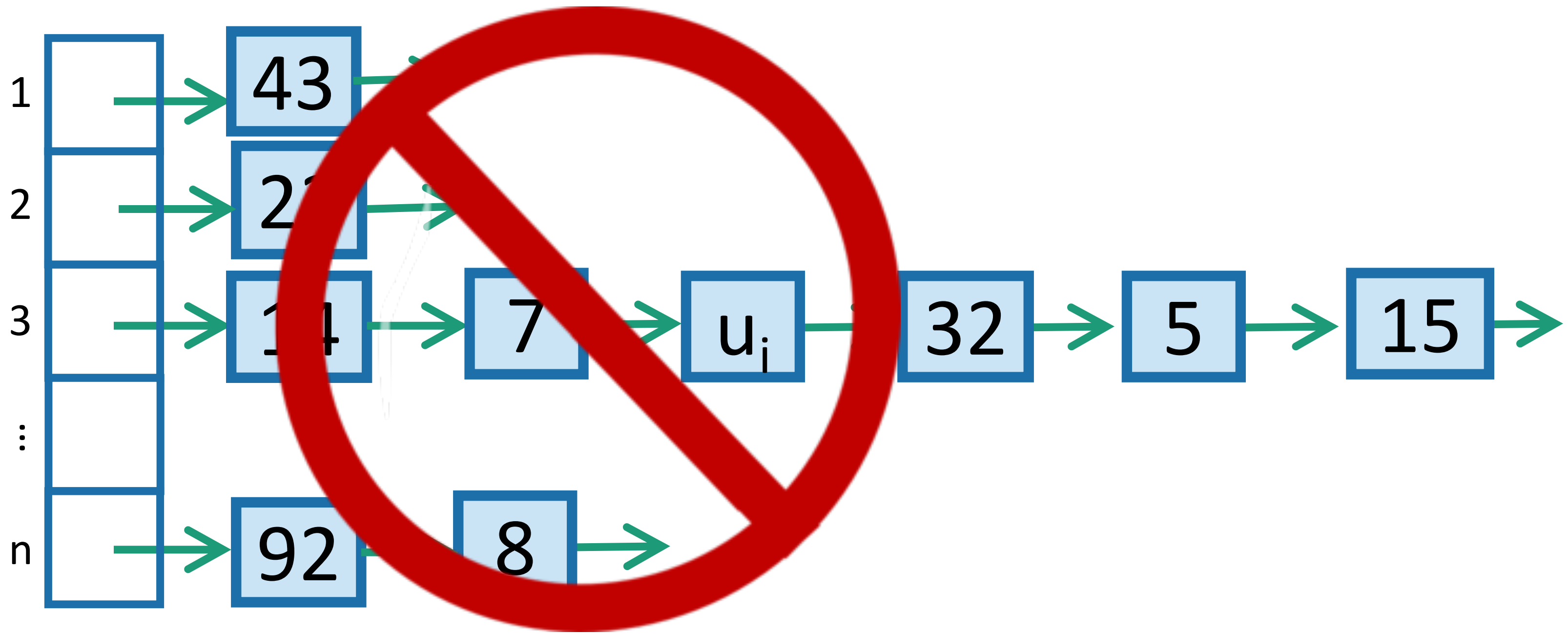
Plucky the Pedantic
Penguin

Why not? What if there's some strategy that foils a random function with high probability?

We'll need to do some analysis…

What do we want?

It's **bad** if lots of items land in u_i's bucket.
So we want **not that**.



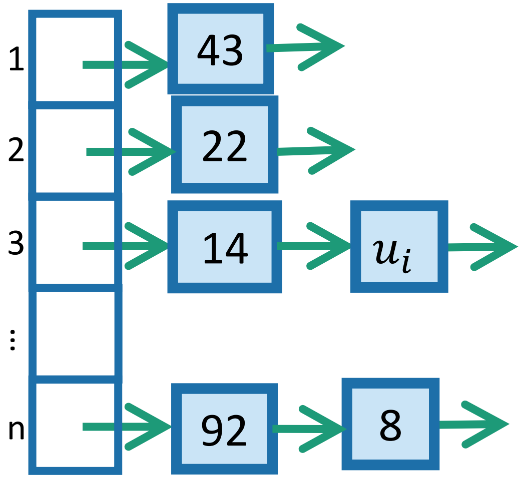
More precisely

We could replace "2" here with any constant; it would still be good. But "2" will be convenient.

- We want:
  - For all ways a bad guy could choose $u_1, u_2, \ldots, u_n$, to put into the hash table, and for all $i \in \{1, \ldots, n\}$,

    E[number of items in u_i 's bucket] ≤ 2.
- If that were the case:
  - For each INSERT/DELETE/SEARCH operation involving u_i,

    E[time of operation] = O(1)



This is what we wanted at the beginning of lecture!

So we want:

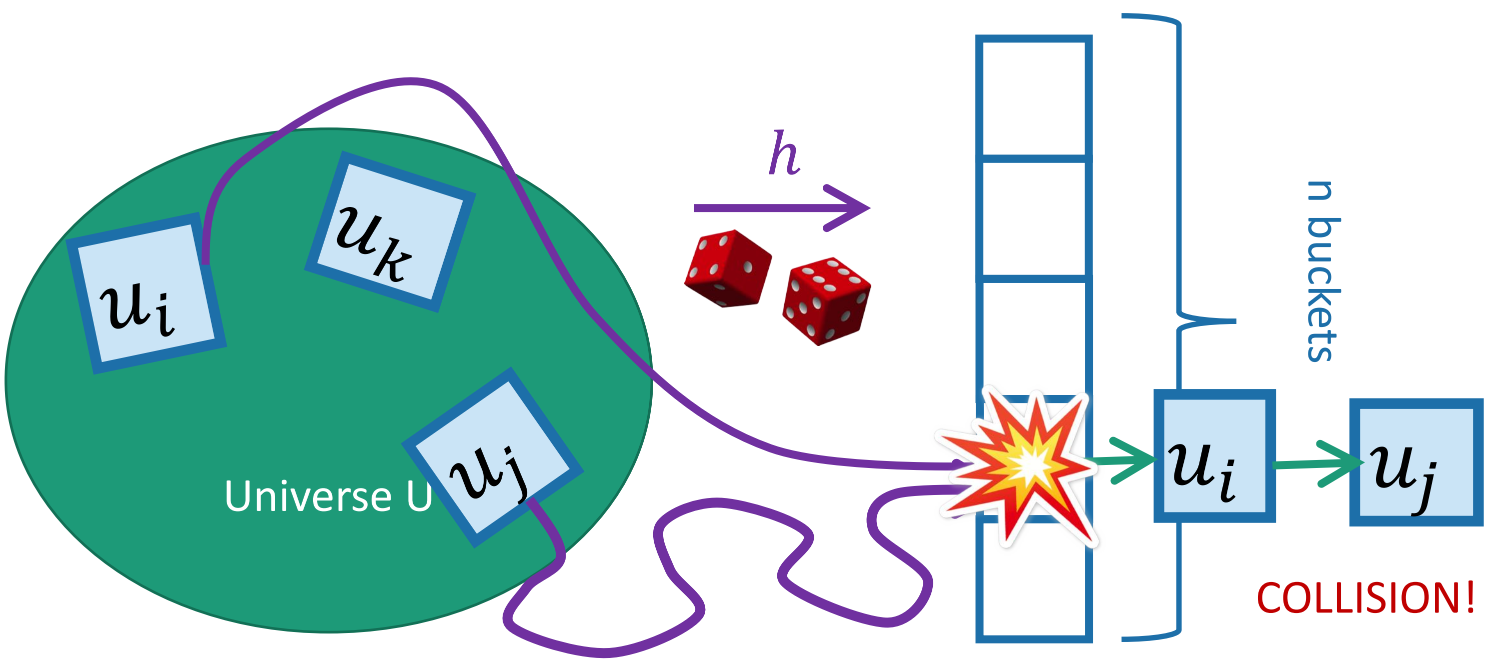
- For all i=1, …, n,

  E[number of items in u_i ‘s bucket] ≤ 2.

h is uniformly random

Expected number of items in u_i's bucket?

- $E[\ ] = \sum_{j=1}^{n} P\{\, h(u_i) = h(u_j) \}$
- $\qquad = 1 + \sum_{j \neq i} P\{\, h(u_i) = h(u_j) \}$
- $\qquad = 1 + \sum_{j \neq i} 1/n$
- $\qquad = 1 + \frac{n-1}{n} \leq 2.$ That's what we wanted!



That’s great!

- We just showed:
  - For all ways a bad guy could choose $u_1, u_2, \ldots, u_n,$ to put into the hash table, and for all $i \in \{1, \ldots, n\}$,

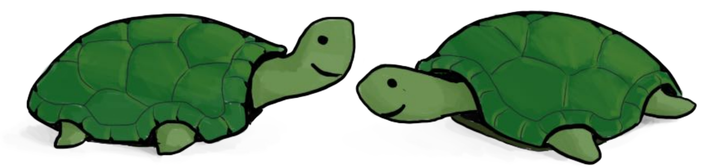
    E[number of items in u_i ‘s bucket] ≤ 2.
- Which implies:
  - No matter what sequence of operations and items the bad guy chooses,

    E[time of INSERT/DELETE/SEARCH] = O(1)
- So our solution is:

Pick a uniformly random hash function?

What's wrong with this plan?

- Hint: How would you implement (and store) and uniformly random function $h: U \to \{1, \dots, n\}$?



Think-Pair-Share Terrapins

- If h is a uniformly random function:
  - That means that **h(1)** is a **uniformly random** number between 1 and n.
  - **h(2)** is also a **uniformly random** number between 1 and n, independent of h(1).
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  - …
  - **h(M)** is also a **uniformly random** number between 1 and n, independent of h(1), h(2), …, h(M-1).

A uniformly random hash function is not a good idea.

- In order to store/evaluate a uniformly random hash function, we'd use a lookup table:

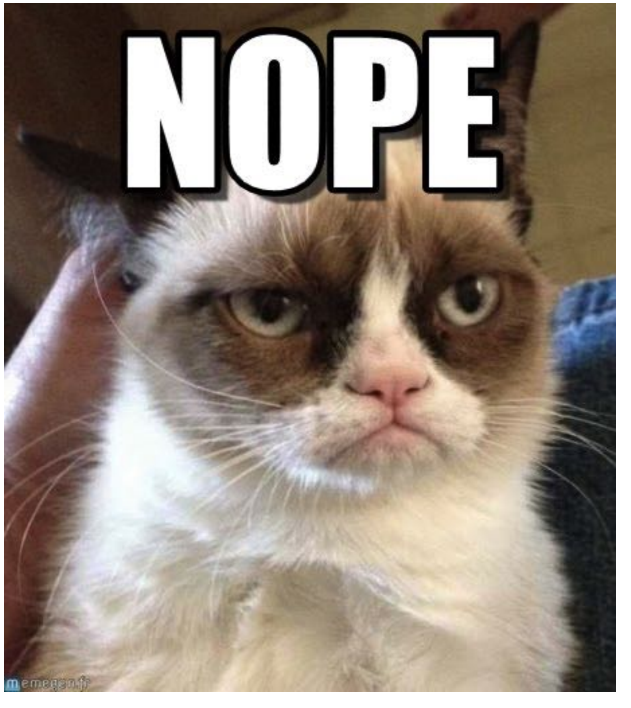
All of the M things in the universe

x	h(x)
AAAAAA	1
AAAAAB	5
AAAAAC	3
AAAAAD	3
…	
ZZZZZY	7
ZZZZZZ	3

- Each value of h(x) takes log(n) bits to store.
- Storing M such values requires Mlog(n) bits.
- In contrast, direct addressing (initializing a bucket for every item in the universe) requires only M bits….

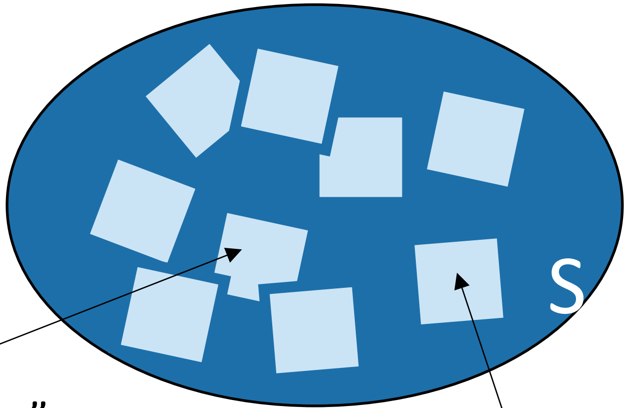
Could we store a uniformly random h without using a lookup table?

- Maybe there's a different way to store h that uses less space?



Aside: description length

- Say I have a set S with s things in it.
- I get to write down the elements of S however I like, in binary using b bits.
- Then $b \geq \log(s)$:
  - There are 2^b binary strings of length b.
  - I need to have at least as many strings as I have items in S.
  - So $s \leq 2^b$ aka $b \geq \log(s)$

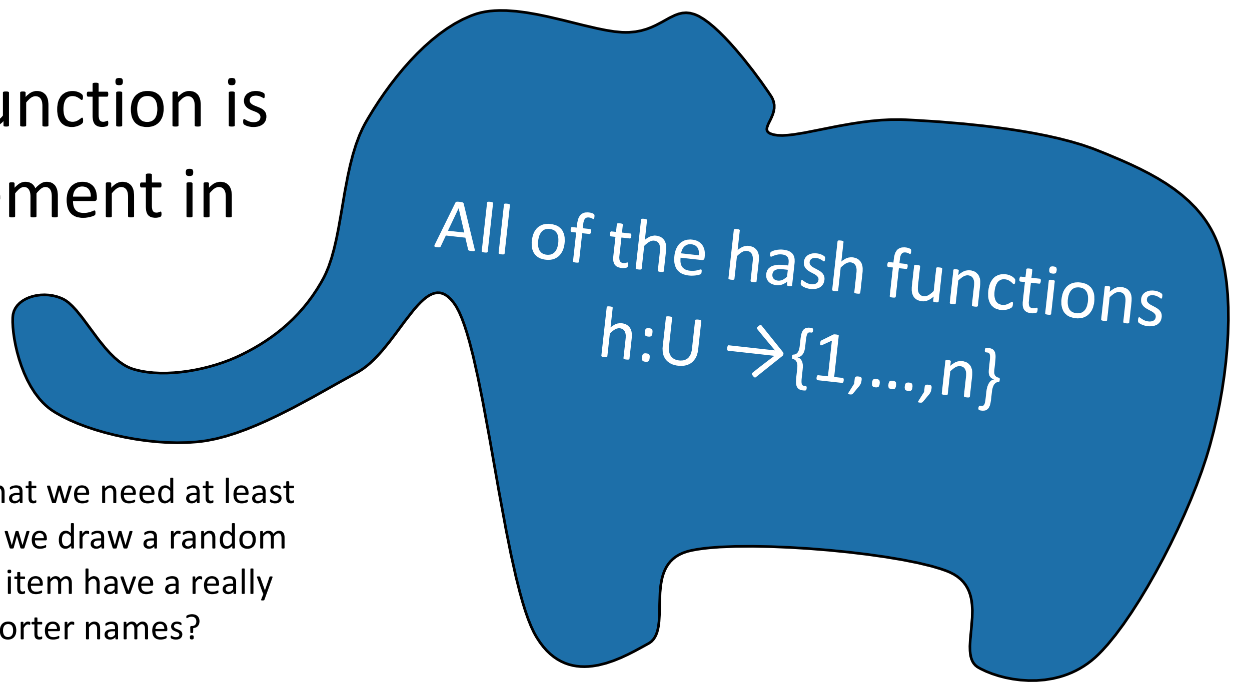


I'll call this one "Fido"
Or, 011011

This one is named "Hercules"
Or, 101111

We need Mlog(n) bits to store a random hash function h:U -> {1,…,n}

- Say that this elephant-shaped blob represents the set of all hash functions.
- It has size n^M. (Really big!)
- To write down a random hash function, we need $\log(n^M) = M\log(n)$ bits.
- A random hash function is just a random element in this set.

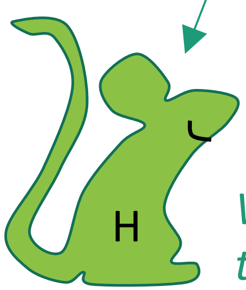


Technically we should argue that we need at least Mlog(n) bits on average when we draw a random hash function… why can't one item have a really long name and others have shorter names?

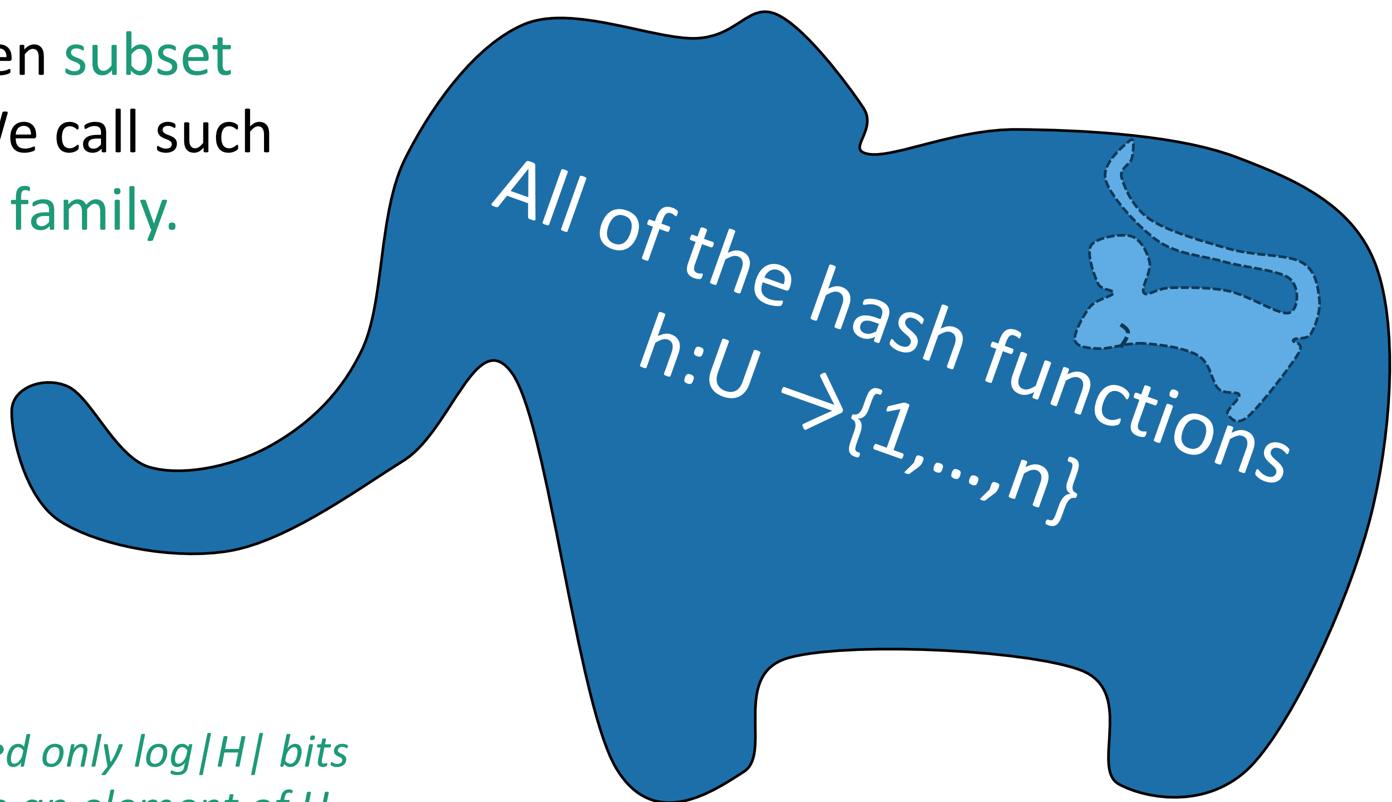
Solution

- Pick from a smaller set of functions.

A cleverly chosen subset of functions. We call such a subset a hash family.



We need only log|H| bits to store an element of H.

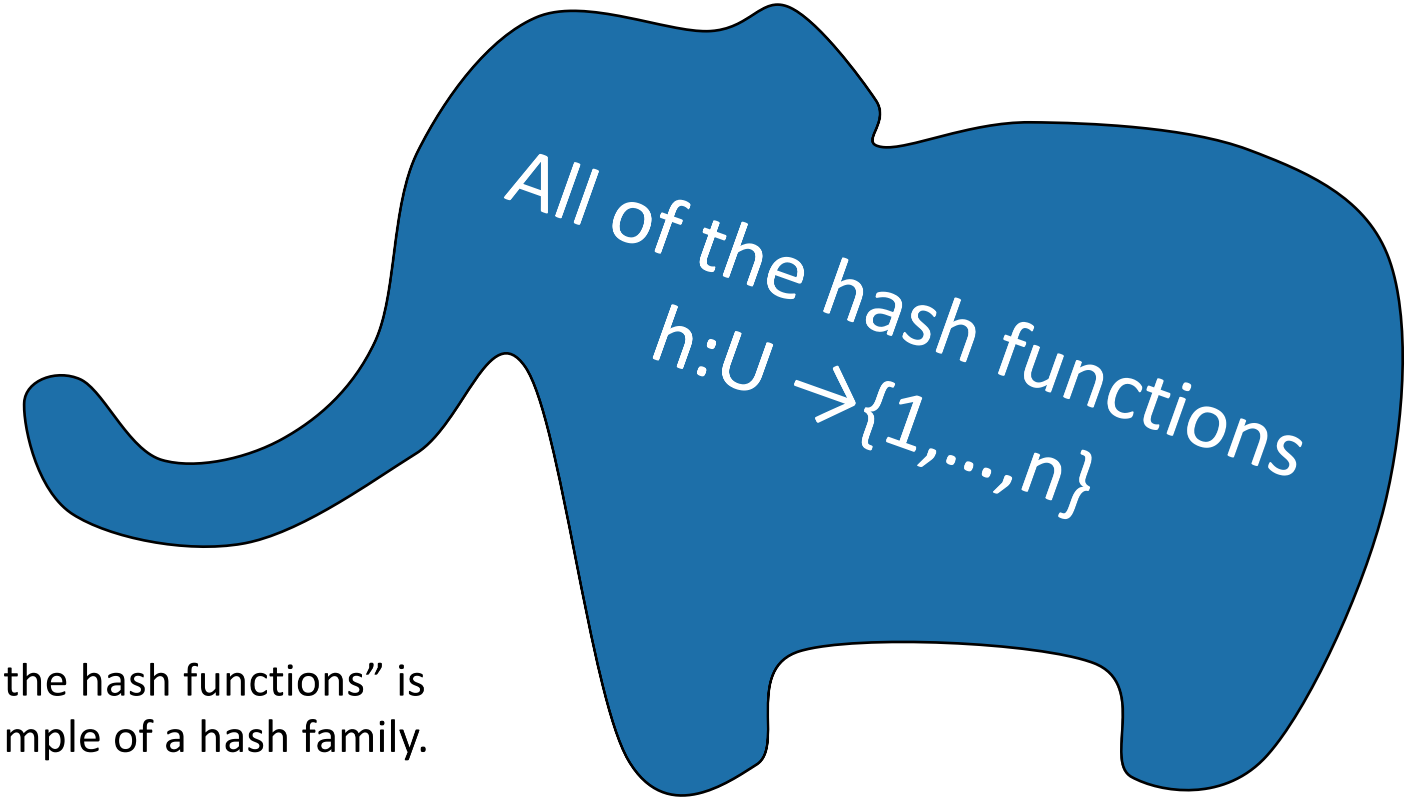


Outline

- **Hash tables** are another sort of data structure that allows fast INSERT/DELETE/SEARCH.
  - like self-balancing binary trees
  - The difference is we can get better performance in expectation by using randomness.
- **Hash families** are the magic behind hash tables.
- **Universal hash families** are even more magic.

Hash families

- A hash family is a collection of hash functions.

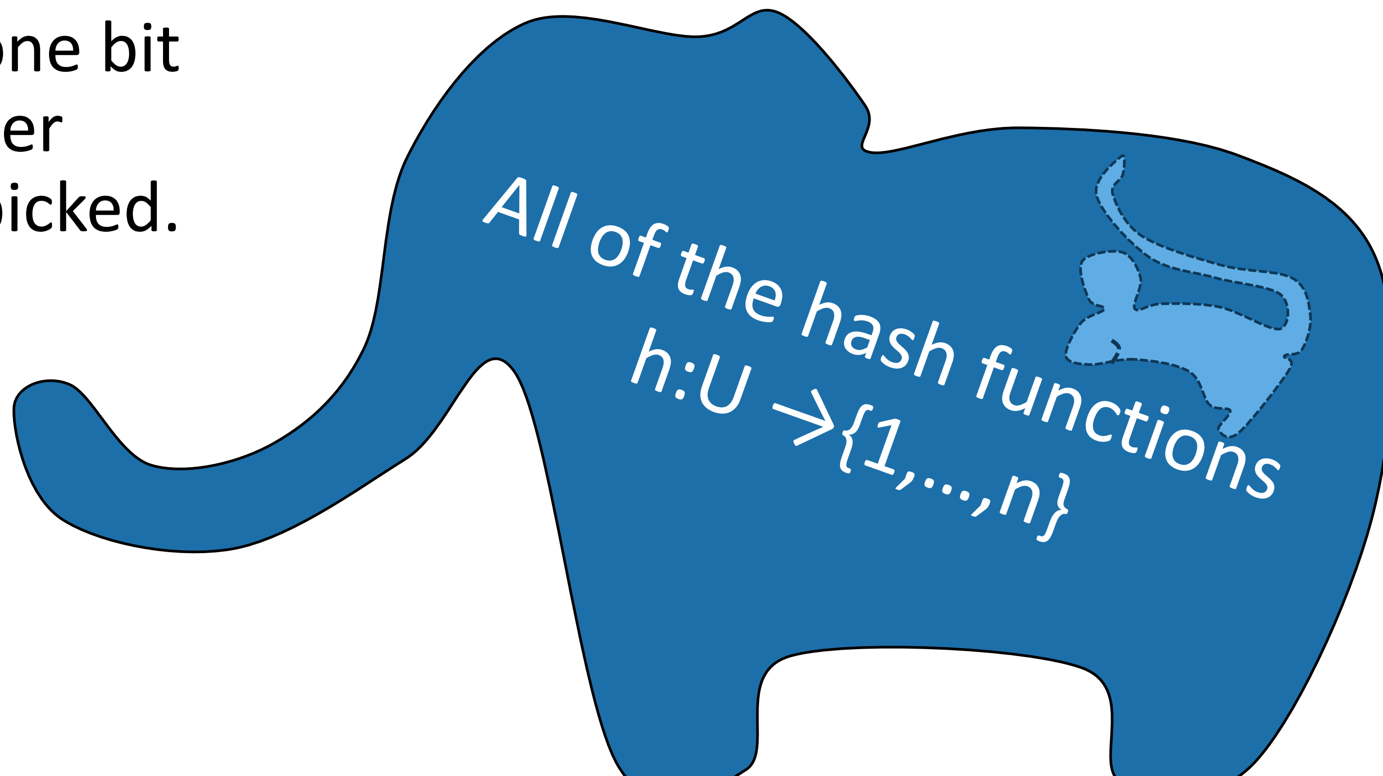


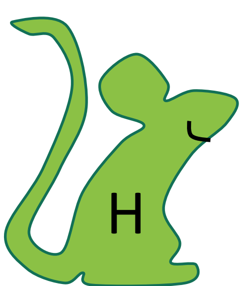
”All of the hash functions” is an example of a hash family.

Example:
a smaller hash family

This is still a terrible idea!
Don't use this example!
For pedagogical purposes only!

- H = { function which returns the least sig. digit,
  function which returns the most sig. digit }
- Pick h in H at random.
- Store just one bit to remember which we picked.





The game

h_0 = Most_significant_digit
h_1 = Least_significant_digit
H = {h_0, h_1}

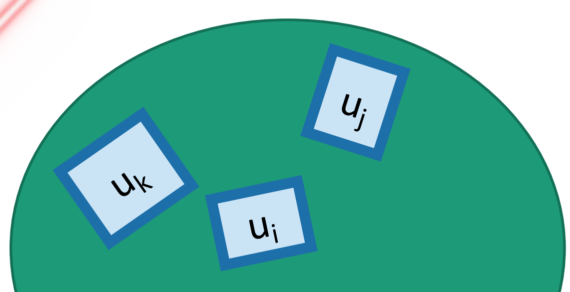
2. You, the algorithm, chooses a **random** hash function $h: U \rightarrow \{0, \ldots, 9\}$. Choose it randomly from H.

I picked h_1

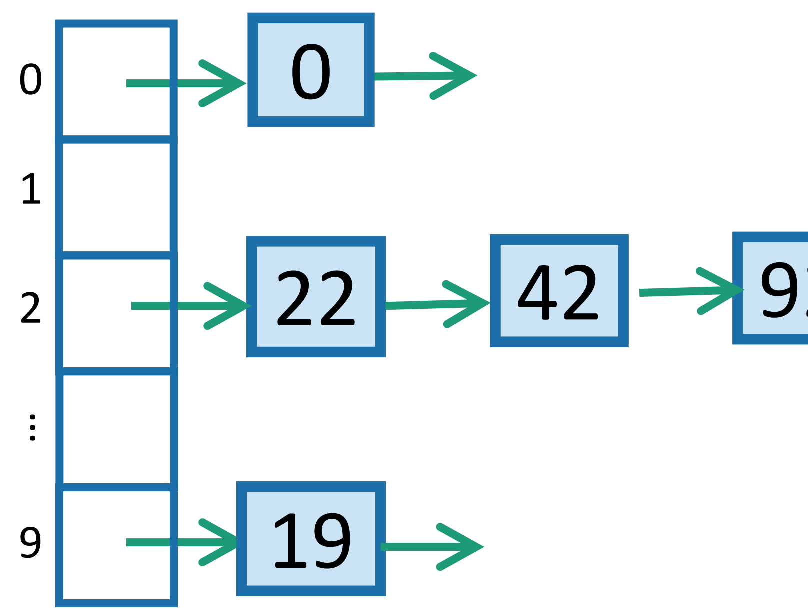
1. An adversary (who knows H) chooses any n items $u_1, u_2, \ldots, u_n \in U$, and any sequence of INSERT/DELETE/SEARCH operations on those items.

19 22 42 92 0

INSERT 19, INSERT 22, INSERT 42, INSERT 92, INSERT 0, SEARCH 42, DELETE 92, SEARCH 0, INSERT 92



3. HASH IT OUT #hashpuns



This is not a very good hash family

- H = { function which returns least sig. digit,

  function which returns most sig. digit }
- On the previous slide, the adversary could have been a lot more adversarial…

The game

h_0 = Most_significant_digit
h_1 = Least_significant_digit
H = $\{h_0, h_1\}$

2. You, the algorithm, chooses a **random** hash function $h: U \to \{0, \ldots, 9\}$. Choose it randomly from H.

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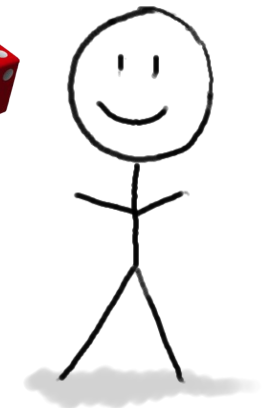
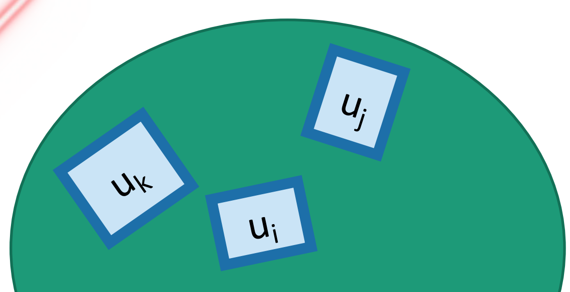
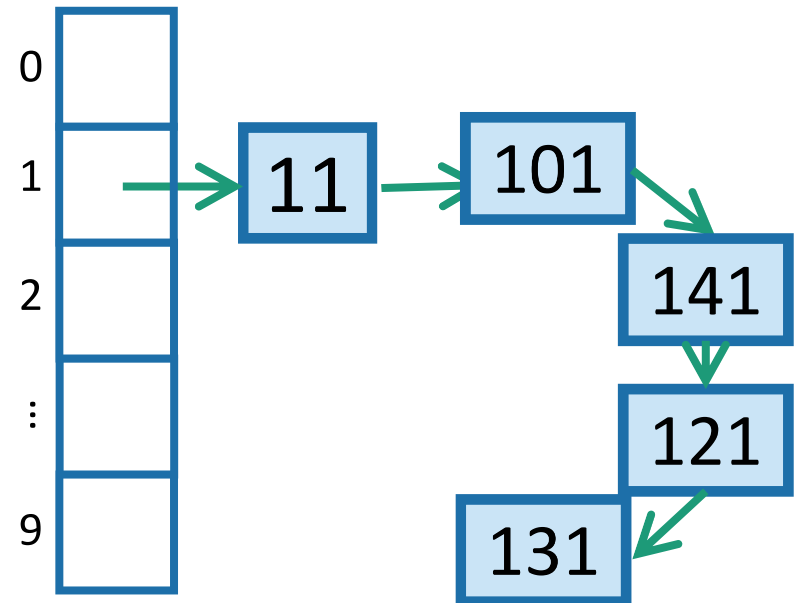


1. An adversary (who knows H) chooses any n items $u_1, u_2, \ldots, u_n \in U$, and any sequence of INSERT/DELETE/SEARCH operations on those items.

3. HASH IT OUT #hashpuns

101 11 121 141 131





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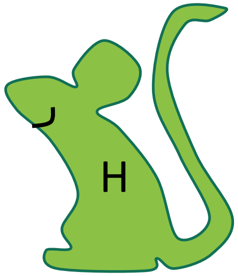
  - like self-balancing binary trees
  - The difference is we can get better performance in expectation by using randomness.

- **Hash families** are the magic behind hash tables.

- **Universal hash families** are even more magic.

How to pick the hash family?

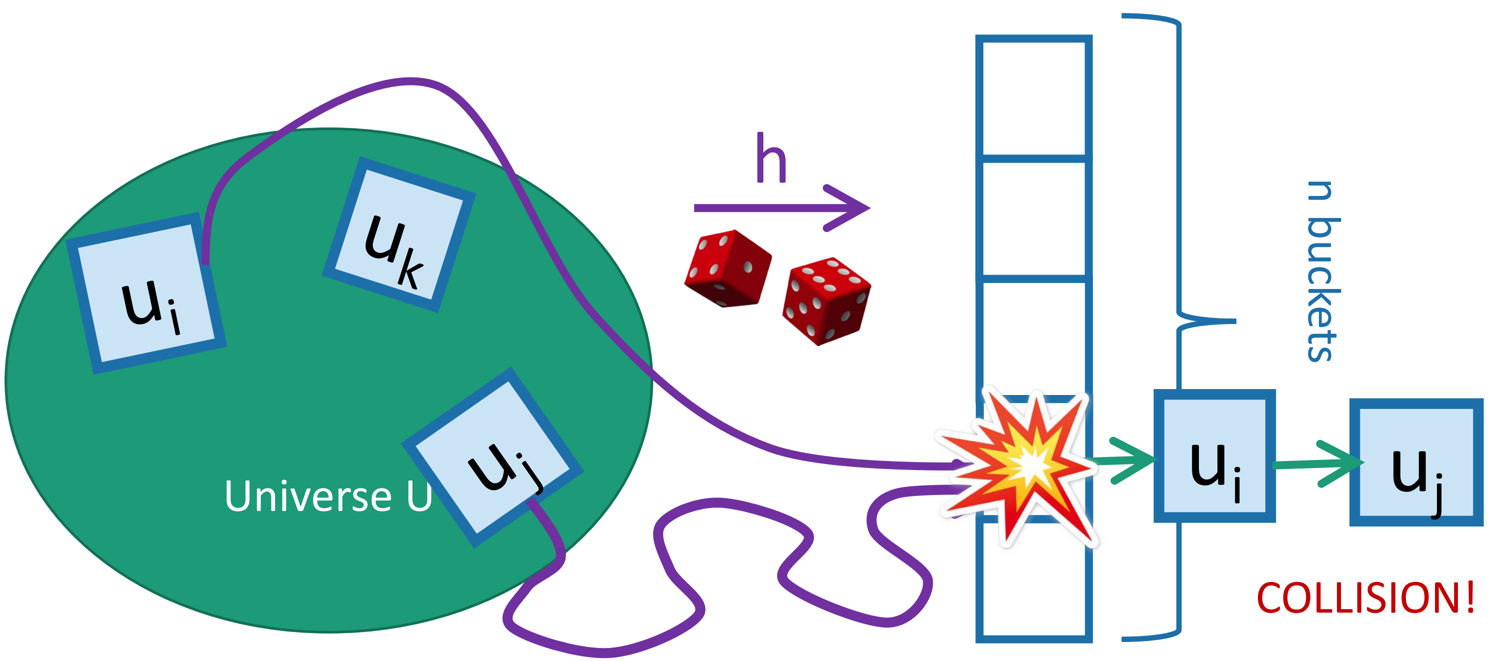
- Definitely not like in that example.
- Let's go back to that computation from earlier….



Expected number of items in u_i's bucket?

- $E[\ ] = \sum_{j=1}^{n} P\{\, h(u_i) = h(u_j) \}$
- $\quad = 1 + \sum_{j \neq i} P\{\, h(u_i) = h(u_j) \}$
- $\quad = 1 + \sum_{j \neq i} 1/n$
- $\quad = 1 + \frac{n-1}{n} \leq 2.$

All that we needed was that this is 1/n



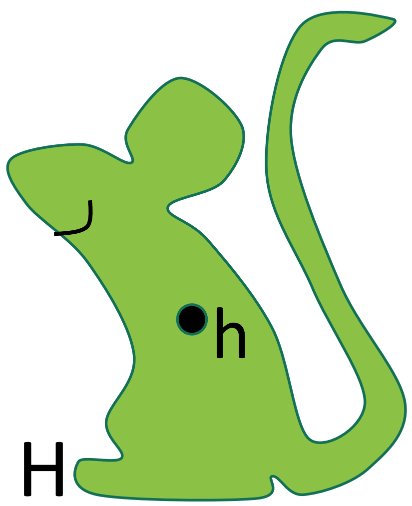
Strategy

- Pick a small hash family H, so that when I choose h randomly from H,

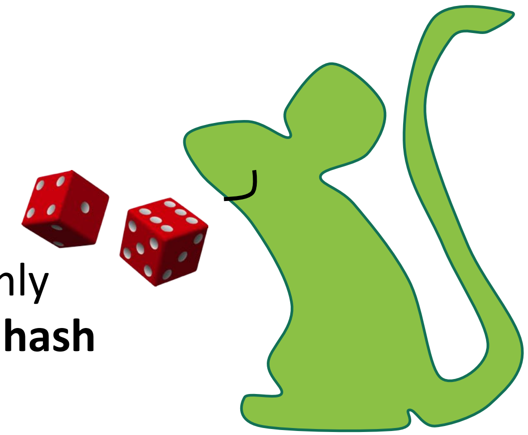
$$\text{for all } u_i, u_j \in U \quad \text{with } u_i \neq u_j,$$
$$P_{h \in H}\{\, h(u_i) = h(u_j) \} \leq \frac{1}{n}$$

In English: fix any two elements of U. The probability that they collide under a random h in H is small.

- A hash family H that satisfies this is called a **universal hash family.**
- Then we still get O(1)-sized buckets in expectation.
- But now the space we need is log(|H|) bits.
  - Hopefully pretty small!

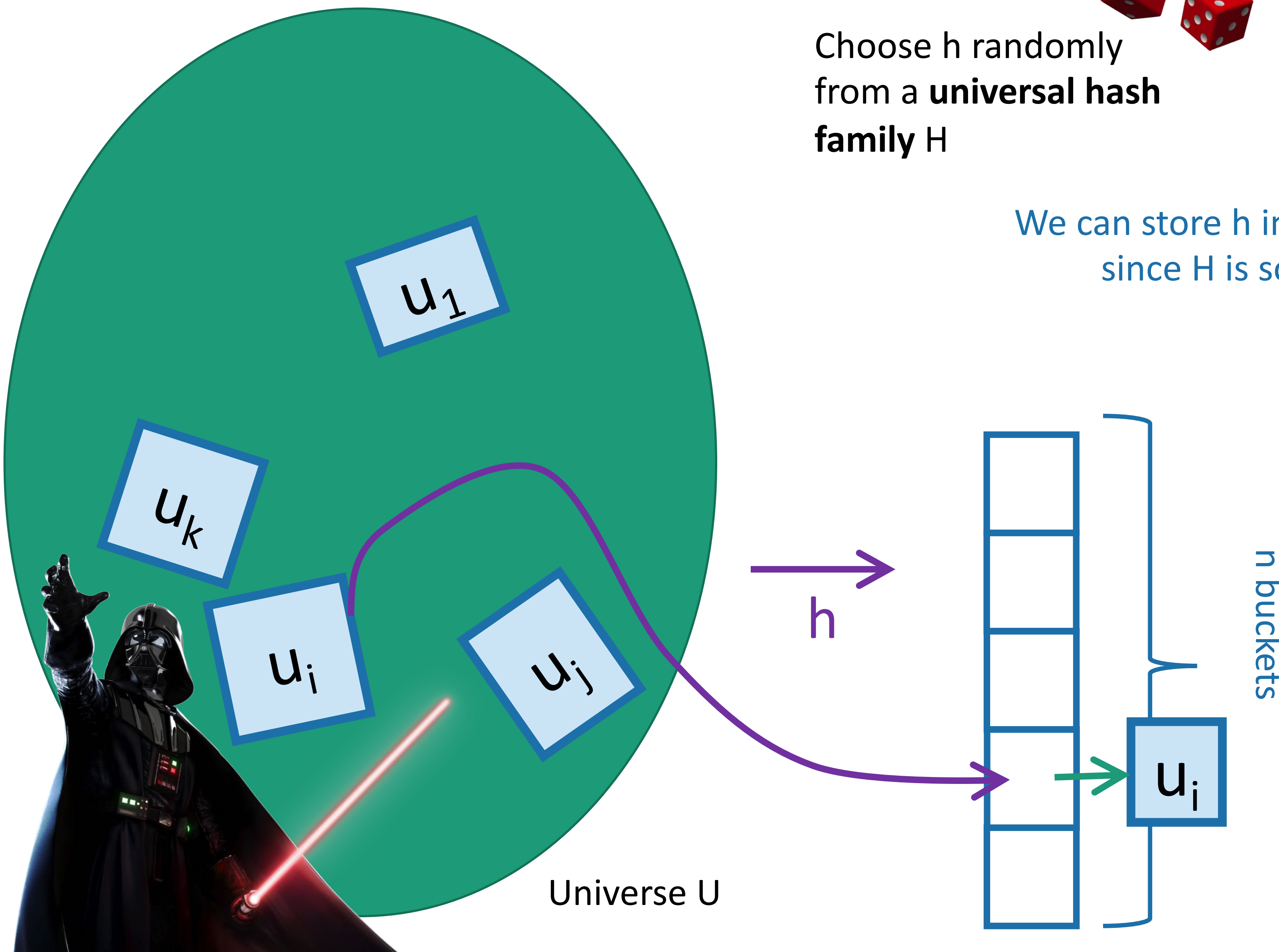


So the whole scheme will be



Choose h randomly from a **universal hash family** H

We can store h in small space since H is so small.



Probably these buckets will be pretty balanced.

Universal hash family

- H is a ***universal hash family*** if, when h is chosen uniformly at random from H,

$$\text{for all } u_i, u_j \in U \quad \text{with } u_i \neq u_j,$$
$$P_{h\in H}\{\, h(u_i) = h(u_j)\} \leq \frac{1}{n}$$

- Pick a small hash family H, so that when I choose h randomly from H,

for all $u_i, u_j \in U$ with $u_i \neq u_j$,

$$P_{h \in H}\{\, h(u_i) = h(u_j)\} \leq \frac{1}{n}$$

Example

- H is all of the functions $h: U \to \{1, \ldots, n\}$
  - We saw this earlier – it corresponds to picking a uniformly random hash function.
  - Unfortunately this H is really really large.

- Pick a small hash family H, so that when I choose h randomly from H,

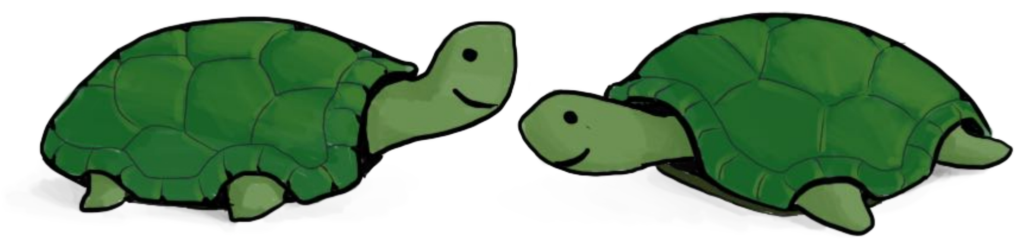
Non-example

for all $u_i, u_j \in U$ with $u_i \neq u_j$,

$$P_{h \in H}\{\, h(u_i) = h(u_j)\} \leq \frac{1}{n}$$

- h_0 = Most_significant_digit
- h_1 = Least_significant_digit
- H = $\{h_0, h_1\}$

Prove that this choice of H is NOT a universal hash family!



- Pick a small hash family H, so that when I choose h randomly from H,

for all $u_i, u_j \in U$ with $u_i \neq u_j$,

$$P_{h \in H}\{\, h(u_i) = h(u_j)\} \leq \frac{1}{n}$$

Non-example

- h_0 = Most_significant_digit
- h_1 = Least_significant_digit
- H = {h_0, h_1}

NOT a universal hash family:

$$P_{h \in H}\{h(101) = h(111)\} = 1 > \frac{1}{10}$$

A small universal hash family??

- Here's one:
  - Pick a prime $p \geq M$.
  - Define

$$f_{a,b}(x) = ax + b \quad mod\ p$$

$$h_{a,b}(x) = f_{a,b}(x) \quad mod\ n$$

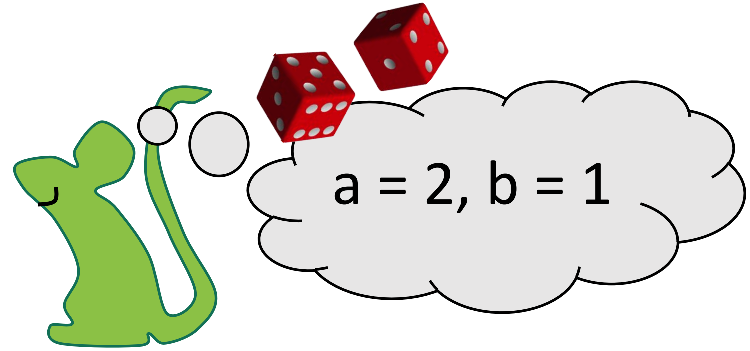
  - Define:

$$H = \{\ h_{a,b}(x)\ :\ a \in \{1, \ldots, p-1\}, b \in \{0, \ldots, p-1\}\ \}$$

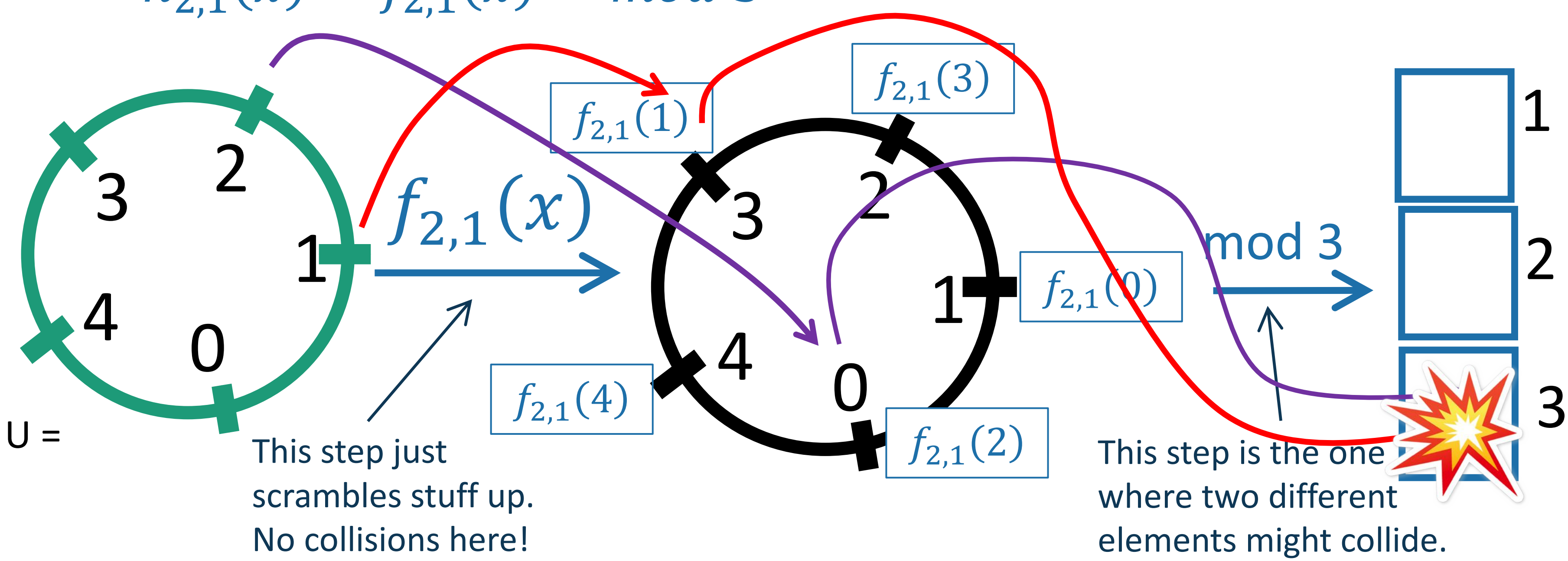
- Claim:

  H is a universal hash family.

Say what?

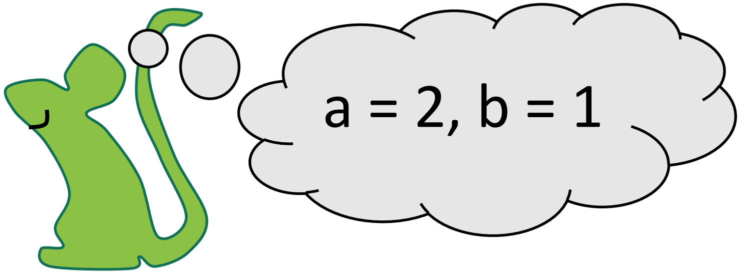


- Example: M = p = 5, n = 3
- To draw h from H:
  - Pick a random a in {1,…,4}, b in {0,…,4}
- As per the definition:
  - $f_{2,1}(x) = 2x + 1 \quad mod\ 5$
  - $h_{2,1}(x) = f_{2,1}(x) \quad mod\ 3$



Ignoring why this is a good idea

- Can we store h with small space?

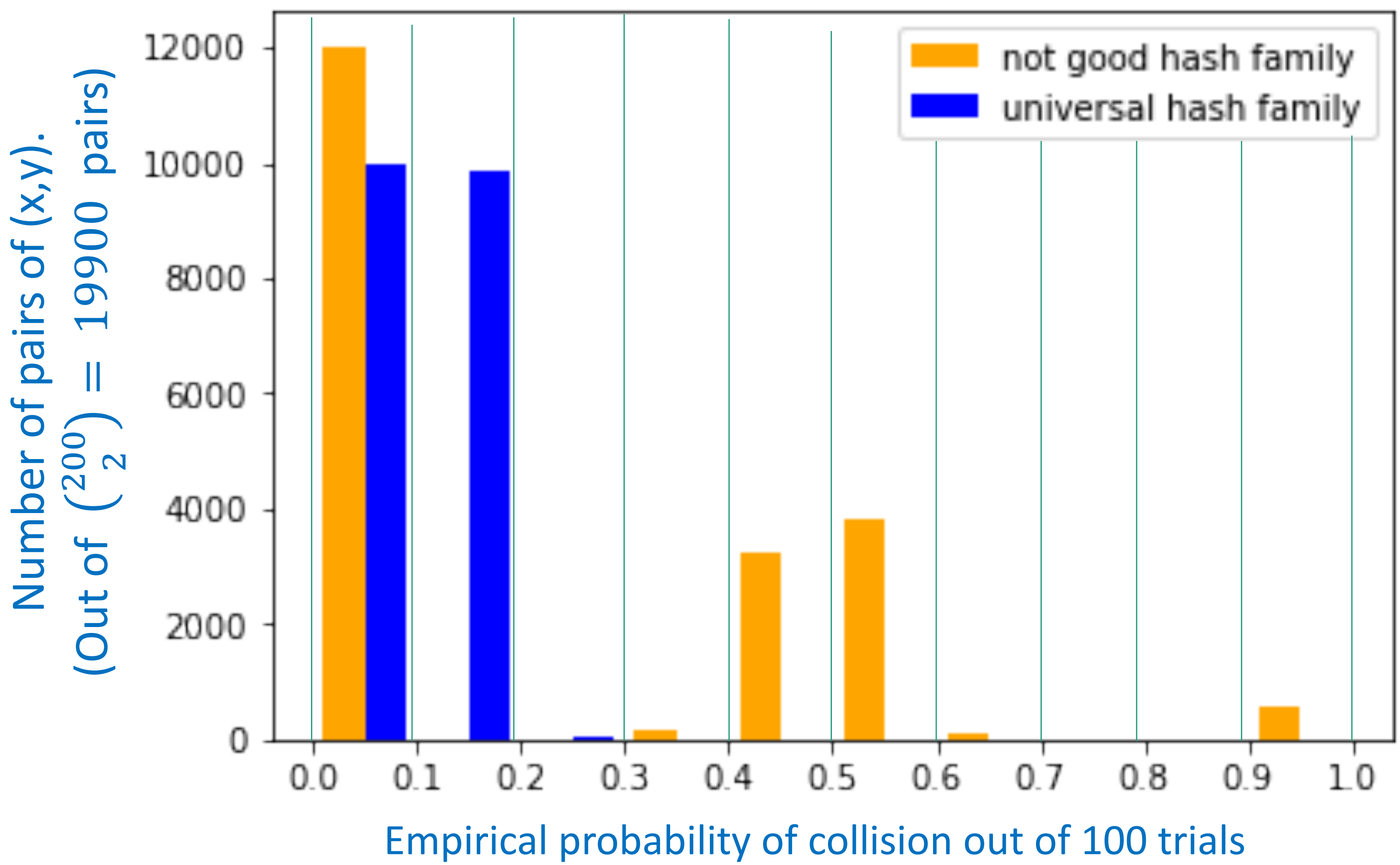


- Just need to store two numbers:
  - a is in {1,…,p-1}
  - b is in {0,…,p-1}
  - So about 2log(p) bits
  - By our choice of p, that's O(log(M)) bits.

Compare: direct addressing was M bits!
Twitter example: log(M) = 140 log(128) = **980** vs M = **128^{140}**

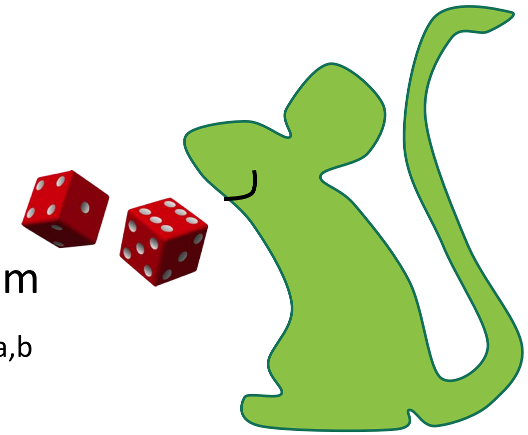
But let’s check that it **does** work

M=200, n=10

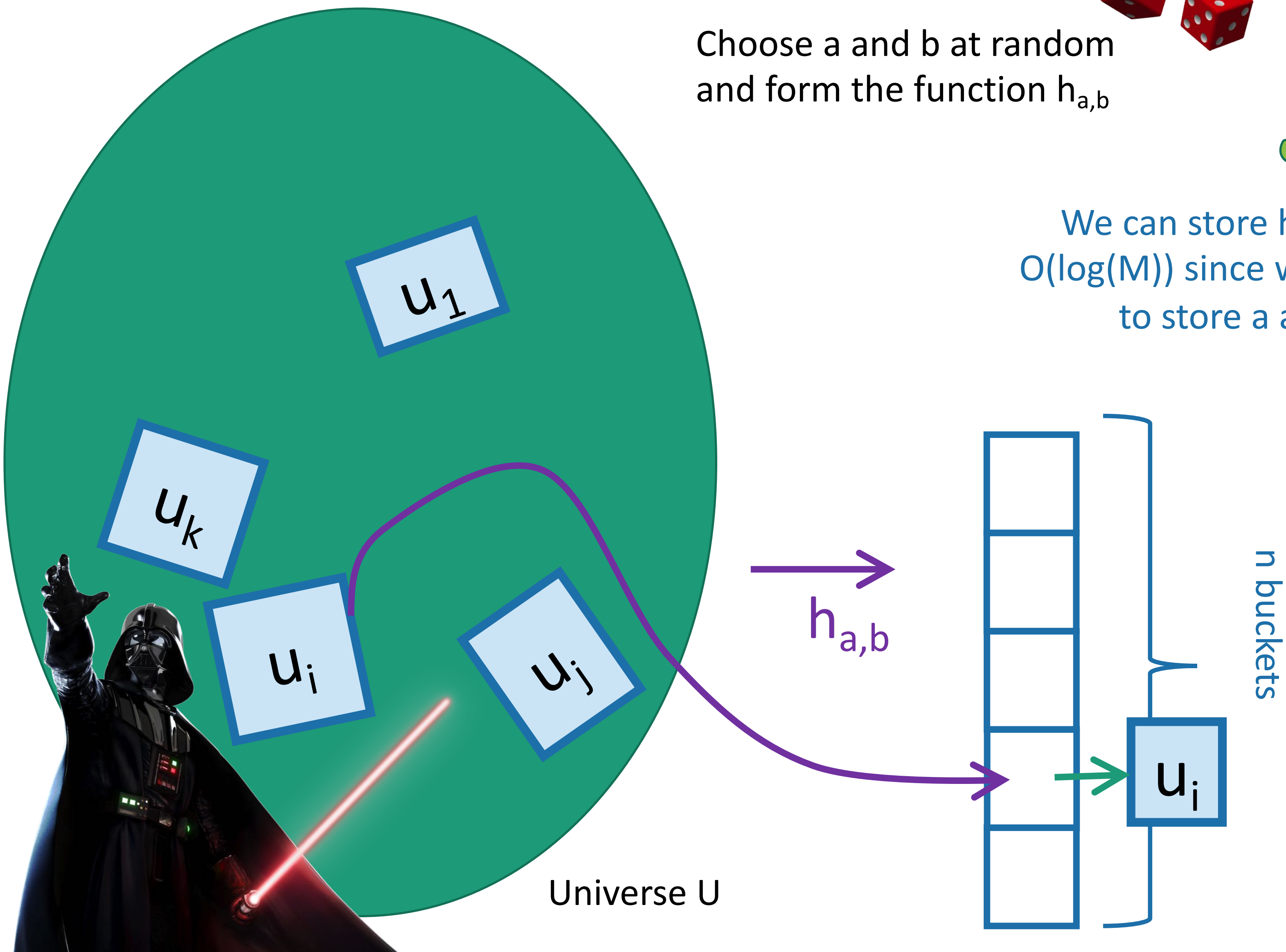


So the whole scheme will be

Choose a and b at random and form the function $h_{a,b}$



We can store h in space O(log(M)) since we just need to store a and b.



Probably these buckets will be pretty balanced.

Outline

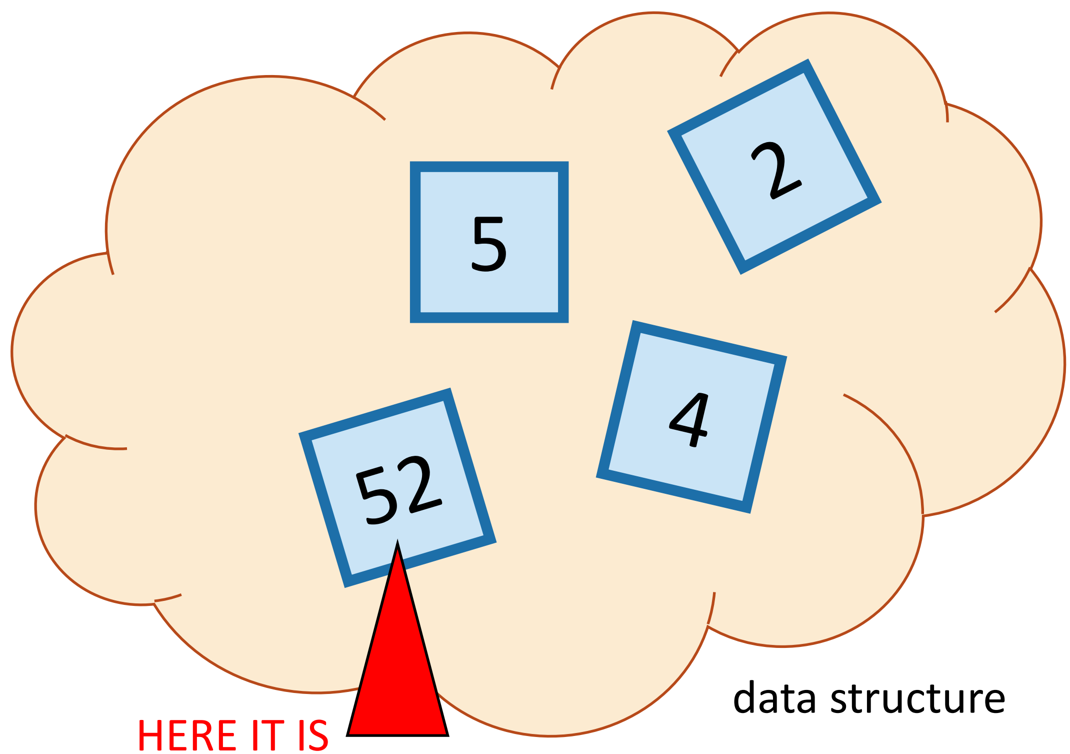
- **Hash tables** are another sort of data structure that allows fast INSERT/DELETE/SEARCH.
  - like self-balancing binary trees
  - The difference is we can get better performance in expectation by using randomness.
- **Hash families** are the magic behind hash tables.
- **Universal hash families** are even more magic.

Recap

Want O(1) INSERT/DELETE/SEARCH

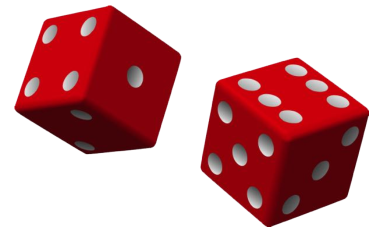
- We are interesting in putting nodes with keys into a data structure that supports fast INSERT/DELETE/SEARCH.

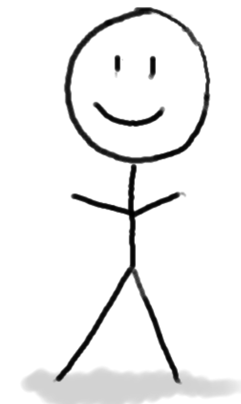
- INSERT 5
- DELETE 4
- SEARCH 52



We studied this game

2. You, the algorithm, chooses a **random** hash function $h: U \rightarrow \{1, \ldots, n\}$.

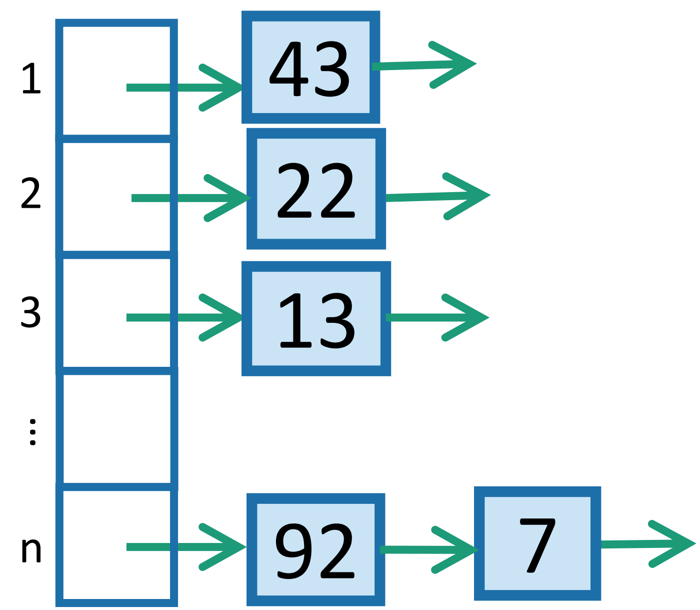




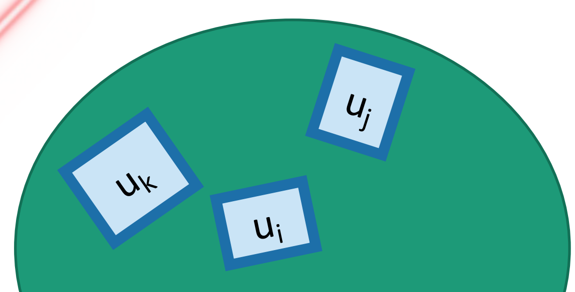
1. An adversary chooses any n items $u_1, u_2, \ldots, u_n \in U$, and any sequence of L INSERT/DELETE/SEARCH operations on those items.

13 22 43 92 7

3. **HASH IT OUT**



INSERT 13, INSERT 22, INSERT 43, INSERT 92, INSERT 7, SEARCH 43, DELETE 92, SEARCH 7, INSERT 92



Uniformly random h was good

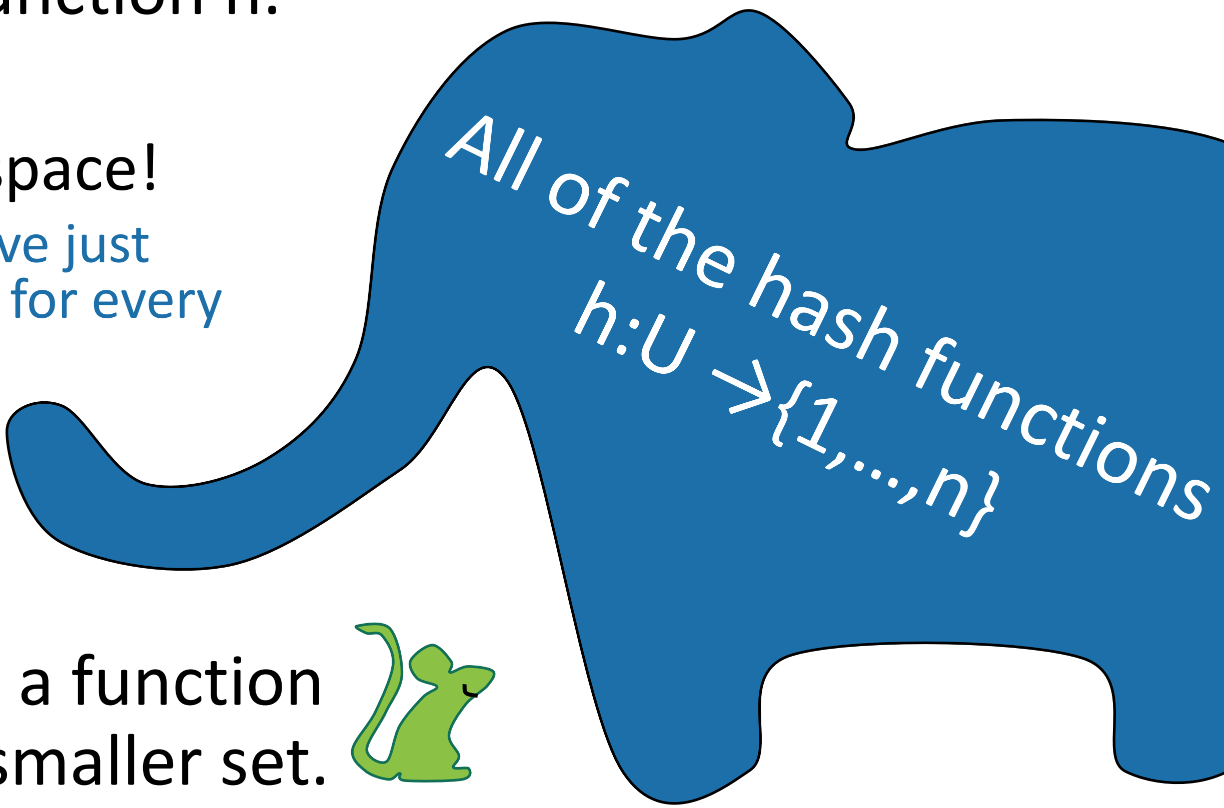
- If we choose h uniformly at random,

$$\text{for all } u_i, u_j \in U \quad \text{with } u_i \neq u_j,$$
$$P_{h \in H}\{\, h(u_i) = h(u_j)\} \leq \frac{1}{n}$$

- That was enough to ensure that all INSERT/DELETE/SEARCH operations took O(1) time in expectation, even on adversarial inputs.

Uniformly random h was bad

- If we actually want to implement this, we have to store the hash function h.



- That takes a lot of space!
    - We may as well have just initialized a bucket for every single item in U.

- Instead, we chose a function randomly from a smaller set.

We needed a smaller set

that still has this property

- If we choose h uniformly at random in H,

$$\text{for all } u_i, u_j \in U \quad \text{with } u_i \neq u_j,$$

$$P_{h \in H}\{\, h(u_i) = h(u_j) \} \leq \frac{1}{n}$$

This was all we needed to make sure that the buckets were balanced in expectation!

- We call any set with that property a

universal hash family.

- We gave an example of a really small one ☺

Hashing a universe of size M into n buckets, where at most n of the items in M ever show up.

Conclusion:

- We can build a hash table that supports INSERT/DELETE/SEARCH in O(1) expected time
- Requires O(n log(M)) bits of space.
  - O(n) buckets
  - O(n) items with log(M) bits per item
  - O(log(M)) to store the hash function