Cây khung nhỏ nhất

Minimum Spanning Trees

Nội dung

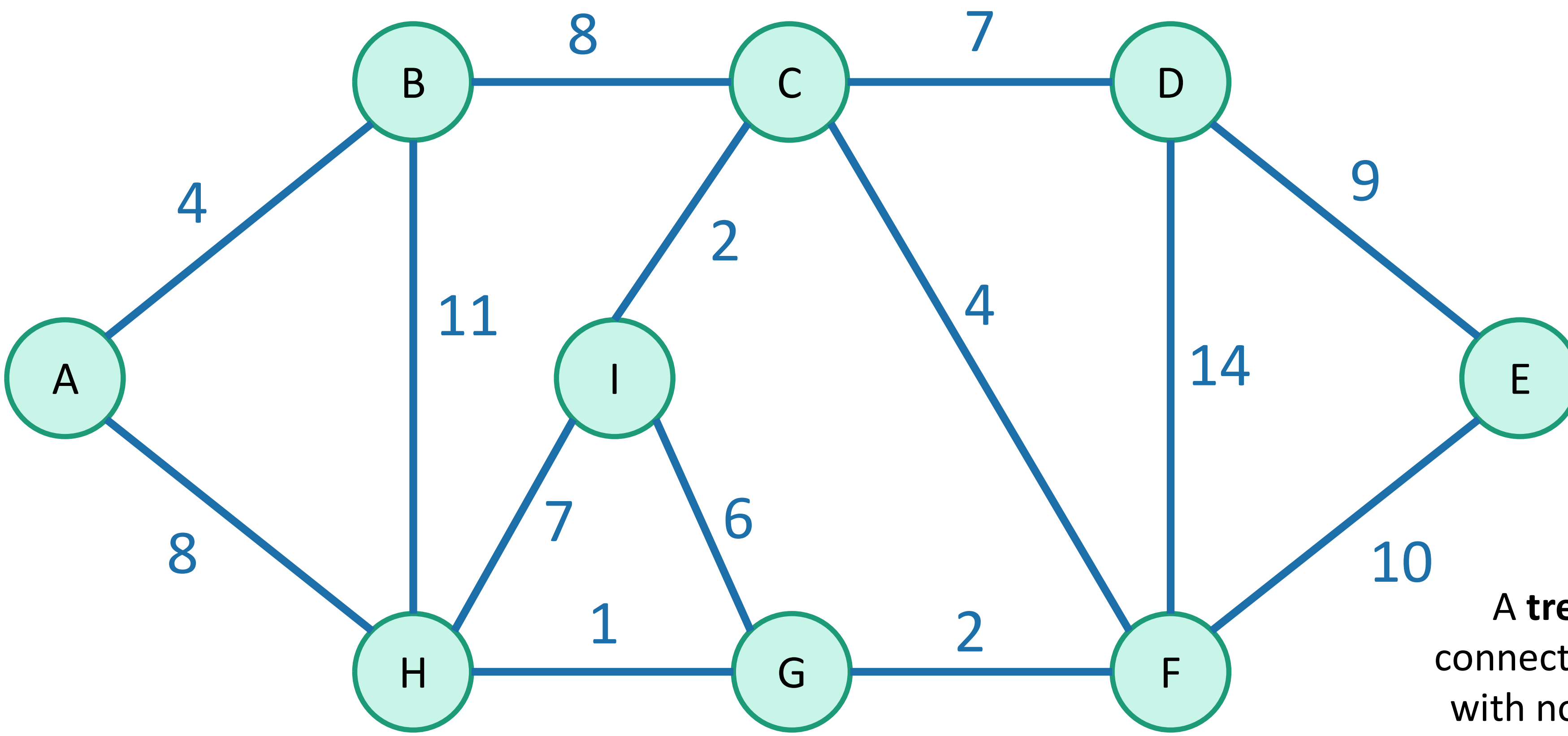
- Khái niệm về cây khung nhỏ nhất (Minimum Spanning Tree)
- Một số khái niệm về lý thuyết đồ thị
- Thuật toán tìm cây khung nhỏ nhất
    - Prim's algorithm
    - Kruskal's algorithm

Sử dụng một phần tài liệu bài giảng CS161 Stanford University

For today, we will focus on connected graphs!

Minimum Spanning Tree

Say we have an undirected weighted graph



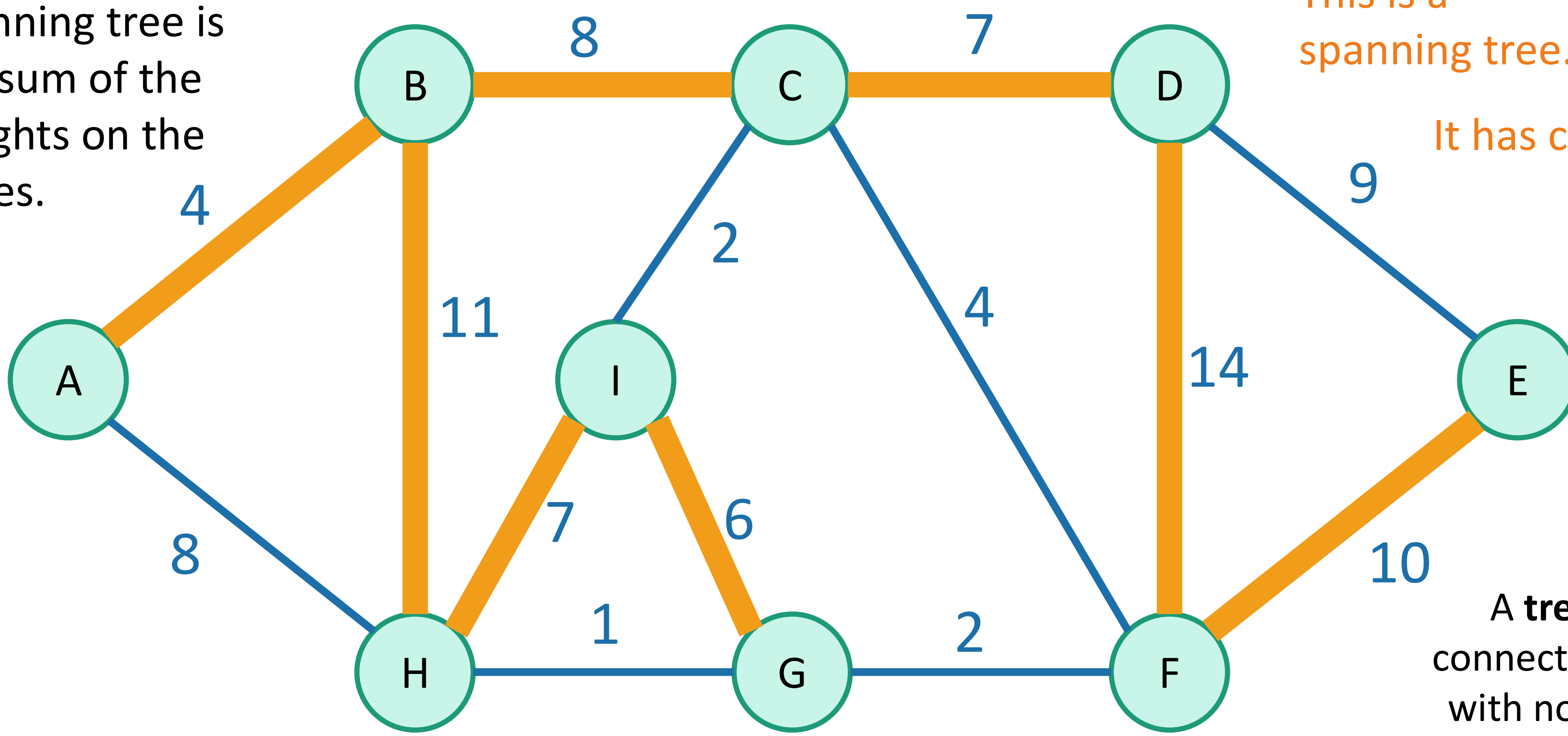
A **tree** is a connected graph with no cycles!

A **spanning tree** is a **tree** that connects all of the vertices.

Minimum Spanning Tree

Say we have an undirected weighted graph

The **cost** of a spanning tree is the sum of the weights on the edges.



This is a spanning tree.

It has cost 67

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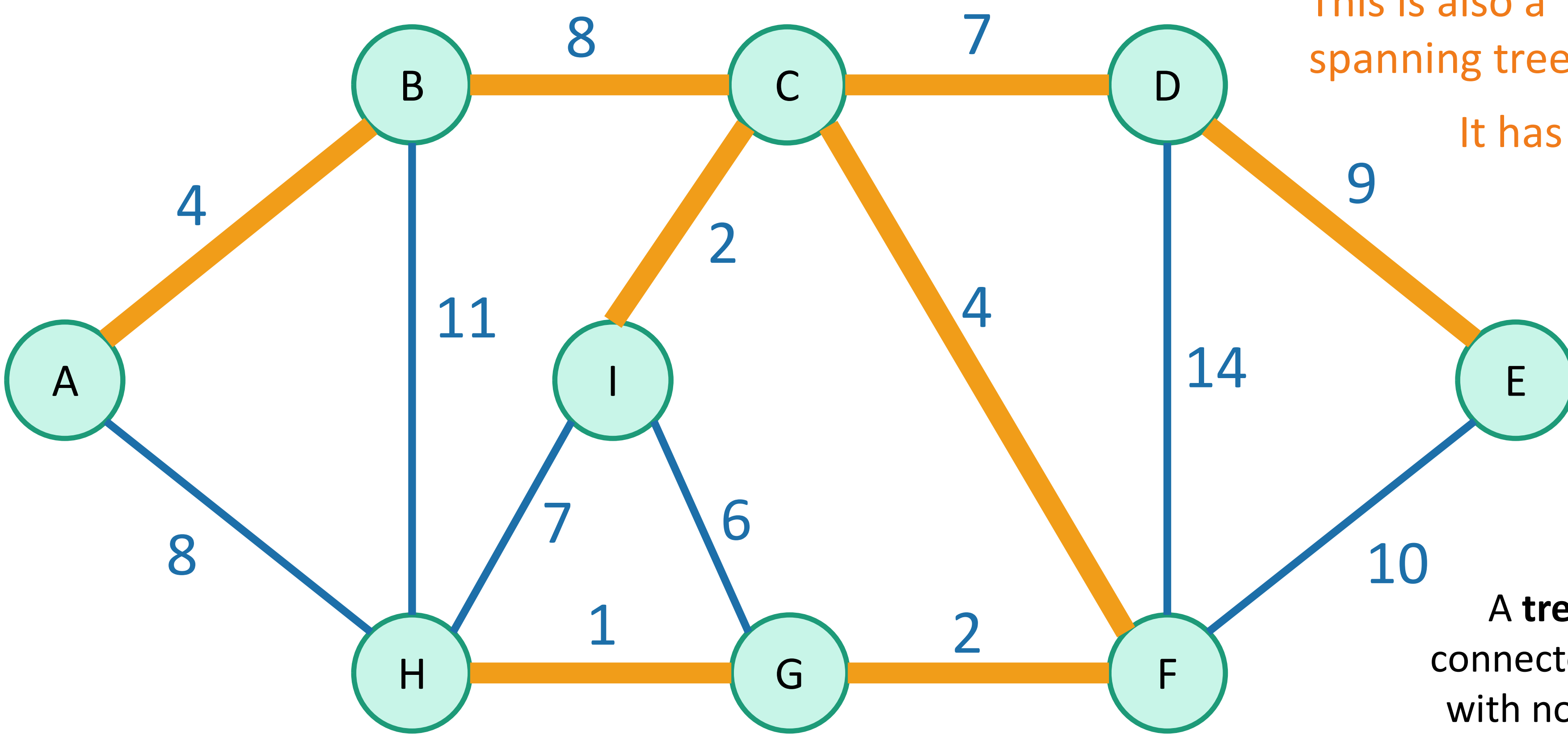
Minimum Spanning Tree

Say we have an undirected weighted graph



This is also a spanning tree.

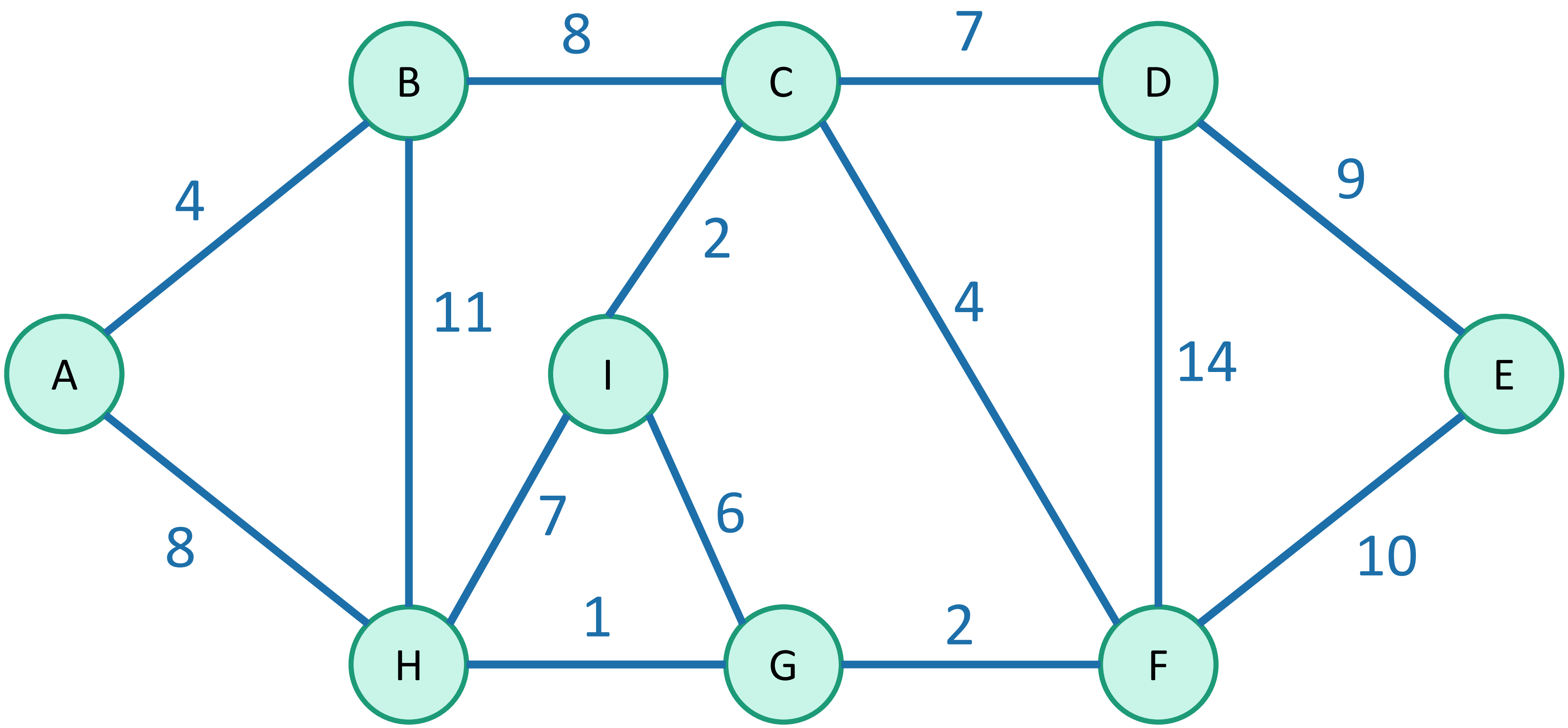
It has cost 37

A **tree** is a connected graph with no cycles!

A **spanning tree** is a **tree** that connects all of the vertices.

Minimum Spanning Tree

Say we have an undirected weighted graph



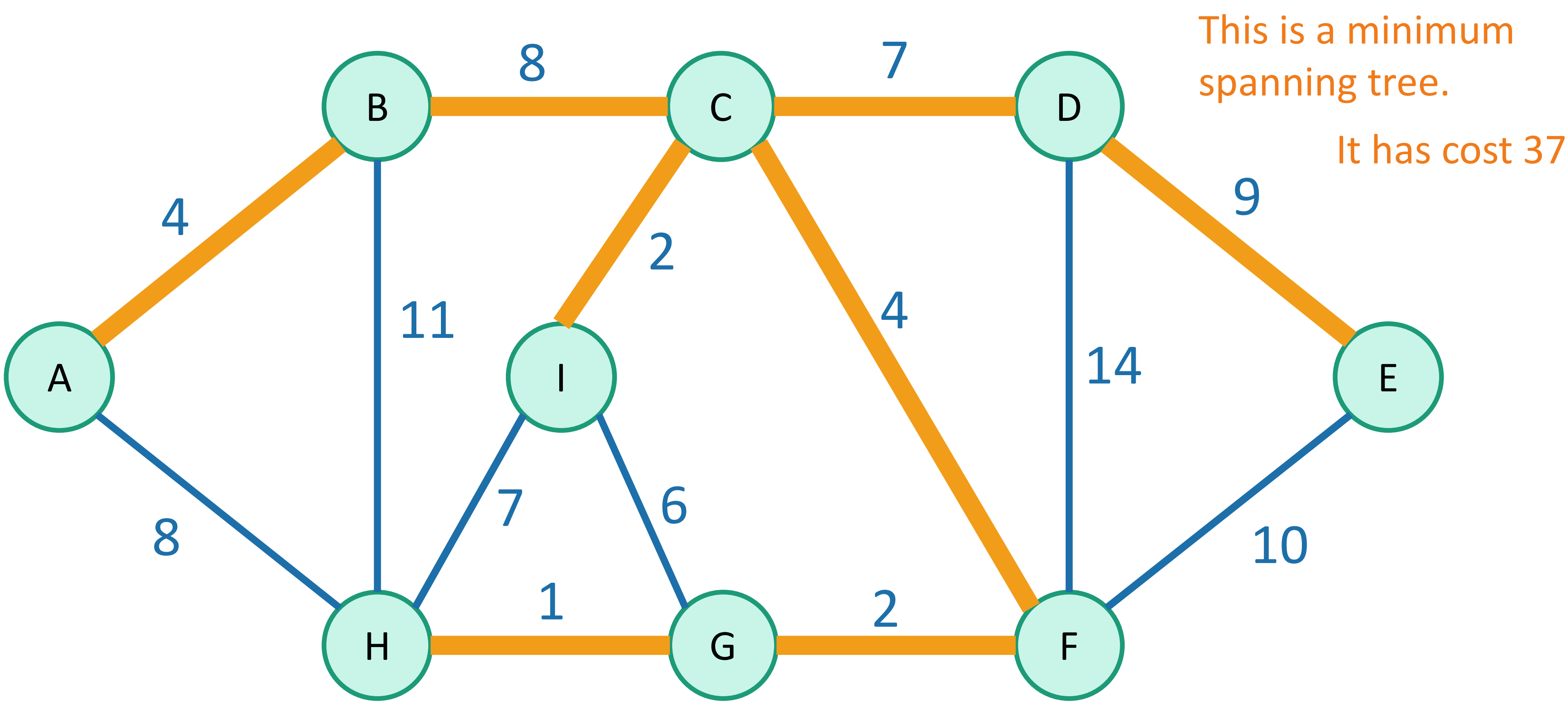
minimum

A **spanning tree** is a **tree** that connects all of the vertices.

of minimum cost

Minimum Spanning Tree

Say we have an undirected weighted graph



minimum

of minimal cost

A **spanning tree** is a **tree** that connects all of the vertices.

Why MSTs?

- Network design
  - Connecting cities with roads/electricity/telephone/…
- cluster analysis
  - eg, genetic distance
- image processing
  - eg, image segmentation
- Useful primitive
  - for other graph algs

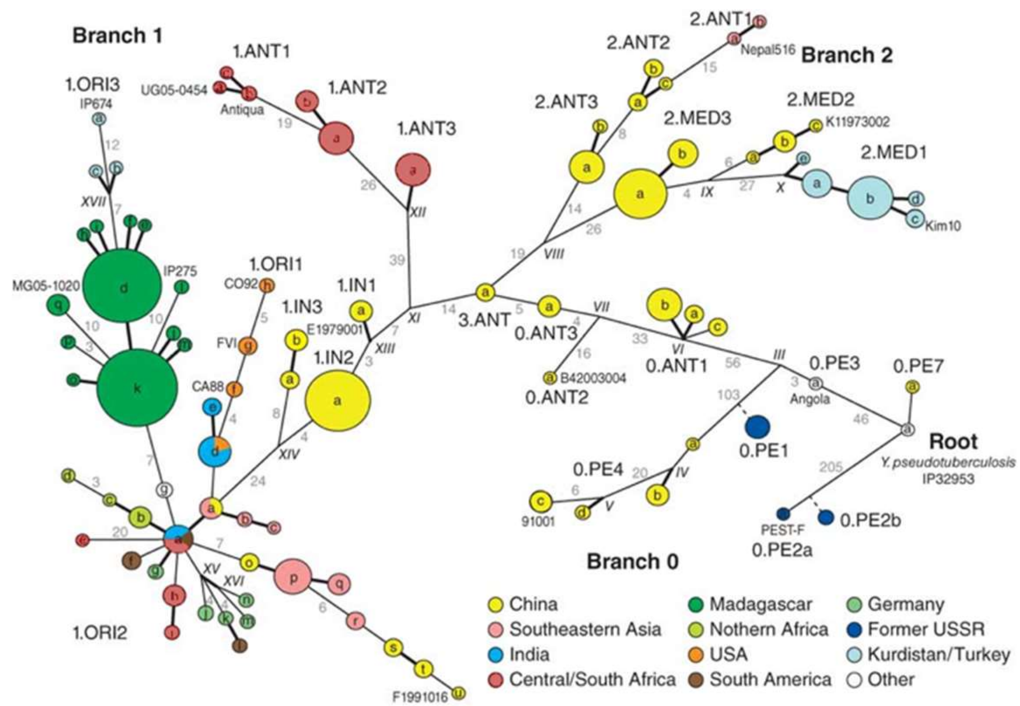
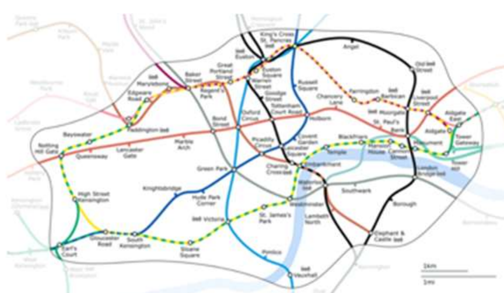


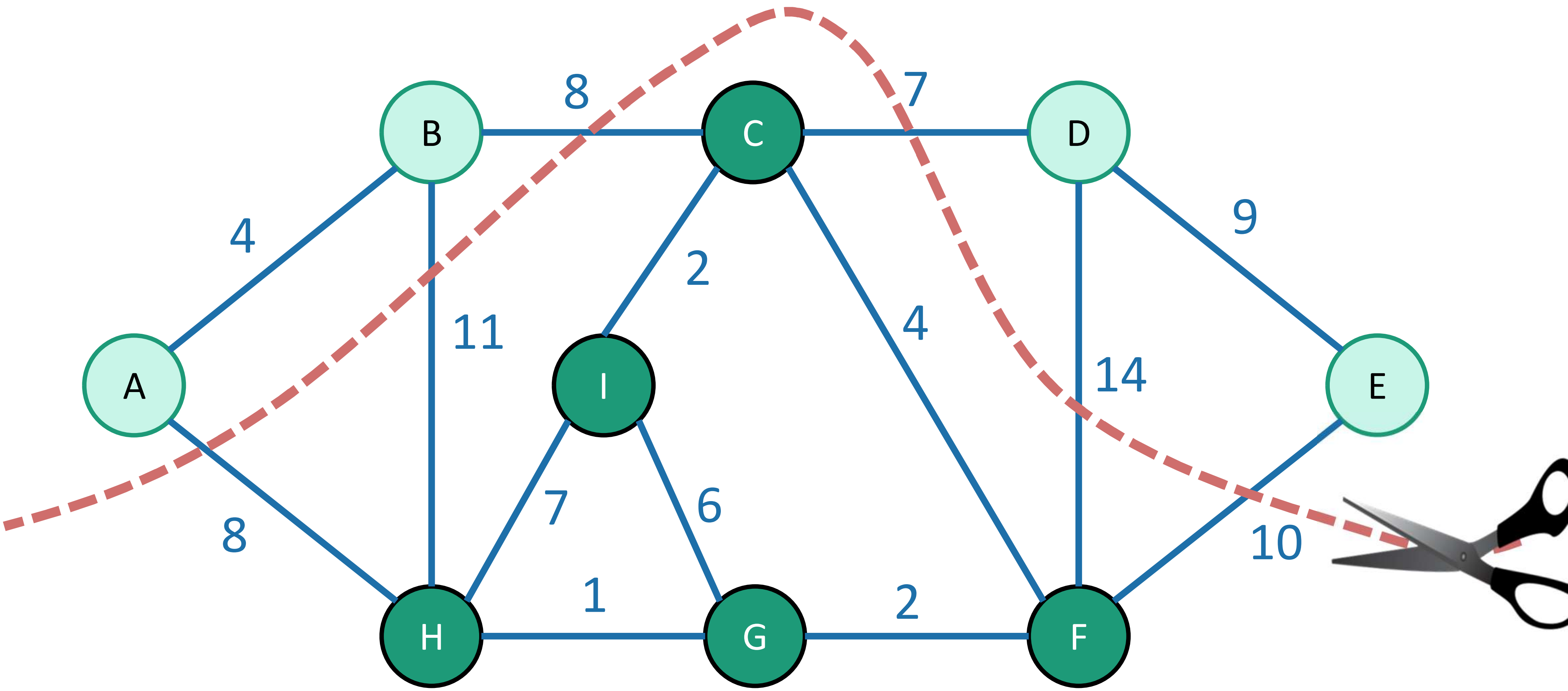
Figure 2: Fully parsimonious minimal spanning tree of 933 SNPs for 282 isolates of *Y. pestis* colored by location.

Morelli et al. Nature genetics 2010



Lát cắt trên đồ thị (Cuts in graphs)

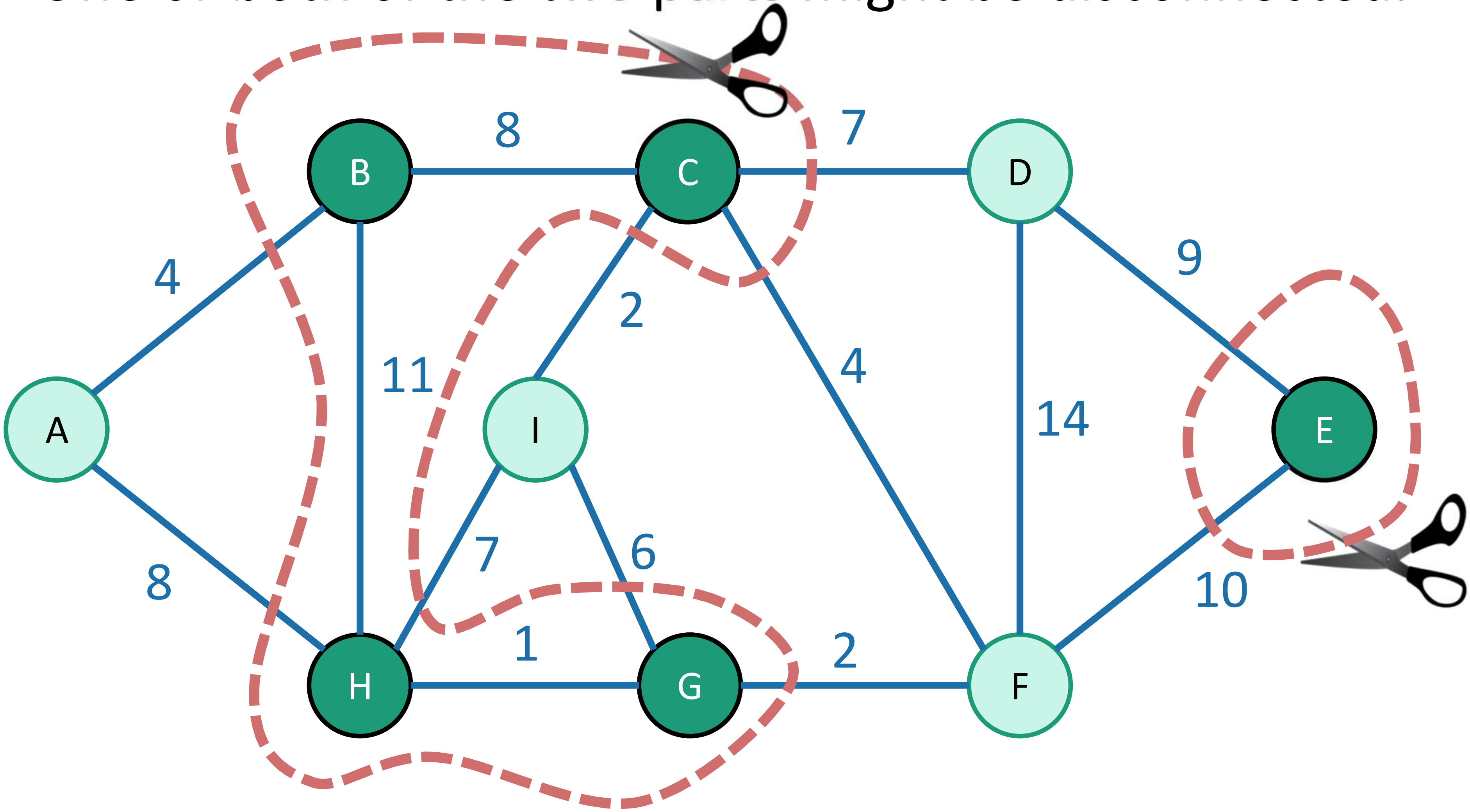
- A cut is a partition of the vertices into two parts:



This is the cut **“{A,B,D,E} and {C,I,H,G,F}”**

Cuts in graphs

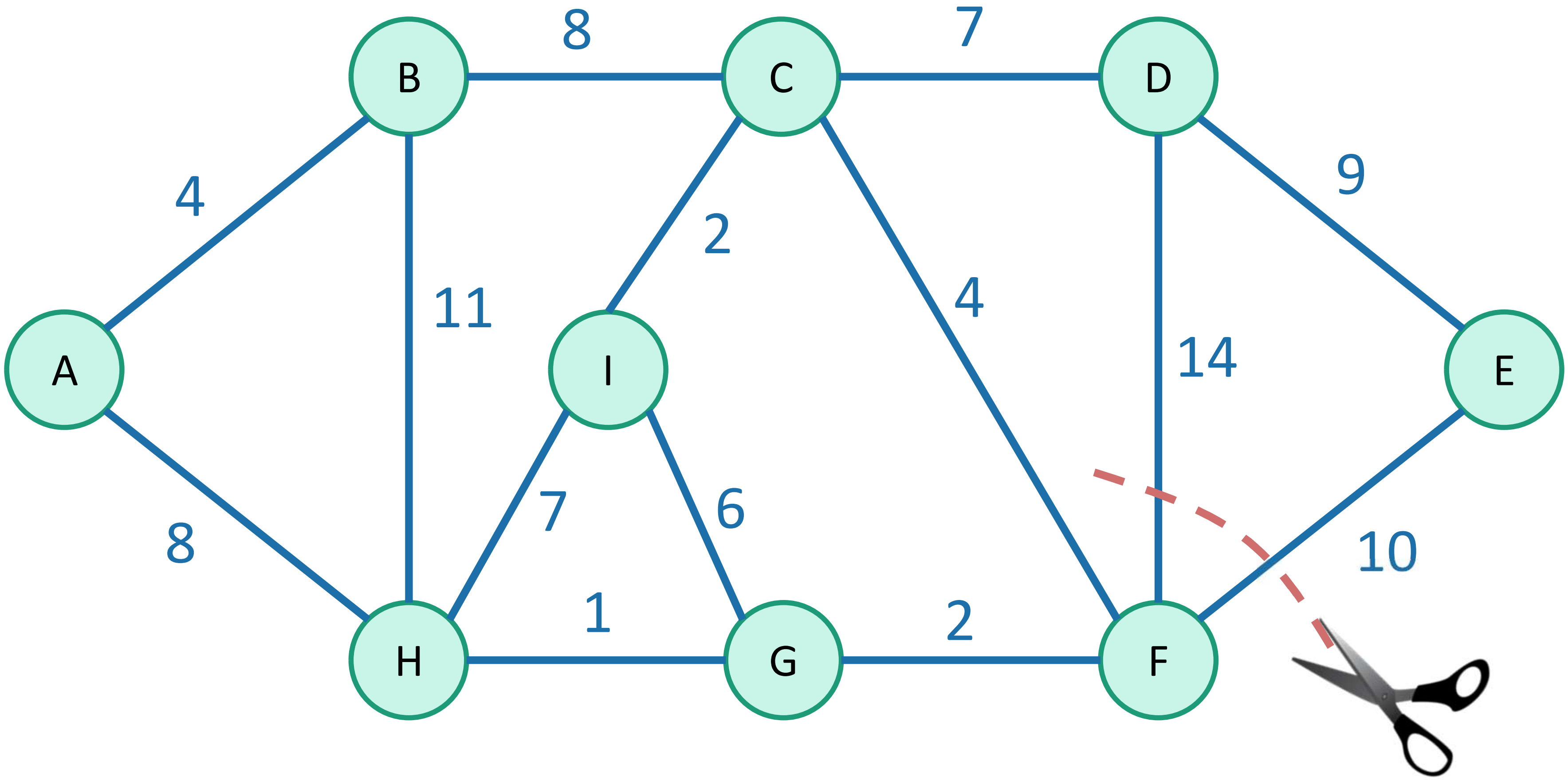
- One or both of the two parts might be disconnected.



This is the cut **"{B,C,E,G,H} and {A,D,I,F}"**

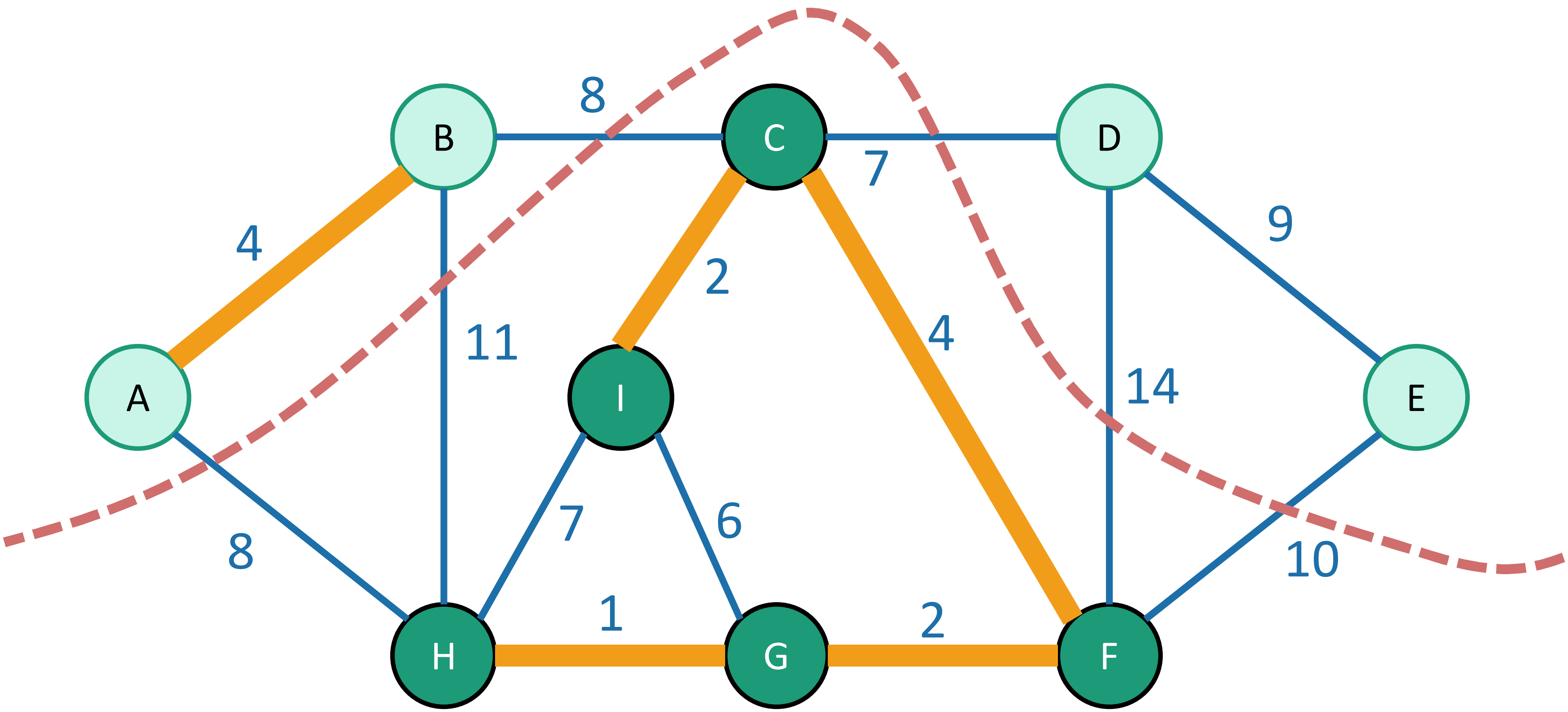
Cuts in graphs

- This is ***not*** a cut. Cuts are partitions of vertices.



Let S be a set of edges in G

- We say a cut **respects** S if no edges in S cross the cut.
- An edge crossing a cut is called **light** if it has the smallest weight of any edge crossing the cut.



S is the set of **thick orange** edges

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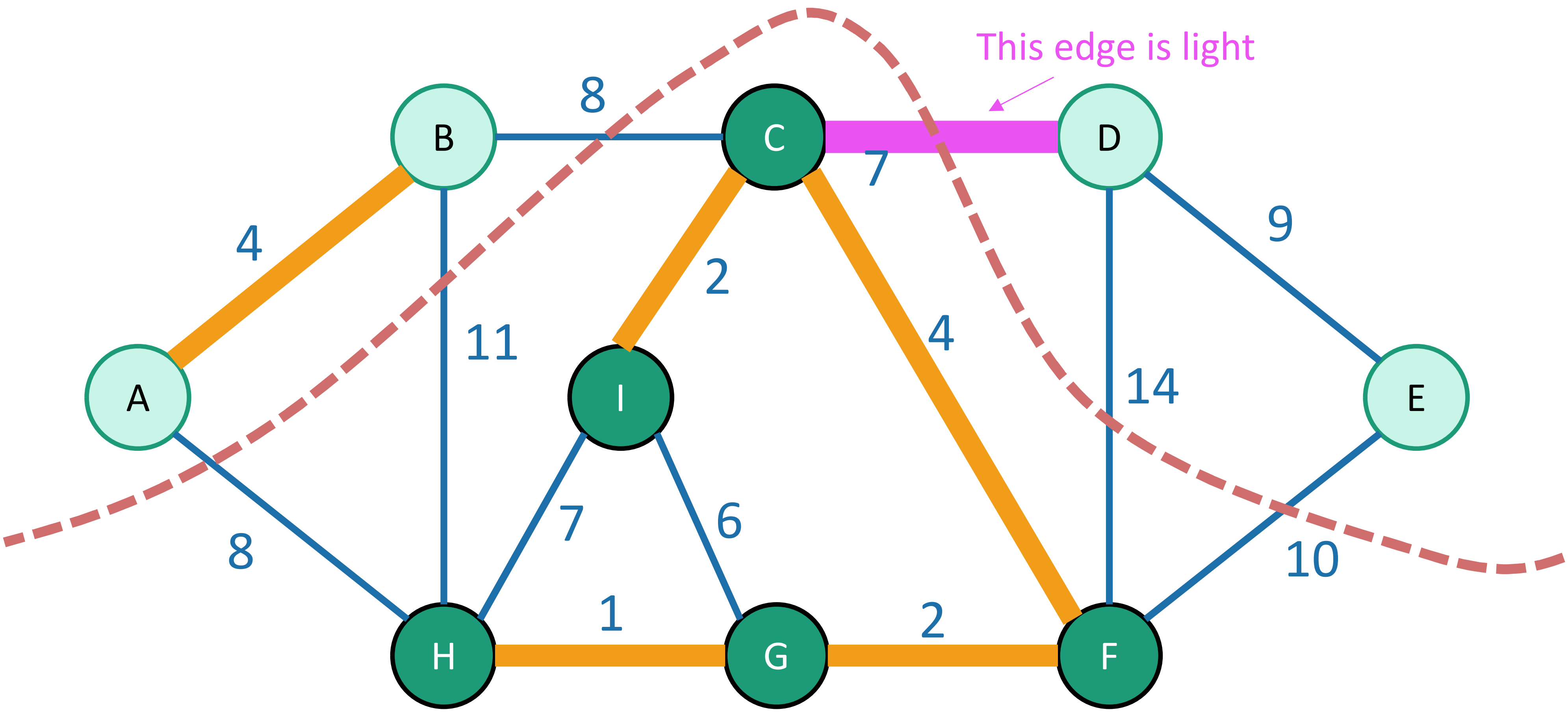

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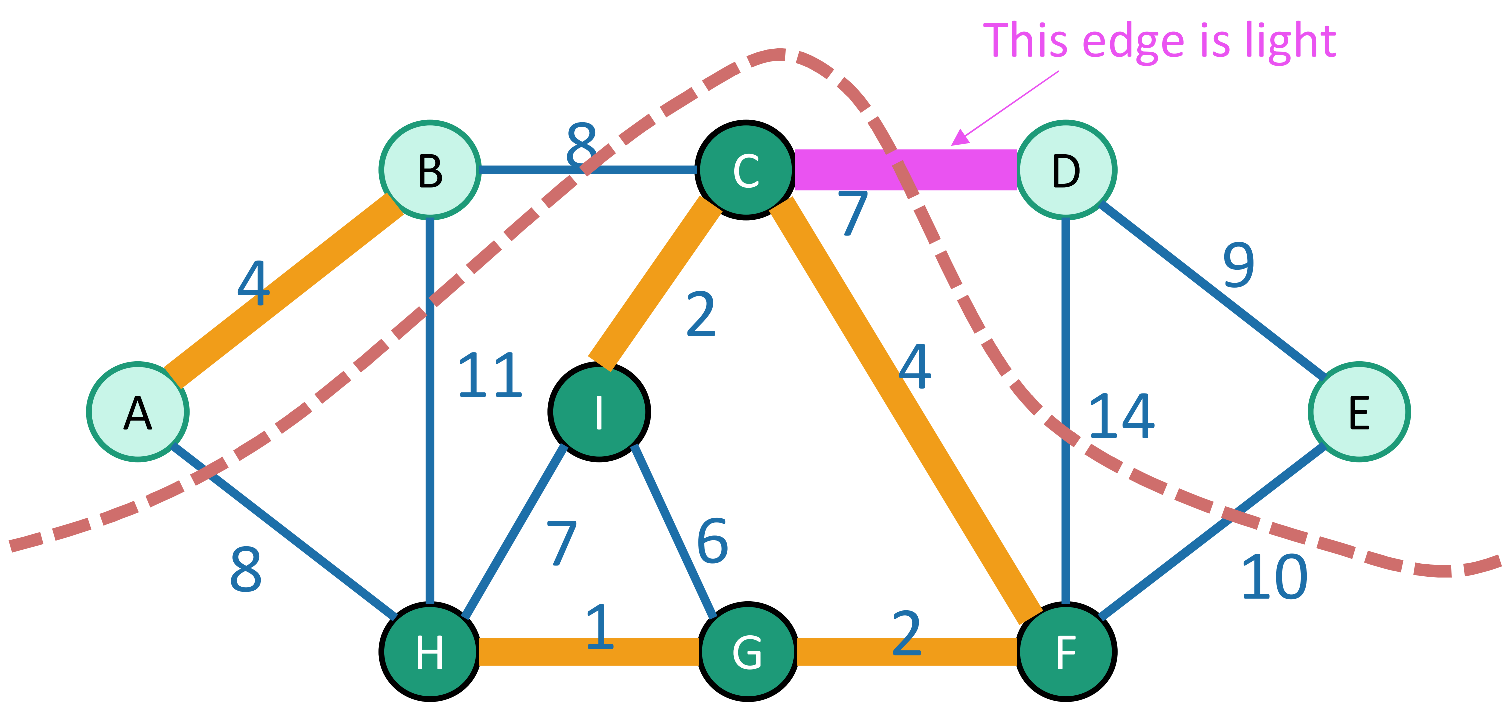
Bổ đề (Lemma)

- Let S be a set of edges, and consider a cut that respects S.
- Suppose there is an MST containing S.
- Let {u,v} be a light edge.
- Then there is an MST containing S U {{u,v}}



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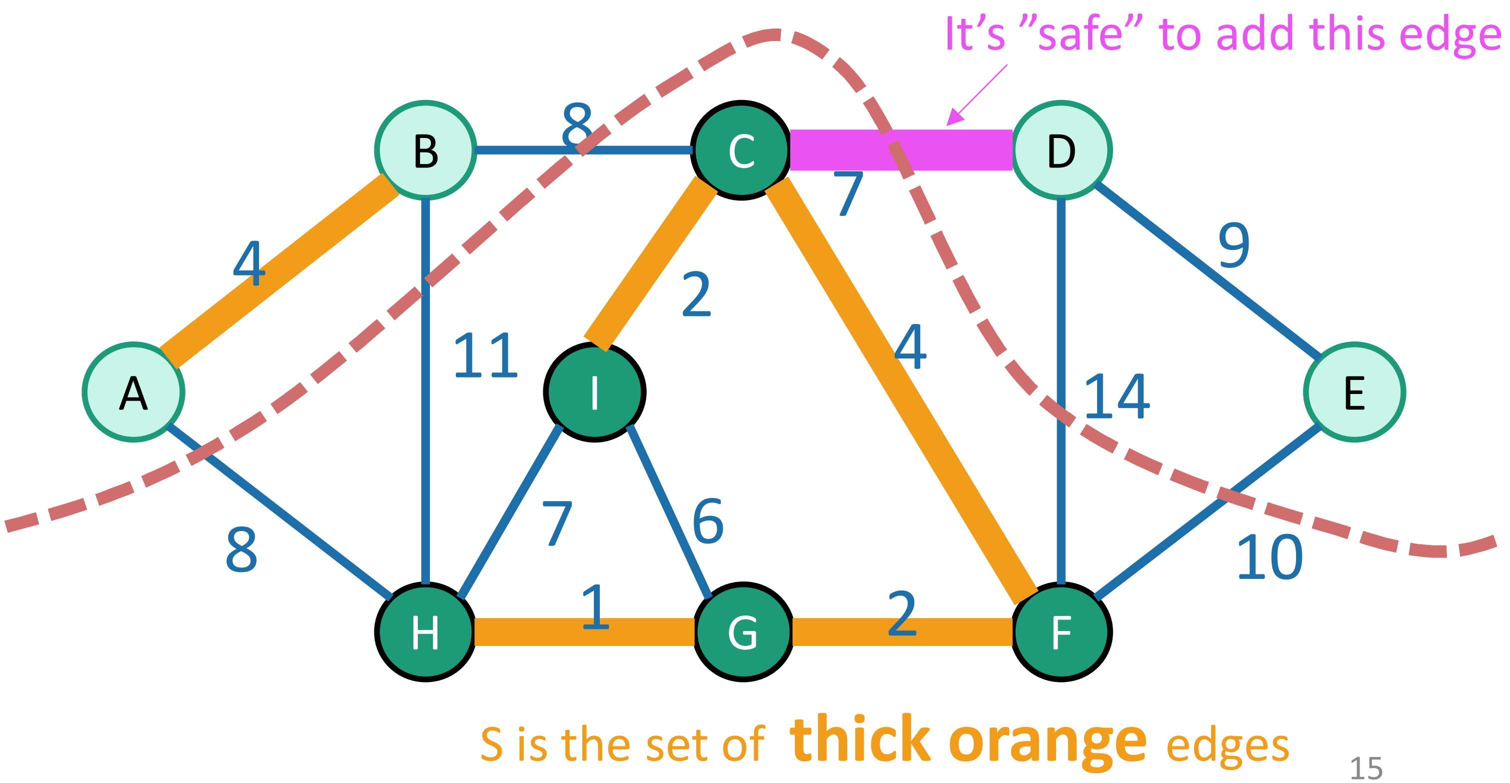
Lemma

- Let S be a set of edges, and consider a cut that respects S.
- Suppose there is an MST containing S.
- Let {u,v} be a light edge.
- Then there is an MST containing S U {{u,v}}

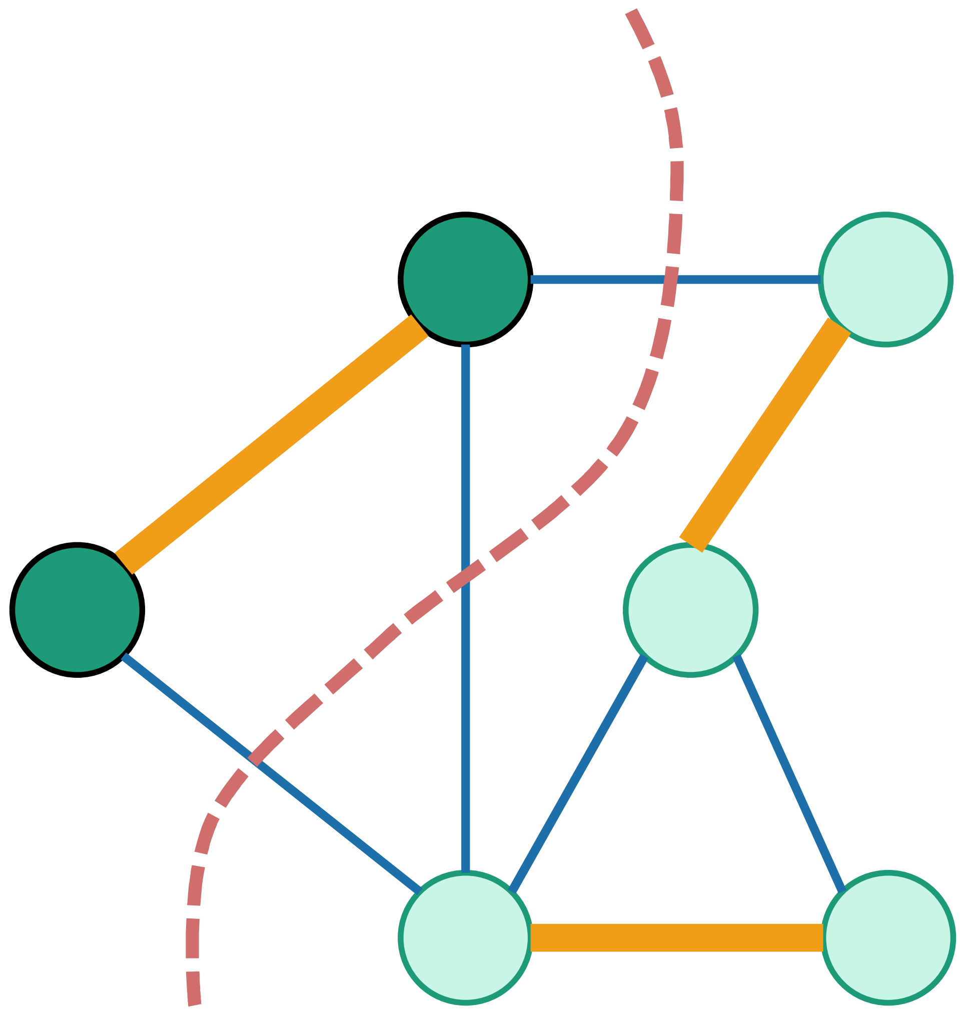
Aka:

If we haven't ruled out the possibility of success so far, then adding a light edge still won't rule it out.



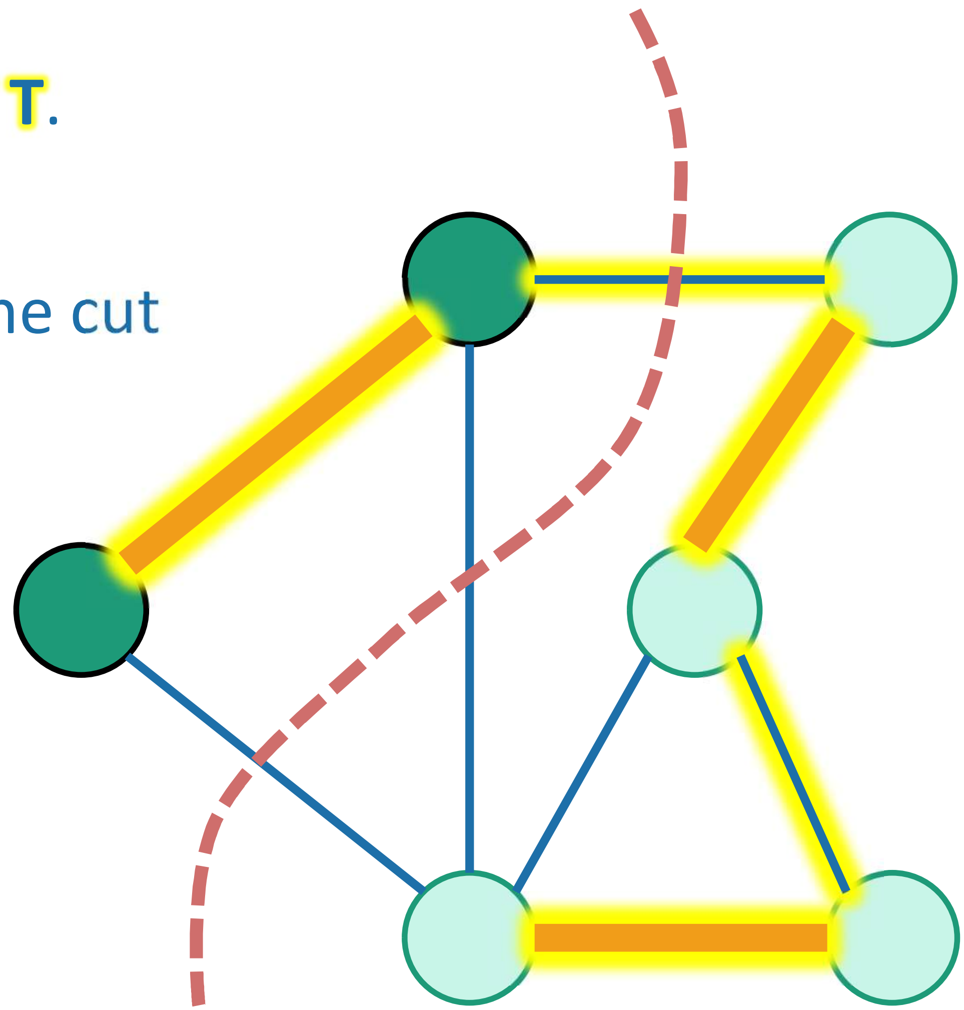
Proof of Lemma

- Assume that we have:
  - a **cut** that respects **S**



Proof of Lemma

- Assume that we have:
  - a **cut** that respects **S**
  - **S** is part of some **MST T**.
- Say that **{u,v}** is light.
  - lowest cost crossing the cut



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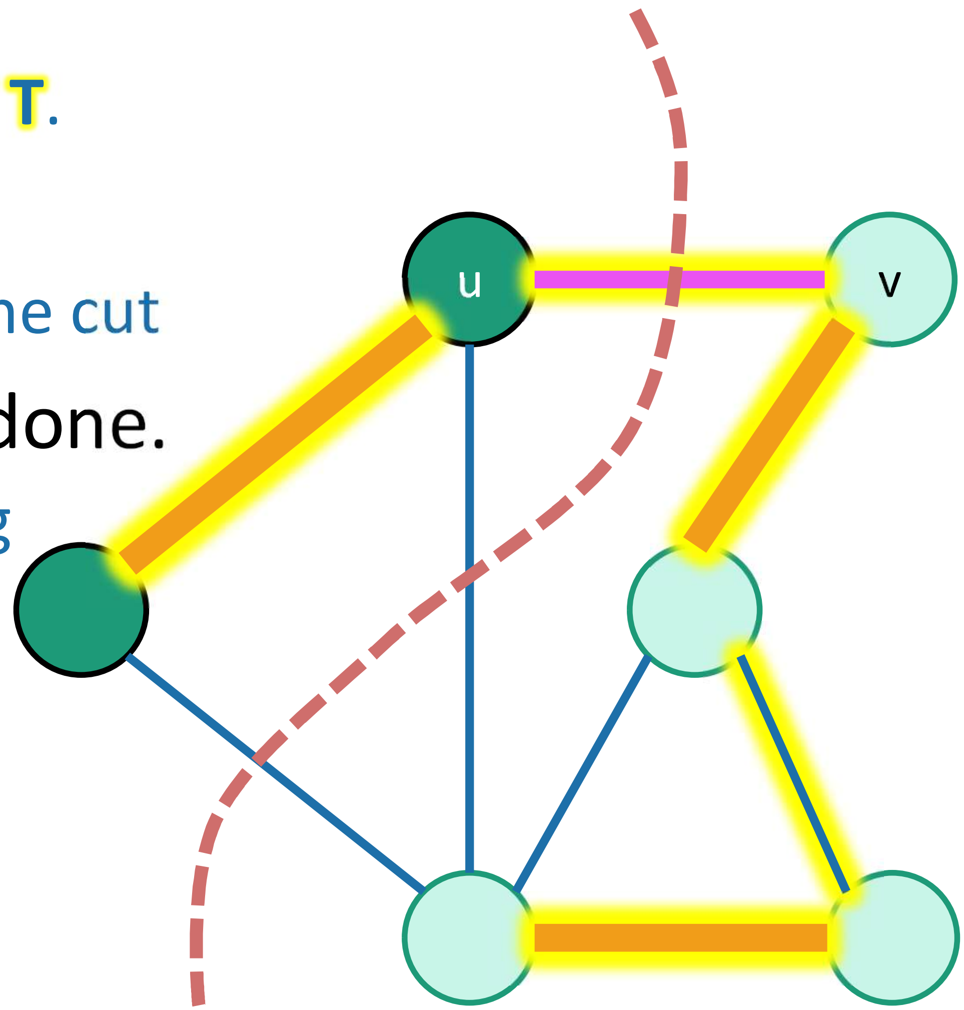
  - **S** is part of some **MST T**.
- Say that **{u,v}** is light.
  - lowest cost crossing the cut
- If **{u,v}** is in T, we are done.
  - T is an MST containing both {u,v} and S.



Proof of Lemma

Claim: Adding any additional edge to a spanning tree will create a cycle.

Proof: Both endpoints are already in the tree and connected to each other.

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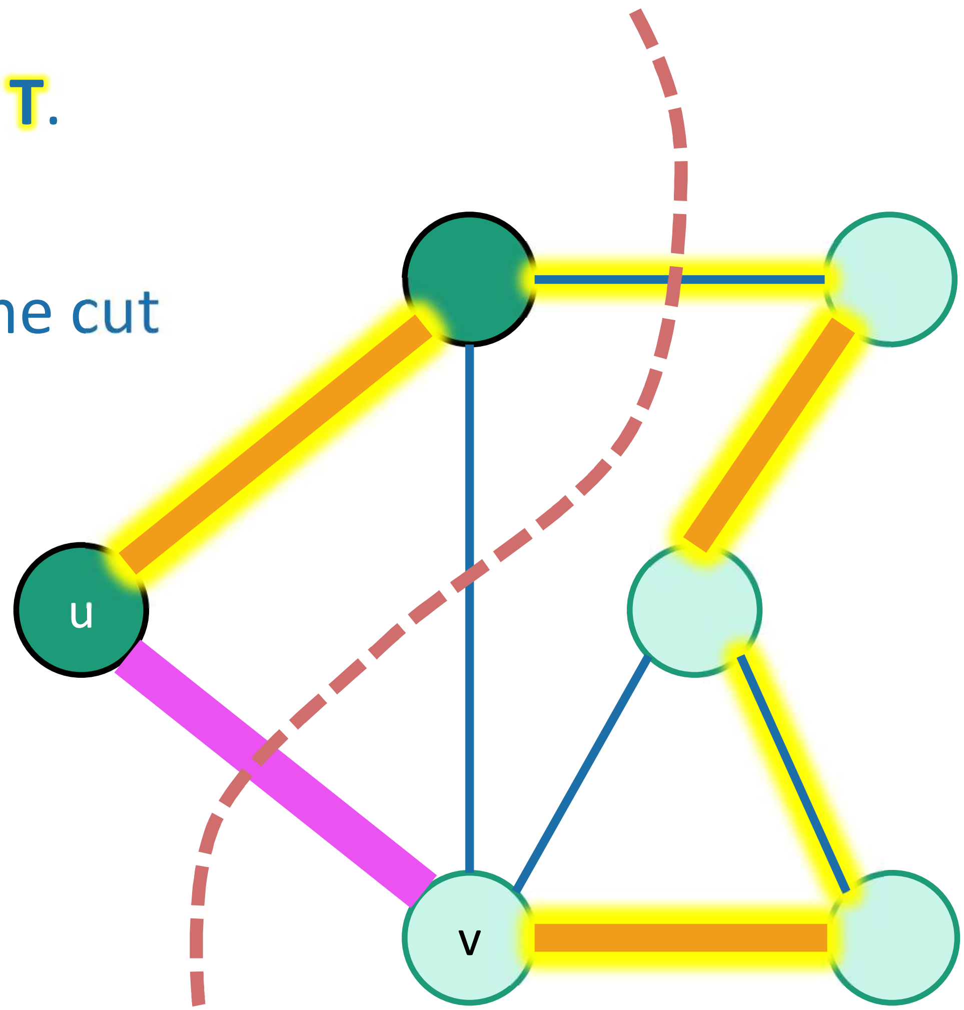
- Say **{u,v}** is not in **T**.
- Note that adding **{u,v}** to **T** will make a cycle.



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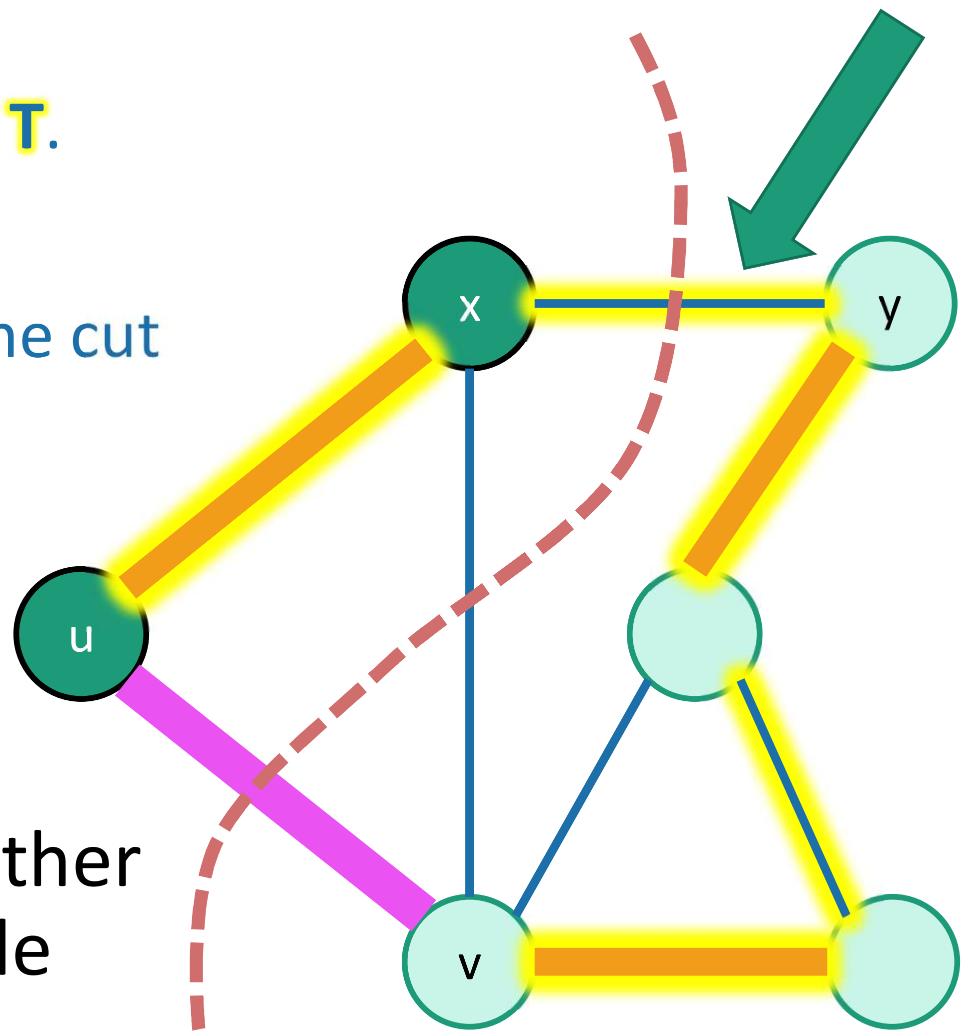
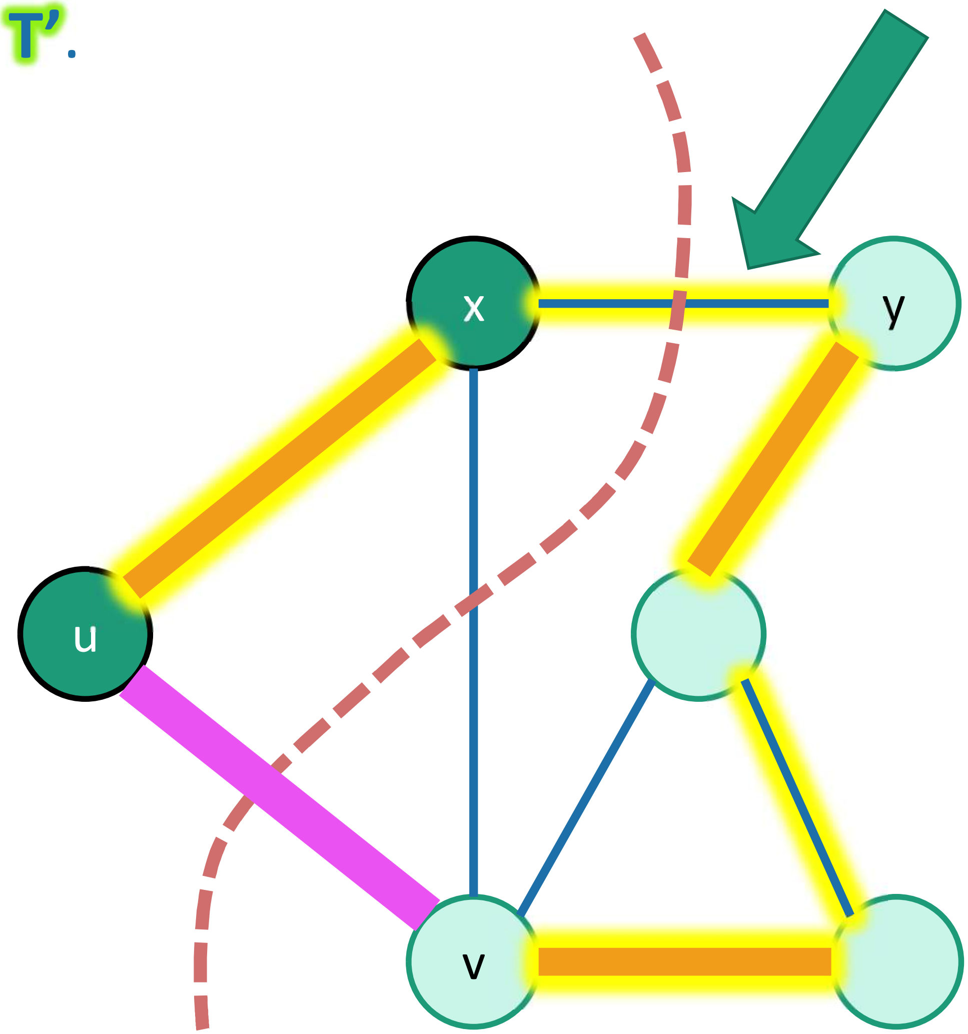
- Say **{u,v}** is not in **T**.
- Note that adding **{u,v}** to **T** will make a cycle.
- There is at least one other edge, **{x,y}**, in this cycle crossing the cut.



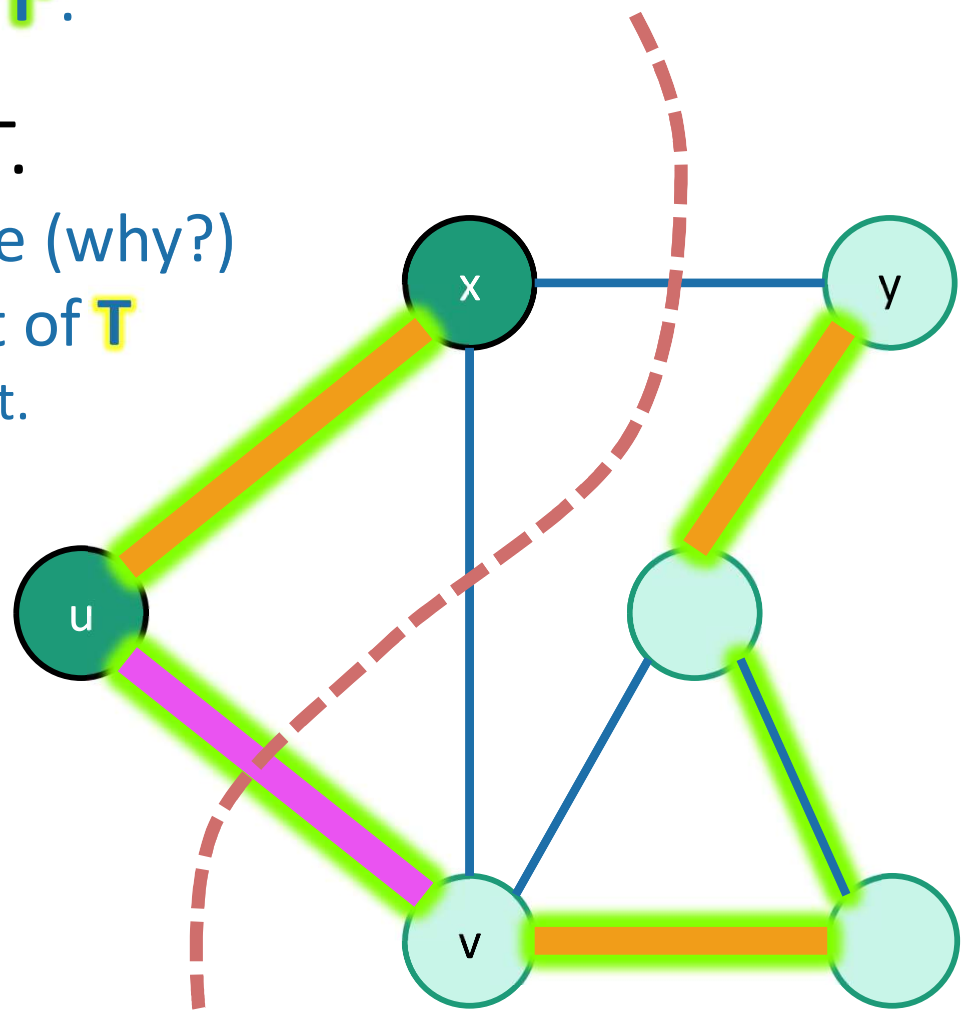
Proof of Lemma ctd.

- Consider swapping {u,v} for {x,y} in **T**.
  - Call the resulting tree **T'**.



Proof of Lemma ctd.

- Consider swapping {u,v} for {x,y} in **T**.
  - Call the resulting tree **T'**.
- **Claim**: **T'** is still an MST.
  - It is still a spanning tree (why?)
  - It has cost at most that of **T**
    - because {u,v} was light.
  - **T** had minimal cost.
  - So **T'** does too.
- So **T'** is an MST containing S and {u,v}.
  - This is what we wanted.



Lemma

- Let S be a set of edges, and consider a cut that respects S.
- Suppose there is an MST containing S.
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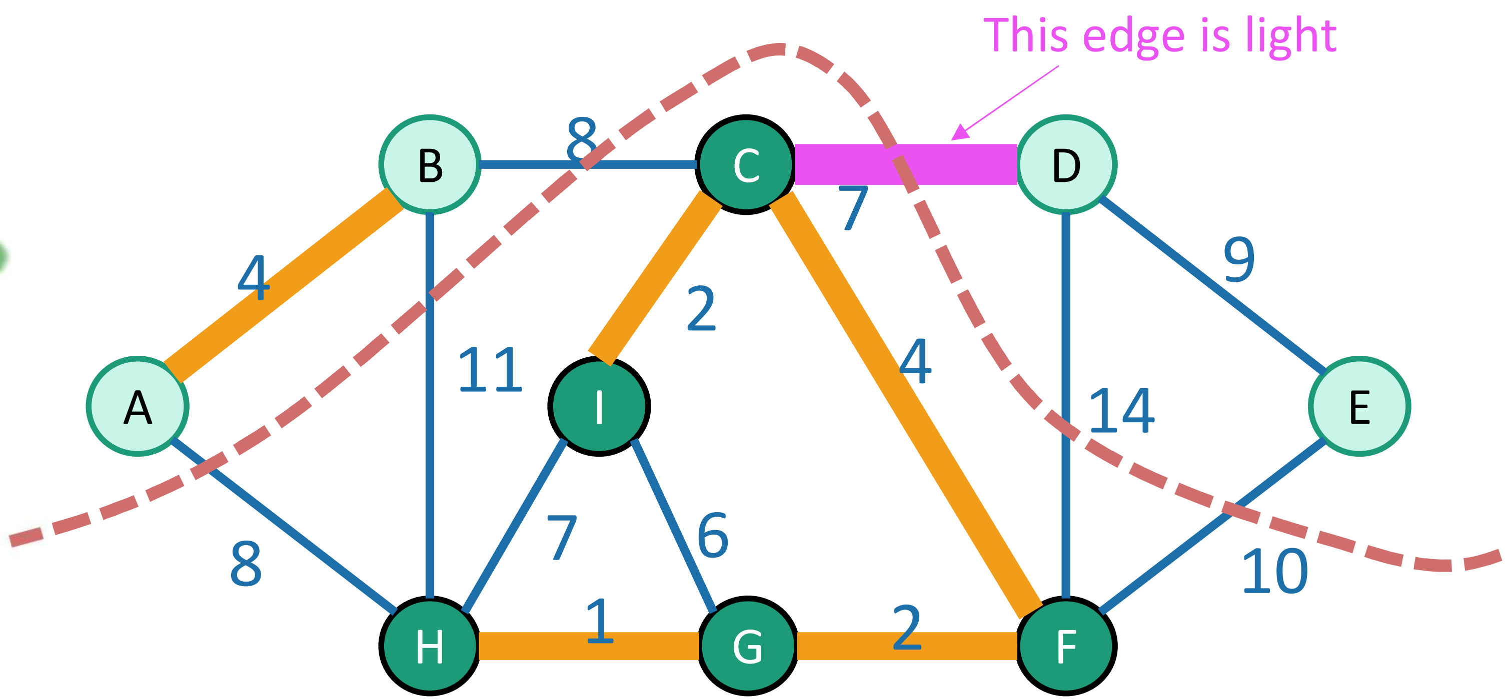


S is the set of **thick orange** edges

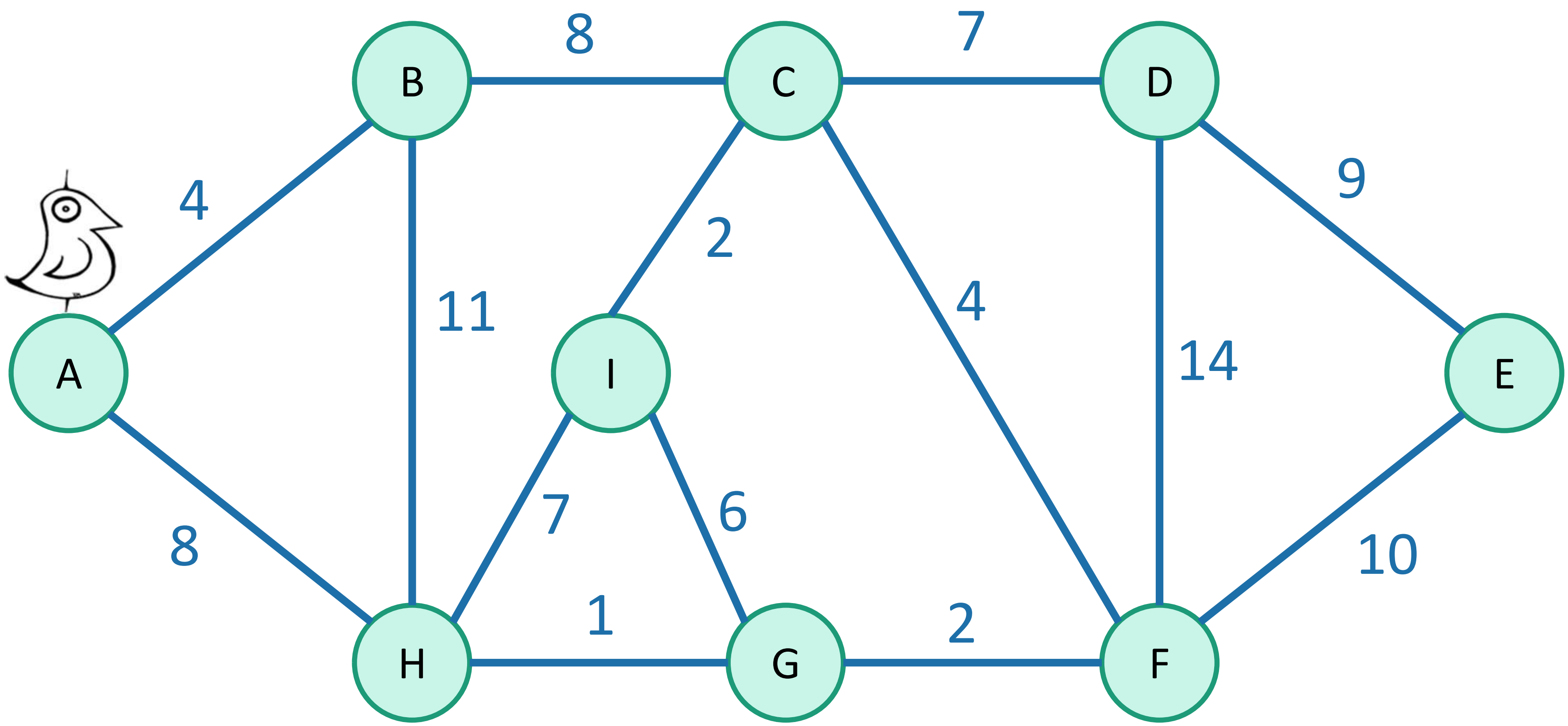
Áp dụng vào tìm MST

- How do we find one?

- The strategy:
  - Make a **series of choices,** adding edges to the tree.
  - Show that each edge we add is **safe to add**:
    - we do not rule out the possibility of success
    - we will choose **light edges** crossing **cuts** and **use the Lemma**.
  - **Keep going** until we have an MST.

Idea 1

Start growing a tree, greedily add the shortest edge we can to grow the tree.



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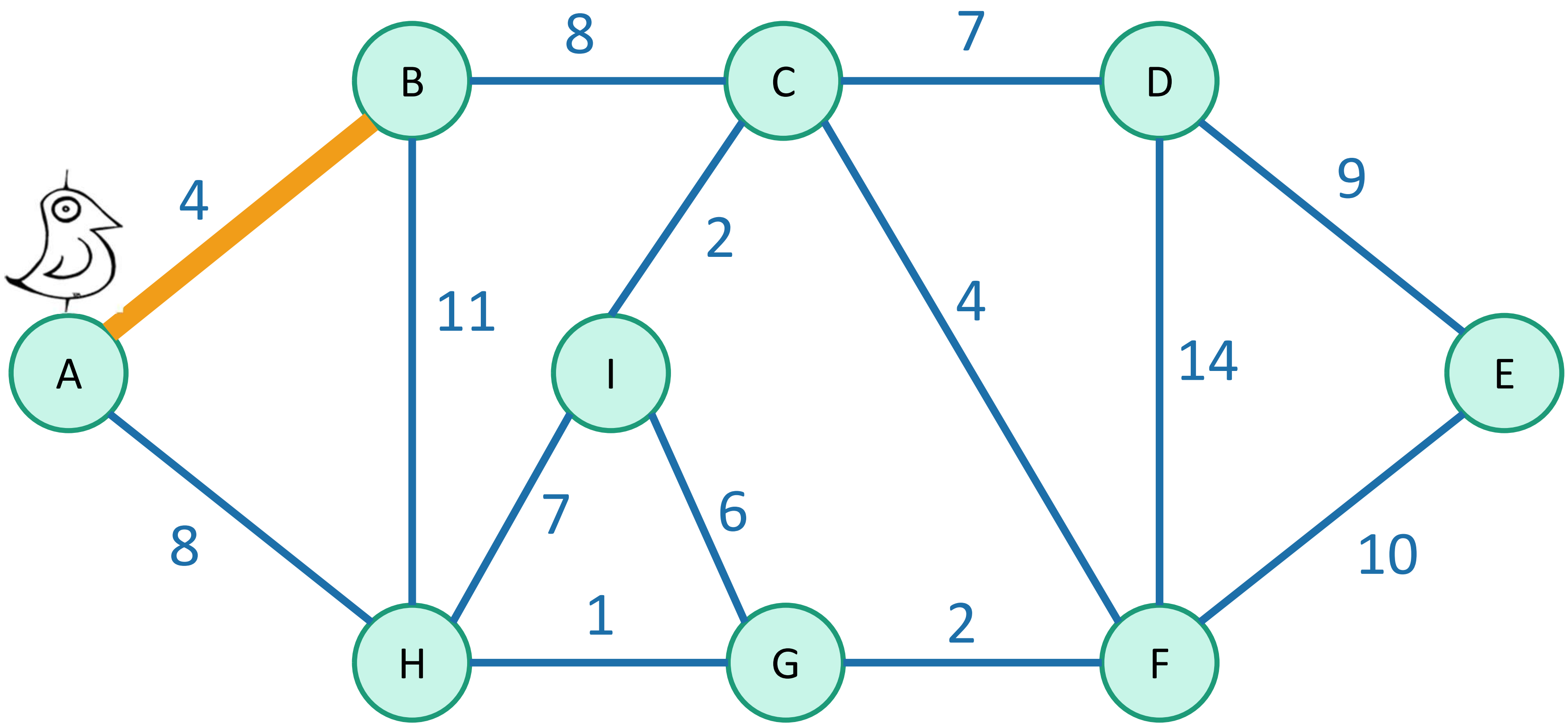

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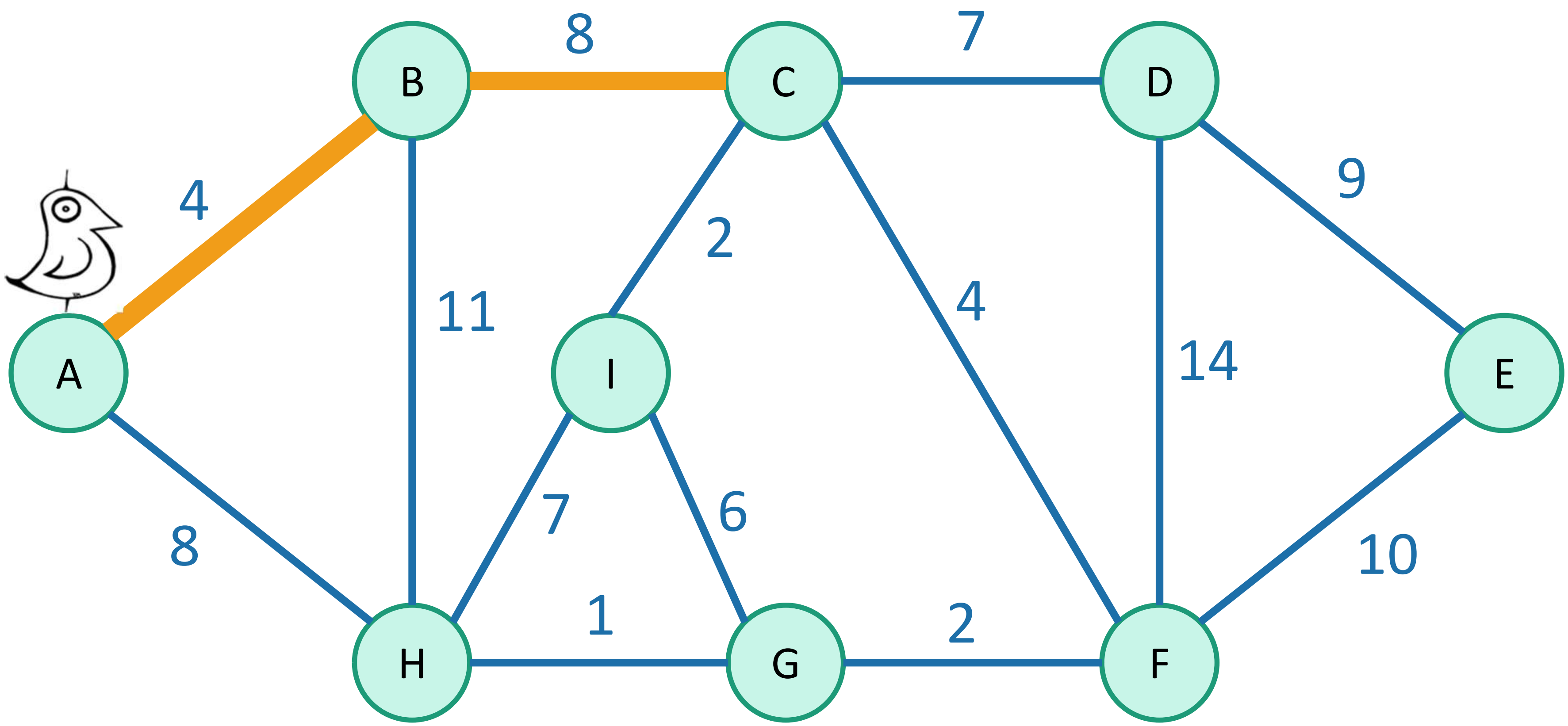

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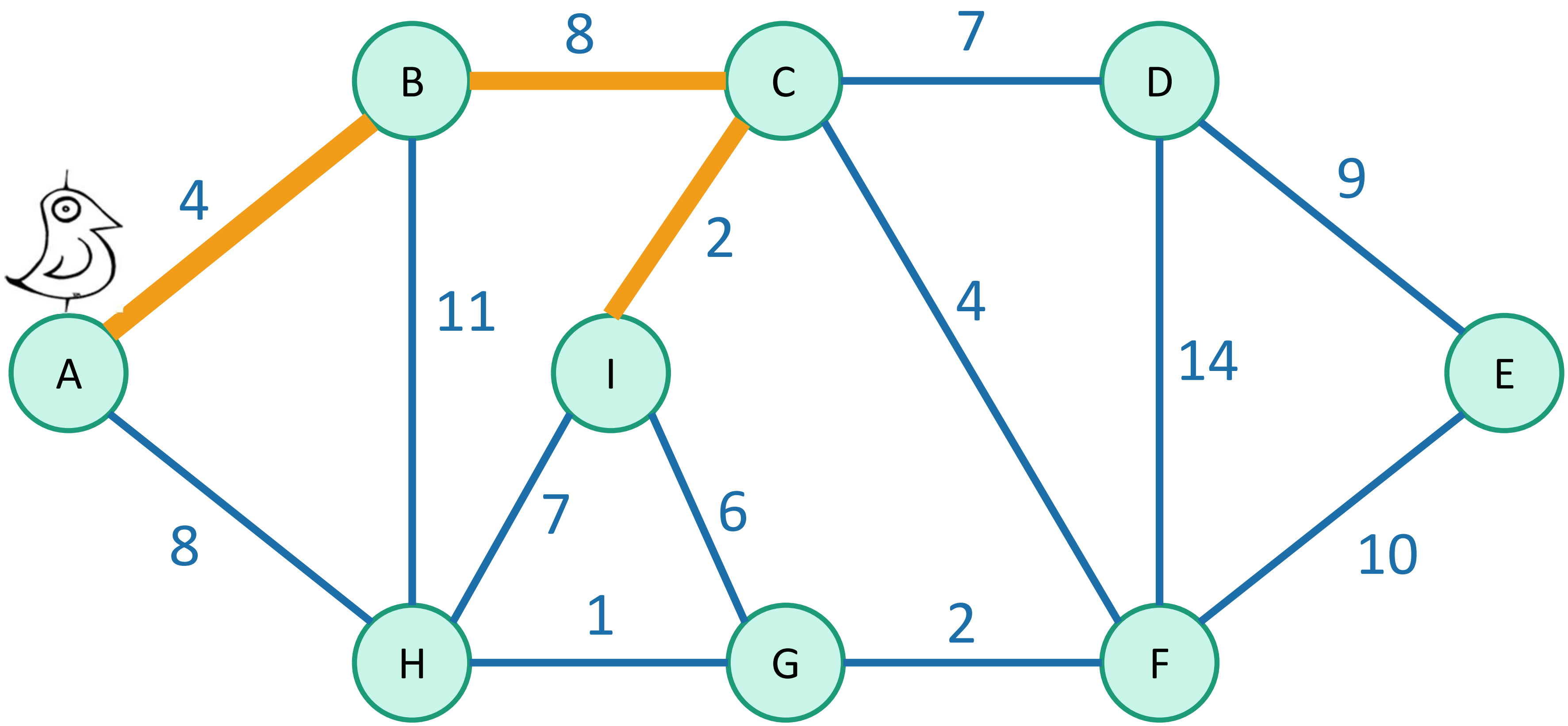

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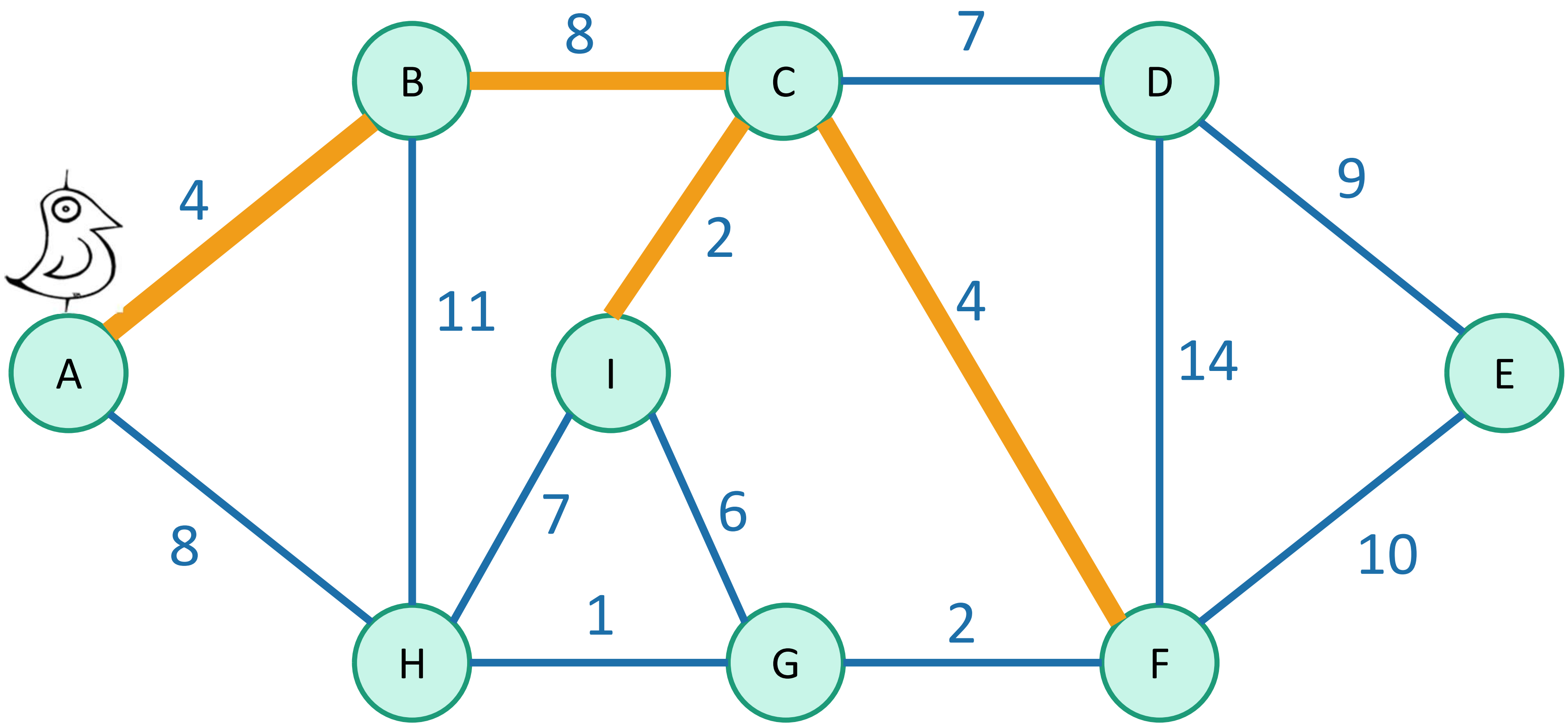

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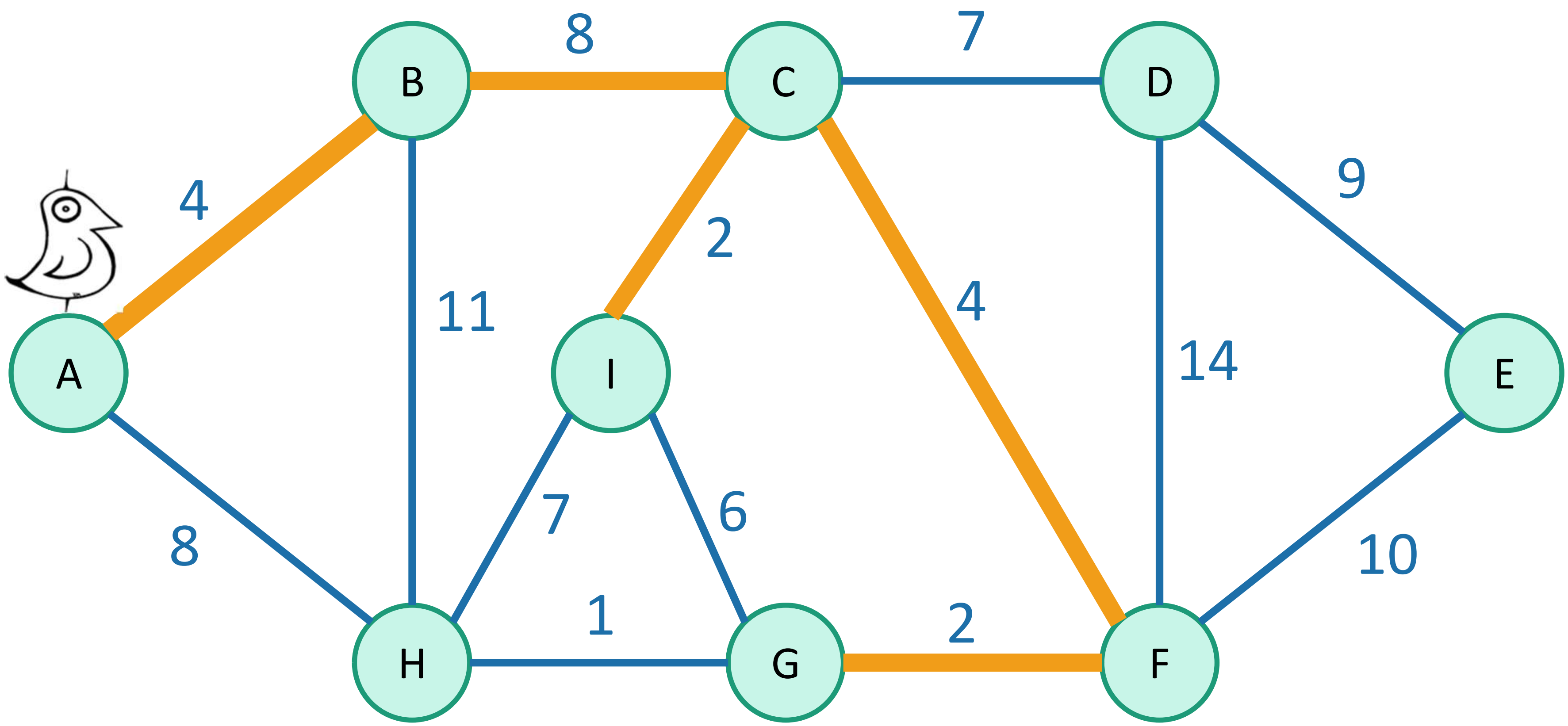

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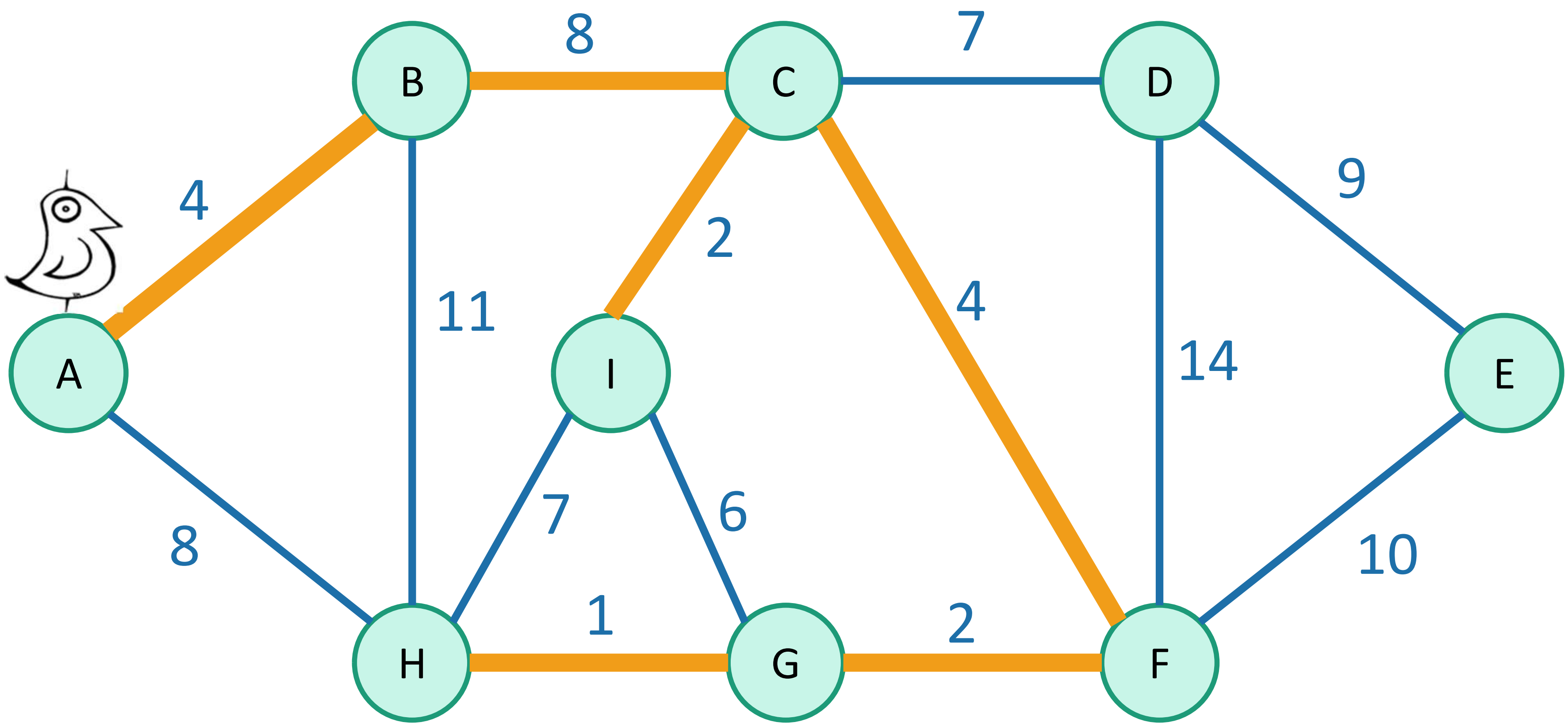

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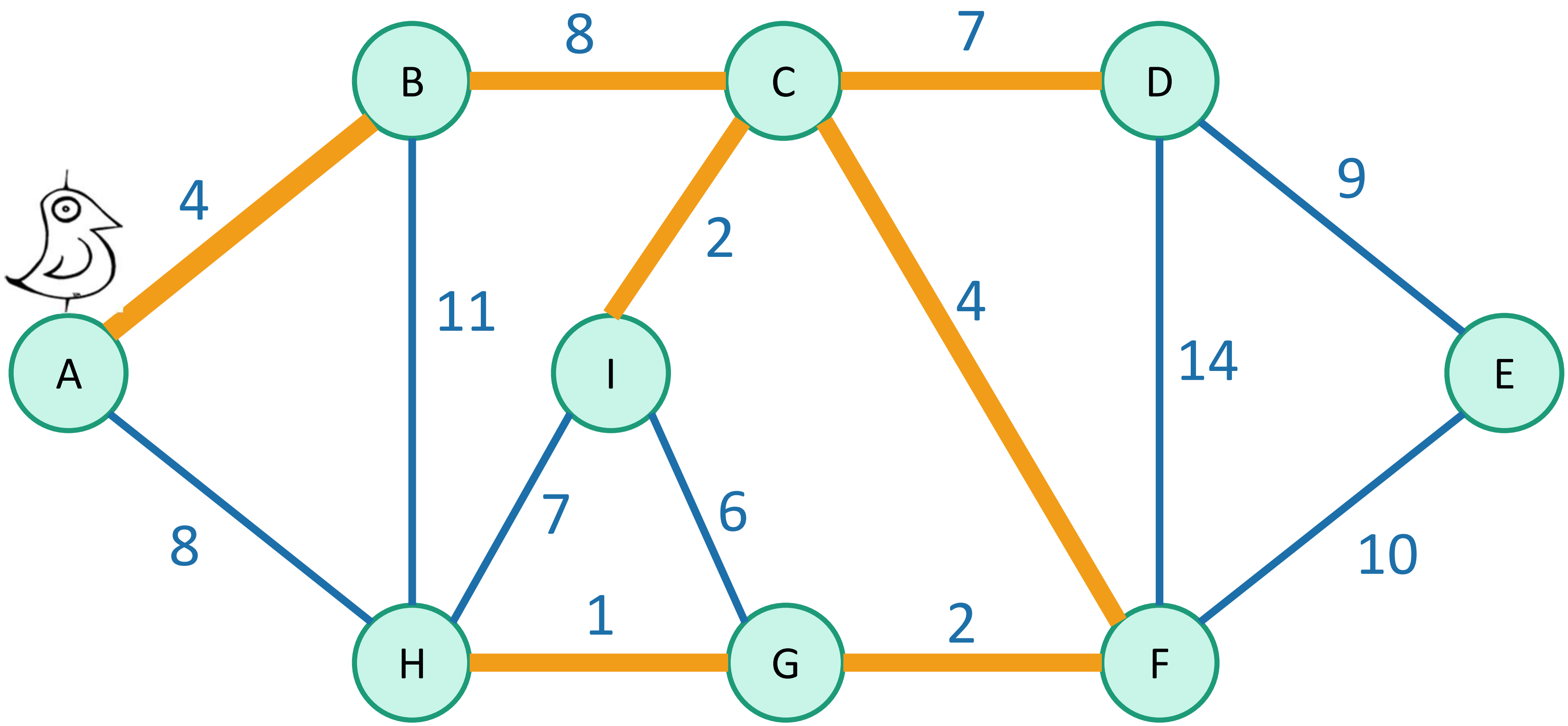

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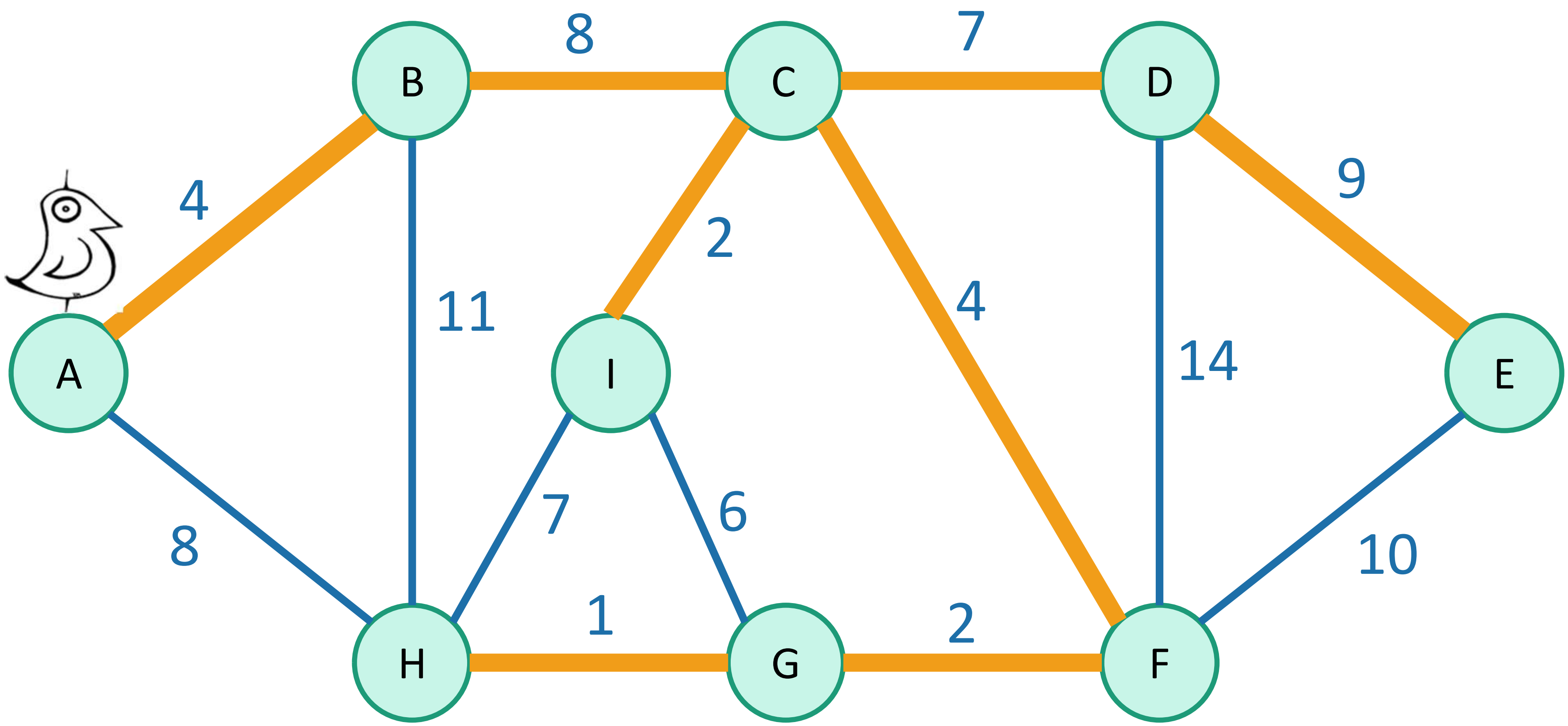

We've discovered

Prim's algorithm!

Jarnik [1930]
Prim [1957]
Dijkstra [1959]

- slowPrim(G = (V,E), starting vertex s):
  - Let (s,u) be the lightest edge coming out of s.
  - MST = { (s,u) }
  - verticesVisited = { s, u }
  - **while** |verticesVisited| < |V|:
    - find the lightest edge {x,v} in E so that:
      - x is in verticesVisited
      - v is not in verticesVisited
    - add {x,v} to MST
    - add v to verticesVisited
  - **return** MST

At most n iterations of this while loop.

Time at most m to go through all the edges and find the lightest.

Naively, the running time is O(nm):

- For each of ≤n iterations of the while loop:
  - Go through all the edges.

Two questions

1. Does it work?
   - That is, does it actually return a MST?

2. How do we actually implement this?
   - the pseudocode above says “slowPrim”…

Lemma

- Let S be a set of edges, and consider a cut that respects S.
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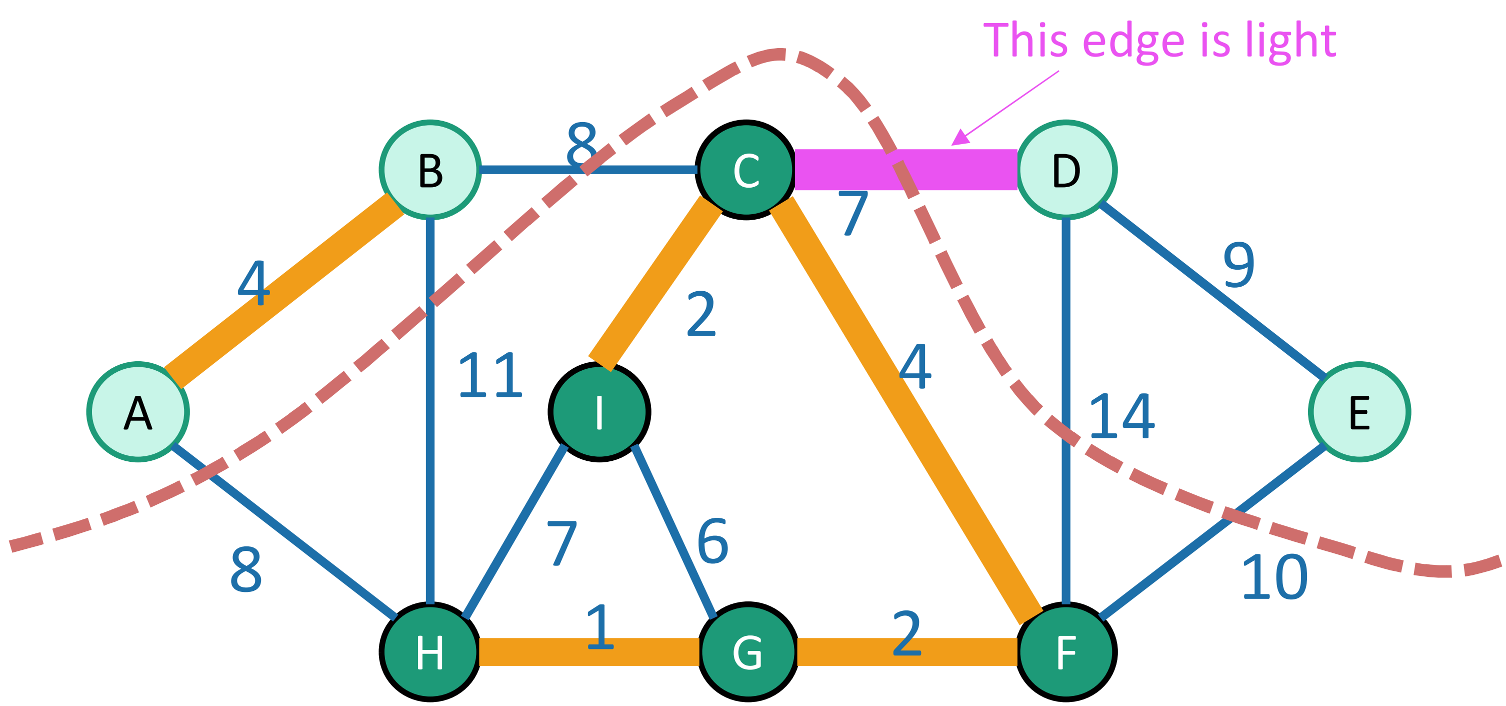
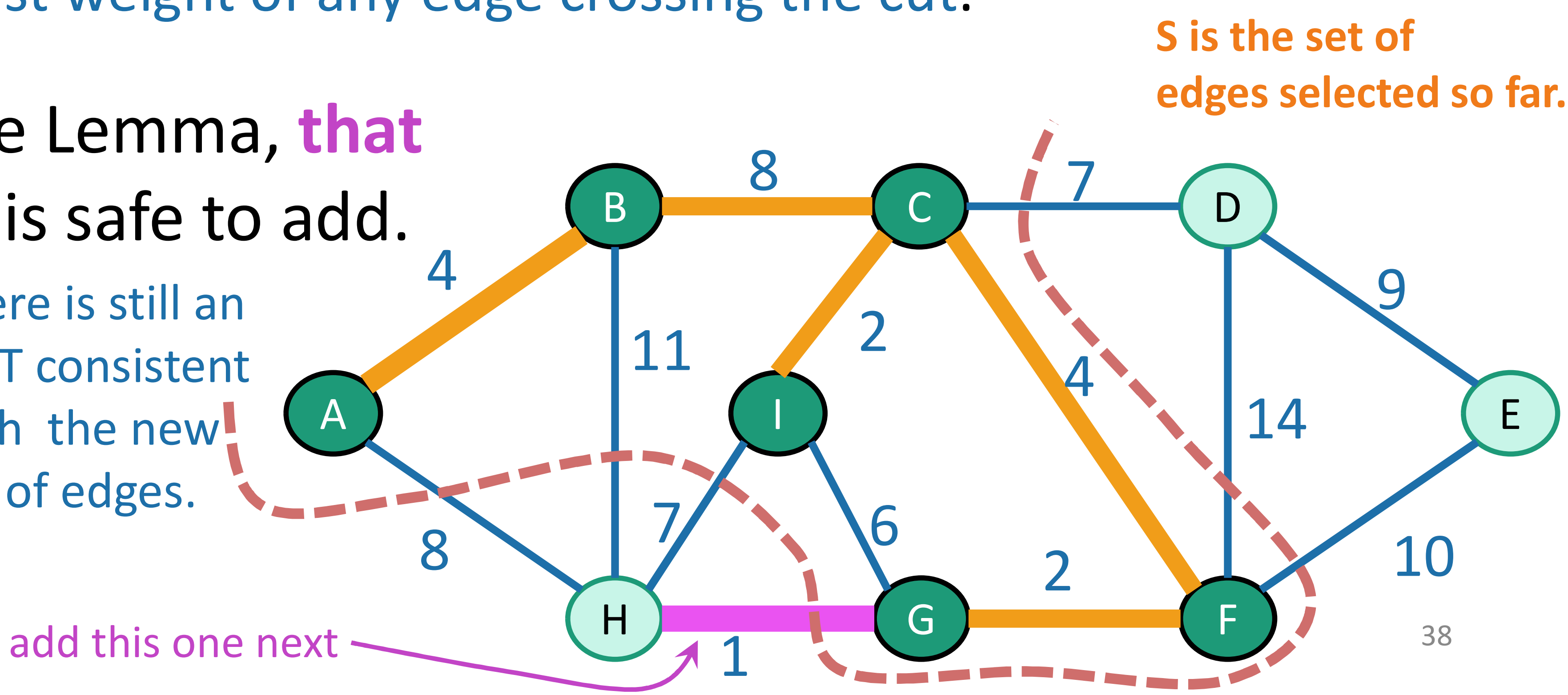


S is the set of **thick orange** edges

Partway through Prim

- Assume that our choices **S** so far don't rule out success
  - There is an MST consistent with these choices
- Consider the cut {**visited**, **unvisited**}
  - This cut respects S.
- The edge we add next is a **light edge**.
  - Least weight of any edge crossing the cut.
- By the Lemma, **that edge** is safe to add.
  - There is still an MST consistent with the new set of edges.



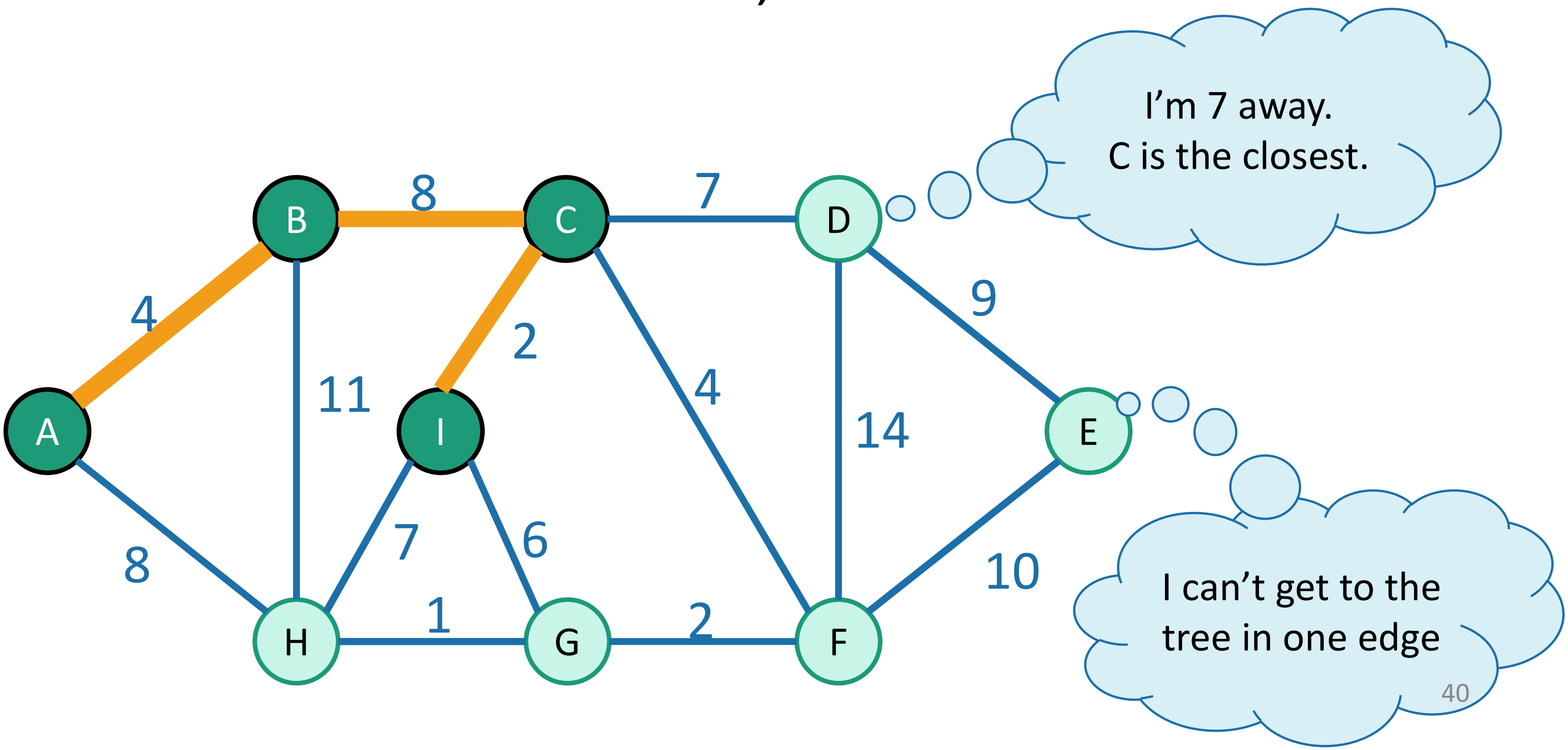
Two questions

1. Does it work?
   - That is, does it actually return a MST?
     - **Yes!**

2. How do we actually implement this?
   - the pseudocode above says “slowPrim”…

How do we actually implement this?

- Each vertex keeps:
  - the **distance** from itself to the **growing spanning tree** if you can get there in one edge.
  - **how to get there**.
- Choose the closest vertex, add it.



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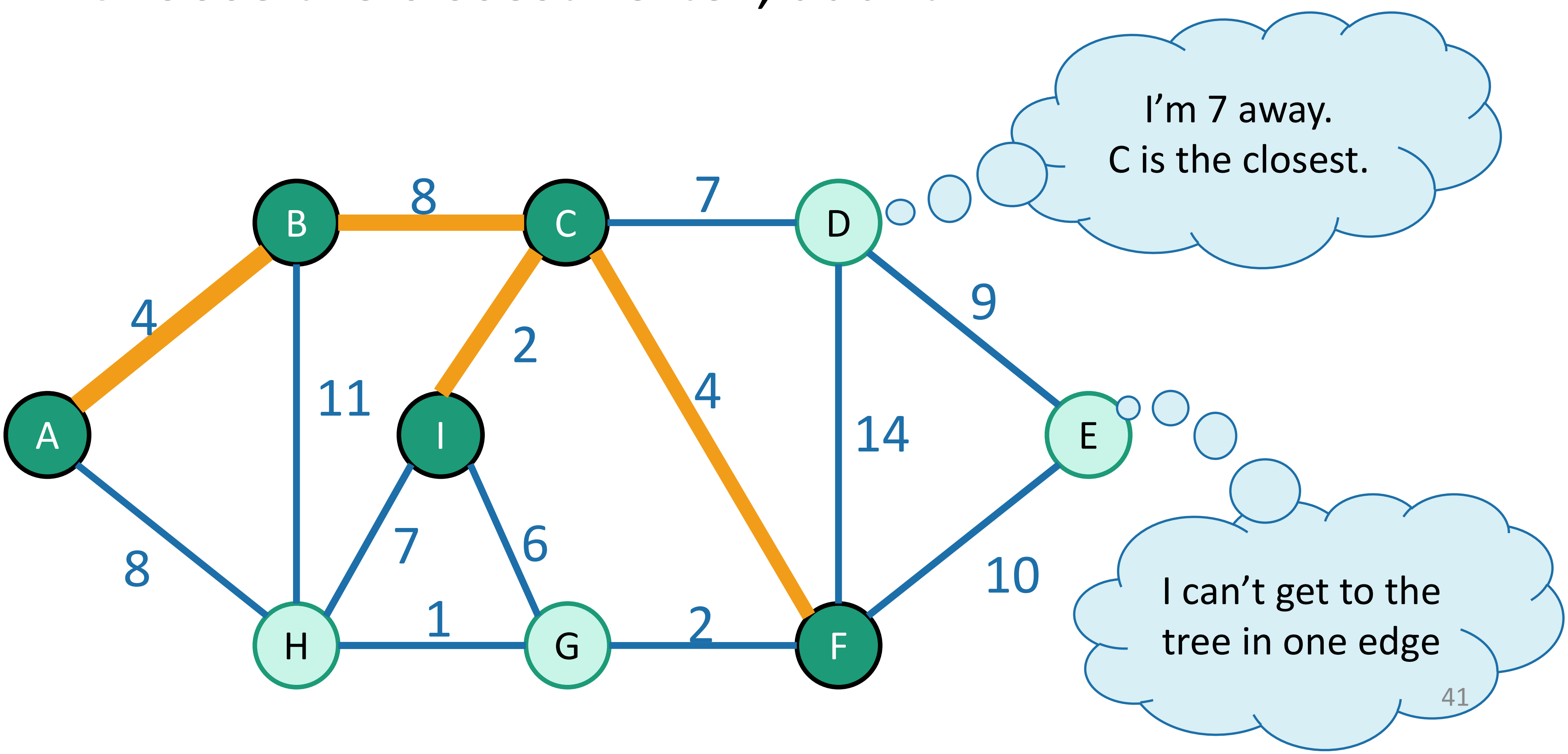
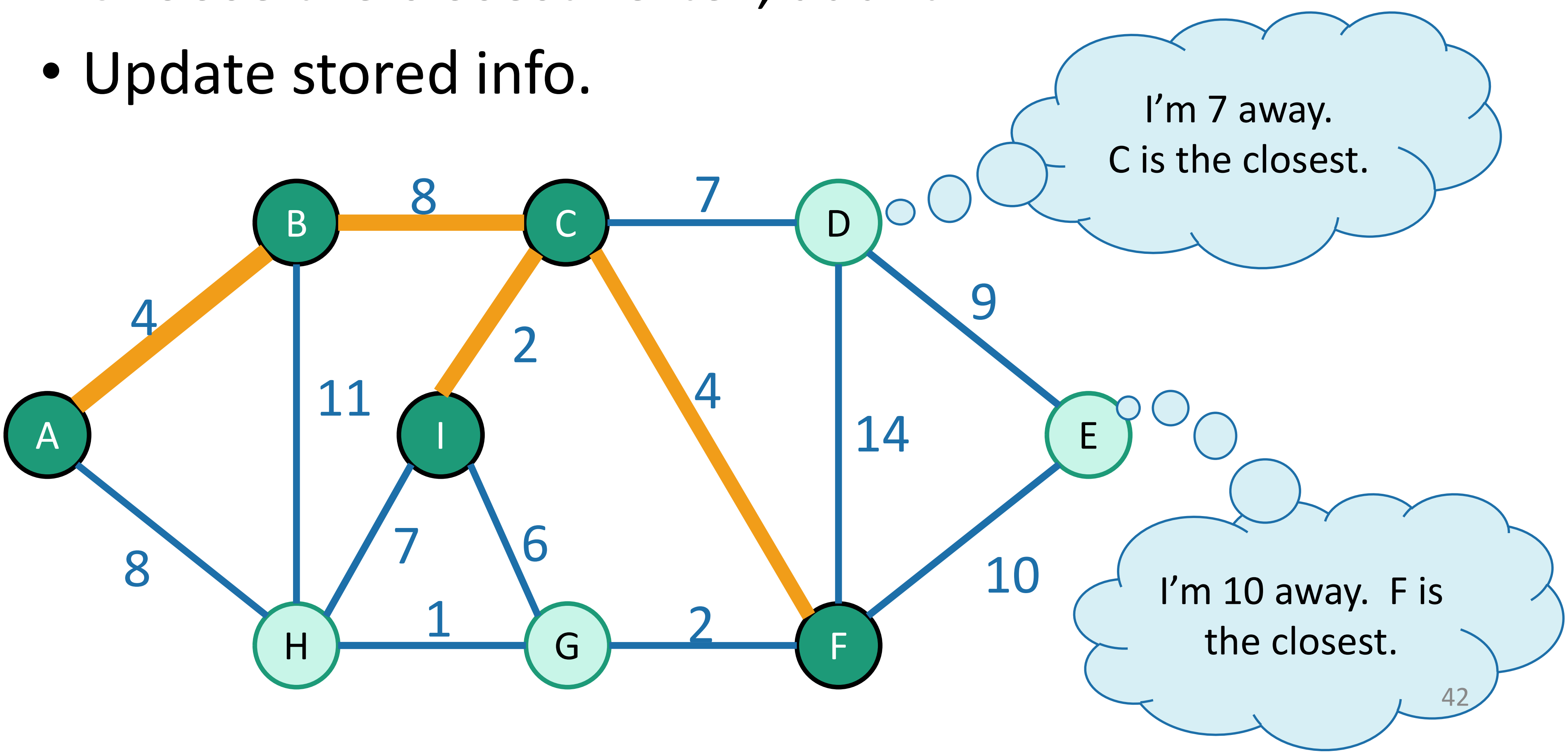


How do we actually implement this?

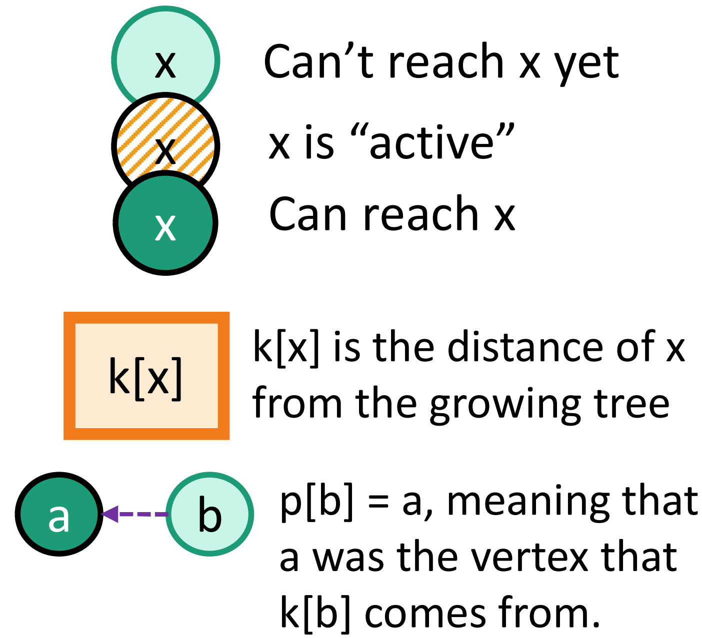
- Each vertex keeps:
  - the **distance** from itself to the **growing spanning tree**
    if you can get there in one edge.
  - **how to get there**.
- Choose the closest vertex, add it.
- Update stored info.

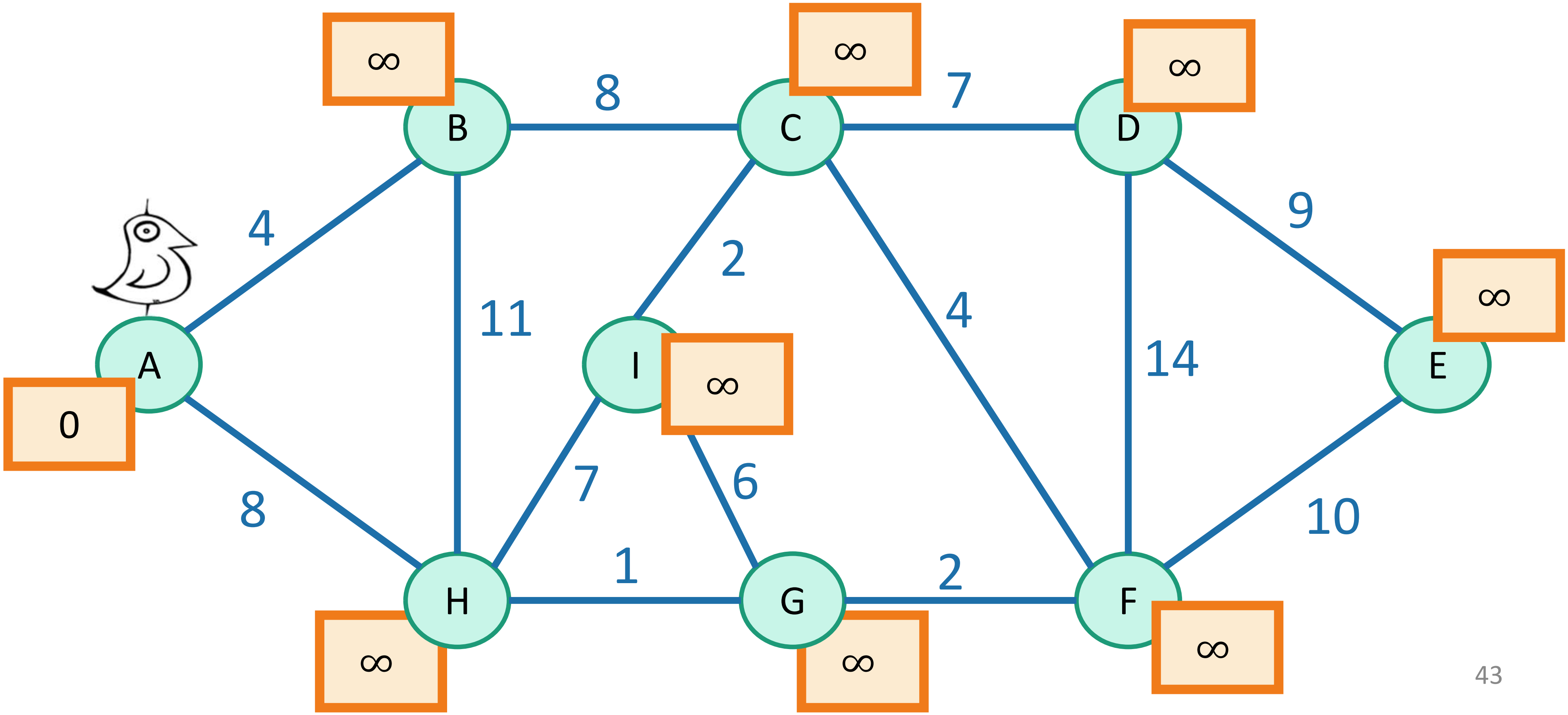


Efficient implementation

Every vertex has a key and a parent

Until all the vertices are **reached:**





Efficient implementation

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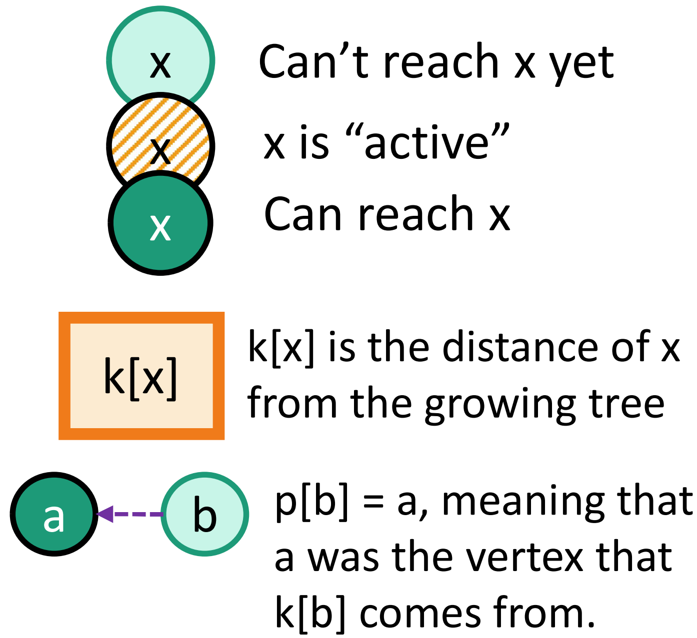
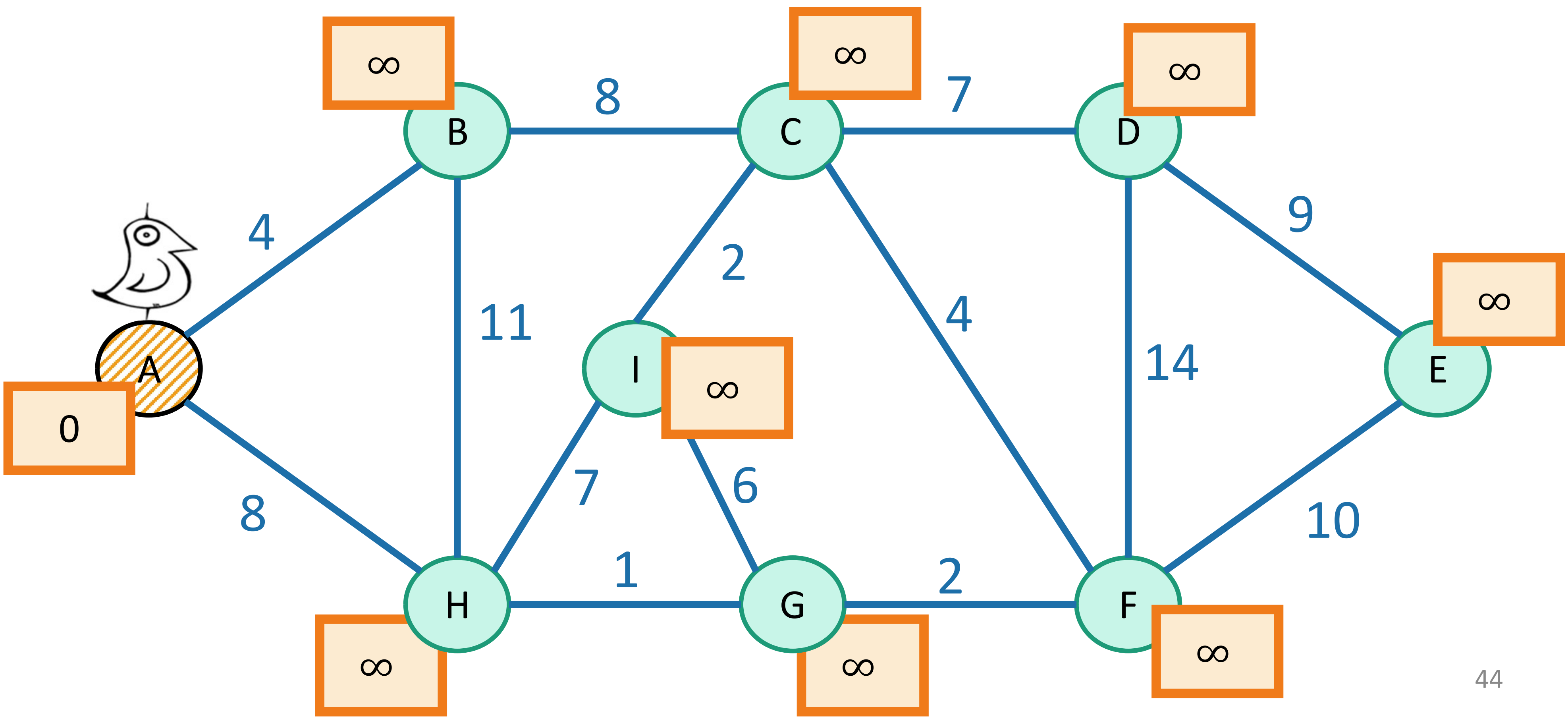
Until all the vertices are **reached:**

- Activate the **unreached** vertex u with the smallest key.





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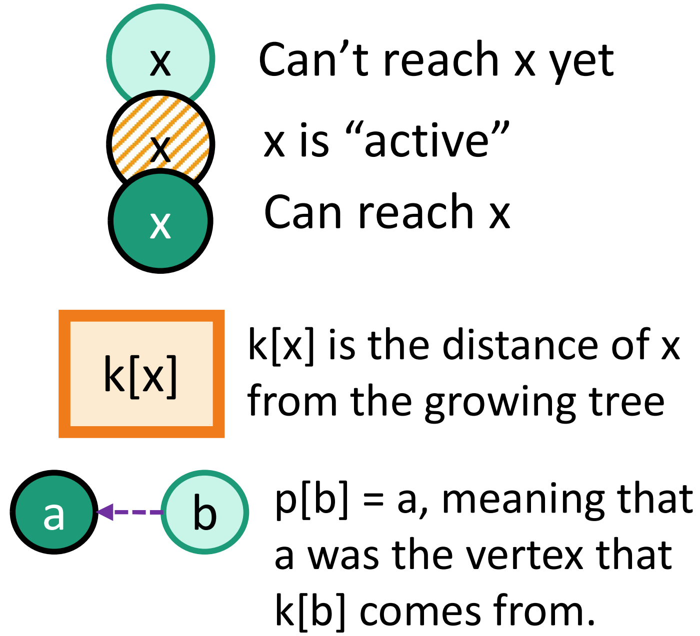
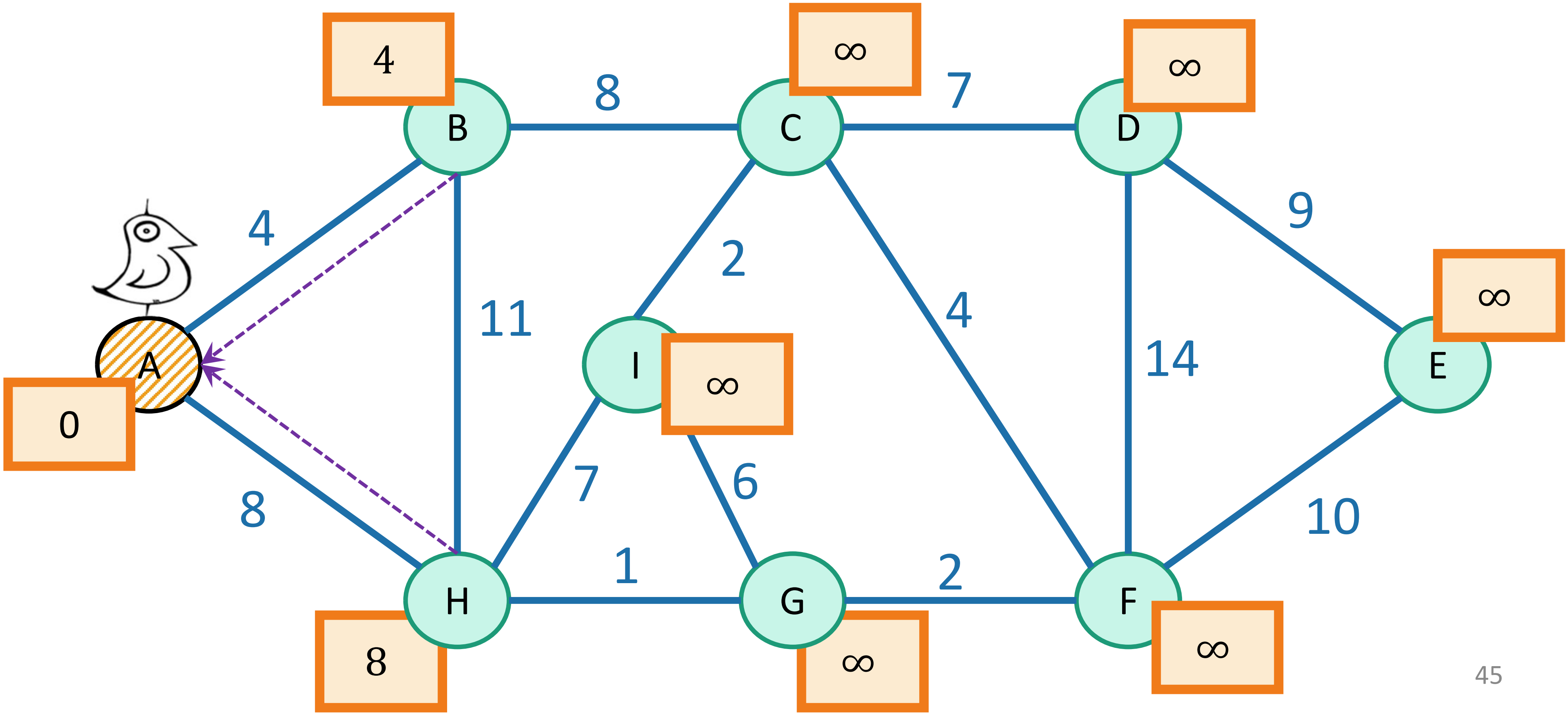
Until all the vertices are **reached:**

- Activate the **unreached** vertex u with the smallest key.
- **for each** of u's unreached neighbors v:
  - k[v] = min(k[v], weight(u,v))
  - if k[v] updated, p[v] = u





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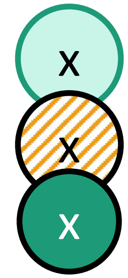
- Mark u as **reached,** and **add (p[u],u) to MST.**



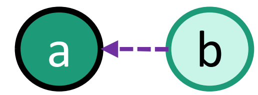
Can't reach x yet

x is "active"

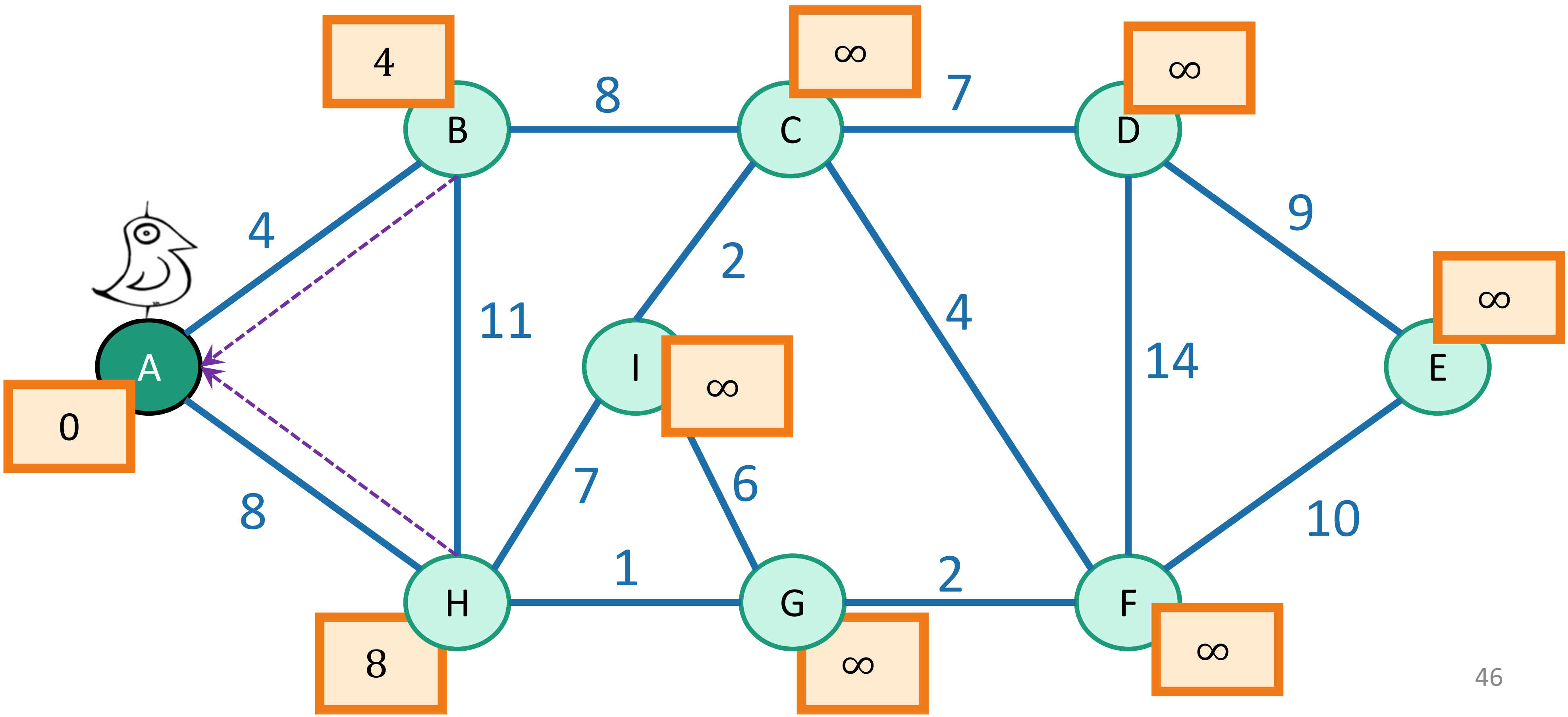
Can reach x

k[x]

k[x] is the distance of x from the growing tree



p[b] = a, meaning that a was the vertex that k[b] comes from.



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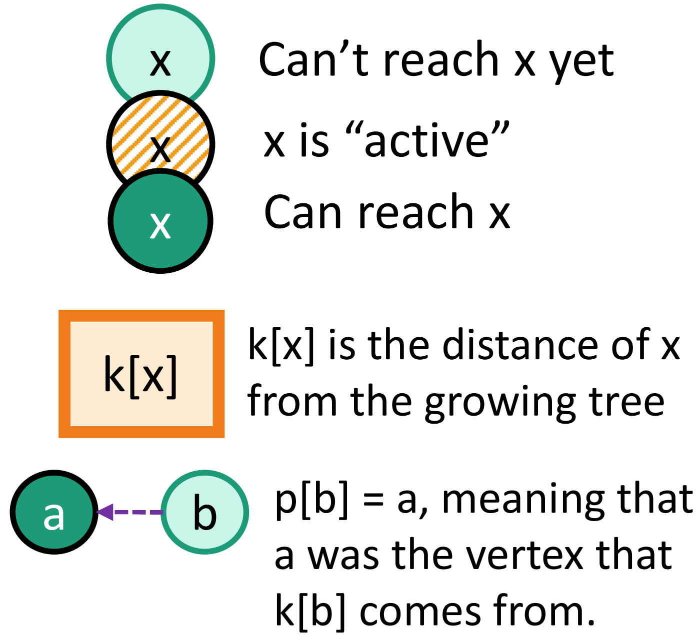

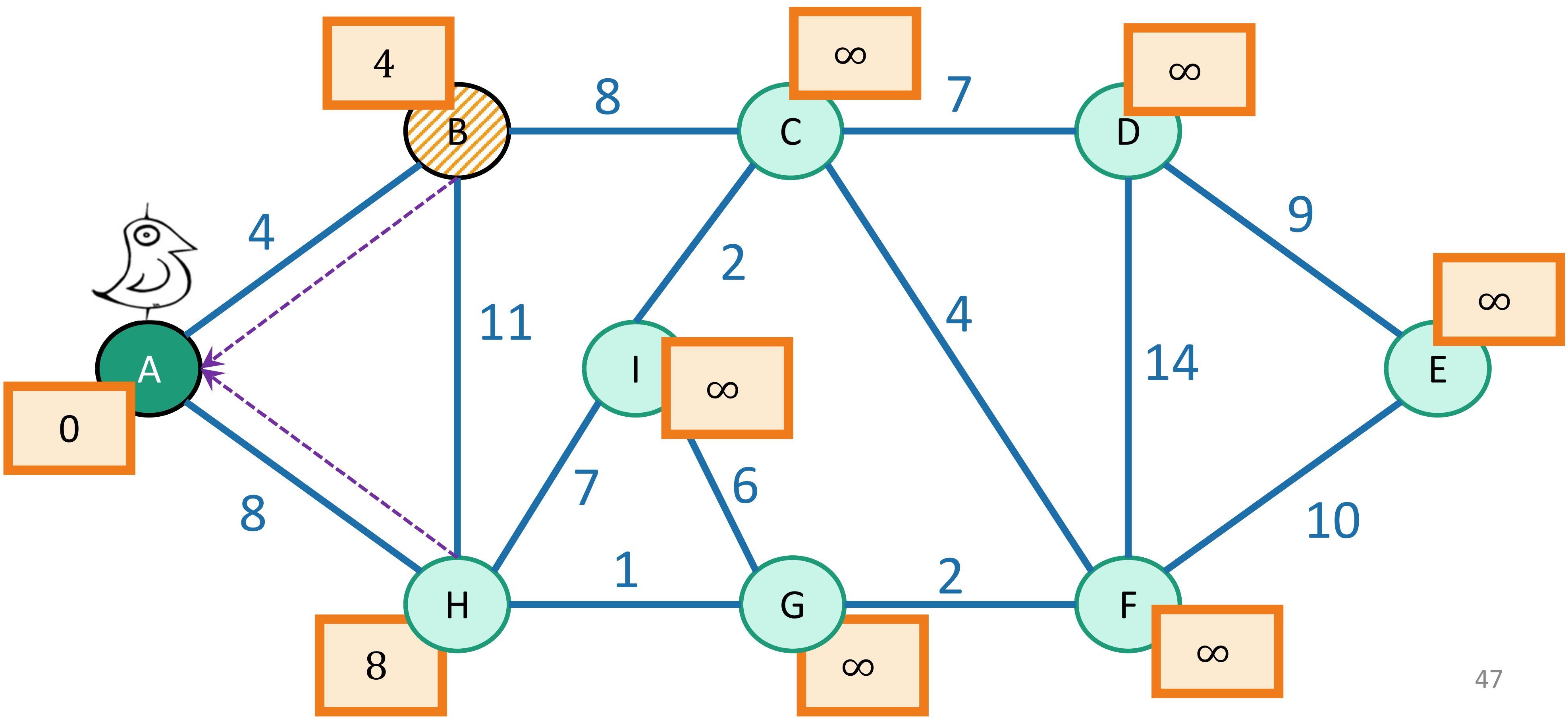

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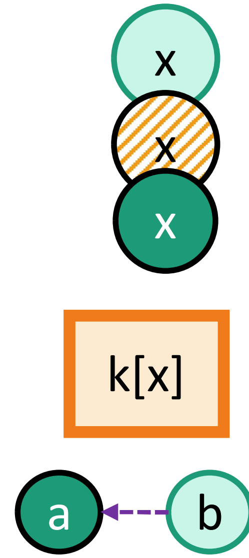

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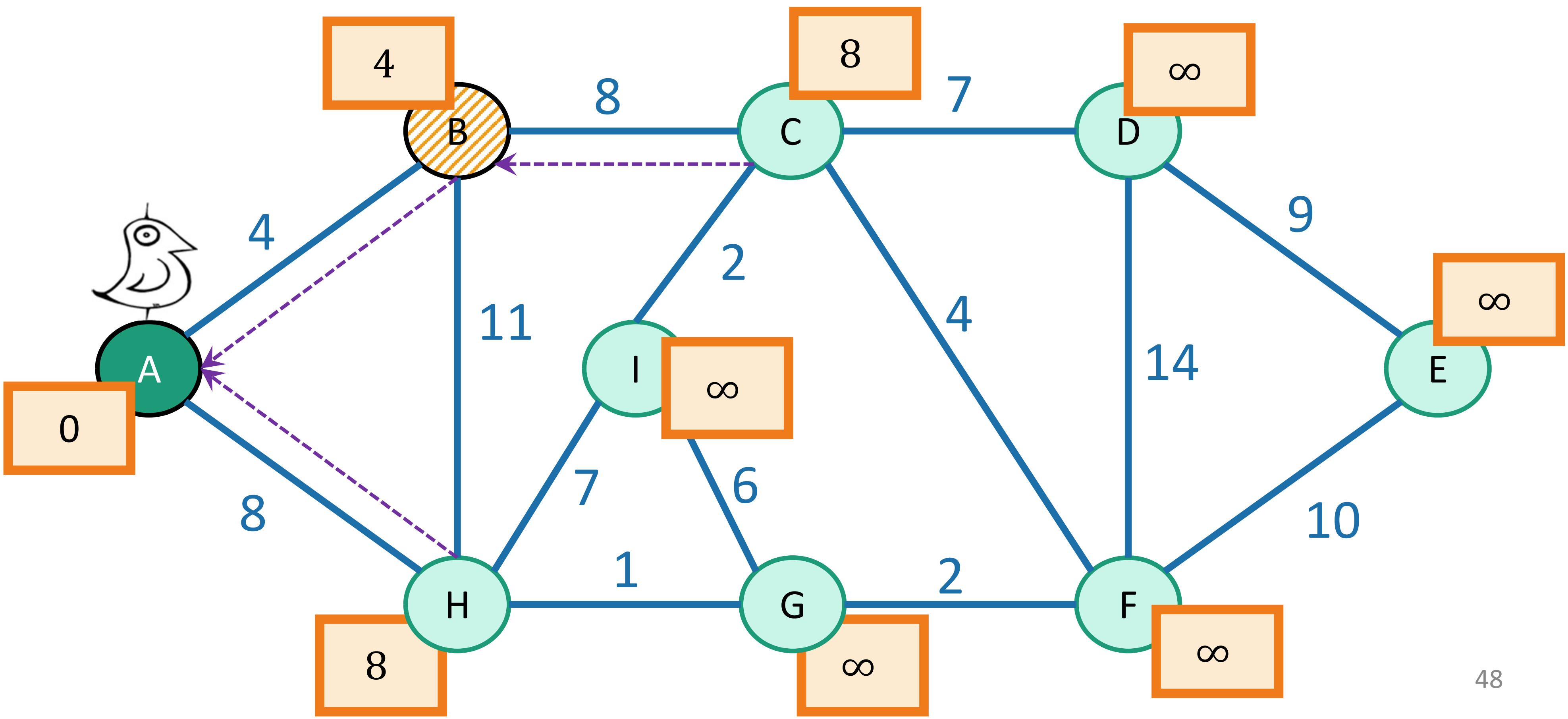

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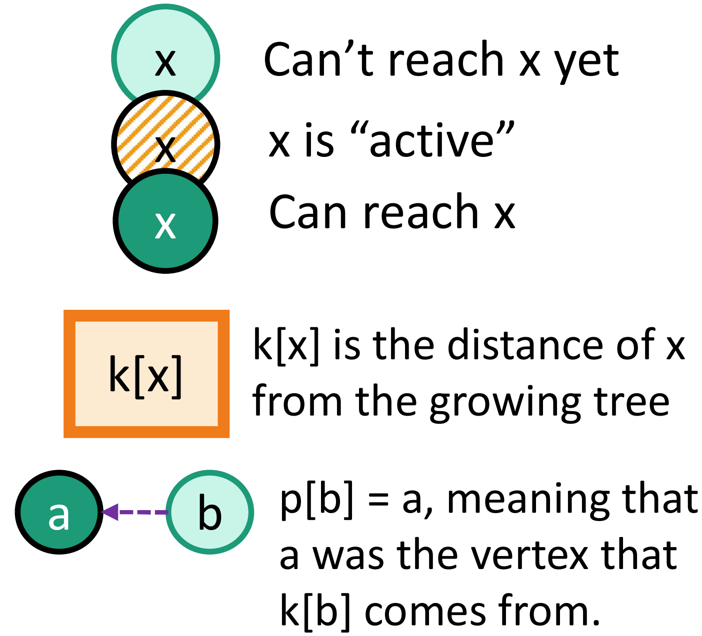

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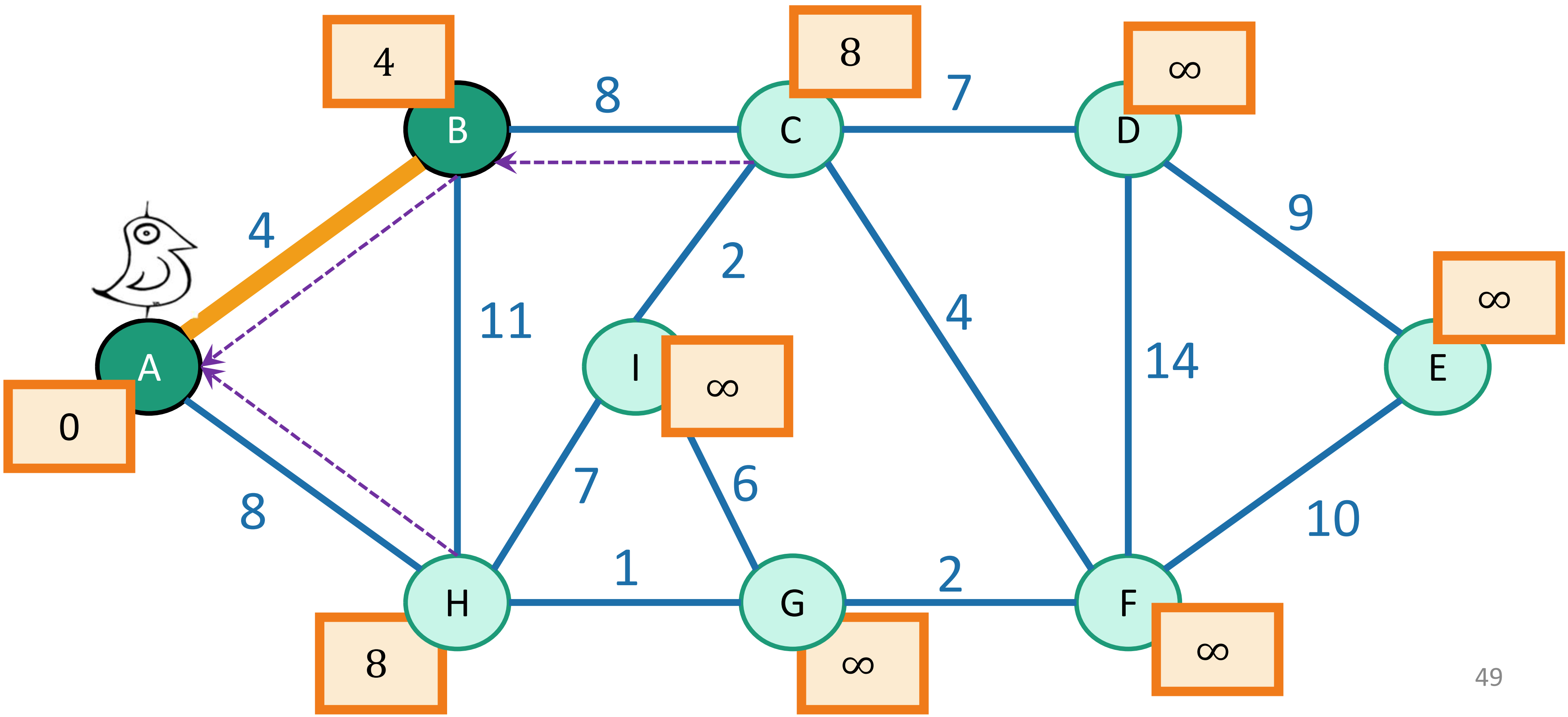

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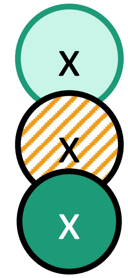

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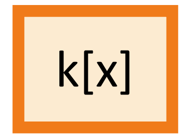

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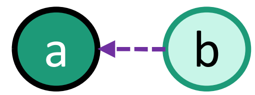

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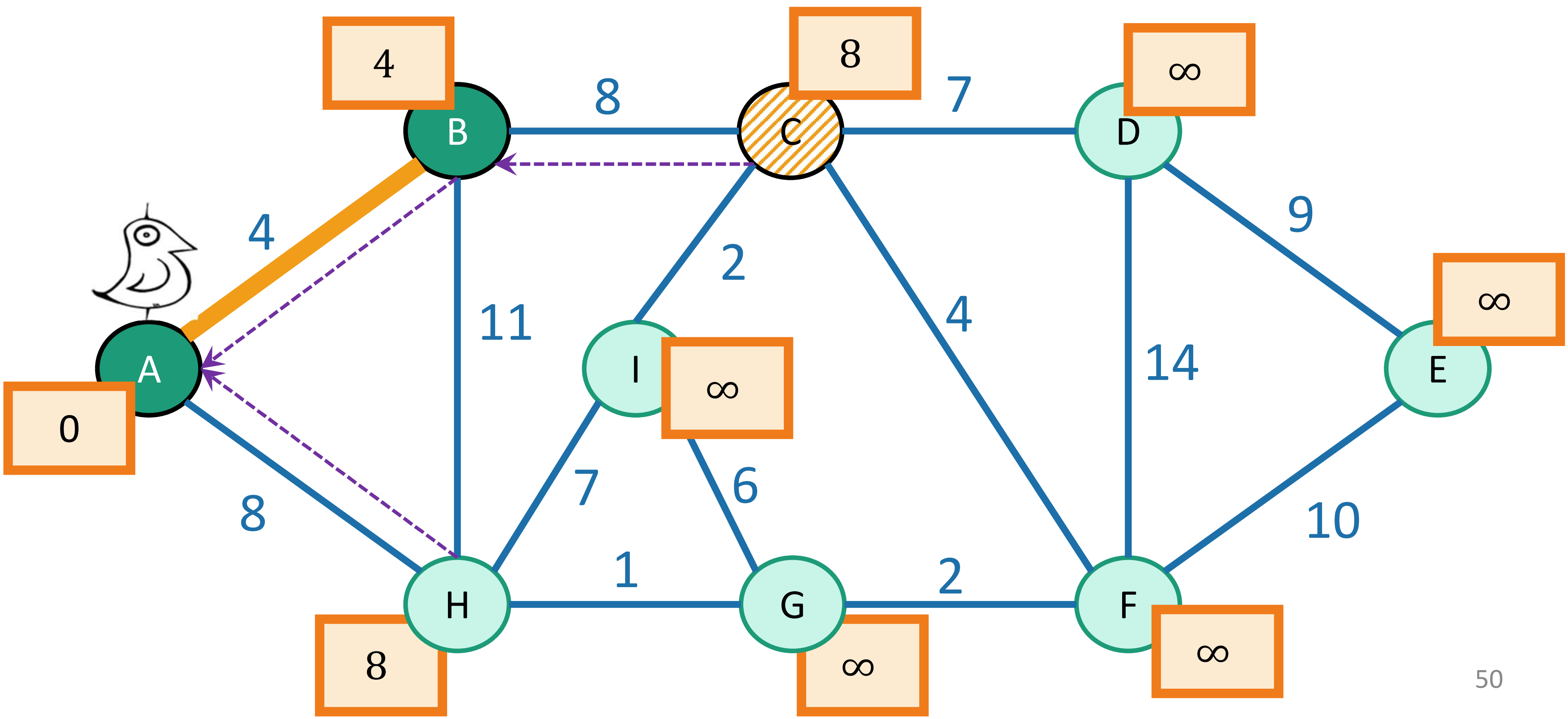

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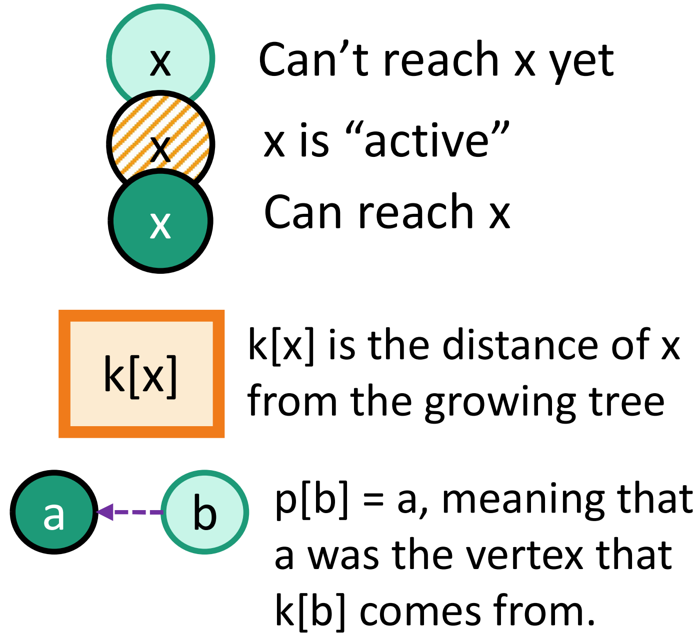

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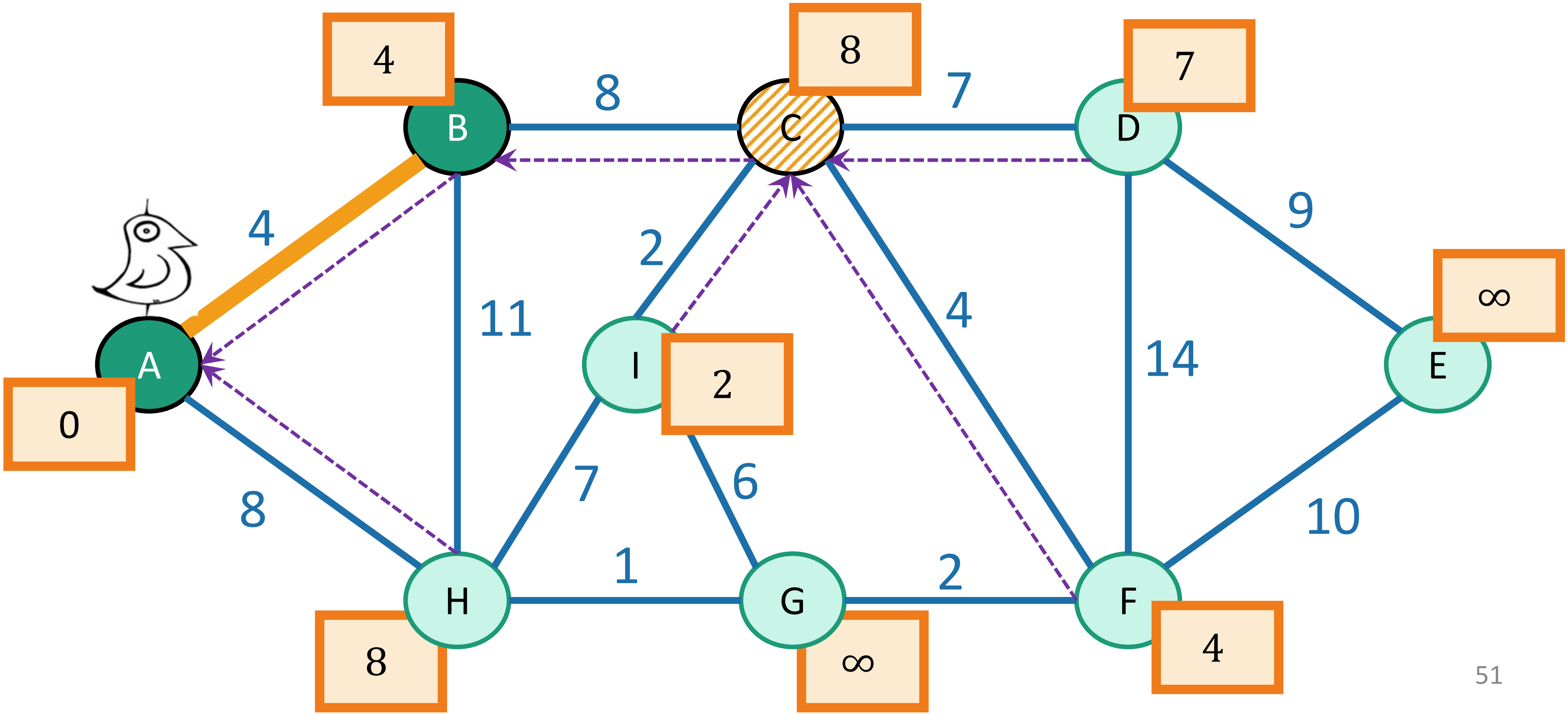

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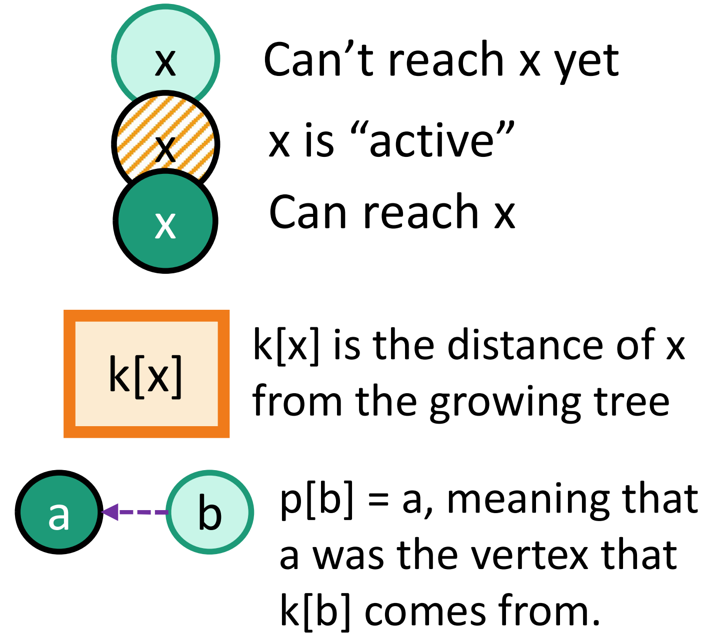

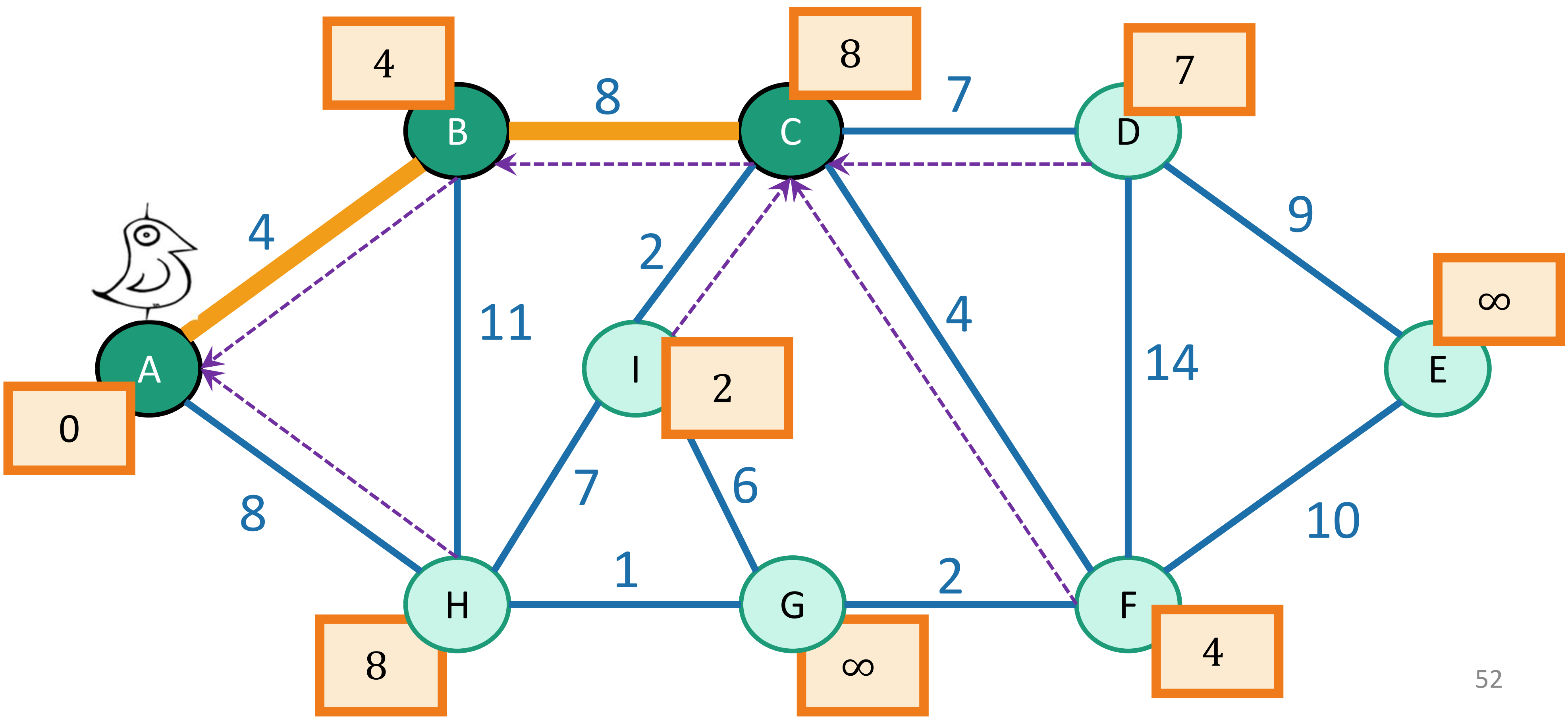

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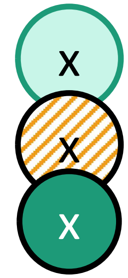

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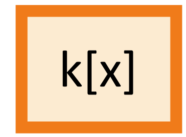

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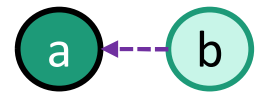

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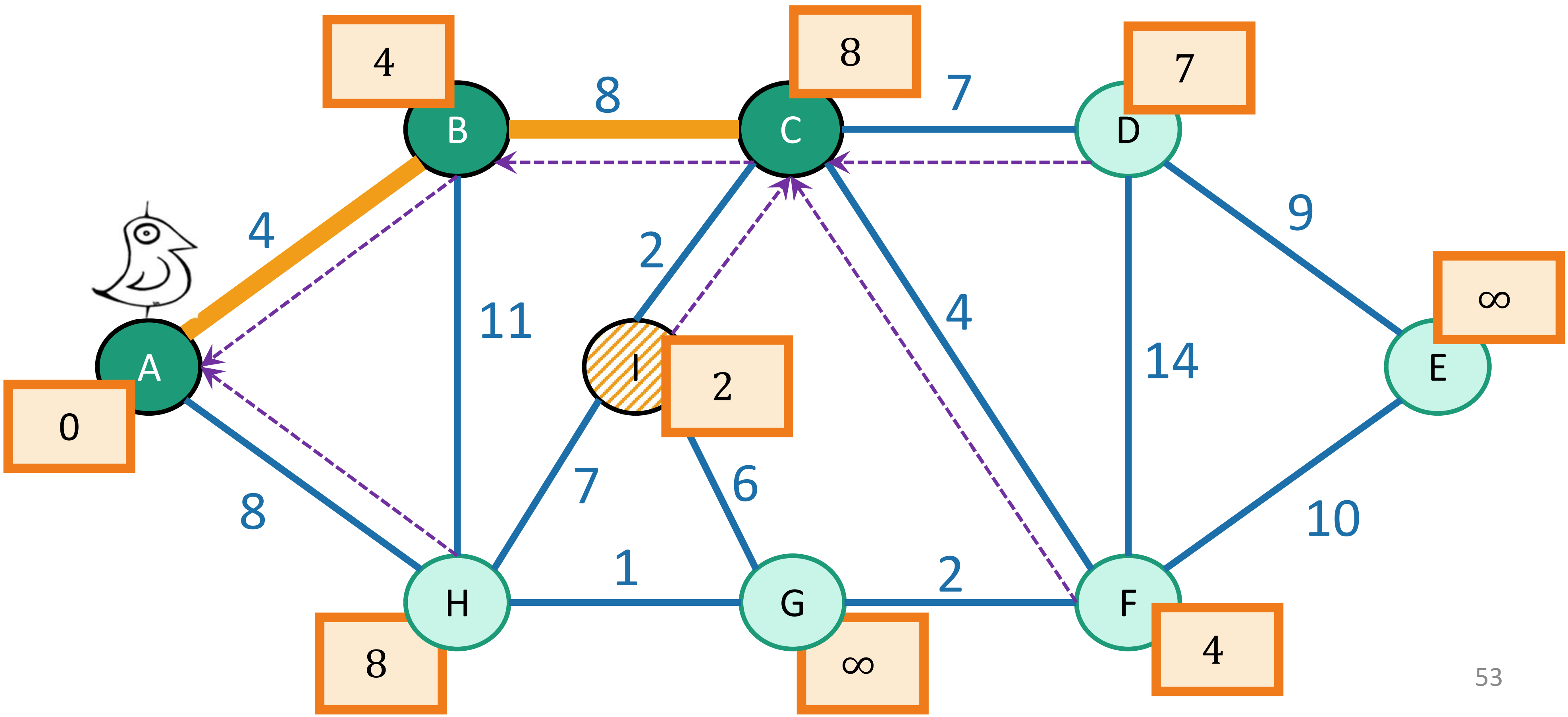

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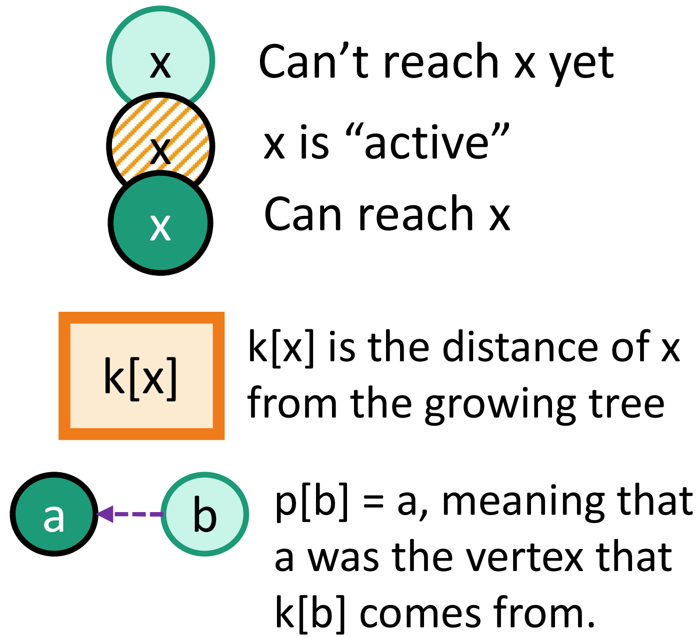

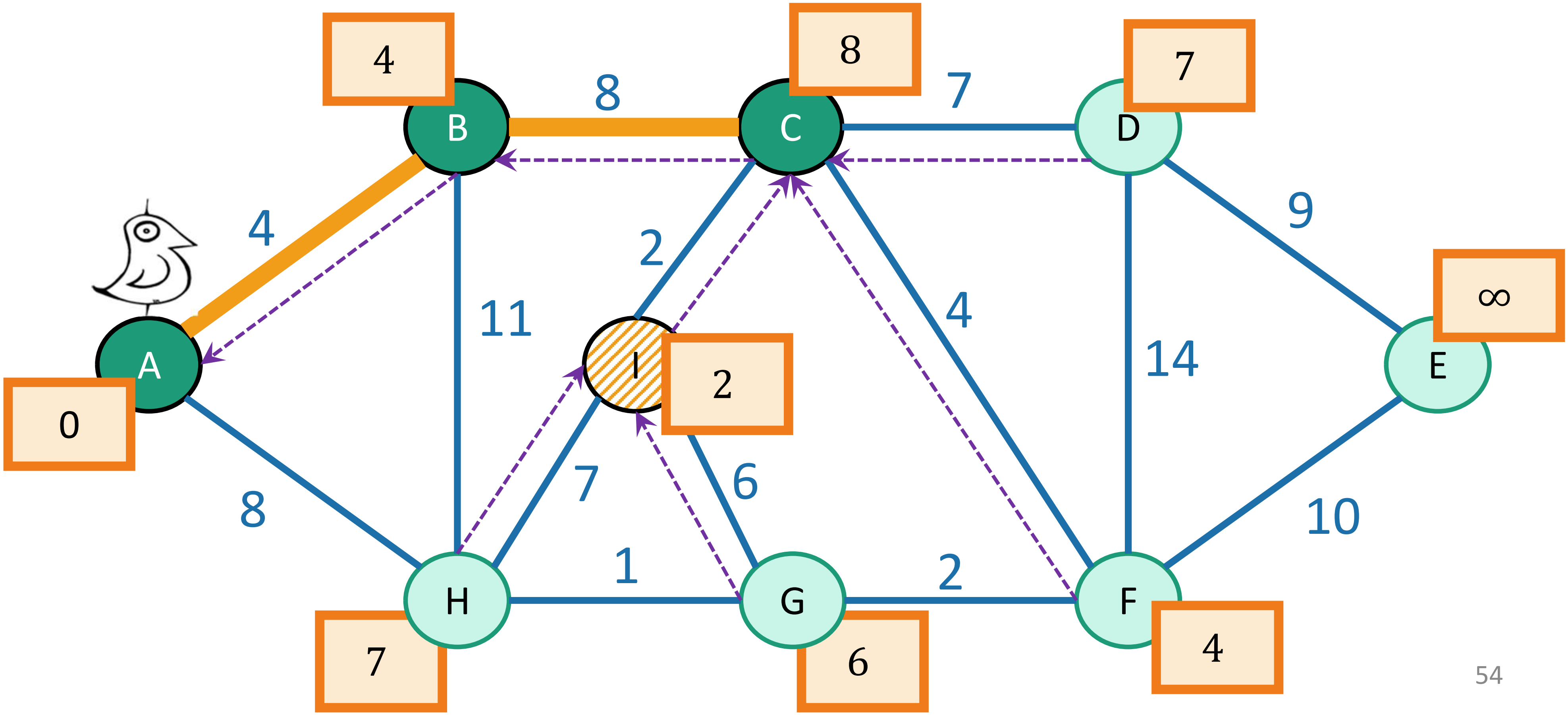

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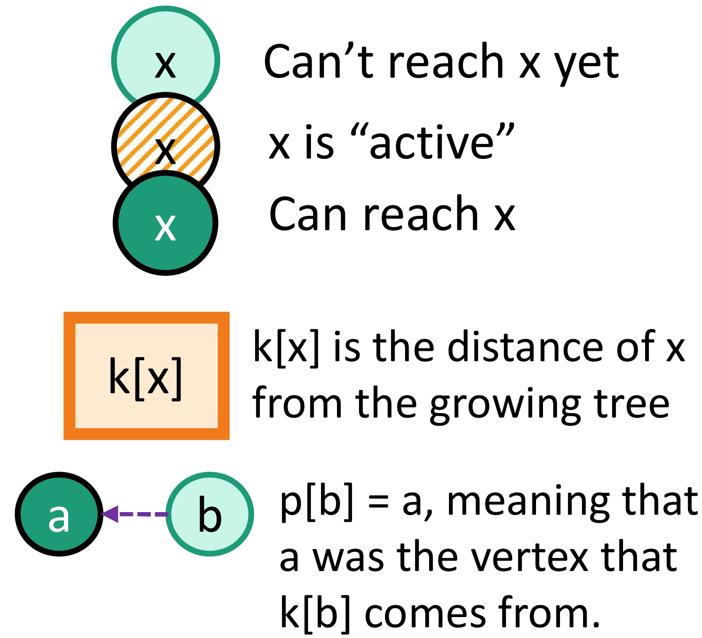

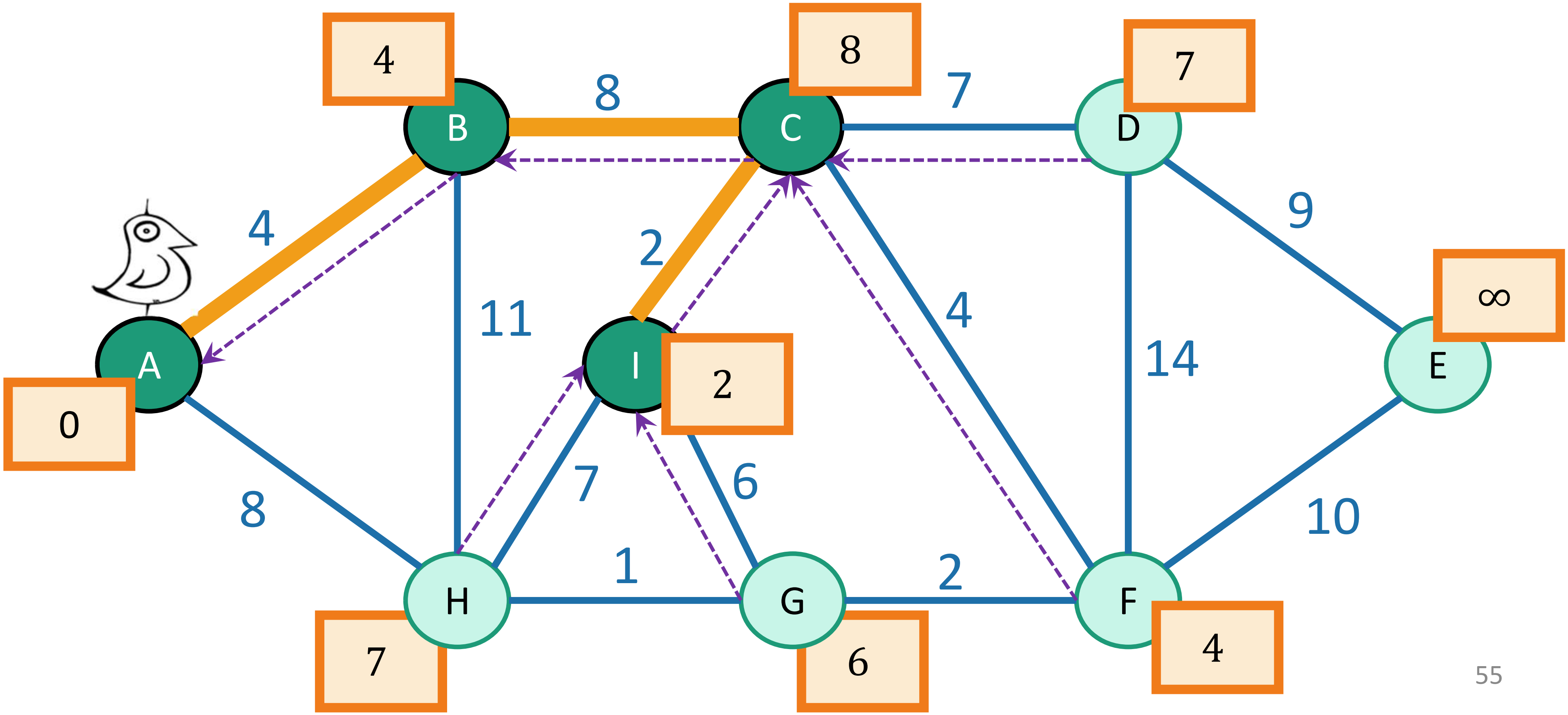

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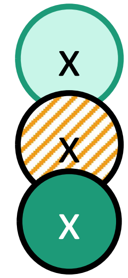

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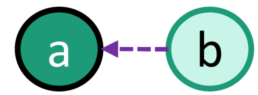

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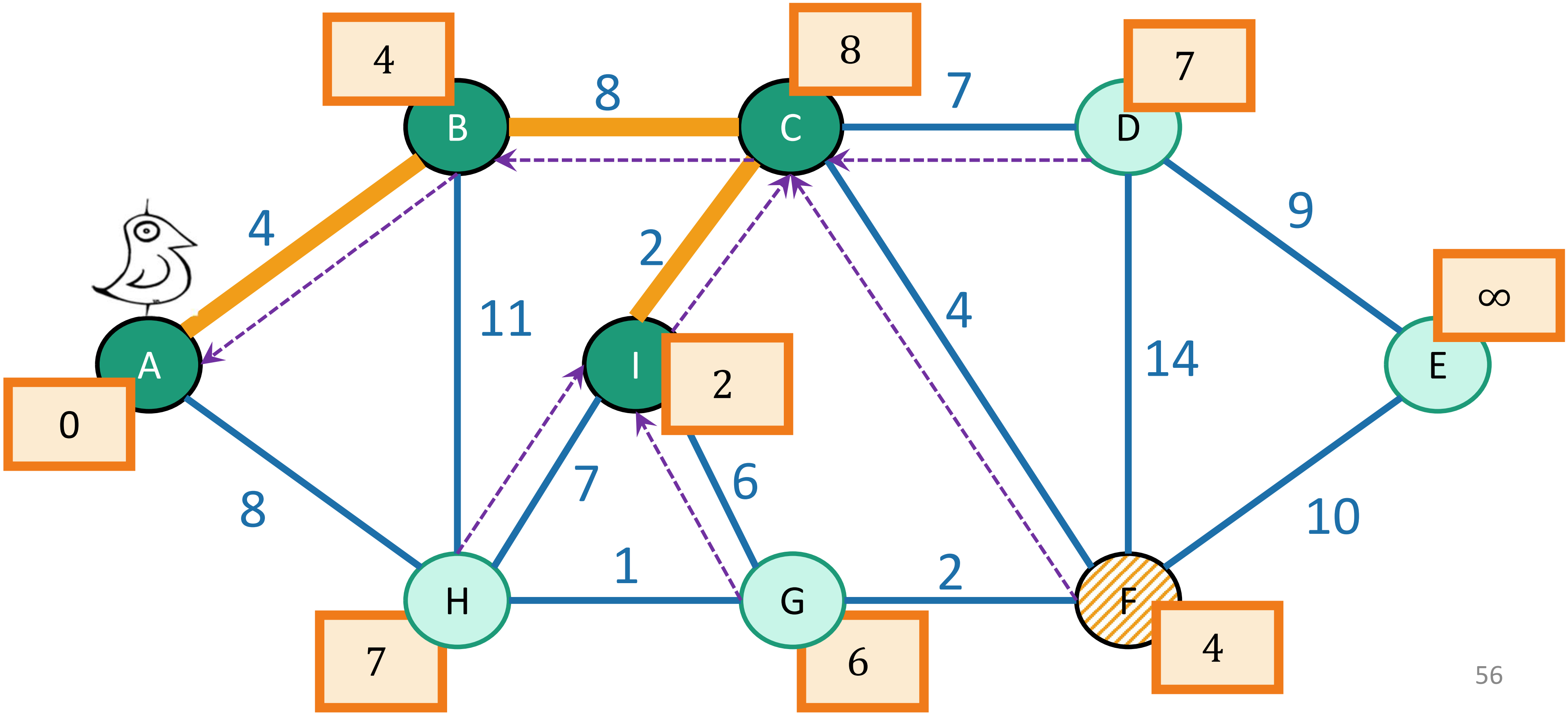

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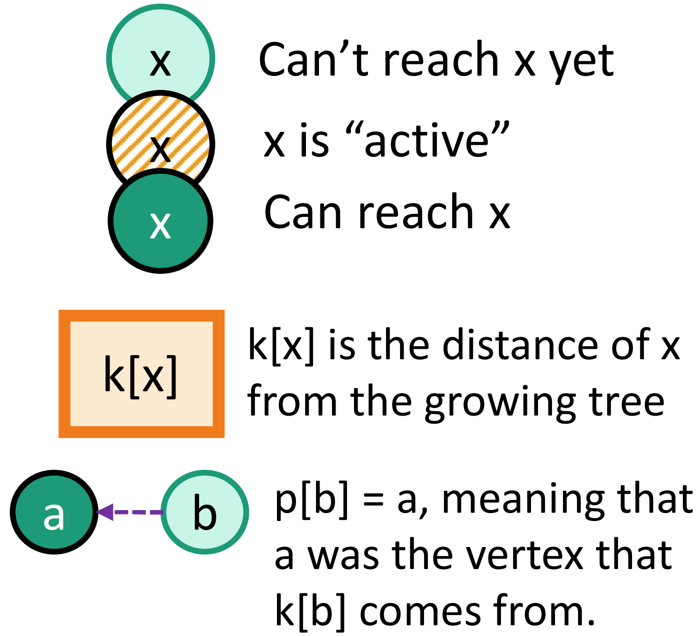

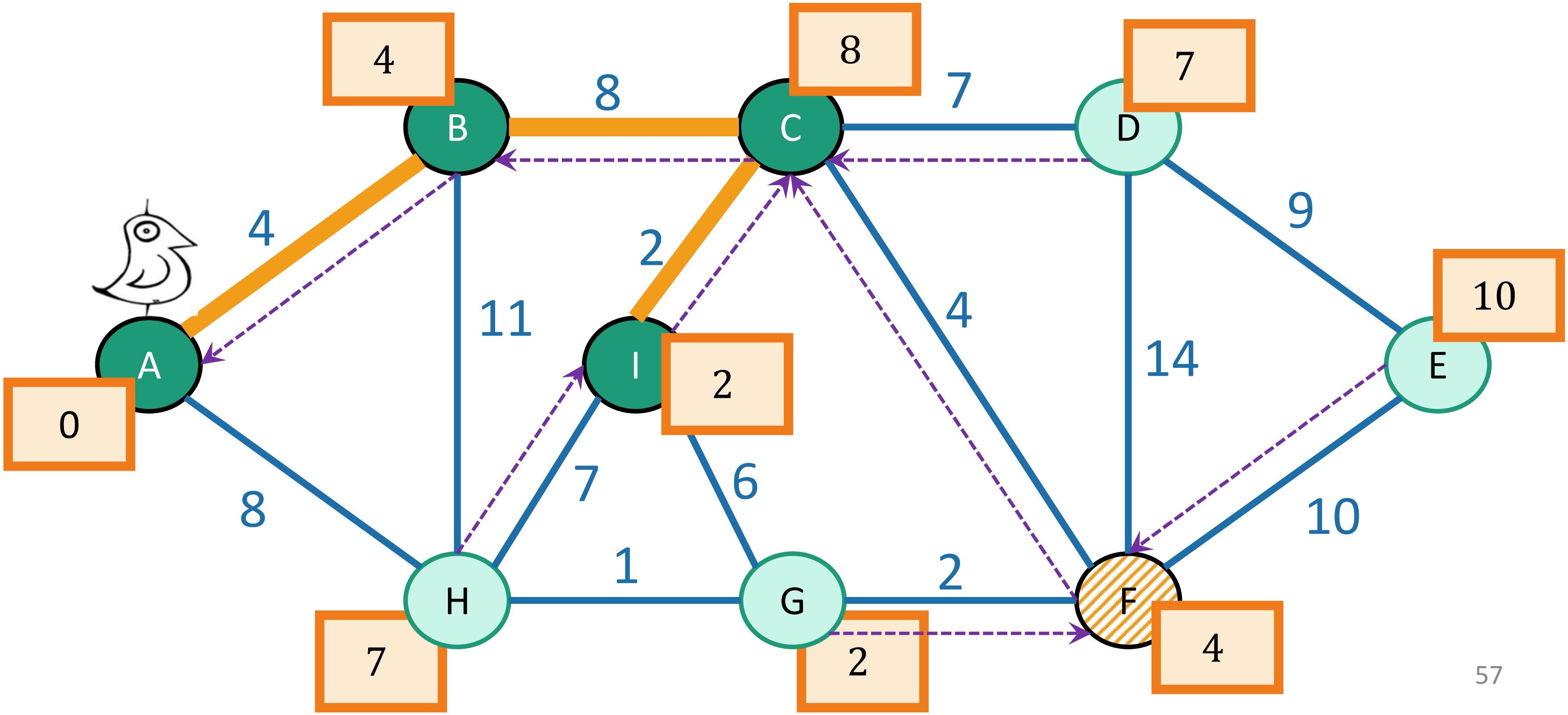

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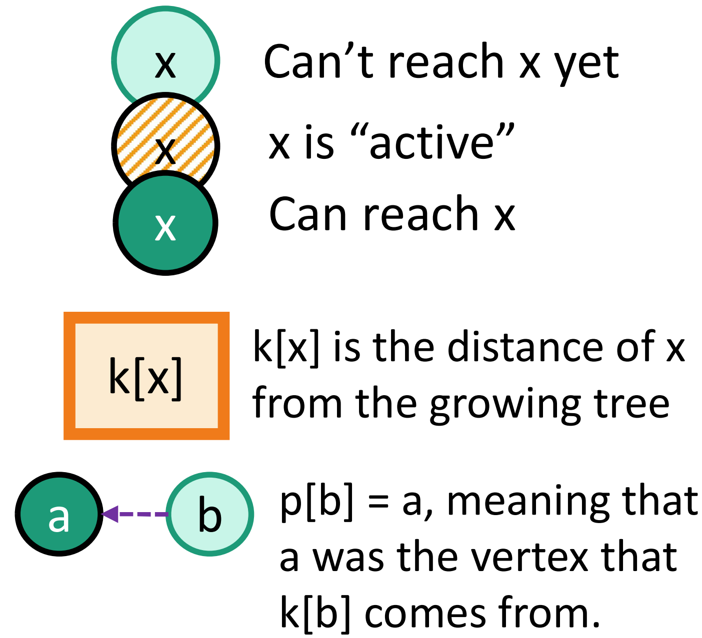

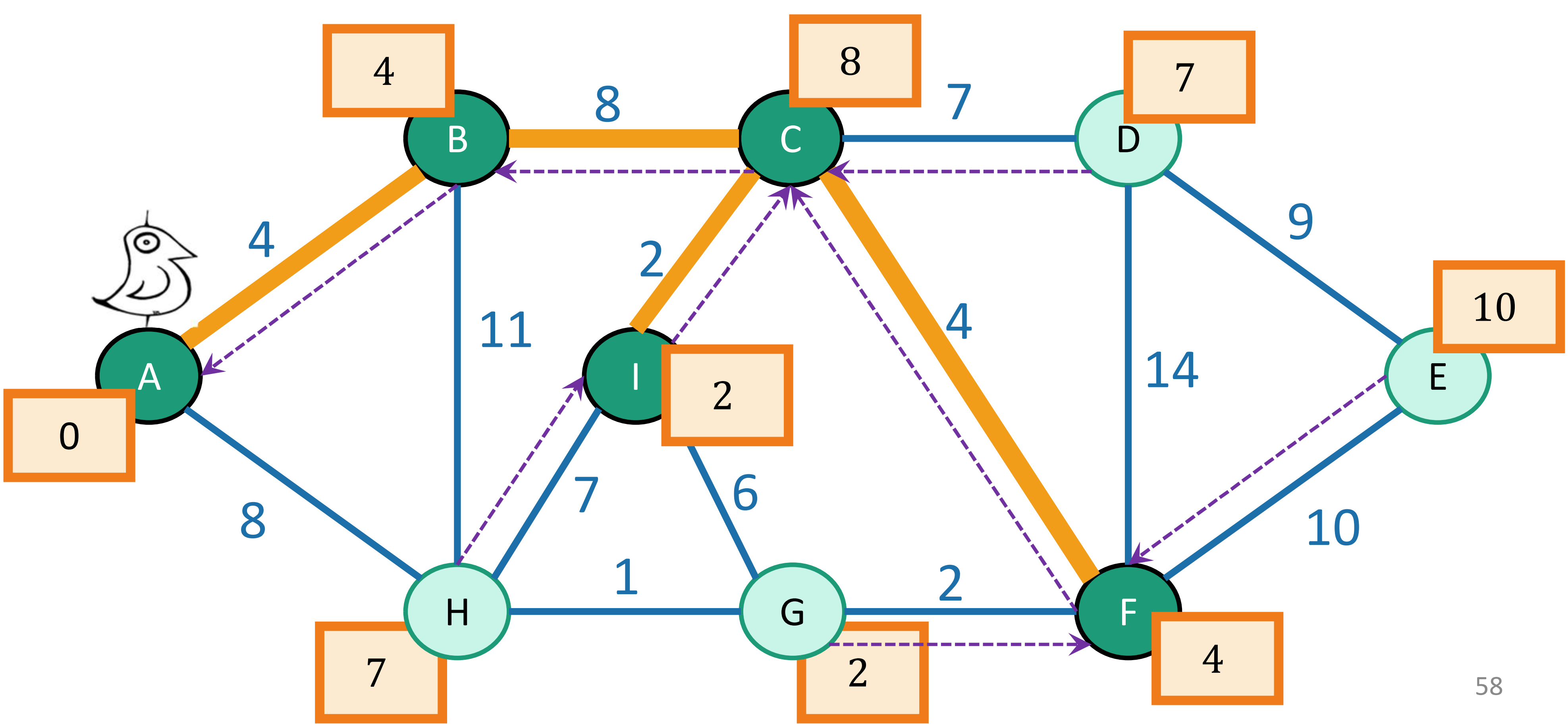

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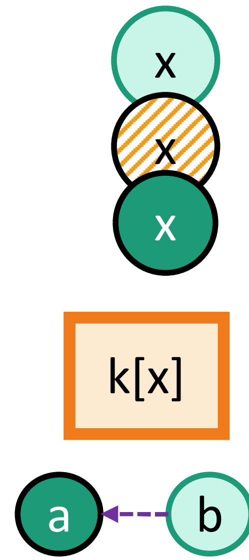

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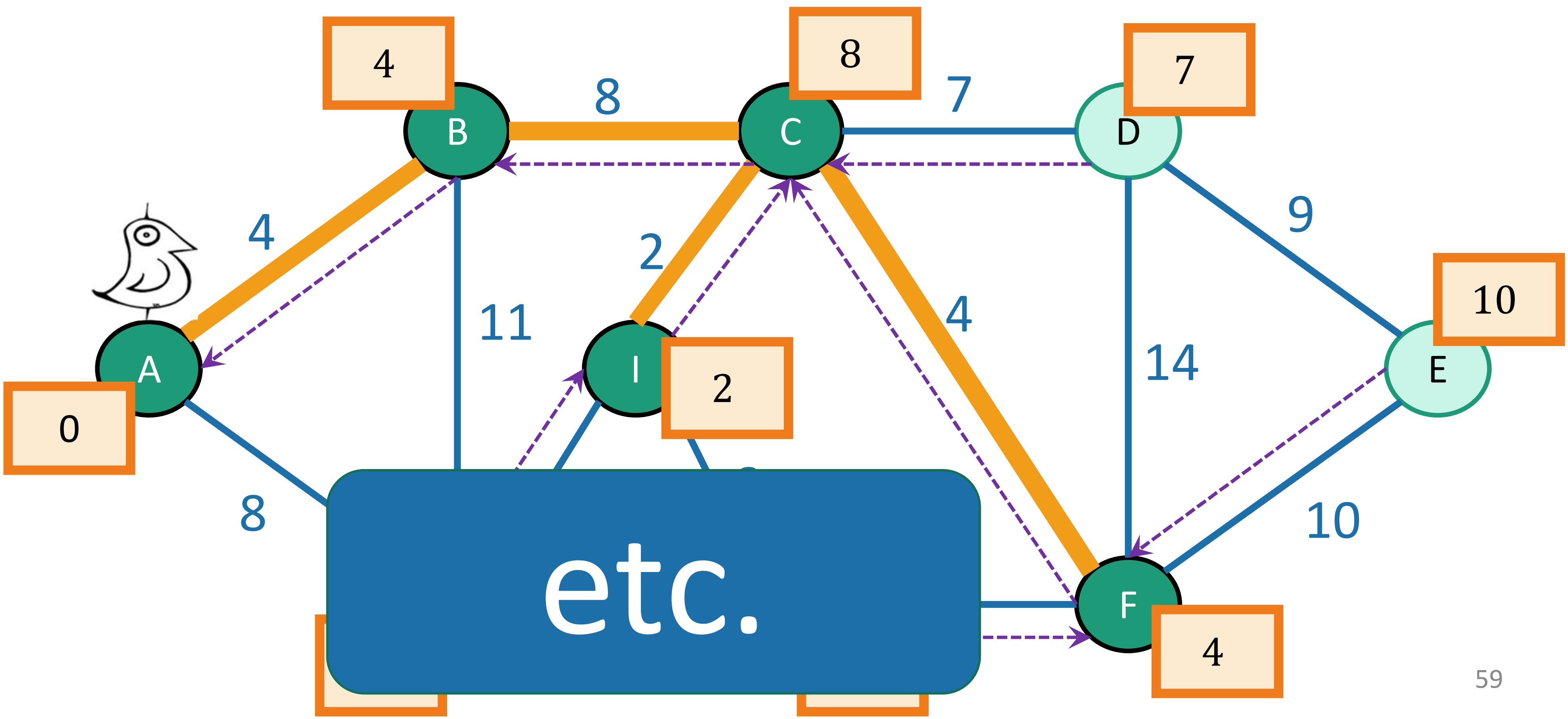
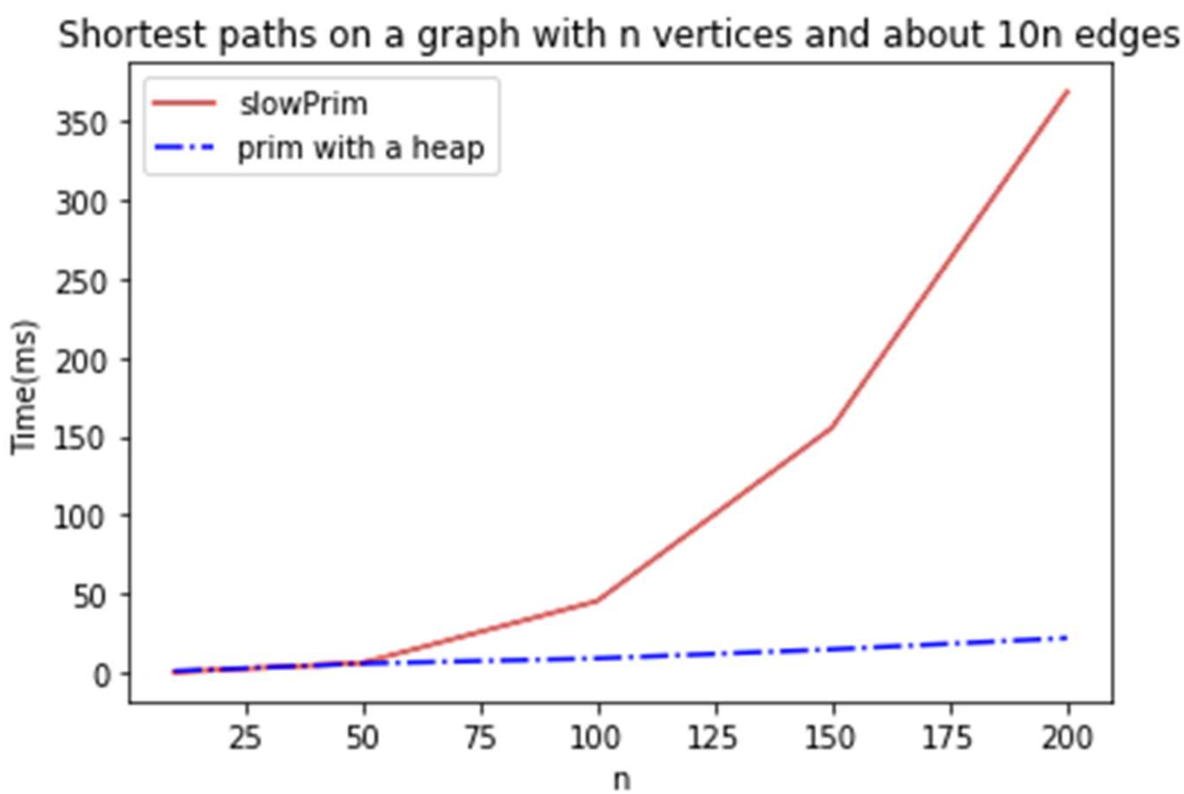


One thing that is similar:
Running time

- Exactly the same as Dijkstra:
  - O(mlog(n)) using a Red-Black tree as a priority queue.
  - O(m + nlog(n)) amortized time if we use a Fibonacci Heap.



Two questions

1. Does it work?
    - That is, does it actually return a MST?
        - **Yes!**

2. How do we actually implement this?
    - the pseudocode above says “slowPrim”…
        - **Implement it basically the same way we’d implement Dijkstra!**

What have we learned?

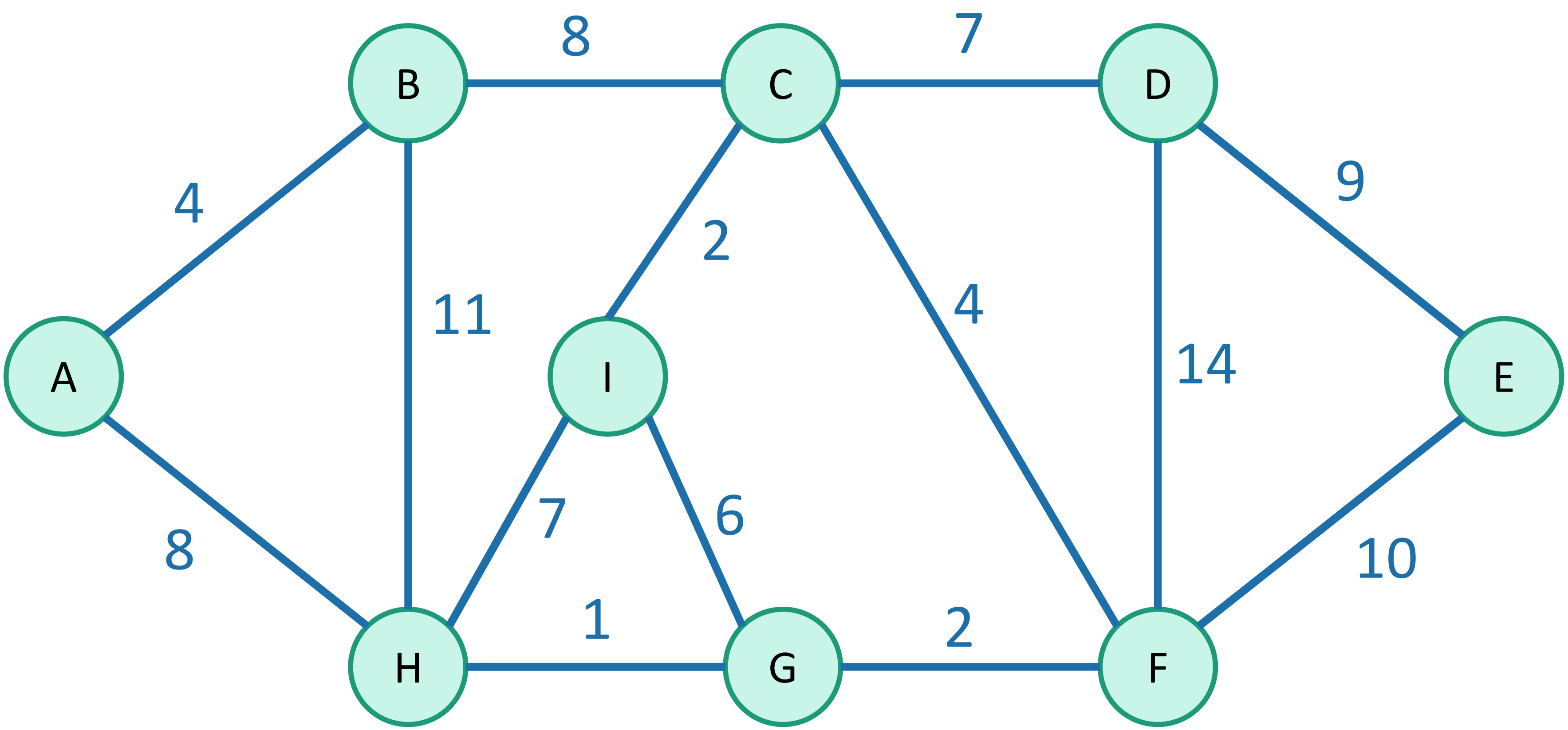
- Prim's algorithm greedily grows a tree
  - smells a lot like Dijkstra's algorithm
- It finds a Minimum Spanning Tree!
  - in time **O(mlog(n))** if we implement it with a Red-Black Tree.
  - In amortized time **O(m + nlog(n))** with a Fibonacci heap.
- To prove it worked, we followed the same recipe for greedy algorithms we saw last time.
  - Show that, at every step, we **don't rule out success.**

That’s not the only greedy algorithm for MST!

That’s not the only greedy algorithm

what if we just always take the cheapest edge?

whether or not it’s connected to what we have so far?



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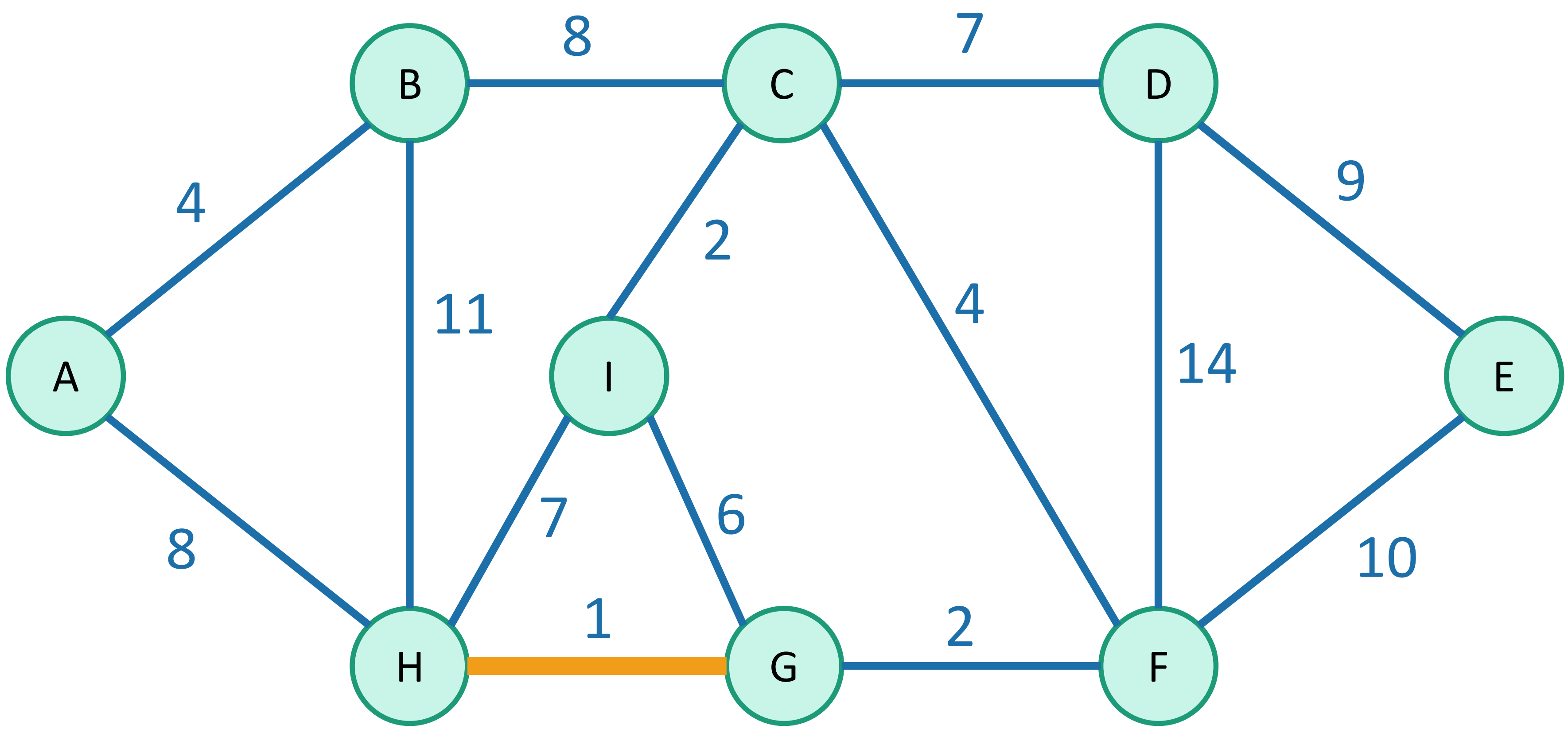

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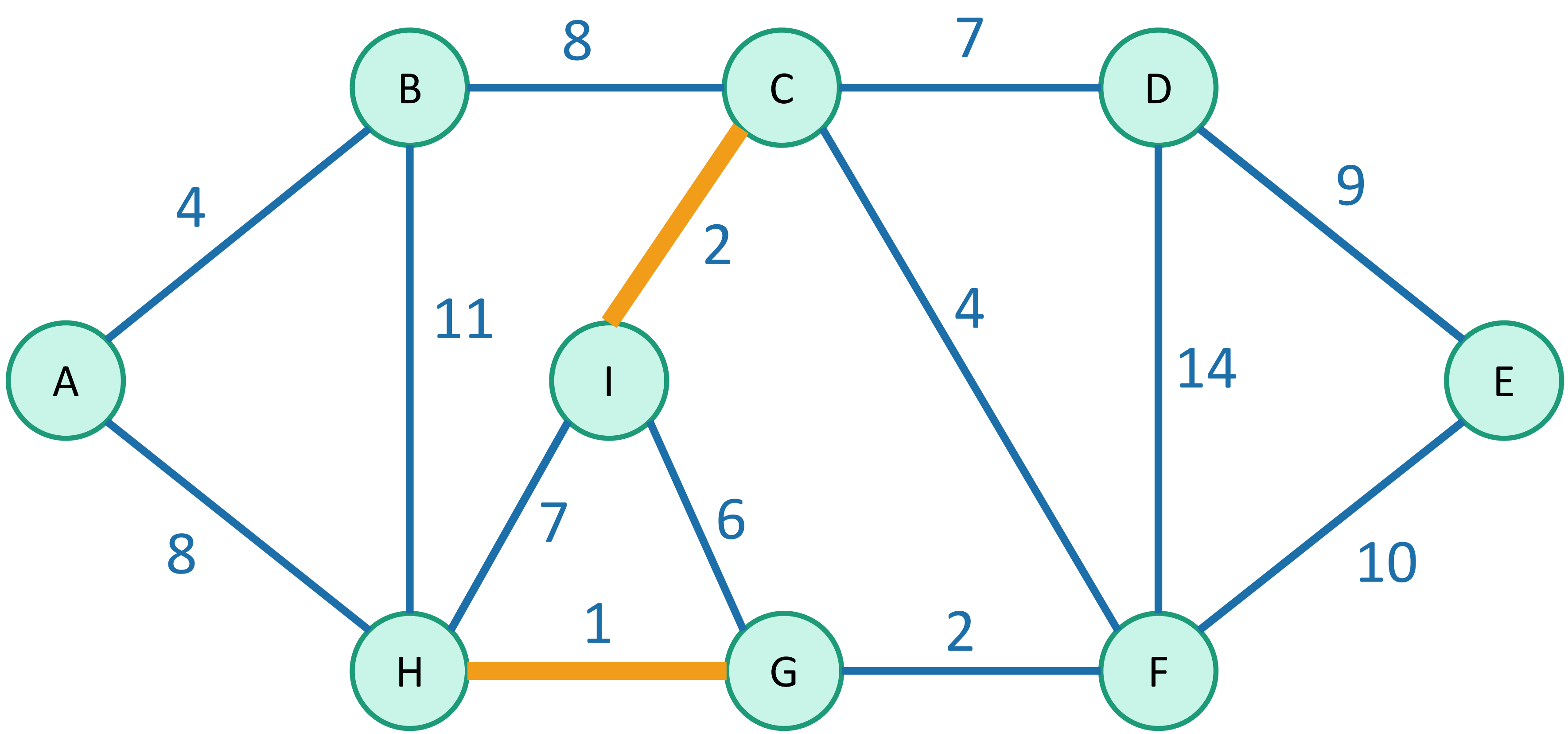

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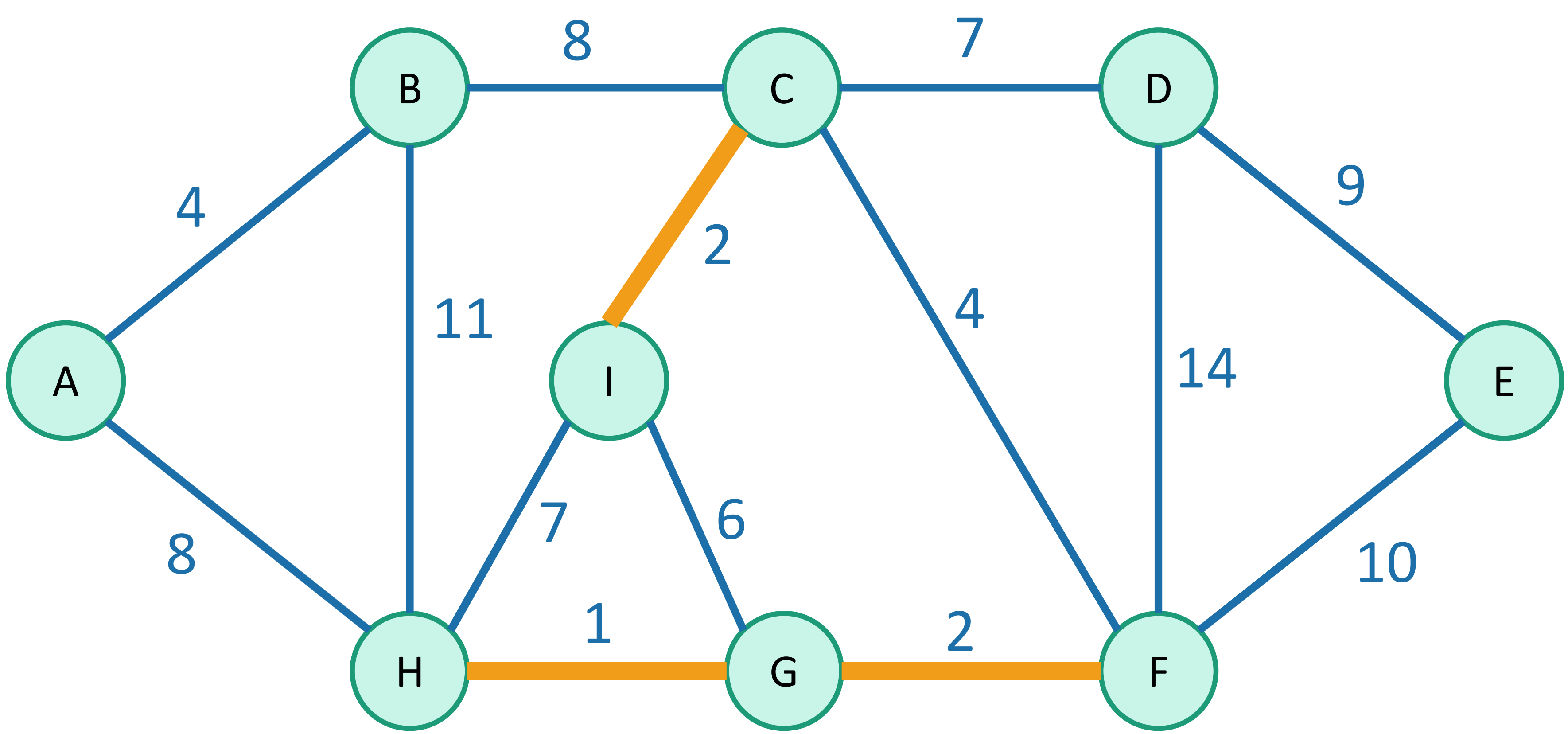

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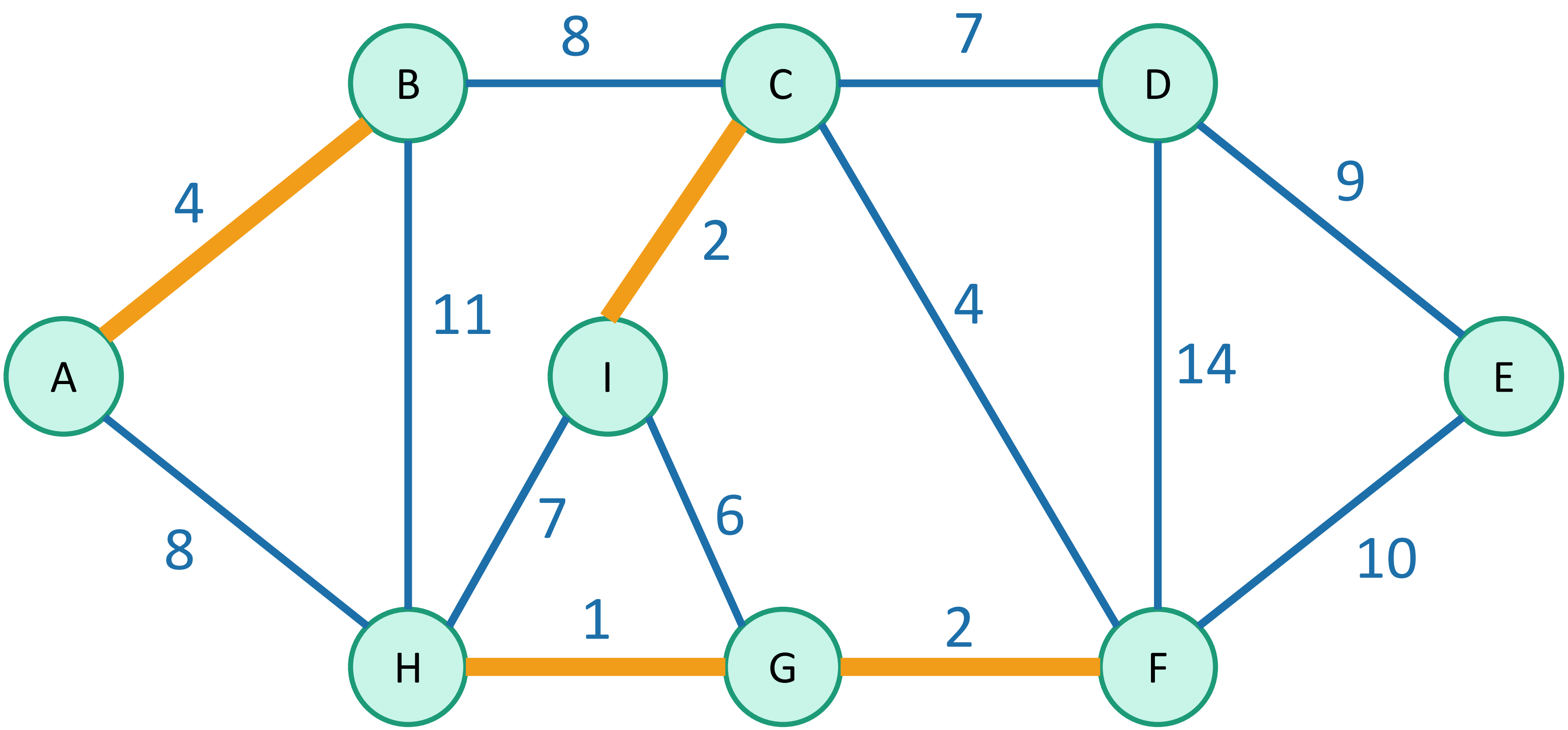

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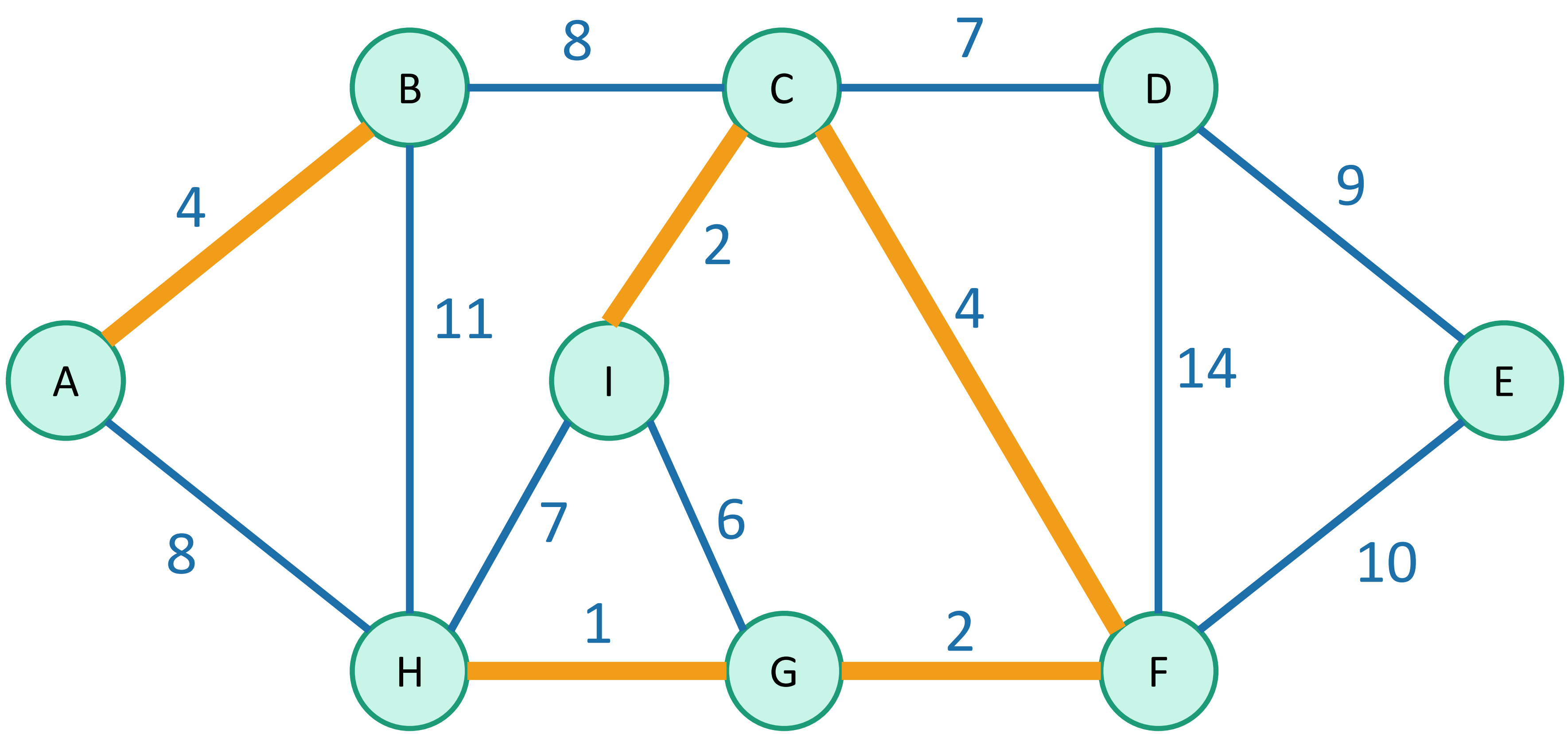

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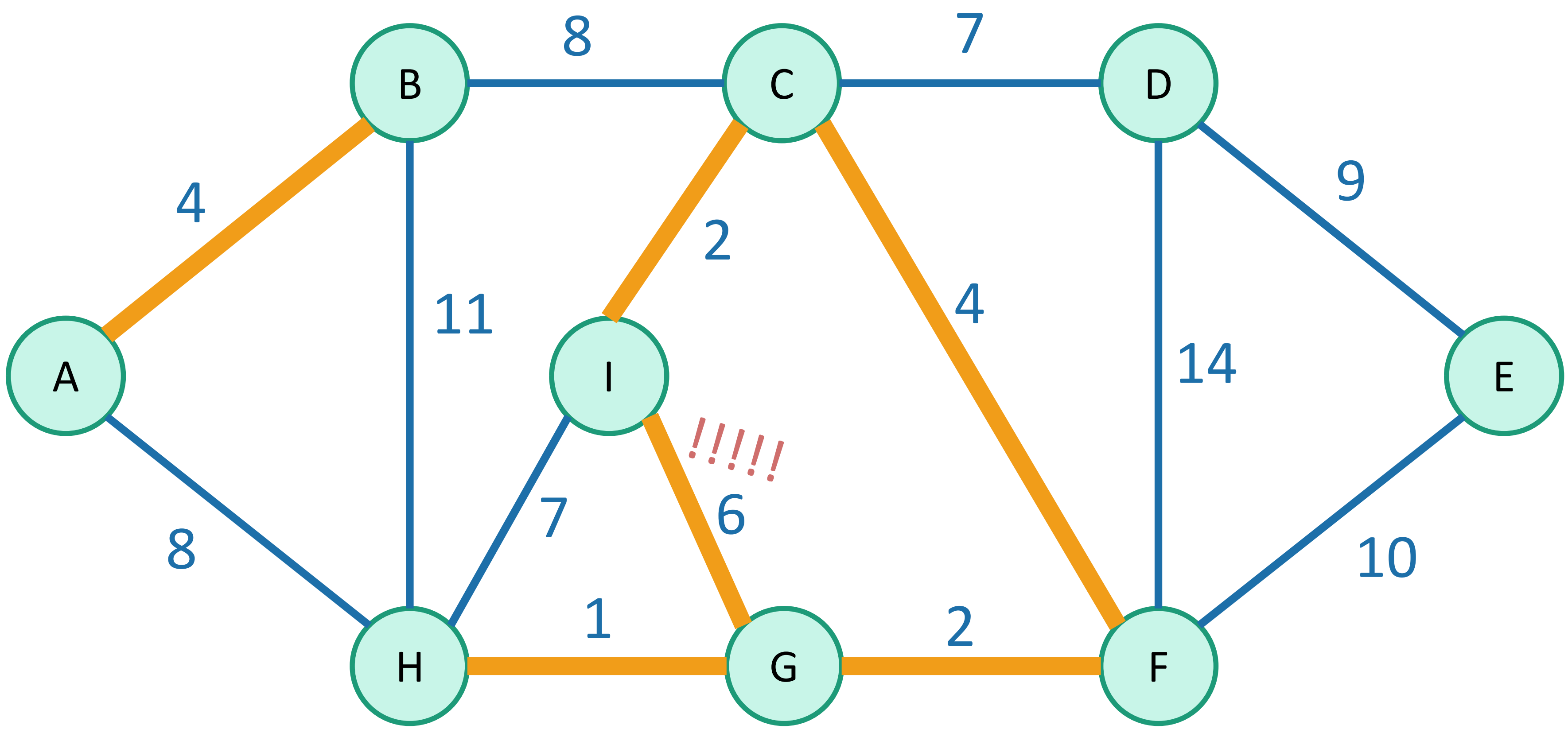
That won’t cause a cycle



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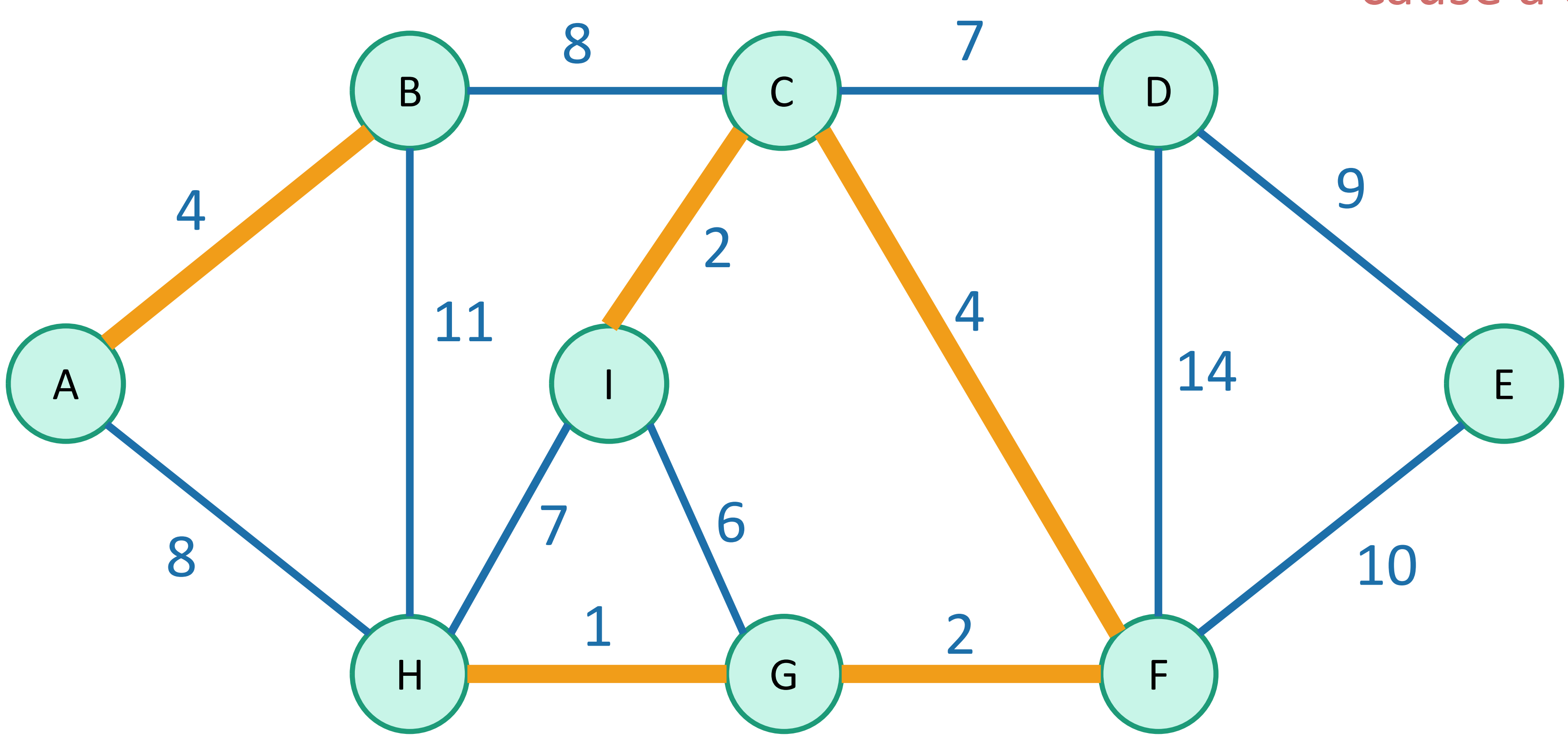

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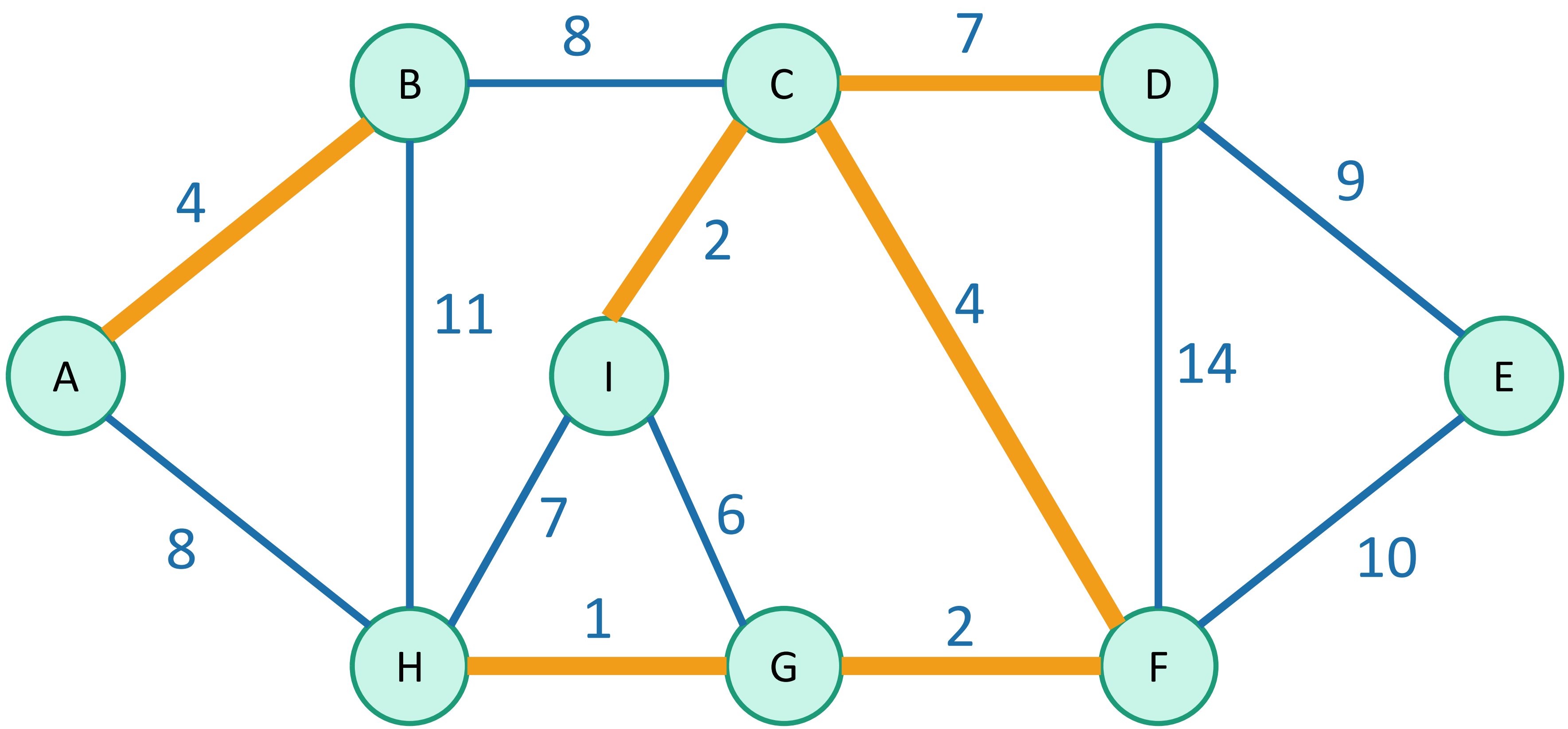

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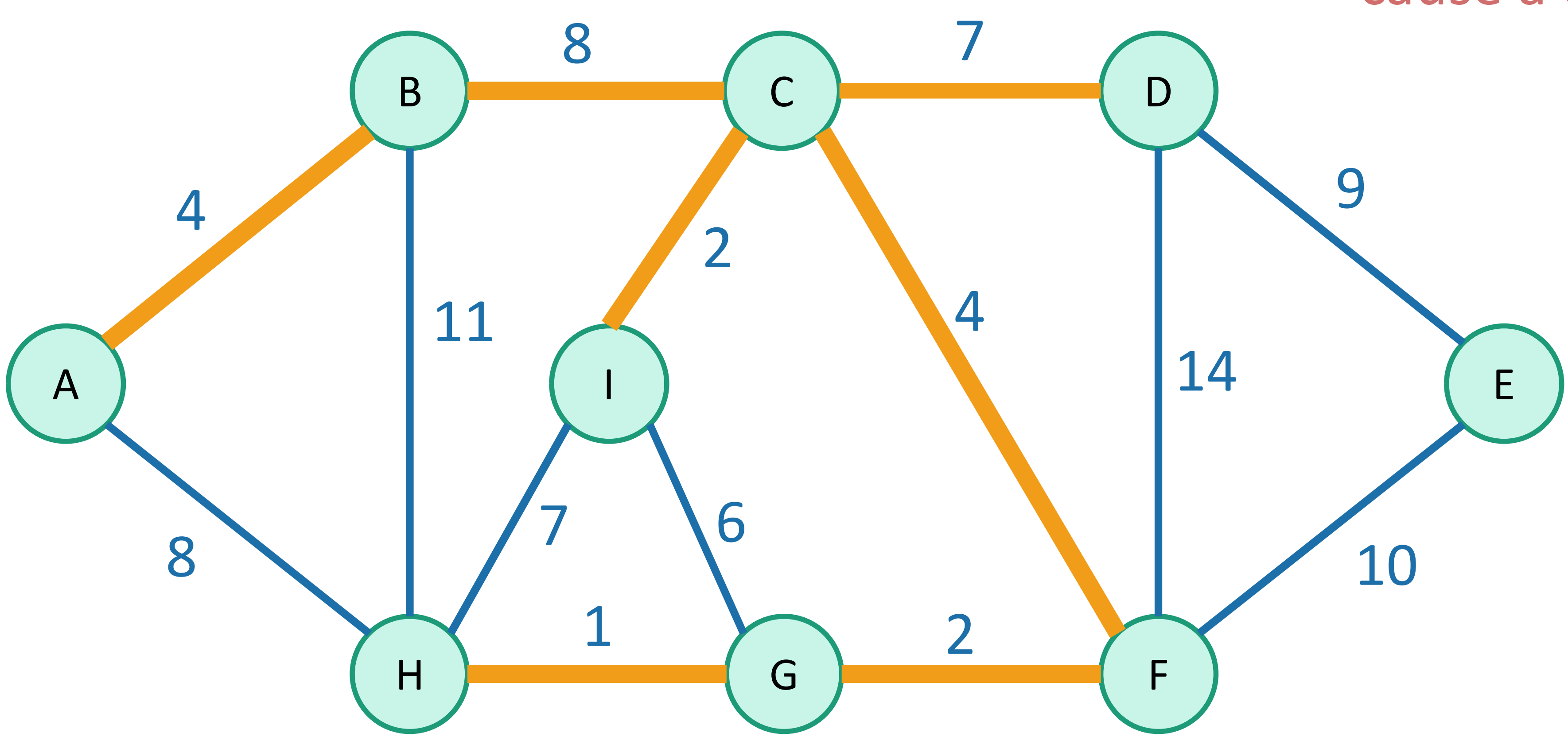

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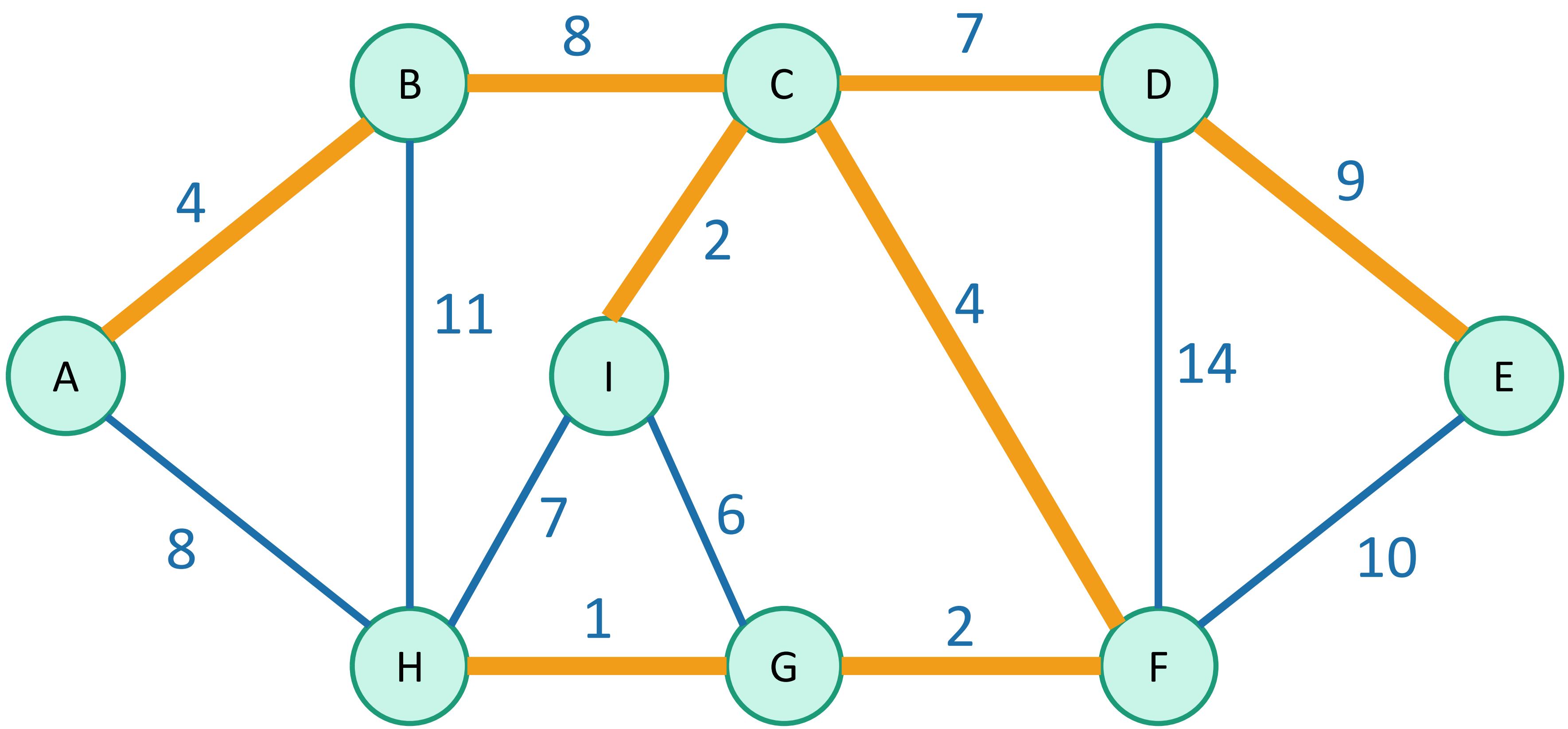


We've discovered Kruskal's algorithm!

- **slowKruskal**(G = (V,E)):
  - Sort the edges in E by non-decreasing weight.
  - MST = {}
  - **for** e in E (in sorted order):
    - **if** adding e to MST won't cause a cycle:
      - add e to MST.
  - **return** MST

m iterations through this loop

How do we check this?

How **would** you figure out if added e would make a cycle in this algorithm?

<u>Naively, the running time is ???:</u>

- For each of m iterations of the for loop:
  - Check if adding e would cause a cycle…

Two questions

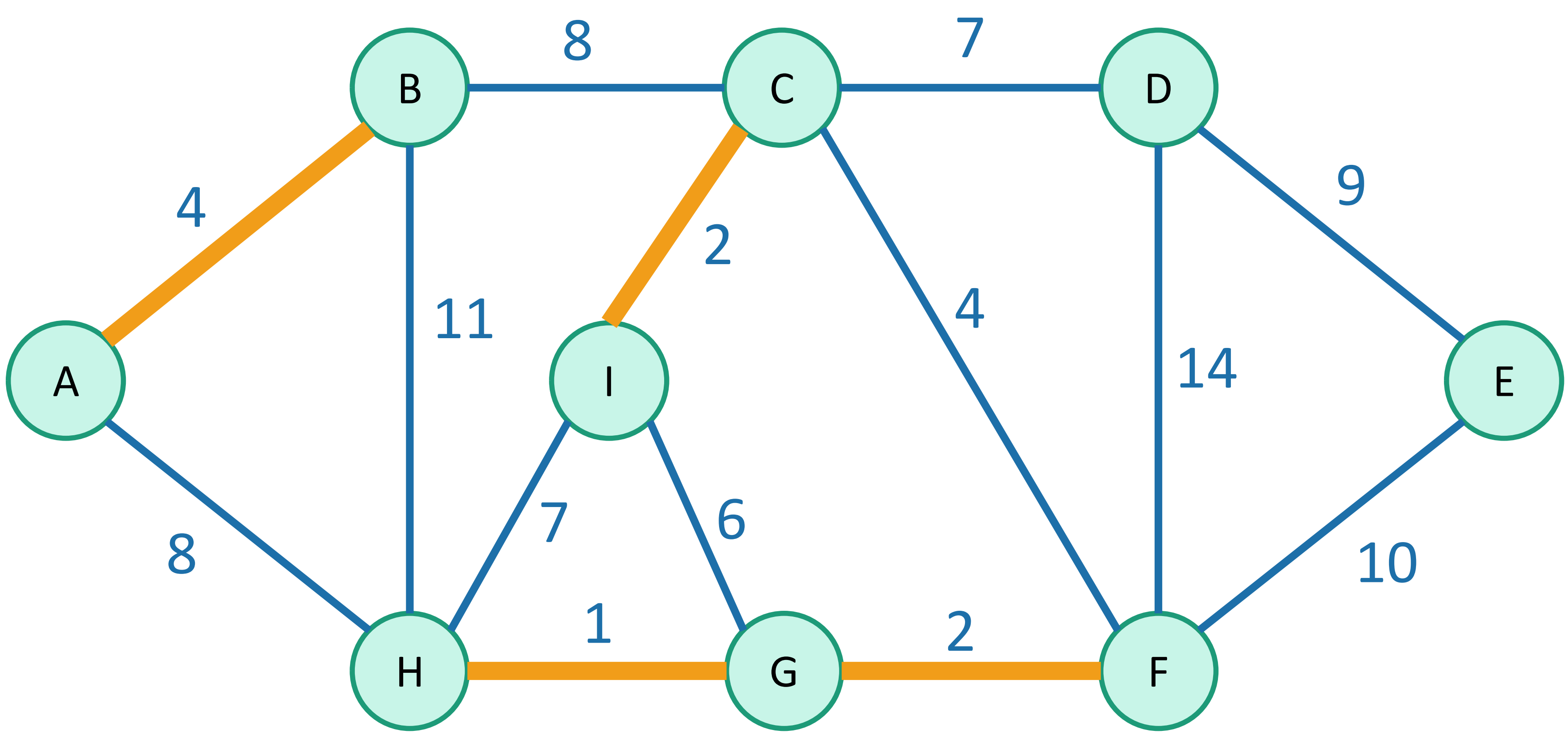
1. Does it work?
   - That is, does it actually return a MST?

2. How do we actually implement this?
   - the pseudocode above says “slowKruskal”…

Let’s do this one first

At each step of Kruskal's, we are maintaining a forest.

A **forest** is a collection of disjoint trees



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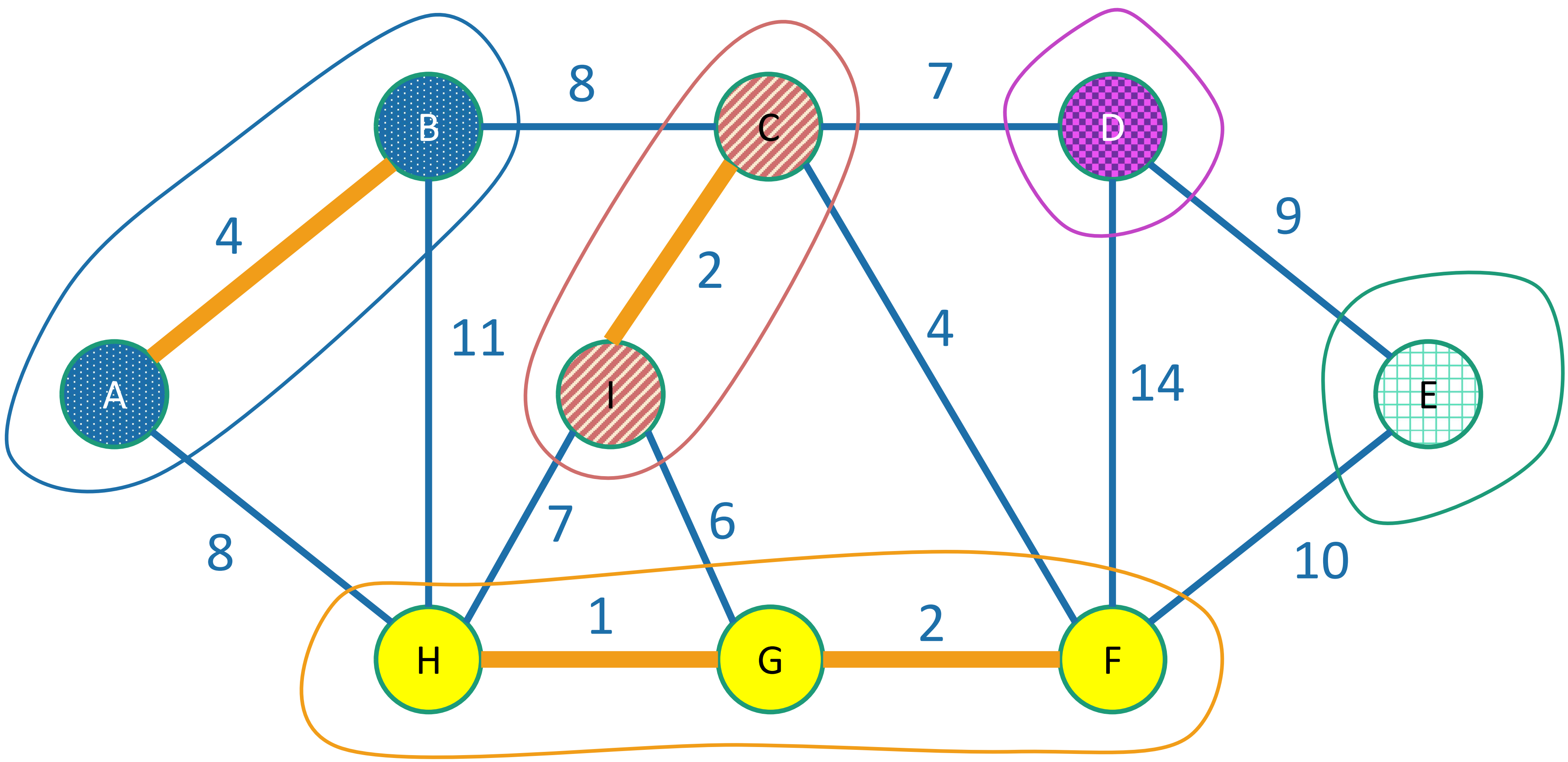

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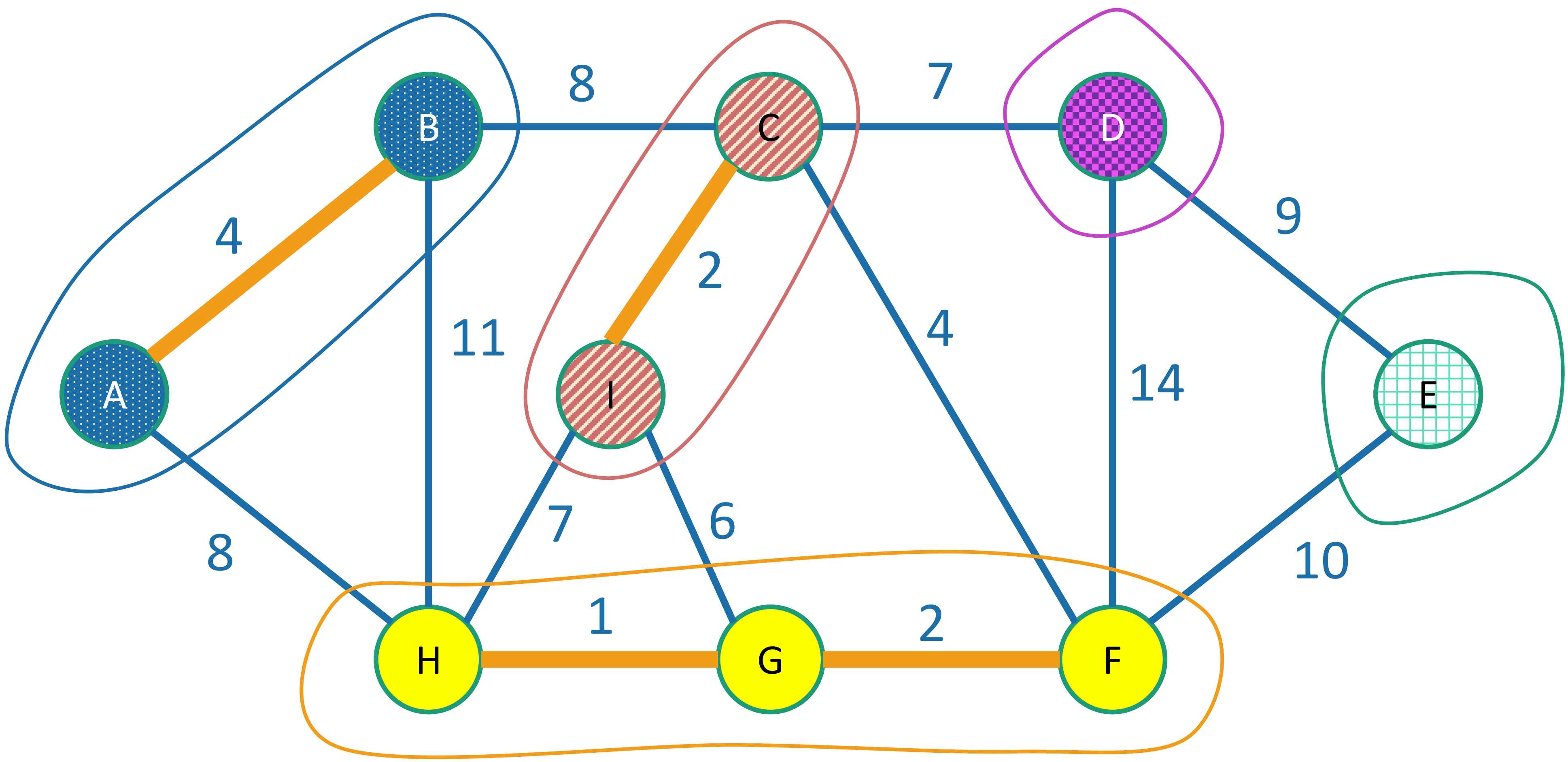

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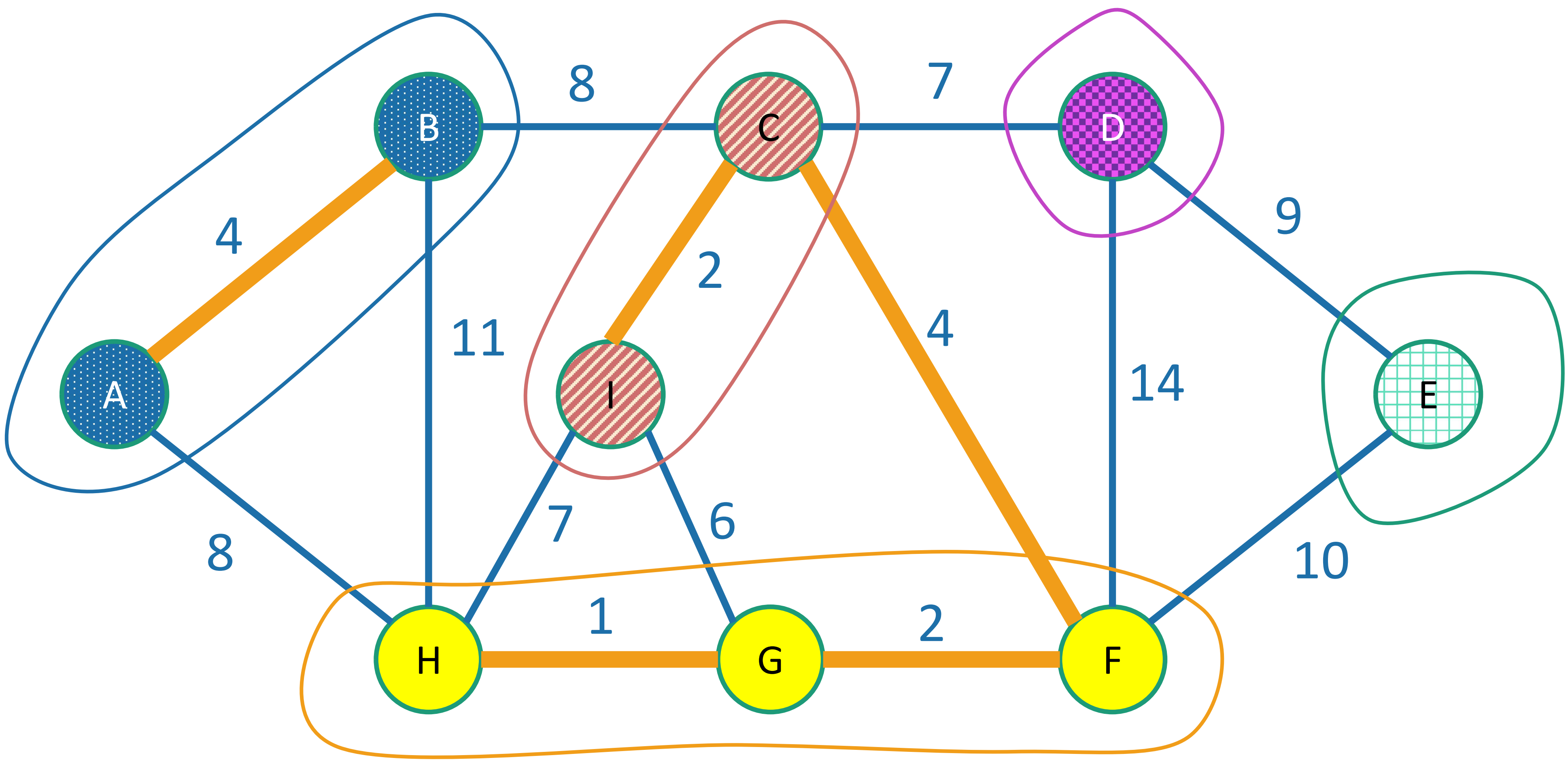

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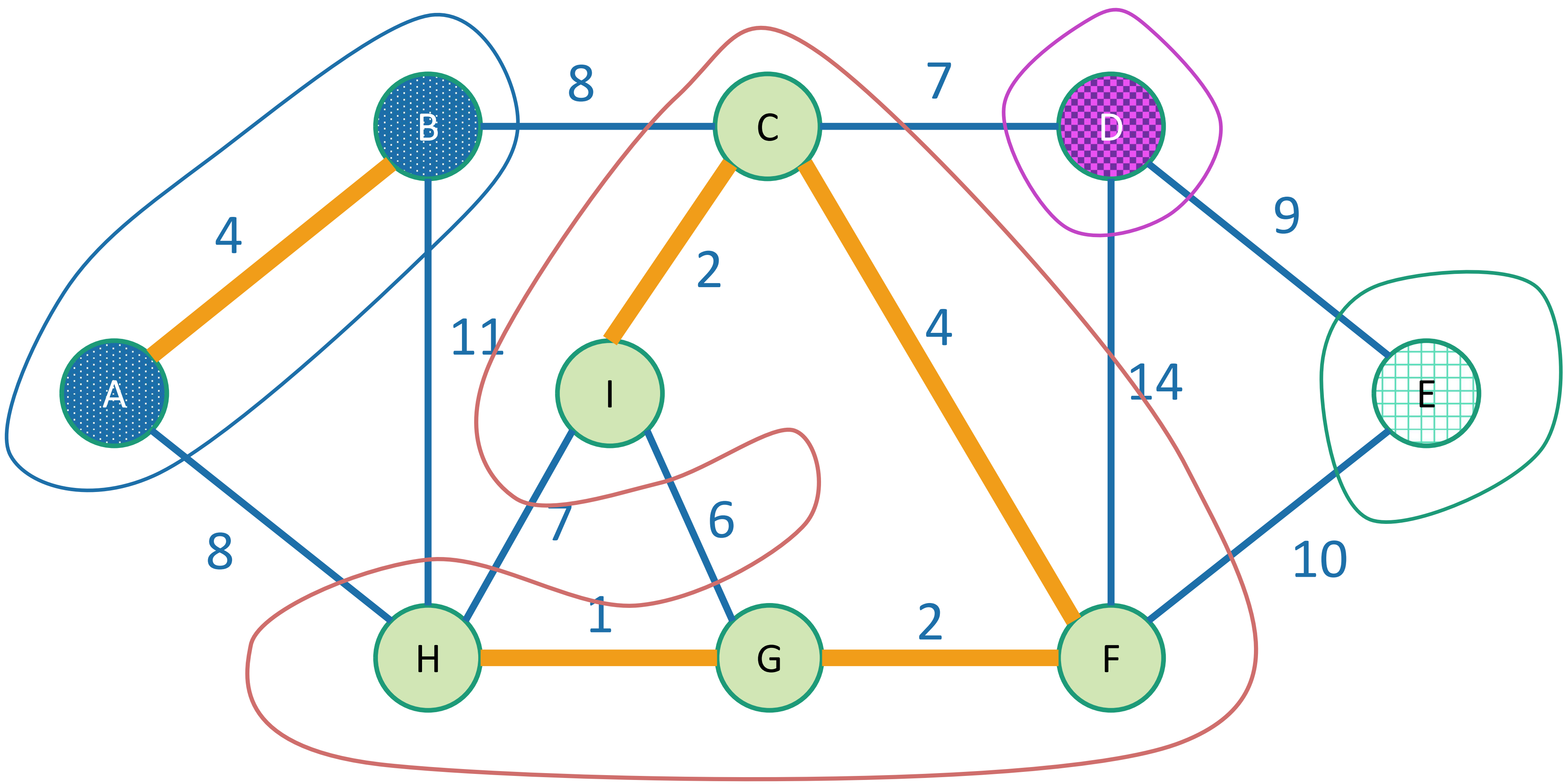


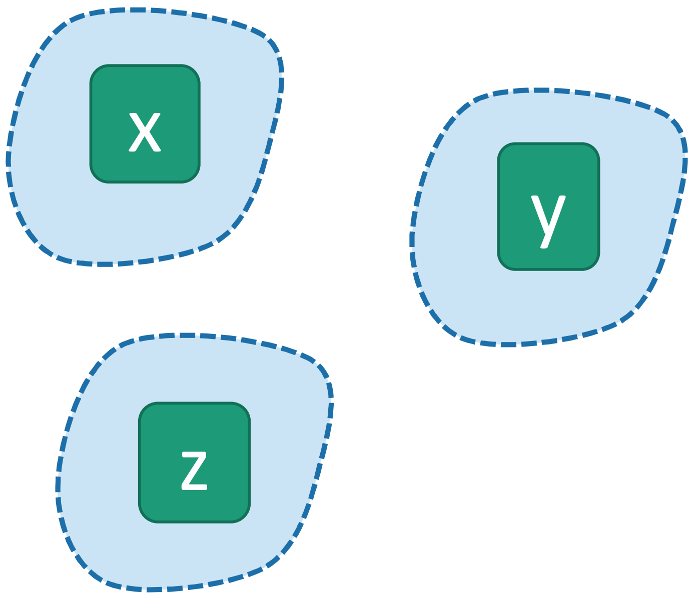
We never add an edge within a tree since that would create a cycle.

Union-find data structure
also called disjoint-set data structure

- Used for storing collections of sets
- Supports:
  - **makeSet(u):** create a set {u}
  - **find(u):** return the set that u is in
  - **union(u,v):** merge the set that u is in with the set that v is in.

```
makeSet(x)
makeSet(y)
makeSet(z)

union(x,y)
```

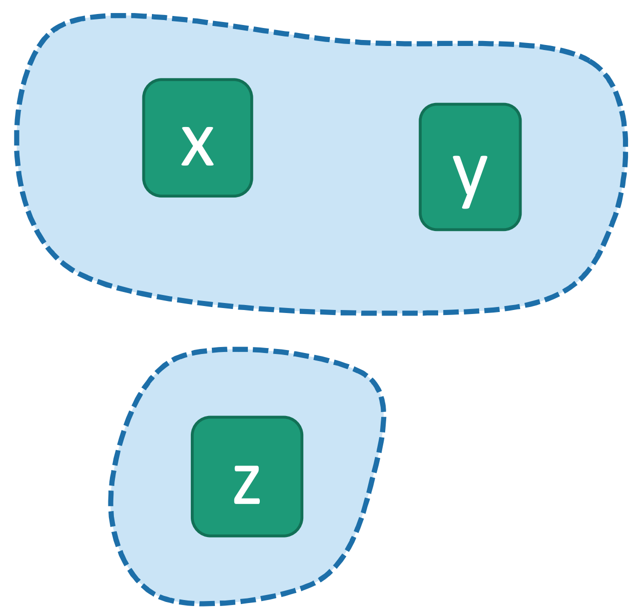


Union-find data structure
also called disjoint-set data structure

- Used for storing collections of sets
- Supports:
  - **makeSet(u):** create a set {u}
  - **find(u):** return the set that u is in
  - **union(u,v):** merge the set that u is in with the set that v is in.

```
makeSet(x)
makeSet(y)
makeSet(z)

union(x,y)
```



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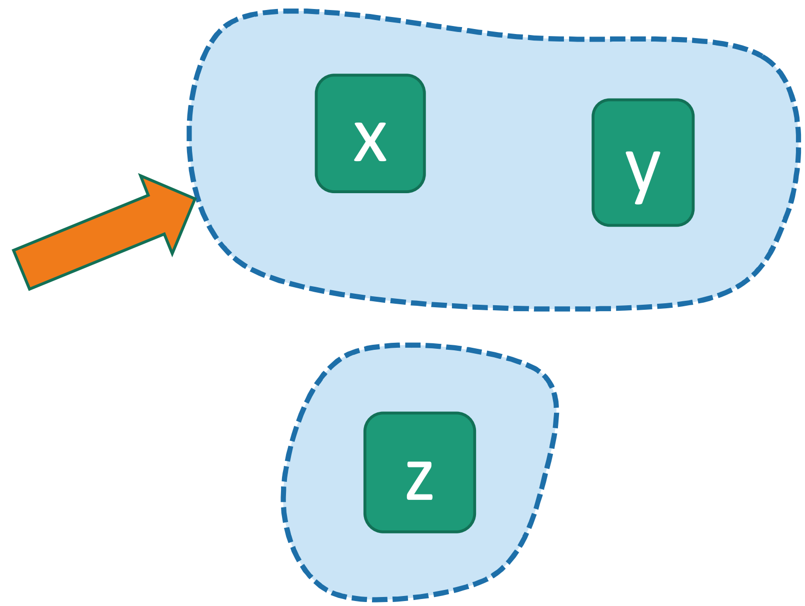
find(x)
```

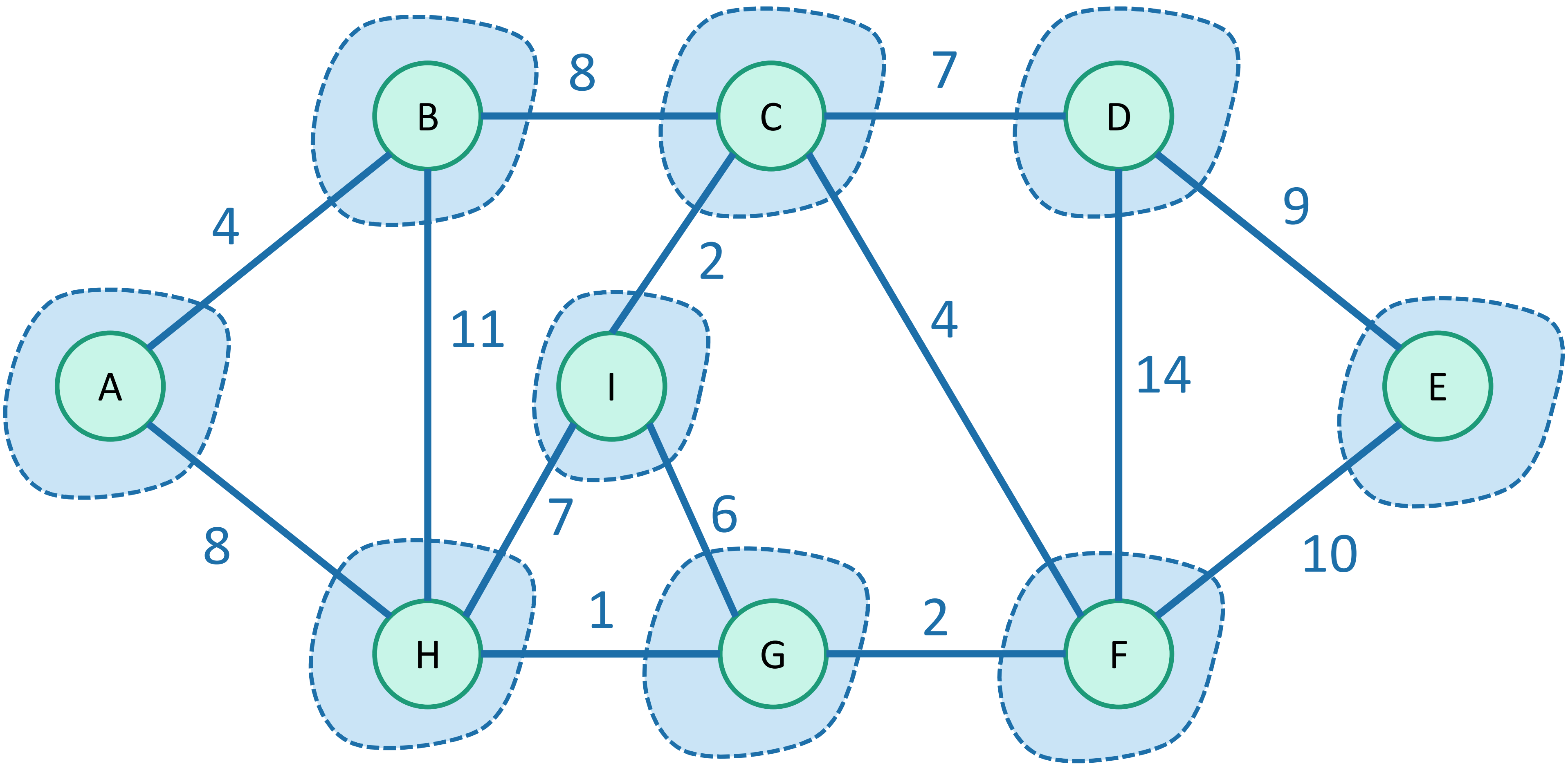


Kruskal pseudo-code

- **kruskal**(G = (V,E)):
  - Sort E by weight in non-decreasing order
  - MST = {} // initialize an empty tree
  - **for** v in V:
    - **makeSet**(v) // put each vertex in its own tree in the forest
  - **for** (u,v) in E: // go through the edges in sorted order
    - **if find**(u) != **find**(v): // if u and v are not in the same tree
      - add (u,v) to MST
      - **union**(u,v) // merge u's tree with v's tree
  - **return** MST

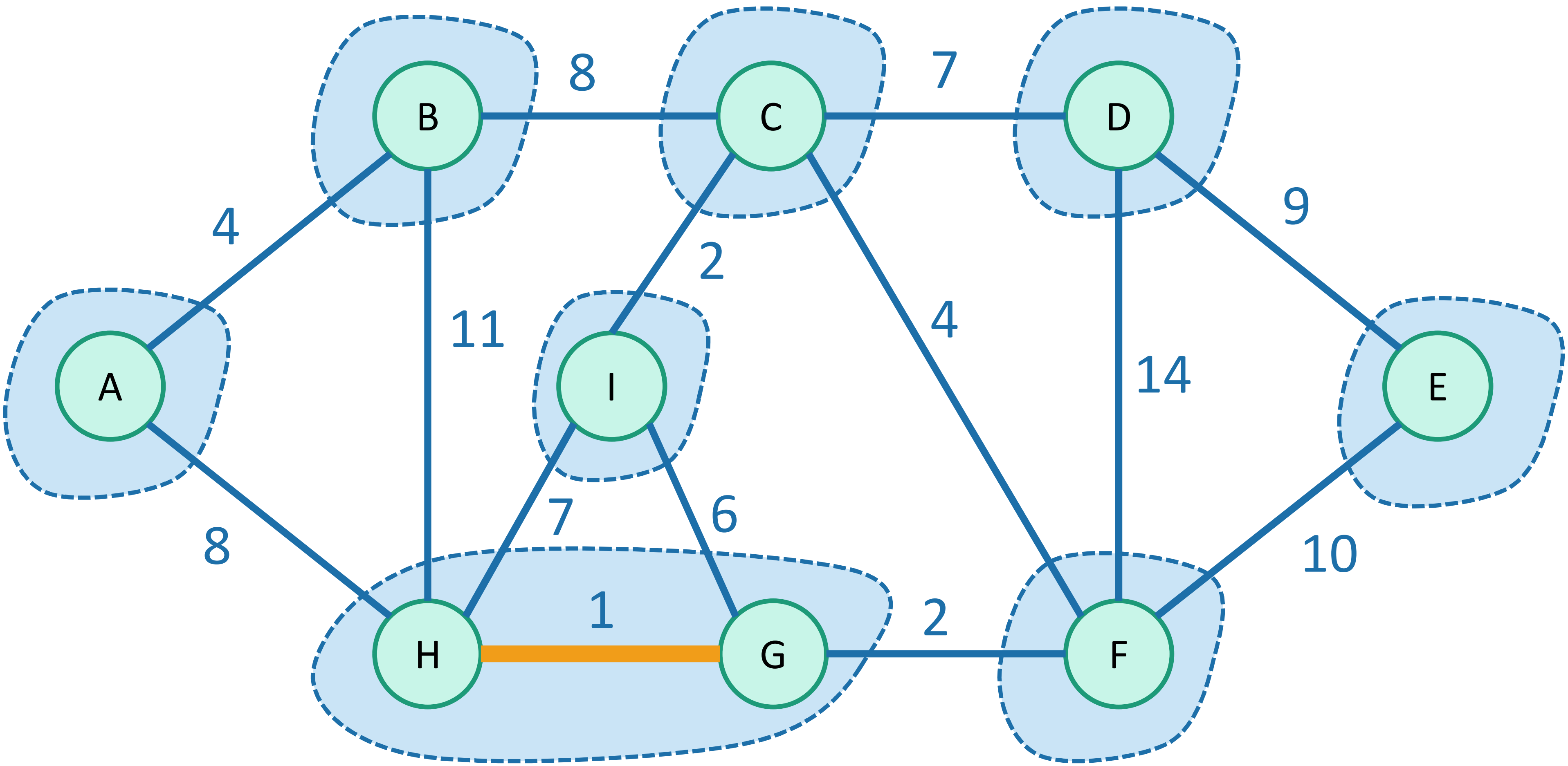
Once more…

To start, every vertex is in its own tree.



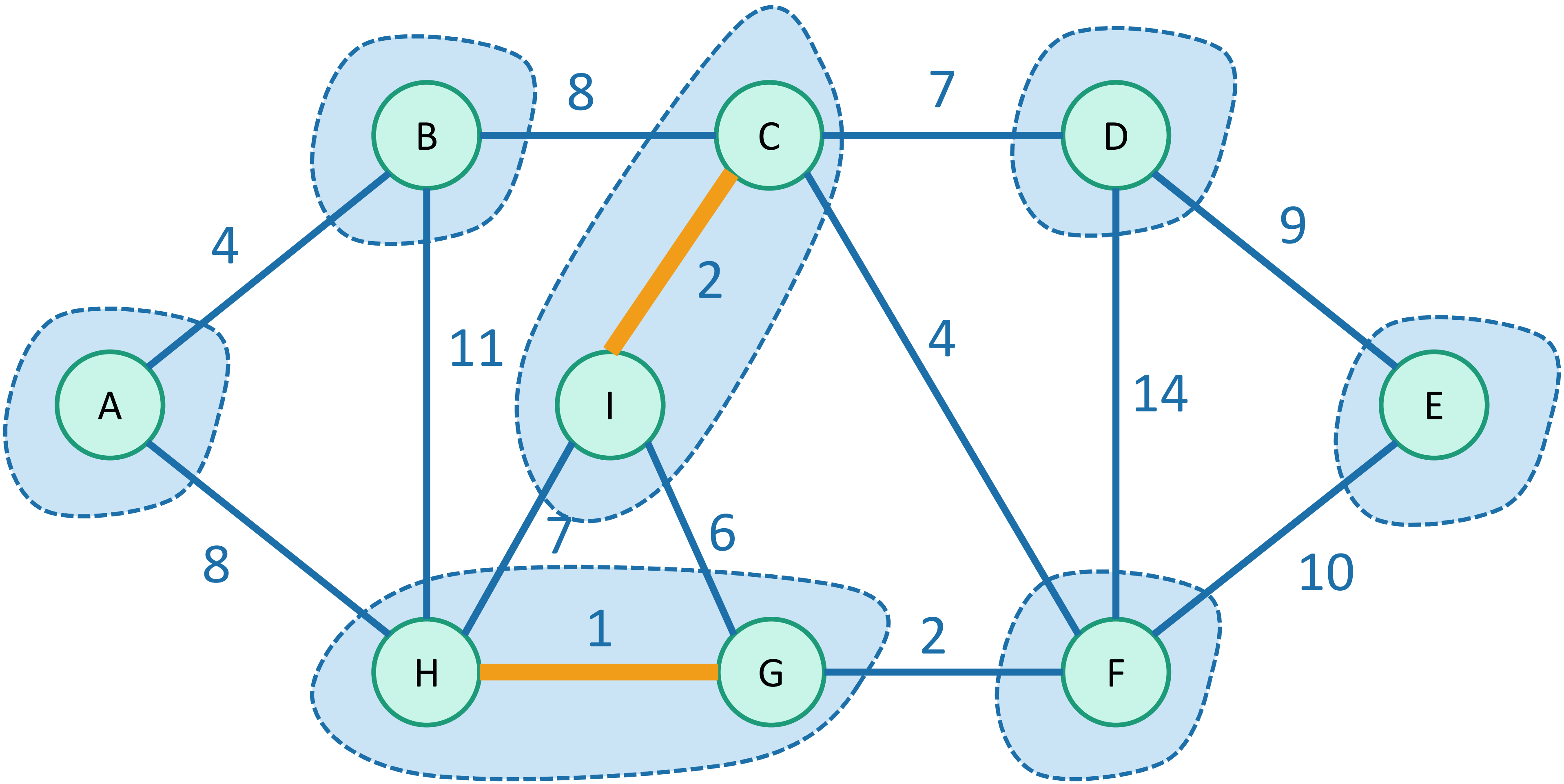
Once more…

Then start merging.



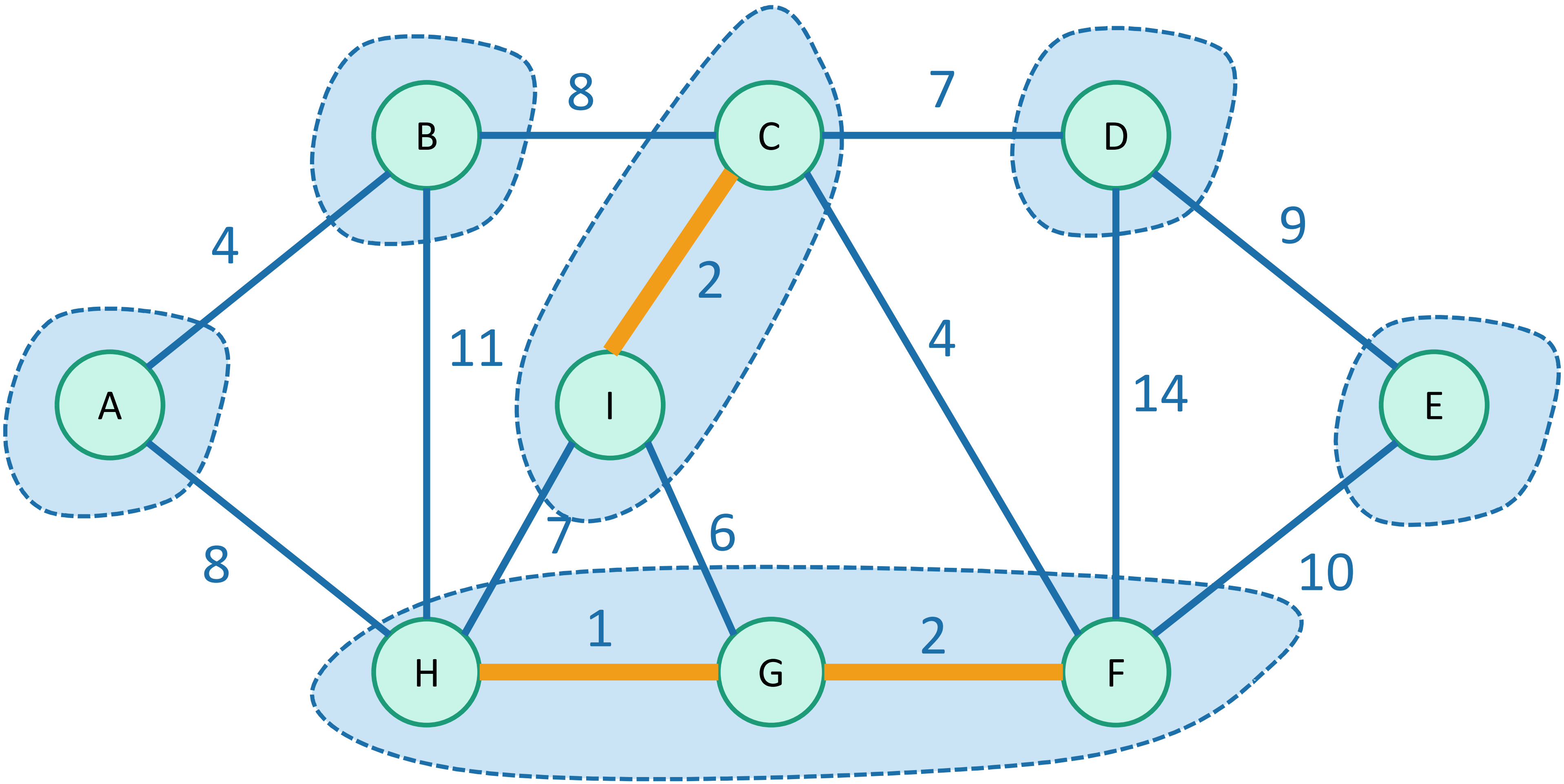
Once more…

Then start merging.



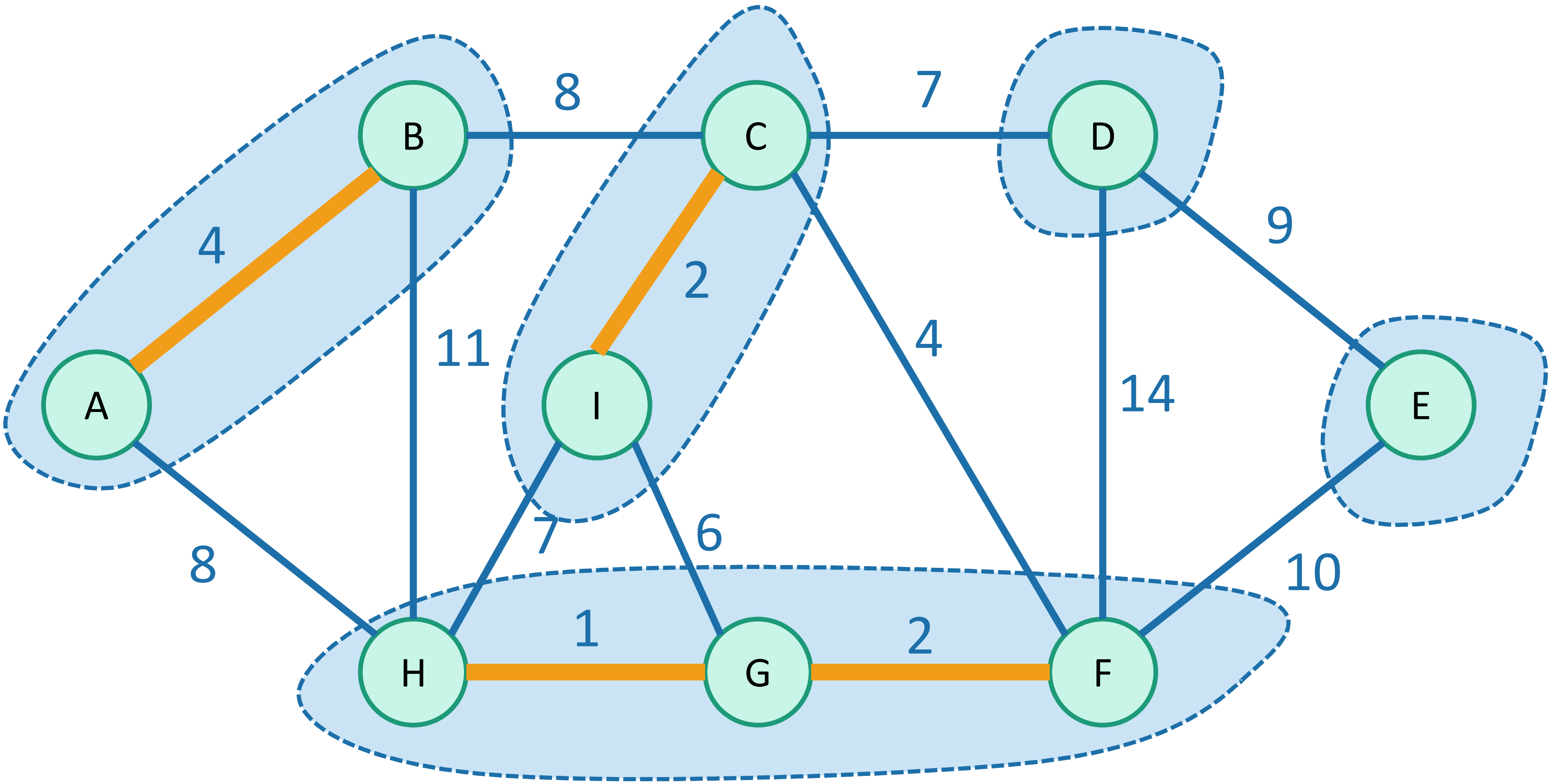
Once more…

Then start merging.



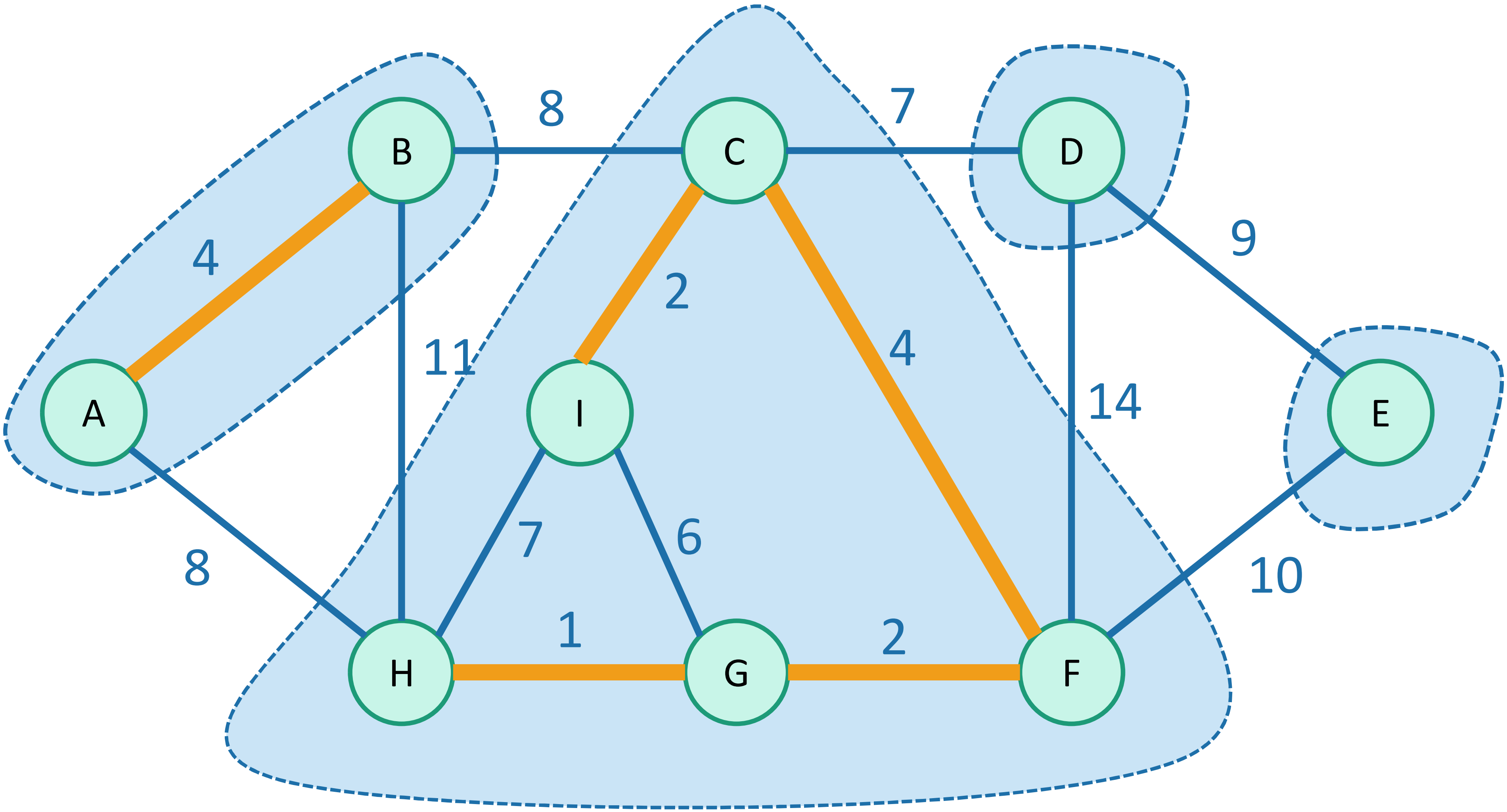
Once more…

Then start merging.



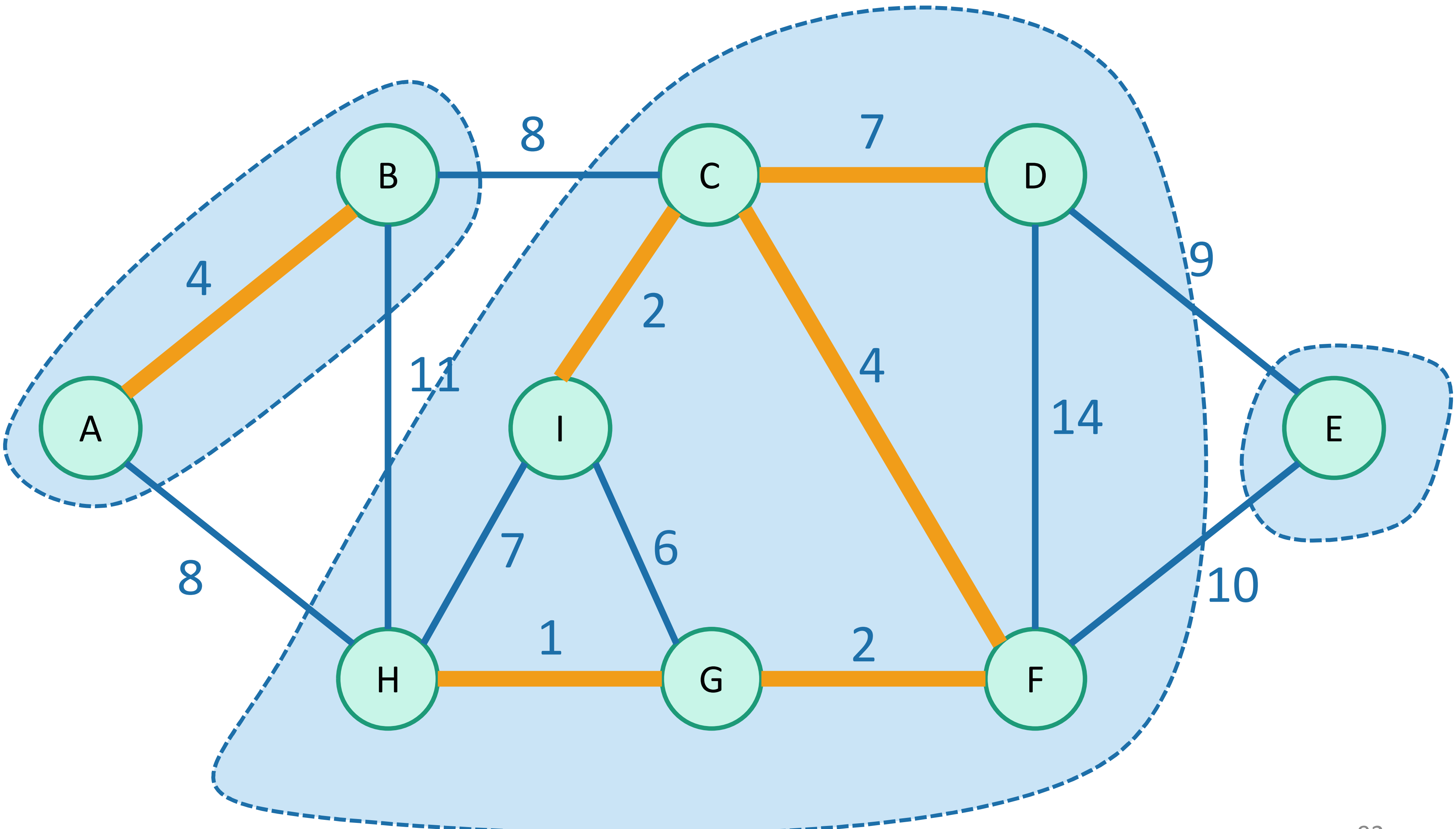
Once more…

Then start merging.



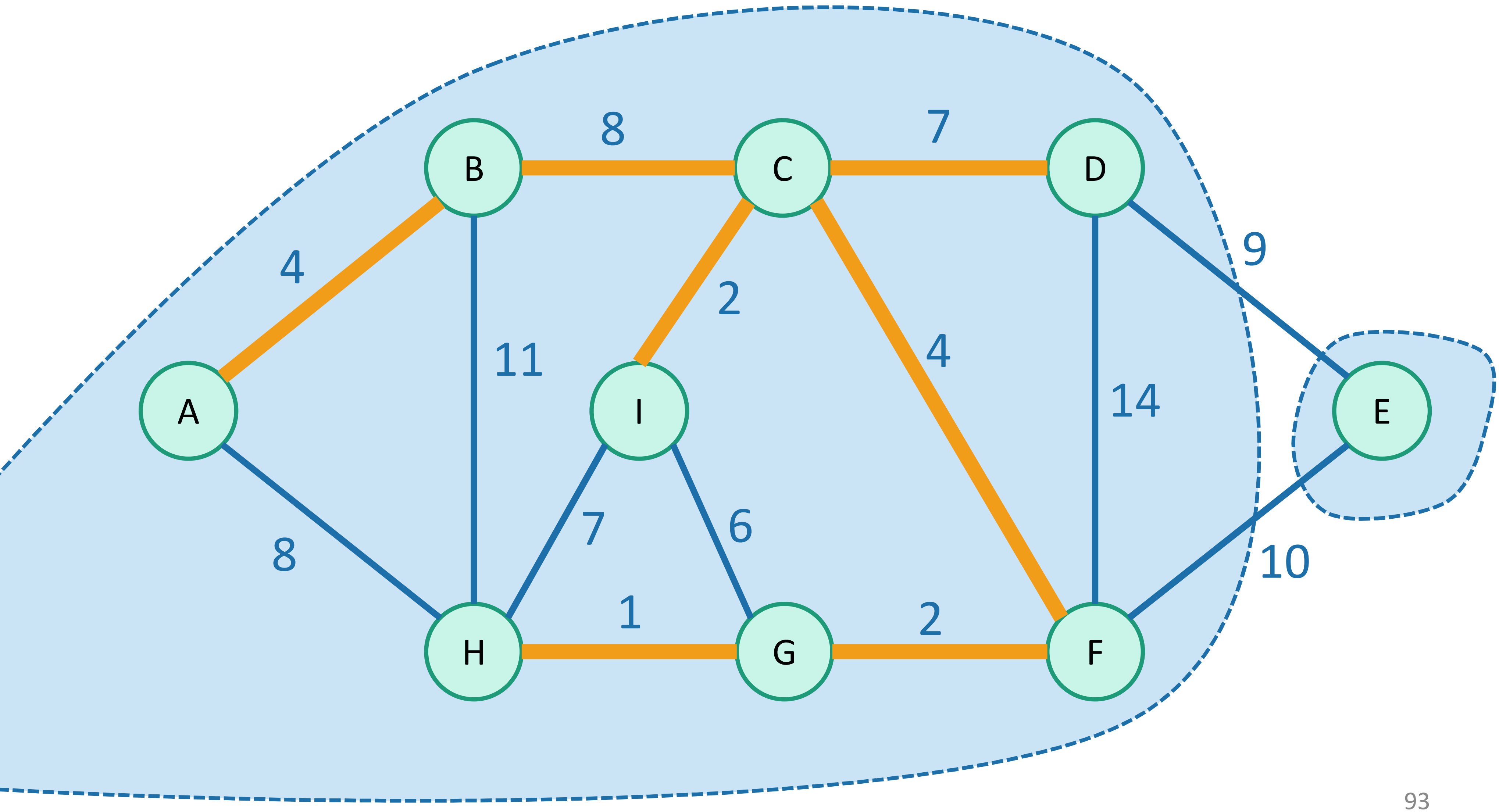
Once more…

Then start merging.



Once more…

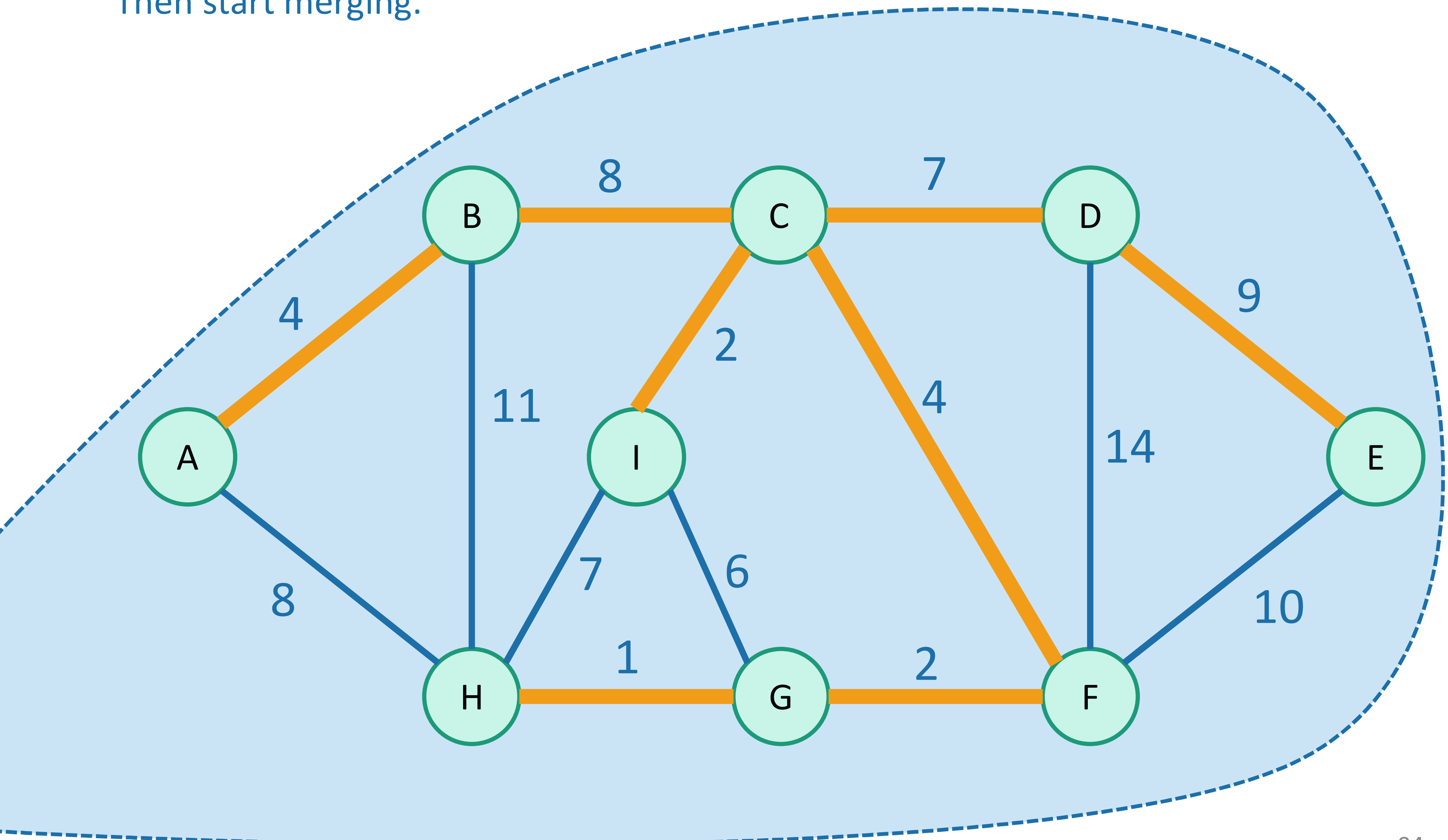
Then start merging.



Stop when we have one big tree!

Once more…

Then start merging.



Running time

- Sorting the edges takes O(m log(n))
  - In practice, if the weights are small integers we can use radixSort and take time O(m)
- For the rest:
  - n calls to **makeSet**
    - put each vertex in its own set
  - 2m calls to **find**
    - for each edge, **find** its endpoints
  - n-1 calls to **union**
    - we will never add more than n-1 edges to the tree,
    - so we will never call **union** more than n-1 times.
- Total running time:
  - Worst-case O(mlog(n)), just like Prim with a RBtree.
  - Closer to O(m) if you can do radixSort

In practice, each of makeSet, find, and union run in constant time*
(There is a simpler way which does find and union in time O(log n)).

*technically, they run in *amortized time* O($\alpha(n)$), where $\alpha(n)$ is the *inverse Ackerman function*.
$\alpha(n) \leq 4$ provided that n is smaller than the number of atoms in the universe.

Two questions

1. Does it work?
   - That is, does it actually return a MST?

Now that we understand this “tree-merging” view, let’s do this one.

2. How do we actually implement this?
   - the pseudocode above says “slowKruskal”…
     - **Worst-case running time O(mlog(n)) using a union-find data structure.**

Does it work?

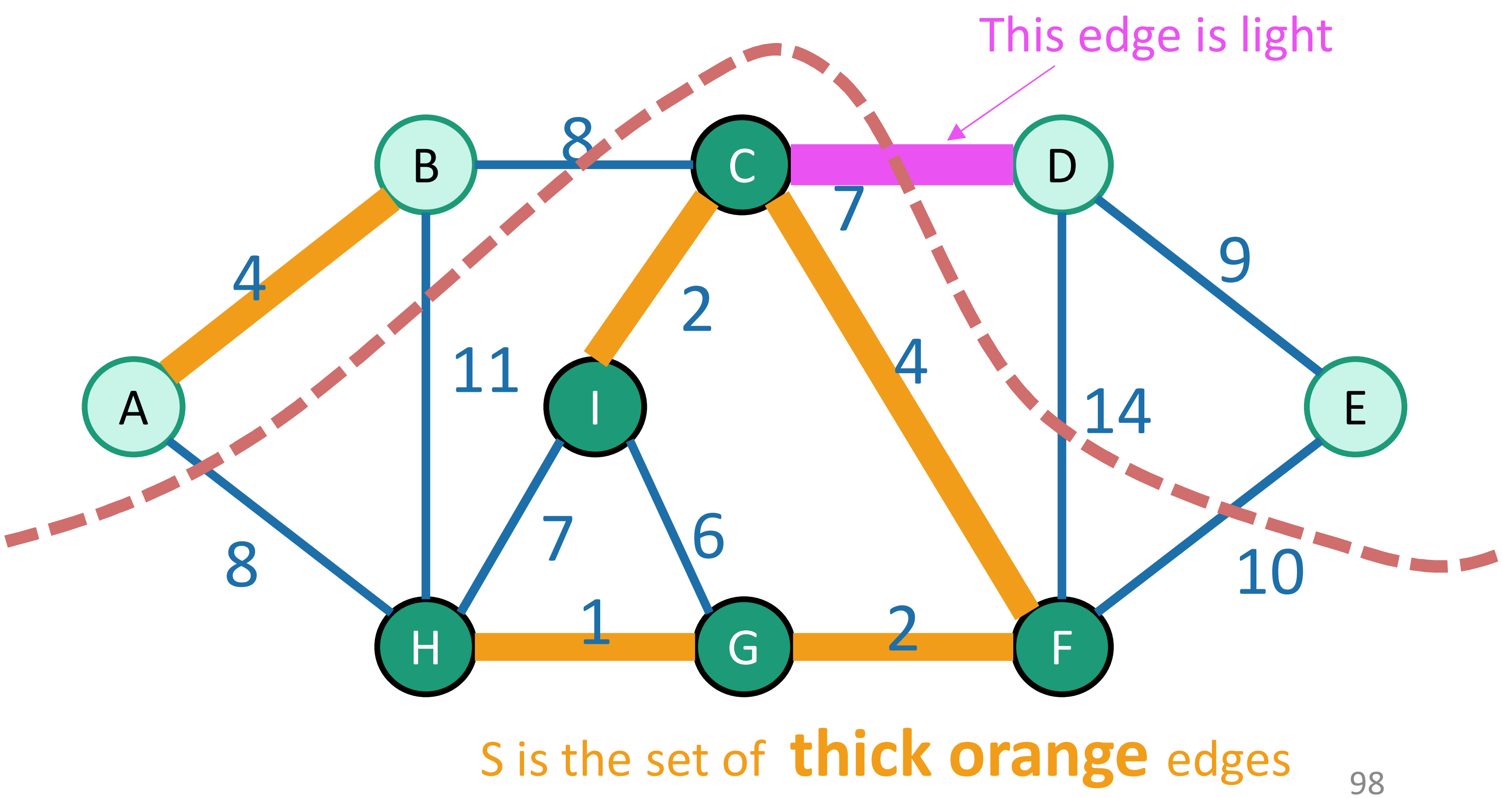
- We need to show that our greedy choices **don't rule out success.**
- That is, at every step:
  - There exists an MST that contains all of the edges we have added so far.

- Now it is time to use our lemma!

again!

Lemma

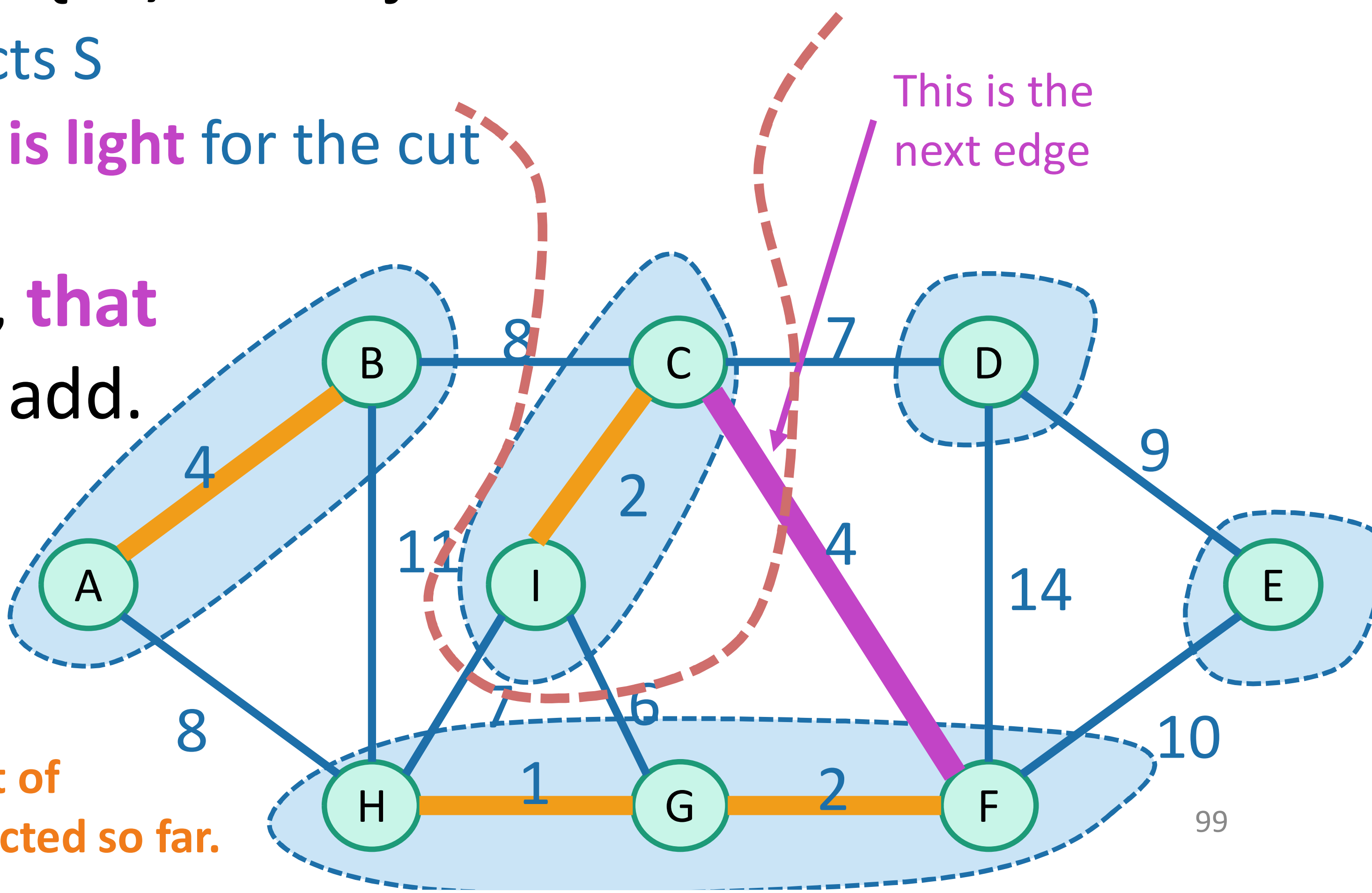
- Let S be a set of edges, and consider a cut that respects S.
- Suppose there is an MST containing S.
- Let {u,v} be a light edge.
- Then there is an MST containing S U {{u,v}}



Partway through Kruskal

- Assume that our choices **S** so far don't rule out success.
  - There is an MST extending them
- The **next edge** we add will merge two trees, **T1, T2**
- Consider the cut **{T1, V – T1}.**
  - This cut respects S
  - Our **new edge is light** for the cut

- By the Lemma, **that edge** is safe to add.
  - There is still an MST extending the new set



S is the set of edges selected so far.

Two questions

1. Does it work?
   - That is, does it actually return a MST?
     - **Yes**

2. How do we actually implement this?
   - the pseudocode above says “slowKruskal”…
     - **Using a union-find data structure!**

Recap

- Two algorithms for Minimum Spanning Tree
  - Prim's algorithm
  - Kruskal's algorithm

- Both are (more) examples of greedy algorithms!
  - Make a **series of choices**.
  - Show that at each step, your choice **does not rule out success.**
  - At the end of the day, you haven't ruled out success, so **you must be successful**.

Compare and contrast

Prim might be a better idea on dense graphs if you can't radixSort edge weights

- Prim:
  - Grows a tree.
  - Time O(mlog(n)) with a red-black tree
  - Time O(m + nlog(n)) with a Fibonacci heap
- Kruskal:
  - Grows a forest.
  - Time O(mlog(n)) with a union-find data structure
  - If you can do radixSort on the edge weights, morally O(m)

Kruskal might be a better idea on sparse graphs if you can radixSort edge weights