Lát cắt cực tiểu, luồng cực đại

Min cuts, max flows

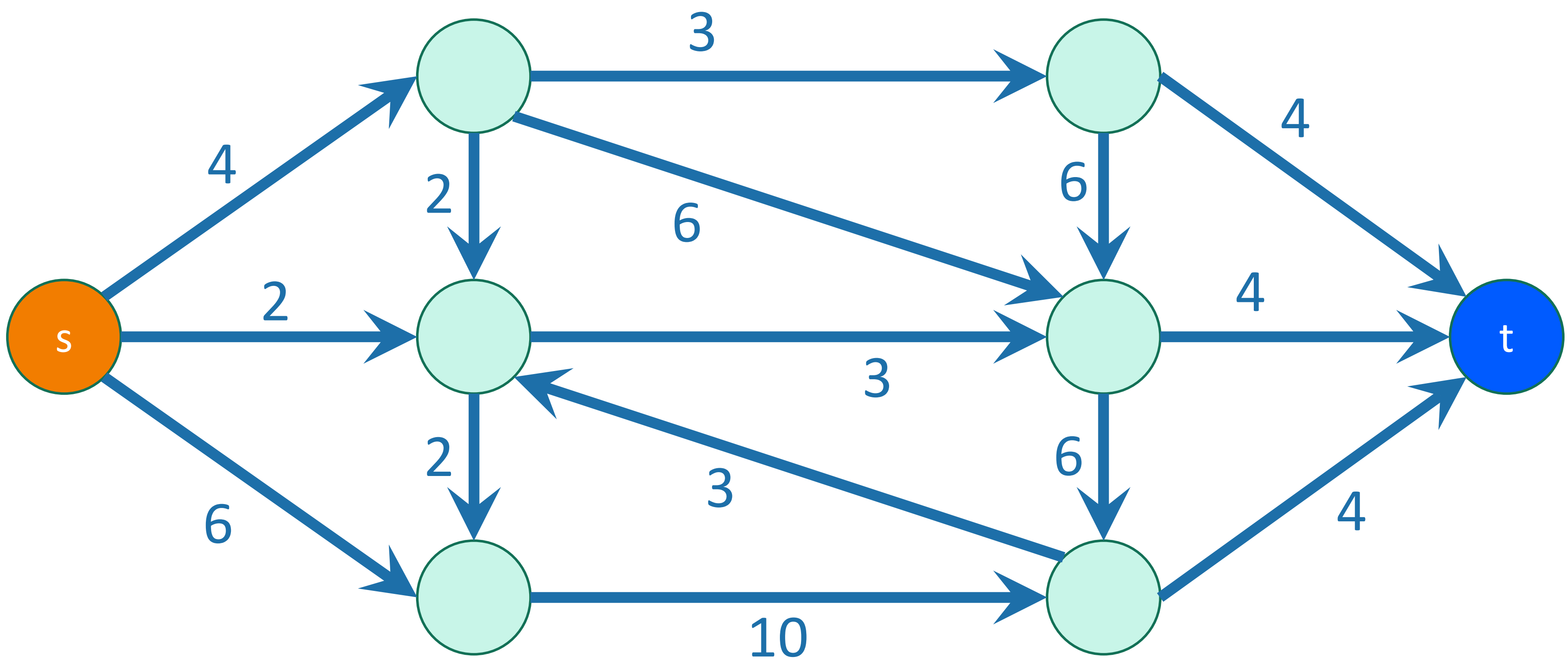
Nội dung

- Lát cắt cực tiểu (minimum s-t cuts)
- Luồng cực đại (maximum s-t flows)
- Thuật toán Ford-Fulkerson
  - Finds min cuts and max flows!
- Một số ví dụ

Sử dụng một phần tài liệu bài giảng CS161 Stanford University

Khái niệm

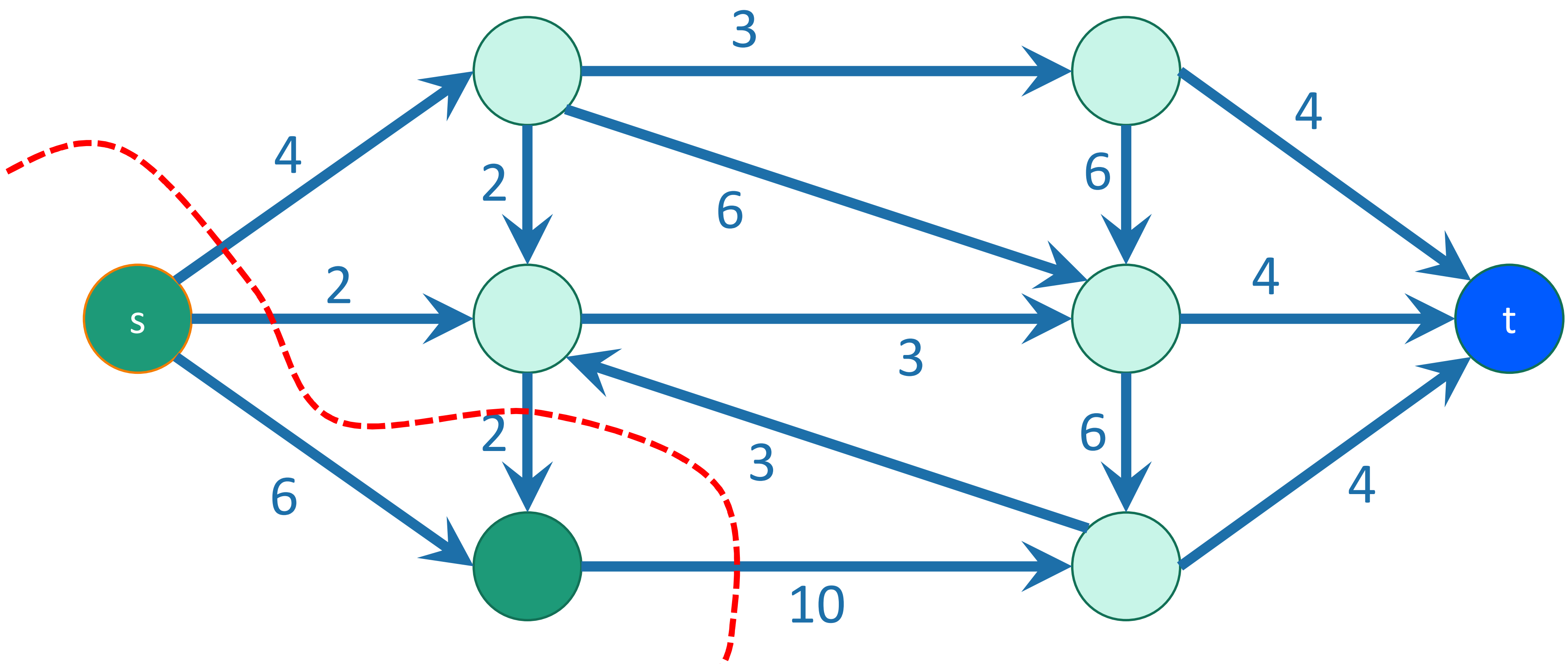
- Graphs are directed and edges have "capacities" (weights)
- We have a special "source" vertex s and "sink" vertex t.
  - s has only outgoing edges
  - t has only incoming edges



Lát cắt

An s-t cut

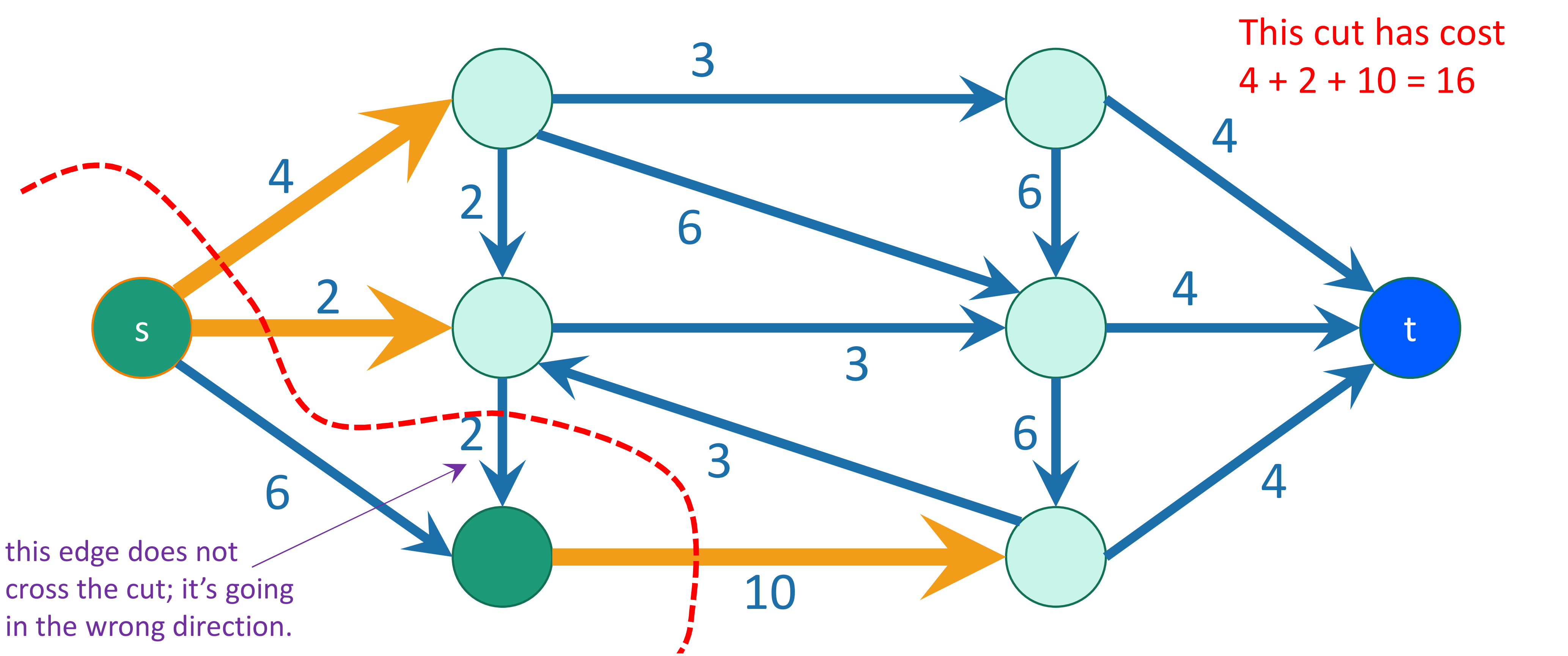
is a cut which separates s from t



An s-t cut

is a cut which separates s from t

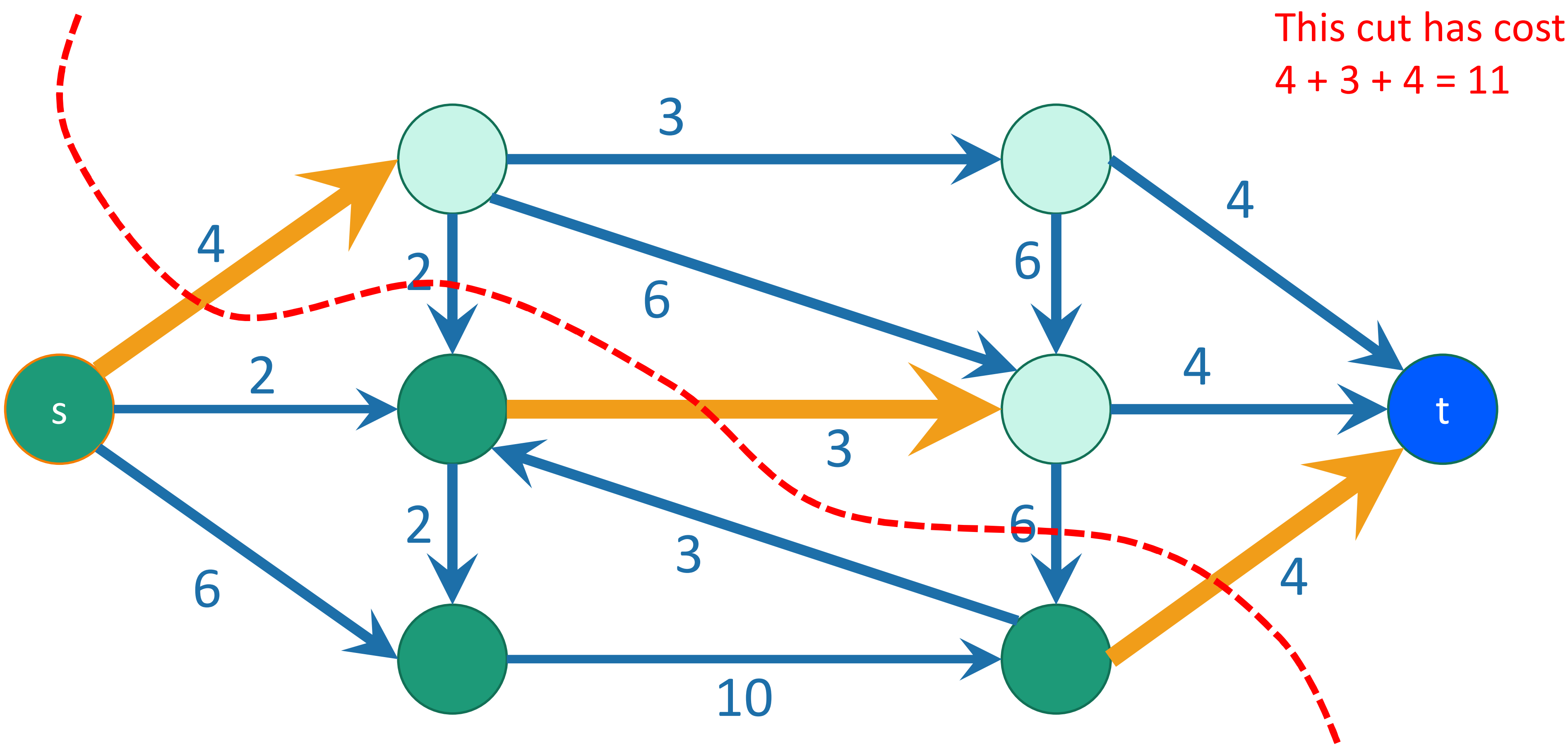
- An edge **crosses the cut** if it goes from s's side to t's side.
- The **cost** (or capacity) of a cut is the sum of the capacities of the edges that cross the cut.



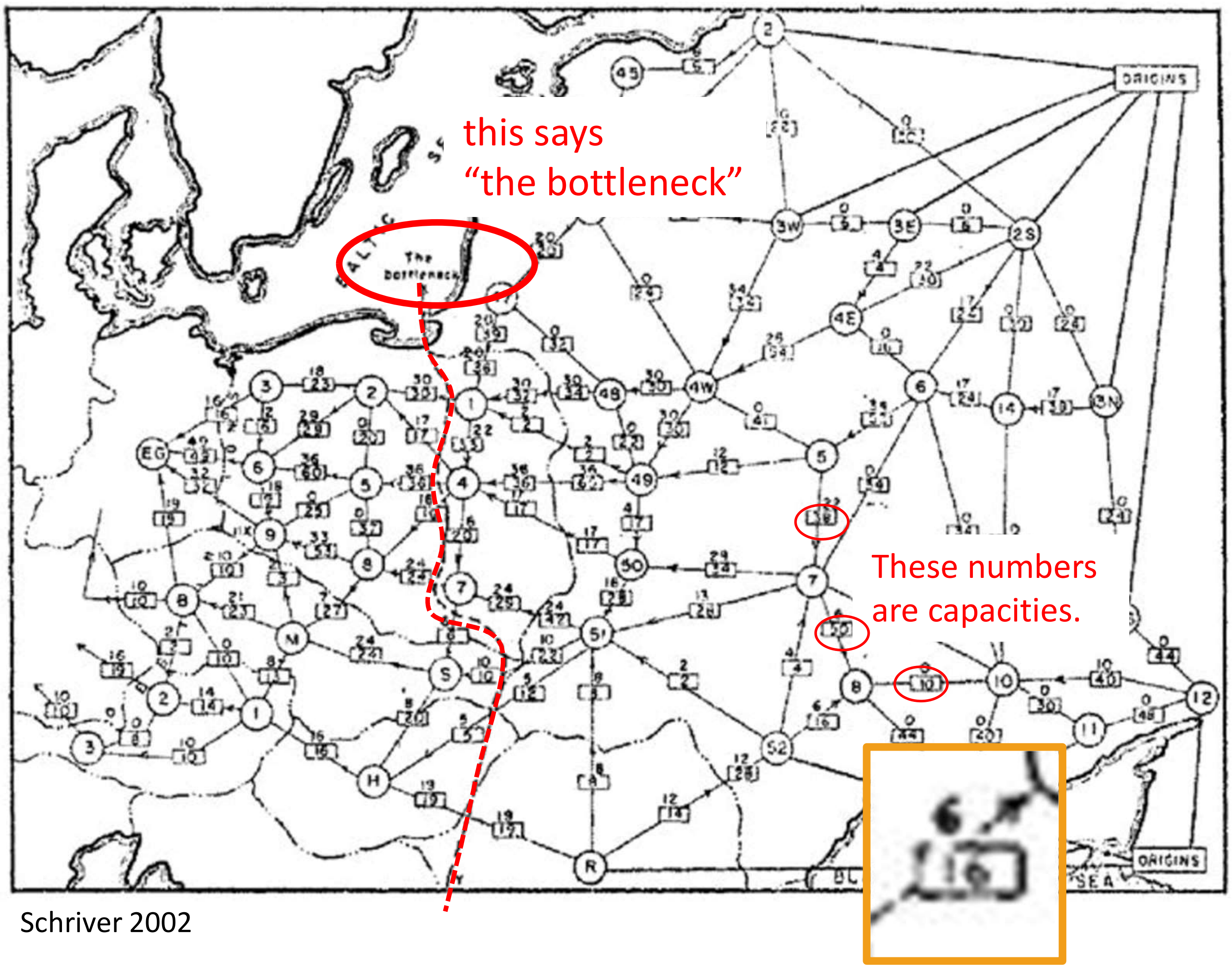
Lát cắt cực tiểu

A minimum s-t cut is a cut which separates s from t with minimum cost

- Question: how do we find a minimum s-t cut?



Example where this comes up

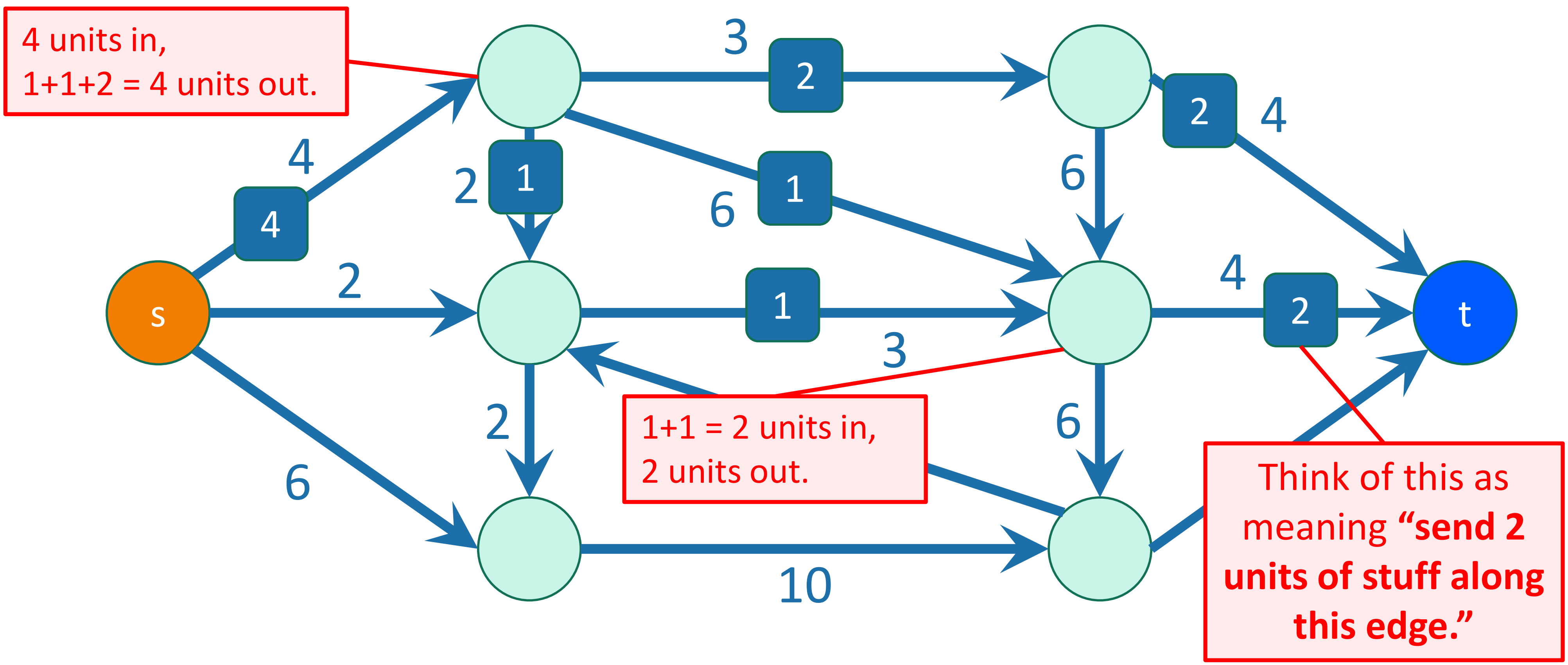


Schriver 2002

- 1955 map of rail networks from the Soviet Union to Eastern Europe.
  - Declassified in 1999.
  - 44 edges, 105 vertices
- The US wanted to cut off routes from suppliers in Russia to Eastern Europe as efficiently as possible.
- In 1955, Ford and Fulkerson gave an algorithm which finds the optimal s-t cut.

Luồng (Flows)

- In addition to a capacity, each edge has a flow
    - (unmarked edges in the picture have flow 0)
- The flow on an edge must be **at most** its capacity.
- At each vertex, the incoming flows must equal the outgoing flows.



Flows

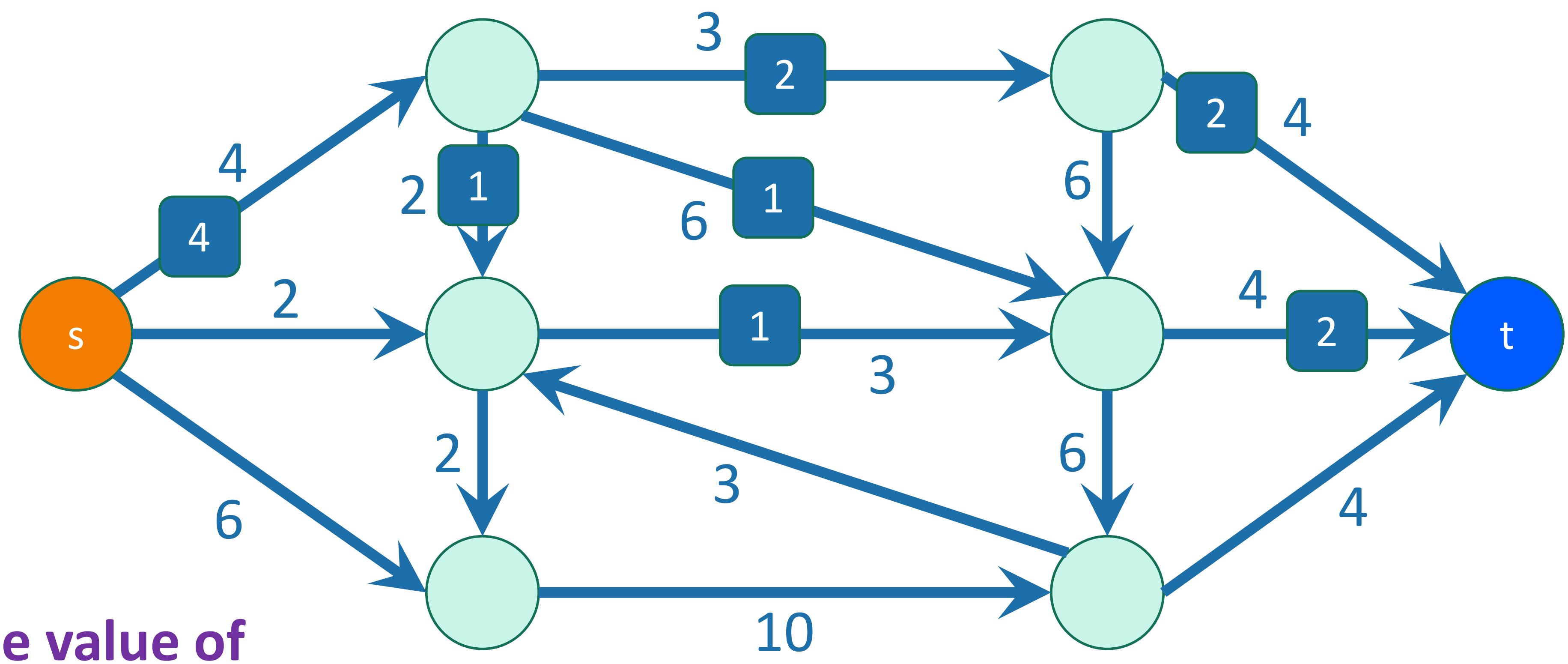
- The value of a flow is:
  - The amount of stuff coming out of s
  - The amount of stuff flowing into t
  - These are the same!

Because of conservation of flows at vertices,

stuff you put in

=

stuff you take out.

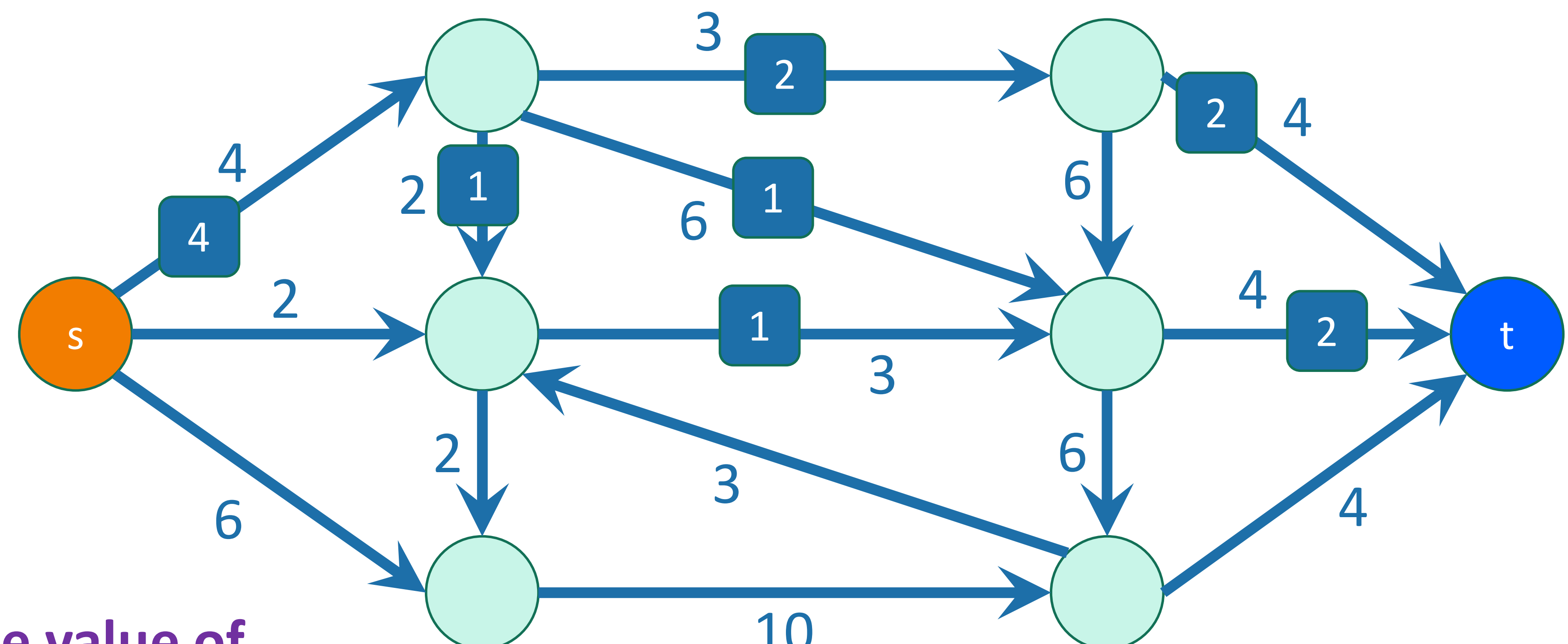


The value of this flow is 4.

Luồng cực đại

A maximum flow is a flow of maximum value

- This example flow is pretty wasteful, I'm not utilizing the capacities very well.

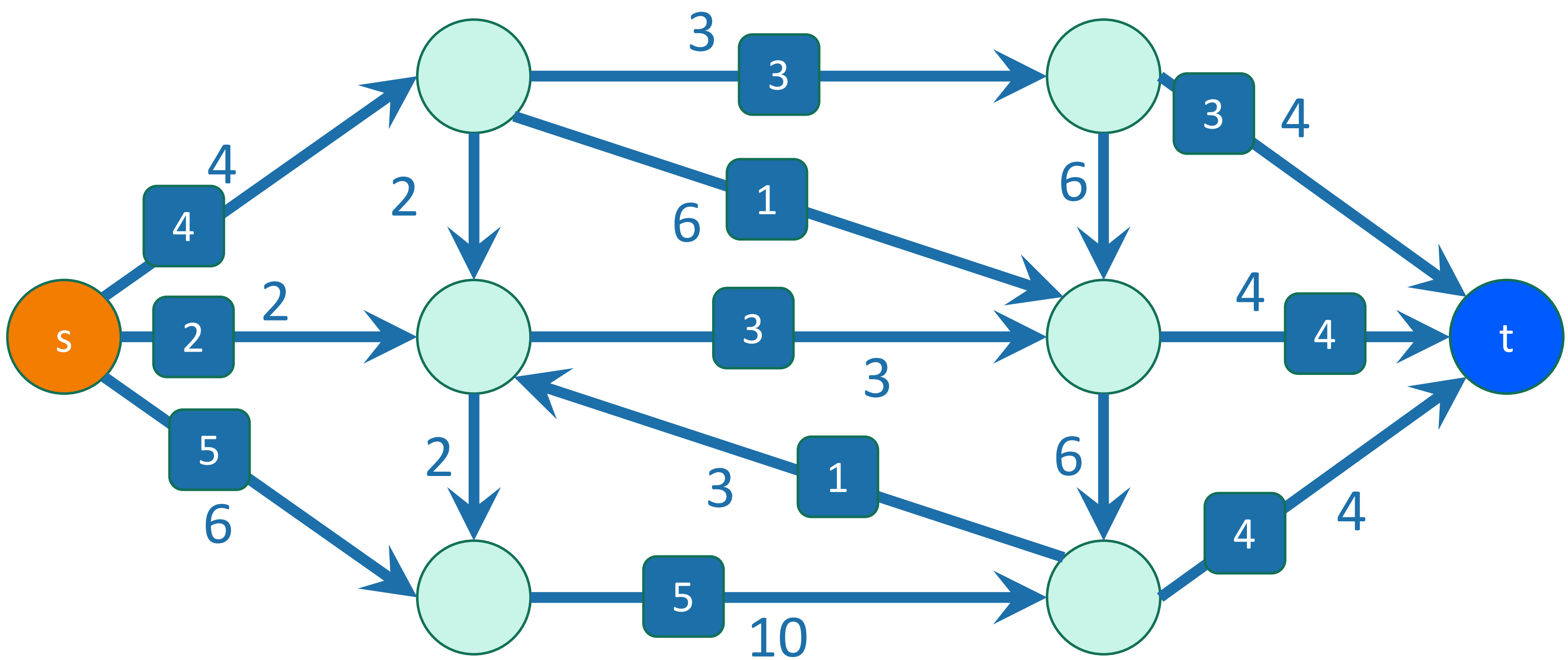


The value of this flow is 4.

A maximum flow

is a flow of maximum value

- This one is maximum; it has value 11.

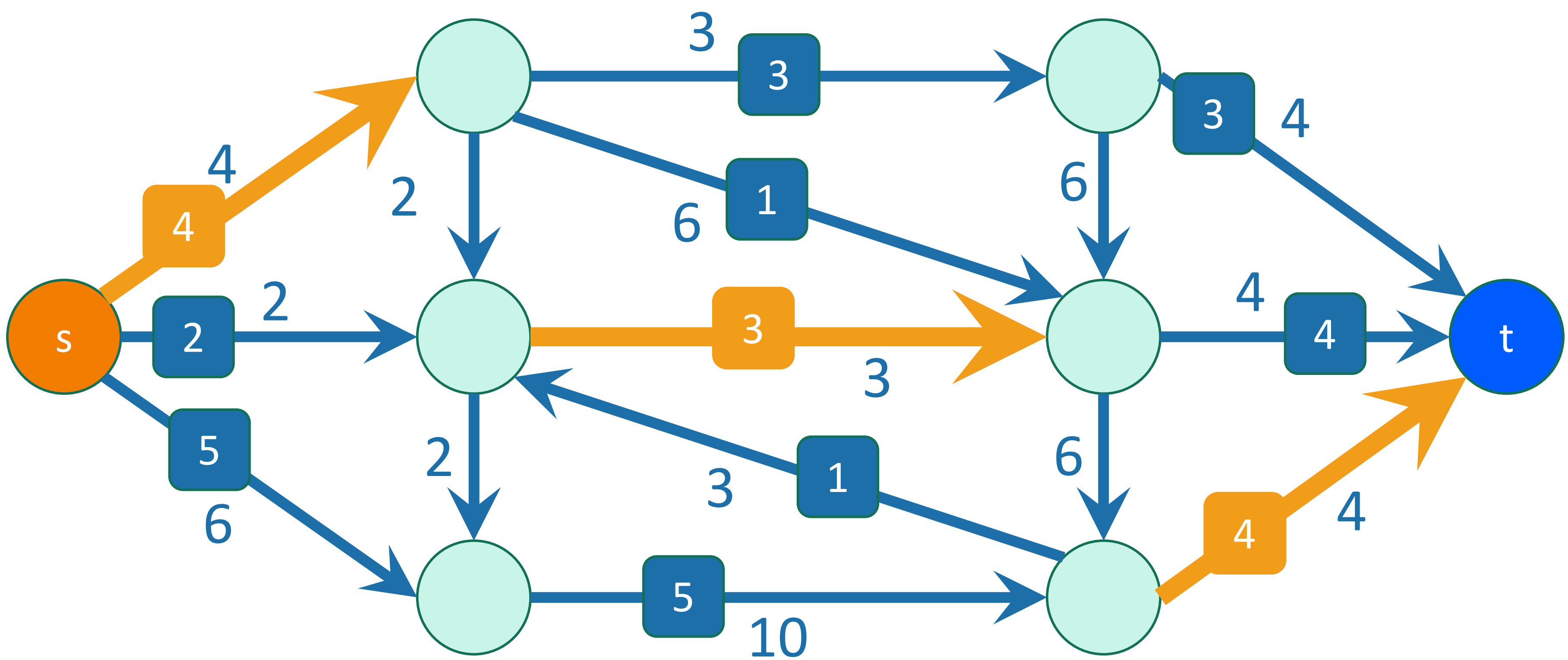


Định lý

Max-flow min-cut theorem

The value of a max flow from s to t
is equal to
the cost of a min s-t cut.

Intuition: in a max flow, the min cut better fill up, and this is the bottleneck.



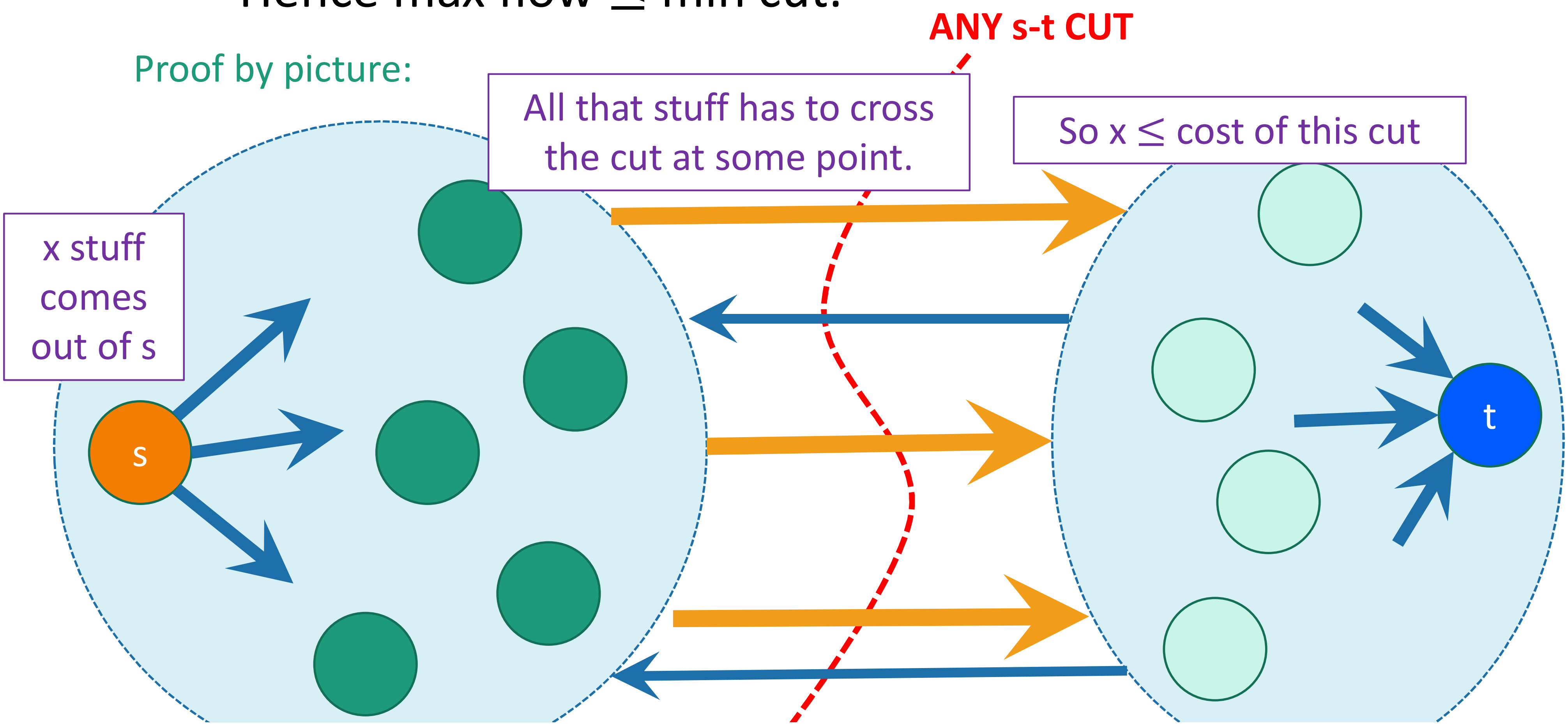
Proof outline

- Lemma 1: max flow $\leq$ min cut.
  - Proof-by-picture
- What we actually want: max flow $=$ min cut.
  - Proof-by-algorithm…the Ford-Fulkerson algorithm!
  - The Ford-Fulkerson algorithm actually finds the max flow and the min cut.

One half of Min-Cut Max-Flow Theorem

- **Lemma 1:**
  - For ANY s-t flow and ANY s-t cut, the value of the flow is at most the cost of the cut.
  - Hence max flow $\leq$ min cut.

Proof by picture:



Ford-Fulkerson Algorithm

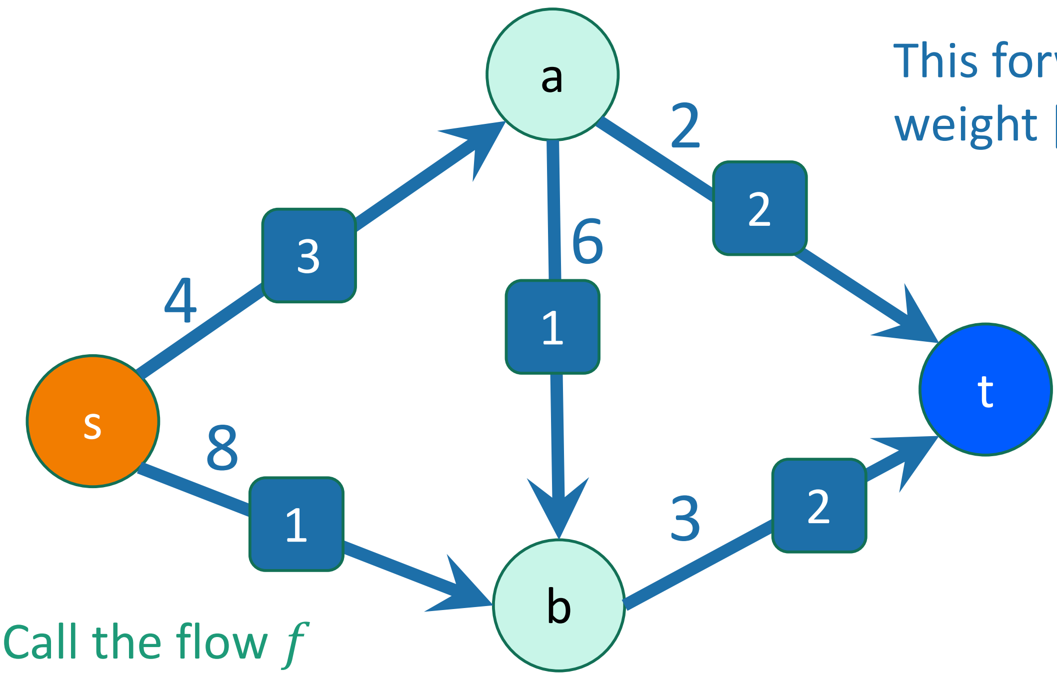
Ford-Fulkerson algorithm

- Outline of algorithm:
  - We will be updating a flow f
  - Start with $f = 0$
  - We will maintain a **"residual graph"** G_f
  - A path from s to t in G_f will give us a way to improve our flow.
  - We will continue until there are no s-t paths left in G_f.

Assume for today that we don't have edges like this, although this assumption can be removed.

Mạng có dư (Residual networks)

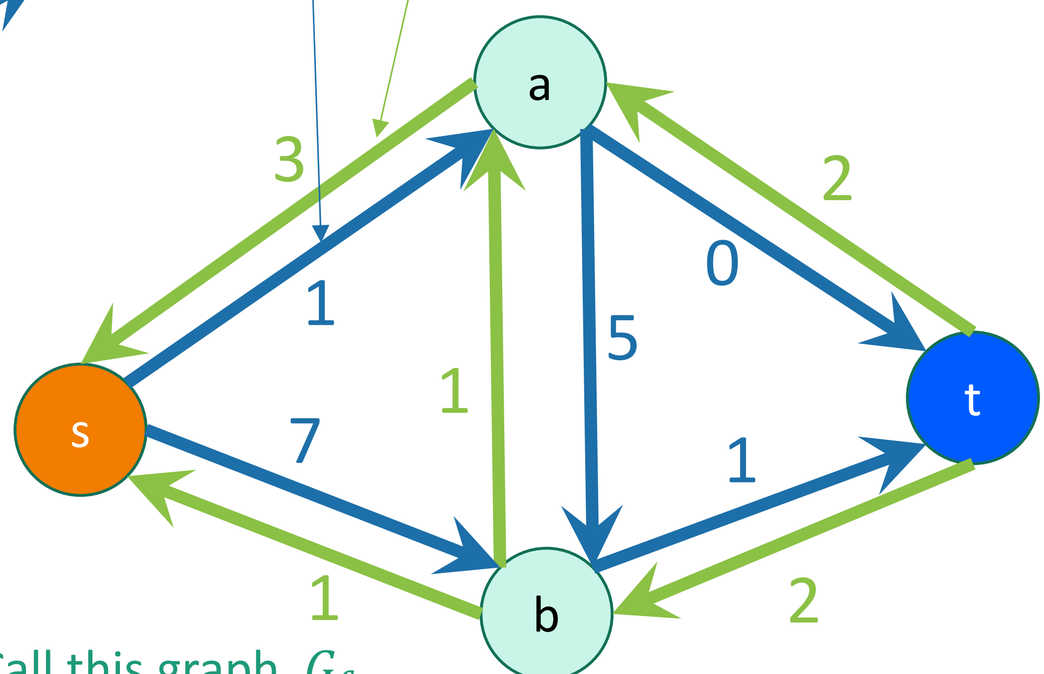
Say we have a flow



Call the flow f
Call the graph G

This forward edge has weight [capacity] – [flow].

This backward edge has weight [flow].



Call this graph G_f

Create a new **residual network** from this flow:

Mạng có dư (Residual networks)

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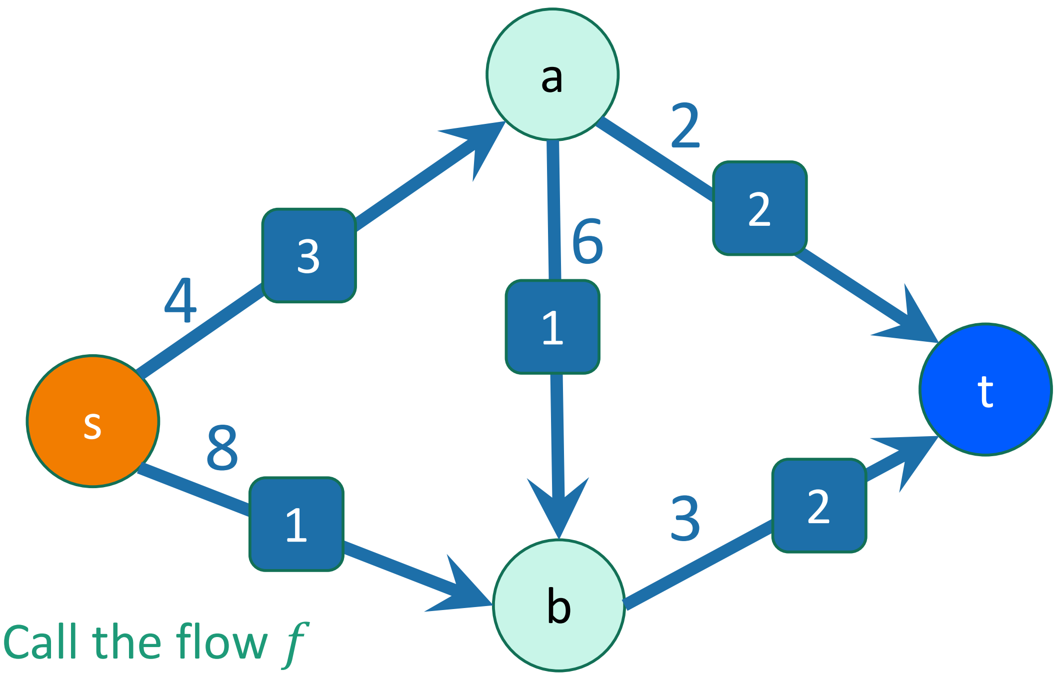
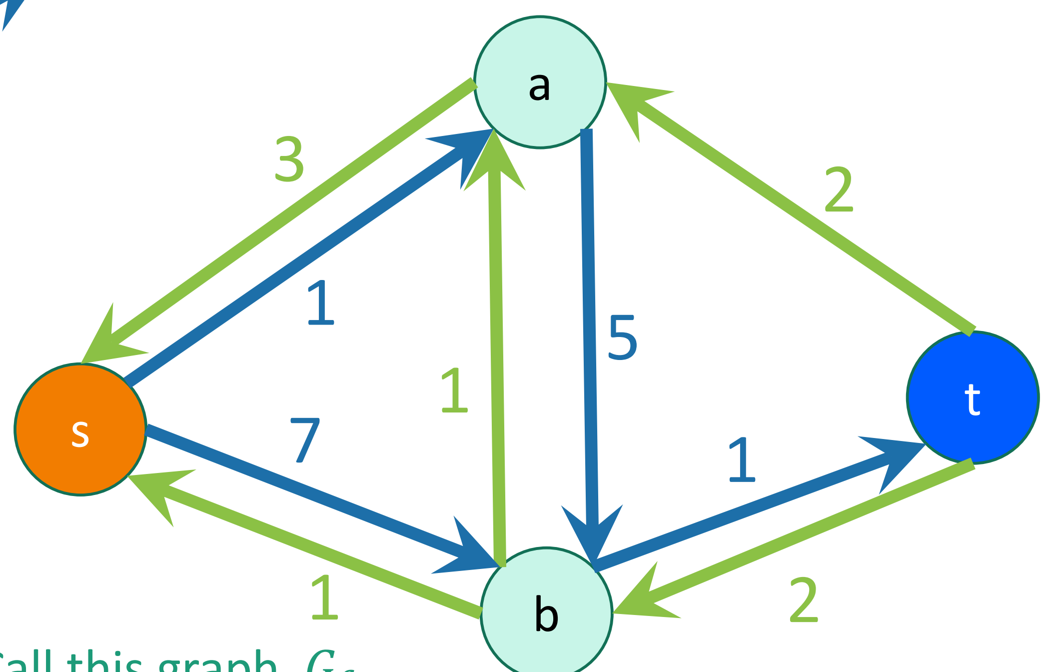


Call the flow f
Call the graph G

Forward edges are the amount that's left.
Backwards edges are the amount that's been used.

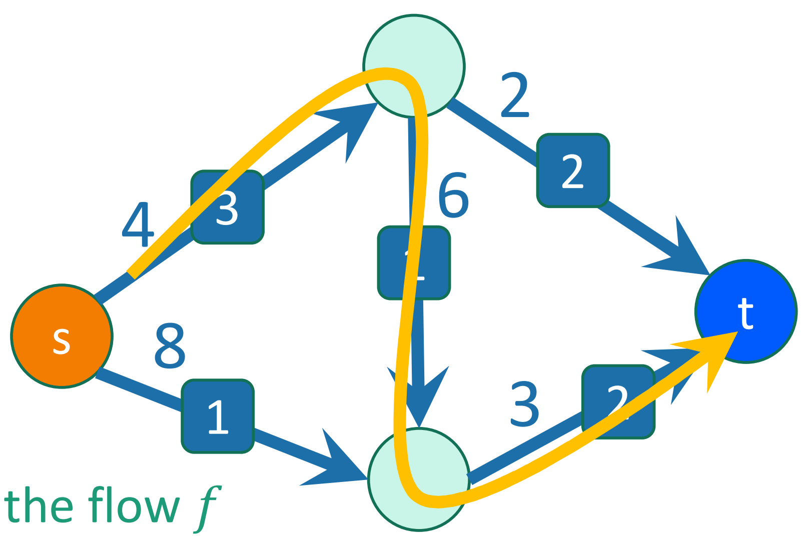
Create a new **residual network** from this flow:



Call this graph G_f

Residual networks tell us how to improve the flow

- **Definition**: A path from s to t in the residual network is called an **augmenting path (đường tăng)**.
- **Claim**: If there is an augmenting path in G_f, we can increase the flow along that path in G.



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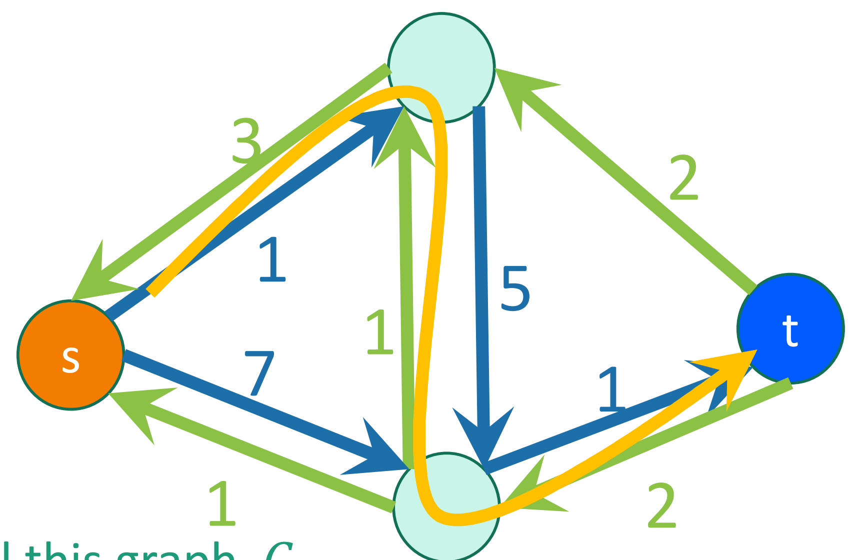

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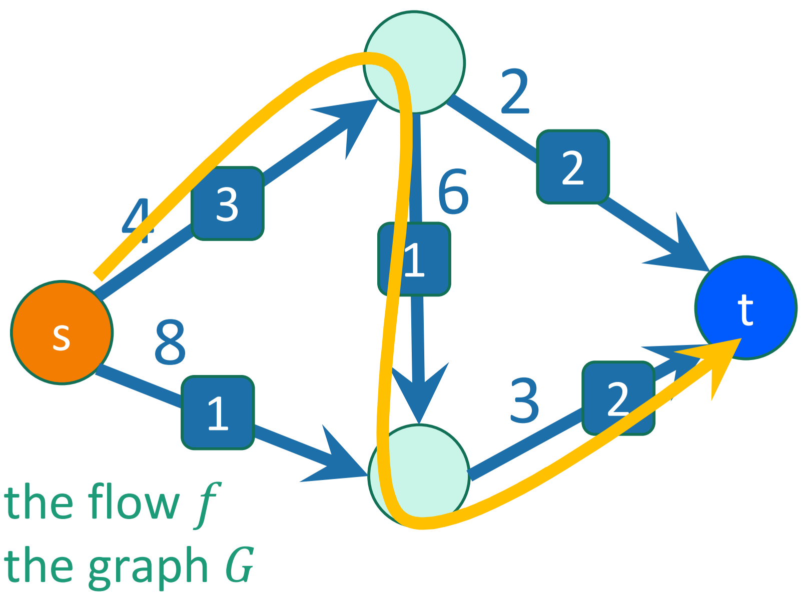
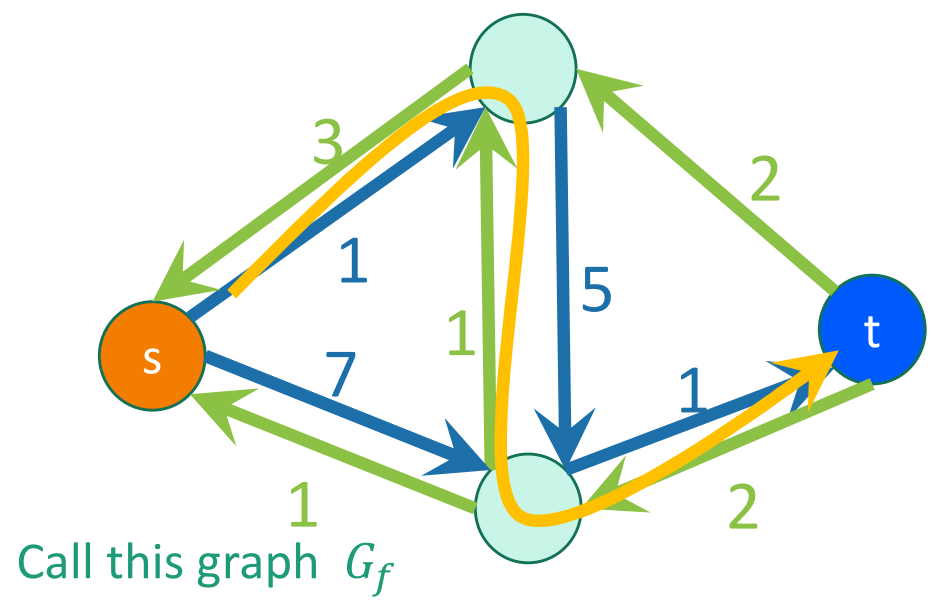
claim:

if there is an augmenting path, we can increase the flow along that path.

- Easy case: every edge on the path in G_f is a **forward edge** in G





- Forward edges indicate how much stuff can still go through.
- Just increase the flow on all the edges!

claim:

if there is an augmenting path, we can increase the flow along that path.

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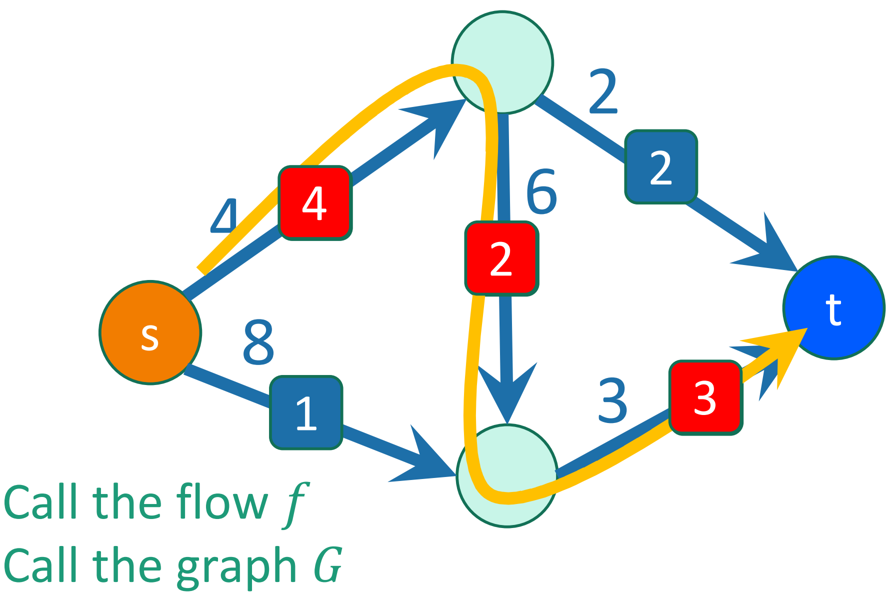

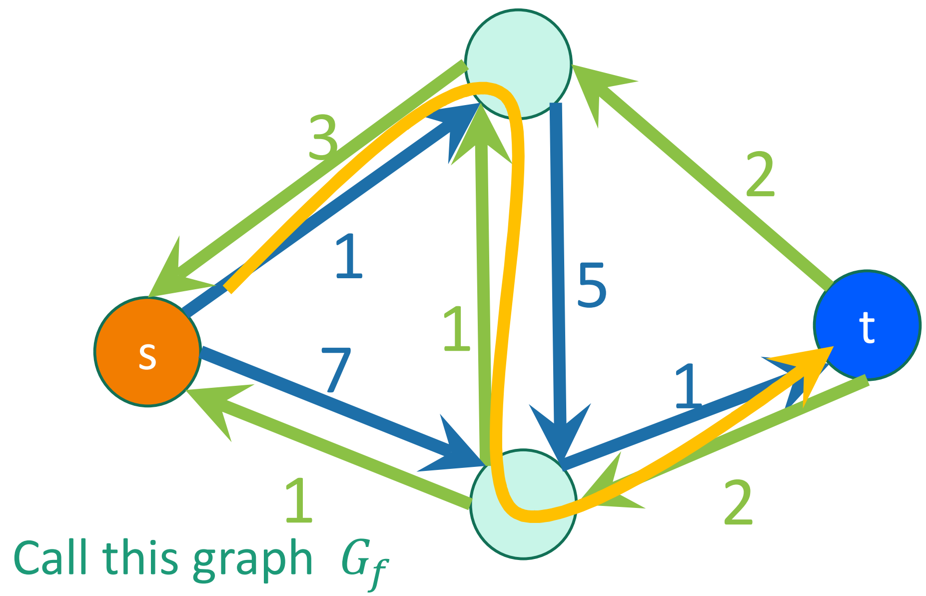

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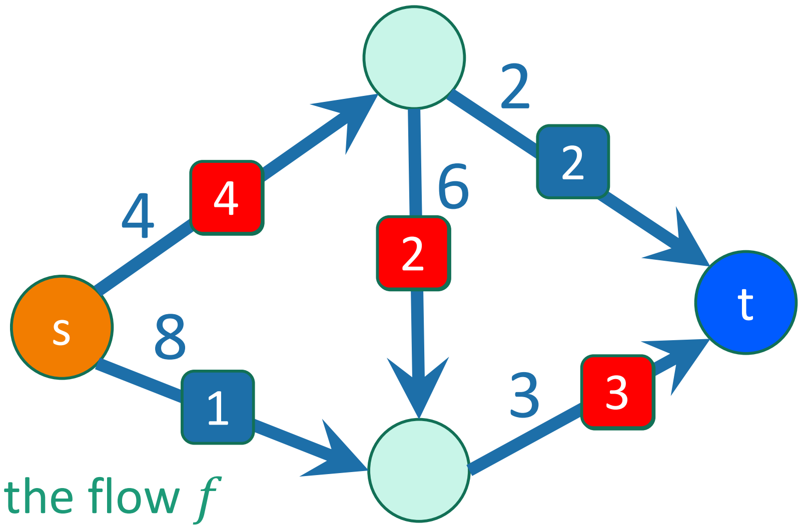

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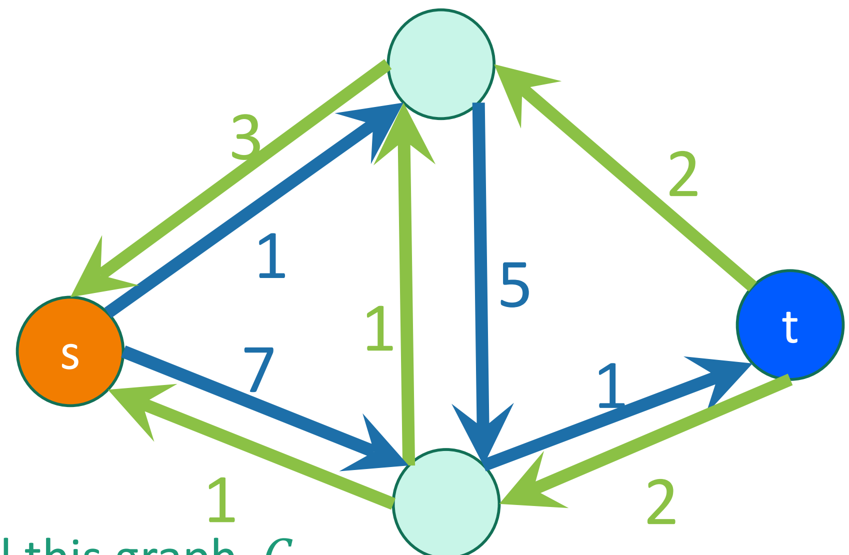

Call this graph G_f

- Forward edges indicate how much stuff can still go through.
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Then update the residual graph.

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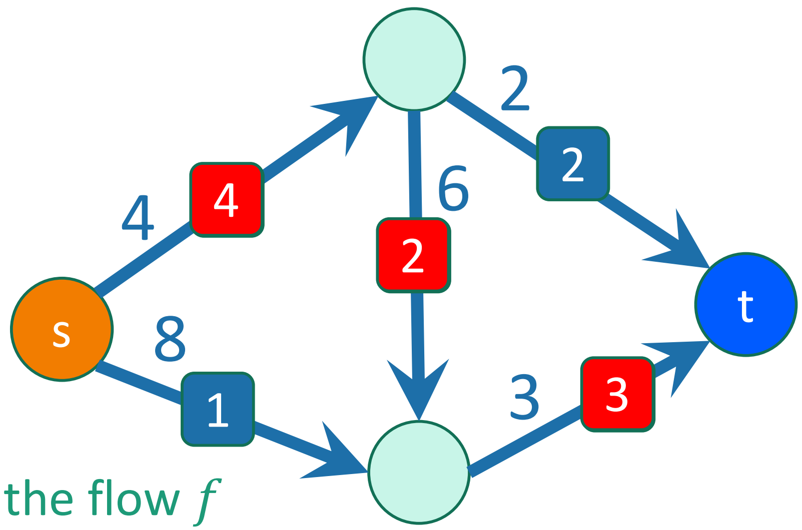

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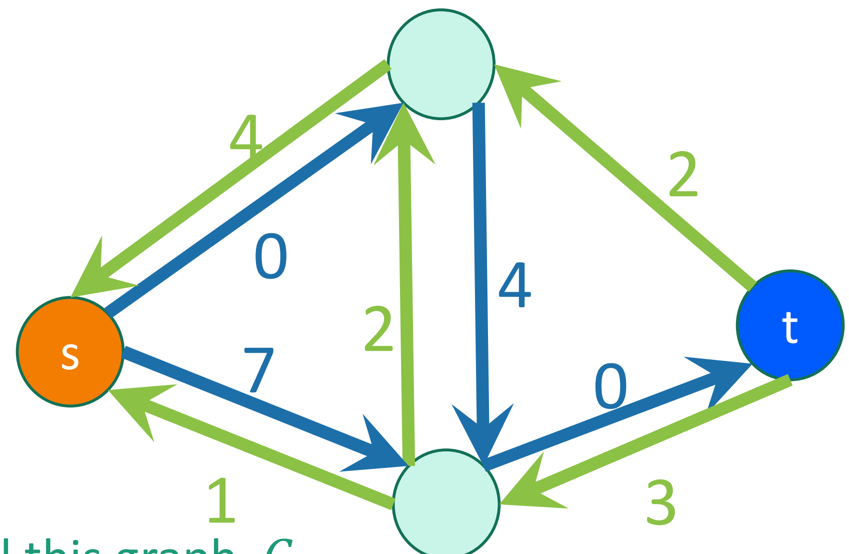

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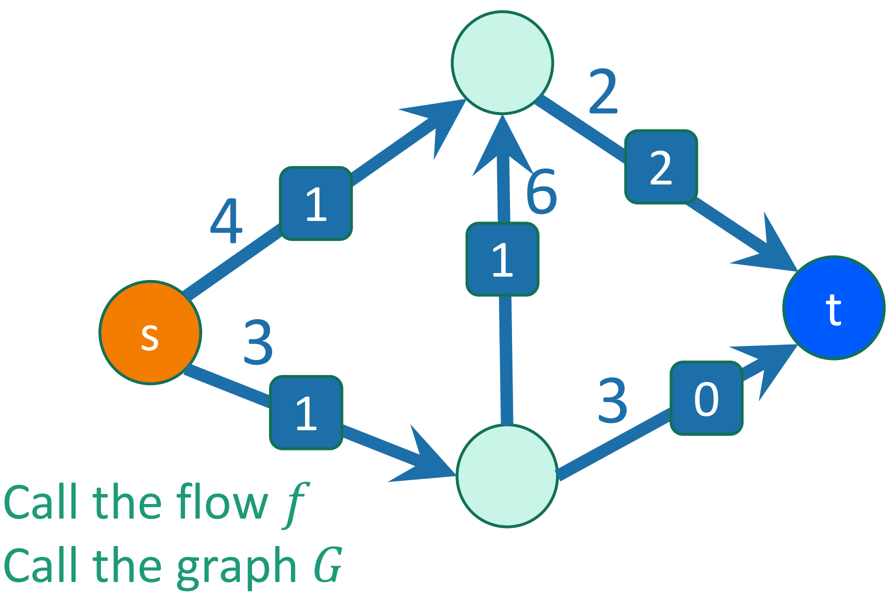
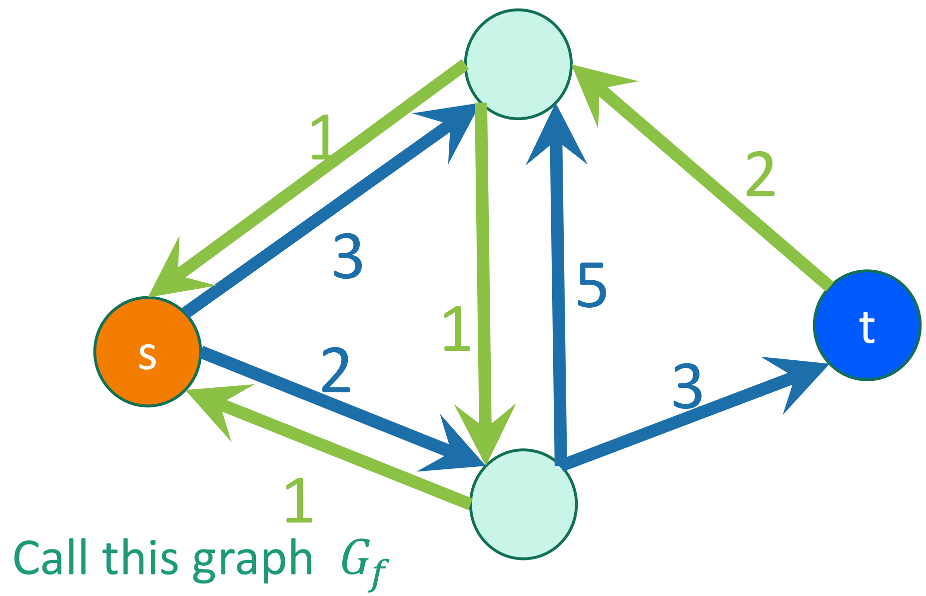
if there is an augmenting path, we can increase the flow along that path.

- Harder case: there are **backward edges** in G in the path.
    - Here's a slightly different example of a flow:





I changed some of the weights and edge directions.

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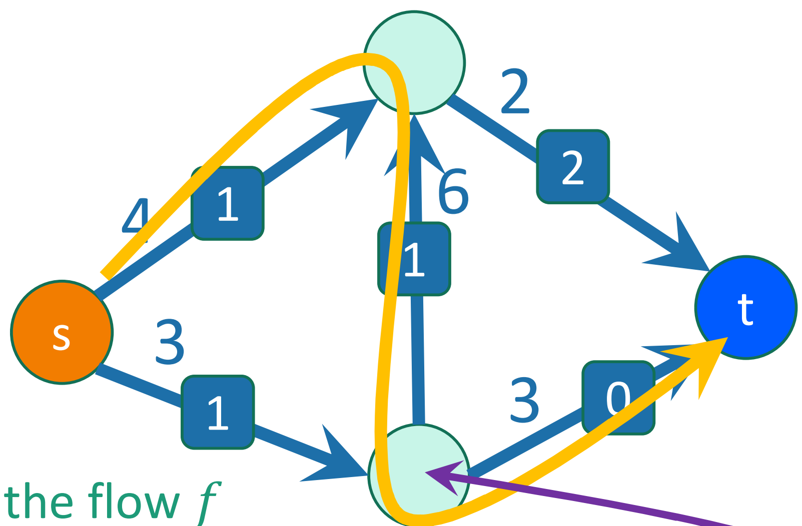

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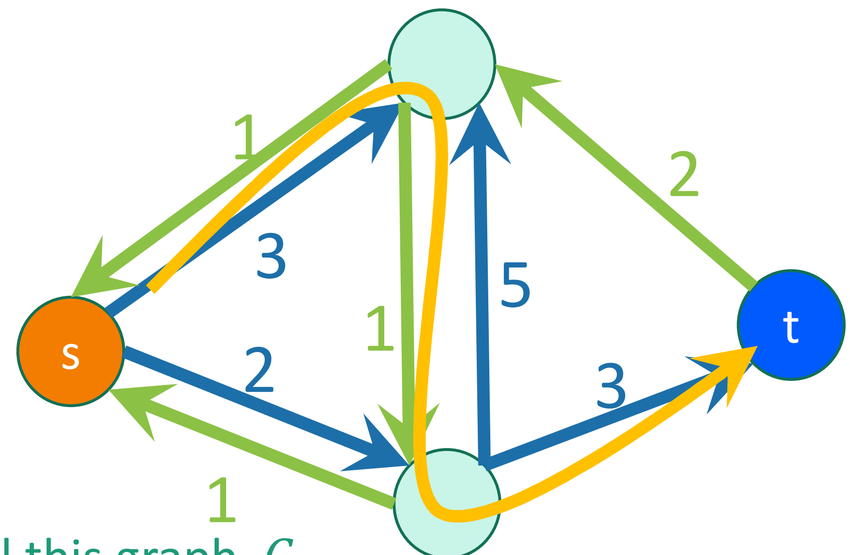


Call this graph G_f

Now we should NOT increase the flow at all the edges along the path!

- For example, that will mess up the conservation of stuff at this vertex.

I changed some of the weights and edge directions.

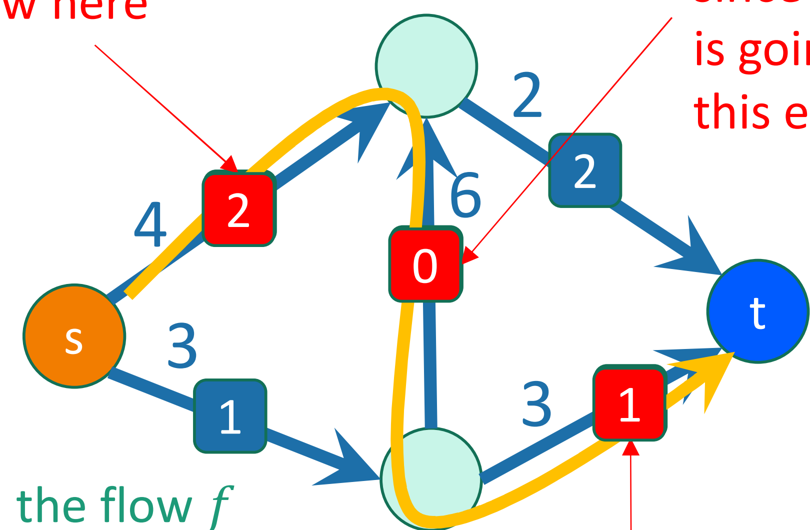
claim:

if there is an augmenting path, we can increase the flow along that path.

- In this case we do something a bit different:

We will add flow here

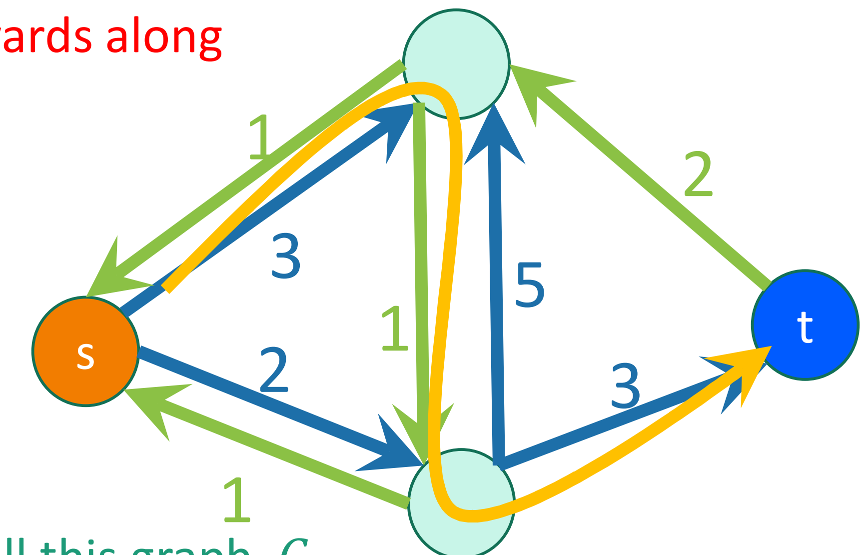
We will remove flow here, since our augmenting path is going backwards along this edge.



We will add flow here

Call the flow f

Call the graph G



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claim:

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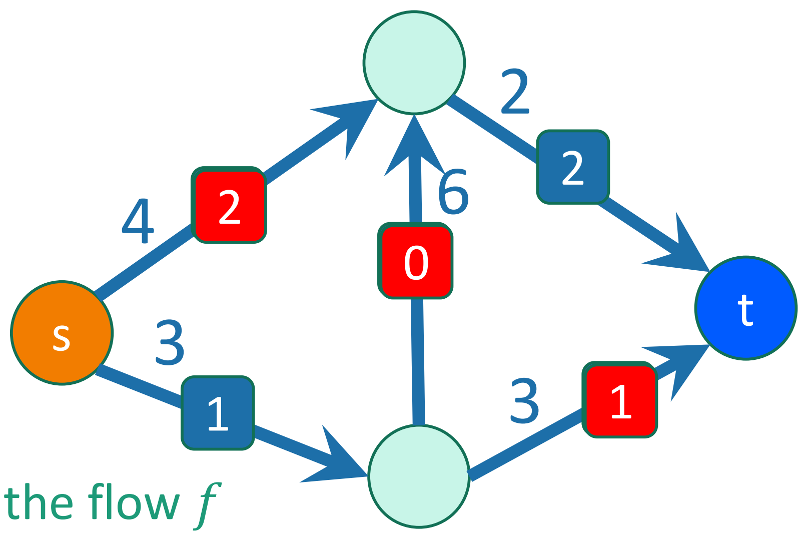
- In this case we do something a bit different:

Then we'll update the residual graph:



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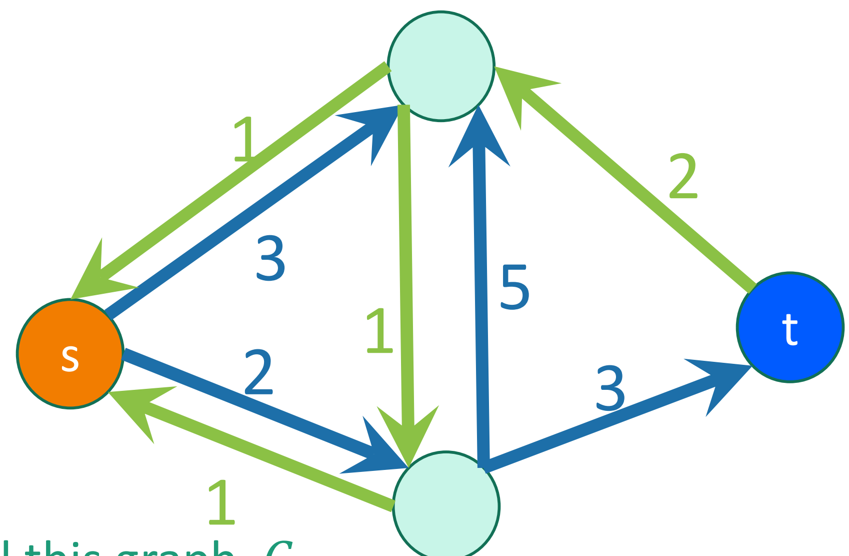

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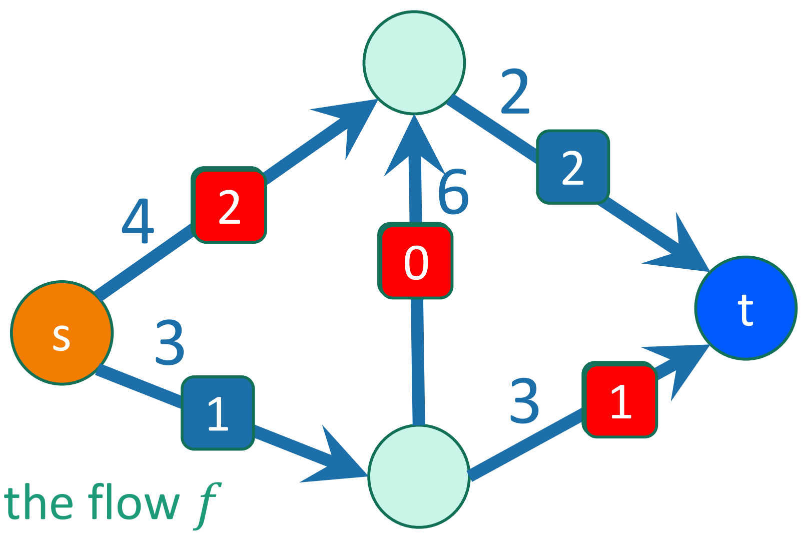

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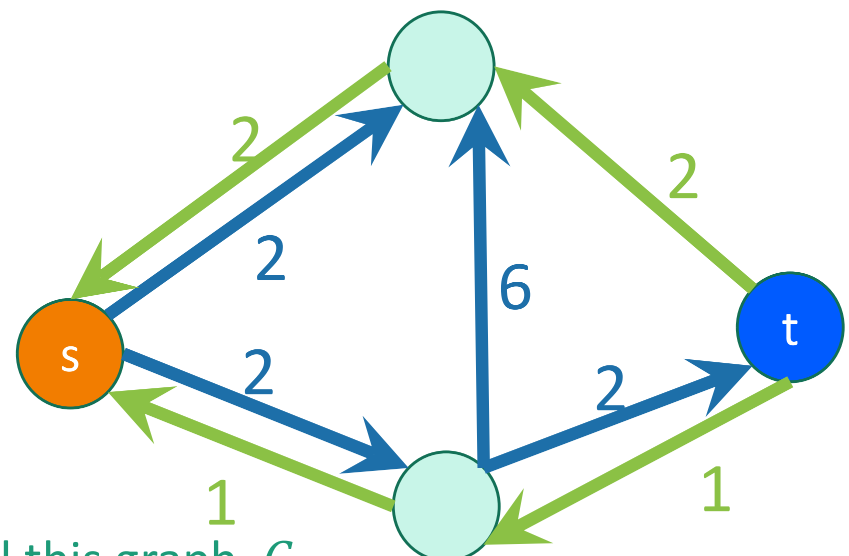
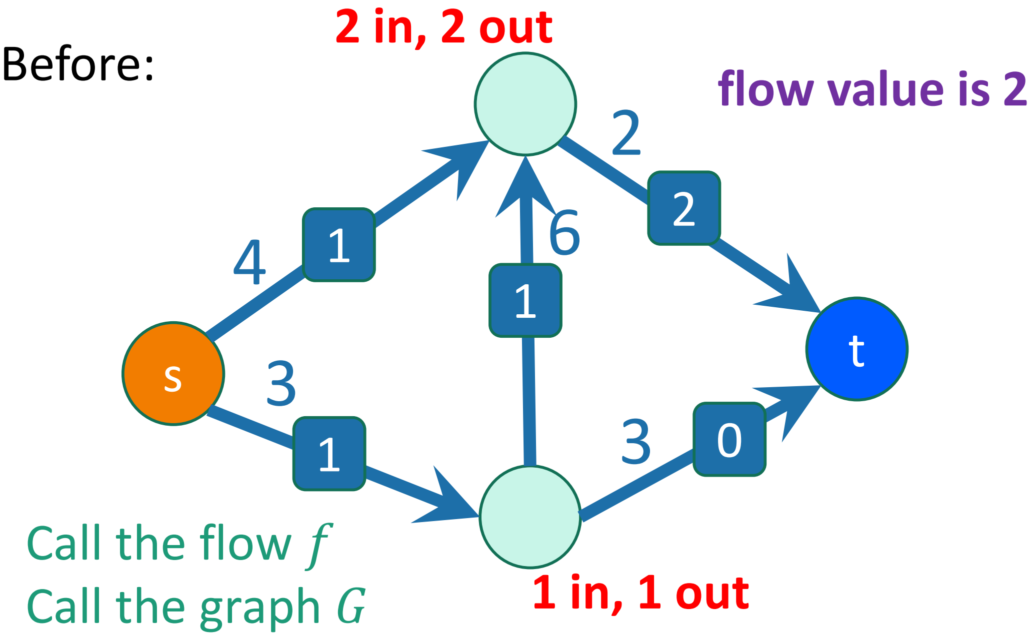
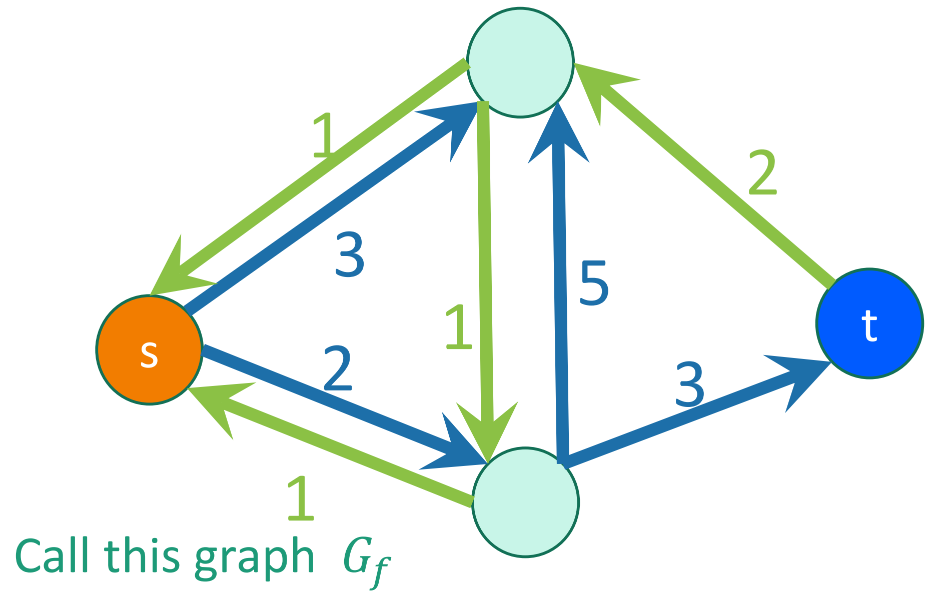


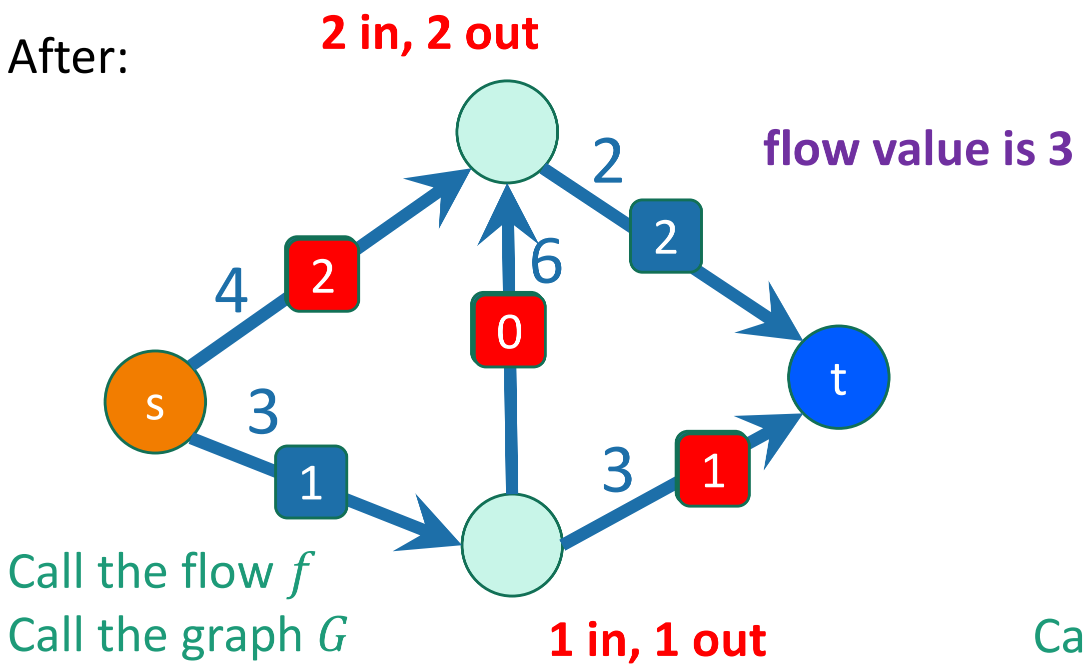
Call this graph G_f

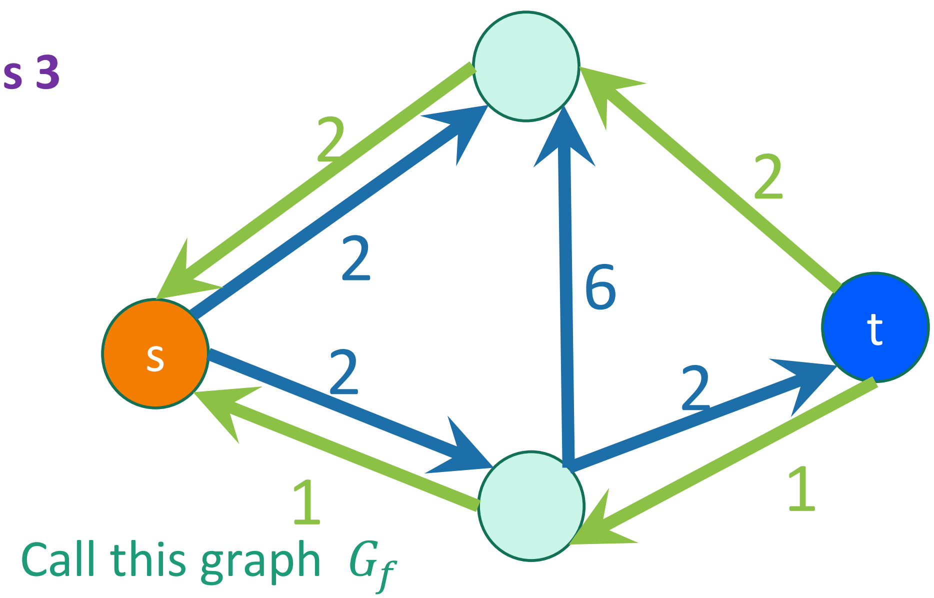
Before:





After:





Still a legit flow, but with a bigger value!

claim:

if there is an augmenting path, we can increase the flow along that path.

proof:

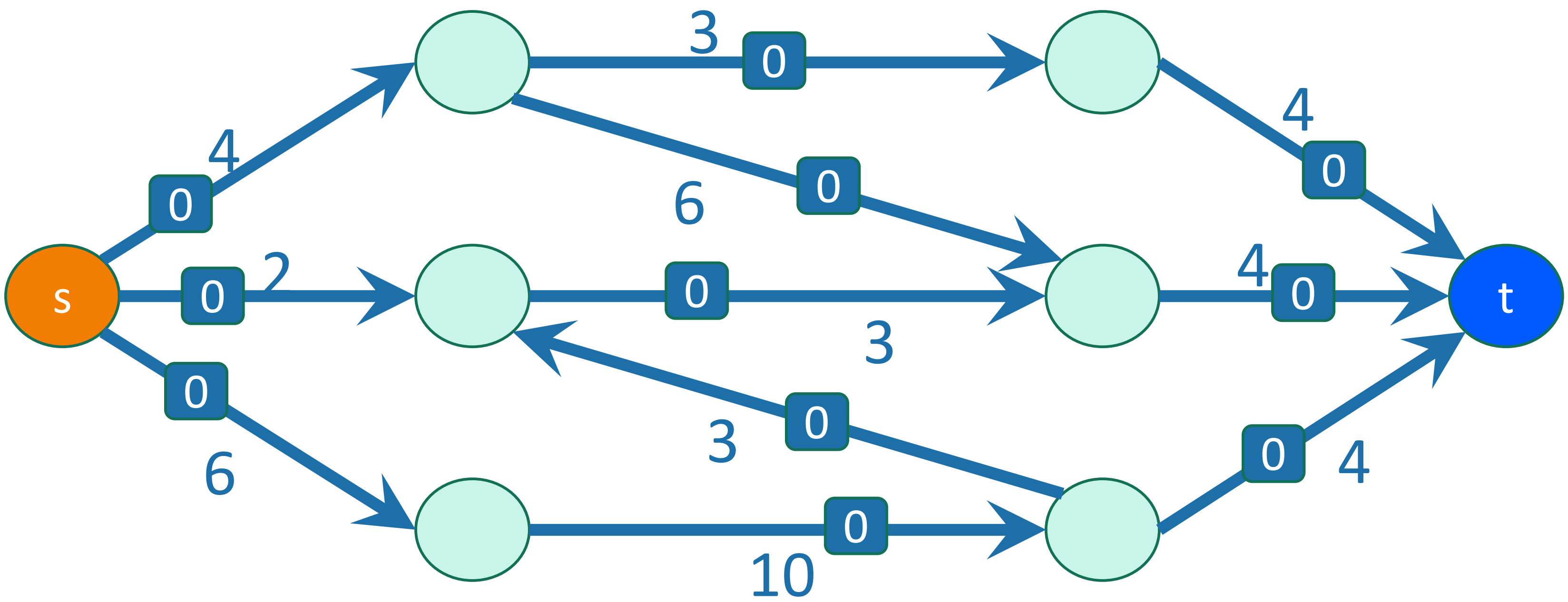
- increaseFlow(path P in G_f , flow f):
  - x = min weight on any edge in P
  - **for** (u,v) in P:
    - **if** (u,v) in E, $\boldsymbol{f'(u,v) \leftarrow f(u,v) + x}$.
    - **if** (v,u) in E, $\boldsymbol{f'(v,u) \leftarrow f(v,u) - x}$
  - **return** f'

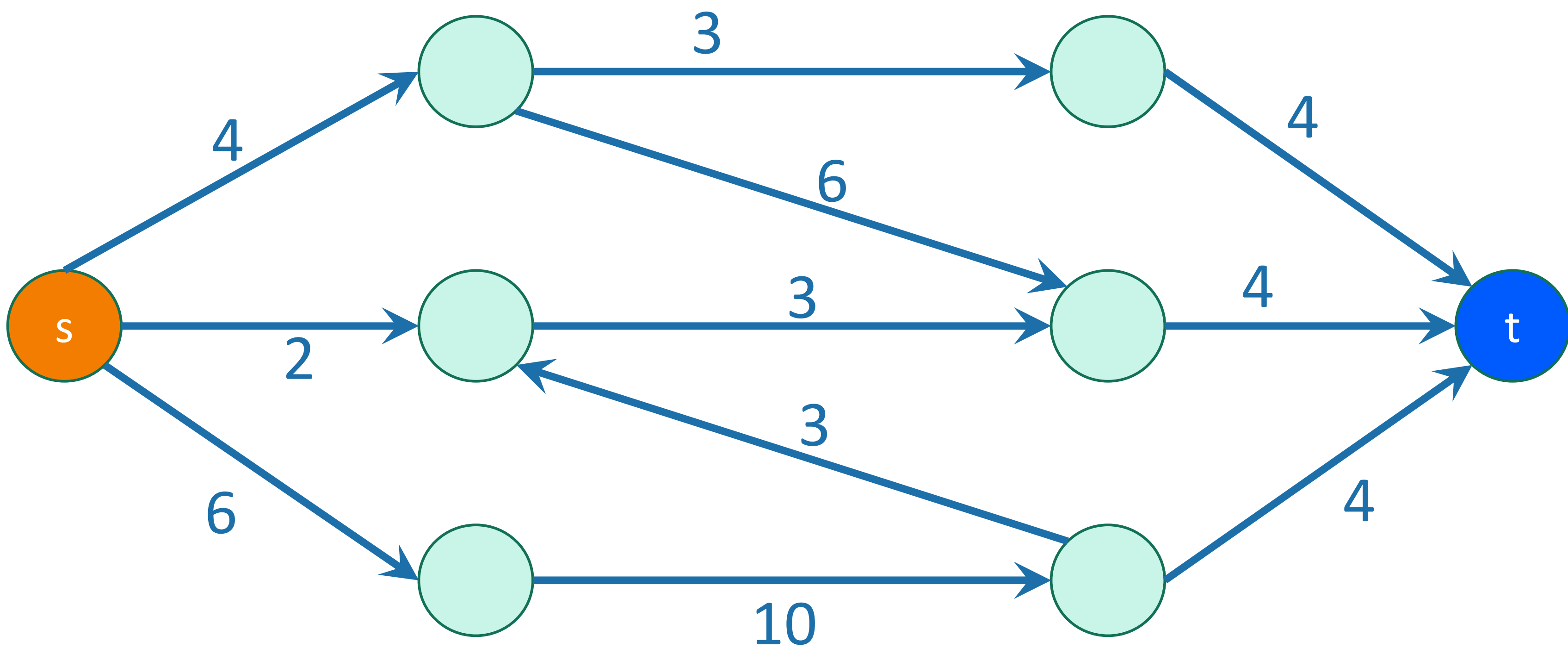
Check that this always makes a bigger (and legit) flow!

Ford-Fulkerson Algorithm

- **Ford-Fulkerson**(G):
  - $f \leftarrow$ all zero flow.
  - $G_f \leftarrow G$
  - **while** t is reachable from s in G_f
    - Find a path P from s to t in G_f // eg, use BFS
    - $f \leftarrow$ **increaseFlow**(P,f)
    - update G_f
  - **return** f

Example of Ford-Fulkerson





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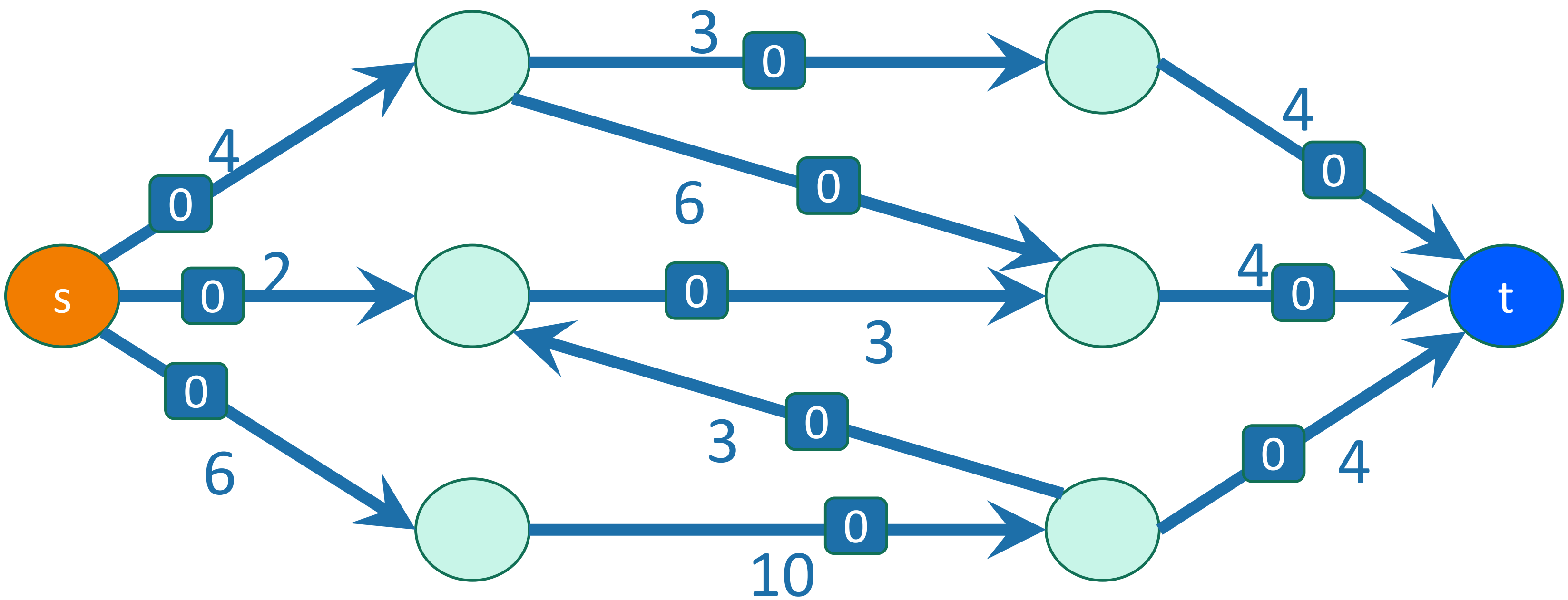

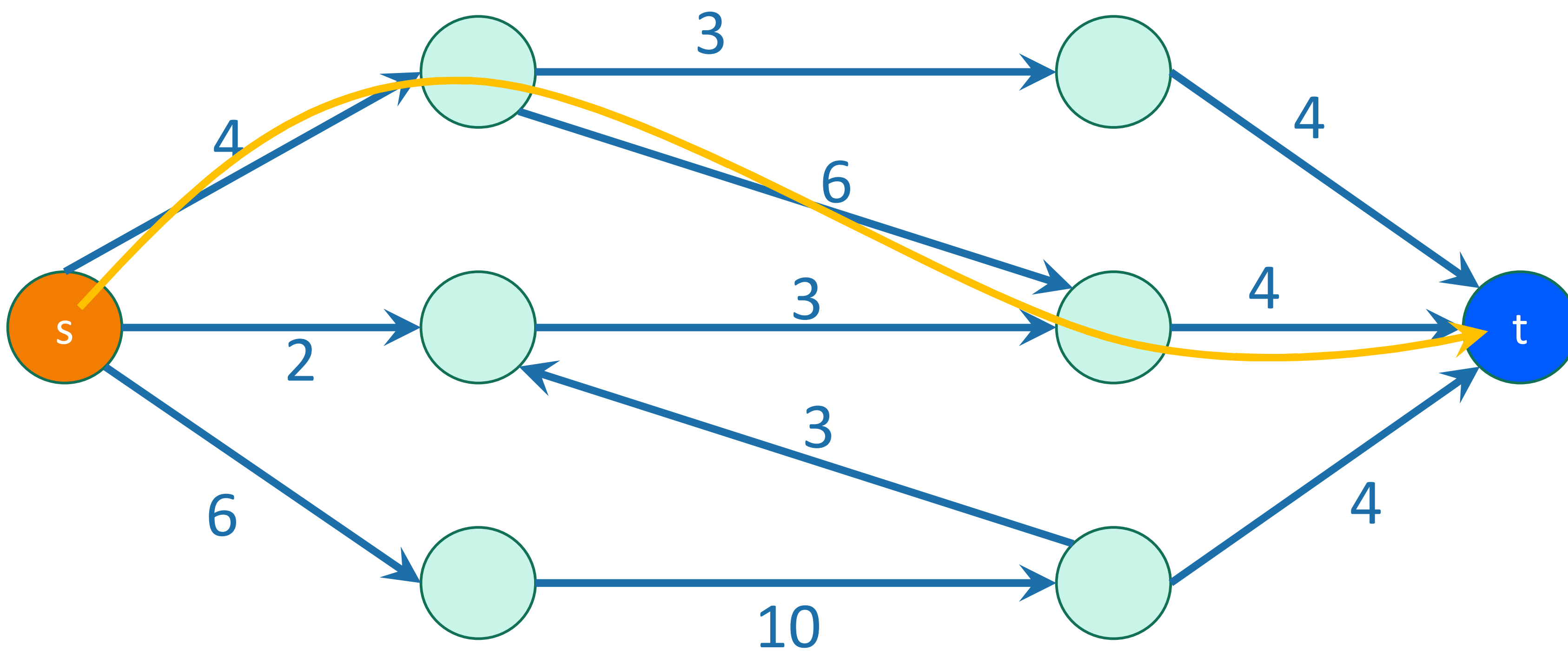

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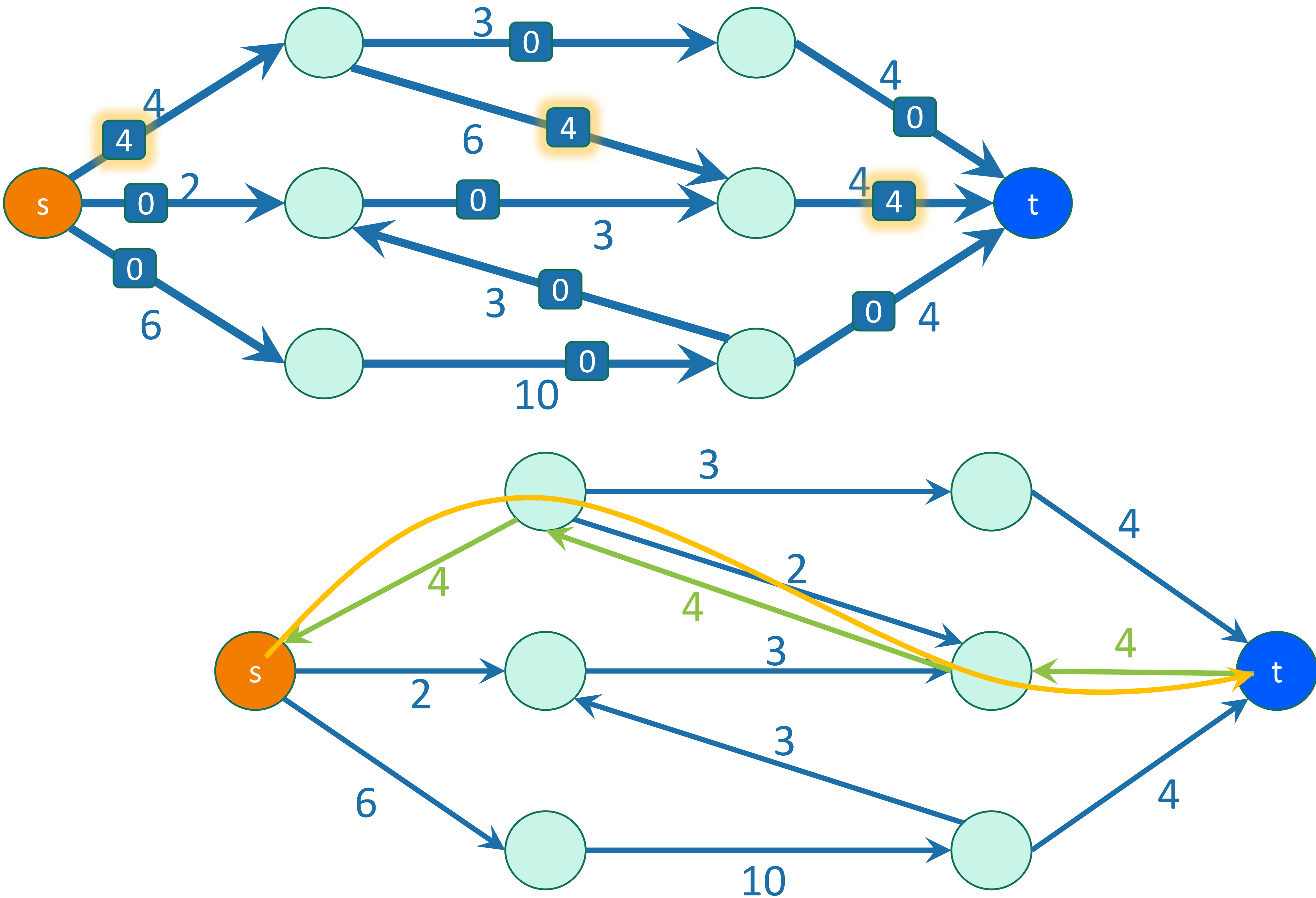

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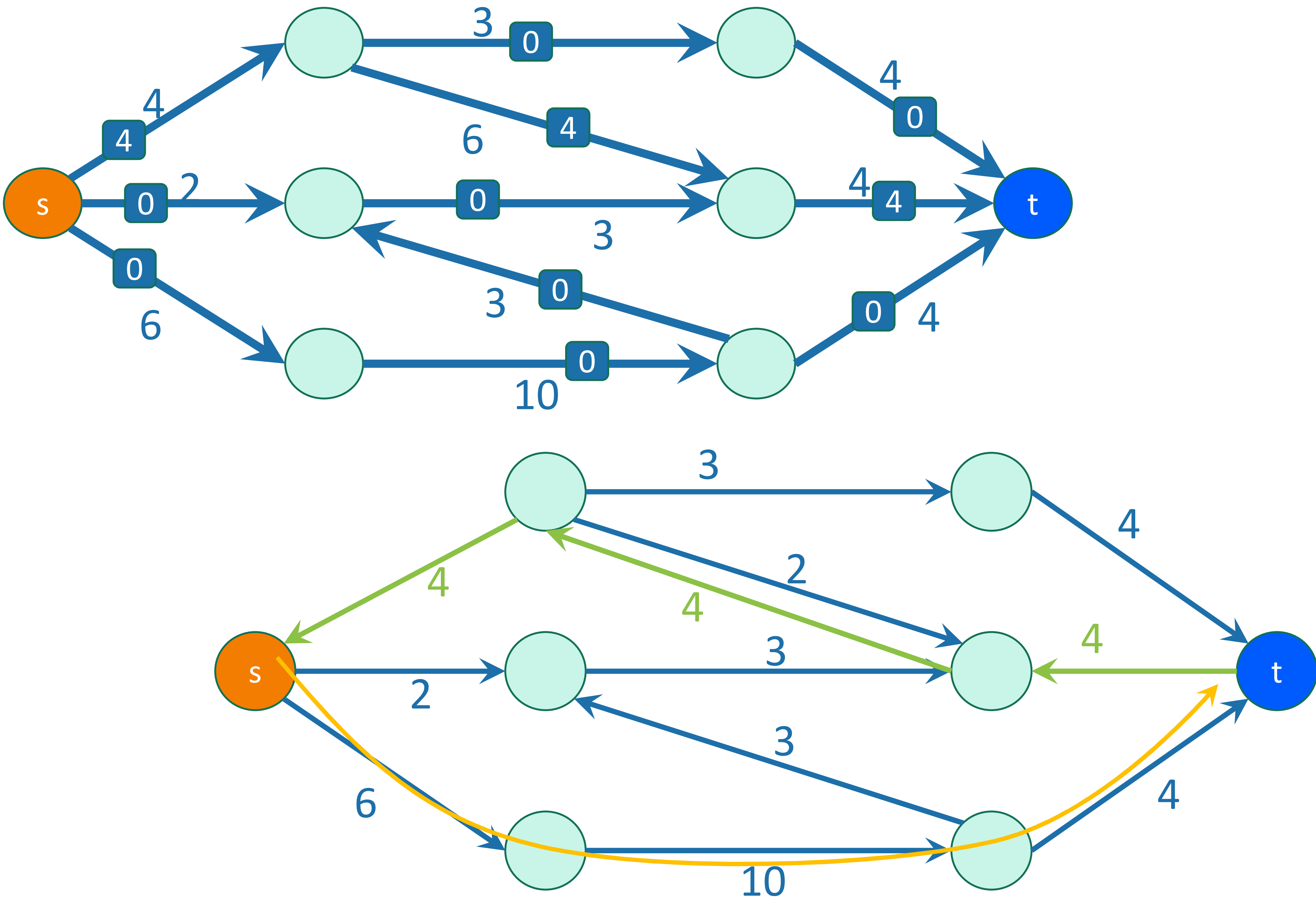
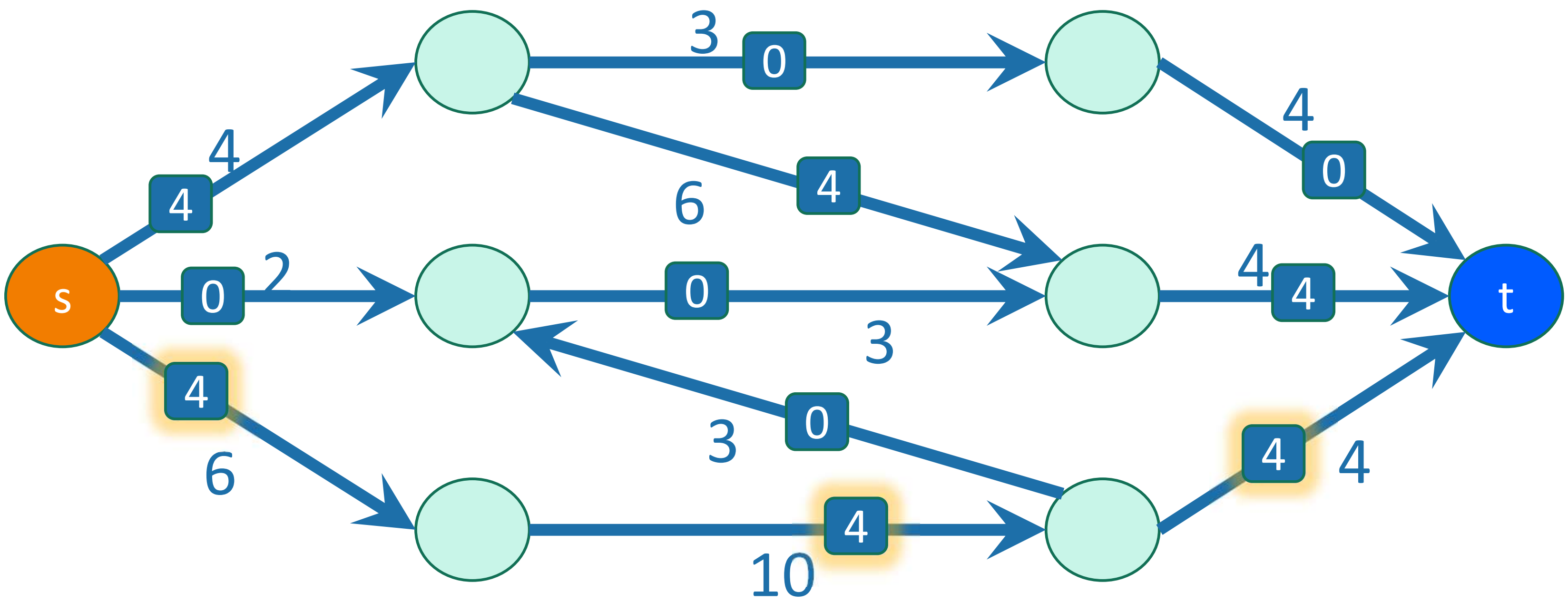
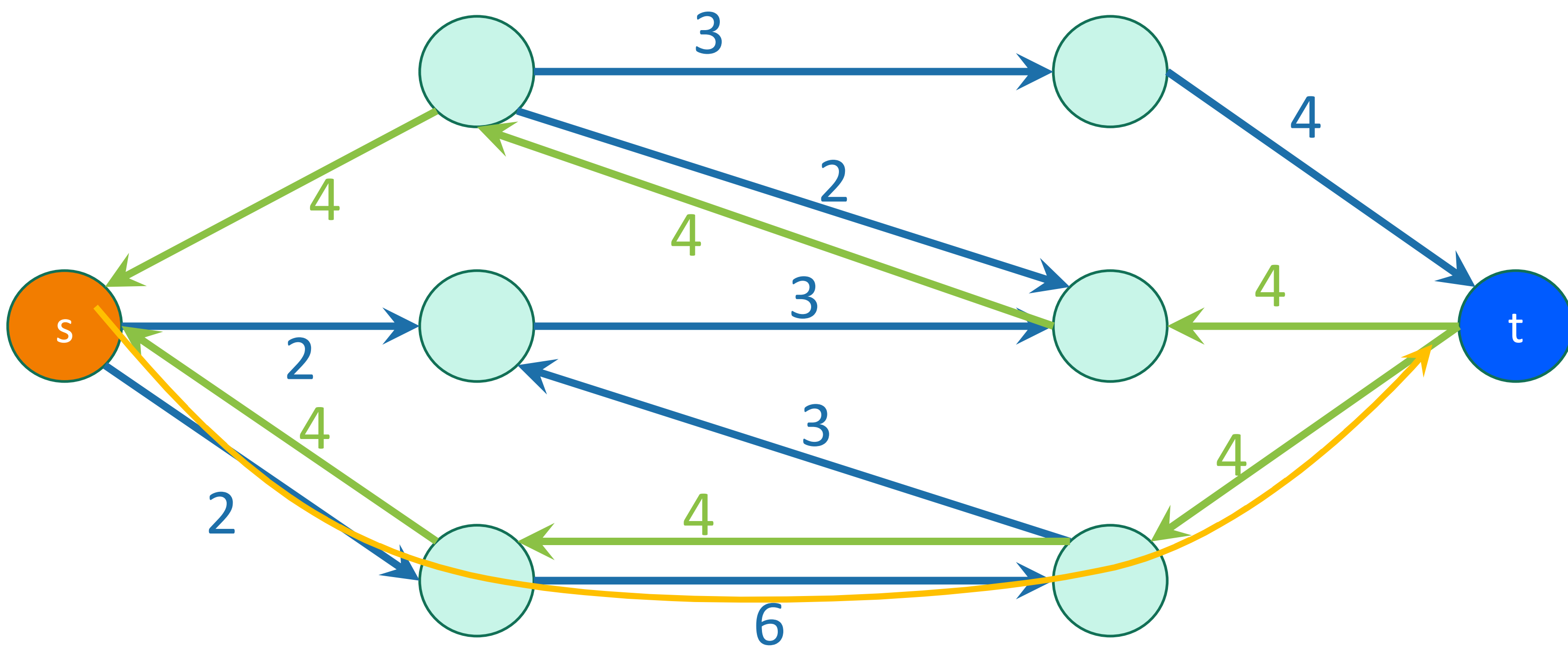


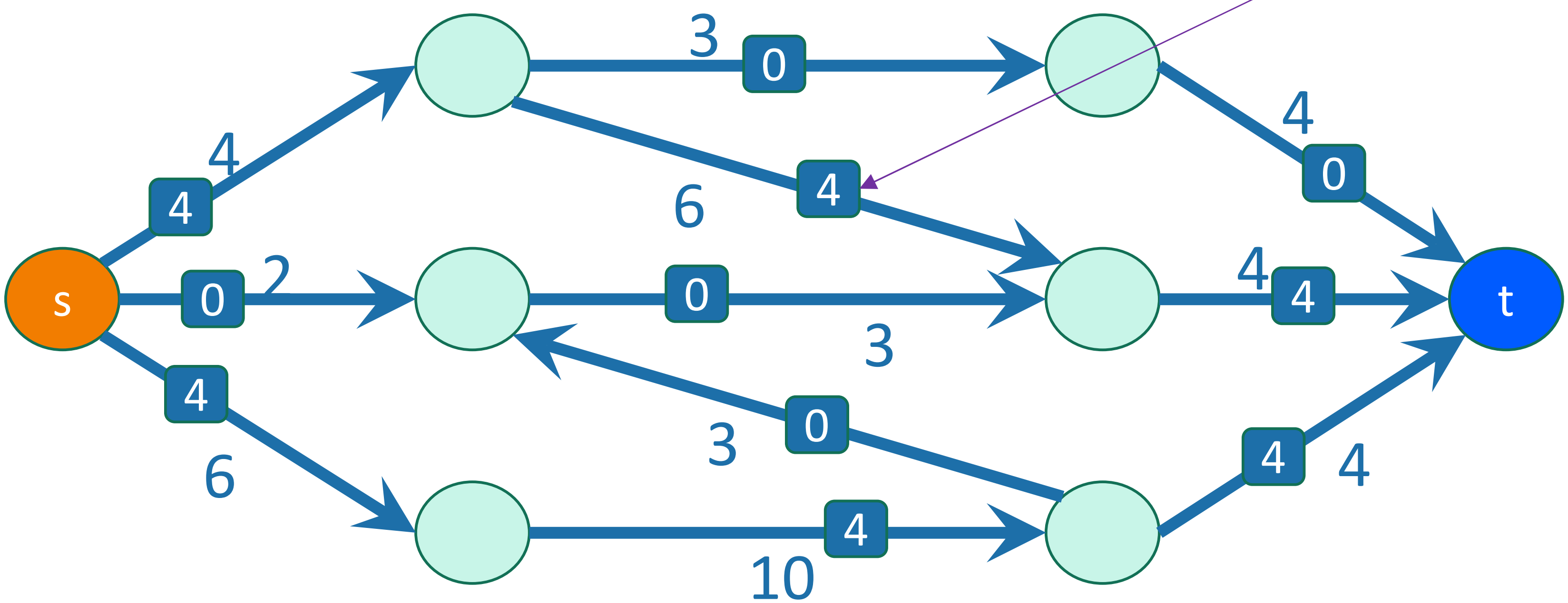
Example of Ford-Fulkerson



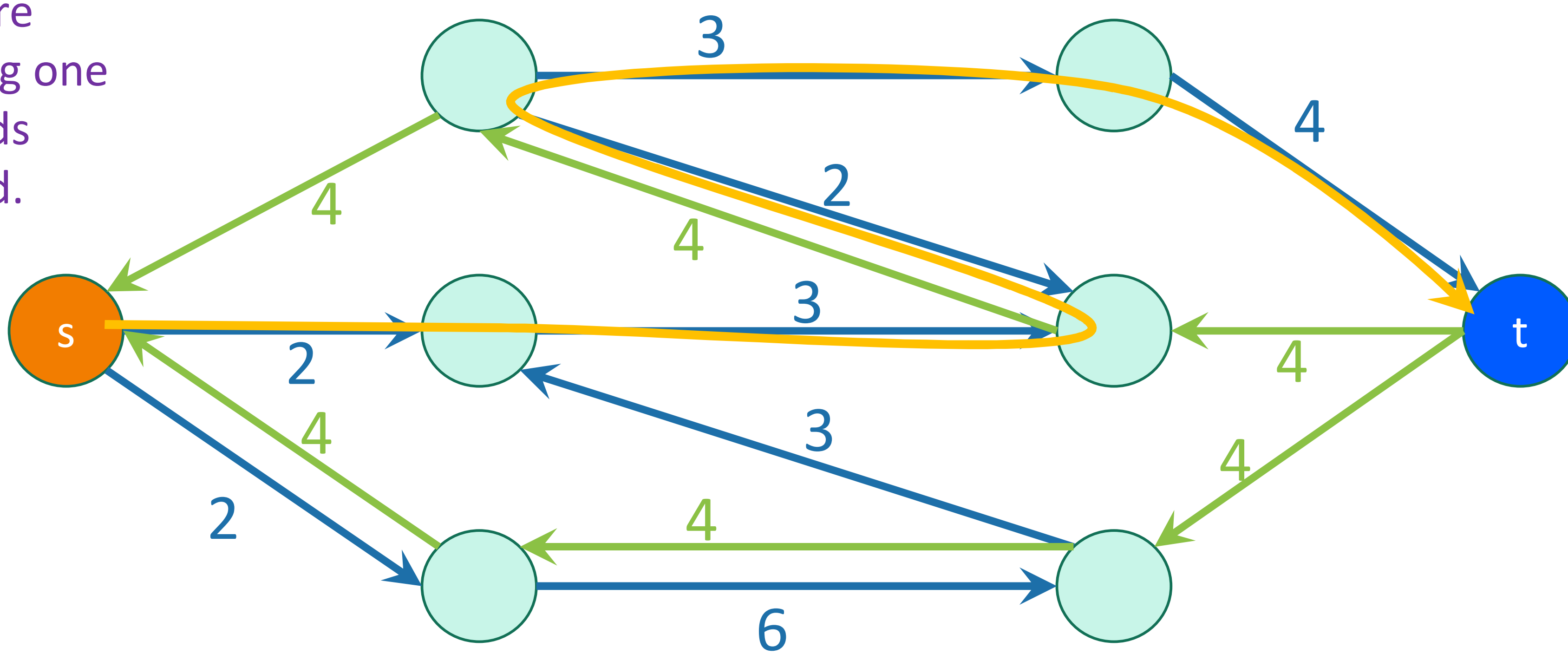


Example of Ford-Fulkerson

We will **remove** flow from this edge.



Notice that we're going back along one of the backwards edges we added.



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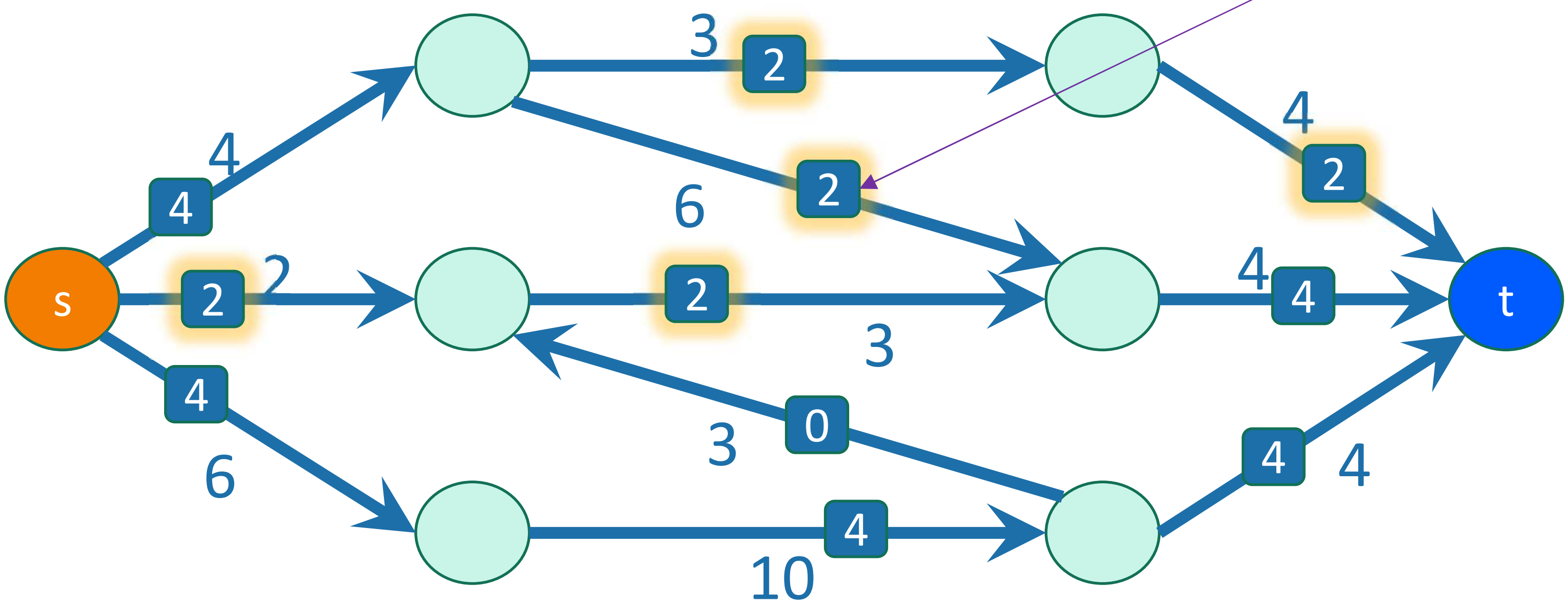
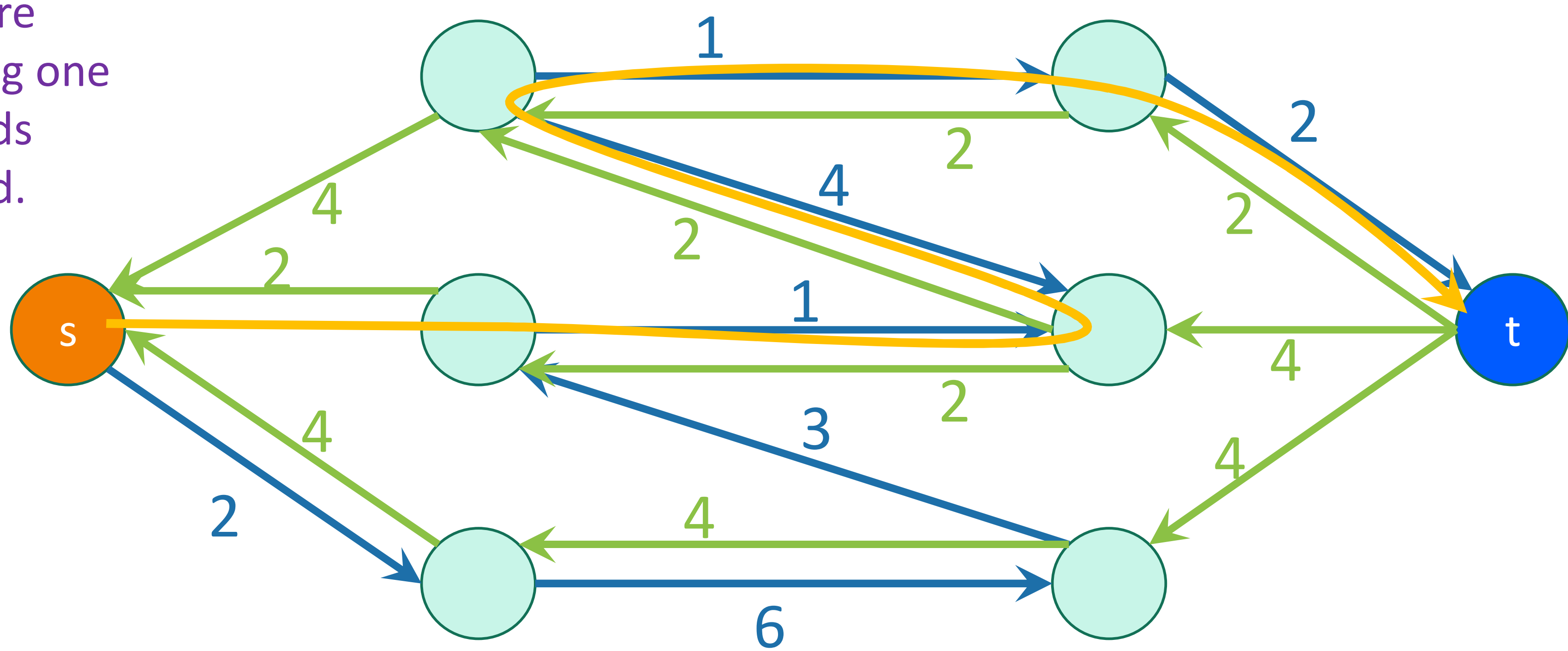
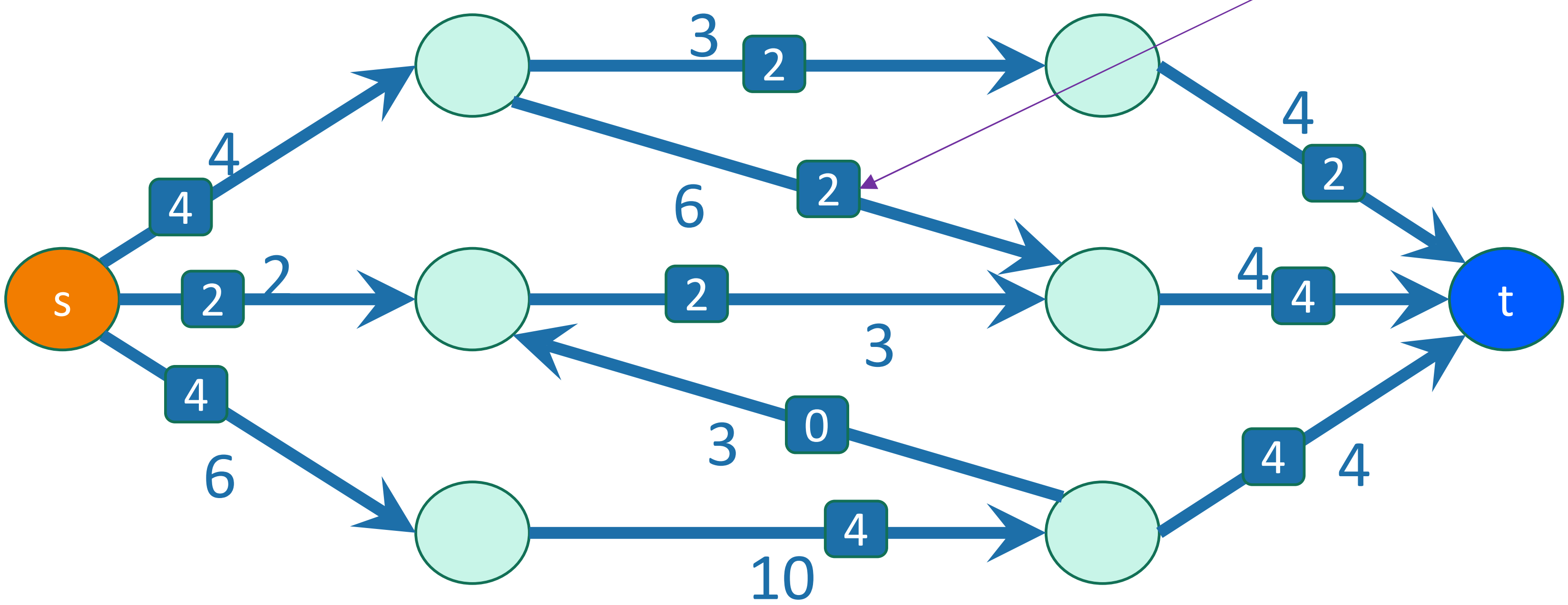


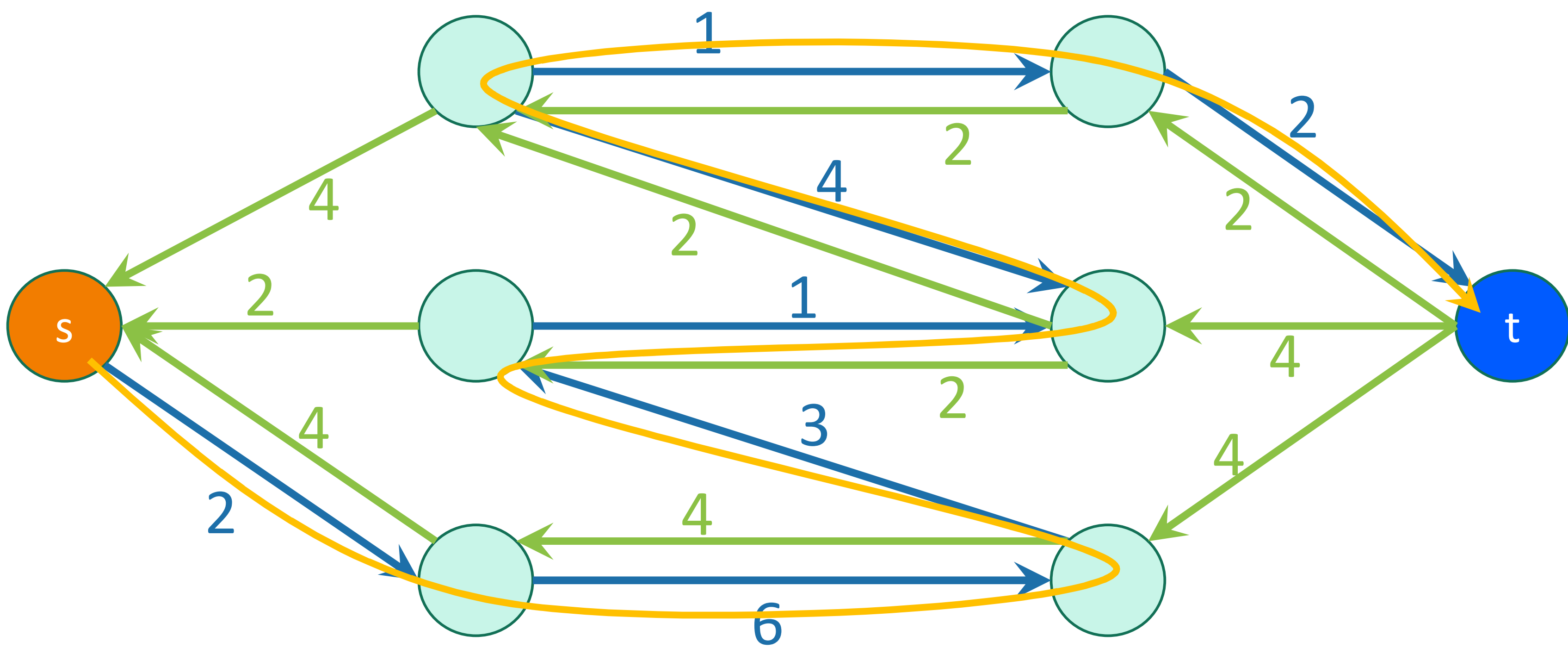
Notice that we're going back along one of the backwards edges we added.



Example of Ford-Fulkerson

We will remove flow from this edge AGAIN.





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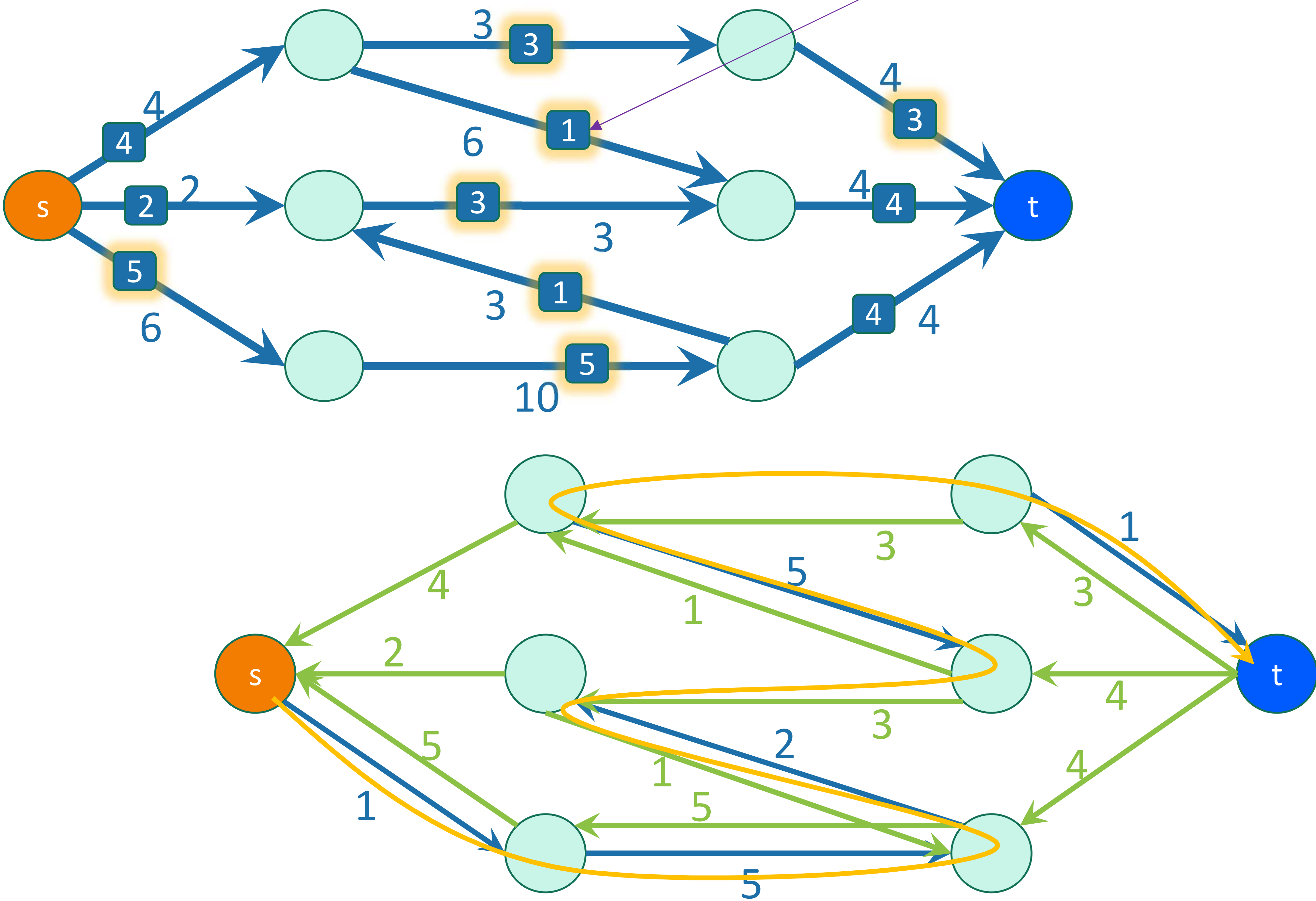
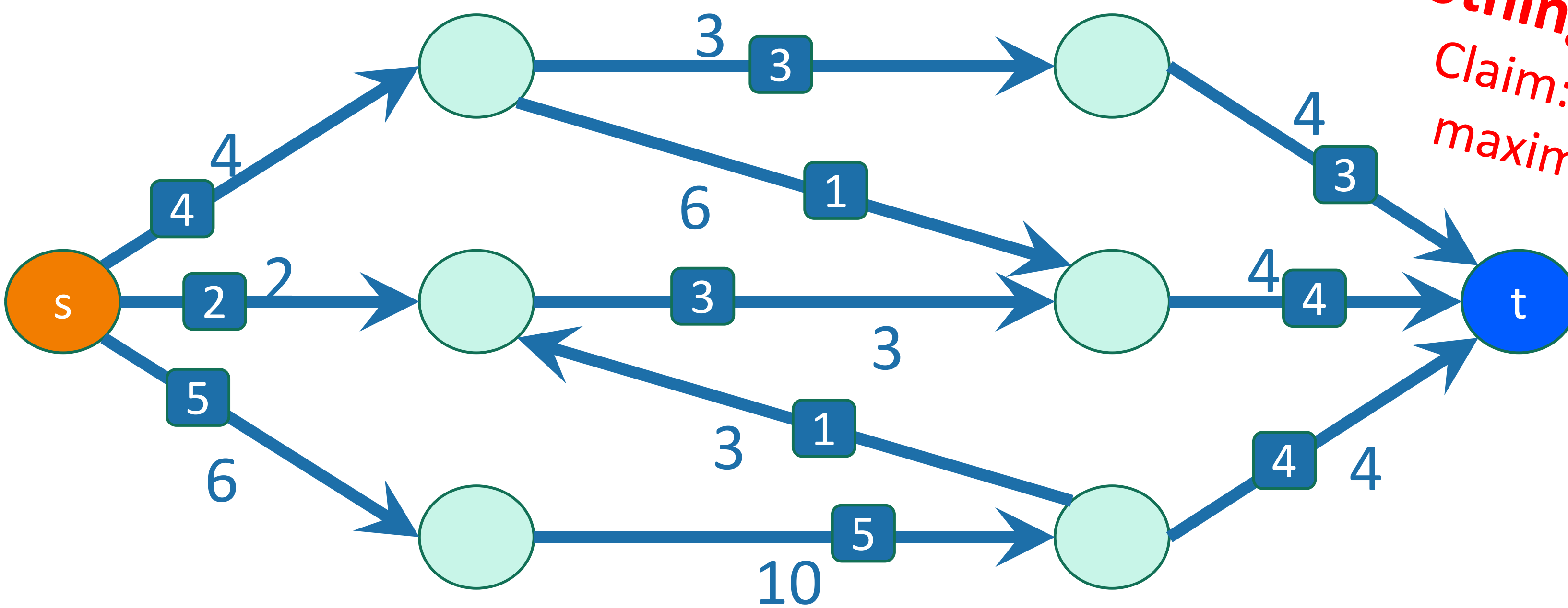
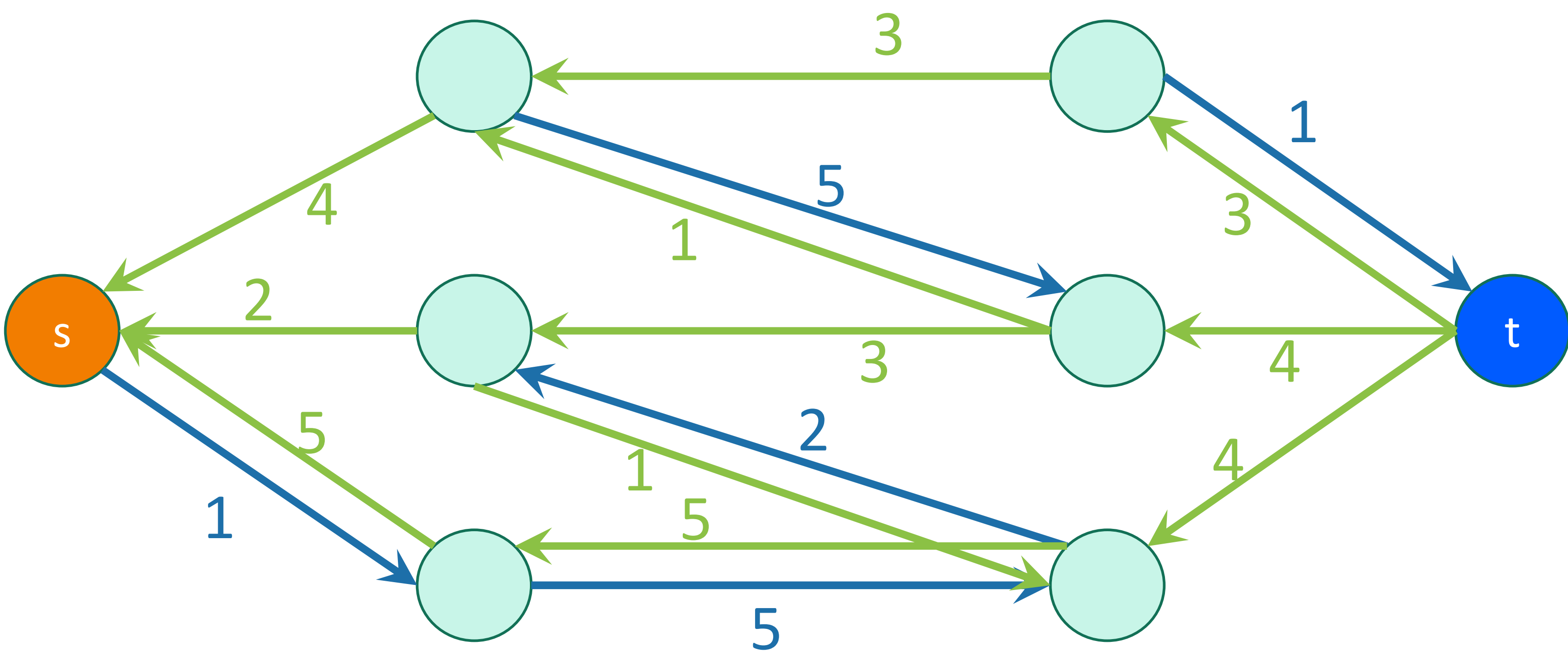


Example of Ford-Fulkerson

Now we have nothing left to do!

Claim: This is the maximum flow.





Example of Ford-Fulkerson

Now we have nothing left to do!

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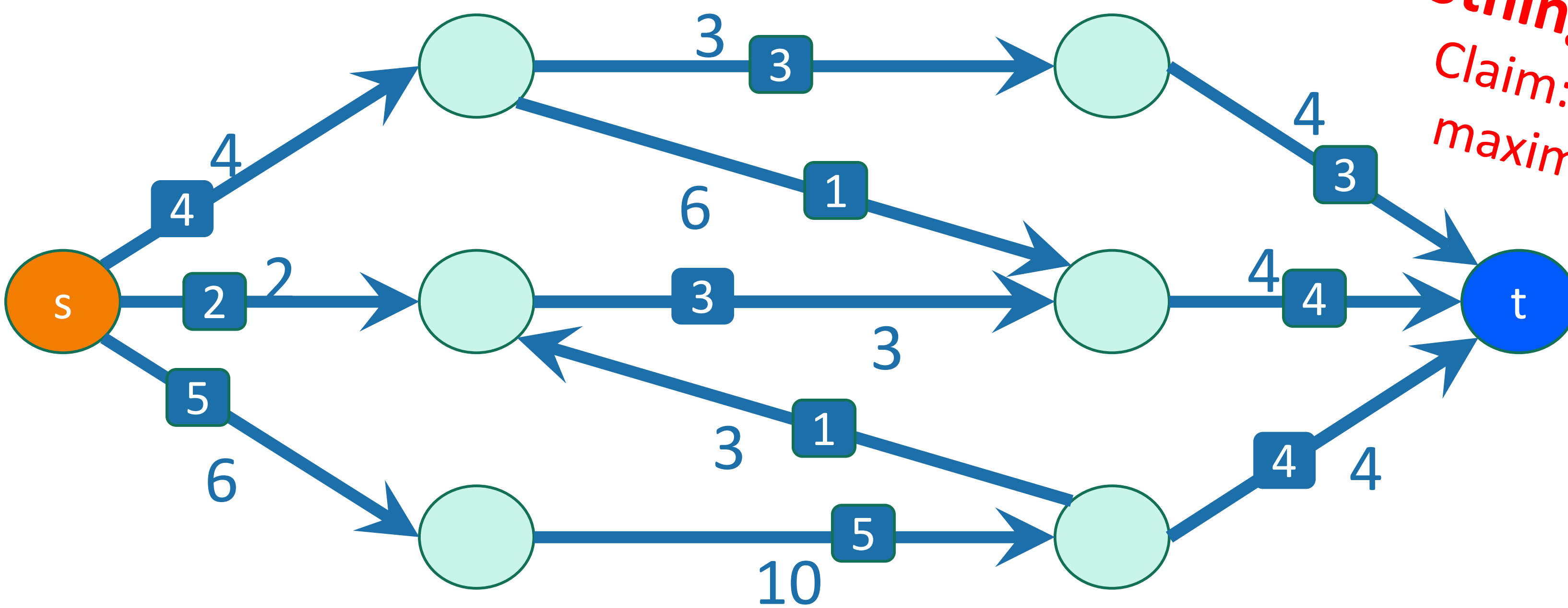

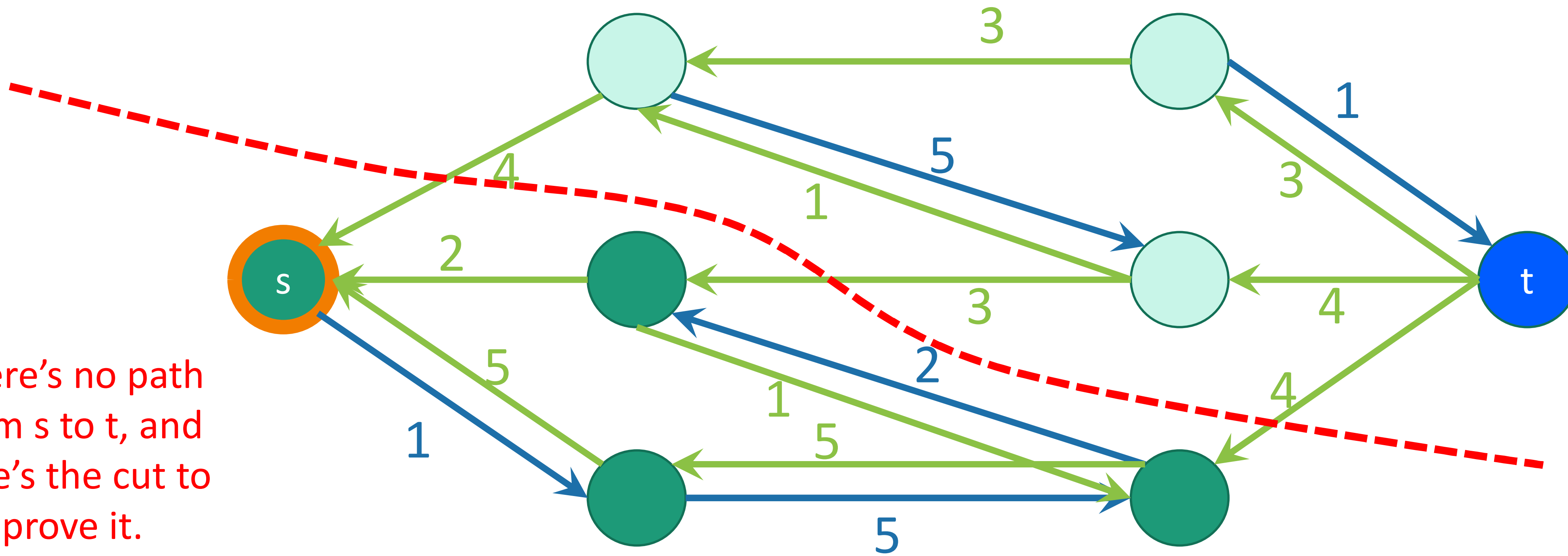

There's no path from s to t, and here's the cut to prove it.

Example of Ford-Fulkerson

Now we have nothing left to do!

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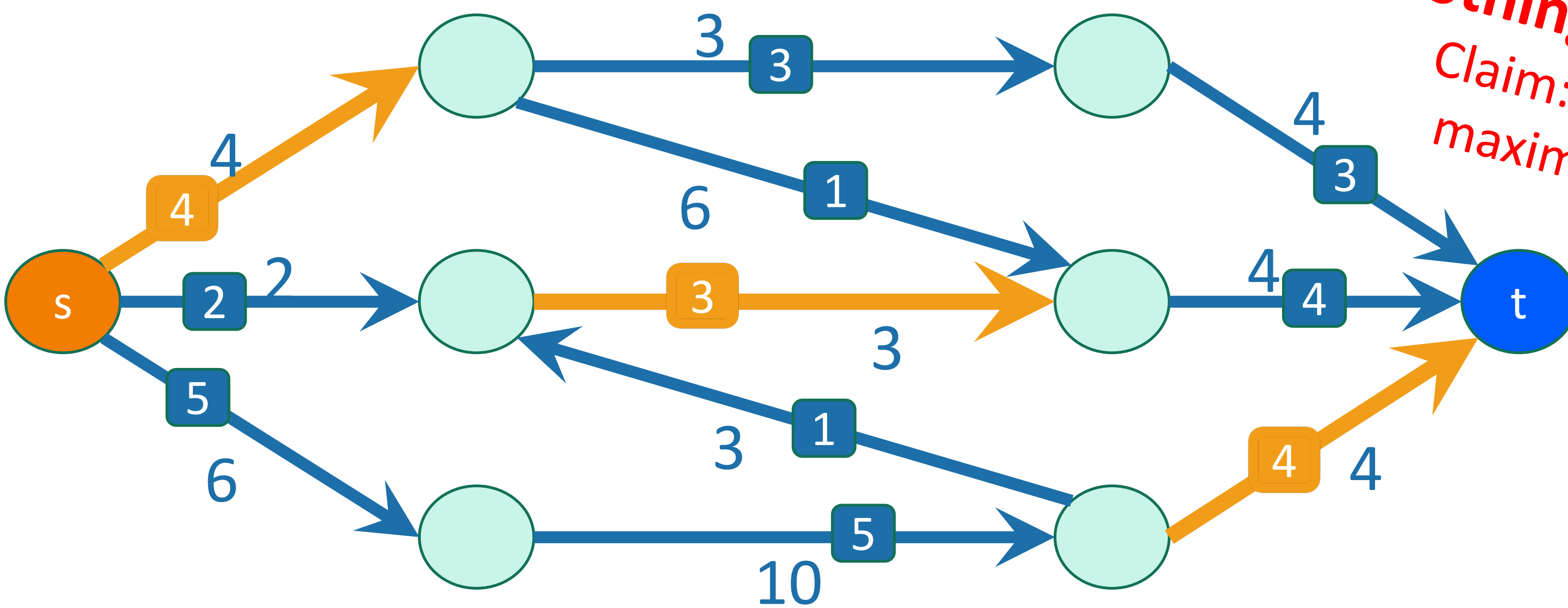
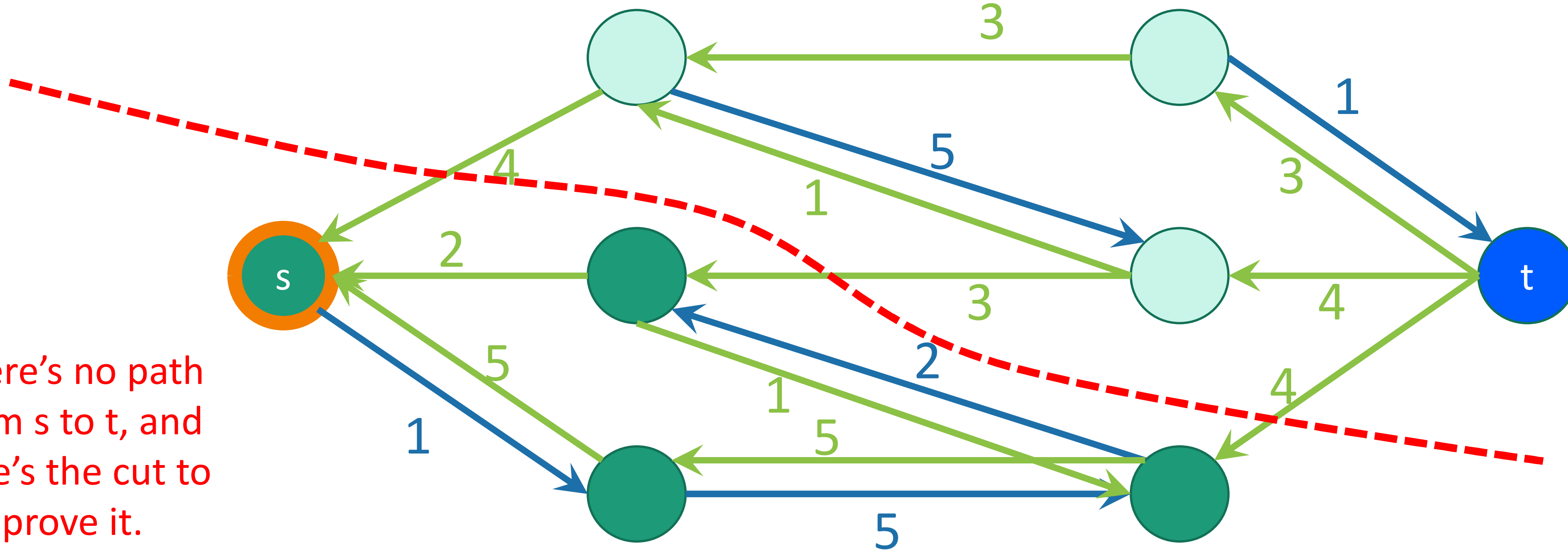




There's no path from s to t, and here's the cut to prove it.

Useful corollary

- Using Ford-Fulkerson alg. you can find :
  - An s-t cut of cost X
  - An s-t flow with value X

- Then the minimum s-t cut and the maximum s-t flow must *both* be equal to X.

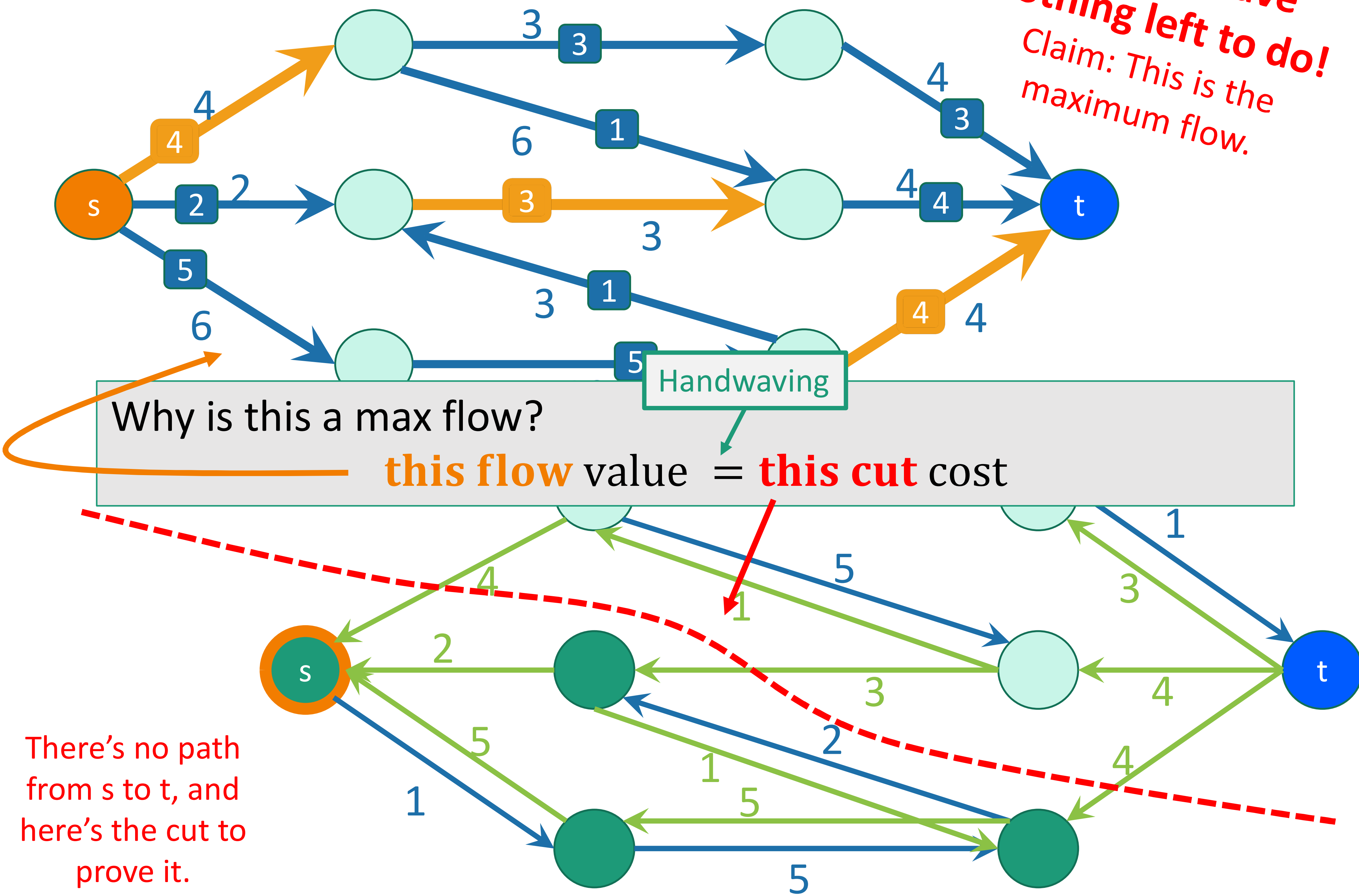
$$X \;\geq\; \text{Min cut cost} \;\geq\; \text{Max flow value} \;\geq\; X$$

$\Rightarrow$ All of these things must be equal!

Example of Ford-Fulkerson

Now we have nothing left to do!

Claim: This is the maximum flow.



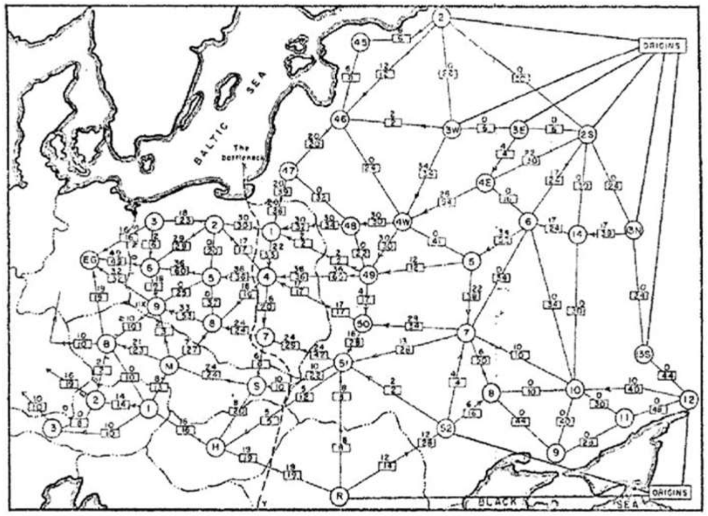
Why is this a max flow?

this flow value = **this cut** cost

There's no path from s to t, and here's the cut to prove it.

What have we learned?

- Max s-t flow is equal to min s-t cut!
  - The USSR and the USA were trying to solve the same problem…
- Useful corollary:
  - To certify that you have a max flow, it’s enough to find a cut with the same cost.
  - To certify that you have a min cut, it’s enough to find a flow with the same value.
- The Ford-Fulkerson algorithm can find the min-cut/max-flow.
  - Repeatedly improve your flow along an augmenting path.

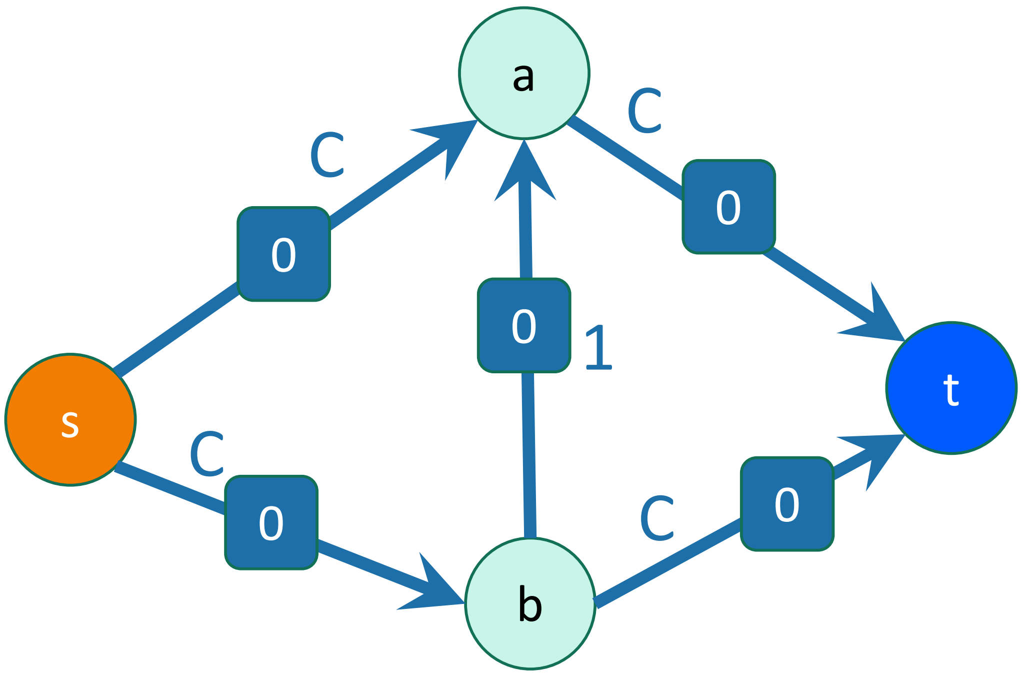


Our usual questions
about Ford-Fulkerson

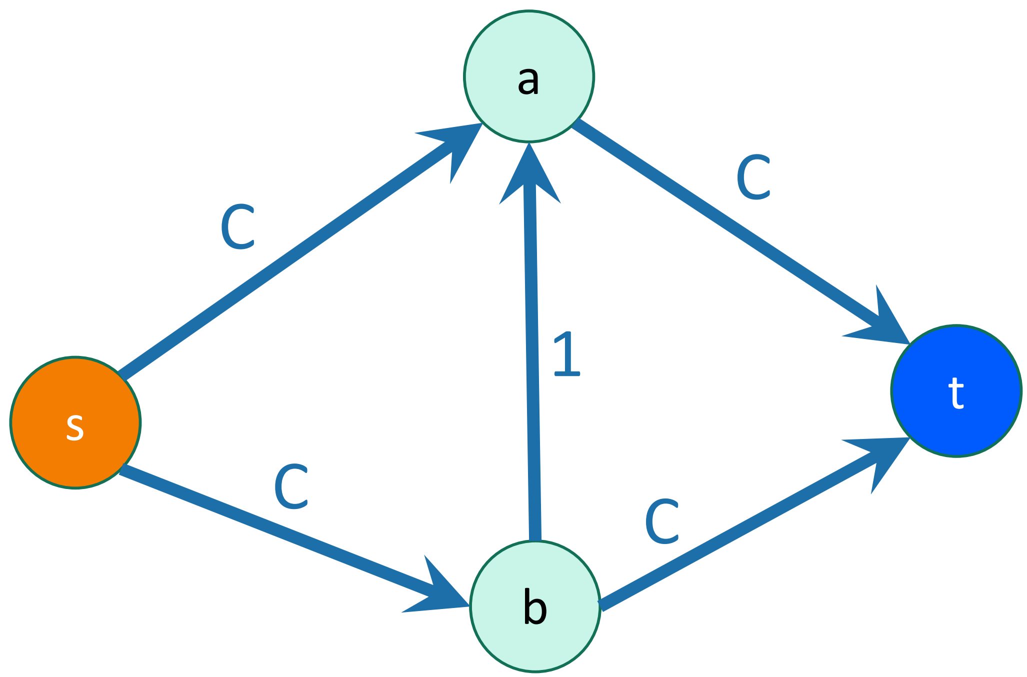
- Does it work?
  - Yep, just showed that
- Is it fast?
  - Depends on how we pick the augmenting paths!

Why should we be concerned?

Suppose we just picked paths arbitrarily.



Choose a really
big number C.



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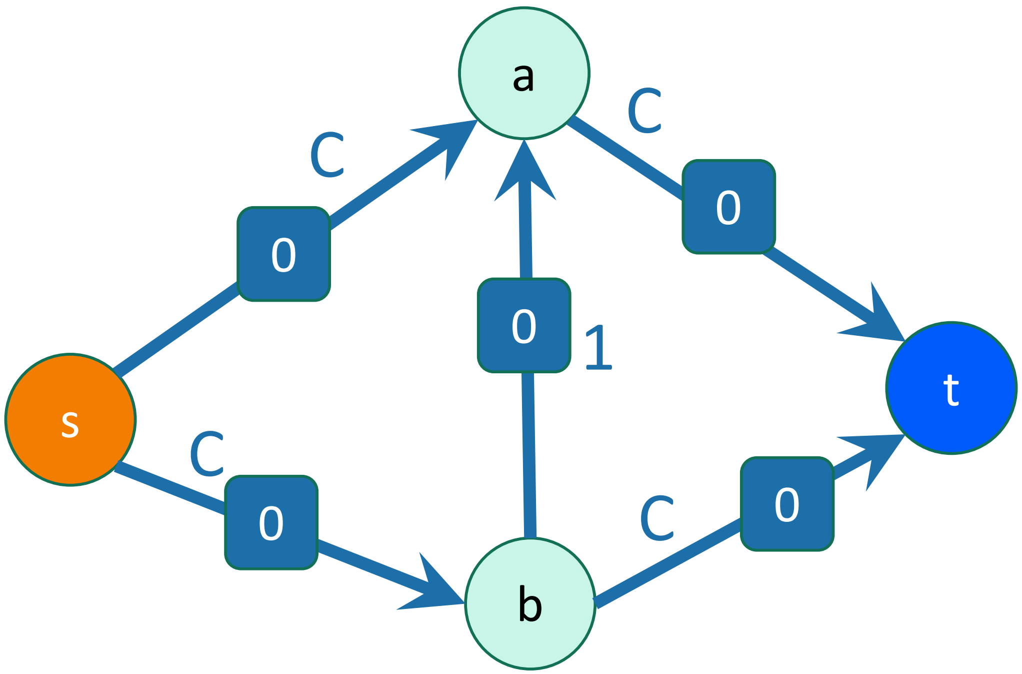

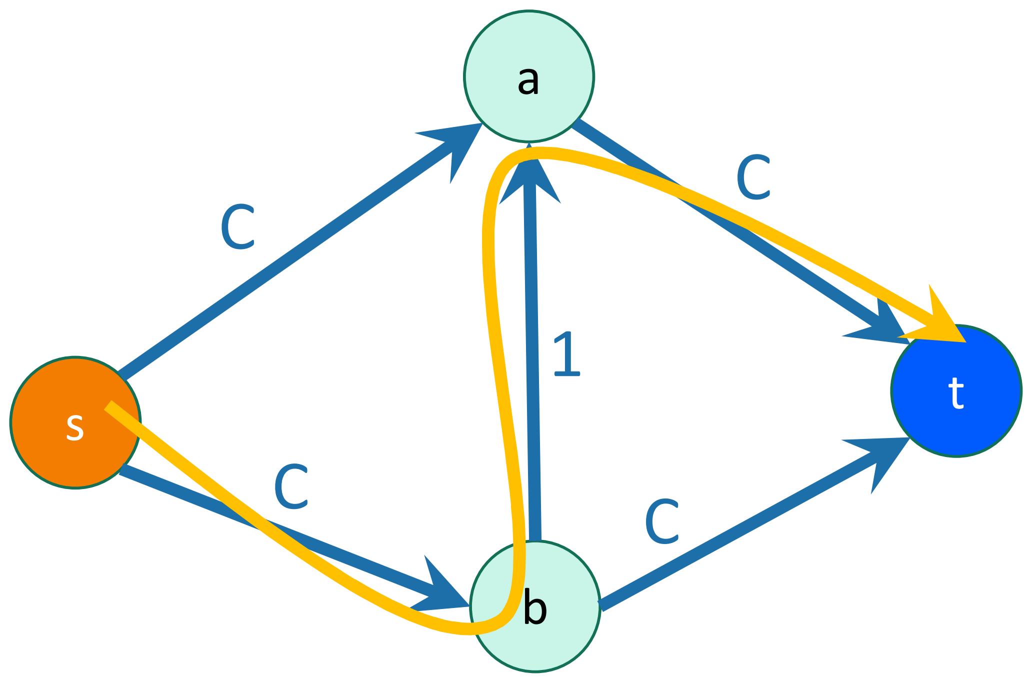

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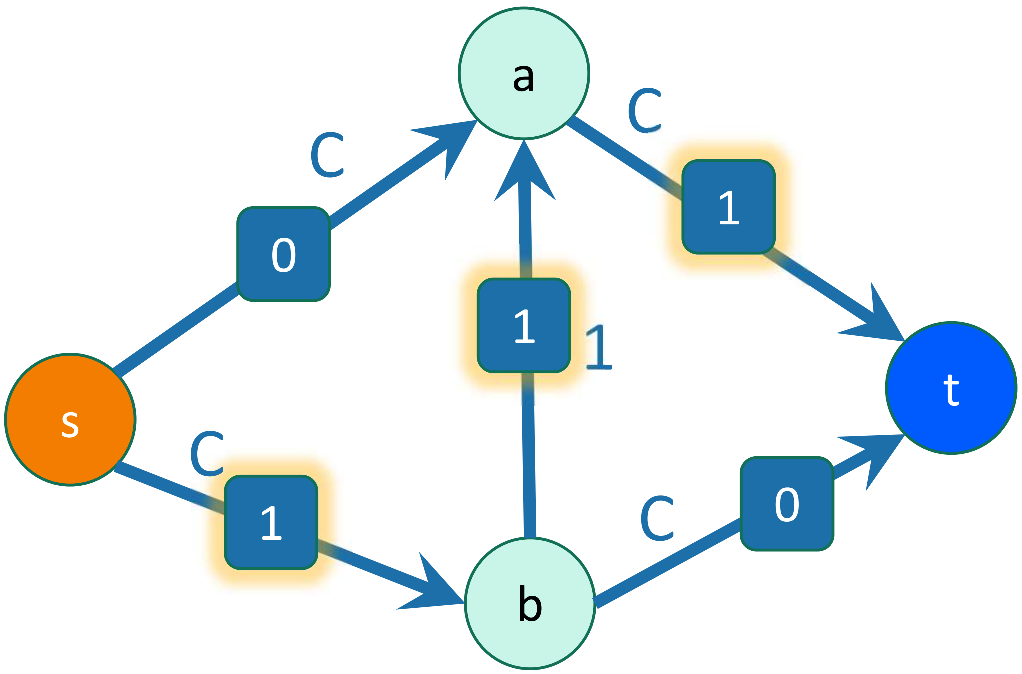

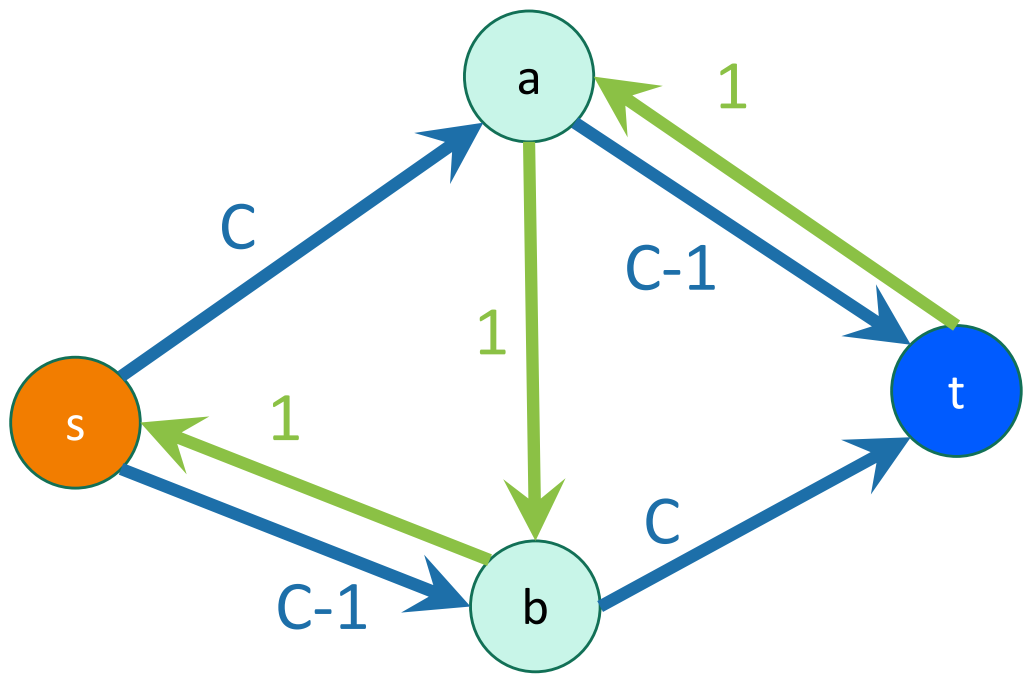

The edge (b,a) disappeared from the residual graph!

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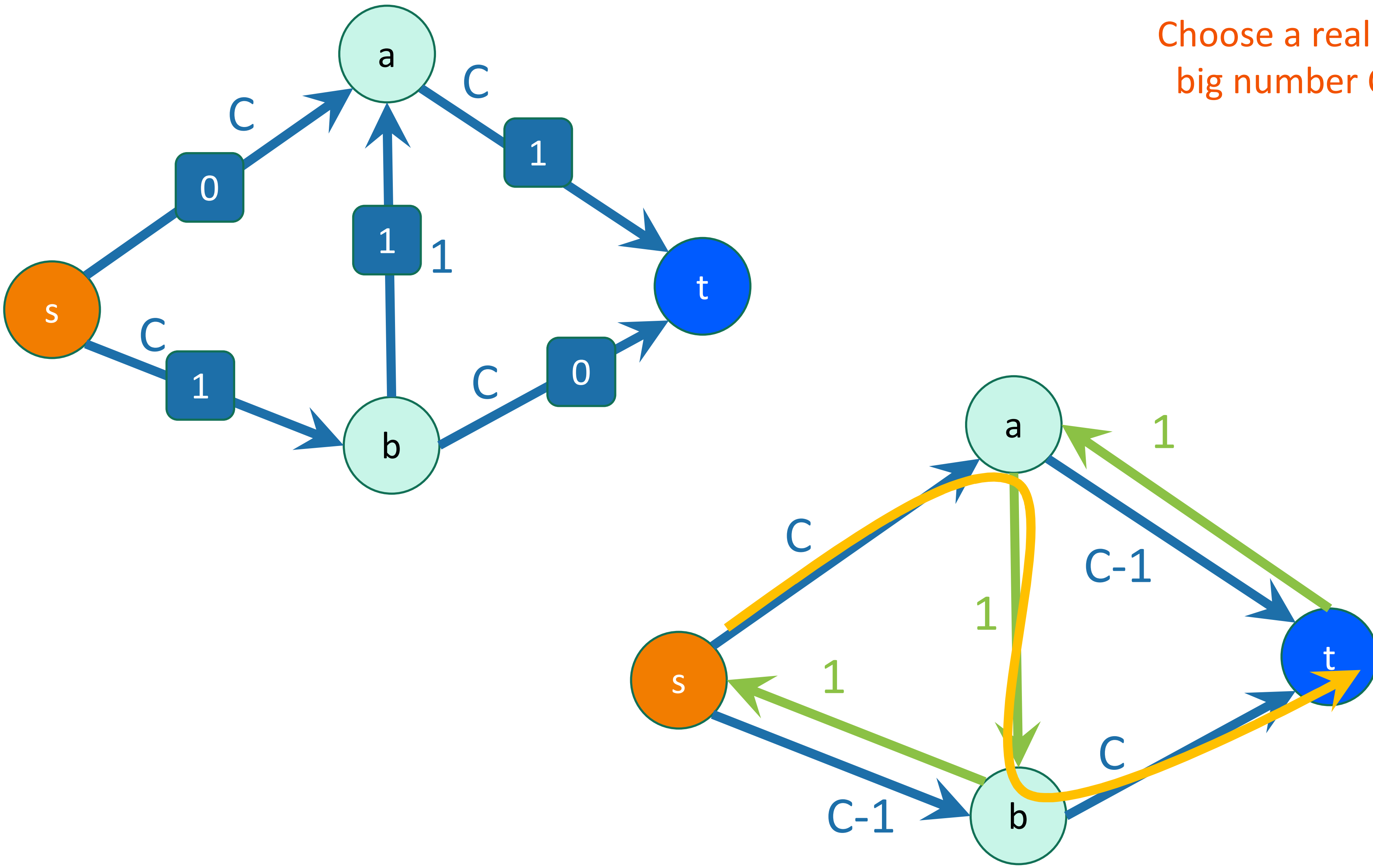

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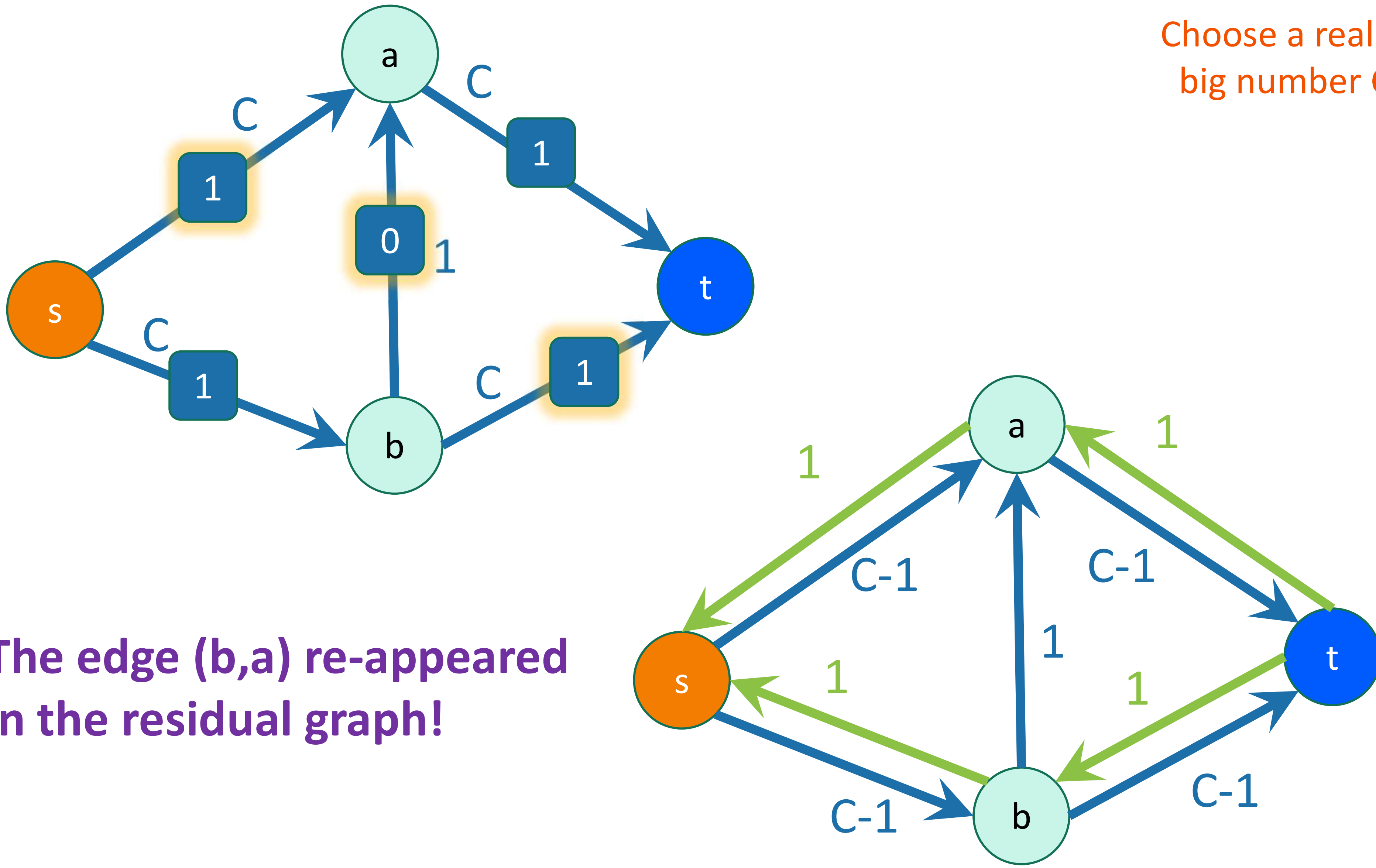
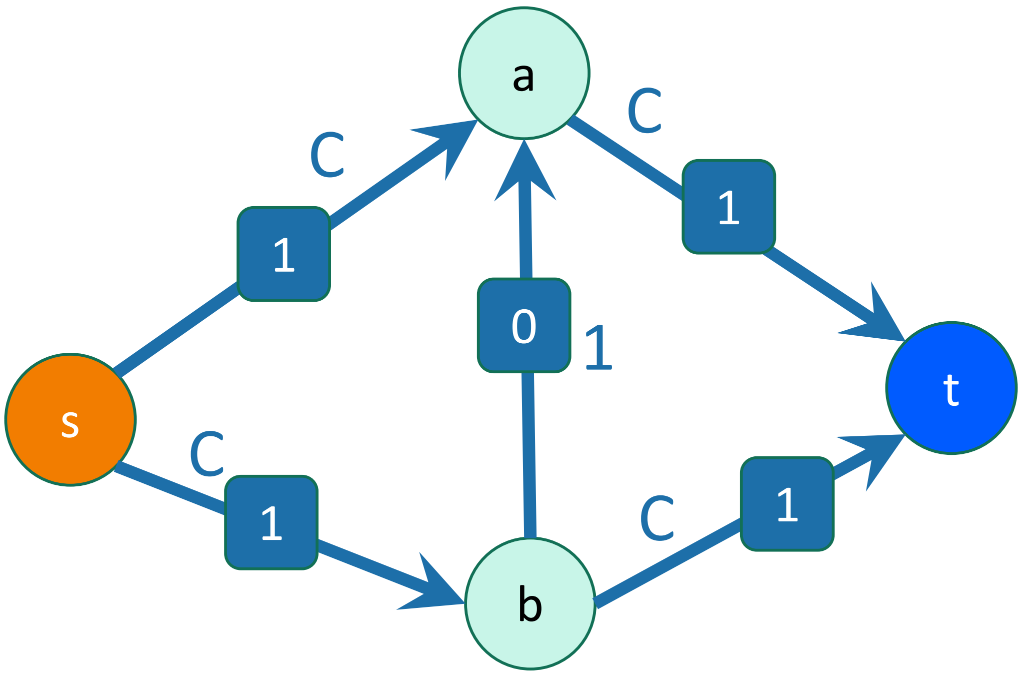


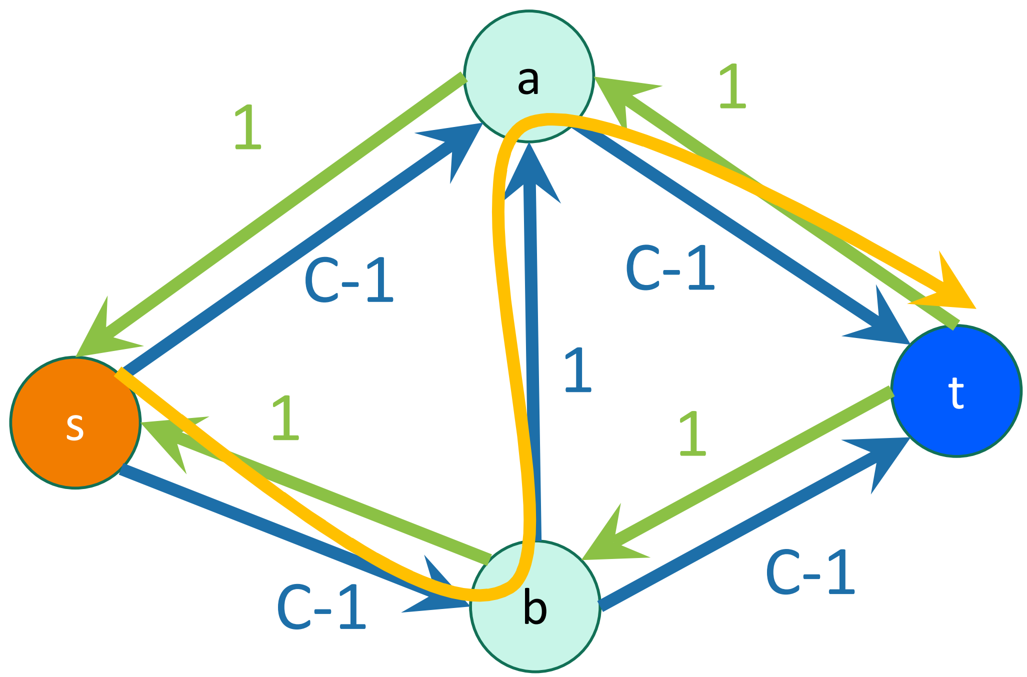
The edge (b,a) re-appeared in the residual graph!

Why should we be concerned?

Suppose we just picked paths arbitrarily.

Choose a really big number C.





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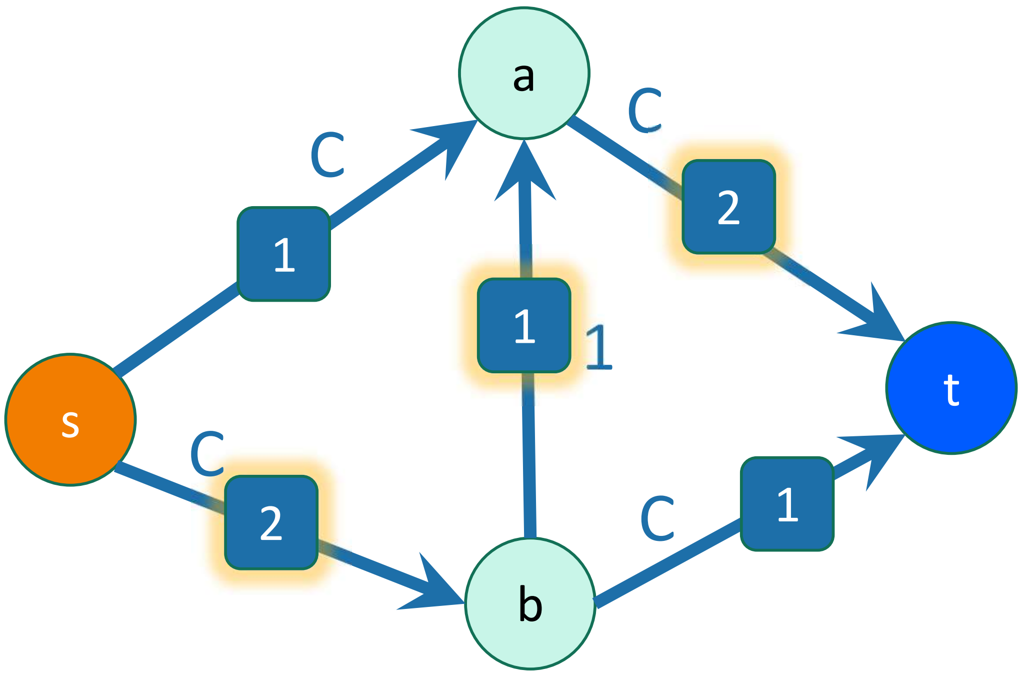

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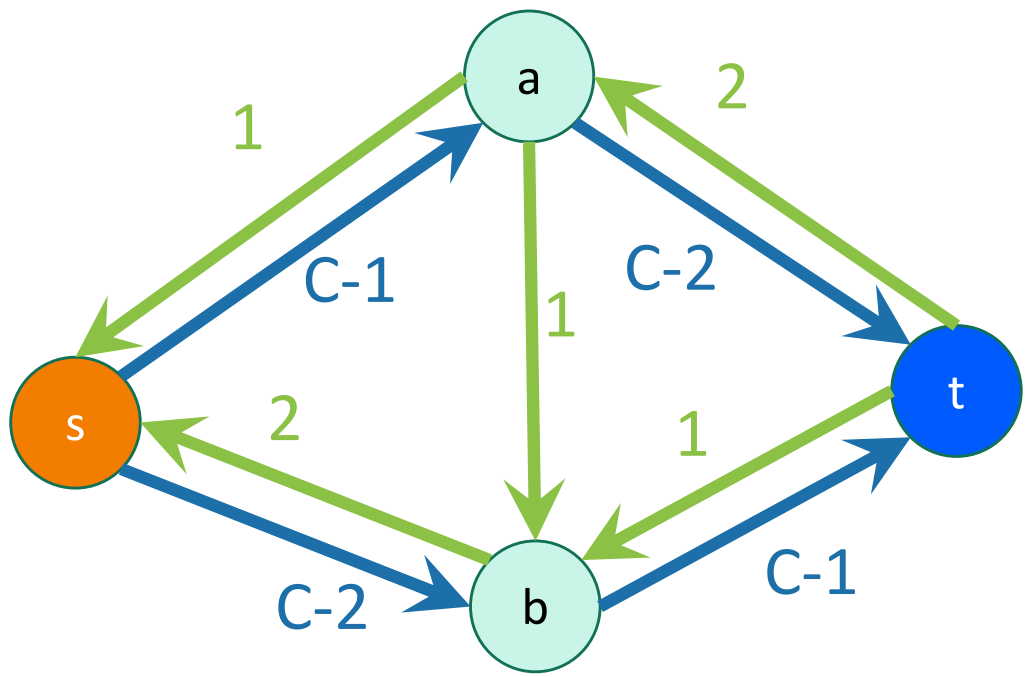

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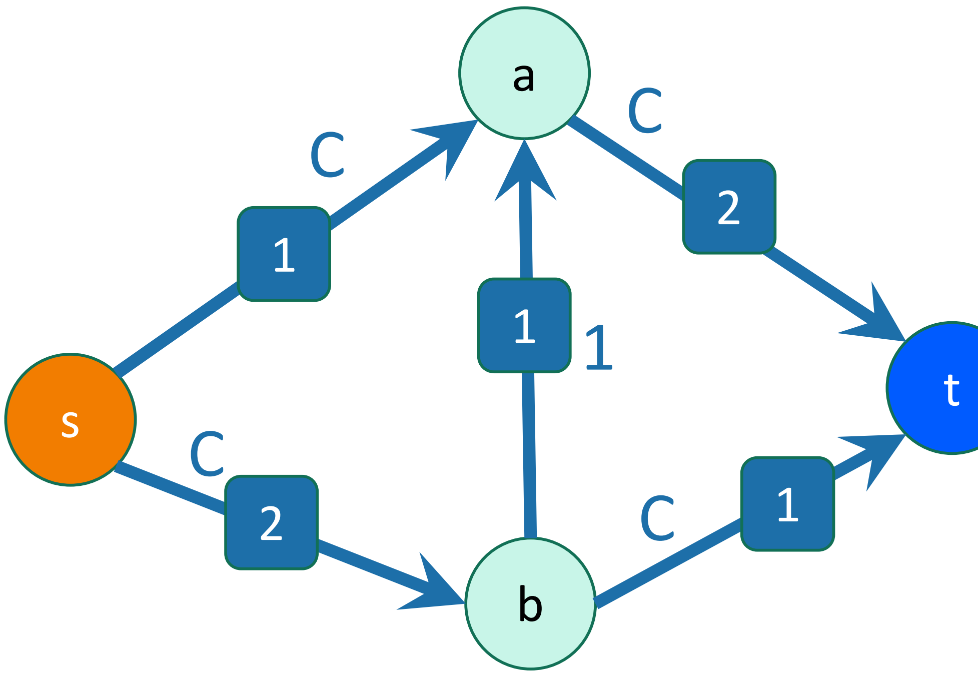
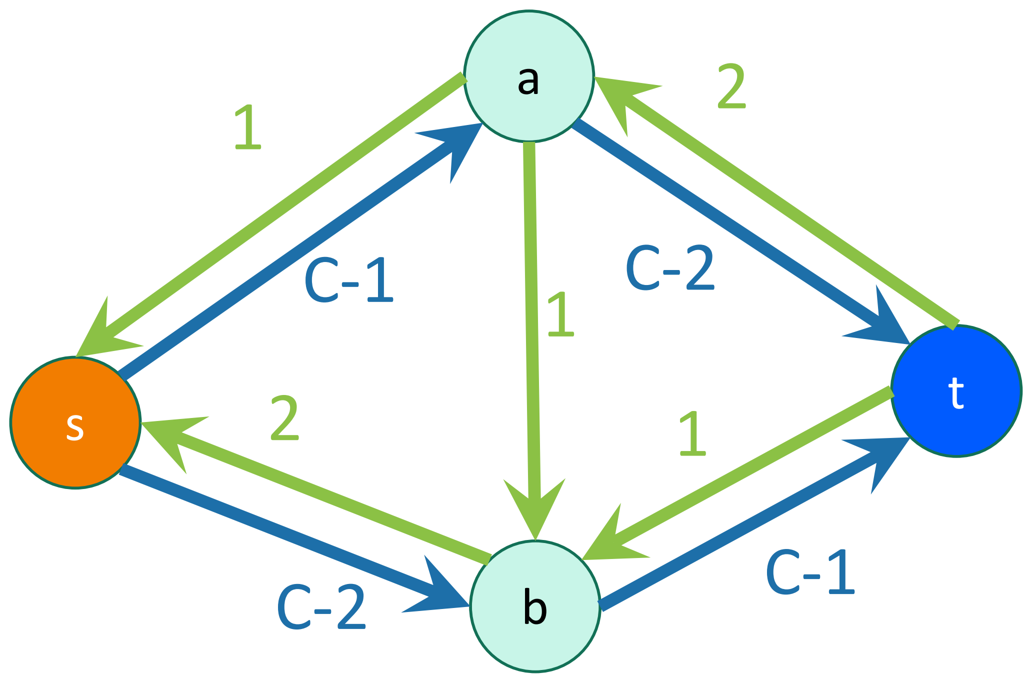


This will go on for C steps, adding flow along (b,a) and then subtracting it again.



The edge (b,a) disappeared from the residual graph!

Edmonds-Karp Algorithm

- If we run the Ford-Fulkerson algorithm, using BFS to pick augmenting paths, it's called the **Edmonds-Karp Algorithm.**

- It turns out that this will run in time $O(nm^2)$ on a graph with n vertices and m edges.

One more useful observation

- If all the capacities are integers, then the flows in any max flow are also all integers.
  - When we update flows in Ford-Fulkerson, we're only ever adding or subtracting integers.
  - Since we started with 0 (an integer), everything stays an integer.

But wait, there's more!

- Max flows and min cuts aren't just for railway routing.
- The Ford-Fulkerson algorithm is the basis for many other graph algorithms.
- For the rest of today, we'll see a few:
  - Maximum bipartite matching
  - Integer assignment problems

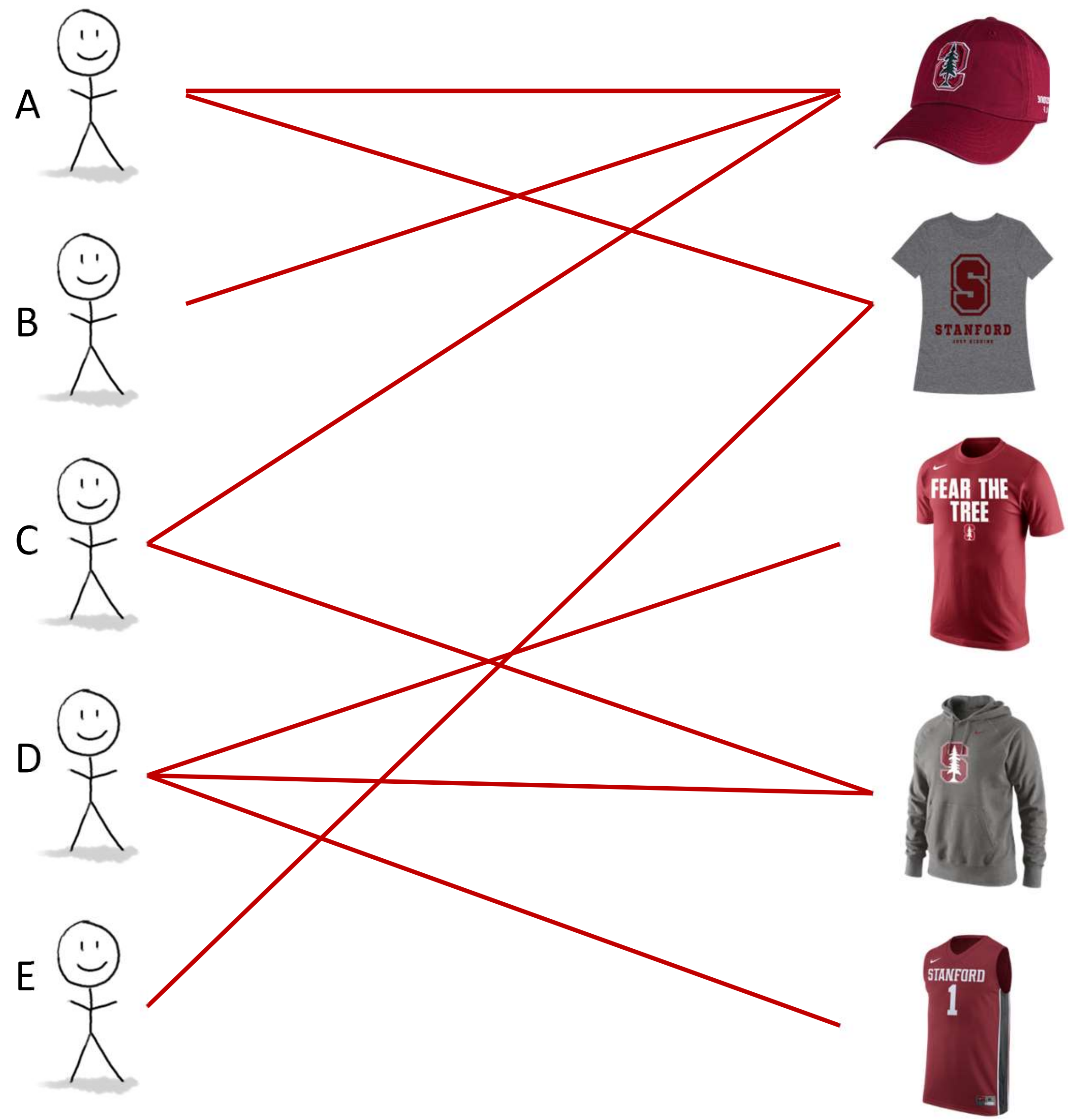
For more on applications, check out this book chapter from "Algorithms" by Jeff Erickson:
https://jeffe.cs.illinois.edu/teaching/algorithms/book/11-maxflowapps.pdf

Ví dụ:

Maximum matching in bipartite graphs

- Different students only want certain items of Stanford swag (depending on fit, style, etc).

- **How can we make as many students as possible happy?**



Stanford Students

Stanford Swag

Maximum matching in bipartite graphs

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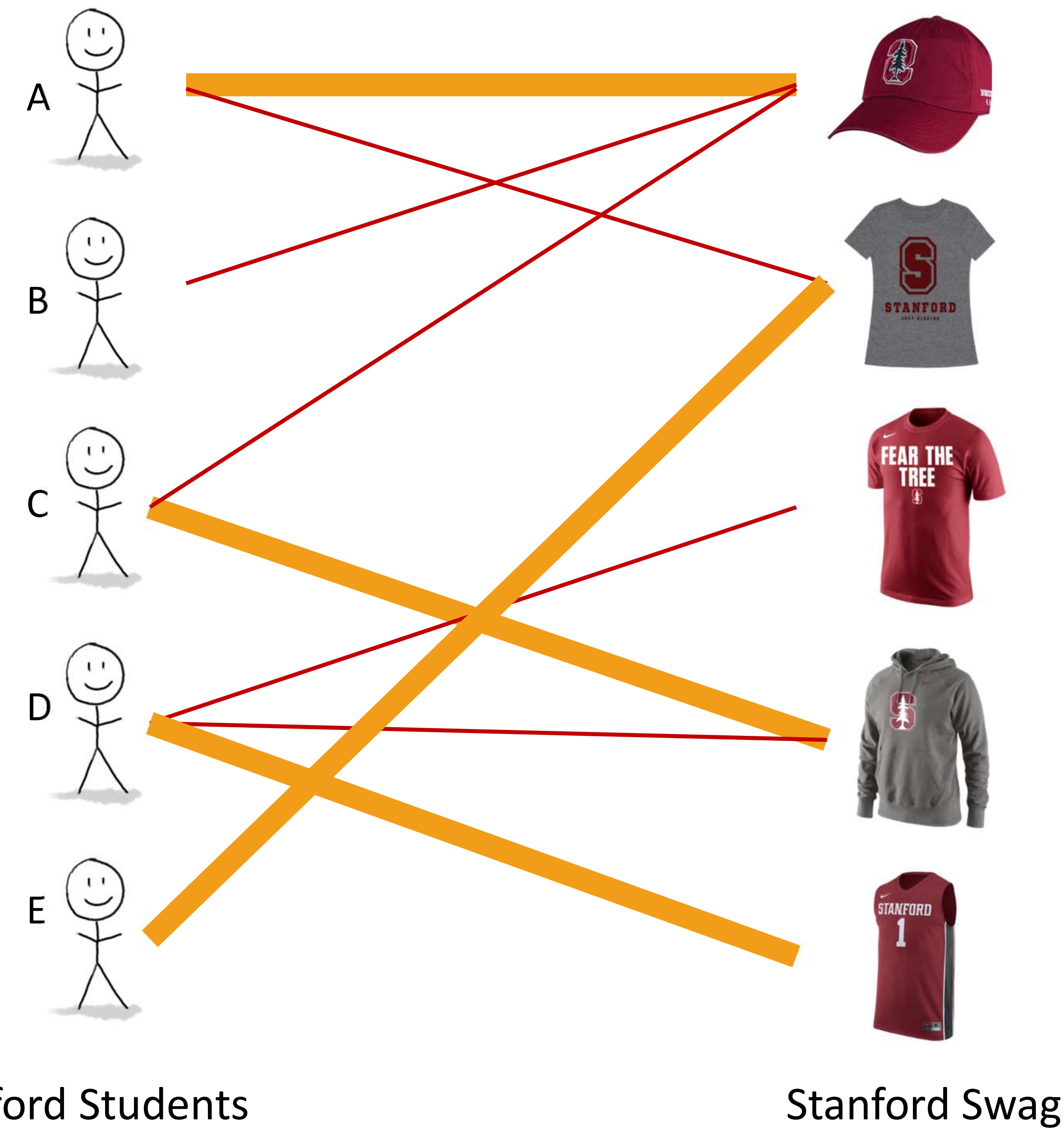
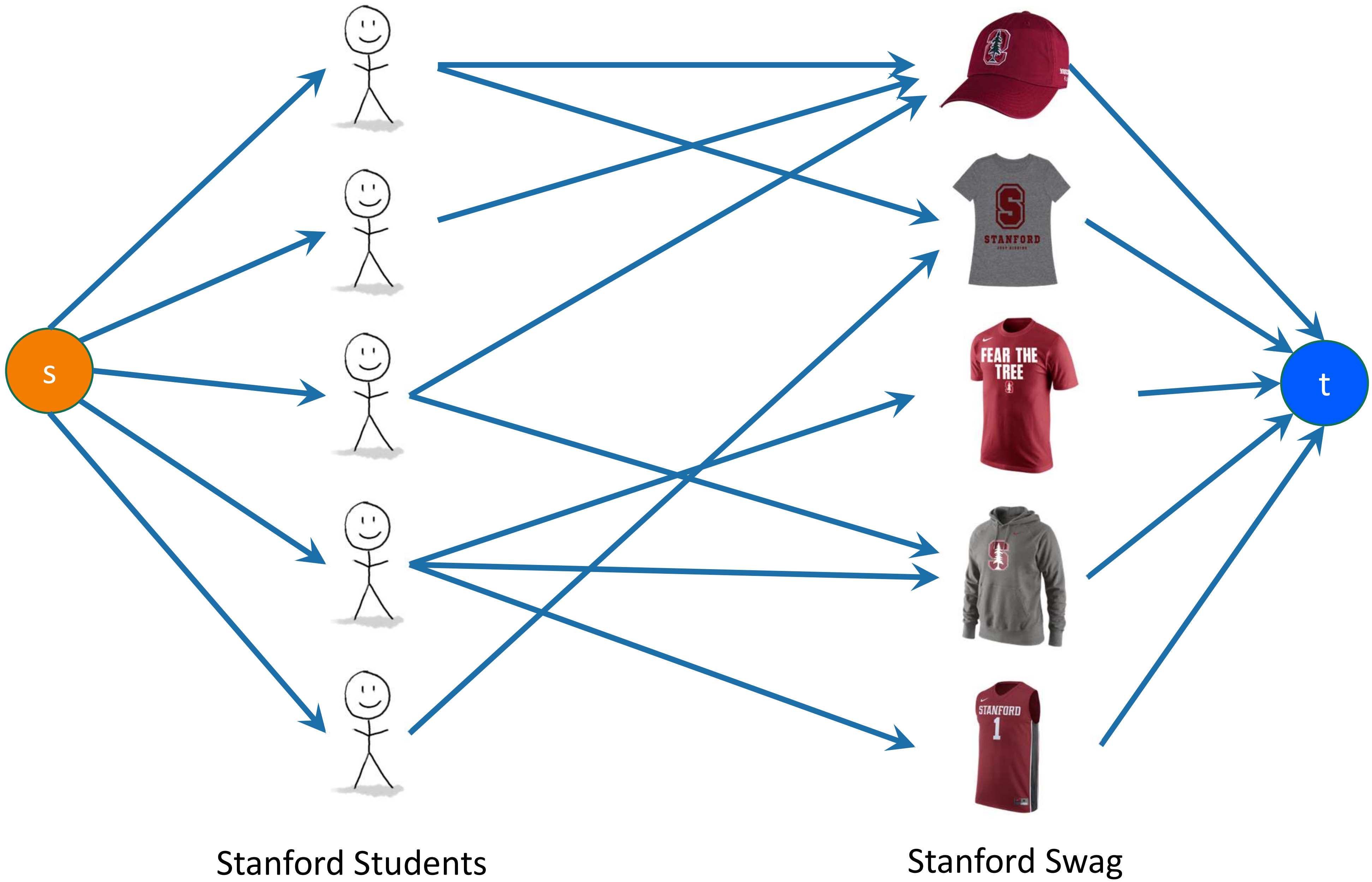


Solution via max flow

All edges have capacity 1.



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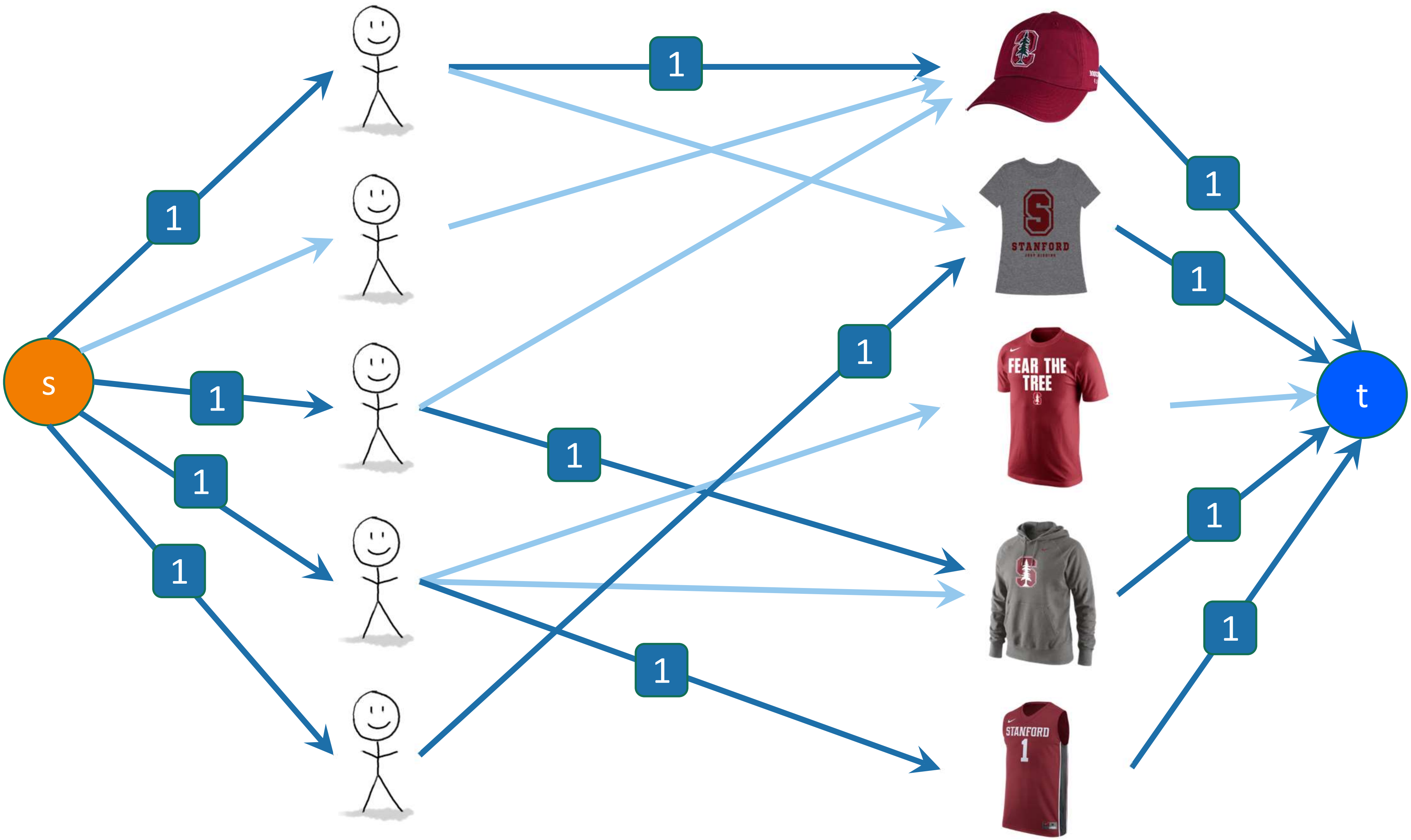


Stanford Students

Stanford Swag

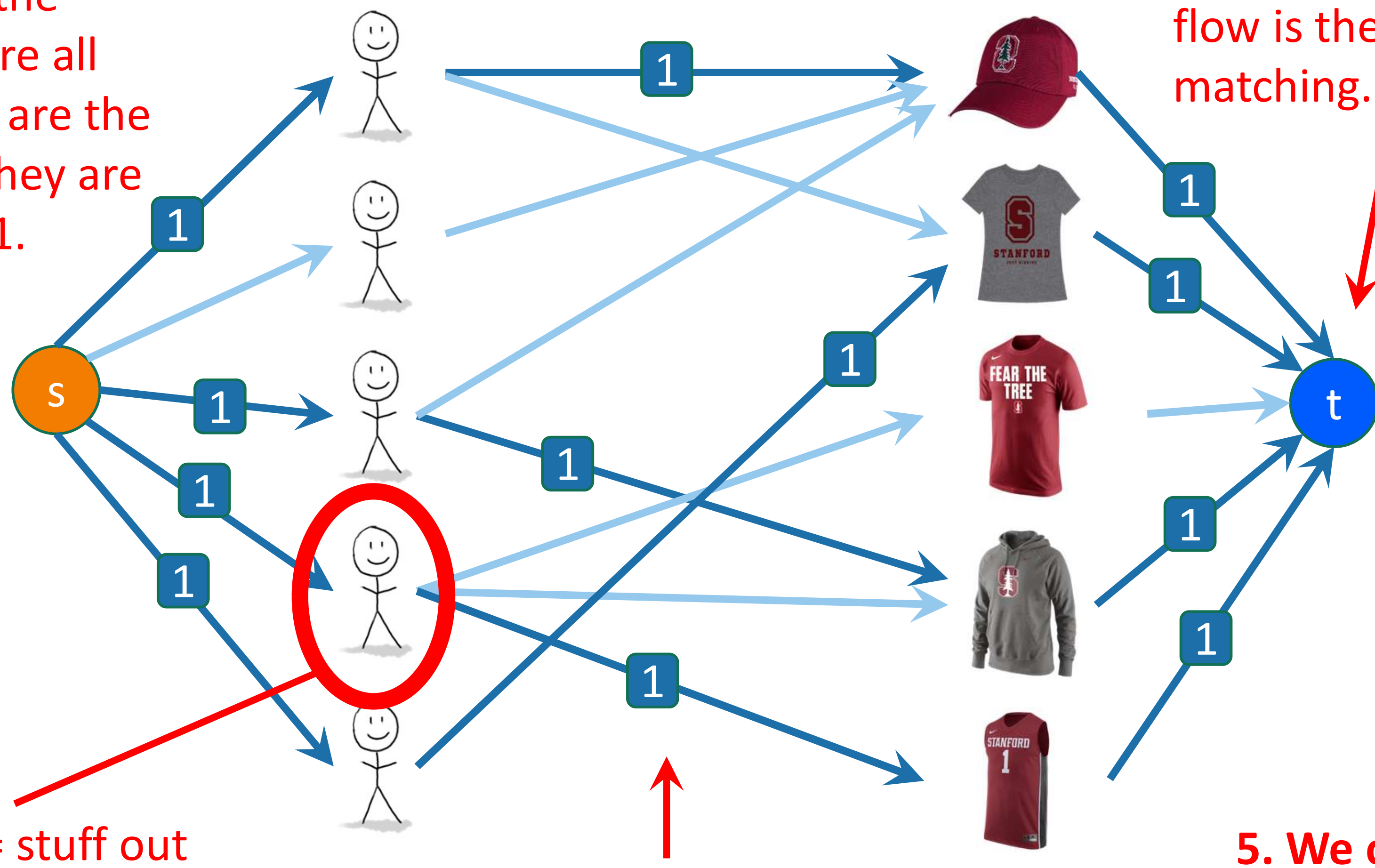
Solution via max flow

why does this work?

All edges have capacity 1.

1. Because the capacities are all integers, so are the flows – so they are either 0 or 1.

4. The value of the flow is the size of the matching.



Value of this flow is 4.

2. Stuff in = stuff out means that the number of items assigned to each student 0 or 1. (And vice versa).

3. Thus, the edges with flow on them form a matching. (And, any matching gives a flow).

5. We conclude that the max flow corresponds to a max matching.

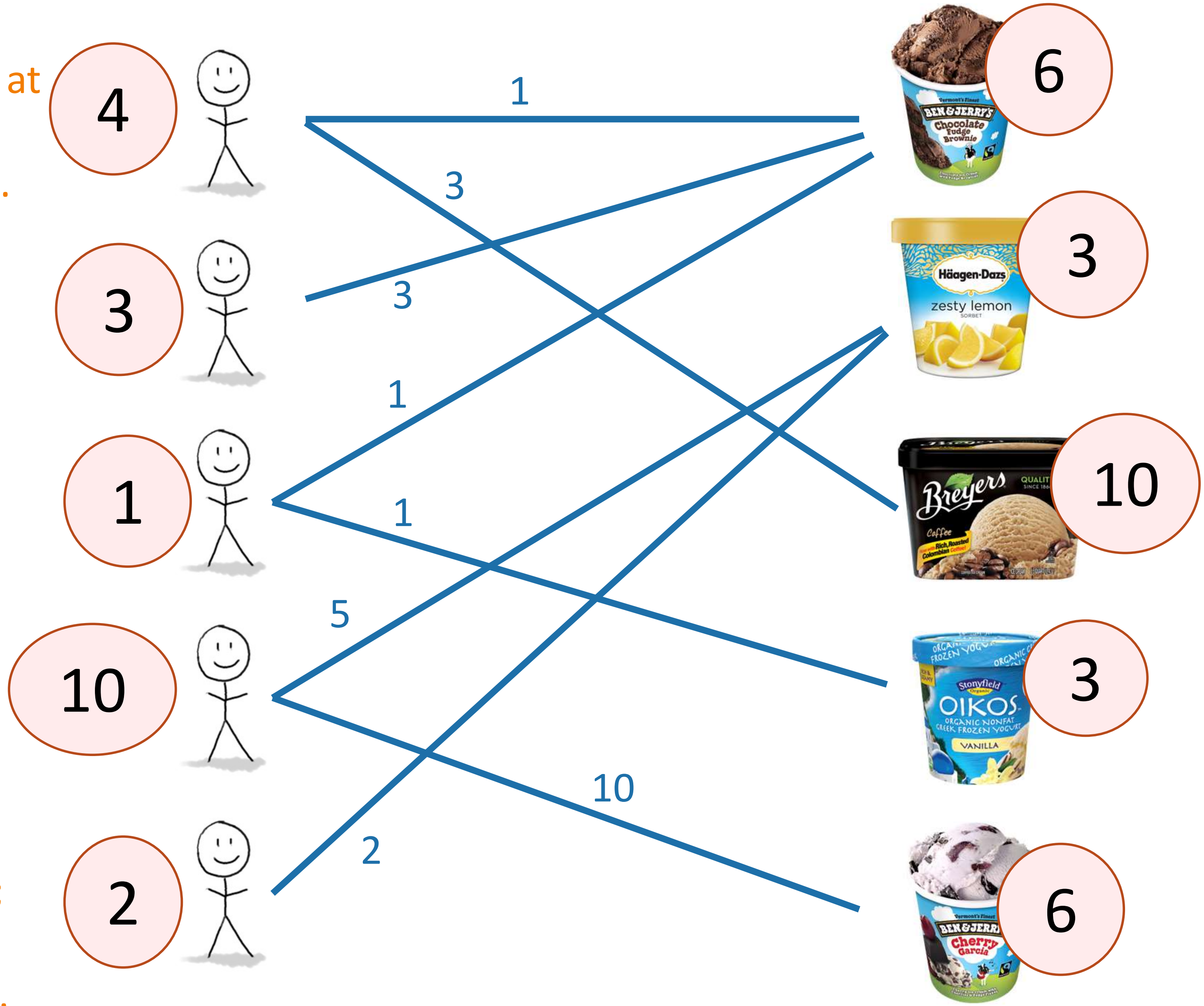
A slightly more complicated example: assignment problems

- One set X
  - Example: Stanford students
- Another set Y
  - Example: tubs of ice cream
- Each x in X can participate in c(x) matches.
  - Student x can only eat 4 scoops of ice cream.
- Each y in Y can only participate in c(y) matches.
  - Tub of ice cream y only has 10 scoops in it.
- Each pair (x,y) can only be matched c(x,y) times.
  - Student x only wants 3 scoops of flavor y
  - Student x’ doesn’t want any scoops of flavor y’
- **Goal: assign as many matches as possible.**

Example

How can we serve as much ice cream as possible?

This person wants 4 scoops of ice cream, at most 1 of chocolate and at most 3 coffee.

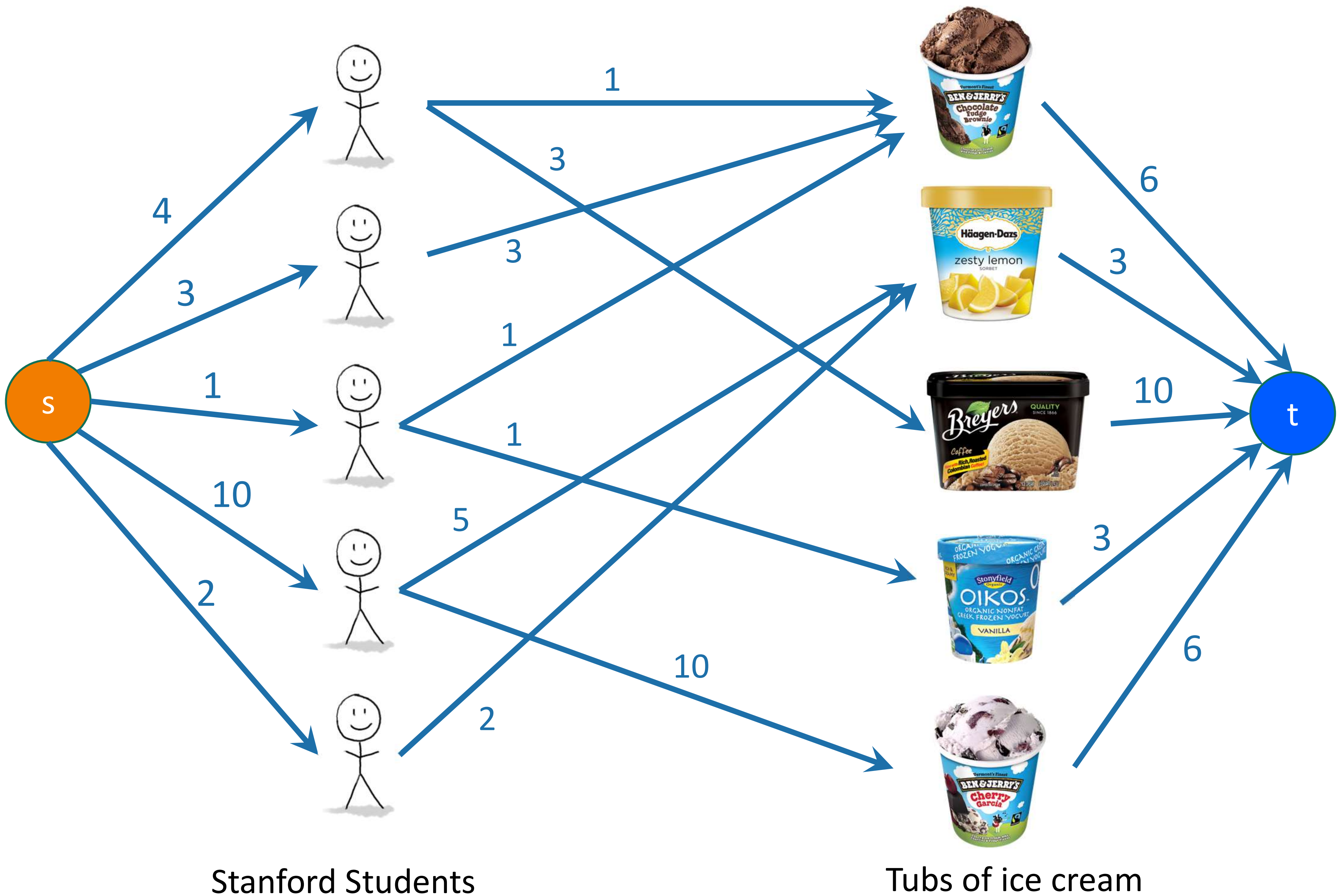


This person is vegan and not that hungry; they only want two scoops of the sorbet.

Stanford Students

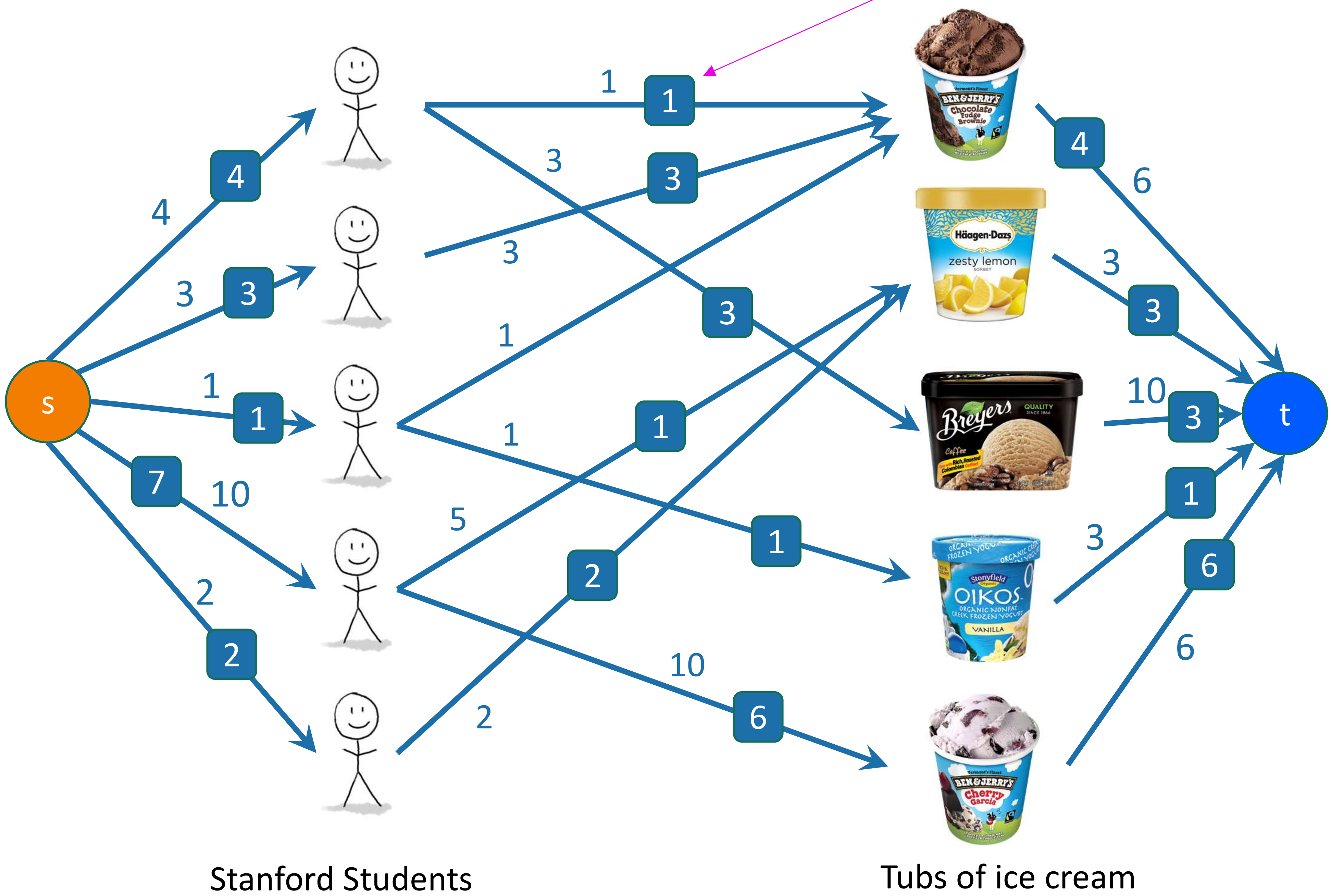
Tubs of ice cream

Solution via max flow

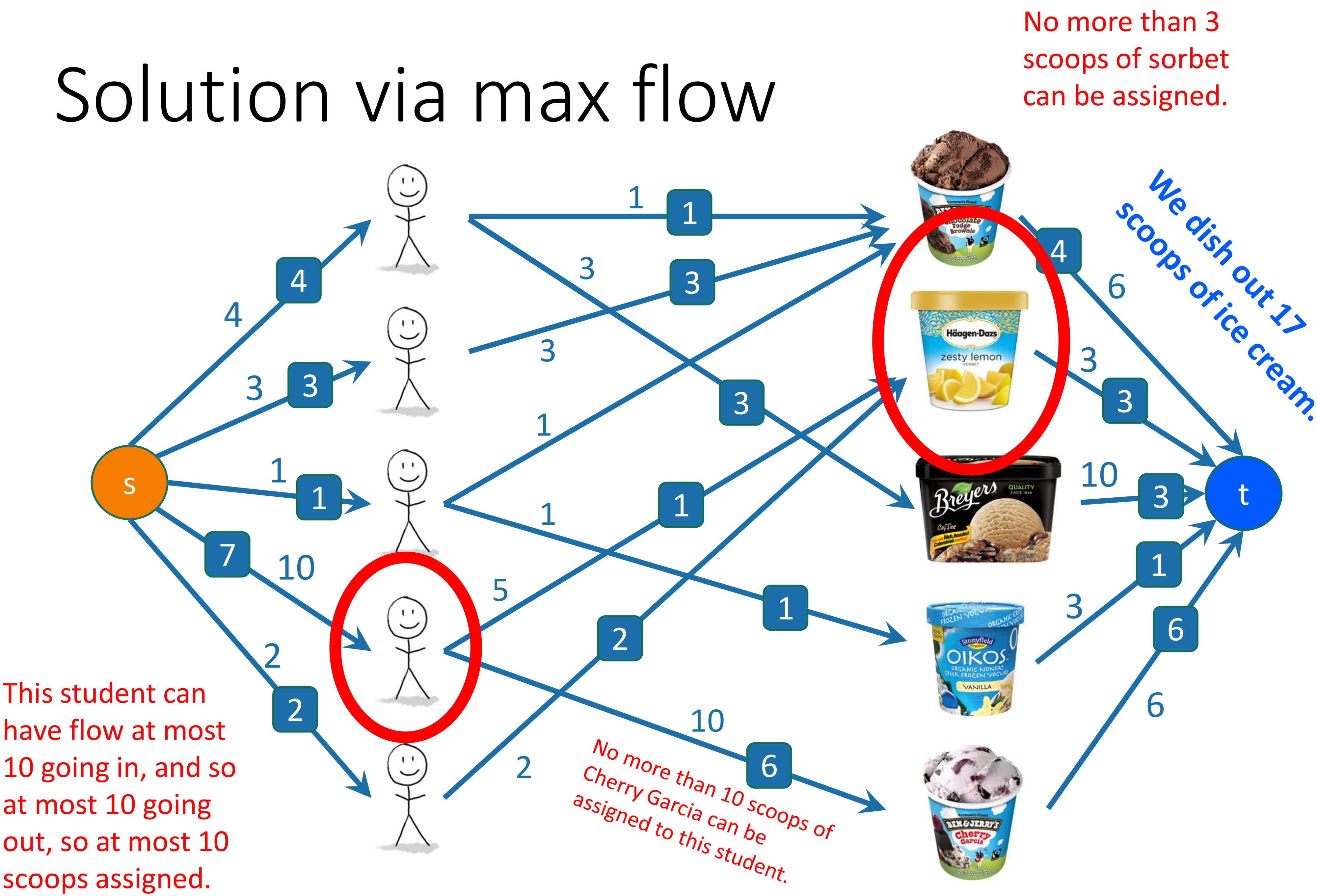


Solution via max flow

Give this person $\boxed{1}$ scoop of this ice cream.



Solution via max flow



As before, flows correspond to assignments, and max flows correspond to max assignments.

Tổng kết

- Khái niệm về lát cắt s-t (nguồn - đích) và luồng s-t.
- Định lý **Min-Cut Max-Flow:** tối thiểu hóa lát cắt đồng nghĩa với tối đa hóa luồng.
- Thuật toán Ford-Fulkerson:
  - Tìm một đường đi "tăng cường"
  - Bổ sung lưu lượng của luồng bằng đường đi này
  - Lặp lại cho đến khi không còn đường đi nào khác.
- Là cơ sở cho nhiều giải thuật khác
  - Ví dụ, bài toán phân công (assignment problems).