Đồ thị: Đường đi ngắn nhất

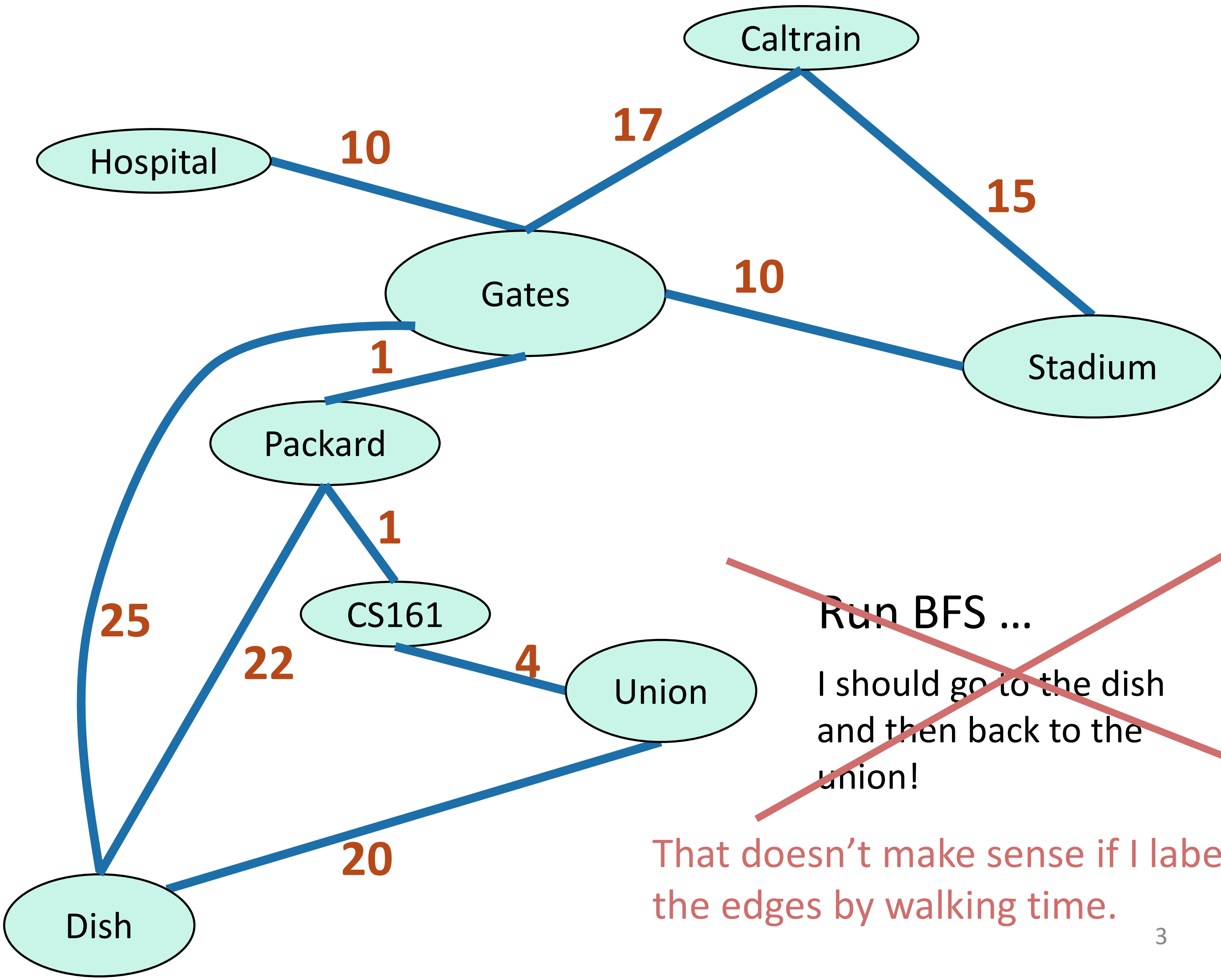
Dijkstra’s algorithm

Nội dung

- Duyệt đồ thị có trọng số
- Thuật toán tìm đường đi ngắn nhất - Dijkstra's algorithm

Sử dụng một phần tài liệu bài giảng CS161 Stanford University

Shortest path from Gates to the Union?



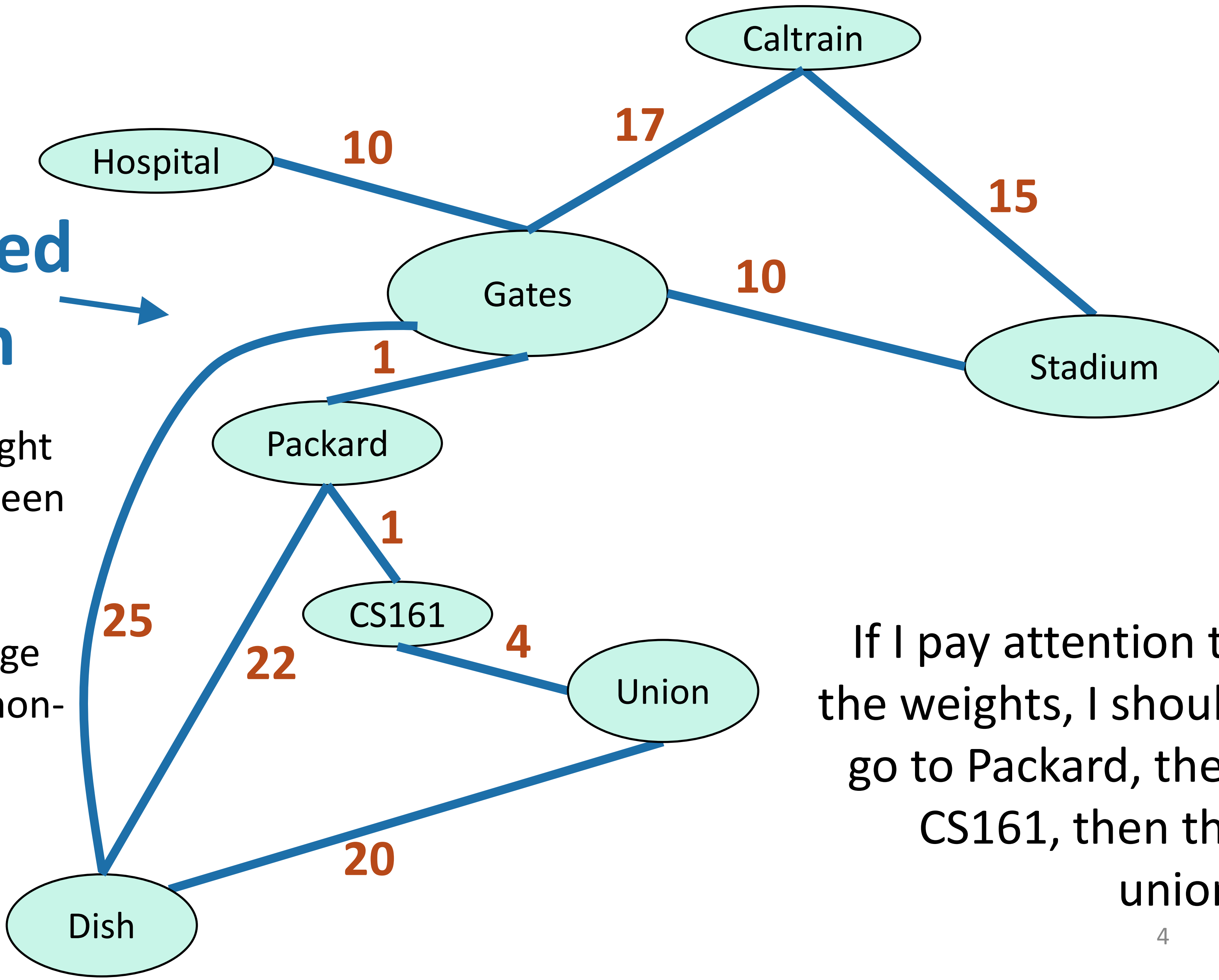
~~Run BFS …~~

~~I should go to the dish and then back to the union!~~

That doesn't make sense if I label the edges by walking time.

Shortest path from Gates to the Union?

weighted graph



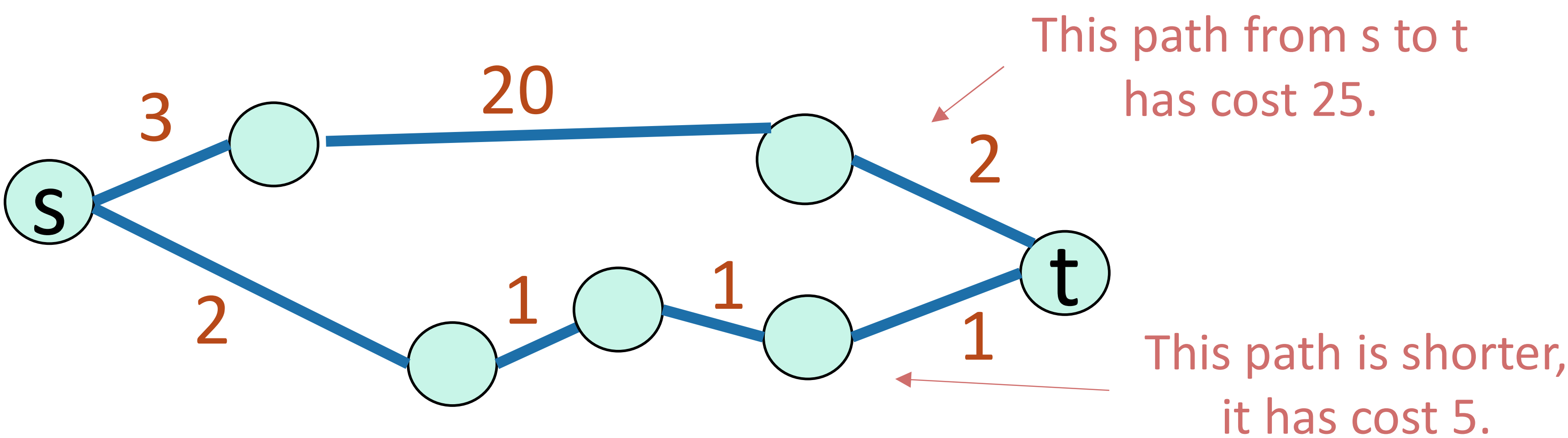
w(u,v) = weight of edge between u and v.

For now, edge weights are non-negative.

If I pay attention to the weights, I should go to Packard, then CS161, then the union.

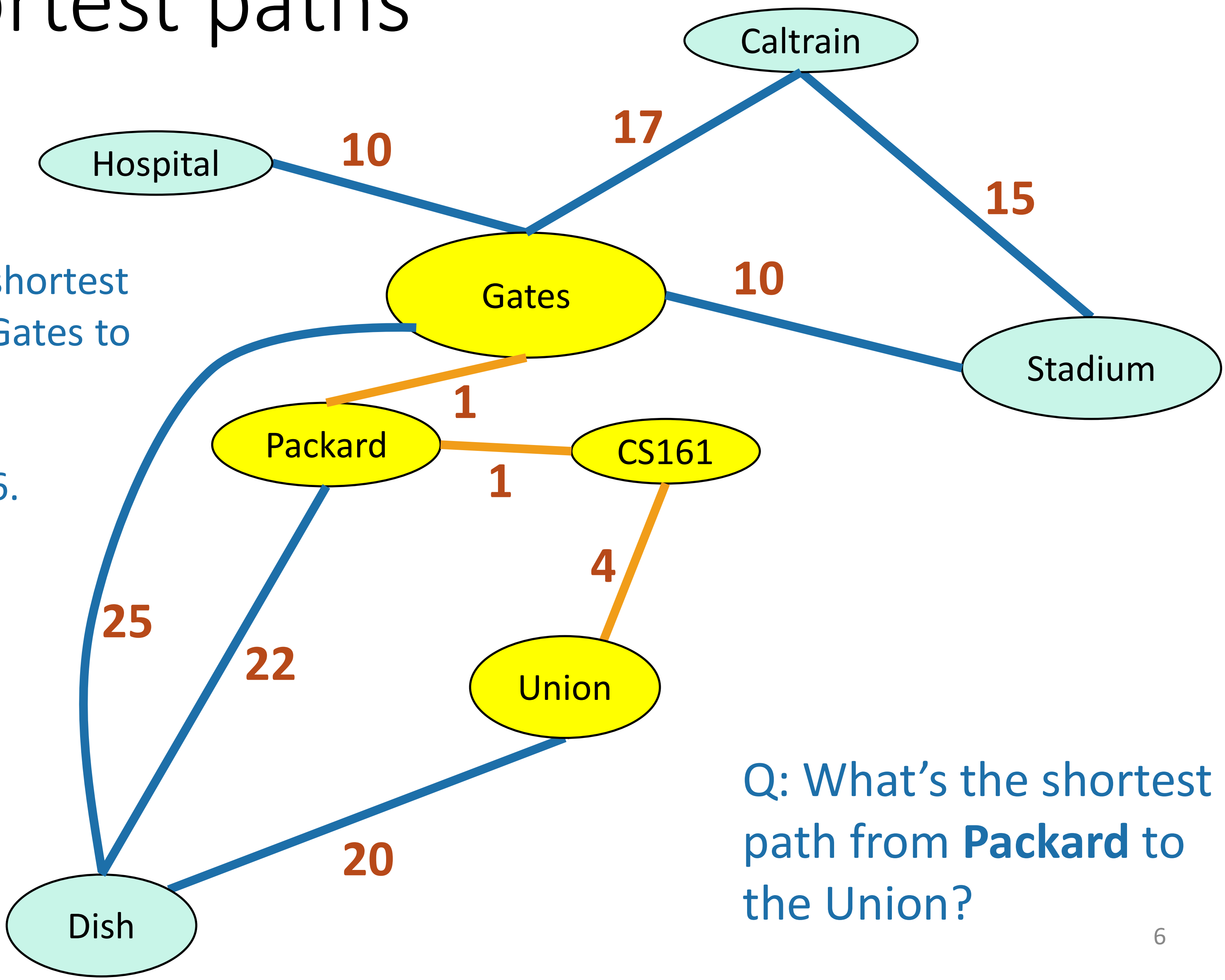
Shortest path problem

- What is the shortest path between u and v in a weighted graph?
  - the **cost** of a path is the sum of the weights along that path
  - The **shortest path** is the one with the minimum cost.



- The **distance** d(u,v) between two vertices u and v is the cost of the the shortest path between u and v.
- For this lecture **all graphs are directed**, but to save on notation I'm just going to draw undirected edges.

Shortest paths



This is the shortest path from Gates to the Union.

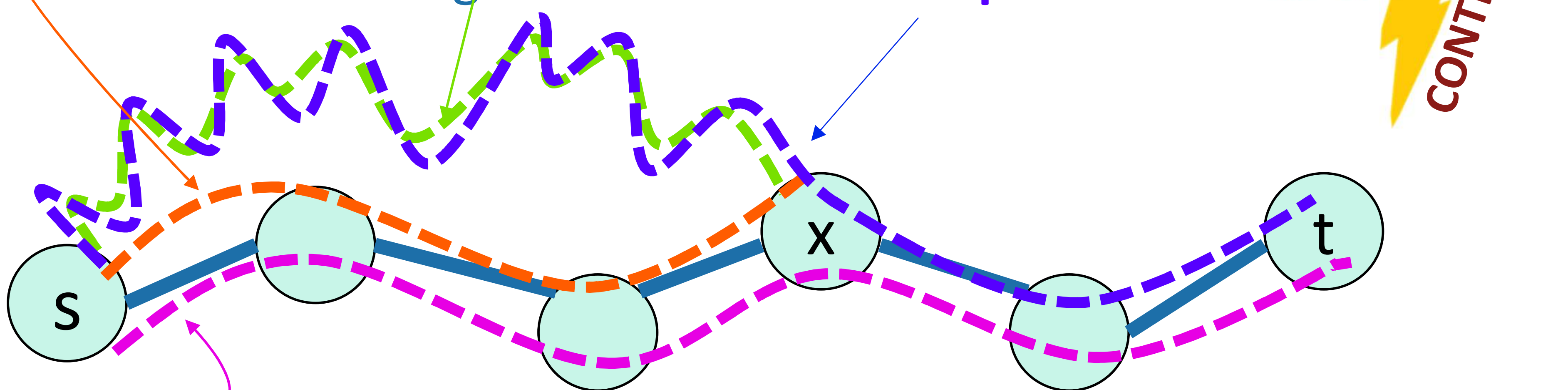
It has cost 6.

Q: What's the shortest path from **Packard** to the Union?

Warm-up

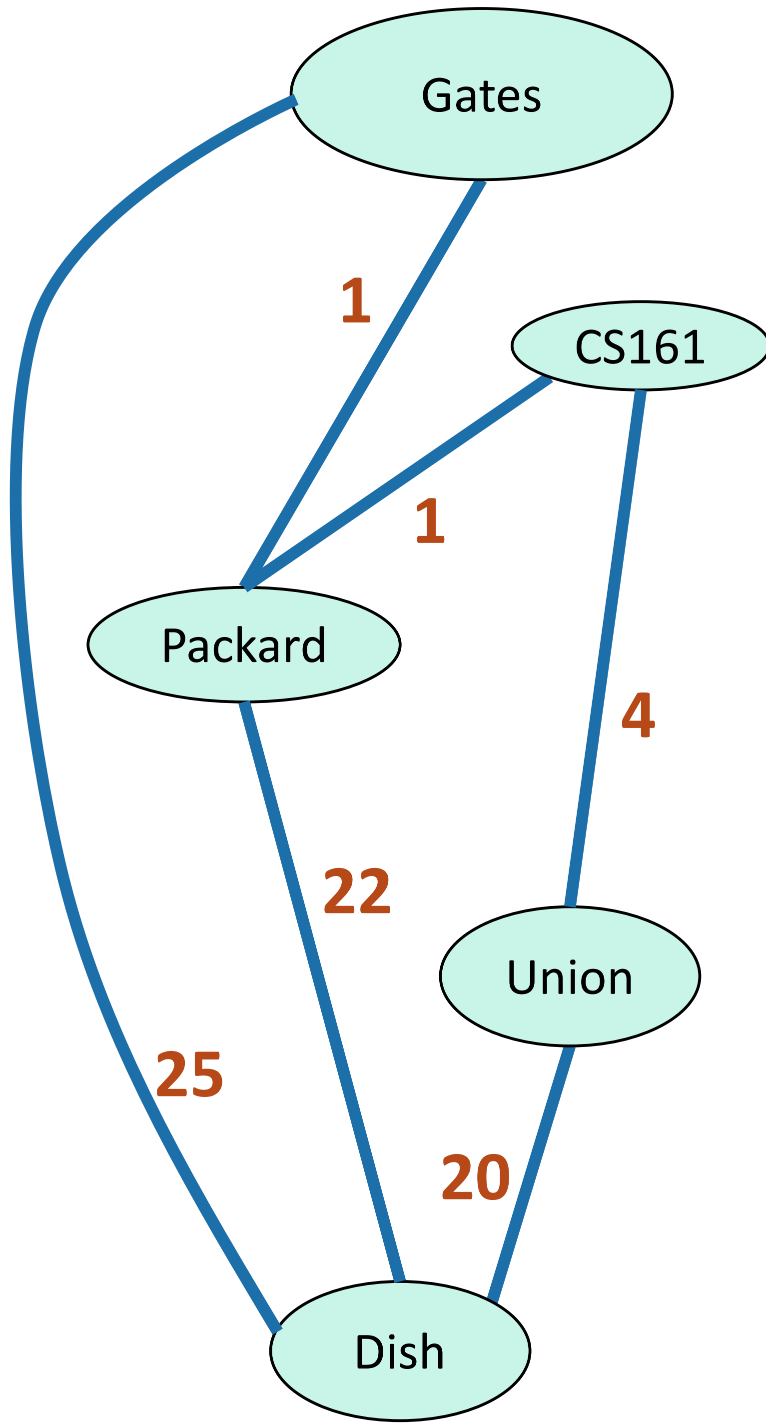
- A sub-path of a shortest path is also a shortest path.

- Say **this** is a shortest path from s to t.
- Claim: **this** is a shortest path from s to x.
  - Suppose not, **this** one is a shorter path from s to x.
  - But then that gives an **even shorter path** from s to t!

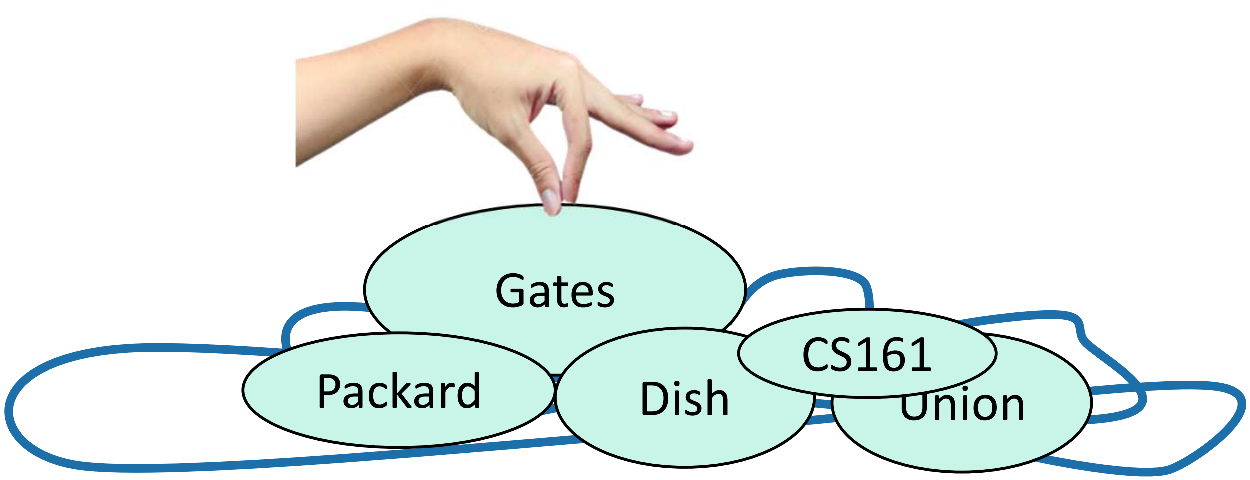


Dijkstra's algorithm

- Finds shortest paths from Gates to everywhere else.

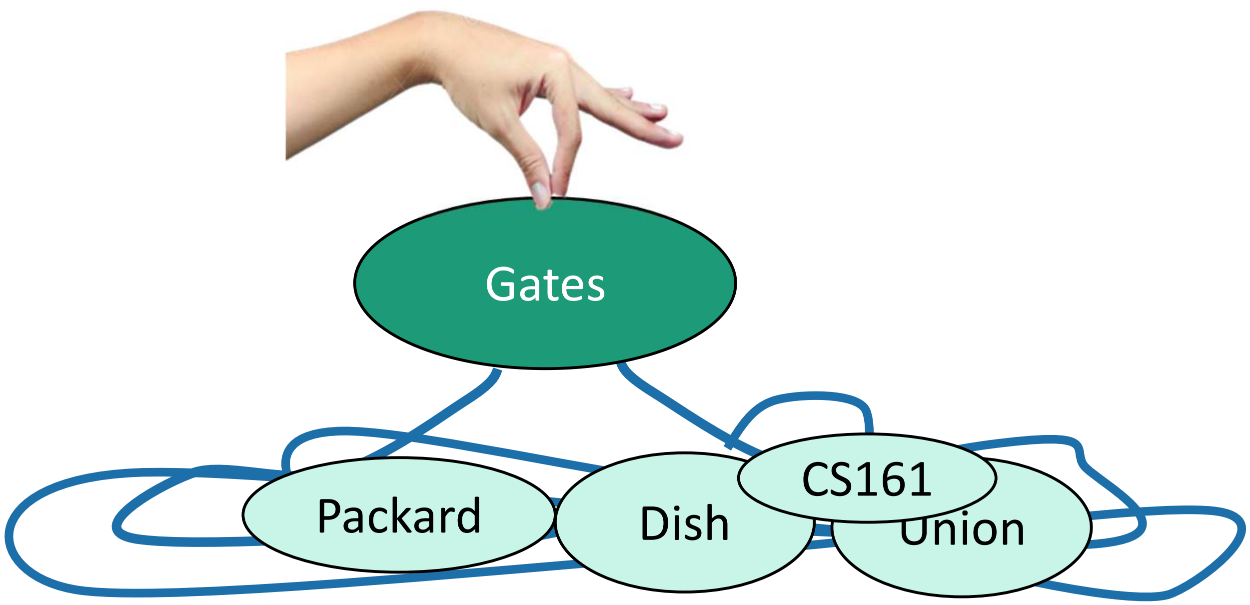


Dijkstra
intuition



Dijkstra
intuition

A vertex is done when it's not on the ground anymore.



Dijkstra
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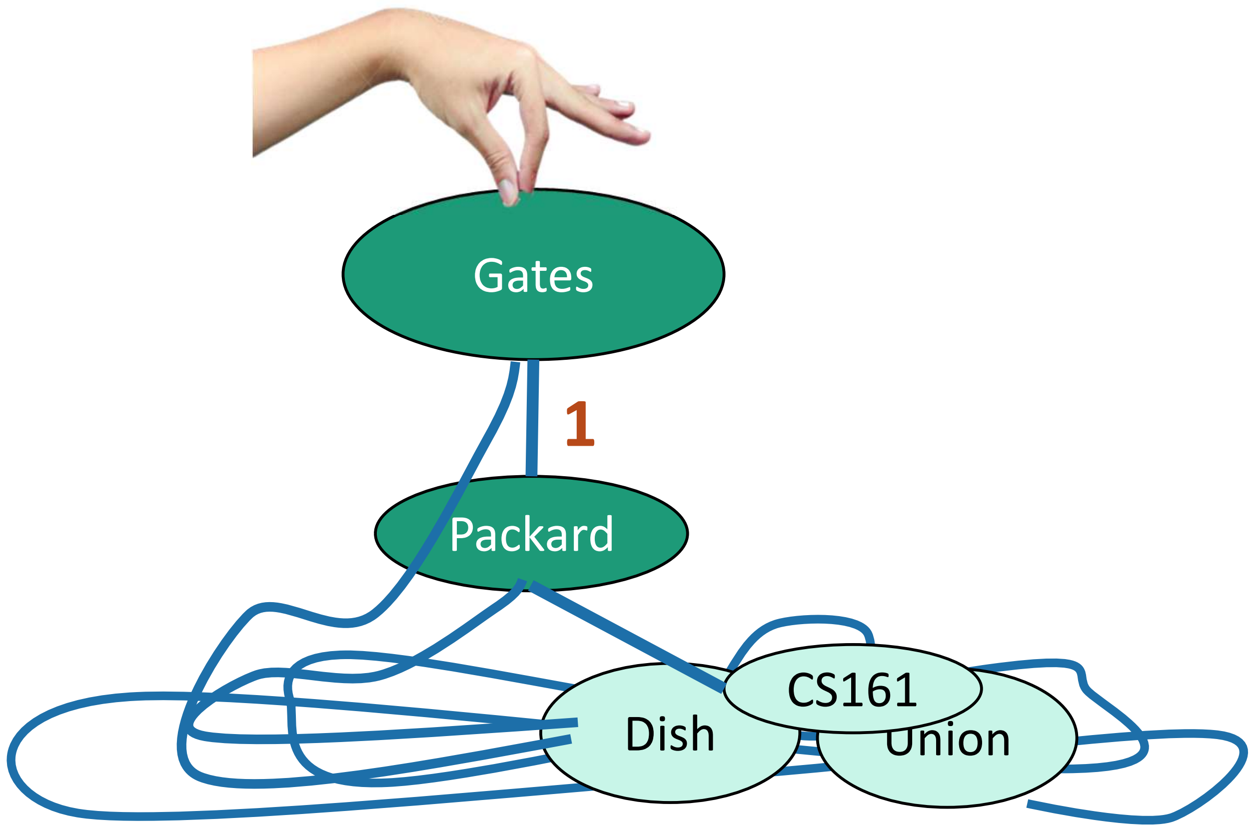

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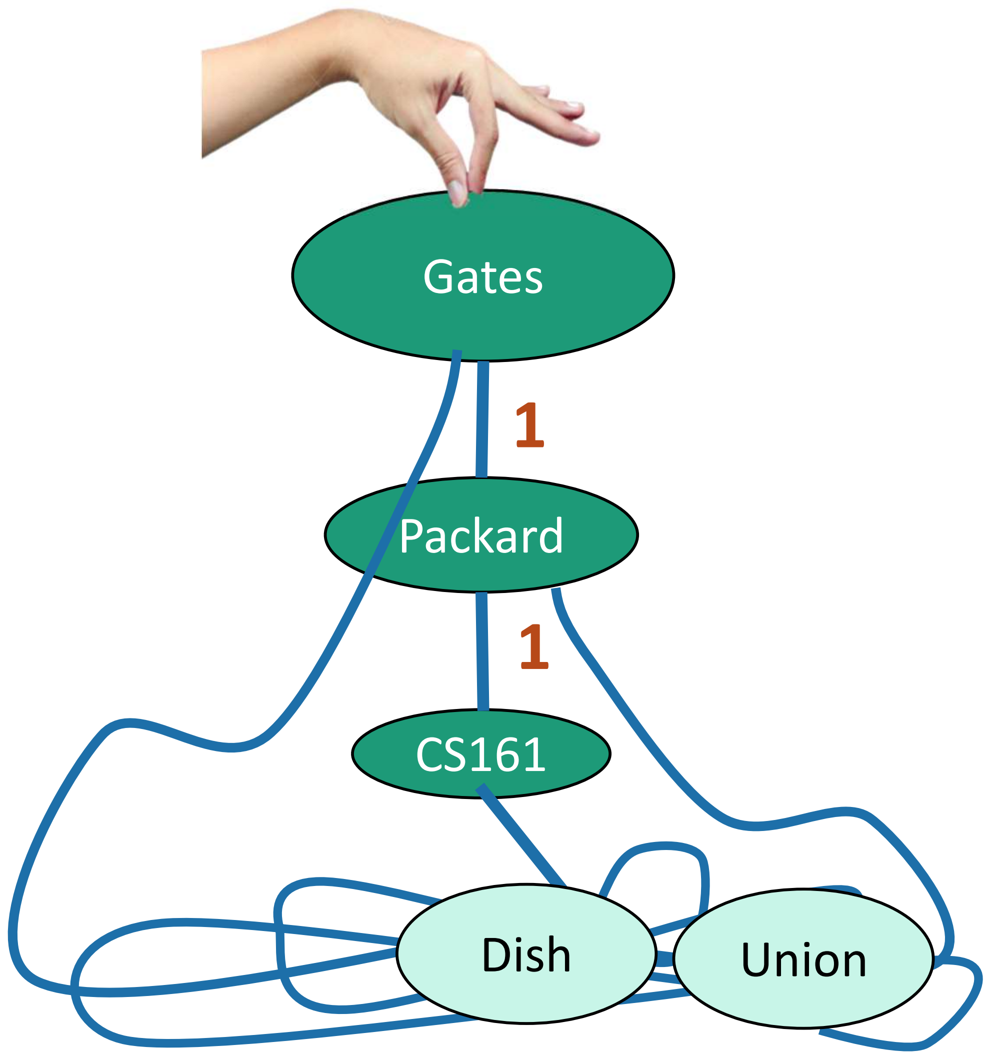

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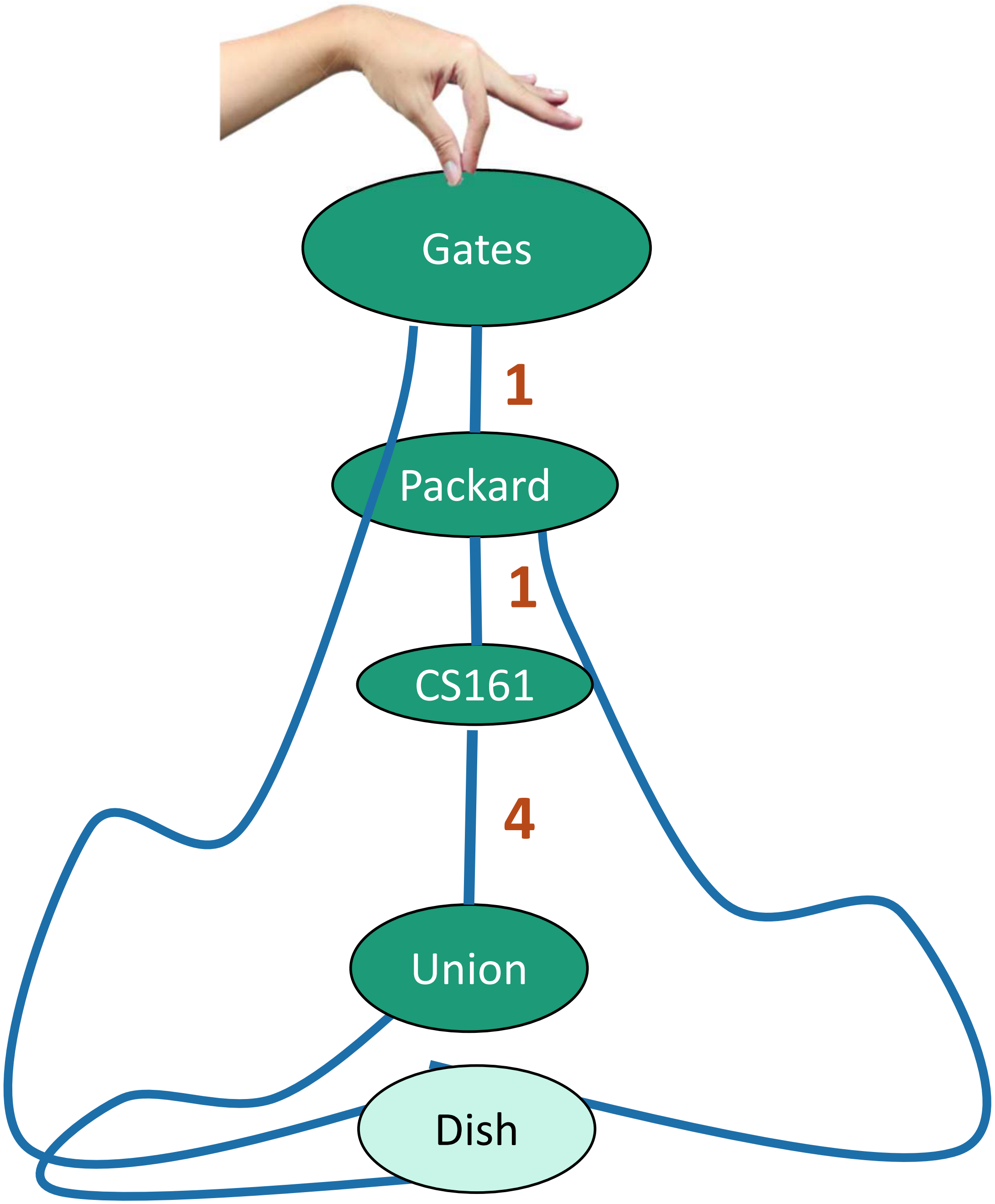

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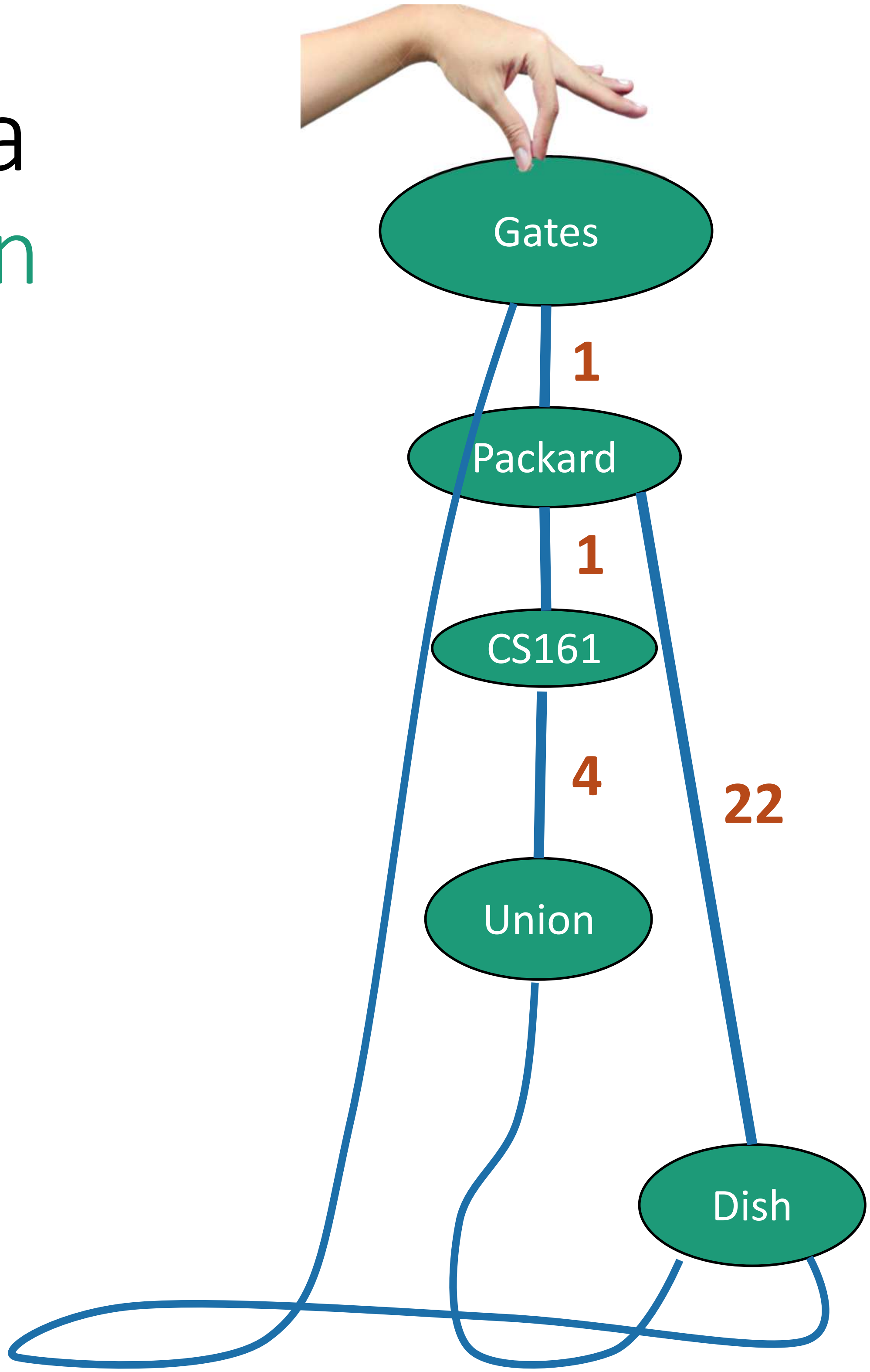
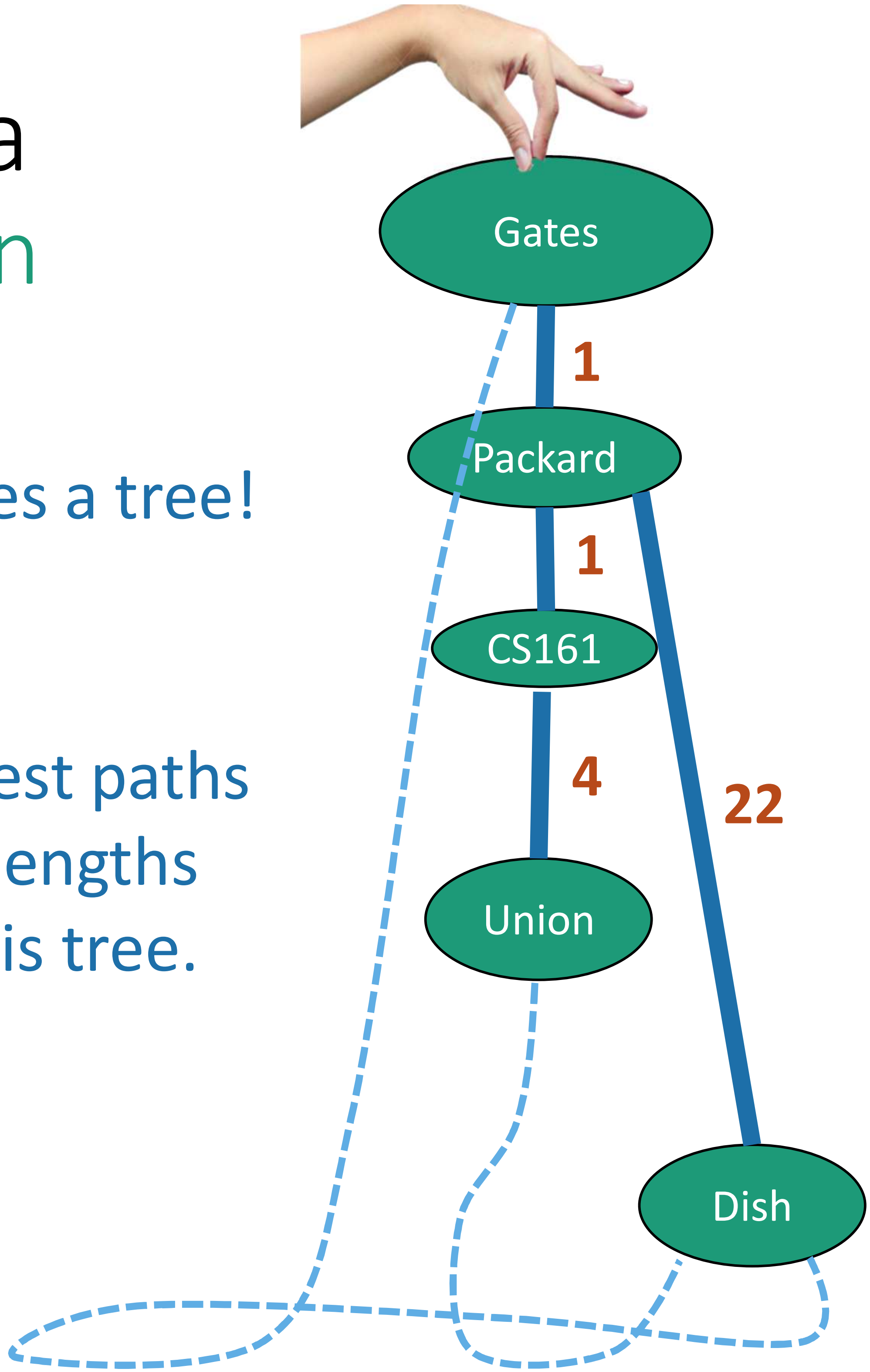


Dijkstra
intuition

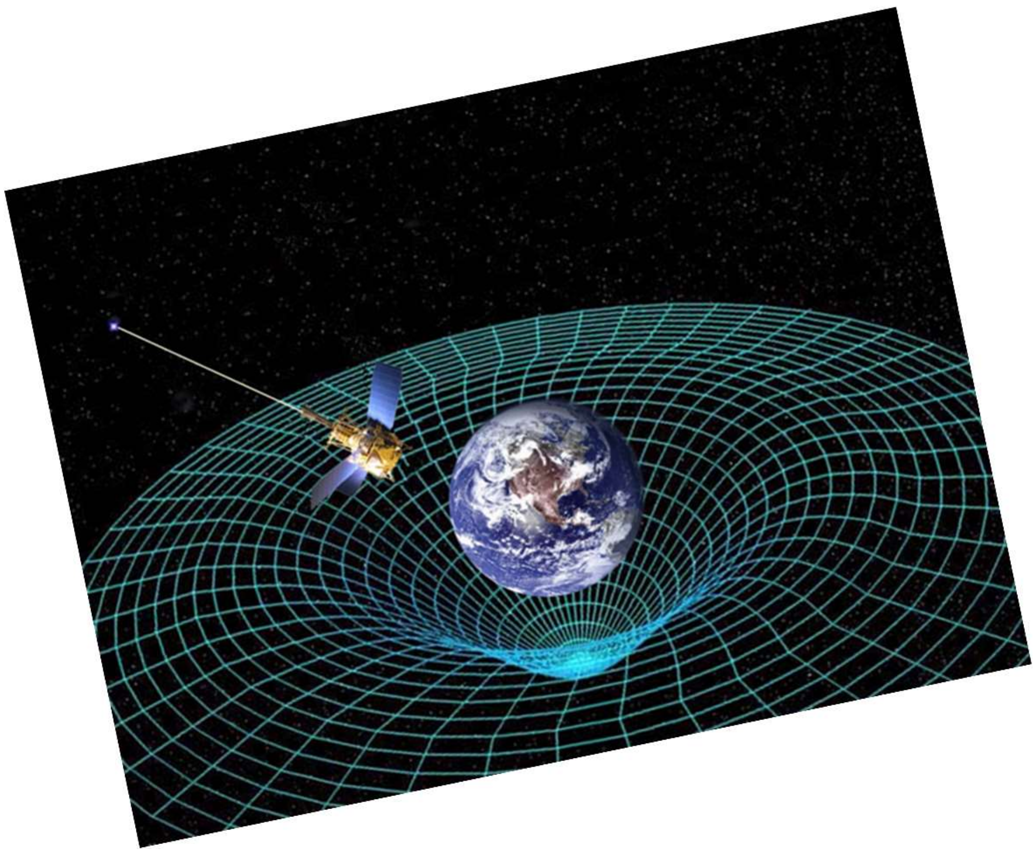
This creates a tree!

The shortest paths are the lengths along this tree.



How do we actually implement this?

- **Without** string and gravity?



Dijkstra by example

How far is a node from Gates?

I'm not sure yet

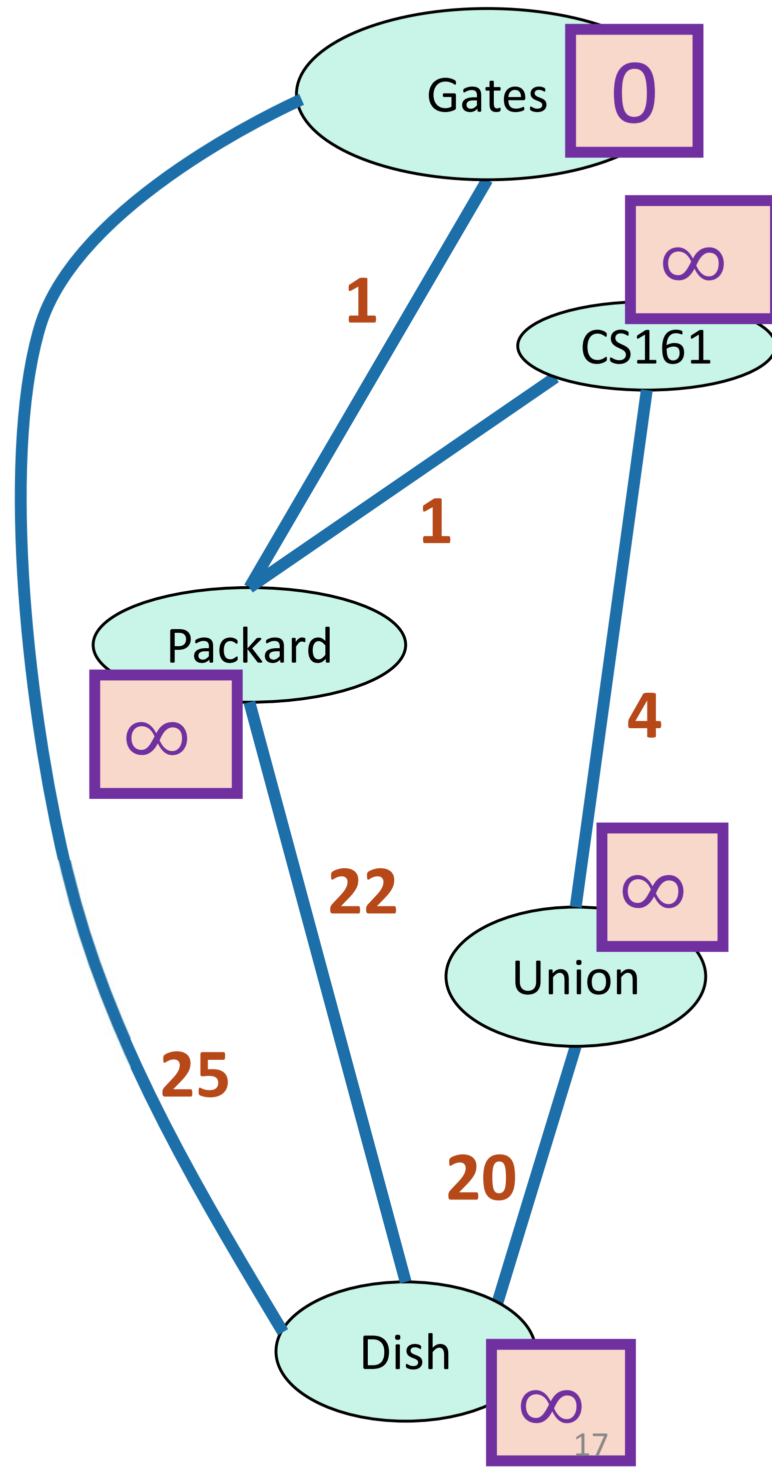
I'm sure

x

x = d[v] is my best **over-estimate** for dist(Gates,v).

Initialize d[v] = ∞
for all non-starting vertices v,
and d[Gates] = 0

- Pick the **not-sure** node u with the smallest estimate **d[u].**



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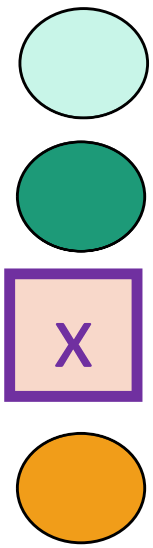

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Current node u

- Pick the **not-sure** node u with the smallest estimate **d[u].**
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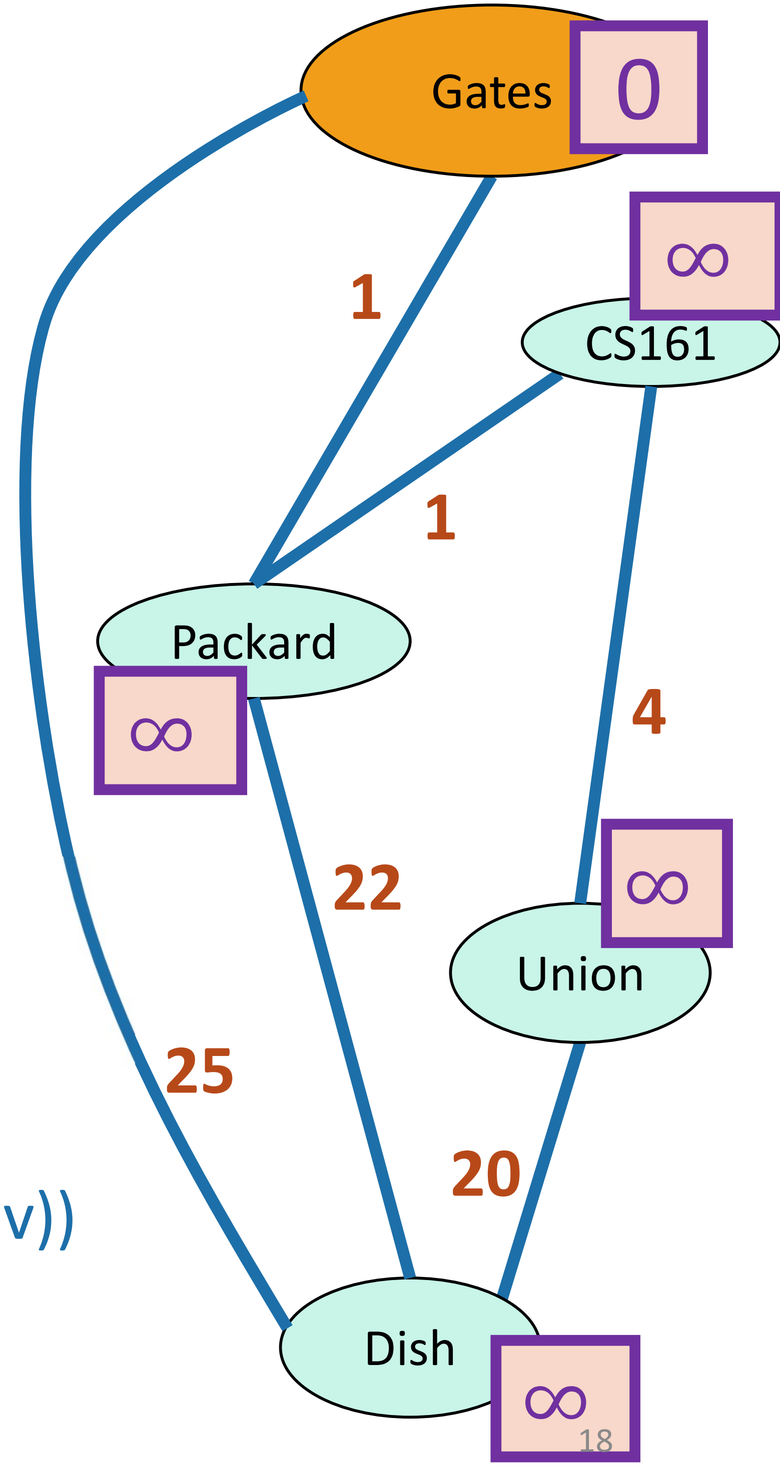
  - d[v] = min(d[v] , d[u] + edgeWeight(u,v))



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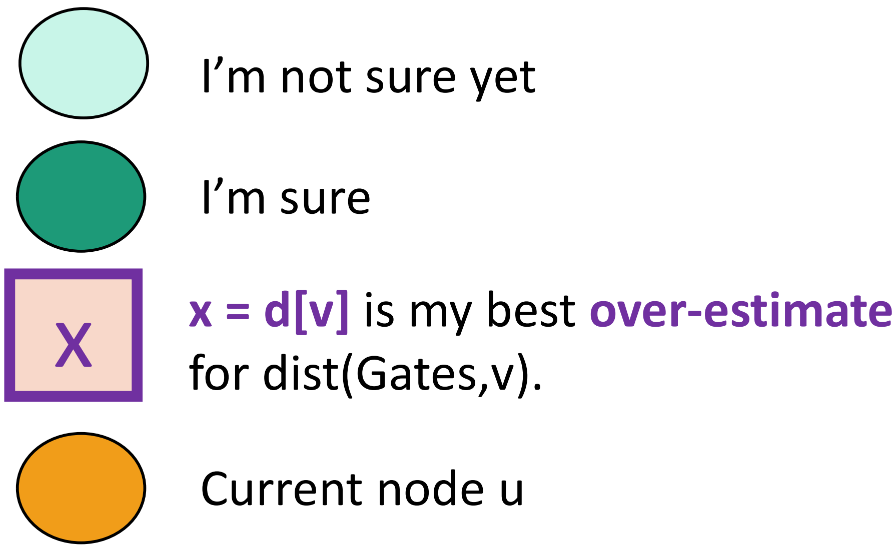

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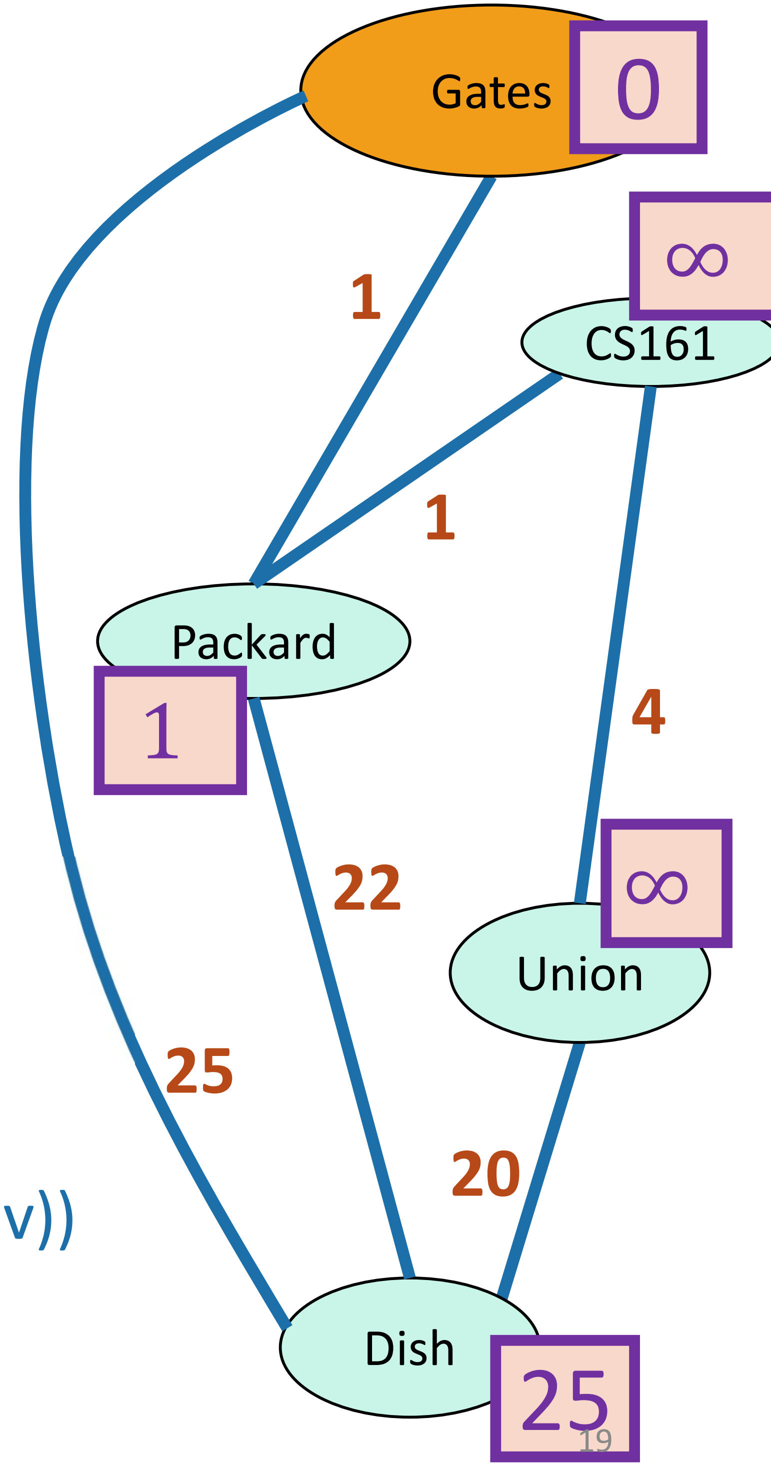

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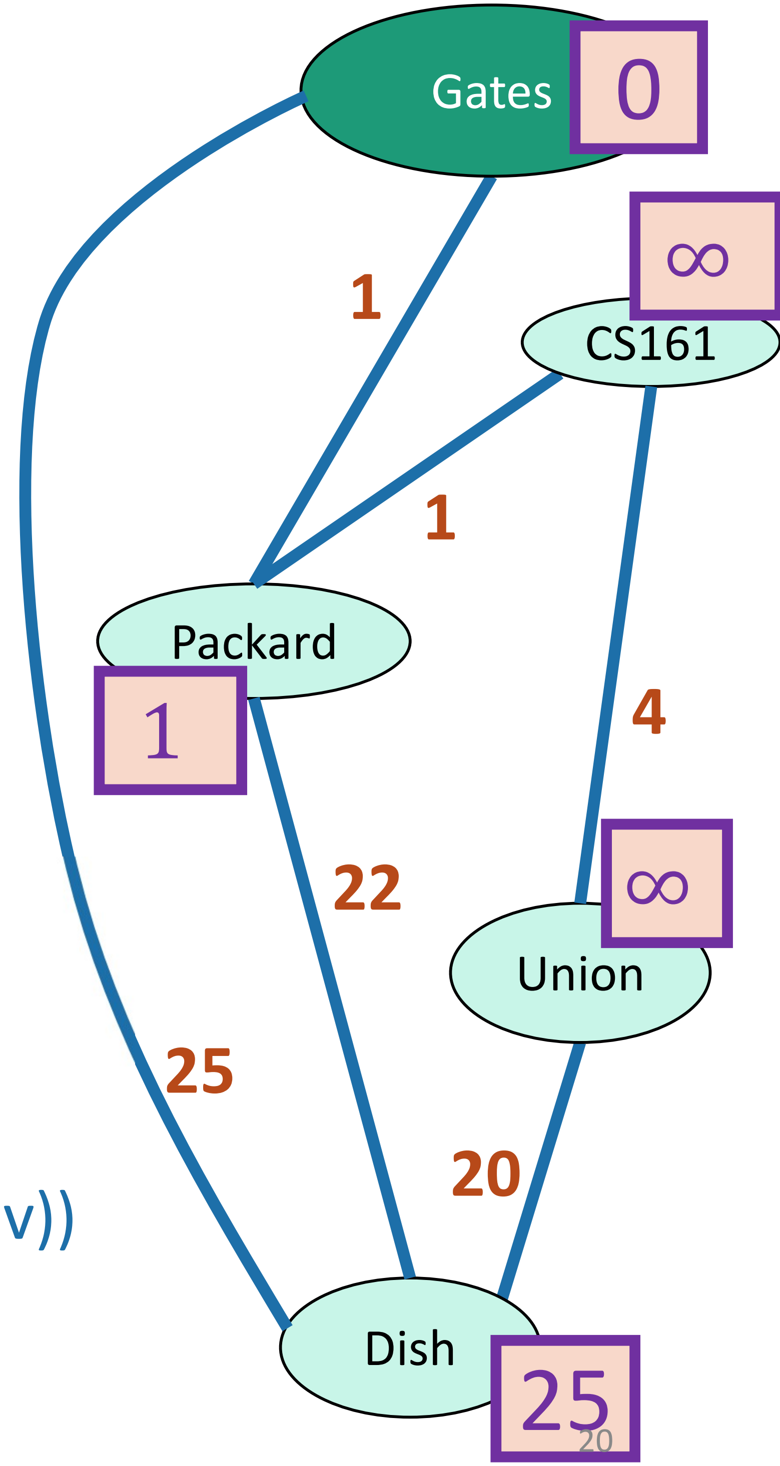

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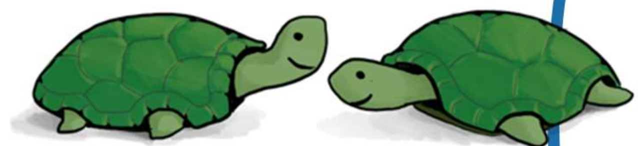
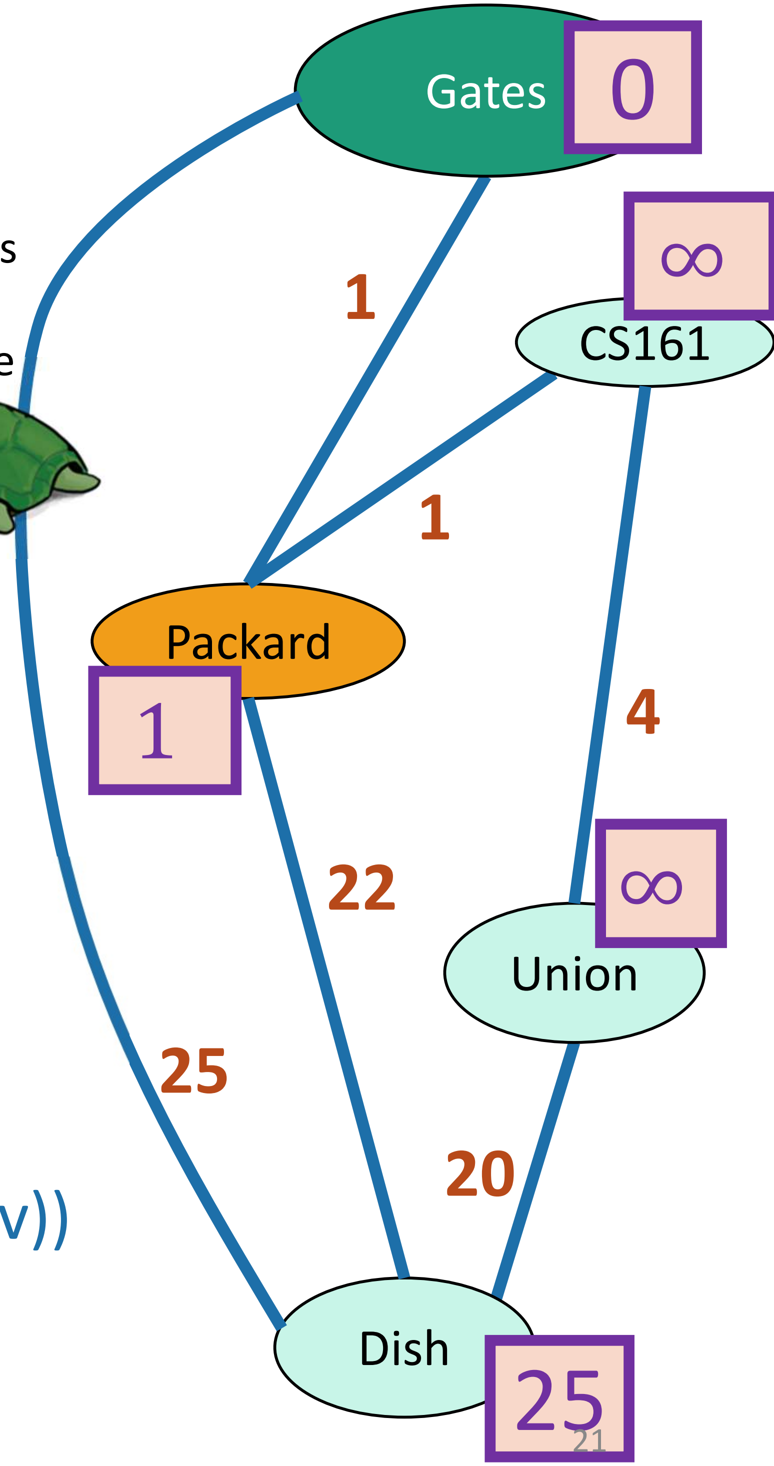
Current node u

Packard has three neighbors. What happens when we update them?
1 min. think; 1 min. share





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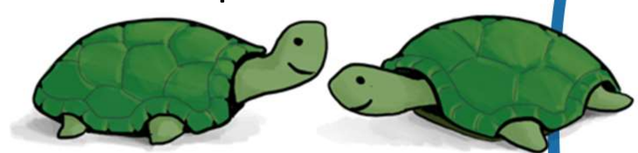

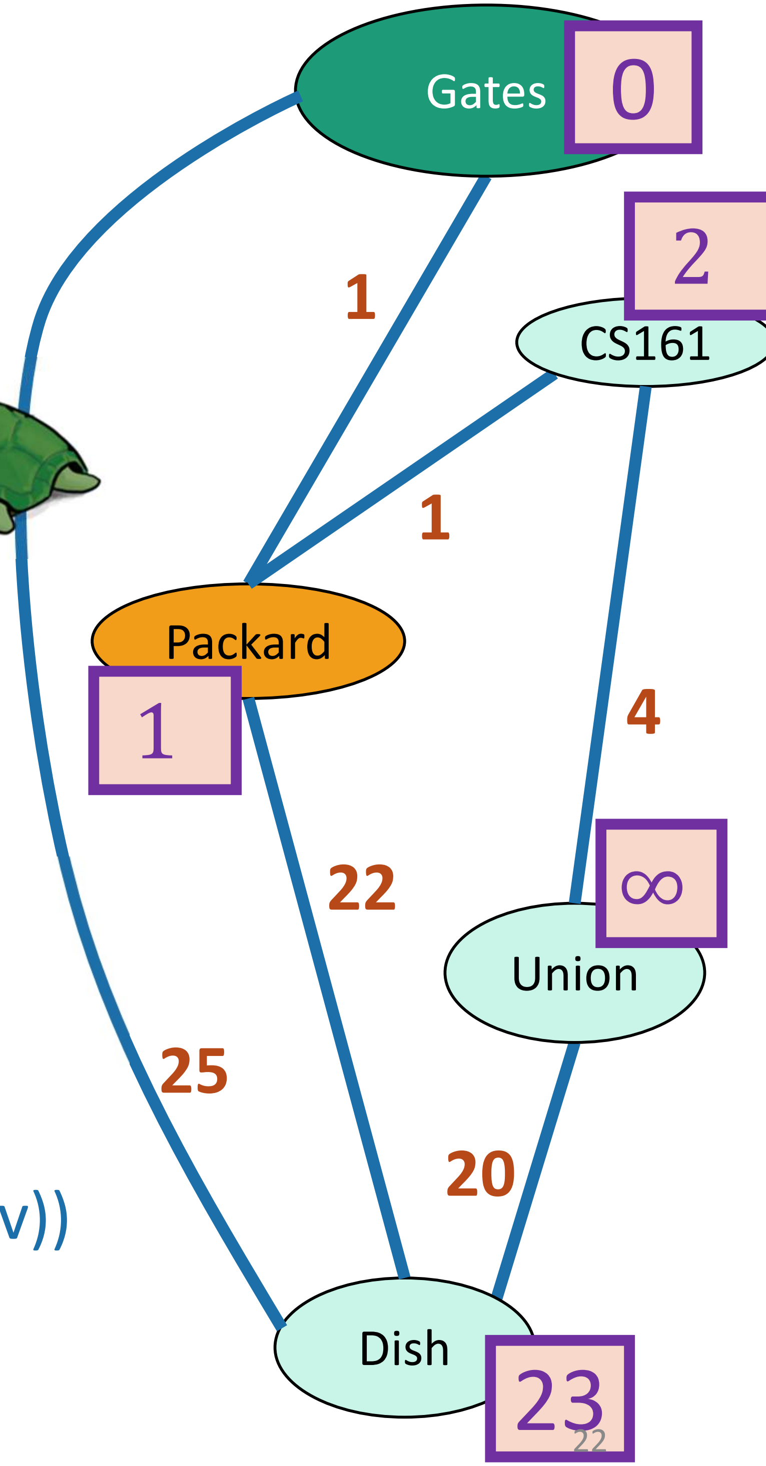

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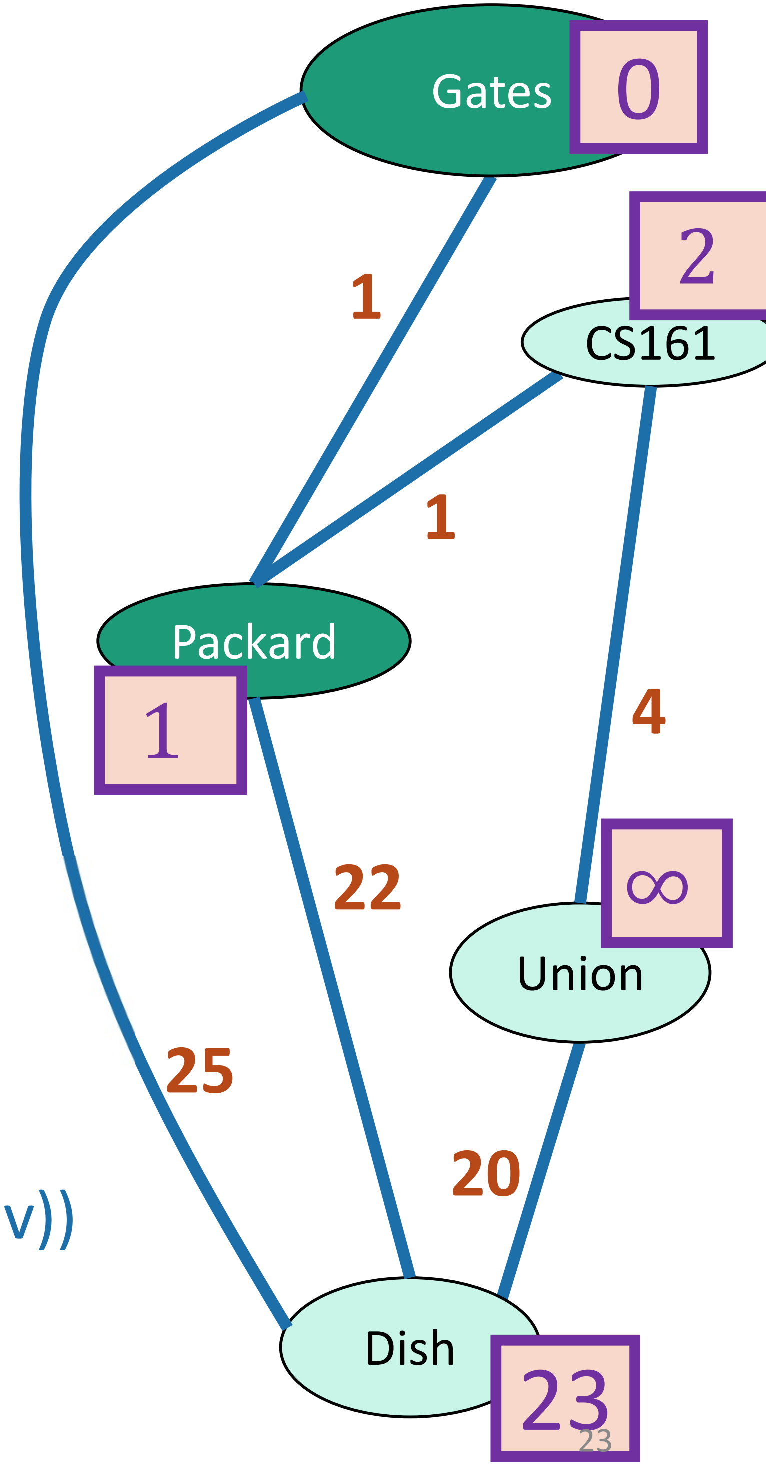

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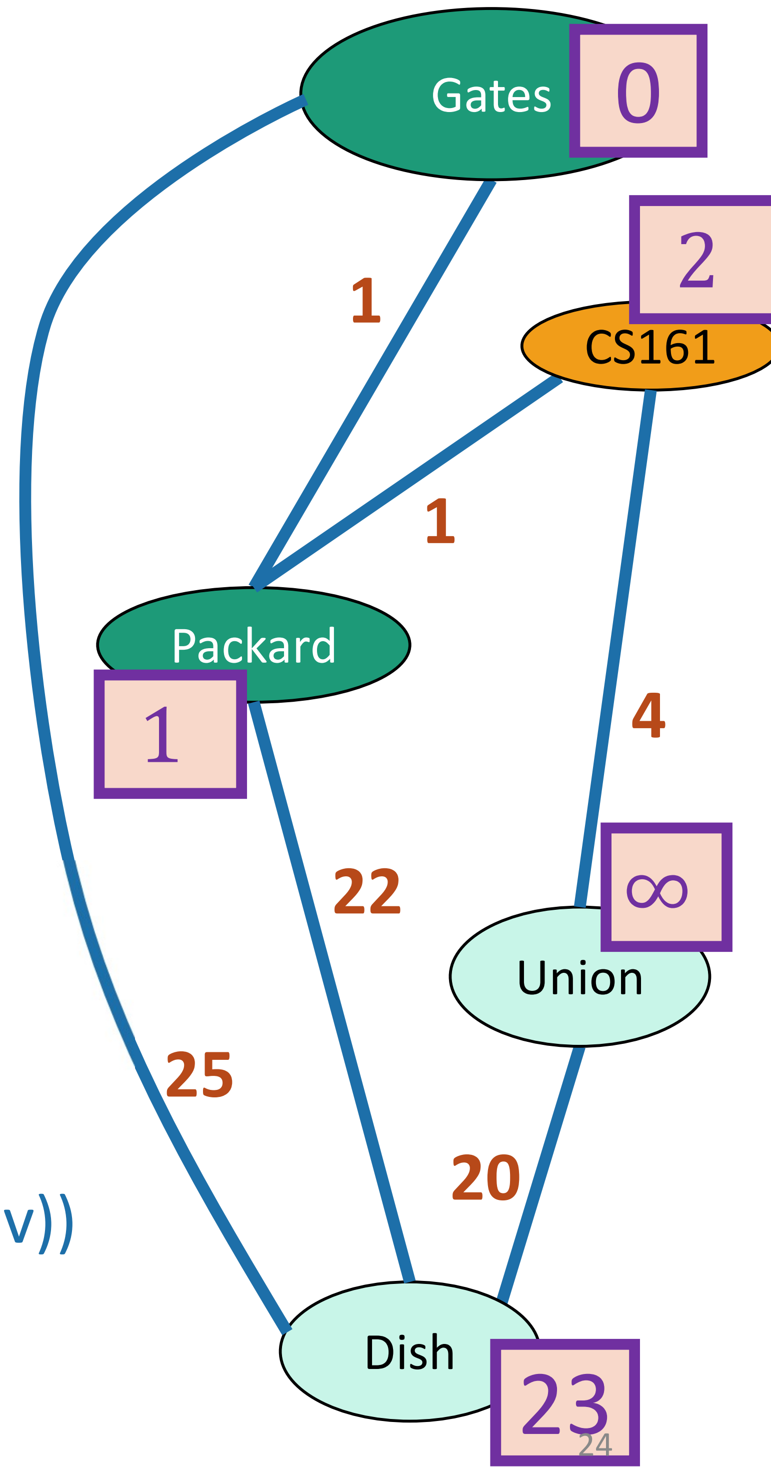

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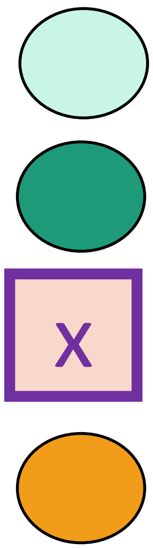

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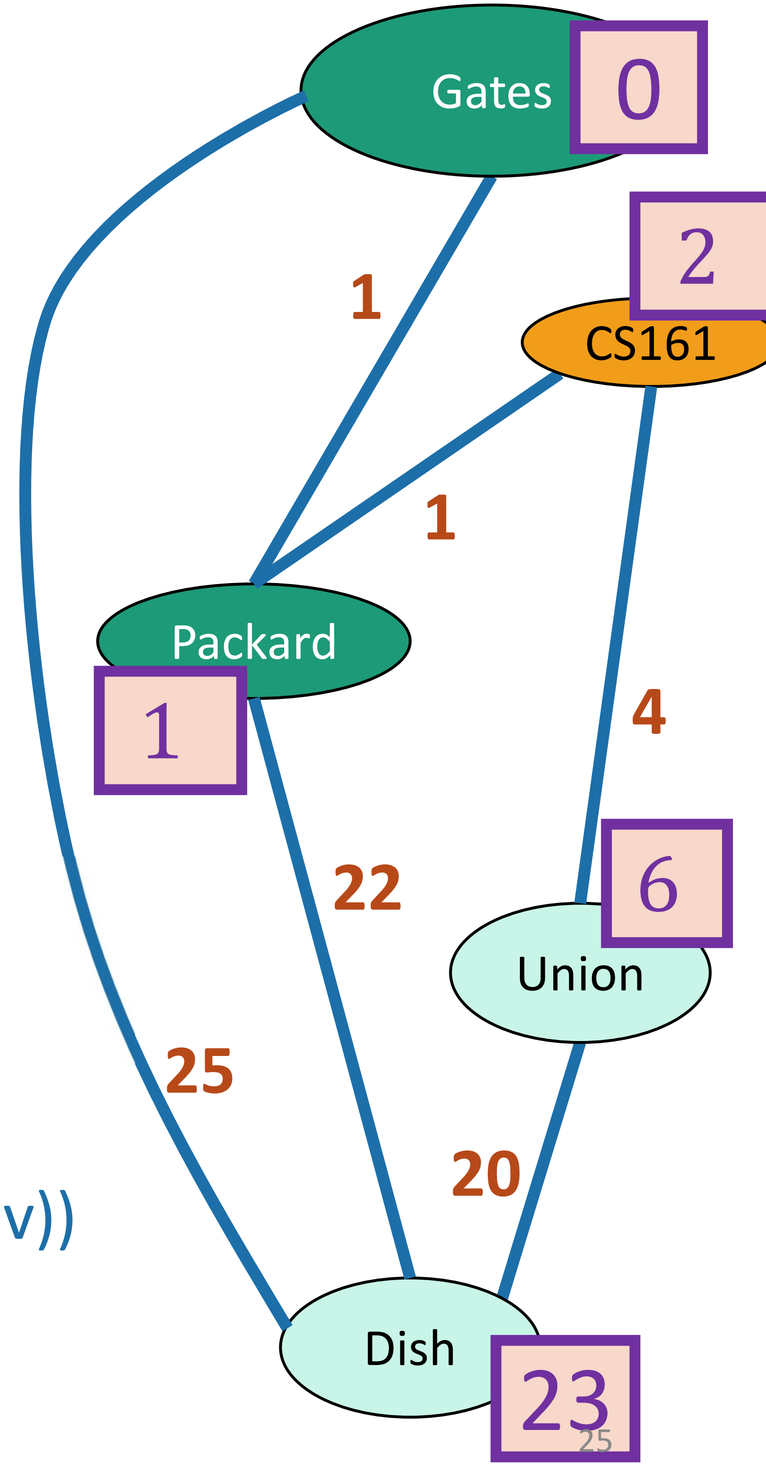

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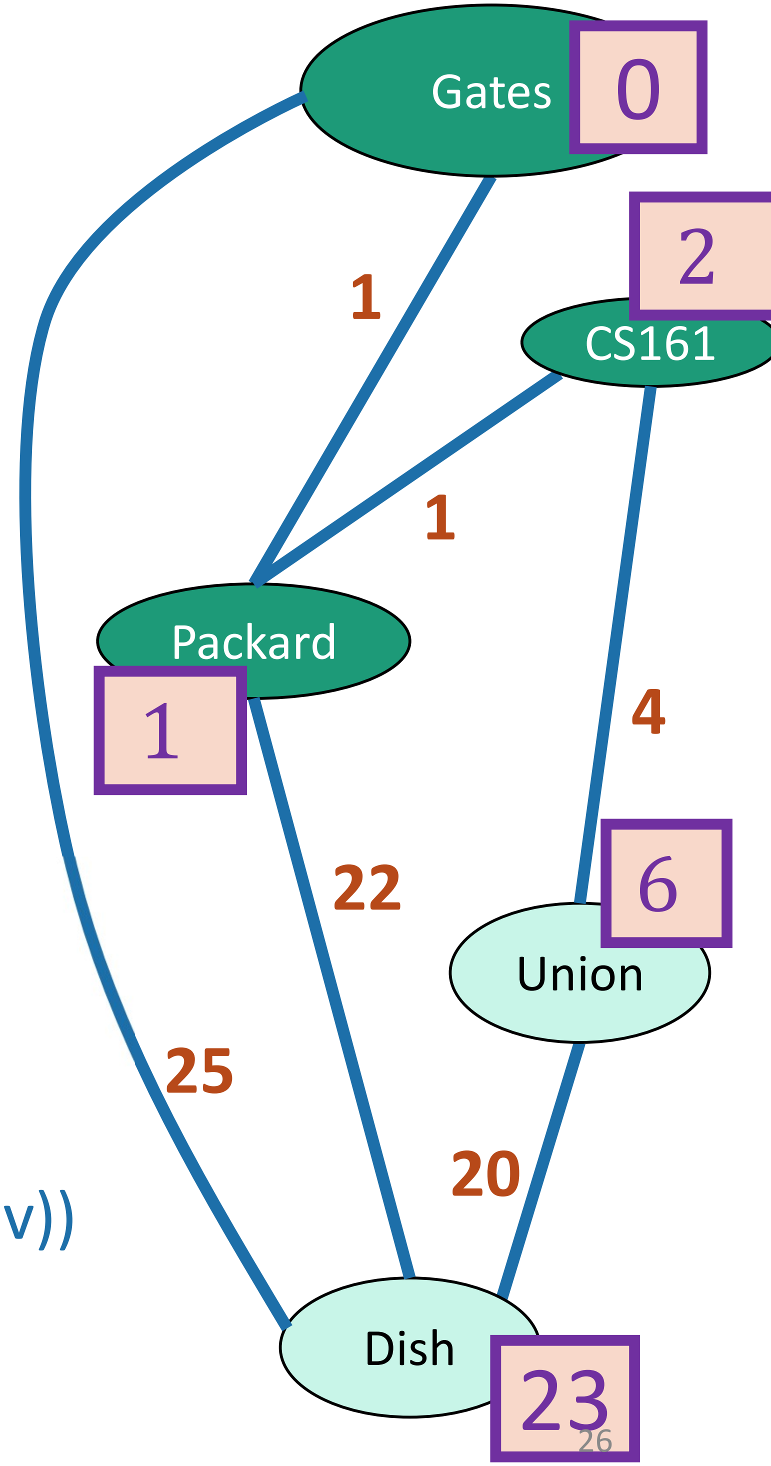

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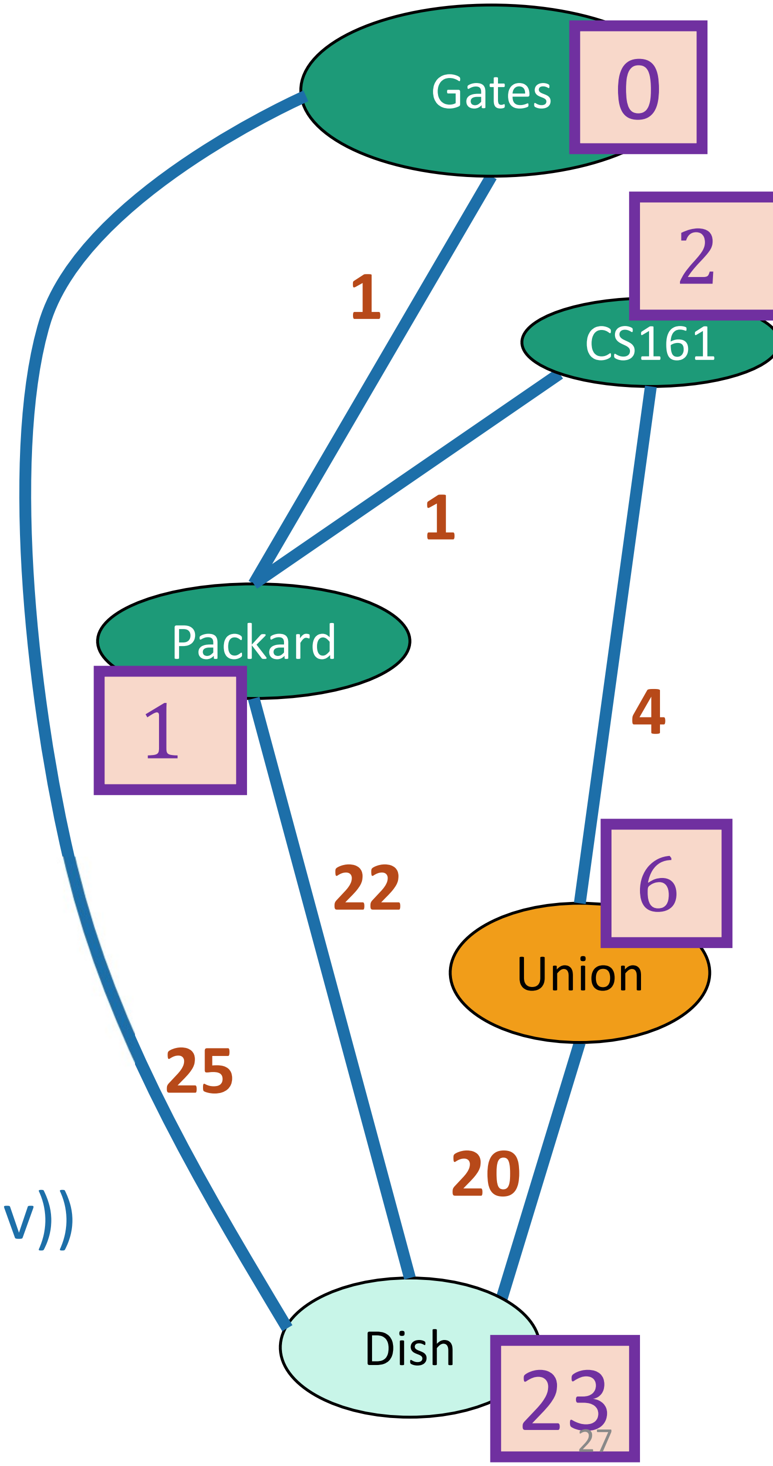

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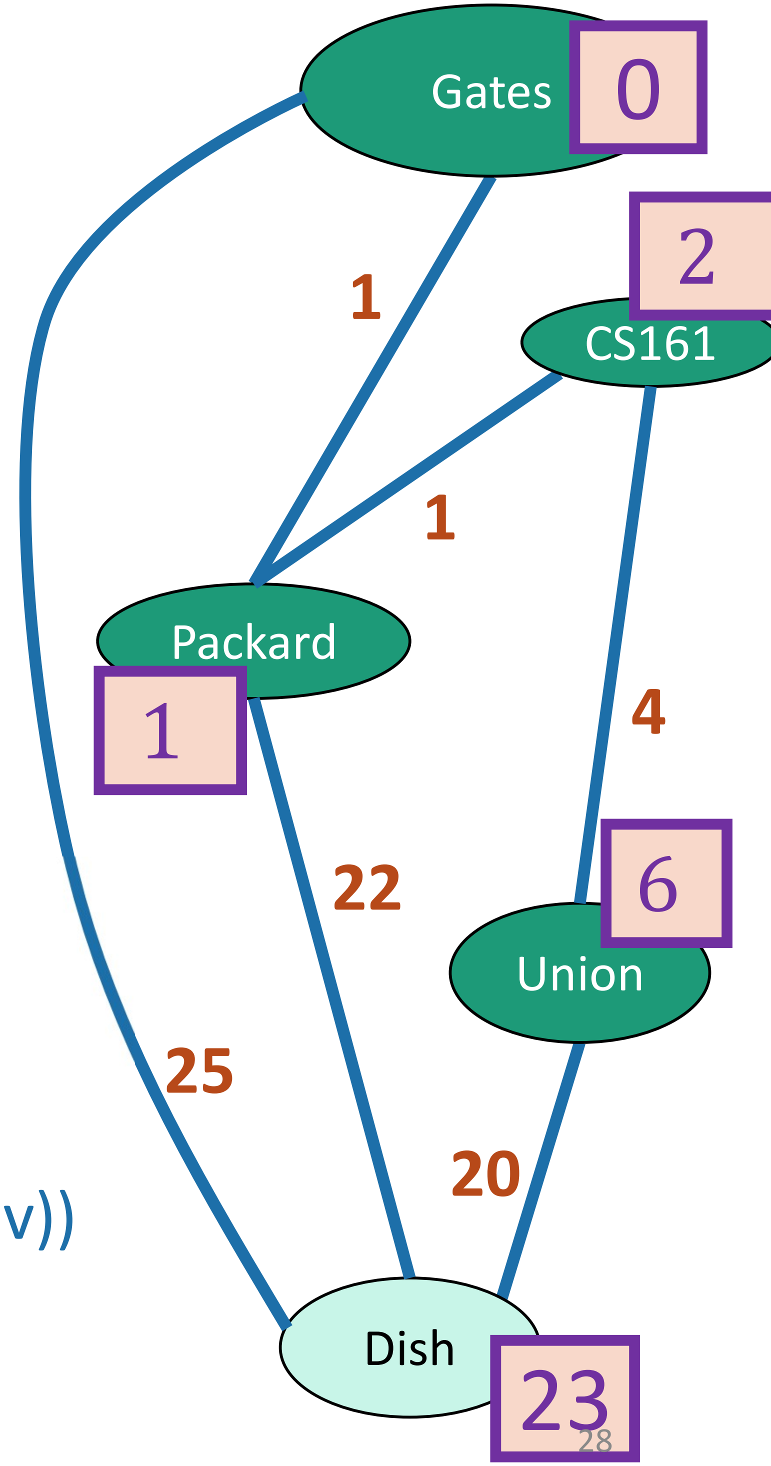

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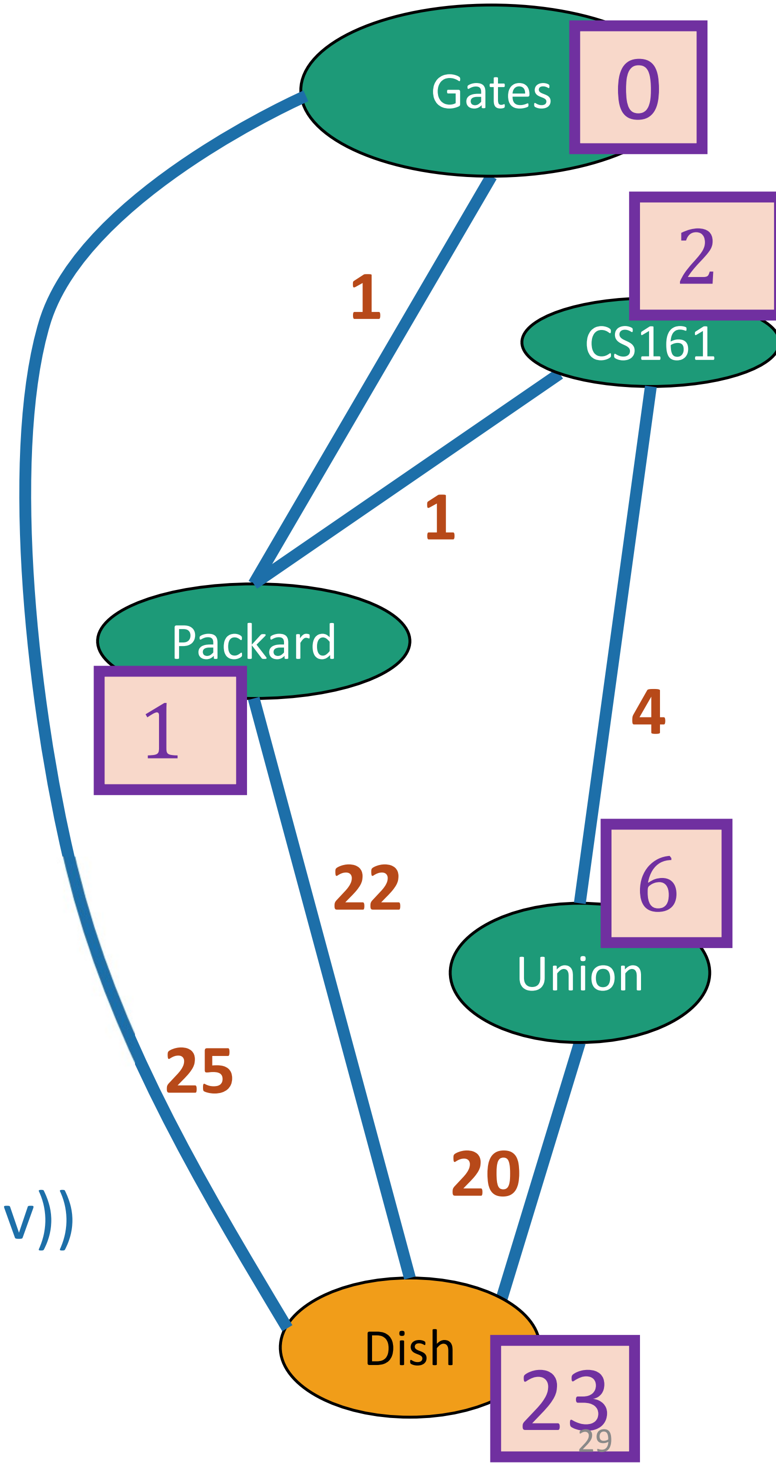

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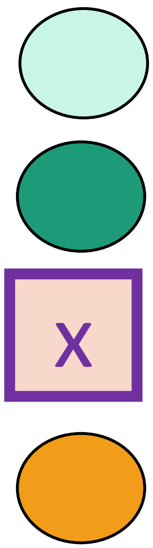

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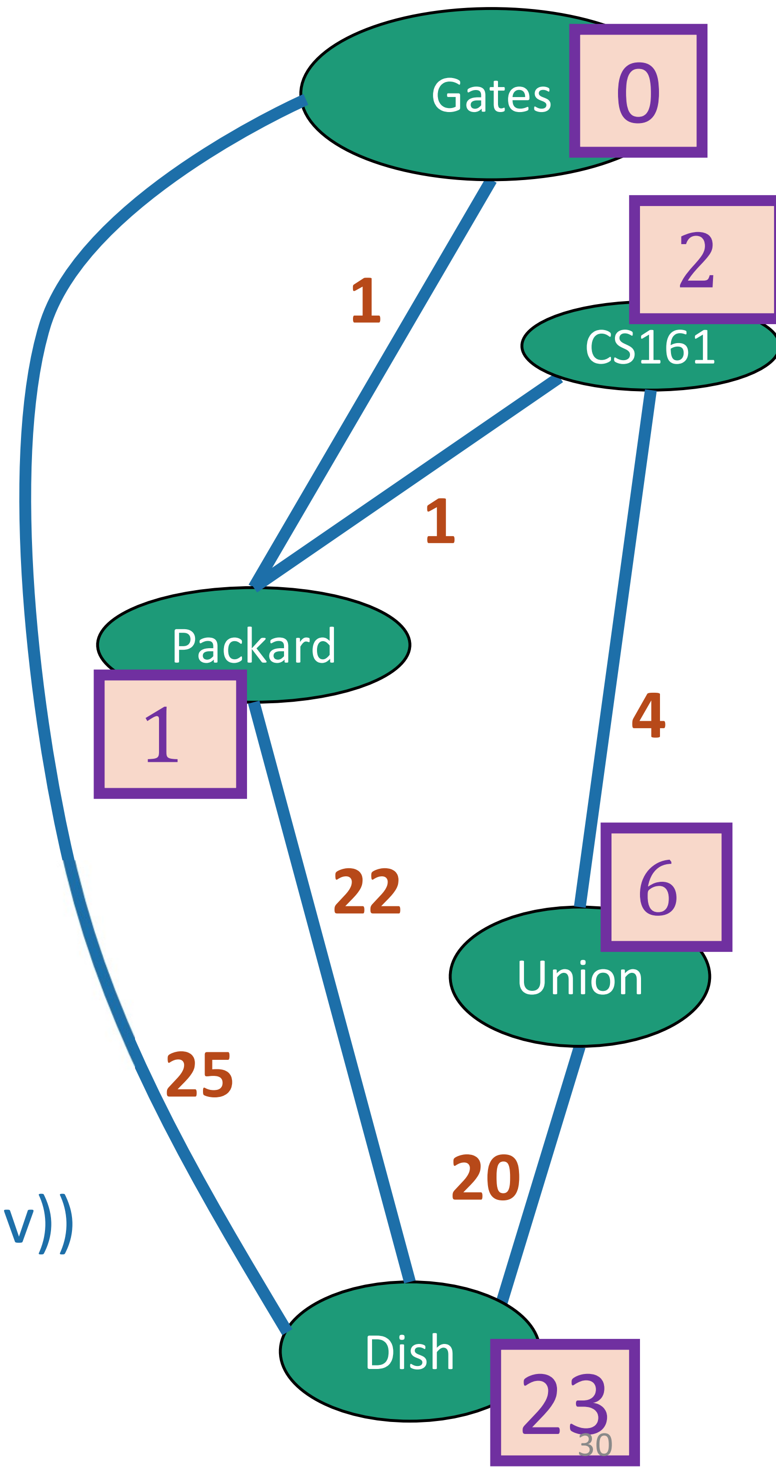

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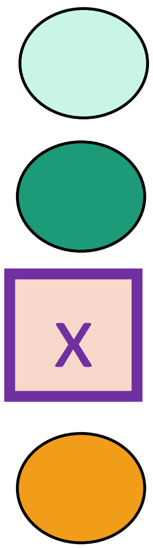

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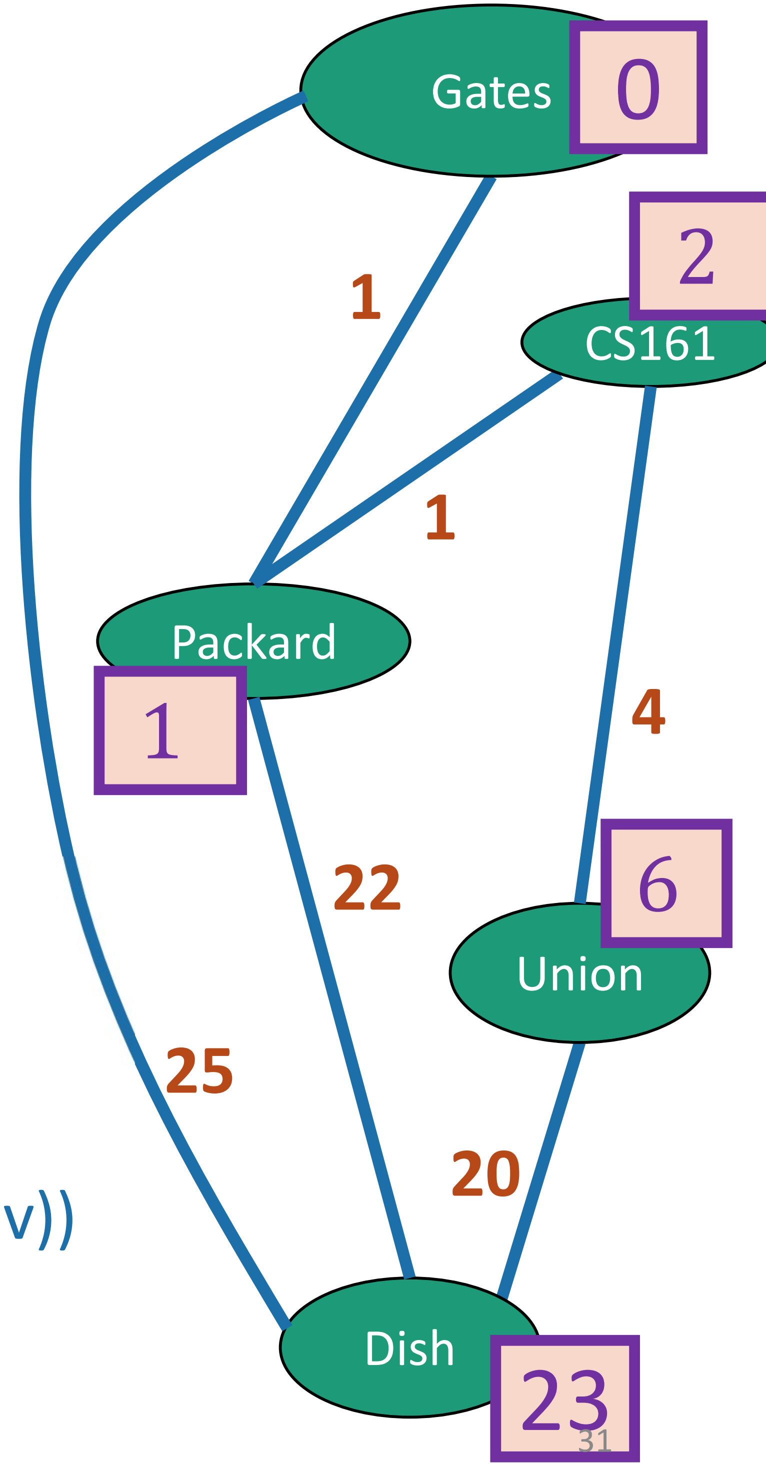
  - d[v] = min(d[v] , d[u] + edgeWeight(u,v))
- Mark u as **sure**.
- Repeat

- After all nodes are **sure**, say that d(Gates, v) = d[v] for all v



Dijkstra’s algorithm

Dijkstra(G,s):

- Set all vertices to **not-sure**
- d[v] = ∞ for all v in V
- d[s] = 0
- **While** there are **not-sure** nodes:
  - Pick the **not-sure** node u with the smallest estimate **d[u].**
  - **For** v in u.neighbors:
    - d[v] ← min(d[v] , d[u] + edgeWeight(u,v))
  - Mark u as **sure**.
- Now d(s, v) = d[v]

Why does this work?

- **Theorem**:
  - Suppose we run Dijkstra on G =(V,E), starting from s.
  - At the end of the algorithm, the estimate **d[v]** is the actual distance d(s,v).

Let's rename "Gates" to "s", our starting vertex.

- Proof outline:
  - **Claim 1**: For all v, **d[v] $\geq$ d(s,v).**
  - **Claim 2**: When a vertex v is marked **sure**, **d[v] = d(s,v)**.

- **Claims 1 and 2** imply the **theorem.**

Claim 2

Claim 1 + def of algorithm

  - When v is marked **sure**, **d[v] = d(s,v).**
  - **d[v] $\geq$ d(s,v)** and never increases, so after v is **sure**, **d[v]** stops changing.
  - This implies that at any time *after* v is marked **sure**, **d[v] = d(s,v).**
  - All vertices are **sure** at the end, so all vertices end up with **d[v] = d(s,v).**

Running time?

Dijkstra(G,s):

- Set all vertices to **not-sure**
- d[v] = ∞ for all v in V
- d[s] = 0
- **While** there are **not-sure** nodes:
  - Pick the **not-sure** node u with the smallest estimate **d[u].**
  - **For** v in u.neighbors:
    - d[v] ← min(d[v] , d[u] + edgeWeight(u,v))
  - Mark u as **sure**.
- Now dist(s, v) = d[v]

- n iterations (one per vertex)
- How long does one iteration take?

Depends on how we implement it…

We need a data structure that:

- Stores unsure vertices v
- Keeps track of d[v]
- Can find u with minimum d[u]
  - `findMin()`
- Can remove that u
  - `removeMin(u)`
- Can update (decrease) d[v]
  - `updateKey(v,d)`

Just the inner loop:

- Pick the **not-sure** node u with the smallest estimate **d[u].**
- Update all u's neighbors v:
  - d[v] ← min(d[v] , d[u] + edgeWeight(u,v))
- Mark u as **sure**.

Total running time is big-oh of:

$$\sum_{u \in V} \left(T(\text{findMin}) + \left(\sum_{v \in u.neighbors} T(\text{updateKey}) \right) + T(\text{removeMin}) \right)$$

= n(T(`findMin`) + T(`removeMin`)) + m T(`updateKey`)

If we use an array

- T(findMin) = O(n)
- T(removeMin) = O(n)
- T(updateKey) = O(1)

- Running time of Dijkstra

  = O(n(T(`findMin`) + T(`removeMin`)) + m T(`updateKey`))

  = $O(n^2) + O(m)$

  = $O(n^2)$

If we use a BST

- T(findMin) = O(log(n))
- T(removeMin) = O(log(n))
- T(updateKey) = O(log(n))

- Running time of Dijkstra

  $= O(n(\, T(\texttt{findMin}) + T(\texttt{removeMin})\,) + m\, T(\texttt{updateKey}))$

  $= O(n\log(n)) + O(m\log(n))$

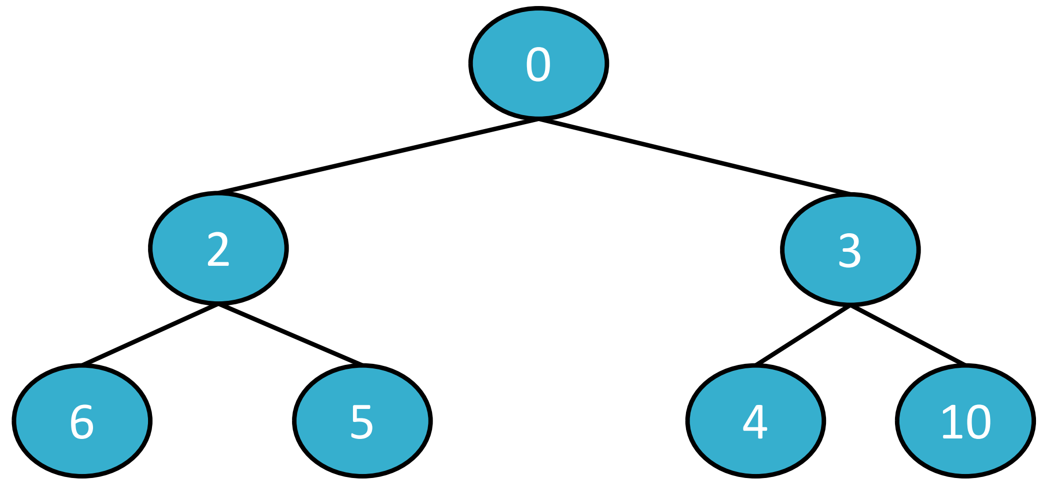
  $= O((n+m)\log(n))$

Better than an array if the graph is sparse!

aka if m is much smaller than n^2

Heaps support these operations

- findMin
- removeMin
- updateKey



- A **heap** is a tree-based data structure that has the property that every node has a smaller key than its children.

Say we use a Heap

- T(findMin) = O(1)
- T(removeMin) = O(log(n))
- T(updateKey) = O(1)

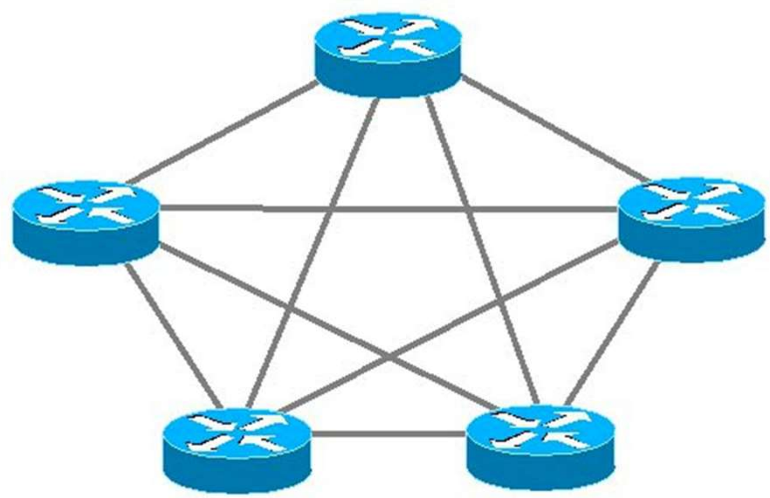
- Running time of Dijkstra

  = O(n(T(`findMin`) + T(`removeMin`)) + m T(`updateKey`))

  = O(nlog(n) + m)

Ứng dụng thực tế

- eg, OSPF (Open Shortest Path First), a routing protocol for IP networks, uses Dijkstra.



Hạn chế

- Chỉ áp dụng cho đồ thị có trọng số không âm (non-negative edge weights).
- Nếu một cạnh thay đổi trọng số, phải chạy lại thuật toán từ đầu.