# Continuous Random Variables: Basics

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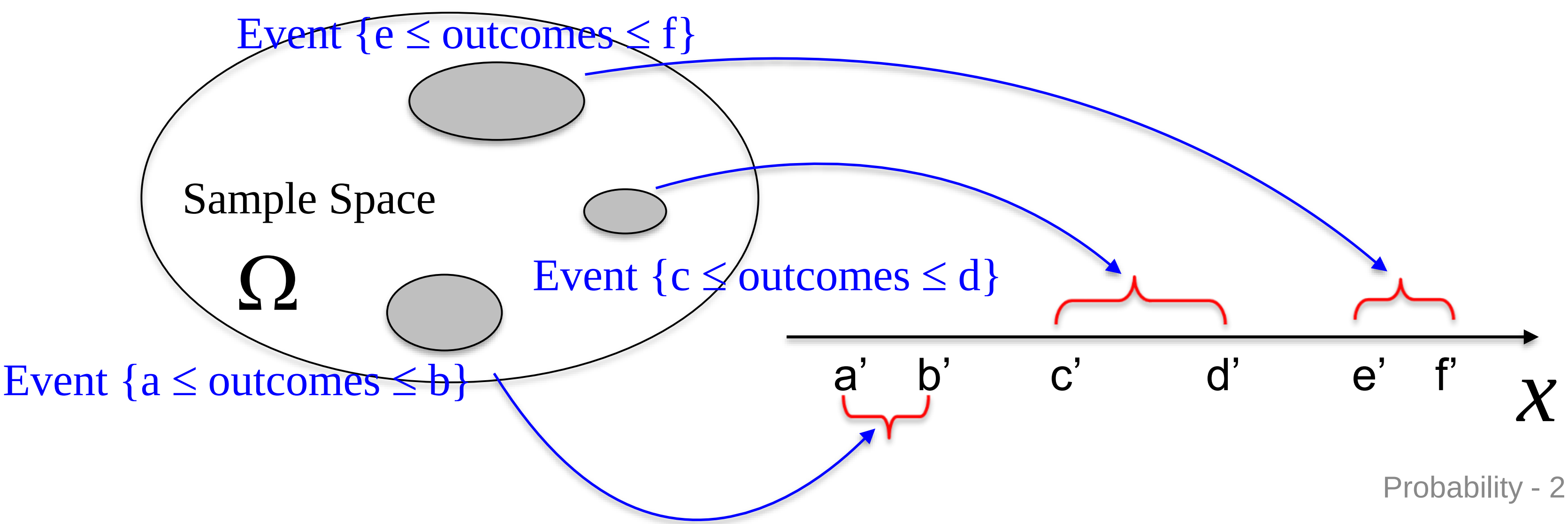
References:
- Dimitri P. Bertsekas and John N. Tsitsiklis, *Introduction to Probability*, Sections 3.1-3.3
- Walpole R. E., Myers R. H., Myers S. L. and Ye K., *Probability & Statistics for Engineers & Scientists*, Ch. 6
- Slides are credited from Prof. Berlin Chen, NTNU.

# Continuous Random Variables

- Random variables with a continuous range of possible values are quite common
  - The velocity of a vehicle
  - The temperature of a day
  - The blood pressure of a person
  - etc.



# Probability Density Functions (1/2)

- A random variable $X$ is called **continuous** if its probability law can described in terms of a nonnegative function $f_X \left( f_X \geq 0 \right)$, called the **probability density function** (**PDF**) of $X$, which satisfies

$$\mathbf{P}\left(X \in B\right) = \int_B f_X \, dx$$

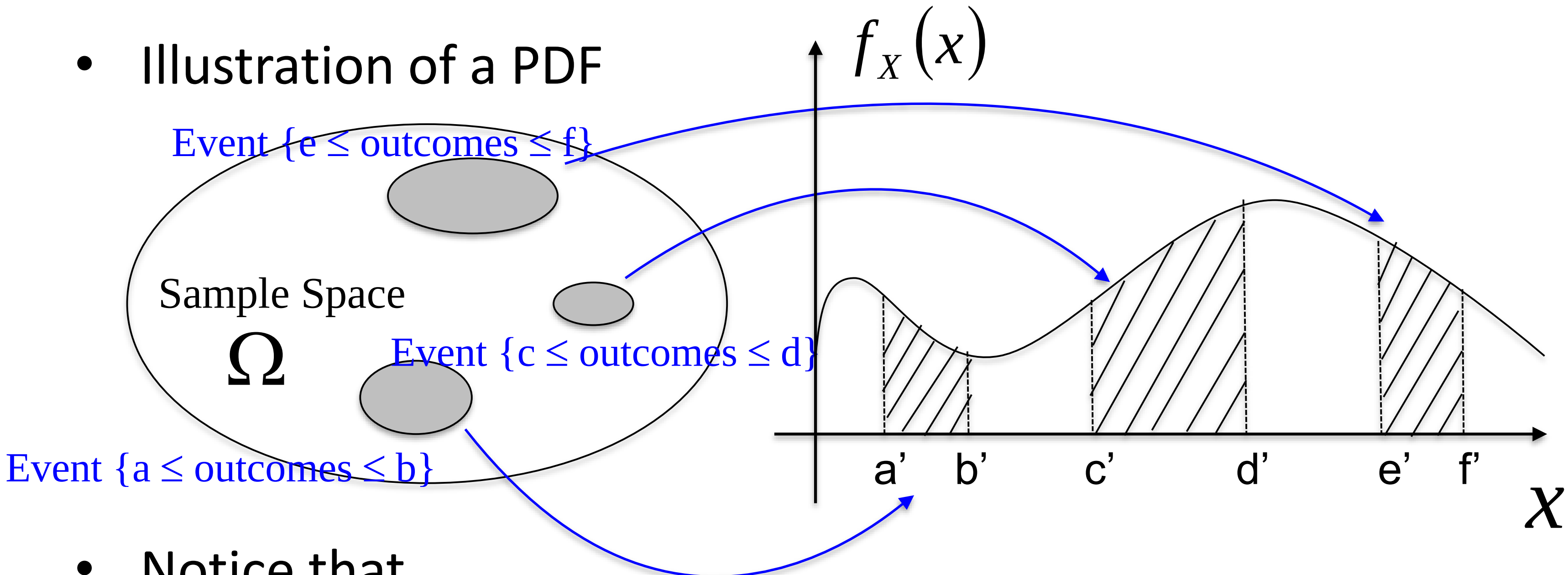
for every subset $B$ of the real line

– The probability that the value of $X$ falls within an interval is

$$\mathbf{P}\left(a \leq X \leq b\right) = \int_a^b f_X \, dx$$

# Probability Density Functions (2/2)

- Illustration of a PDF




- Notice that
  - For any single value $a$ , we have $\mathbf{P}(X = a) = \int_a^a f_X(x)dx = 0$
  - Including or excluding the endpoints of an interval has no effect on its probability
  $$\mathbf{P}(a \le X \le b) = \mathbf{P}(a < X \le b) = \mathbf{P}(a \le X < b) = \mathbf{P}(a < X < b)$$
  - Normalization probability
  $$\int_{-\infty}^{\infty} f_X(x)dx = \mathbf{P}(-\infty < X < \infty) = 1$$

# Interpretation of the PDF

- For an interval $[x, x+\delta]$ with very small length $\delta$, we have

$$P([x, x+\delta]) = \int_x^{x+\delta} f_X(t)\,dt \approx f_X(x) \cdot \delta$$

  - Therefore, $f_X(x)$ can be viewed as the "probability mass per unit length" near $x$

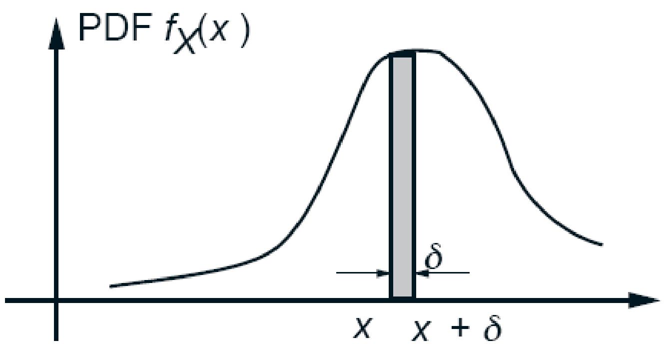


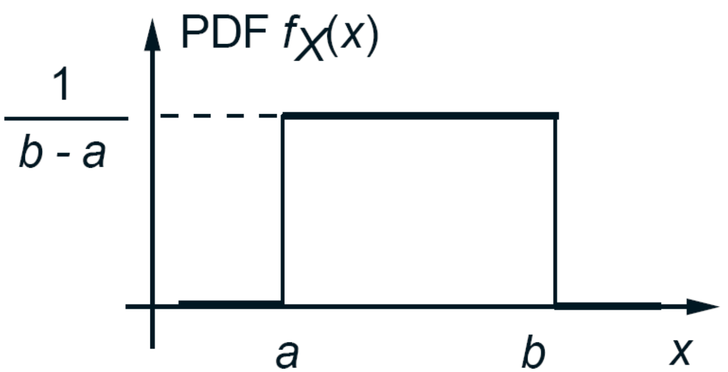

**Figure 3.2:** Interpretation of the PDF $f_X(x)$ as "probability mass per unit length" around $x$. If $\delta$ is very small, the probability that $X$ takes value in the interval $[x, x+\delta]$ is the shaded area in the figure, which is approximately equal to $f_X(x) \cdot \delta$.

- $f_X(x)$ is not the probability of any particular event, it is also not restricted to be less than or equal to one

# **Continuous Uniform** Random Variable

- A random variable $X$ is that takes values in an interval $[a\ , b]$ , and all subintervals of the same length are equally likely ( $X$ is **uniform** or **uniformly distributed**)

$$f_X(x)\begin{cases} \dfrac{1}{b-a}, & \text{if } a \le x \le b \\ 0, & \text{otherwise} \end{cases}$$




- Normalization property

$$\int_{-\infty}^{\infty} f_X(x)dx = \int_a^b \frac{1}{b-a}dx = 1$$

# Random Variable with **Piecewise Constant** PDF

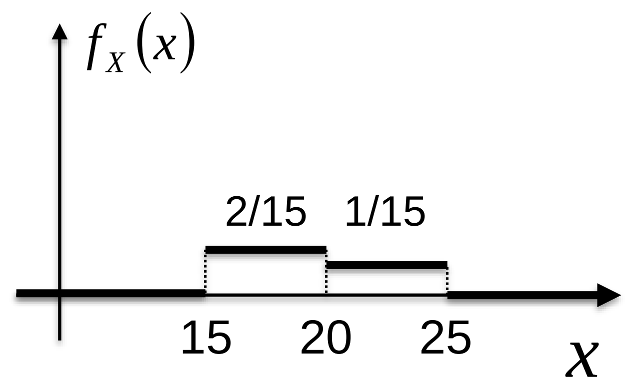
- **Example 3.2.** Alvin's driving time to work is between 15 and 20 minutes if the day is sunny, and between 20 and 25 minutes if the day is rainy, with all times being equally likely in each case. Assume that a day is sunny with probability 2/3 and rainy with probability 1/3. What is the PDF of driving time, viewed as a random variable $X$ ?

$$f_X(x) = \begin{cases} c_1, & \text{if } 15 \le x \le 20 \\ c_2, & \text{if } 20 \le x \le 25 \\ 0, & \text{otherwise.} \end{cases}$$

$$\mathbf{P}(\text{sunny day}) = \frac{2}{3} = \int_{15}^{20} f_X(x)dx = \int_{15}^{20} c_1 dx = 5 \cdot c_1$$

$$\mathbf{P}(\text{rainy day}) = \frac{1}{3} = \int_{20}^{25} f_X(x)dx = \int_{20}^{25} c_2 dx = 5 \cdot c_2$$

$$\therefore c_1 = \frac{2}{15}, \ c_2 = \frac{1}{15}$$



# Functions of A Continuous Random Variable

- If $X$ is a continuous random variable with given PDF, and real-valued function $Y = g(X)$ is also a random variable
  - $Y$ could be a continuous variable, e.g.:

$$y = g(x) = x^2$$

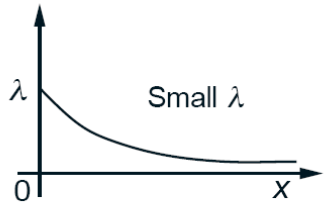
  - $Y$ could be a discrete variable, e.g.:

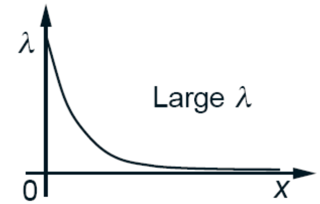
$$y = g(x) = \begin{cases} 1 & \text{for } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

# **Exponential** Random Variable

- An **exponential** random variable $X$ has a PDF of the form

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } x \geq 0, \\ 0, & \text{otherwise,} \end{cases}$$




  – $\lambda$ is a positive parameter characterizing the PDF
- Normalization Property

$$\int_{-\infty}^{\infty} f_X(x)dx = \int_0^{\infty} \lambda e^{-\lambda x} dx = -e^{-\lambda x}\Big|_0^{\infty} = 1$$

- The probability that $X$ exceeds a certain value decreases exponentially

$$\mathbf{P}(X \geq a) = \int_a^{\infty} \lambda e^{-\lambda x} dx = -e^{-\lambda x}\Big|_a^{\infty} = 0 - \left(-e^{-\lambda a}\right) = e^{-\lambda a}$$

An exponential random variable can be a good model for the amount of time until an incident of interest takes place.

# An Example: **Exponential** PDF

- Suppose that a system contains a certain type of component whose time, in years, to failure is given by *T*. The random variable is *T* modeled nicely by the exponential distribution with mean time to failure equal to 5 . If 5 of theses components are installed in different systems, what is the probability that at least 2 are still functioning at the end of 8 years? (假設某系統包含某種類型的元件，該類型元件在故障前使用的時間 *T* 以年作為計算單位，該連續隨機變數 *T* 為指數函數，該類型元件平均故障時間為 5 年，如果這些元件有5個分別安裝在不同的系統中，在 8 年結束時至少還有 2 個仍在運行的機率是多少?)

$$\mathbf{P}(X \geq a) = \int_a^\infty \lambda e^{-\lambda x} dx = -e^{-\lambda x}\Big|_a^\infty = 0 - \left(-e^{-\lambda a}\right) = e^{-\lambda a}$$

  – Solution:
    - $\lambda = 1/\mu_t \rightarrow \lambda = 1/5$

$$\mathbf{P}(T > 8) = \int_8^\infty \lambda e^{\frac{-t}{5}} dt = -e^{\frac{-t}{5}}\Big|_8^\infty = 0 - \left(-e^{\frac{-8}{5}}\right) = e^{\frac{-8}{5}} \approx 0.2$$

$$\mathbf{P}(X \geq 2) = \sum_{x=2}^{5} \binom{n}{x} 0.2^x (1-0.2)^{5-x} = 1 - \sum_{x=0}^{1} \binom{n}{x} 0.2^x (1-0.2)^{5-x} = 1 - 0.7373 = 0.2627$$

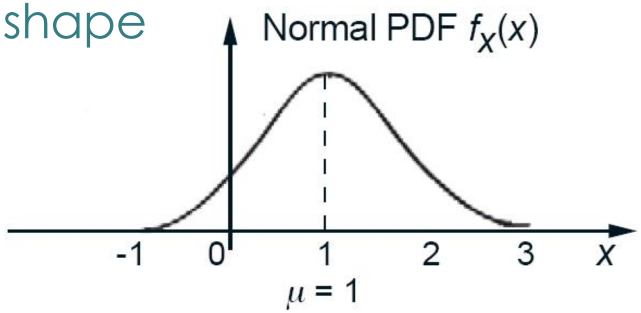
- the **average #** of outcomes **per unit time** (arrival rate)
- Also could be per unit length, area, or volume, rather than time

# **Normal** (or **Gaussian**) Random Variable

- An continuous random variable $X$ is said to be **normal** (or **Gaussian**) if it has a PDF of the form

bell shape

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad -\infty \le x \le \infty$$




  – Where the parameters $\mu$ and $\sigma^2$ are respectively its mean and variance (to be shown latter on!)

- Normalization Property

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = 1$$

(?? See the end of chapter problems)

# Normality is Preserved by Linear Transformations

- If $X$ is a random variable with mean $\mu$ and variance $\sigma^2$, and if $a$ $(a \neq 0)$ and $b$ are scalars, then the random variable

$$Y = aX + b$$

a linear function of $X$

is also normal with mean and variance

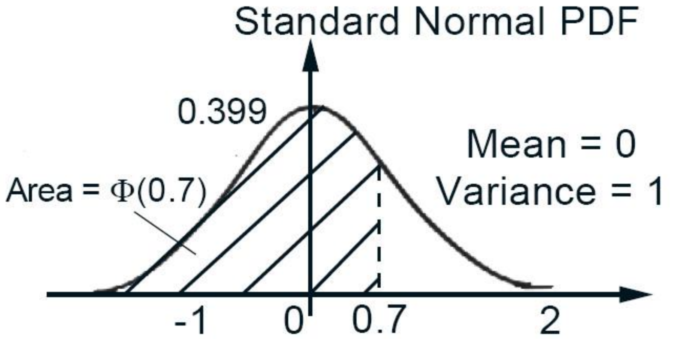
$$\mathbf{E}[Y] = a\mu + b$$

$$\operatorname{var}(Y) = a^2\sigma^2$$

# **Standard Normal** Random Variable

- An random variable $Y$ with zero mean $\mu = 0$ and unit variance $\sigma^2 = 1$ is said to be a **standard normal**

$$f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}, \quad -\infty \le y \le \infty$$




- Normalization Property

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy = 1$$

- The standard normal is symmetric around $y = 0$

# The PDF of a Random Variable Can be Arbitrarily Large

- **Example 3.3. A PDF can be arbitrarily large.** Consider a random variable $X$ with PDF

$$f_X(x) = \begin{cases} \dfrac{1}{2\sqrt{x}}, & \text{if } 0 < x \le 1, \\ 0, & \text{otherwise,} \end{cases}$$

  - The PDF value becomes infinite large as $x$ approaches zero

- Normalization Property

$$\int_0^1 f_X(x)dx = \int_0^1 \frac{1}{2\sqrt{x}}\, dx = \sqrt{x}\Big|_0^1 = 1$$

# Expectation of a Continuous Random Variable (1/2)

- Let $X$ be a continuous random variable with PDF $f_X$
  - The expectation of $X$ is defined by

$$\mathbf{E}[X] = \int_{-\infty}^{\infty} x \cdot f_X(x)dx$$

  - The expectation of a function $g(X)$ has the form

$$\mathbf{E}[g(X)] = \int_{-\infty}^{\infty} g(x) \cdot f_X(x)dx$$

(?? See the end of chapter problems)

  - The variance of $X$ is defined by

$$\text{var}(X) = \mathbf{E}\left[(X - \mathbf{E}[X])^2\right] = \int_{-\infty}^{\infty} (X - \mathbf{E}[X])^2 \cdot f_X(x)dx$$

    - We also have

$$\text{var}(X) = \mathbf{E}\left[X^2\right] - (\mathbf{E}[X])^2 \geq 0$$

# Expectation of a Continuous Random Variable (2/2)

- If $Y = aX + b$ , where $a$ and $b$ are given scalars, then

$$\mathbf{E}[Y] = a\mathbf{E}[X] + b$$

$$\mathrm{var}(Y) = a^2 \,\mathrm{var}(X)$$

# Illustrative Examples (1/3)

- Mean and Variance of Uniform random variable $X$

$$f_X(x)\begin{cases} \dfrac{1}{b-a}, & \text{if } a \le x \le b \\ 0, & \text{otherwise} \end{cases}$$

$$\mathbf{E}[X] = \int_a^b x \cdot f_X(x)dx = \int_a^b x \cdot \frac{1}{b-a}dx$$
$$= \frac{1}{b-a} \cdot \frac{1}{2} x^2 \Big|_a^b$$
$$= \frac{b+a}{2}$$

$$\mathbf{E}[X^2] = \int_a^b x^2 \cdot f_X(x)dx$$
$$= \frac{1}{b-a} \cdot \frac{1}{3} x^3 \Big|_a^b$$
$$= \frac{b^2+ab+a^2}{3}$$

$$\therefore \operatorname{var}(X) = \mathbf{E}[X^2] - (\mathbf{E}[X])^2 = \frac{b^2+ab+a^2}{3} - \left(\frac{b+a}{2}\right)^2$$
$$= \frac{(b-a)^2}{12}$$

# Illustrative Examples (2/3)

- Mean and Variance of the **Exponential** Random Variable $X$

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } x \geq 0, \\ 0, & \text{otherwise,} \end{cases}$$

$$\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$$

$$\begin{aligned} \mathbf{E}[X] &= \int_0^\infty x \cdot f_X(x) dx = \int_0^\infty x \cdot \lambda e^{-\lambda x} dx \\ &= -xe^{-\lambda x}\Big|_0^\infty + \int_0^\infty e^{-\lambda x} dx \\ &= 0 - \frac{1}{\lambda} e^{-\lambda x}\Big|_0^\infty = \frac{1}{\lambda} \end{aligned}$$

$$\left( \because \frac{d\left(-xe^{-\lambda x}\right)}{dx} = \lambda x e^{-\lambda x} - e^{-\lambda x} \right)$$

Integration by parts

$$\begin{aligned} \mathbf{E}\left[X^2\right] &== \int_0^\infty x^2 \cdot \lambda e^{-\lambda x} dx \\ &= -x^2 e^{-\lambda x}\Big|_0^\infty + \int_0^\infty 2xe^{-\lambda x} dx \\ &= 0 + \boxed{\frac{1}{\lambda}} \left( \int_0^\infty 2x \cdot \boxed{\lambda} e^{-\lambda x} dx \right) = \frac{2}{\lambda} \mathbf{E}[X] = \frac{2}{\lambda^2} \end{aligned}$$

$$\left( \because \frac{d\left(-x^2 e^{-\lambda x}\right)}{dx} = x^2 \lambda e^{-\lambda x} - 2xe^{-\lambda x} \right)$$

$$\therefore \operatorname{var}(X) = \mathbf{E}\left[X^2\right] - \left(\mathbf{E}[X]\right)^2 = \frac{1}{\lambda^2}$$

Integration by parts, c.f. http://en.wikipedia.org/wiki/Integration_by_parts

# Illustrative Examples (3/3)

- Mean and Variance of the **Normal** Random Variable $X$

$$f_X(x)=\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(x-\mu)^2}{2\sigma^2}},\quad -\infty\le x\le\infty$$

? (see Sec. 3.6)

$$\text{Let } Y=\frac{X-\mu}{\sigma}\Rightarrow f_Y(y)=\frac{1}{\sqrt{2\pi}}e^{\frac{-y^2}{2}},\quad -\infty\le y\le\infty$$

$$\mathbf{E}[Y]=\int_{-\infty}^{\infty}y\cdot\frac{1}{\sqrt{2\pi}}e^{\frac{-y^2}{2}}dy=-\frac{1}{\sqrt{2\pi}}e^{\frac{-y^2}{2}}\Big|_{-\infty}^{\infty}=0$$

$$\left(\because\frac{d\left(-ye^{\frac{-y^2}{2}}\right)}{dy}=y^2e^{\frac{-y^2}{2}}-e^{\frac{-y^2}{2}}\right)$$

$$\Rightarrow\mathbf{E}[X]=\sigma\mathbf{E}[Y]+\mu=0+\mu=\mu$$

$$\text{var}[Y]=\int_{-\infty}^{\infty}(y-\mathbf{E}[Y])^2\cdot\frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}dy$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}y^2e^{\frac{-y^2}{2}}dy=\left[\frac{1}{\sqrt{2\pi}}\cdot -ye^{\frac{-y^2}{2}}\Big|_{-\infty}^{\infty}\right]+\underbrace{\left[\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{\frac{-y^2}{2}}dy\right]}_{1}$$

$$=0+1$$

$$=1\qquad \because\text{var}(Y)=\frac{\text{var}(X)}{\sigma^2}\ \therefore\text{var}(X)=\sigma^2$$

# Cumulative Distribution Functions

- The cumulative distribution function (CDF) of a random variable $X$ is denoted by $F_X(x)$ and provides the probability $\mathbf{P}(X \le x)$

$$F_X(x) = \mathbf{P}(X \le x) = \begin{cases} \sum_{k \le x} p_X(k), & \text{if } X \text{ is discrete} \\ \int_{-\infty}^{x} f_X(t)\,dt, & \text{if } X \text{ is continuous} \end{cases}$$

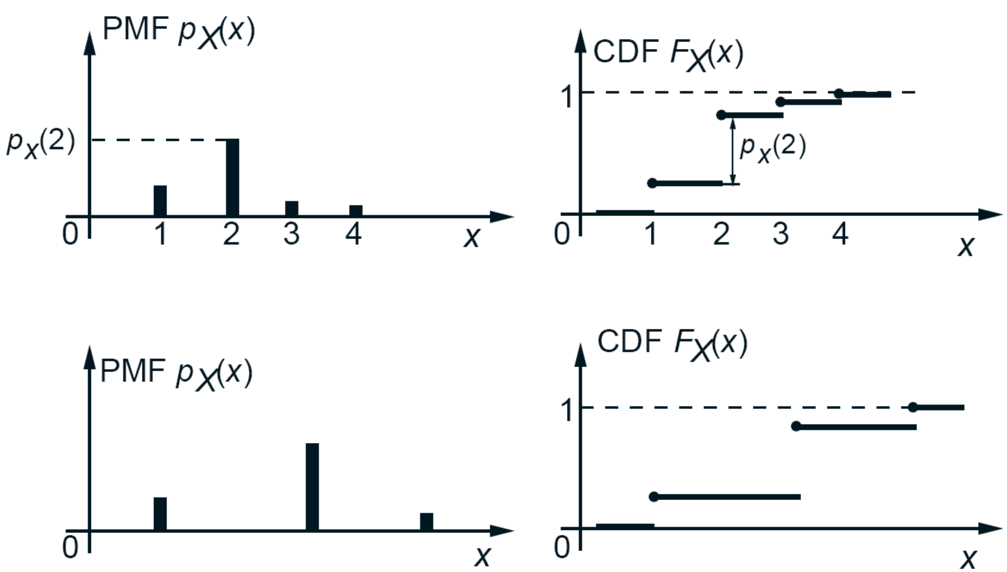
  - The CDF $F_X(x)$ accumulates probability up to $x$
  - The CDF $F_X(x)$ provides a unified way to describe all kinds of random variables mathematically

# Properties of a CDF (1/3)

- The CDF $F_X(x)$ is monotonically non-decreasing

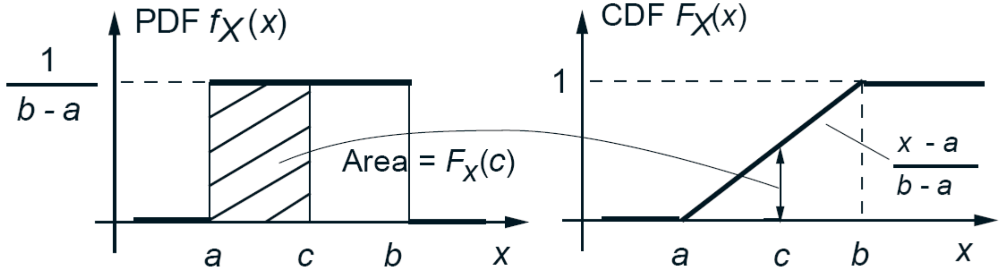
$$\text{if } x_i \le x_j \text{ , then } F_X(x_i) \le F_X(x_j)$$

- The CDF $F_X(x)$ tends to 0 as $x \to -\infty$ , and to 1 as $x \to \infty$
- If $X$ is discrete, then $F_X(x)$ is a piecewise constant function of $x$



# Properties of a CDF (2/3)

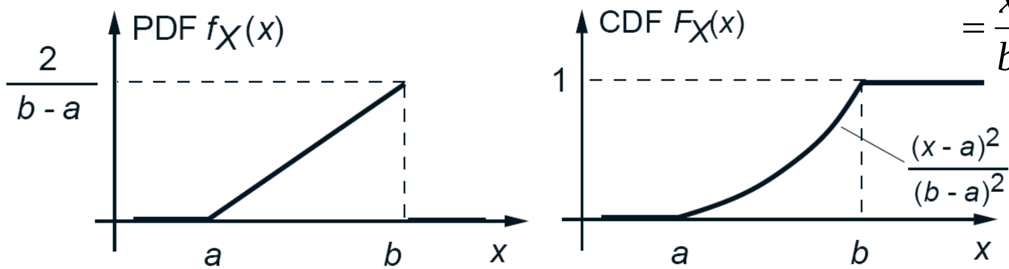
- If $X$ is continuous, then $F_X(x)$ is a continuous function of $x$




$$f_X(x)=\frac{1}{b-a},\quad for\ a\le x\le b$$

$$F_X(X\le x)=\int_a^x f_X(t)dt=\int_a^x \frac{1}{b-a}dt$$

$$=\frac{x-a}{b-a}$$




$$f_X(x)=c(x-a),\quad for\ a\le x\le b$$

$$\Rightarrow \int_a^b c(x-a)dx=\frac{c}{2}(x-a)^2\Big|_a^b=1$$

$$\Rightarrow c=\frac{2}{(b-a)^2}$$

$$\Rightarrow f_X(b)=\frac{2(b-a)}{(b-a)^2}=\frac{2}{b-a}$$

$$F_X(X\le x)=\int_a^x f_X(t)dt=\int_a^x \frac{2(t-a)}{(b-a)^2}dt$$

$$=\frac{(x-a)^2}{(b-a)^2}$$

# Properties of a CDF (3/3)

- If $X$ is discrete and takes integer values, the PMF and the CDF can be obtained from each other by summing or differencing

$$F_X(k) = \mathbf{P}(X \le k) = \sum_{i=-\infty}^{k} p_X(i),$$

$$p_X(k) = \mathbf{P}(X \le k) - \mathbf{P}(X \le k-1) = F_X(k) - F_X(k-1)$$

- If $X$ is continuous, the PDF and the CDF can be obtained from each other by integration or differentiation

$$F_X(x) = \mathbf{P}(X \le x) = \int_{-\infty}^{x} f_X(t)\,dt,$$

$$f_X(x) = \frac{dF_X(x)}{dx}$$

  - The second equality is valid for those $x$ for which the CDF has a derivative (or at which PDF is continuous)

# An Illustrative Example (1/2)

- **Example 3.6. The Maximum of Several Random Variables.** You are allowed to take a certain test three times, and your final score will be the maximum of the test scores. Thus,

$$X = \max\{X_1, X_2, X_3\}$$

A function of discrete random variables

Where $X_1$, $X_2$, $X_3$ are the tree test scores and $X$ is the final score

- Assume that your score in each test takes one of the values from 1 to 10 with equal probability 1/10, independently of the scores in other tests.
- What is the PMF $p_X$ of the final score?

Trick: compute first the CDF and then the PMF!

# An Illustrative Example (2/2)

$$
\begin{aligned}
\because F_X(k) &= \mathbf{P}(X \le k) \\
&= \mathbf{P}(X_1 \le k, X_2 \le k, X_3 \le k) \\
&= \mathbf{P}(X_1 \le k)\mathbf{P}(X_2 \le k)\mathbf{P}(X_3 \le k) \\
&= \left(\frac{k}{10}\right)^3
\end{aligned}
$$

$$
\therefore p_X(k) = \mathbf{P}(X \le k) - \mathbf{P}(X \le k-1) = \left(\frac{k}{10}\right)^3 - \left(\frac{k-1}{10}\right)^3
$$

# Another Example on Continuous RV

- Suppose that the error in the reaction temperature, in °C (實驗反應溫度), for a controlled laboratory experiment is a continuous random variable X having the probability density function

$$f_X(x) = \begin{cases} \dfrac{x^2}{3}, & \text{if } -1 < x < 2, \\ 0, & \text{otherwise,} \end{cases}$$

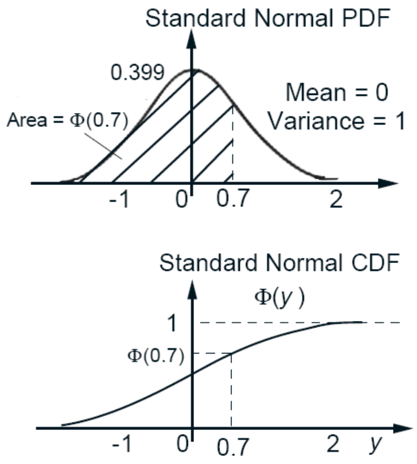
- Problems:
  - a. Find $\mathbf{P}(0 < X \le 1)$ .
  - b. Find the $F(x)$, and use it to evaluate $\mathbf{P}(0 < X \le 1)$
  - Solution:
    - a. $\mathbf{P}(0 < x \le 1) = \int_0^1 f_X(x)dx = \int_0^1 \frac{x^2}{3}dx = \left.\frac{x^3}{9}\right|_0^1 = \frac{1}{9}$
    - b. $F_X(x) = \mathbf{P}(X \le x) = \int_{-\infty}^{x} f_X(t)dt = \left.\frac{t^3}{9}\right|_{-1}^{x} = \frac{x^3+1}{9}$

$$F_X(x) = \begin{cases} 0, & x < -1 \\ \dfrac{x^3+1}{9}, & -1 \le x < 2, \\ 1, & x \ge 2. \end{cases}$$

$$\mathbf{P}(0 < x \le 1) = F(1) - F(0) = \frac{2}{9} - \frac{1}{9} = \frac{1}{9}$$

# CDF of the **Standard Normal**

- The CDF of the standard normal $Y$, denoted as $\Phi(y)$, is recorded in a table and is very useful tool for calculating various probabilities, including normal variables




$$\Phi(y) = \mathbf{P}(Y \le y) = \mathbf{P}(Y < y) = \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt$$

  - The table only provides the value of $\Phi(y)$ for $y \ge 0$
  - Because the symmetry of the PDF, the CDF at negative values of $Y$ can be computed from corresponding positive ones

$$\begin{aligned}\Phi(-0.5) &= \mathbf{P}(Y \le -0.5) = 1 - \mathbf{P}(Y \le 0.5) \\ &= 1 - \Phi(0.5) = 1 - 0.6915 \\ &= 0.3085\end{aligned}$$

$\Phi(-y) = 1 - \Phi(y)$, for all $y$

# Table of the CDF of **Standard Normal**

| z | 0.00 | 0.01 | 0.02 | 0.03 | 0.04 | 0.05 | 0.06 | 0.07 | 0.08 | 0.09 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.0 | 0.5000 | 0.5040 | 0.5080 | 0.5120 | 0.5159 | 0.5199 | 0.5239 | 0.5279 | 0.5319 | 0.5359 |
| 0.1 | 0.5398 | 0.5438 | 0.5478 | 0.5517 | 0.5557 | 0.5596 | 0.5636 | 0.5675 | 0.5714 | 0.5753 |
| 0.2 | 0.5793 | 0.5832 | 0.5871 | 0.5910 | 0.5948 | 0.5987 | 0.6026 | 0.6064 | 0.6103 | 0.6141 |
| 0.3 | 0.6179 | 0.6217 | 0.6255 | 0.6293 | 0.6331 | 0.6368 | 0.6406 | 0.6443 | 0.6480 | 0.6517 |
| 0.4 | 0.6554 | 0.6591 | 0.6628 | 0.6664 | 0.6700 | 0.6736 | 0.6772 | 0.6808 | 0.6844 | 0.6879 |
| 0.5 | 0.6915 | 0.6950 | 0.6985 | 0.7019 | 0.7054 | 0.7088 | 0.7123 | 0.7157 | 0.7190 | 0.7224 |
| 0.6 | 0.7257 | 0.7291 | 0.7324 | 0.7357 | 0.7389 | 0.7422 | 0.7454 | 0.7486 | 0.7517 | 0.7549 |
| 0.7 | 0.7580 | 0.7611 | 0.7642 | 0.7673 | 0.7704 | 0.7734 | 0.7764 | 0.7794 | 0.7823 | 0.7854 |
| 0.8 | 0.7881 | 0.7910 | 0.7939 | 0.7967 | 0.7995 | 0.8023 | 0.8051 | 0.8078 | 0.8106 | 0.8133 |
| 0.9 | 0.8159 | 0.8186 | 0.8212 | 0.8238 | 0.8264 | 0.8289 | 0.8315 | 0.8340 | 0.8365 | 0.8389 |
| 1.0 | 0.8413 | 0.8438 | 0.8461 | 0.8485 | 0.8508 | 0.8531 | 0.8554 | 0.8577 | 0.8599 | 0.8621 |
| 1.1 | 0.8643 | 0.8665 | 0.8686 | 0.8708 | 0.8729 | 0.8749 | 0.8770 | 0.8790 | 0.8804 | 0.8830 |
| 1.2 | 0.8849 | 0.8869 | 0.8888 | 0.8907 | 0.8925 | 0.8944 | 0.8962 | 0.8980 | 0.8997 | 0.9015 |
| 1.3 | 0.9032 | 0.9049 | 0.9066 | 0.9082 | 0.9099 | 0.9115 | 0.9131 | 0.9147 | 0.9162 | 0.9177 |
| 1.4 | 0.9192 | 0.9207 | 0.9222 | 0.9236 | 0.9251 | 0.9265 | 0.9279 | 0.9292 | 0.9306 | 0.9319 |
| 1.5 | 0.9332 | 0.9345 | 0.9357 | 0.9370 | 0.9382 | 0.9394 | 0.9406 | 0.9418 | 0.9429 | 0.9441 |
| 1.6 | 0.9452 | 0.9463 | 0.9474 | 0.9484 | 0.9495 | 0.9505 | 0.9515 | 0.9525 | 0.9535 | 0.9545 |
| 1.7 | 0.9554 | 0.9564 | 0.9573 | 0.9582 | 0.9591 | 0.9599 | 0.9608 | 0.9616 | 0.9625 | 0.9633 |
| 1.8 | 0.9641 | 0.9649 | 0.9656 | 0.9664 | 0.9671 | 0.9678 | 0.9686 | 0.9693 | 0.9699 | 0.9706 |
| 1.9 | 0.9713 | 0.9719 | 0.9726 | 0.9732 | 0.9738 | 0.9744 | 0.9750 | 0.9756 | 0.9761 | 0.9767 |
| 2.0 | 0.9773 | 0.9778 | 0.9783 | 0.9788 | 0.9793 | 0.9798 | 0.9803 | 0.9808 | 0.9812 | 0.9817 |
| 2.1 | 0.9821 | 0.9826 | 0.9830 | 0.9834 | 0.9838 | 0.9842 | 0.9846 | 0.9850 | 0.9854 | 0.9857 |
| 2.2 | 0.9861 | 0.9865 | 0.9868 | 0.9871 | 0.9874 | 0.9878 | 0.9881 | 0.9884 | 0.9887 | 0.9890 |
| 2.3 | 0.9893 | 0.9896 | 0.9898 | 0.9901 | 0.9904 | 0.9906 | 0.9909 | 0.9911 | 0.9913 | 0.9916 |
| 2.4 | 0.9918 | 0.9920 | 0.9922 | 0.9924 | 0.9927 | 0.9929 | 0.9931 | 0.9932 | 0.9934 | 0.9936 |
| 2.5 | 0.9938 | 0.9940 | 0.9941 | 0.9943 | 0.9945 | 0.9946 | 0.9948 | 0.9949 | 0.9951 | 0.9952 |
| 2.6 | 0.9953 | 0.9955 | 0.9956 | 0.9957 | 0.9959 | 0.9960 | 0.9961 | 0.9962 | 0.9963 | 0.9964 |
| 2.7 | 0.9965 | 0.9966 | 0.9967 | 0.9968 | 0.9969 | 0.9970 | 0.9971 | 0.9972 | 0.9973 | 0.9974 |
| 2.8 | 0.9974 | 0.9975 | 0.9976 | 0.9977 | 0.9977 | 0.9978 | 0.9979 | 0.9980 | 0.9980 | 0.9981 |
| 2.9 | 0.9981 | 0.9982 | 0.9982 | 0.9983 | 0.9984 | 0.9984 | 0.9985 | 0.9985 | 0.9986 | 0.9986 |

# CDF Calculation of the **Normal**

- The CDF of a normal random variable $X$ with mean $\mu$ and variance $\sigma^2$ is obtained using the standard normal table as

$$\mathbf{P}(X \le x) = \mathbf{P}\left(\frac{X-\mu}{\sigma} \le \frac{x-\mu}{\sigma}\right) = \mathbf{P}\left(Y \le \frac{x-\mu}{\sigma}\right) = \Phi\left(\frac{x-\mu}{\sigma}\right)$$

$$\left(\begin{array}{l} \text{Let } Y = \dfrac{x-\mu}{\sigma}. \text{ Since } X \text{ is normal and } Y \text{ is a linear function of } X, \\ Y \text{ hence is also normal (with mean 0 and variance 1).} \\ \mathbf{E}[Y] = \dfrac{\mathbf{E}[X]-\mu}{\sigma} = 0, \ \mathrm{var}(Y) = \dfrac{\mathrm{var}(X)}{\sigma^2} = 1 \end{array}\right)$$

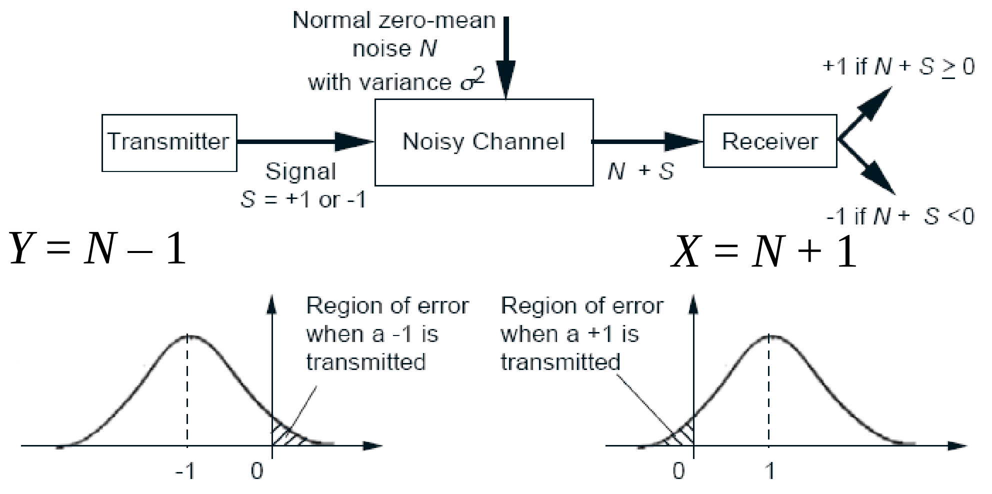
# Illustrative Examples (1/3)

- **Example 3.7. Using the Normal Table.** The annual snowfall at a particular geographic location is modeled as a normal random variable with a mean of $\mu = 60$ inches, and a standard deviation of $\sigma = 20$. What is the probability that this year's snowfall will be least 80 inches?

$$
\begin{aligned}
\mathbf{P}(X \geq 80) &= 1 - \mathbf{P}(X \leq 80) \\
&= 1 - \mathbf{P}\left(Y \leq \frac{80-60}{20}\right) \\
&= 1 - \Phi(1) \\
&= 1 - 0.8413 \\
&= 0.1587
\end{aligned}
$$

# Illustrative Examples (2/3)

- **Example 3.8. Signal Detection.**
  - A binary message is transmitted as a signal that is either –1 or +1. The communication channel corrupts the transmission with additive normal noise with a mean $\mu = 0$ and variance $\sigma^2 = 1$. The receiver concludes that the signal –1 (or +1) was transmitted if the value received is < 0 (or $\geq$ 0, respectively).
  - What is the probability of error?



# Illustrative Examples (3/3)

- Probability of error when sending signal –1

$$P(Y \geq 0) = P(N - 1 \geq 0) = P(N \geq 1)$$

mean of *N*

$$= P\left(\frac{N - 0}{\sigma} \geq \frac{1}{\sigma}\right) = 1 - \Phi\left(\frac{1}{\sigma}\right)$$

Standard deviation of *N*

- Probability of error when sending signal +1

$$P(X < 0) = P(N + 1 < 0) = P(N < -1)$$

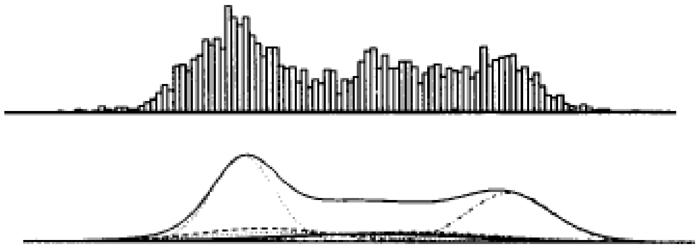
$$= P\left(\frac{N - 0}{\sigma} < \frac{-1}{\sigma}\right) = \Phi\left(-\frac{1}{\sigma}\right) = 1 - \Phi\left(\frac{1}{\sigma}\right)$$

# More Factors about Normal

- The Normal random variable plays an important role in a broad range of probabilistic models
  - It models well the additive effect of many independent factors, in a variety of engineering, physical, and statistical contexts
    - The sum of a large number of independent and identically distributed (not necessarily normal) random variables has an approximately normal CDF, regardless of the CDF of the individual random variables (See Chapter 7)

$$W = X_1 + X_2 + \cdots + X_n \; (X_1, X_2, \cdots, X_n \text{ are i.i.d})$$

  - We can approximate any probability distribution (the PDF of a random variable) with the linear combination of an enough number of normal distributions



$$f_Y(y) = \alpha_1 f_{X_1}(y) + \alpha_2 f_{X_2}(y) + \cdots + \alpha_K f_{X_K}(y)$$

$$(X_1, X_2, \cdots, X_K \text{ are normal}, \textstyle\sum_{k=1}^{K} \alpha_k = 1)$$

# Relation between the **Geometric** and **Exponential** (1/2)

- The CDF of the geometric

$$F_{geo}(n) = \sum_{k=1}^{n} (1-p)^{k-1} p = p \frac{1-(1-p)^n}{1-(1-p)} = 1-(1-p)^n$$

$$\text{for } n = 1, 2, \ldots$$

- The CDF of the exponential

$$F_{\exp}(x) = \int_0^x \lambda e^{-\lambda x} dx = -e^{-\lambda x}\Big|_0^x = 1 - e^{-\lambda x}$$

$$\text{for } x > 0$$

- Compare the above two CDFs and let

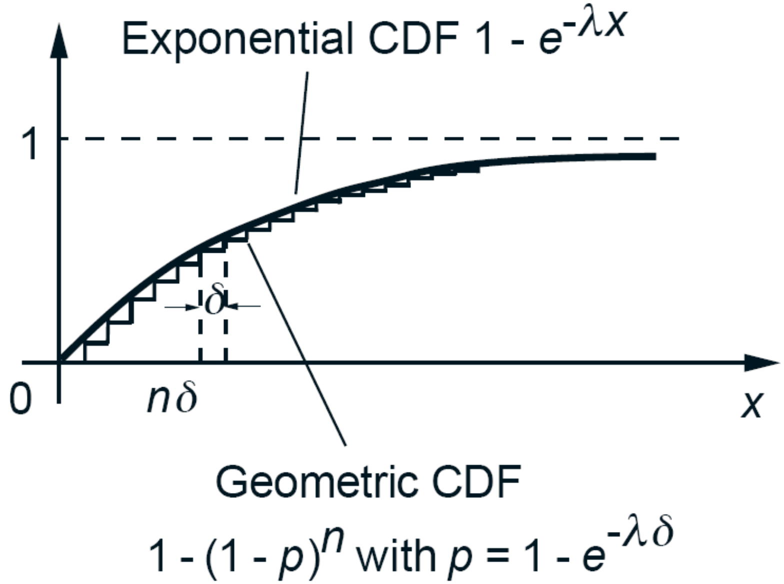
$$e^{-\lambda x} = (1-p)^n$$

$$\Rightarrow x = n \cdot \frac{-1}{\lambda} \ln(1-p) \quad \left( \text{let } \delta = \frac{-1}{\lambda} \ln(1-p) > 0 \right)$$

$$\Rightarrow x = n \cdot \delta \quad \left( \therefore 1-p = e^{-\lambda\delta} \text{ or } p = 1 - e^{-\lambda\delta} \right)$$

# Relation between the **Geometric** and **Exponential** (2/2)

$$\because F_{\exp}(\delta n) = 1 - e^{-\lambda \delta n} = 1 - (1 - p)^n = F_{geo}(n)$$



# Recitation

- SECTION 3.1 Continuous Random Variables and PDFs
  - Problems 2, 3, 4
- SECTION 3.2 Cumulative Distribution Functions
  - Problems 6, 7, 8
- SECTION 3.3 Normal Random Variables
  - Problems 9, 10, 12