Discrete Random Variables: Basics

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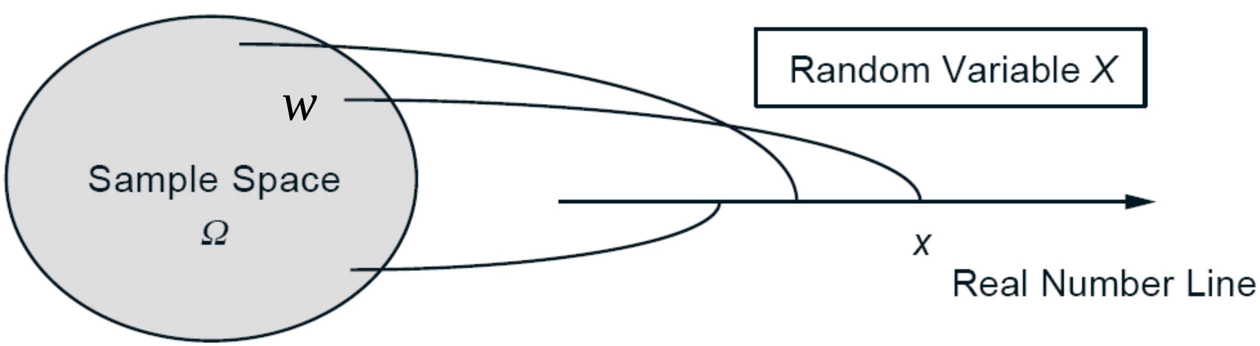
References:
- Dimitri P. Bertsekas and John N. Tsitsiklis, *Introduction to Probability*, Sections 2.1-2.3
- Walpole R. E., Myers R. H., Myers S. L. and Ye K., *Probability & Statistics for Engineers & Scientists*, Ch. 5
- Slides are credited from Prof. Berlin Chen, NTNU.

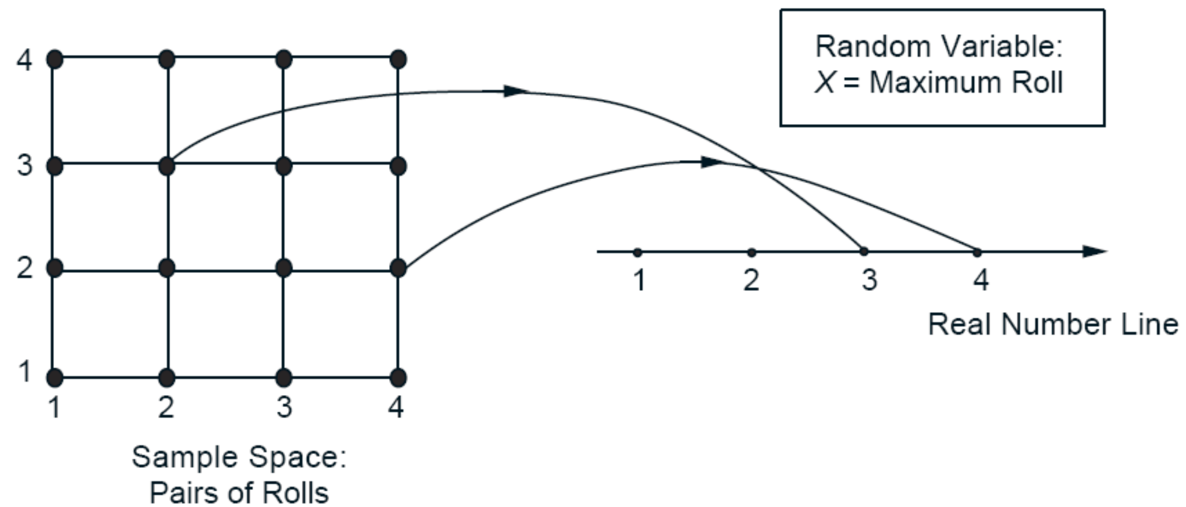
Random Variables

- Given an experiment and the corresponding set of possible outcomes (the sample space), a random variable associates a particular number with each outcome
  - This number is referred to as the (numerical) value of the random variable
  - We can say a random variable is a real-valued function of the experimental outcome



Random Variables: Example

- An experiment consists of two rolls of a 4-sided die, and the random variable is the maximum of the two rolls
  - If the outcome of the experiment is (4, 2), the value of this random variable is 4
  - If the outcome of the experiment is (3, 3), the value of this random variable is 3



  - Can be one-to-one or many-to-one mapping

Main Concepts Related to Random Variables

- For a probabilistic model of an experiment
  - A random variable is **a real-valued function** of the outcome of the experiment

$$X : w \rightarrow x$$

  - A **function of a random variable** defines another random variable

$$Y = g(x)$$

  - We can associate with each random variable certain "averages" of interest such the **mean** and the **variance**
  - A random variable can be **conditioned** on an event or on another random variable
  - There is a notion of **independence** of a random variable from an event or from another random variable

Discrete/Continuous Random Variables

- A random variable is called **discrete** if its **range** (the set of values that it can take) is finite or at most countably infinite

$$\text{finite: } \{1,2,3,4\}, \text{ countably infinite : } \{1,2,\cdots\}$$

- A random variable is called **continuous** (not discrete) if its **range** (the set of values that it can take) is uncountably infinite
  - E.g., the experiment of choosing a point a from the interval [−1, 1]
    - A random variable that associates the numerical value a^2 to the outcome a is not discrete

- In this chapter, we focus exclusively on discrete random variables

Concepts Related to Discrete Random Variables

- For a probabilistic model of an experiment
  - A **discrete random variable** is a real-valued function of the outcome of the experiment that can take a finite or countably infinite number of values
  - A (discrete) random variable has an associated **probability mass function** (PMF), which gives the probability of each numerical value that the random variable can take
  - **A function of a random variable** defines another random variable, whose PMF can be obtained from the PMF of the original random variable

Discrete/Continuous Random Variables

- A (discrete) random variable X is characterized through the probabilities of the values that it can take, which is captured by the probability mass function (PMF) of X, denoted $p_X(x)$

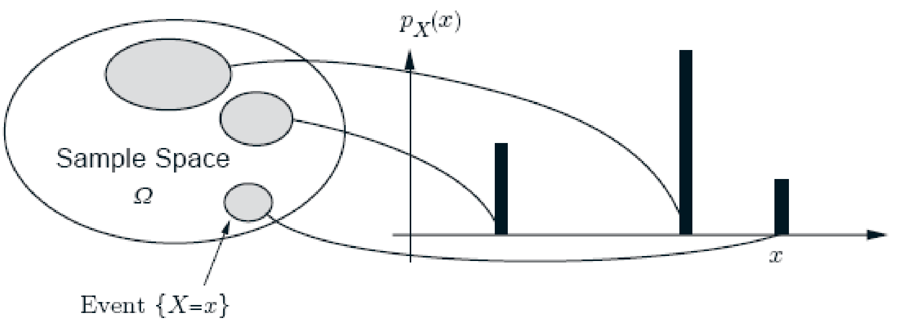
$$p_X(x) = \mathbf{P}(\{X = x\}) \text{ or } p_X(x) = \mathbf{P}(X = x)$$

  - The sum of probabilities of all outcomes that give rise to a value of X equal to x
  - **Upper case** characters (e.g., X) denote random variables, while **lower case** ones (e.g., x) denote the numerical values of a random variable
- The summation of the outputs of the PMF function of a random variable over all it possible numerical values is equal to one $\sum_x p_X(x) = 1$ $\{X = x\}'s$ are disjoint and form a partition of the sample space

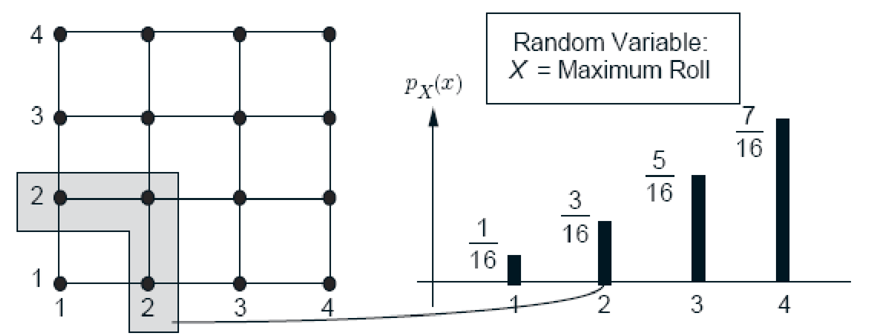
Calculation of the PMF

- For each possible value x of a random variable X :
  1. Collect all the possible outcomes that give rise to the event $\{X = x\}$
  2. Add their probabilities to obtain $p_X(x)$

- An example: the PMF $p_X(x)$ of the random variable X = maximum roll in two independent rolls of a fair 4-sided die



(a)



(b)

Bernoulli Random Variables

- A Bernoulli random variable X takes two values 1 and 0 with probabilities p and $1-p$, respectively
  - **PMF**

$$p_X(x) = \begin{cases} p, & \text{if } x = 1 \\ 1-p, & \text{if } x = 0 \end{cases}$$

- The Bernoulli random variable is often used to model generic probabilistic situations with just two outcomes
  1. The toss of a coin (outcomes: head and tail)
  2. A trial (outcomes: success and failure)
  3. the state of a telephone (outcomes: free and busy)

  …

Binomial Random Variables (1/2)

- A binomial random variable X has parameters n and p
  - PMF

$$p_X(x) = \mathbf{P}(X = k) = \binom{n}{k} p^k (1-p)^{n-k}, \quad k = 0, 1, \ldots, n$$

- The Binomial random variable can be used to model, e.g.
  1. The number of heads in *n* independent tosses of a coin (outcomes: 1, 2, …,*n*), each toss has probability p to be a head
  2. The number of successes in *n* independent trials (outcomes: 1, 2, …,*n*), each trial has probability p to be successful

- Normalization Property

Note taht : $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$

$$\sum_{k=0}^{n} p_X(x) = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} = 1$$

Binomial Random Variables (2/2)

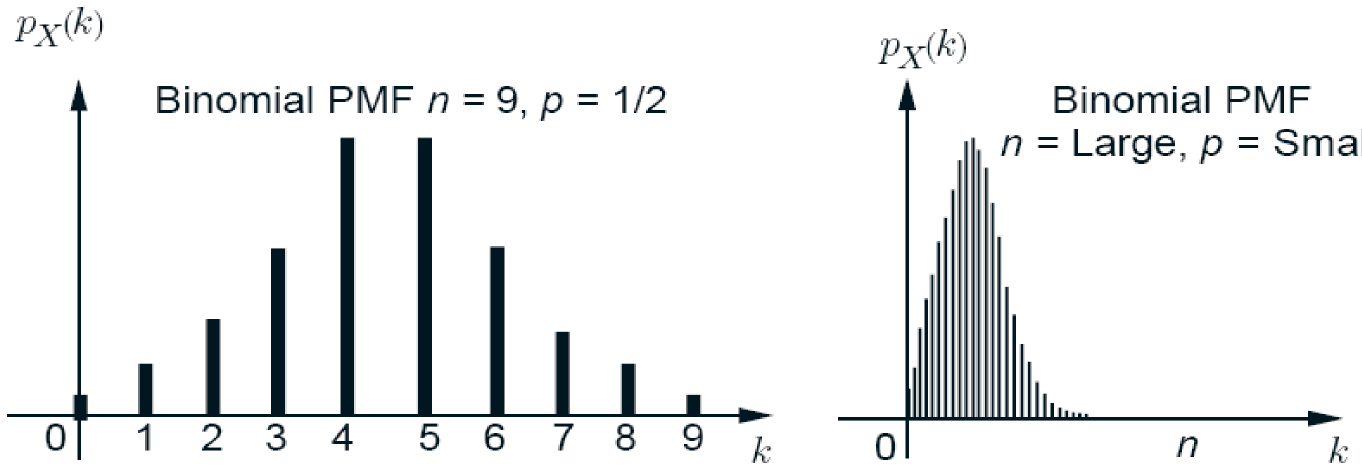


Figure 2.3: The PMF of a binomial random variable. If $p = 1/2$, the PMF is symmetric around $n/2$. Otherwise, the PMF is skewed towards 0 if $p < 1/2$, and towards n if $p > 1/2$.

Illustrated Examples: **Binomial** PMF (1/2)

- The probability that a certain kind of component will survive a shock test is 3/4. Find the probability that exactly 2 of the next 4 components tested survive. (某種元件將在某撞擊測試中通過的機率為 3/4，請找出在測試的 4 個元件中有兩個通過的機率。)

$$p_X(x) = \mathbf{P}(X = k) = \binom{n}{k} p^k (1-p)^{n-k}, \quad k = 0,1,\ldots,n$$

  – Solution:

$$p_X(x) = \mathbf{P}(X = 2) = \binom{4}{2}\left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right)^2 = \left(\frac{4!}{2!\,2!}\right)\left(\frac{3^2}{4^4}\right) = \frac{27}{128}$$

Illustrated Examples: **Binomial** PMF (2/2)

- The probability that a patient recovers from a rare blood disease is 0.4. If 15 people are known to have contracted this disease, what is the probability that (a) at least 10 survive, (b) 3 to 8 survive, and (c) exactly 5 survive? (患者從某種罕見的血液疾病中復原的機率為 0.4，如果有 15 個人已知患有這種疾病，則 (a) 至少 10 人存活；(b) 3 至 8 人存活；(c) 恰好 5 人存活的機率是多少?)
  - Solution:

(a) $$p_X(x) = \mathbf{P}(X \geq 10) = 1 - \mathbf{P}(X < 10) = 1 - \sum_{k=0}^{9} \binom{15}{k} 0.4^k (1-0.4)^{15-k} = 1 - 0.9662$$

$$= 0.0338$$

(b) $$p_X(x) = \mathbf{P}(3 \leq X \leq 8) = \sum_{k=3}^{8} \binom{15}{k} 0.4^k (1-0.4)^{15-k}$$

$$= \sum_{k=0}^{8} \binom{15}{k} 0.4^k (1-0.4)^{15-k} - \sum_{k=0}^{2} \binom{15}{k} 0.4^k (1-0.4)^{15-k}$$

$$= 0.9050 - 0.0271 = 0.8779$$

(c) $$p_X(x) = \mathbf{P}(X = 5) = \binom{15}{5} 0.4^5 (1-0.4)^{15-5} = 0.1859$$

More Random Variables relevant to **Binomial** PMF (1/2)

- **Multinomial** random variable X has parameters $n, x_1, x_2, \ldots, x_k, p_1, p_2, \ldots, p_k$: the number of outcomes in n trials, with each of k possible outcomes for each trial having probabilities $p_1, p_2, \ldots, p_k$
  - PMF

$$p_X(x) = \binom{n}{x_1 x_2, \ldots x_k} p_1^{x_1} p_2^{x_2}, \cdots, p_k^{x_k} \quad \text{, with } \sum_{i=1}^{k} x_i = n \text{ and } \sum_{i=1}^{k} p_i = 1$$

- **Hypergeometric** random variable X has parameters N, n, k : the number of successes, x, in a sample of size n is selected from N items without replacement, where k items are classified as successes (and $N - k$ as failures)

$$p_X(x) = \frac{\binom{k}{x}\binom{N-k}{n-x}}{\binom{N}{n}} \quad , \max\{0, n-(N-k)\} \le x \le \min\{k, n\}$$

Binomial PMF: sampling with replacement of each item to ensure independence between trials

More Random Variables relevant to **Binomial** (2/2)

- **Negative Binomial** random variable X has parameters k, p : in n independent trials, with probability of success p and the probability of failure $1-p$ on each trial, the probability that the kth success occurs on the xth trial
  - PMF

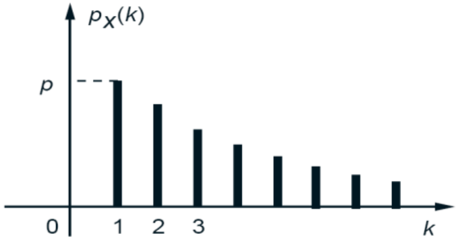
$$p_X(x) = \binom{x-1}{k-1} p^k (1-p)^{x-k} \quad , x = k, k+1, k+2, \ldots$$

- **Geometric** random variable X has parameter p : is a **special case** of the negative binomial with $k = 1$.

Geometric Random Variables

- A geometric random variable X has parameter p $(0 < p < 1)$
  - PMF

$$p_X(k) = (1-p)^{k-1} p, \quad k = 1, 2, \ldots,$$



- The geometric random variable can be used to model, e.g.
  - The number of independent tosses of a coin needed for a head to come up for the first time, each toss has probability p to be a head
  - The number of independent trials until (and including) the first "success", each trial has probability p to be successful

- Normalization Property

$$\sum_{k=1}^{\infty} p_X(k) = \sum_{k=1}^{\infty} (1-p)^{k-1} p = p \sum_{k=0}^{\infty} (1-p)^k = p \frac{1}{1-(1-p)} = 1$$

An Illustrated Example: **Multinomial** PMF

- For a certain airport containing three runways it is known that in the ideal setting the following are the probabilities that the individual runways are accessed by a randomly arriving commercial jet:

  Runway 1: p_1 = 2/9, Runway 2: p_2 = 1/6, Runway 3: p_3 = 11/18

  What is the probability that 6 randomly arriving airplanes are distribution in the following fashion?

  Runway 1: 2 airplanes,

  Runway 2: 1 airplane,

  Runway 3: 3 airplanes

$$p_X(x) = \binom{n}{x_1 x_2, \ldots x_k} p_1^{x_1} p_2^{x_2}, \cdots, p_k^{x_k} \quad \text{, with } \sum_{i=1}^{k} x_i = n \text{ and } \sum_{i=1}^{k} p_i = 1$$

  – Solution:

$$p_X(x) = \binom{6}{2,1,3}\left(\frac{2}{9}\right)^2\left(\frac{1}{6}\right)^1\left(\frac{11}{18}\right)^3 = \left(\frac{6!}{2!\cdot 1!\cdot 3!}\right)\left(\frac{2}{9}\right)^2\left(\frac{1}{6}\right)^1\left(\frac{11}{18}\right)^3 = 0.1127$$

An Example: **Hypergeometric** PMF

- Lots of 40 components each are deemed unacceptable if they contain 3 or more defectives. The procedure for sampling a lot is to select 5 components at random and to reject the lot if a defective is found. What is the probability that exactly 1 defective is found in the sample if there are 3 defectives in the entire lot? (每批貨有 40 個元件，如果含有 3 個或更多個瑕疵品，則該批貨被認為是不可接受的，抽樣的程序是隨機選擇 5 個元件，如果發現有瑕疵，則拒絕該批貨。如果一批貨中有 3 個瑕疵，那麼在樣本中發現正好有 1 個瑕疵的機率是多少?)

$$p_X(x) = \frac{\binom{k}{x}\binom{N-k}{n-x}}{\binom{N}{n}} \quad , \max\{0, n-(N-k)\} \le x \le \min\{k, n\}$$

  - Solution:

$$p_X(x) = \mathbf{P}(X=1) = \frac{\binom{3}{1}\binom{37}{4}}{\binom{40}{5}} = 0.3011, \; 0 \le x \le 3$$

How about $\mathbf{P}(X \le 1)$?

Illustrated Examples: **Negative Binomial** PMF

- In an NBA championship series, the team that wins four games out of seven is the winner. Suppose that teams *A* and *B* face each other in the championship games and that team *A* has probability 0.55 of winning a game over team *B*.
  - (a) What is the probability that team *A* will win the series in 6 games?
  - (b) What is the probability that team *A* will in the series?
  - (c) If team *A* and *B* were facing each other in a regional playoff series, which is decided by winning three out of five games, what is the probability that tea *A* would win the series?

$$p_X(x) = \binom{x-1}{k-1} p^k (1-p)^{x-k} \quad , x = k, k+1, k+2, \ldots$$

  - Solution:

(a) $p_X(x) = \mathbf{P}(X = 6) = \binom{6-1}{4-1}(0.55)^4(1-0.55)^{6-4} = 0.1853$

(b) $p_X(x) = \mathbf{P}(4 \le X \le 7) = \sum_{x=4}^{7} p_X(x) = \mathbf{P}(X = 4) + \mathbf{P}(X = 5) + \mathbf{P}(X = 6) + \mathbf{P}(X = 7)$

$= 0.0915 + 0.1647 + 0.1853 + 0.1668 = 0.6083$

(c) $p_X(x) = \mathbf{P}(3 \le X \le 5) = \sum_{x=3}^{5} p_X(x) = \mathbf{P}(X = 3) + \mathbf{P}(X = 4) + \mathbf{P}(X = 5)$

$= 0.1664 + 0.2246 + 0.2021 = 0.5931$

An Illustrated Example: **Geometric** PMF

- At a "busy time," a telephone exchange is very near capacity, so callers have difficulty placing their calls. It may be of interest to know the number of attempts necessary in order to make a connection. Suppose that we let p = 0.05 be the necessary in order to make a connection during a busy time. We are interested in knowing the probability that 5 attempts are necessary for a successful call. (在「繁忙時間」中電話交換機非常接近滿載，因此打電話者要打通電話可能會有困難。我們想知道成功通話所需要嘗試的次數。假設我們令 p = 0.05 是繁忙時間連接成功的紀律，則成功通話需要 5 次嘗試的機率為何?)

$$p_X(k) = (1-p)^{k-1} p, \quad k = 1, 2, \ldots,$$

  – Solution:

$$p_X(x) = \mathbf{P}(X = 5) = (1-0.05)^{5-1}(0.05) = (0.95)^4(0.05) = 0.041$$

Poisson Random Variables (1/2)

- A Poisson random variable X has parameter λ
  - PMF

$$p_X(k) = e^{-\lambda} \frac{\lambda^k}{k!}, \quad k = 0,1,2,\ldots,$$

- the **average #** of outcomes **per unit time** (arrival rate)
- Also could be per unit length, area, or volume, rather than time

- The Poisson random variable can be used to model, e.g.
  - The number of typos in a book
  - The number of cars involved in an accidents in a city on a given day

- Normalization Property

MacLaurin series

$$\sum_{k=0}^{n} p_X(k) = \sum_{k=0}^{n} e^{-\lambda} \frac{\lambda^k}{k!} = e^{-\lambda} \underbrace{\left(1 + \lambda + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \cdots\right)}_{e^{\lambda}} = 1$$

Poisson Random Variables (2/2)

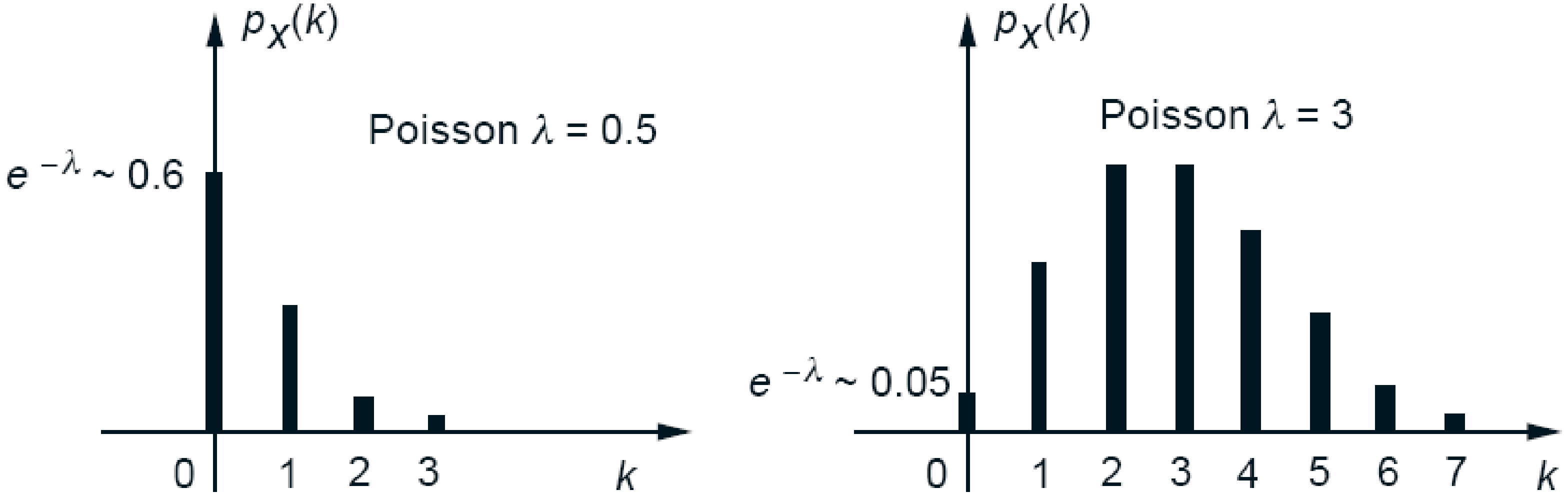


Figure 2.5: The PMF $e^{-\lambda}\frac{\lambda^k}{k!}$ of the Poisson random variable for different values of λ. Note that if $\lambda < 1$, then the PMF is monotonically decreasing, while if $\lambda > 1$, the PMF first increases and then decreases as the value of k increases (this is shown in the end-of-chapter problems).

Relationship between **Binomial** and **Poisson**

- The Poisson PMF with parameter λ is a good approximation for a binomial PMF with parameters n and p, provided that $\lambda = np$, n is very large and and p is very small

$$\lim_{n\to\infty}\binom{n}{k}p^k(1-p)^{n-k}$$

$$=\lim_{n\to\infty}\frac{n!}{(n-k)!k!}p^k(1-p)^{n-k} \qquad (\because \lambda = np \Rightarrow p=\frac{\lambda}{n})$$

$$=\lim_{n\to\infty}\frac{n(n-1)\cdots(n-k+1)}{k!}\left(\frac{\lambda}{n}\right)^k\left(1-\frac{\lambda}{n}\right)^{n-k}$$

$$=\lim_{n\to\infty}\frac{\lambda^k}{k!}\frac{n(n-1)\cdots(n-k+1)}{n^k}\left(1-\frac{\lambda}{n}\right)^{n-k}$$

$$=\lim_{n\to\infty}\frac{\lambda^k}{k!}\left(\frac{n}{n}\right)\left(\frac{n-1}{n}\right)\cdots\left(\frac{n-k+1}{n}\right)\left(1-\frac{\lambda}{n}\right)^{n-k} \qquad (\because \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n=e^x)$$

$$=\lim_{n\to\infty}\frac{\lambda^k}{k!}e^{-\lambda}$$

Illustrated Examples: **Poisson** PMF (1/2)

- During a laboratory experiment, the average number of radioactive particles passing through a counter in 1 millisecond is 4. What is the probability that 6 particles enter the counter in a given millisecond? (在某實驗室的實驗程序中，在一毫秒內通過某個計數器的放射粒子的平均數量為 4，在一個給定的毫秒內有 6 個粒子進入計數器的機率是多少?)

$$p_X(k) = e^{-\lambda}\frac{\lambda^k}{k!}, \quad k = 0,1,2,\ldots,$$

  – Solution:

$$p_X(x) = \mathbf{P}(X = 6) = e^{-4}\frac{4^6}{6!} = \sum_{k=0}^{6} e^{-k}\frac{4^k}{k!} - \sum_{k=0}^{5} e^{-k}\frac{4^k}{k!}$$

$$= 0.8893 - 0.7851 = 0.1042$$

Illustrated Examples: **Poisson** PMF (2/2)

- Ten is the average number of oil tankers arriving each day at a certain port. The facilities at the port can handle at most 15 tankers per day. What is the probability that on a given day tankers have to be turned away? (每天到達某個港口的油輪平均數量為 10 艘。該港口的設施單位每天最多可處理 15 艘油輪，在任意的某一天，油輪必須被迫趕離的機率是多少?)

$$p_X(k) = e^{-\lambda}\frac{\lambda^k}{k!}, \quad k = 0,1,2,\ldots,$$

  - Solution:

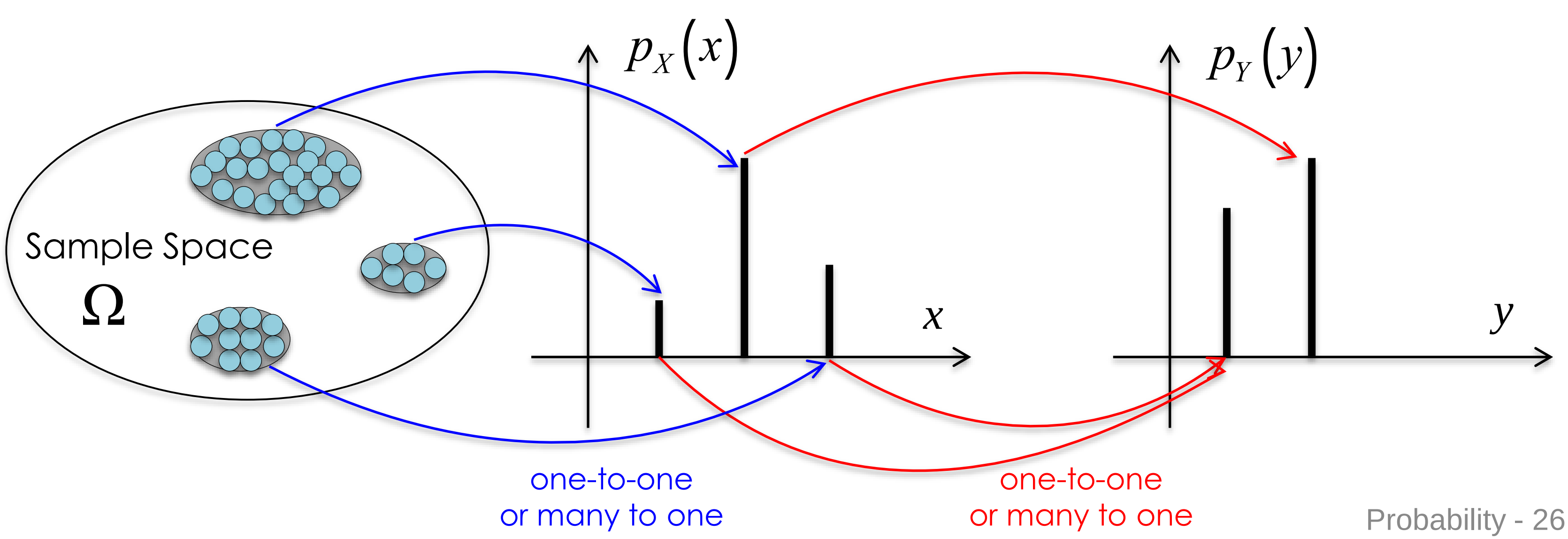
$$p_X(x) = \mathbf{P}(X \geq 15) = 1 - \sum_{k=0}^{15} e^{-k}\frac{10^k}{k!} = 1 - 0.9513 = 0.0487$$

Functions of Random Variables (1/2)

- Given a random variable X, other random variables can be generated by applying various transformations on X
  - Linear $Y = g(x) = aX + b$

    Daily temperature in degree Fahrenheit

    Daily temperature in degree Celsius
  - Nonlinear $Y = g(x) = \log X$



Functions of Random Variables (2/2)

- That is, if Y is an function of X $\left(Y = g(X)\right)$, then Y is also a random variable
  - If X is discrete with PMF $p_X(x)$, then Y is also discrete and its PMF can be calculated using

$$p_Y(y) = \sum_{\{x \mid g(x) = y\}} p_X(x)$$

Functions of Random Variables: An Example

Example 2.1. Let $Y = |X|$ and let us apply the preceding formula for the PMF p_Y to the case where

$$p_X(x) = \begin{cases} 1/9 & \text{if } x \text{ is an integer in the range } [-4, 4], \\ 0 & \text{otherwise.} \end{cases}$$

The possible values of Y are $y = 0, 1, 2, 3, 4$. To compute $p_Y(y)$ for some given value y from this range, we must add $p_X(x)$ over all values x such that $|x| = y$. In particular, there is only one value of X that corresponds to $y = 0$, namely $x = 0$. Thus,

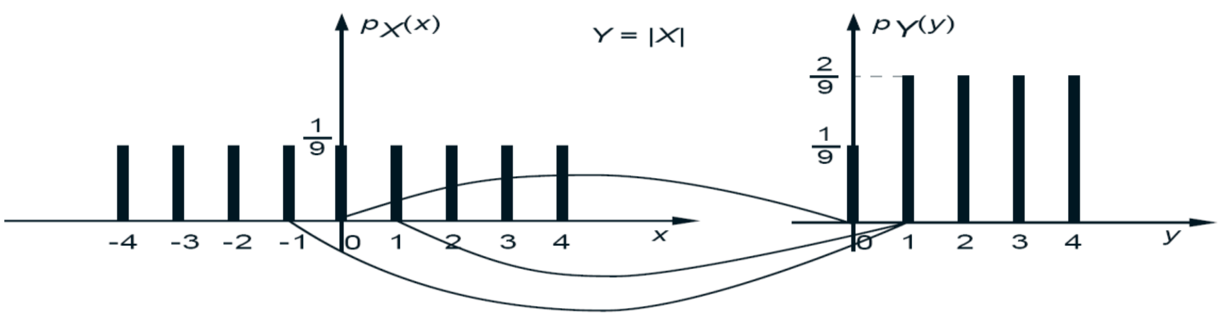
$$p_Y(0) = p_X(0) = \frac{1}{9}.$$

Also, there are two values of X that correspond to each $y = 1, 2, 3, 4$, so for example,

$$p_Y(1) = p_X(-1) + p_X(1) = \frac{2}{9}.$$

Thus, the PMF of Y is

$$p_Y(y) = \begin{cases} 2/9 & \text{if } y = 1, 2, 3, 4, \\ 1/9 & \text{if } y = 0, \\ 0 & \text{otherwise.} \end{cases}$$



Recitation

- SECTION 2.2 Probability Mass Functions
  - Problems 3, 8, 10
- SECTION 2.3 Functions of Random Variables
  - Problems 13, 14