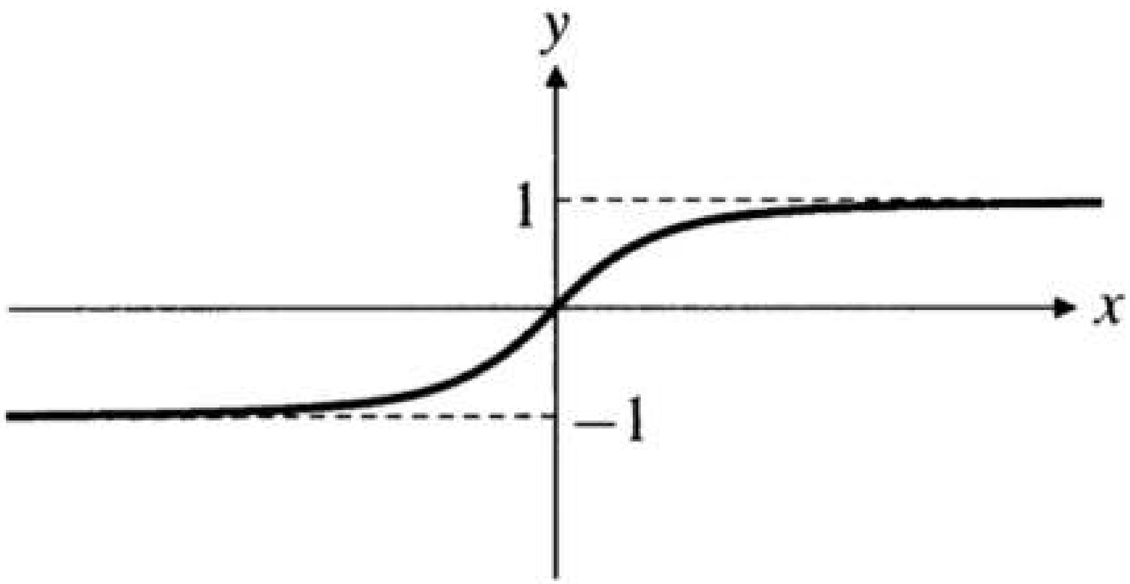
Limits at Infinity and Infinite Limits

Limits at Infinity

$$f(x) = \frac{x}{\sqrt{x^2+1}}$$

x	$f(x) = x/\sqrt{x^2+1}$
$-1{,}000$	-0.9999995
-100	-0.9999500
-10	-0.9950372
-1	-0.7071068
0	0.0000000
1	0.7071068
10	0.9950372
100	0.9999500
$1{,}000$	0.9999995



Limits at Infinity and Infinite Limits

Limits at Infinity

$$f(x) = \frac{x}{\sqrt{x^2+1}}$$

$$\lim_{x \to \infty} f(x) = 1$$

"$f(x)$ approaches 1 as x approaches infinity."

$$\lim_{x \to -\infty} f(x) = -1$$

"$f(x)$ approaches -1 as x approaches negative infinity."

x	$f(x) = x/\sqrt{x^2+1}$
−1,000	−0.9999995
−100	−0.9999500
−10	−0.9950372
−1	−0.7071068
0	0.0000000
1	0.7071068
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100	0.9999500
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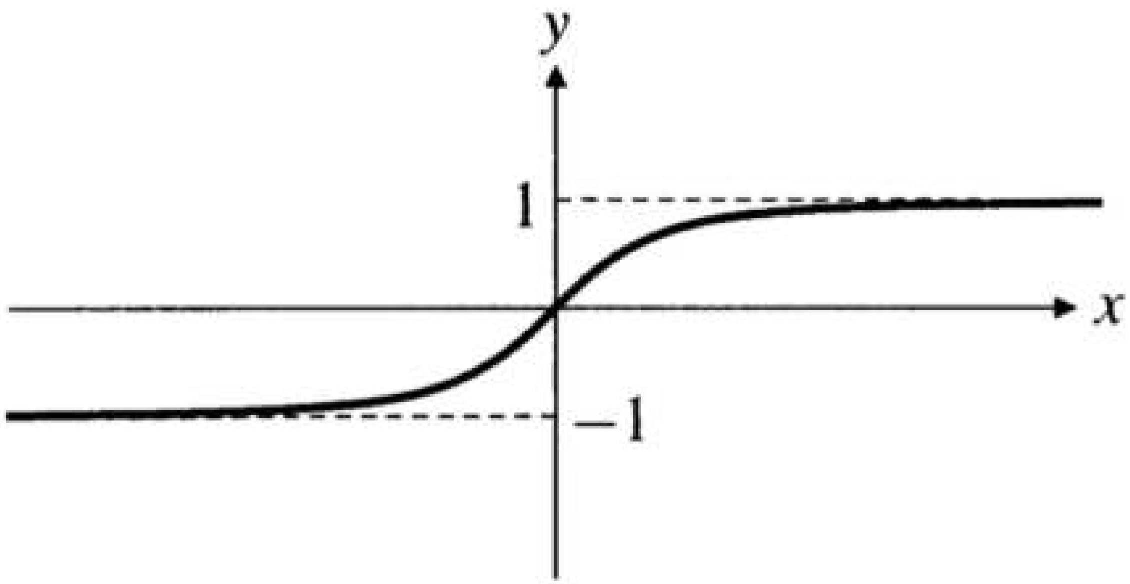


Limits at Infinity and Infinite Limits

Limits at Infinity

DEFINITION

Limits at infinity and negative infinity (informal definition)

If the function f is defined on an interval (a, ∞) and if we can ensure that $f(x)$ is as close as we want to the number L by taking x large enough, then we say that $f(x)$ **approaches the limit L as x approaches infinity**, and we write

$$\lim_{x \to \infty} f(x) = L.$$

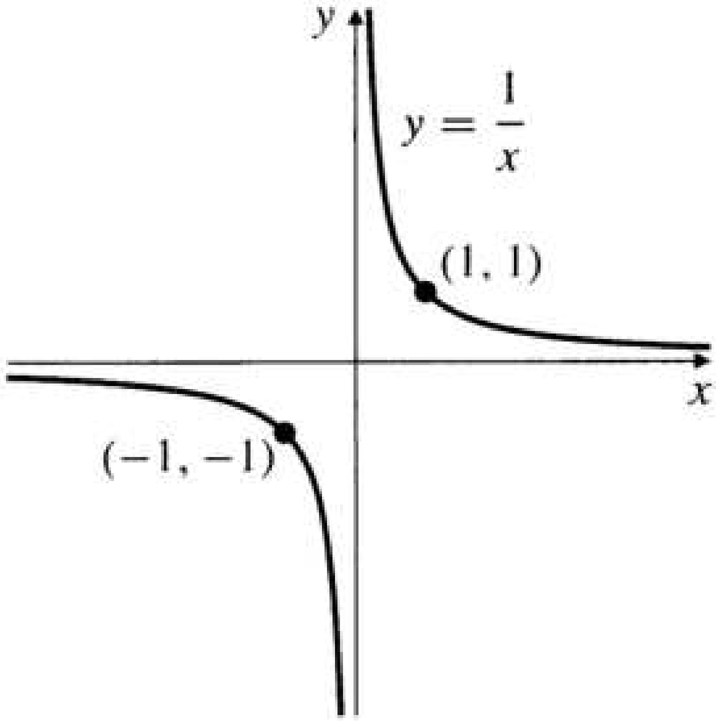
If f is defined on an interval $(-\infty, b)$ and if we can ensure that $f(x)$ is as close as we want to the number M by taking x negative and large enough in absolute value, then we say that $f(x)$ **approaches the limit M as x approaches negative infinity**, and we write

$$\lim_{x \to -\infty} f(x) = M.$$

Limits at Infinity and Infinite Limits

Limits at Infinity

EXAMPLE



$$\lim_{x \to \pm\infty} \frac{1}{x} = 0$$

Limits at Infinity and Infinite Limits

Limits at Infinity

EXAMPLE Evaluate $\lim_{x \to \infty} f(x)$ and $\lim_{x \to -\infty} f(x)$ for $f(x) = \dfrac{x}{\sqrt{x^2+1}}$.

Limits at Infinity and Infinite Limits

Limits at Infinity for Rational Functions

EXAMPLE

(Numerator and denominator of the same degree) Evaluate $\lim_{x\to\pm\infty} \dfrac{2x^2 - x + 3}{3x^2 + 5}$.

EXAMPLE

(Degree of numerator less than degree of denominator) Evaluate $\lim_{x\to\pm\infty} \dfrac{5x + 2}{2x^3 - 1}$.

Limits at Infinity and Infinite Limits

Limits at Infinity for Rational Functions

Let $P_m(x) = a_m x^m + \cdots + a_0$ and $Q_n(x) = b_n x^n + \cdots + b_0$ be polynomials of degree m and n, respectively, so that $a_m \neq 0$ and $b_n \neq 0$. Then

$$\lim_{x \to \pm\infty} \frac{P_m(x)}{Q_n(x)}$$

(a) equals zero if $m < n$,

(b) equals $\dfrac{a_m}{b_n}$ if $m = n$,

(c) does not exist if $m > n$.

Limits at Infinity and Infinite Limits

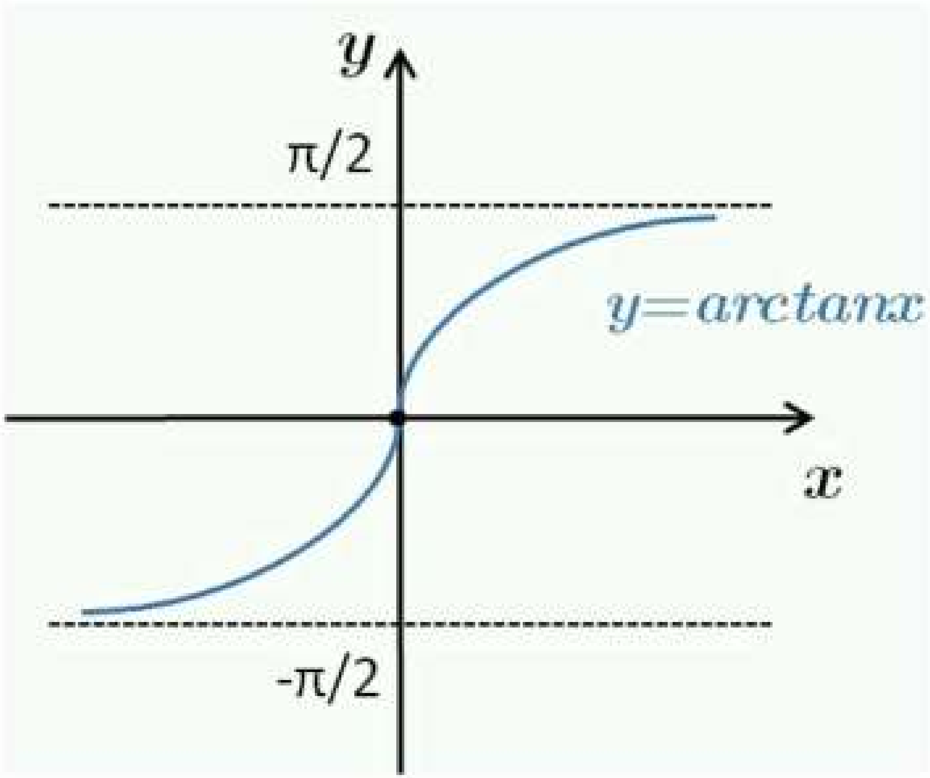
Limits at Infinity

EXAMPLE Find $\lim_{x \to \infty} \left(\sqrt{x^2 + x} - x \right)$.

Limits at Infinity and Infinite Limits

Limits at Infinity

EXAMPLE Consider the function $f(x)=\tan^{-1}x$.



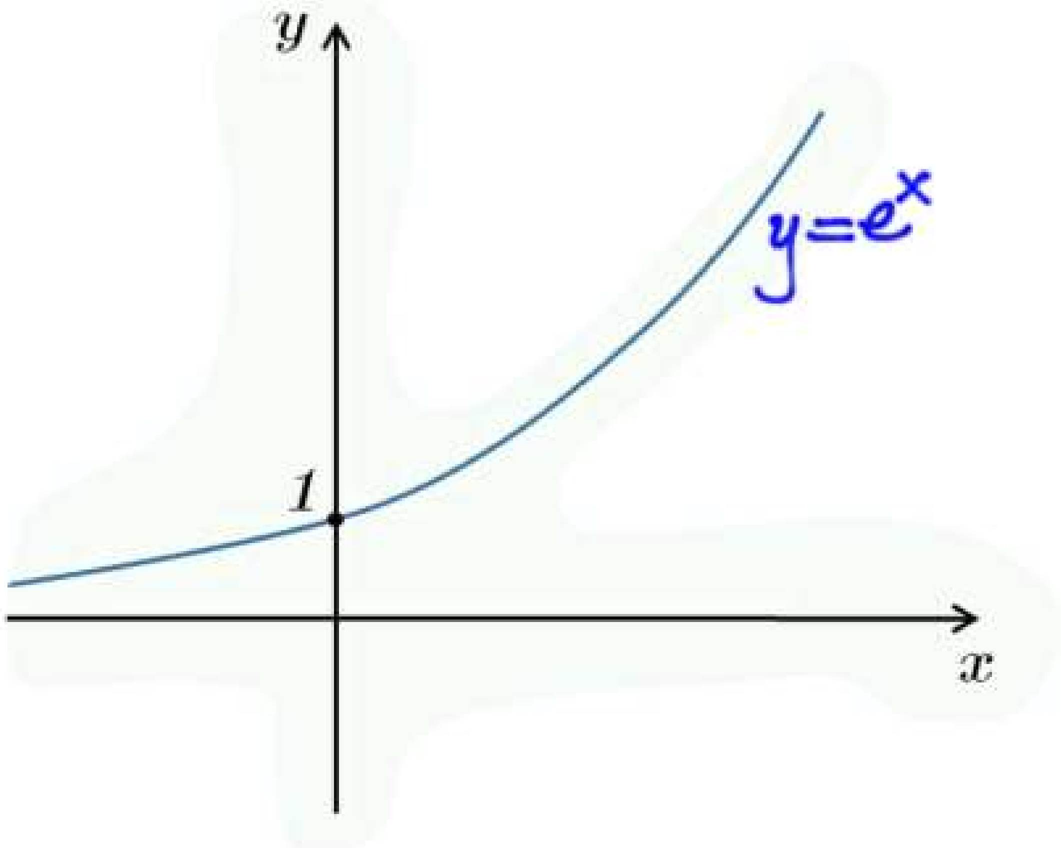
$$\lim_{x\to\infty} f(x) = \pi/2$$

$$\lim_{x\to\infty} f(x) = -\pi/2$$

Limits at Infinity and Infinite Limits

Limits at Infinity

EXAMPLE Consider the function $f(x) = e^x$.



$$\lim_{x \to -\infty} f(x) = 0.$$

Limits at Infinity and Infinite Limits

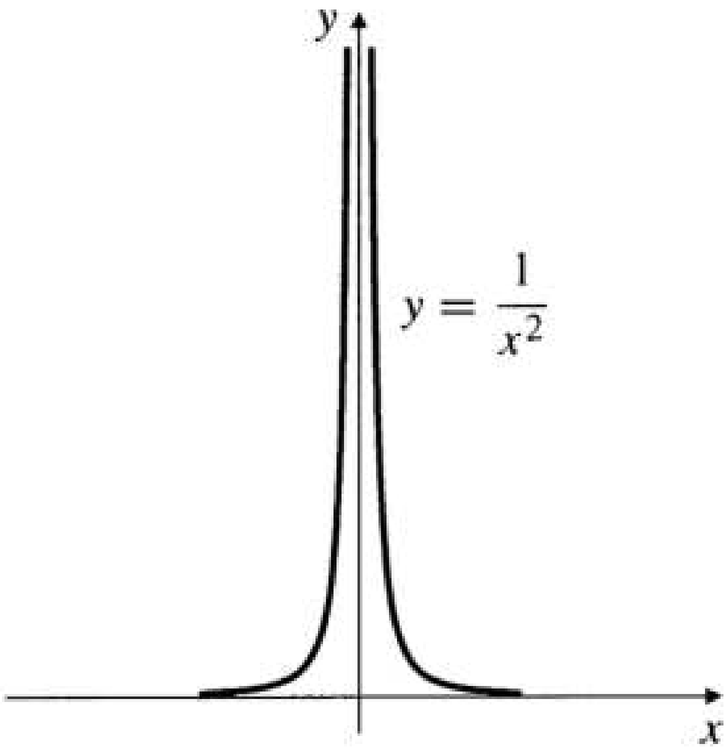
Infinite Limits

EXAMPLE **(A two-sided infinite limit)** Describe the behaviour of the function $f(x) = 1/x^2$ near $x = 0$.

Limits at Infinity and Infinite Limits

Infinite Limits

EXAMPLE **(A two-sided infinite limit)** Describe the behaviour of the function $f(x) = 1/x^2$ near $x = 0$.

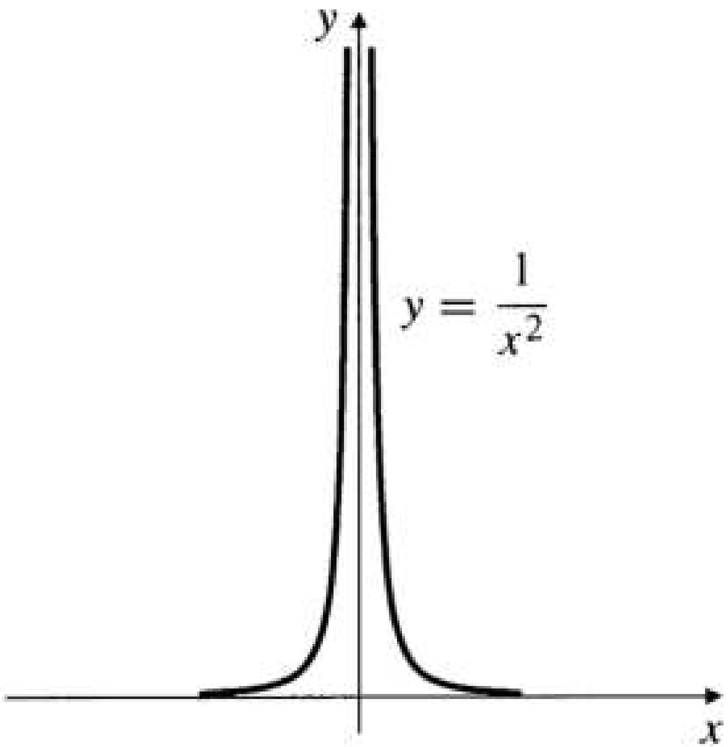


The graph of $y = 1/x^2$

Limits at Infinity and Infinite Limits

Infinite Limits

EXAMPLE **(A two-sided infinite limit)** Describe the behaviour of the function $f(x) = 1/x^2$ near $x = 0$.



The graph of $y = 1/x^2$

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{1}{x^2} = \infty.$$

Limits at Infinity and Infinite Limits

Infinite Limits

EXAMPLE

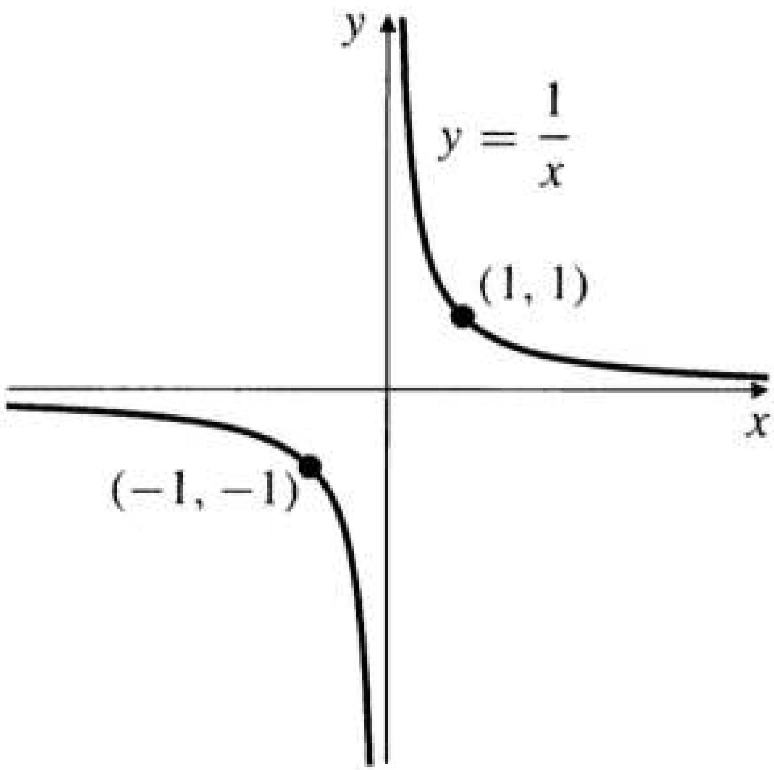
(One-sided infinite limits) Describe the behaviour of the function $f(x) = 1/x$ near $x = 0$.

Limits at Infinity and Infinite Limits

Infinite Limits

EXAMPLE

(One-sided infinite limits) Describe the behaviour of the function $f(x) = 1/x$ near $x = 0$.

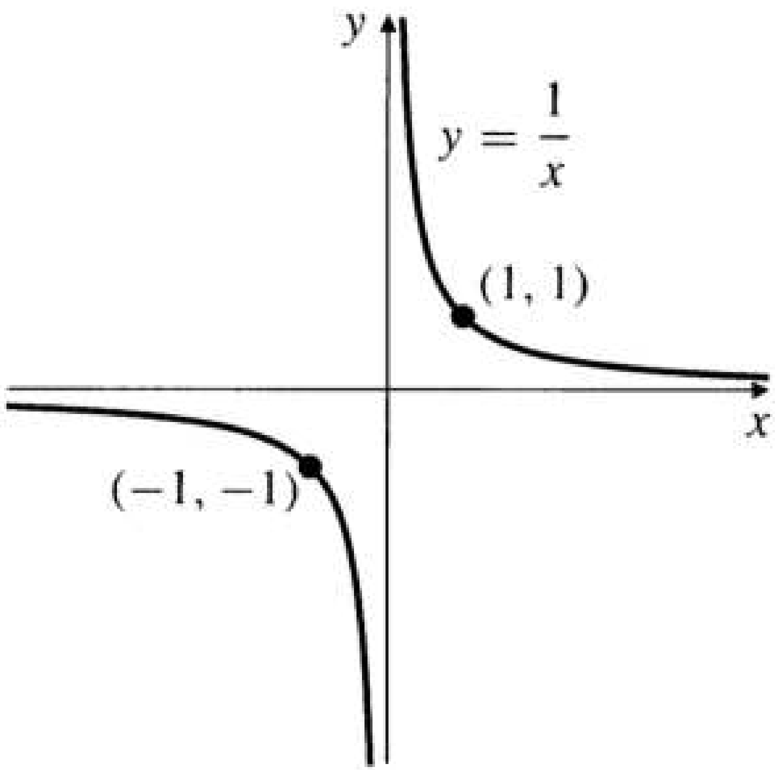


Limits at Infinity and Infinite Limits

Infinite Limits

EXAMPLE

(One-sided infinite limits) Describe the behaviour of the function $f(x) = 1/x$ near $x = 0$.



$$\lim_{x \to 0+} f(x) = \infty.$$

$$\lim_{x \to 0-} f(x) = -\infty.$$

Limits at Infinity and Infinite Limits

Infinite Limits

EXAMPLE **(Polynomial behaviour at infinity)**

(a) $\lim_{x\to\infty}(3x^3 - x^2 + 2) = \infty$

(b) $\lim_{x\to-\infty}(3x^3 - x^2 + 2) = -\infty$

(c) $\lim_{x\to\infty}(x^4 - 5x^3 - x) = \infty$

(d) $\lim_{x\to-\infty}(x^4 - 5x^3 - x) = \infty$

Limits at Infinity and Infinite Limits

Infinite Limits

EXAMPLE **(Polynomial behaviour at infinity)**

(a) $\lim_{x\to\infty} (3x^3 - x^2 + 2) = \infty$

(b) $\lim_{x\to-\infty} (3x^3 - x^2 + 2) = -\infty$

(c) $\lim_{x\to\infty} (x^4 - 5x^3 - x) = \infty$

(d) $\lim_{x\to-\infty} (x^4 - 5x^3 - x) = \infty$

EXAMPLE **(Rational functions with numerator of higher degree)** Evaluate

$$\lim_{x\to\infty} \frac{x^3 + 1}{x^2 + 1}.$$

Limits at Infinity and Infinite Limits

Infinite Limits

EXAMPLE

(a) $$\lim_{x \to 2} \frac{(x-2)^2}{x^2-4} = \lim_{x \to 2} \frac{(x-2)^2}{(x-2)(x+2)} = \lim_{x \to 2} \frac{x-2}{x+2} = 0.$$

Limits at Infinity and Infinite Limits

Infinite Limits

EXAMPLE

(a) $$\lim_{x\to 2}\frac{(x-2)^2}{x^2-4}=\lim_{x\to 2}\frac{(x-2)^2}{(x-2)(x+2)}=\lim_{x\to 2}\frac{x-2}{x+2}=0.$$

(b) $$\lim_{x\to 2}\frac{x-2}{x^2-4}=\lim_{x\to 2}\frac{x-2}{(x-2)(x+2)}=\lim_{x\to 2}\frac{1}{x+2}=\frac{1}{4}.$$

Limits at Infinity and Infinite Limits

Infinite Limits

EXAMPLE

(a) $\lim\limits_{x\to 2} \dfrac{(x-2)^2}{x^2-4} = \lim\limits_{x\to 2} \dfrac{(x-2)^2}{(x-2)(x+2)} = \lim\limits_{x\to 2} \dfrac{x-2}{x+2} = 0.$

(b) $\lim\limits_{x\to 2} \dfrac{x-2}{x^2-4} = \lim\limits_{x\to 2} \dfrac{x-2}{(x-2)(x+2)} = \lim\limits_{x\to 2} \dfrac{1}{x+2} = \dfrac{1}{4}.$

(c) $\lim\limits_{x\to 2+} \dfrac{x-3}{x^2-4} = \lim\limits_{x\to 2+} \dfrac{x-3}{(x-2)(x+2)} = -\infty.$ (The values are negative for $x > 2$, x near 2.)

Limits at Infinity and Infinite Limits

Infinite Limits

EXAMPLE

(a) $$\lim_{x\to 2}\frac{(x-2)^2}{x^2-4}=\lim_{x\to 2}\frac{(x-2)^2}{(x-2)(x+2)}=\lim_{x\to 2}\frac{x-2}{x+2}=0.$$

(b) $$\lim_{x\to 2}\frac{x-2}{x^2-4}=\lim_{x\to 2}\frac{x-2}{(x-2)(x+2)}=\lim_{x\to 2}\frac{1}{x+2}=\frac{1}{4}.$$

(c) $$\lim_{x\to 2+}\frac{x-3}{x^2-4}=\lim_{x\to 2+}\frac{x-3}{(x-2)(x+2)}=-\infty.$$ (The values are negative for $x>2$, x near 2.)

(d) $$\lim_{x\to 2-}\frac{x-3}{x^2-4}=\lim_{x\to 2-}\frac{x-3}{(x-2)(x+2)}=\infty.$$ (The values are positive for $x<2$, x near 2.)

Limits at Infinity and Infinite Limits

Infinite Limits

EXAMPLE

(a) $$\lim_{x\to 2} \frac{(x-2)^2}{x^2-4} = \lim_{x\to 2} \frac{(x-2)^2}{(x-2)(x+2)} = \lim_{x\to 2} \frac{x-2}{x+2} = 0.$$

(b) $$\lim_{x\to 2} \frac{x-2}{x^2-4} = \lim_{x\to 2} \frac{x-2}{(x-2)(x+2)} = \lim_{x\to 2} \frac{1}{x+2} = \frac{1}{4}.$$

(c) $$\lim_{x\to 2+} \frac{x-3}{x^2-4} = \lim_{x\to 2+} \frac{x-3}{(x-2)(x+2)} = -\infty.$$ (The values are negative for $x > 2$, x near 2.)

(d) $$\lim_{x\to 2-} \frac{x-3}{x^2-4} = \lim_{x\to 2-} \frac{x-3}{(x-2)(x+2)} = \infty.$$ (The values are positive for $x < 2$, x near 2.)

(e) $$\lim_{x\to 2} \frac{x-3}{x^2-4} = \lim_{x\to 2} \frac{x-3}{(x-2)(x+2)}$$ does not exist.

Limits at Infinity and Infinite Limits

Infinite Limits

EXAMPLE

(a) $$\lim_{x\to 2}\frac{(x-2)^2}{x^2-4}=\lim_{x\to 2}\frac{(x-2)^2}{(x-2)(x+2)}=\lim_{x\to 2}\frac{x-2}{x+2}=0.$$

(b) $$\lim_{x\to 2}\frac{x-2}{x^2-4}=\lim_{x\to 2}\frac{x-2}{(x-2)(x+2)}=\lim_{x\to 2}\frac{1}{x+2}=\frac{1}{4}.$$

(c) $$\lim_{x\to 2+}\frac{x-3}{x^2-4}=\lim_{x\to 2+}\frac{x-3}{(x-2)(x+2)}=-\infty.$$ (The values are negative for $x>2$, x near 2.)

(d) $$\lim_{x\to 2-}\frac{x-3}{x^2-4}=\lim_{x\to 2-}\frac{x-3}{(x-2)(x+2)}=\infty.$$ (The values are positive for $x<2$, x near 2.)

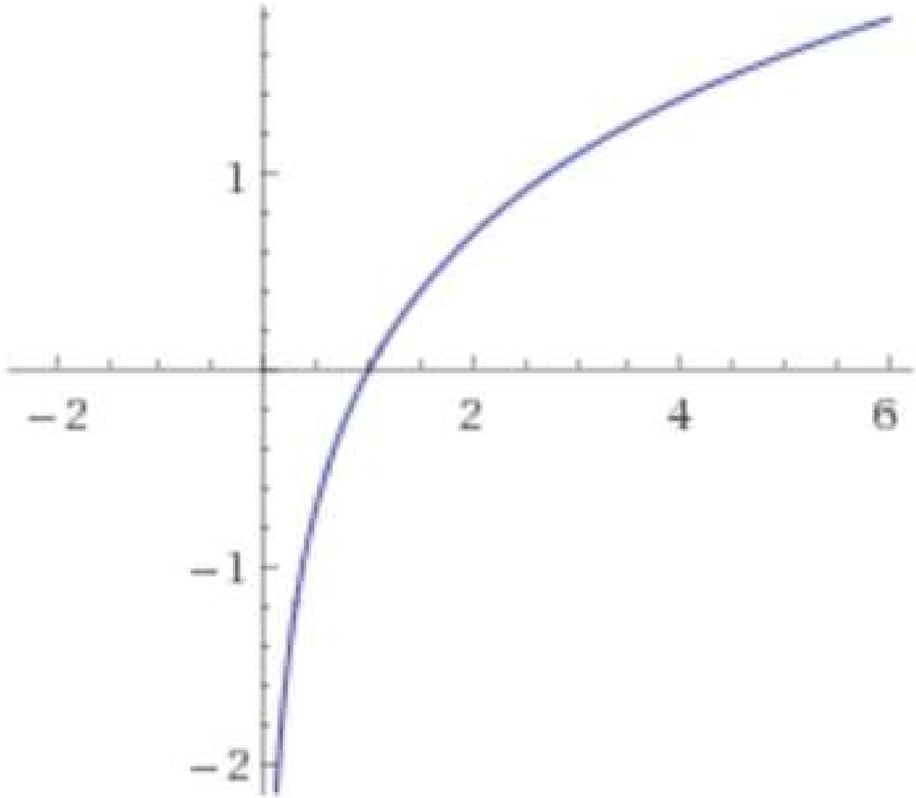
(e) $$\lim_{x\to 2}\frac{x-3}{x^2-4}=\lim_{x\to 2}\frac{x-3}{(x-2)(x+2)}$$ does not exist.

(f) $$\lim_{x\to 2}\frac{2-x}{(x-2)^3}=\lim_{x\to 2}\frac{-(x-2)}{(x-2)^3}=\lim_{x\to 2}\frac{-1}{(x-2)^2}=-\infty.$$

Limits at Infinity and Infinite Limits

Infinite Limits

EXAMPLE Consider the function $f(x) = \ln x$.



$$\lim_{x \to 0^+} f(x) = -\infty$$

Continuity

Suppose the domain of a function $f(x)$ is either an interval or a union of separate intervals.
A point P in the domain of $f(x)$ is called an **interior point** if it belongs to some open interval contained in the domain.

Continuity

Suppose the domain of a function $f(x)$ is either an interval or a union of separate intervals.
A point P in the domain of $f(x)$ is called an **interior point** if it belongs to some open interval contained in the domain. If it is not an interior point then it is called an **end point**.

Continuity

Suppose the domain of a function $f(x)$ is either an interval or a union of separate intervals.
A point P in the domain of $f(x)$ is called an interior point if it belongs to some open interval contained in the domain. If it is not an interior point then it is called an end point.
EXAMPLE. The function $f(x)=\sqrt{1-x^2}$ has domain $[-1,1]$. The interior points forms up the open interval $(-1,1)$. The endpoints are -1 and 1.

Continuity

Suppose the domain of a function $f(x)$ is either an interval or a union of separate intervals.
A point P in the domain of $f(x)$ is called an **interior point** if it belongs to some open interval contained in the domain. If it is not an interior point then it is called an **end point**.
EXAMPLE. The domain of the function $g(x) = \frac{1}{x}$ is the union of the open intervals $(-\infty, 0) \cup (0, \infty)$ and consists entirely of interior points.

Continuity

Continuity at a Point

Continuity at an interior point

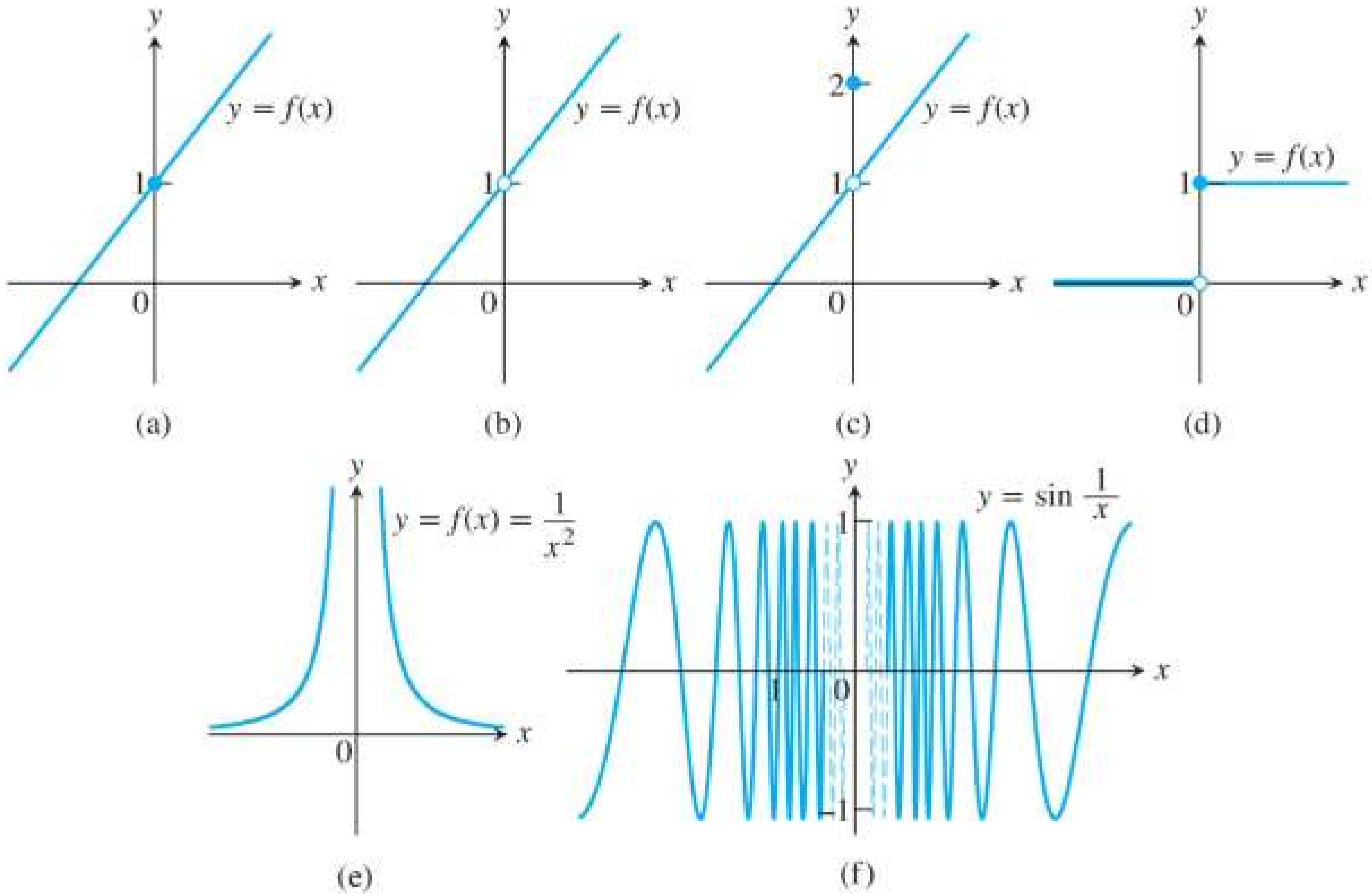
We say that a function f is **continuous** at an interior point c of its domain if

$$\lim_{x \to c} f(x) = f(c).$$

If either $\lim_{x \to c} f(x)$ fails to exist or it exists but is not equal to $f(c)$, then we will say that f is **discontinuous** at c.

Continuity

Continuity at a Point



The function in (a) is continuous at $x = 0$; the functions in (b) through (f)

Continuity

Continuity at a Point

Continuity Test

A function $f(x)$ is continuous at an interior point $x = c$ of its domain if and only if it meets the following three conditions.

1.	$f(c)$ exists	(c lies in the domain of f).
2.	$\lim_{x \to c} f(x)$ exists	(f has a limit as $x \to c$).
3.	$\lim_{x \to c} f(x) = f(c)$	(the limit equals the function value).

Continuity

Continuity at a Point

Continuity Test

A function $f(x)$ is continuous at an interior point $x = c$ of its domain if and only if it meets the following three conditions.

1.	$f(c)$ exists	(c lies in the domain of f).
2.	$\lim_{x \to c} f(x)$ exists	(f has a limit as $x \to c$).
3.	$\lim_{x \to c} f(x) = f(c)$	(the limit equals the function value).

DEFINITION

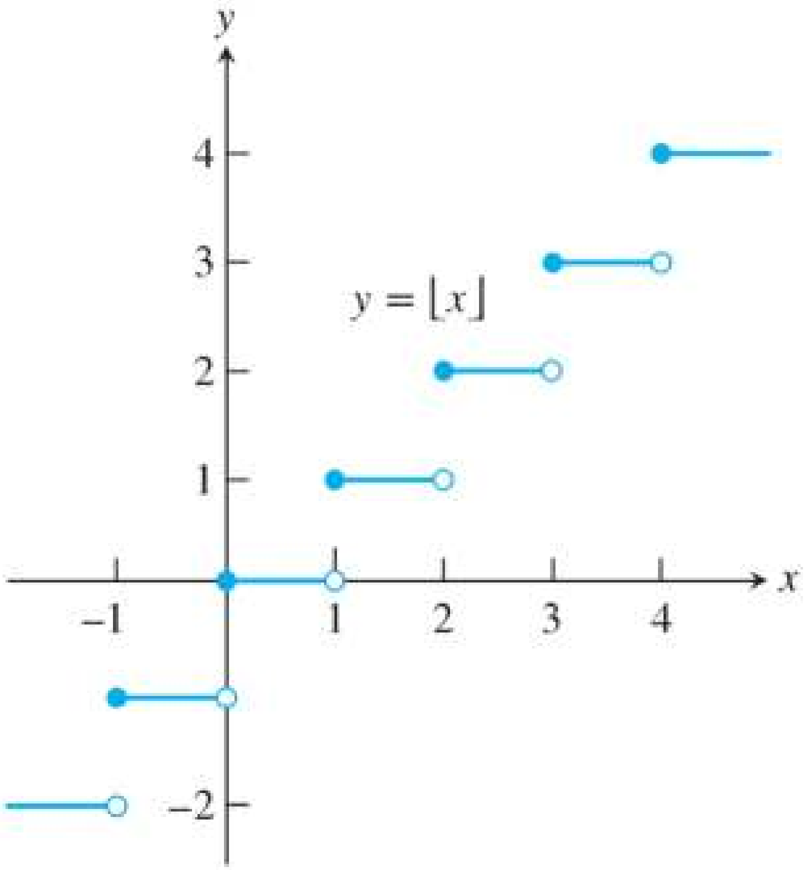
Right and left continuity

We say that f is **right continuous** at c if $\lim\limits_{x \to c+} f(x) = f(c)$.

We say that f is **left continuous** at c if $\lim\limits_{x \to c-} f(x) = f(c)$.

Continuity

Continuity at a Point



The greatest integer function is continuous at every noninteger point. It is right-continuous, but not left-continuous, at every integer point.

Continuity

Continuity at a Point

THEOREM

Function f is continuous at c if and only if it is both right continuous and left continuous at c.

Continuity

Continuity at a Point

THEOREM

Function f is continuous at c if and only if it is both right continuous and left continuous at c.

DEFINITION

Continuity at an endpoint

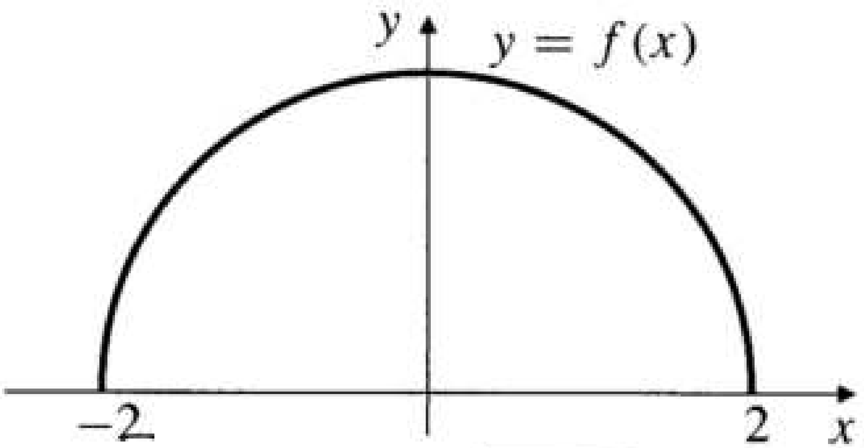
We say that f is continuous at a left endpoint c of its domain if it is right continuous there.

We say that f is continuous at a right endpoint c of its domain if it is left continuous there.

Continuity

Continuity at a Point

EXAMPLE



$f(x) = \sqrt{4 - x^2}$ is continuous at every point of its domain

$$\lim_{x \to 2^-} f(x) = 0 = f(2)$$

$$\lim_{x \to -2^+} f(x) = 0 = f(-2)$$

Continuity

Continuity on an Interval

DEFINITION

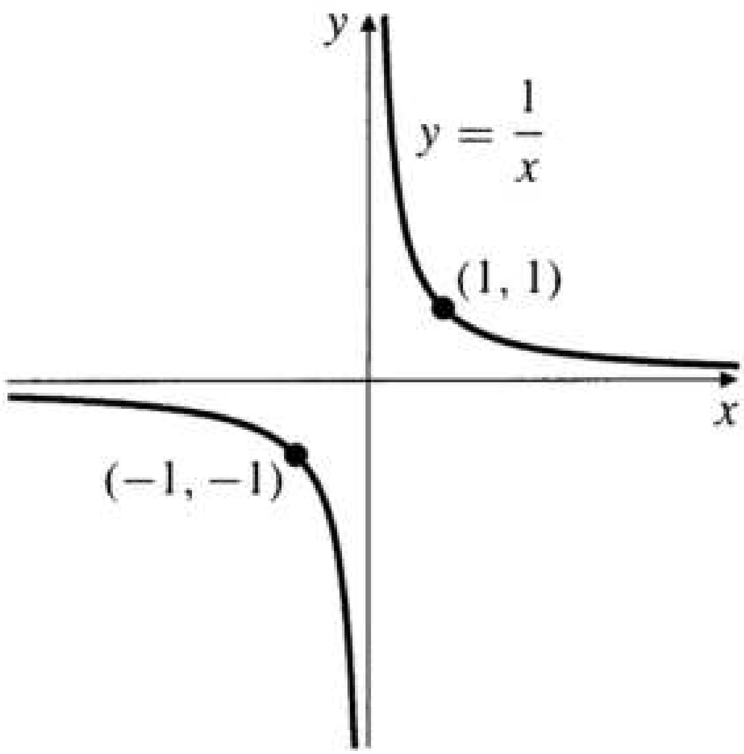
Continuity on an interval

We say that function f is **continuous on the interval** I if it is continuous at each point of I. In particular, we will say that f is a **continuous function** if f is continuous at every point of its domain.

Continuity

Continuity on an Interval

EXAMPLE



$1/x$ is continuous on its domain

There Are Lots of Continuous Functions

The following functions are continuous wherever they are defined:

(a) all polynomials;

(b) all rational functions;

(c) all rational powers $x^{m/n} = \sqrt[n]{x^m}$;

(d) the sine, cosine, tangent, secant, cosecant, and cotangent functions defined in Section P.7; and

(e) the absolute value function $|x|$.

Continuity

There Are Lots of Continuous Functions

THEOREM

Combining continuous functions

If the functions f and g are both defined on an interval containing c and both are continuous at c, then the following functions are also continuous at c:

1. the sum $f + g$ and the difference $f - g$;
2. the product fg;
3. the constant multiple kf, where k is any number;
4. the quotient f/g (provided $g(c) \neq 0$); and
5. the nth root $(f(x))^{1/n}$, provided $f(c) > 0$ if n is even.

There Are Lots of Continuous Functions

THEOREM

Composites of continuous functions are continuous

If $f(g(x))$ is defined on an interval containing c, and if f is continuous at L and $\lim_{x\to c} g(x) = L$, then

$$\lim_{x\to c} f(g(x)) = f(L) = f\left(\lim_{x\to c} g(x)\right).$$

In particular, if g is continuous at c (so $L = g(c)$), then the composition $f \circ g$ is continuous at c:

$$\lim_{x\to c} f(g(x)) = f(g(c)).$$

Continuity

There Are Lots of Continuous Functions

EXAMPLE The following functions are continuous everywhere on their respective domains:

(a) $3x^2 - 2x$ (b) $\dfrac{x-2}{x^2-4}$ (c) $|x^2 - 1|$

(d) $\sqrt{x}$ (e) $\sqrt{x^2 - 2x - 5}$ (f) $\dfrac{|x|}{\sqrt{|x+2|}}$.

Continuity

Continuous Extensions and Removable Discontinuities

If $f(c)$ is not defined, but $\lim_{x \to c} f(x) = L$ exists we can define a new function

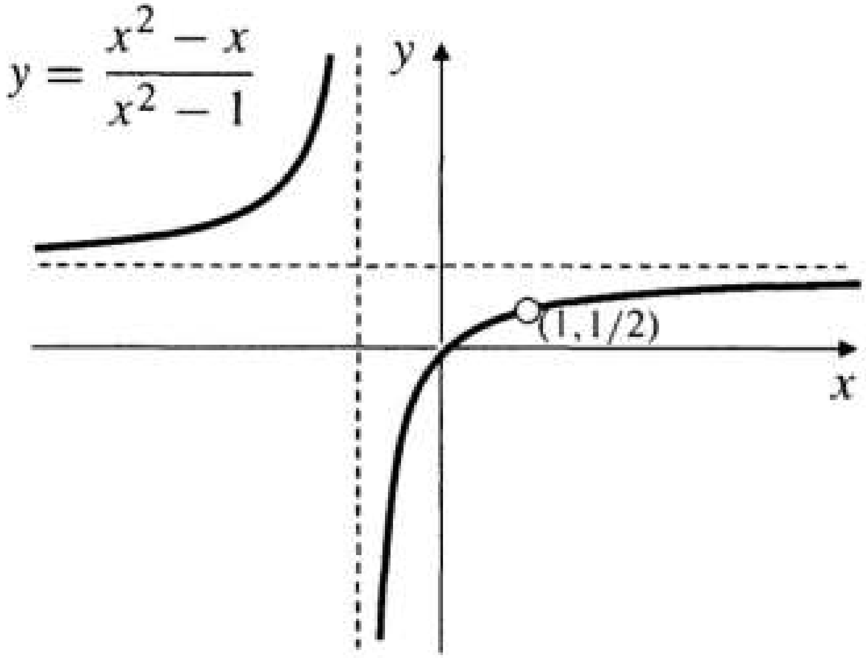
$$F(x) = \begin{cases} f(x) & \text{if } x \text{ is in the domain of } f \\ L & \text{if } x = c. \end{cases}$$

$F(x)$ is continuous at $x = c$. It is called the **continuous extension** of $f(x)$ to $x = c$.

Continuity

Continuous Extensions and Removable Discontinuities

EXAMPLE Show that $f(x) = \dfrac{x^2 - x}{x^2 - 1}$ has a continuous extension to $x = 1$, and find that extension.



Continuity

Continuous Extensions and Removable Discontinuities

THEOREM

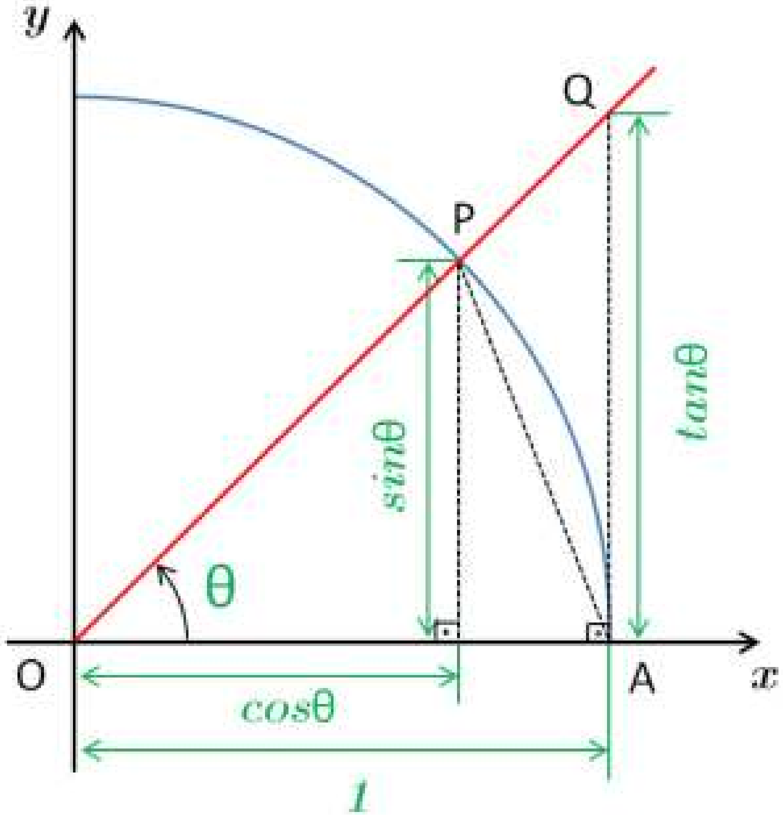
$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$$

Continuity

Continuous Extensions and Removable Discontinuities

THEOREM

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$$



area of $\overset{\Delta}{AOP}$ $\leq$area of sector AOP $\leq$area of $\overset{\Delta}{AOQ}$ and so

$$\frac{\sin \theta}{2} \leq \frac{\theta}{2} \leq \frac{\tan \theta}{2}.$$

Dividing by $\sin \theta / 2$ and taking reciprocals we get

$$\cos \theta \leq \frac{\sin \theta}{\theta} \leq 1.$$

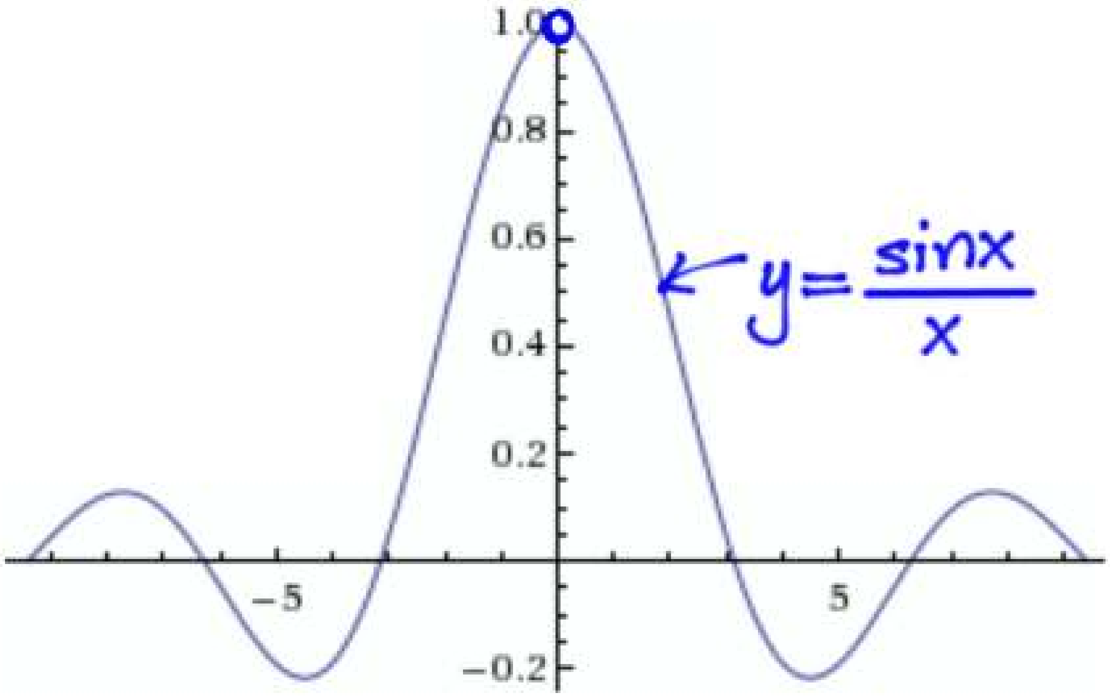
Using the Sandwich theorem, we obtain that

$$\lim_{\theta \to 0^+} \frac{\sin \theta}{\theta} = 1.$$

Continuity

Continuous Extensions and Removable Discontinuities

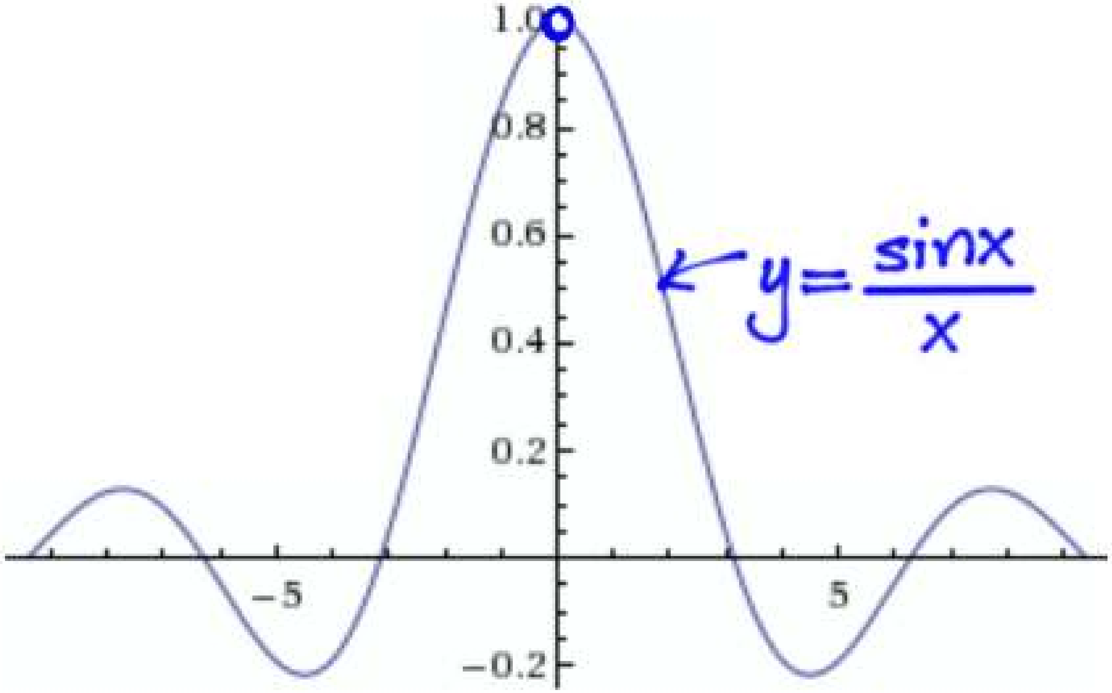
EXAMPLE Show that $f(x) = \frac{\sin x}{x}$ has a continuous extension to $x = 1$ and find that extension.



Continuity

Continuous Extensions and Removable Discontinuities

EXAMPLE Show that $f(x) = \frac{\sin x}{x}$ has a continuous extension to $x = 1$ and find that extension.



$$F(x) = \begin{cases} \frac{\sin x}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

Continuity

Continuous Functions on Closed, Finite Intervals

THEOREM

The Max-Min Theorem

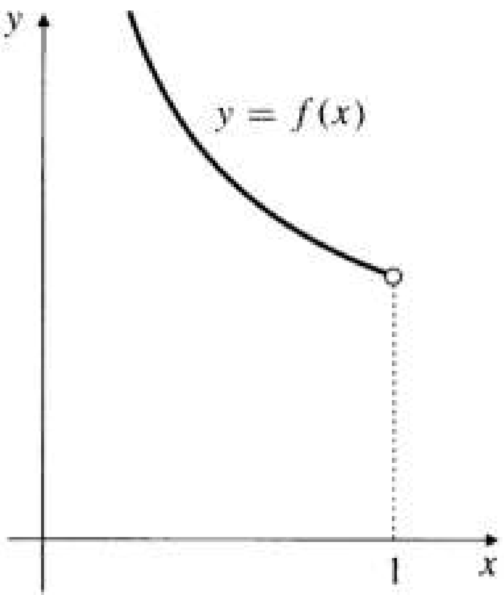
If $f(x)$ is continuous on the closed, finite interval $[a, b]$, then there exist numbers p and q in $[a, b]$ such that for all x in $[a, b]$,

$$f(p) \leq f(x) \leq f(q).$$

Thus f has the absolute minimum value $m = f(p)$, taken on at the point p, and the absolute maximum value $M = f(q)$, taken on at the point q.

Continuity

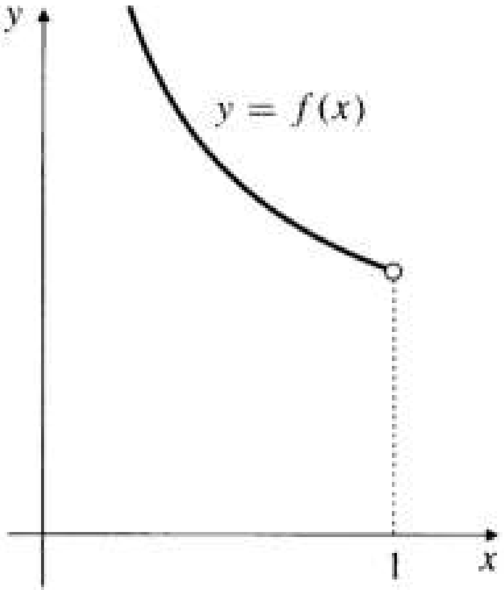
Continuous Functions on Closed, Finite Intervals



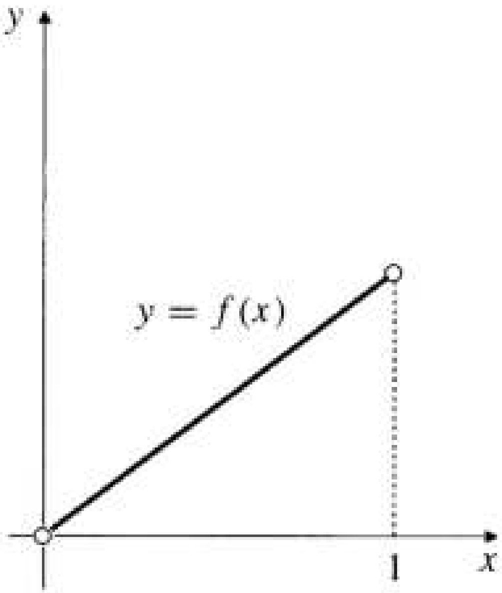
$f(x) = 1/x$ is continuous on the open interval $(0, 1)$. It is not bounded and has neither a maximum nor a minimum value

Continuity

Continuous Functions on Closed, Finite Intervals



$f(x) = 1/x$ is continuous on the open interval $(0, 1)$. It is not bounded and has neither a maximum nor a minimum value



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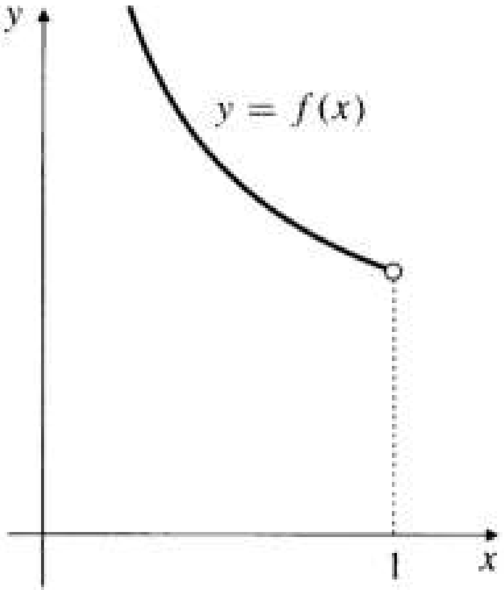

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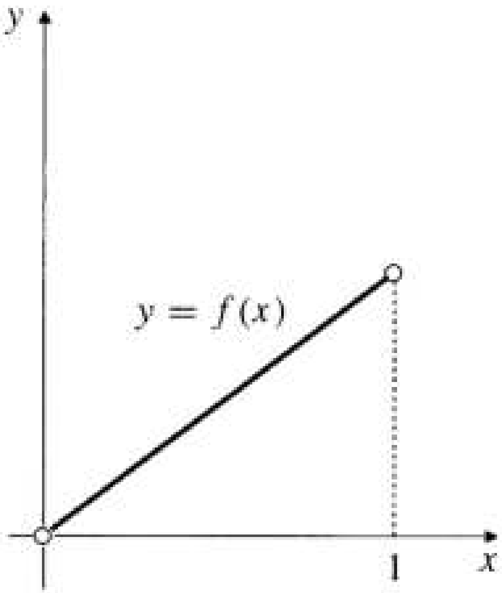

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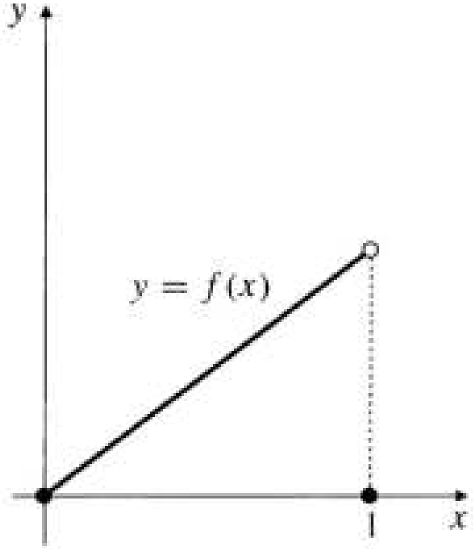

This function is defined on the closed interval $[0, 1]$ but is discontinuous at the endpoint $x = 1$. It has a minimum value but no maximum value

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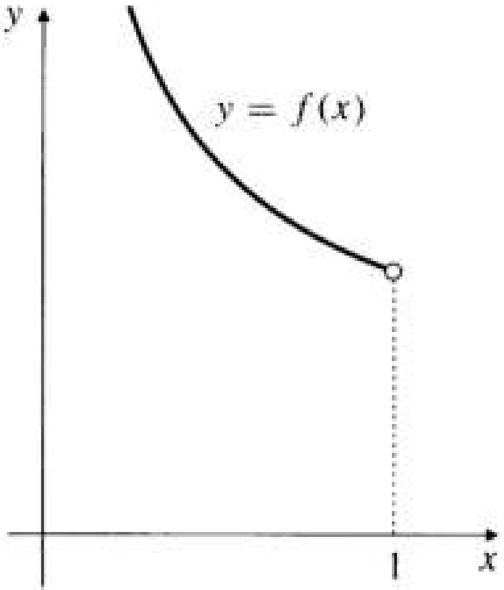

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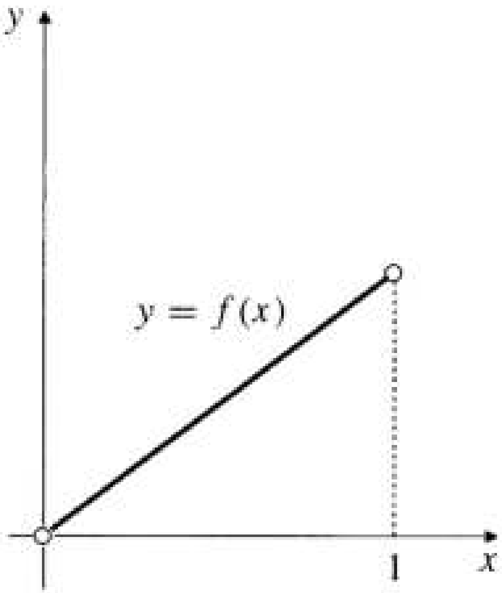

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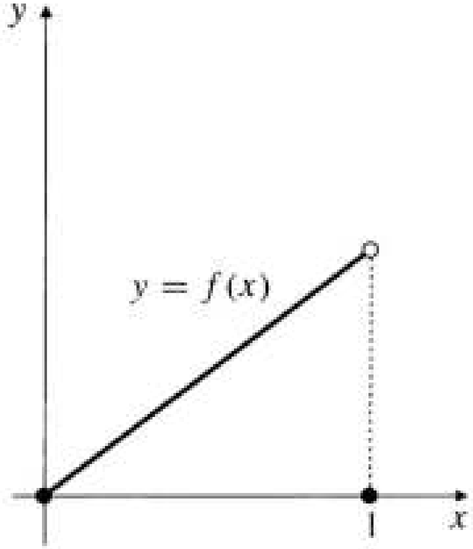

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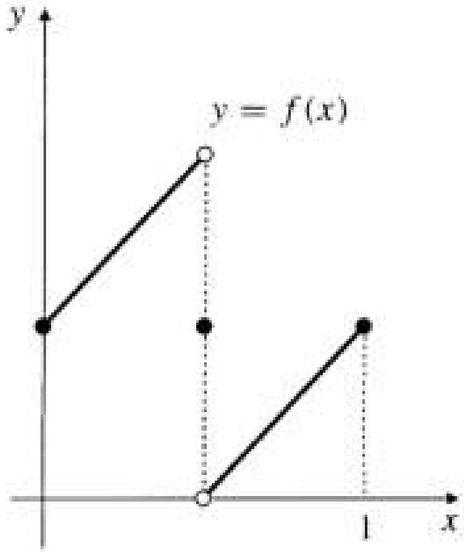

This function is discontinuous at an interior point of its domain, the closed interval $[0, 1]$. It is bounded but has neither maximum nor minimum values

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THEOREM

The Intermediate-Value Theorem

If $f(x)$ is continuous on the interval $[a, b]$ and if s is a number between $f(a)$ and $f(b)$, then there exists a number c in $[a, b]$ such that $f(c) = s$.

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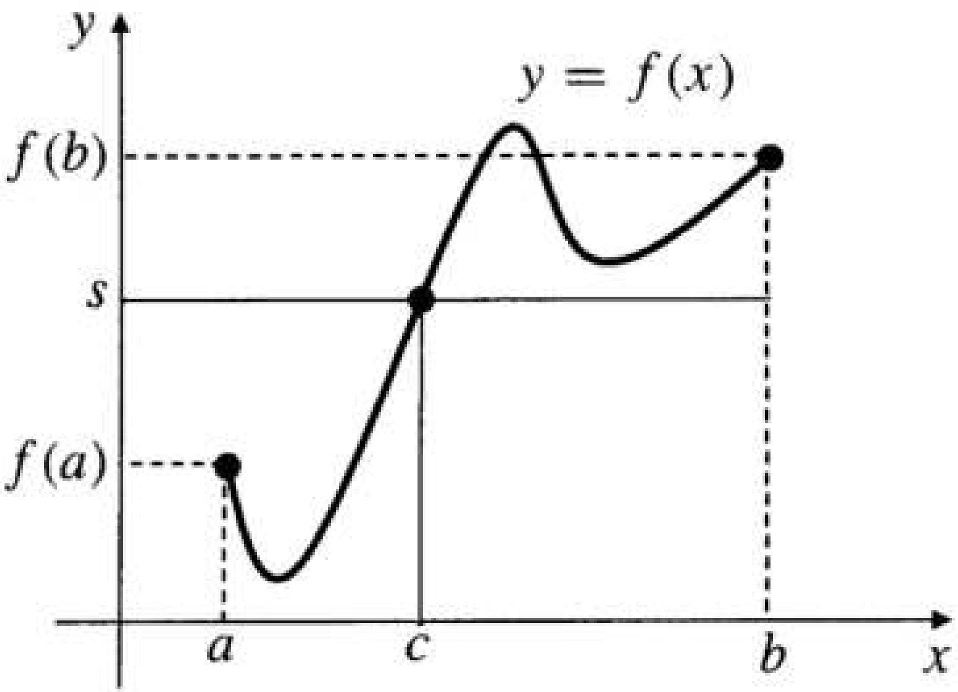

The continuous function f takes on the value s at some point c between a and b

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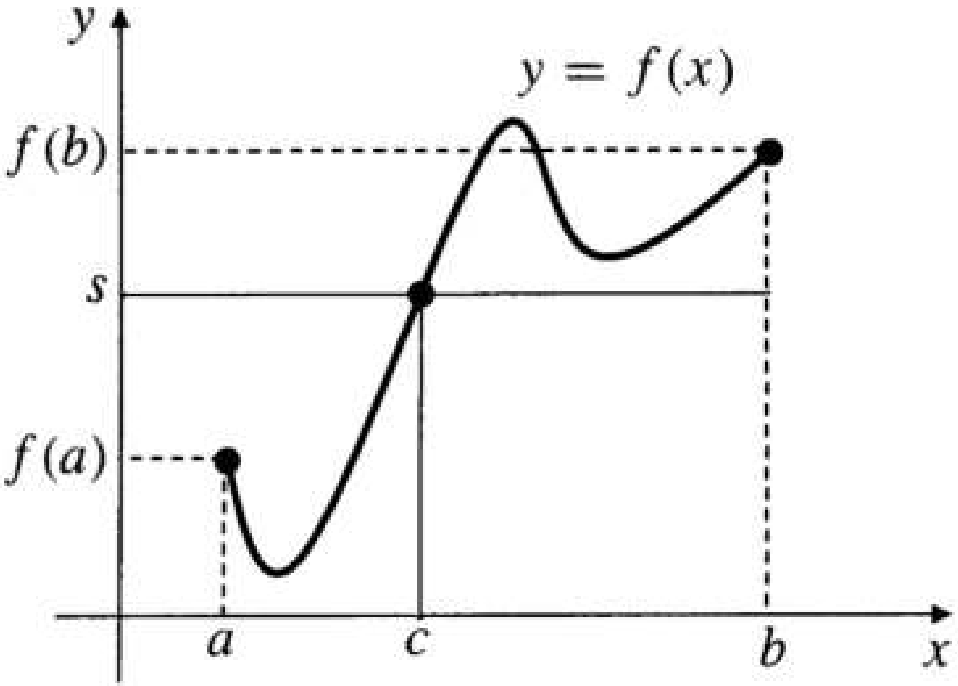

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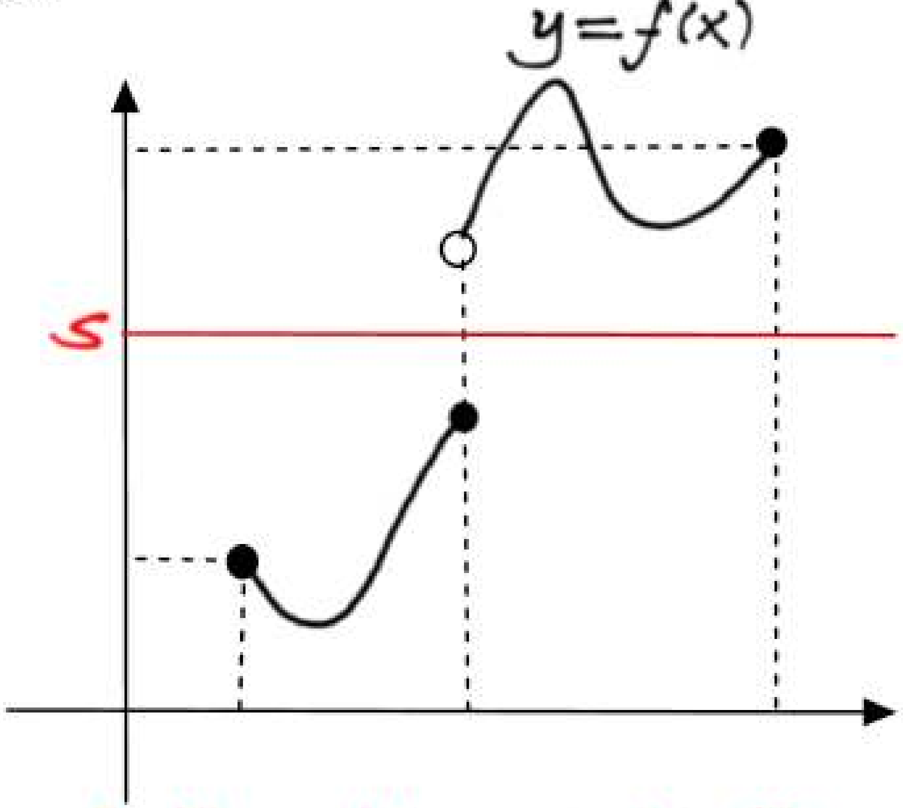

A discontinuous function may fail to take on all intermediate values.

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EXAMPLE

Determine the intervals on which $f(x) = x^3 - 4x$ is positive and negative.

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EXAMPLE

Determine the intervals on which $f(x) = x^3 - 4x$ is positive and negative.

EXAMPLE

Show that the equation $x^3 - x - 1 = 0$ has a solution in the interval $[1, 2]$.

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EXAMPLE

(The Bisection Method) Solve the equation $x^3 - x - 1 = 0$ correct to 3 decimal places by successive bisections.

Continuous Functions on Closed, Finite Intervals

EXAMPLE **(The Bisection Method)** Solve the equation $x^3 - x - 1 = 0$ of correct to 3 decimal places by successive bisections.

Bisection Number	x	$f(x)$	Root in Interval	Midpoint
	1	−1		
	2	5	[1, 2]	1.5
1	1.5	0.8750	[1, 1.5]	1.25
2	1.25	−0.2969	[1.25, 1.5]	1.375
3	1.375	0.2246	[1.25, 1.375]	1.3125
4	1.3125	−0.0515	[1.3125, 1.375]	1.3438
5	1.3438	0.0826	[1.3125, 1.3438]	1.3282
6	1.3282	0.0147	[1.3125, 1.3282]	1.3204
7	1.3204	−0.0186	[1.3204, 1.3282]	1.3243
8	1.3243	−0.0018	[1.3243, 1.3282]	1.3263
9	1.3263	0.0065	[1.3243, 1.3263]	1.3253
10	1.3253	0.0025	[1.3243, 1.3253]	1.3248
11	1.3248	0.0003	[1.3243, 1.3248]	1.3246
12	1.3246	−0.0007	[1.3246, 1.3248]	