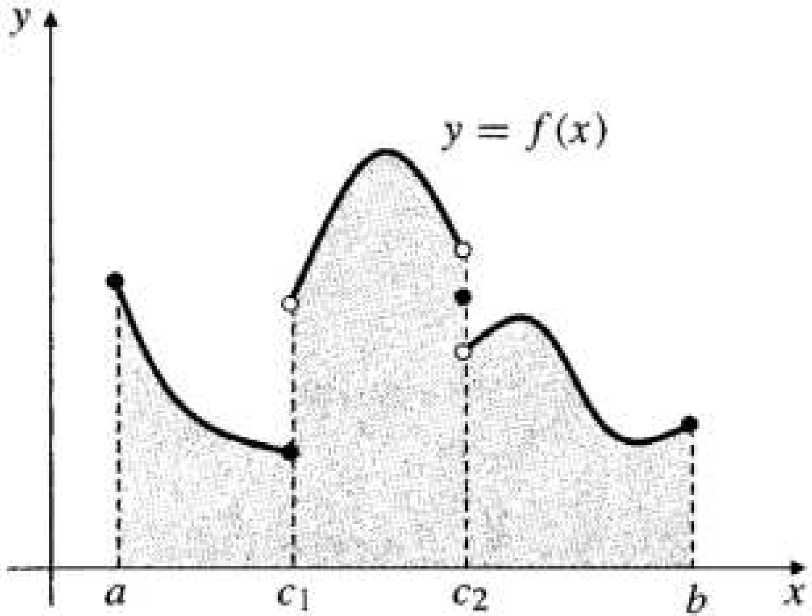
Properties of the Definite Integral

Piecewise continuous functions

Let $c_0 < c_1 < c_2 < \cdots < c_n$ be a finite set of points on the real line. A function f defined on $[c_0, c_n]$ except possibly at some of the points c_i, $(0 \le i \le n)$, is called **piecewise continuous** on that interval if for each i $(1 \le i \le n)$ there exists a function F_i continuous on the *closed* interval $[c_{i-1}, c_i]$ such that

$$f(x) = F_i(x) \qquad \text{on the } \textit{open} \text{ interval} \quad (c_{i-1}, c_i).$$

In this case,



Properties of the Definite Integral

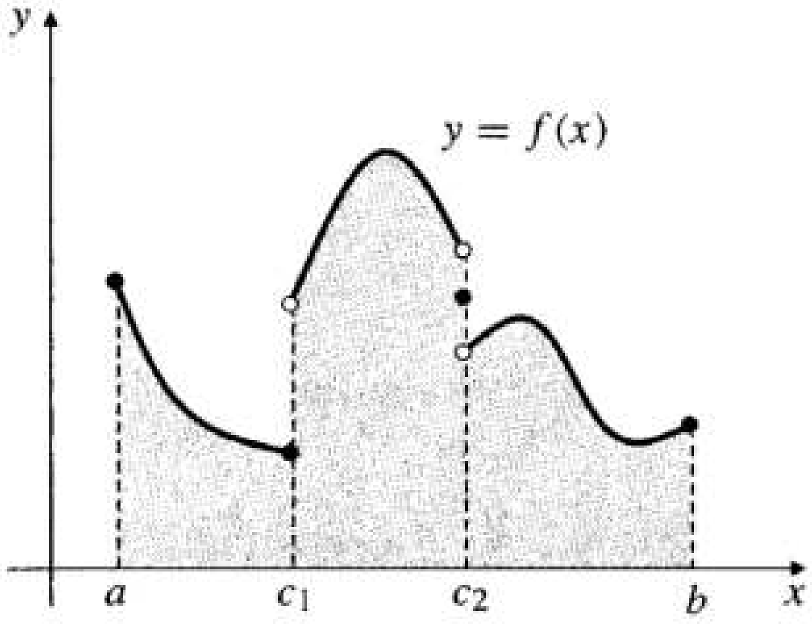
Piecewise continuous functions

Let $c_0 < c_1 < c_2 < \cdots < c_n$ be a finite set of points on the real line. A function f defined on $[c_0, c_n]$ except possibly at some of the points c_i, $(0 \le i \le n)$, is called **piecewise continuous** on that interval if for each i $(1 \le i \le n)$ there exists a function F_i continuous on the *closed* interval $[c_{i-1}, c_i]$ such that

$$f(x) = F_i(x) \qquad \text{on the } \textit{open} \text{ interval} \quad (c_{i-1}, c_i).$$

In this case,

$$\boxed{\int_{c_0}^{c_n} f(x)\,dx = \sum_{i=1}^{n} \int_{c_{i-1}}^{c_i} F_i(x)\,dx.}$$



Properties of the Definite Integral

EXAMPLE Find $\int_0^3 f(x)\,dx$, where $f(x) = \begin{cases} \sqrt{1-x^2} & \text{if } 0 \le x \le 1 \\ 2 & \text{if } 1 < x \le 2 \\ x-2 & \text{if } 2 < x \le 3. \end{cases}$

The Fundamental Theorem of Calculus

THEOREM **The Fundamental Theorem of Calculus**

Suppose that the function f is continuous on an interval I containing the point a.

PART I. Let the function F be defined on I by

$$F(x) = \int_a^x f(t)\,dt.$$

Then F is differentiable on I, and $F'(x) = f(x)$ there. Thus, F is an antiderivative of f on I:

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

PART II. If $G(x)$ is *any* antiderivative of $f(x)$ on I, so that $G'(x) = f(x)$ on I, then for any b in I we have

$$\int_a^b f(x)\,dx = G(b) - G(a).$$

The Fundamental Theorem of Calculus

DEFINITION

To facilitate the evaluation of definite integrals using the Fundamental Theorem of Calculus, we define the **evaluation symbol**:

$$F(x)\Big|_a^b = F(b) - F(a).$$

The Fundamental Theorem of Calculus

EXAMPLE Evaluate (a) $\int_0^a x^2\,dx$ and (b) $\int_{-1}^2 (x^2 - 3x + 2)\,dx$.

The Fundamental Theorem of Calculus

EXAMPLE Evaluate (a) $\int_0^a x^2\,dx$ and (b) $\int_{-1}^2 (x^2 - 3x + 2)\,dx$.

Solution

(a) $\displaystyle\int_0^a x^2\,dx = \frac{1}{3}x^3\Big|_0^a = \frac{1}{3}a^3 - \frac{1}{3}0^3 = \frac{a^3}{3}$ (because $\dfrac{d}{dx}\dfrac{x^3}{3} = x^2$).

The Fundamental Theorem of Calculus

EXAMPLE Evaluate (a) $\int_0^a x^2\,dx$ and (b) $\int_{-1}^2 (x^2-3x+2)\,dx$.

Solution

(a) $\int_0^a x^2\,dx = \frac{1}{3}x^3\Big|_0^a = \frac{1}{3}a^3 - \frac{1}{3}0^3 = \frac{a^3}{3}$ (because $\frac{d}{dx}\frac{x^3}{3} = x^2$).

(b) $$\int_{-1}^2 (x^2-3x+2)\,dx = \left(\frac{1}{3}x^3 - \frac{3}{2}x^2 + 2x\right)\Bigg|_{-1}^2$$
$$= \frac{1}{3}(8) - \frac{3}{2}(4) + 4 - \left(\frac{1}{3}(-1) - \frac{3}{2}(1) + (-2)\right) = \frac{9}{2}.$$

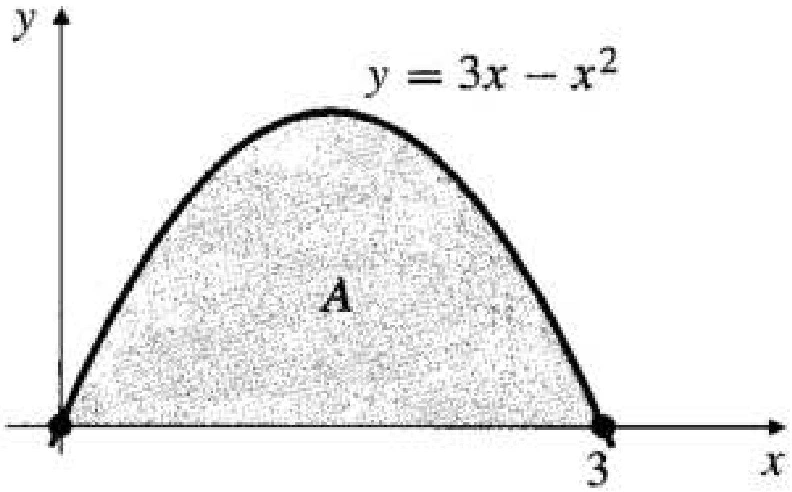
The Fundamental Theorem of Calculus

EXAMPLE

Find the area A of the plane region lying above the x-axis and under the curve $y = 3x - x^2$.

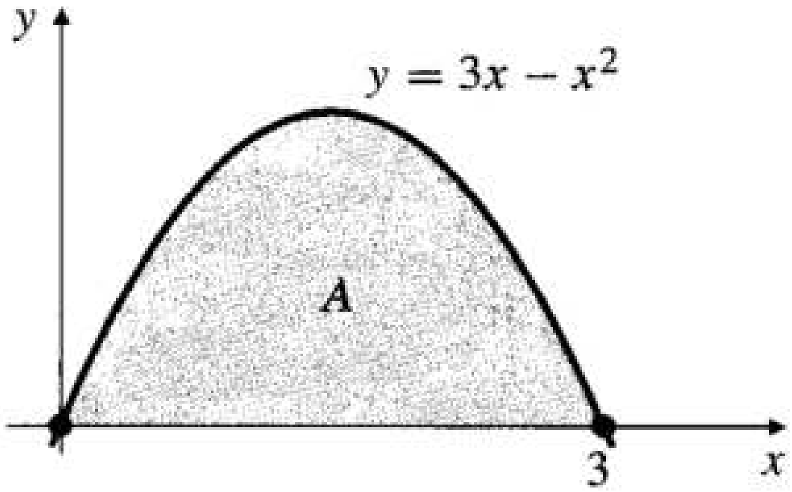
The Fundamental Theorem of Calculus

EXAMPLE Find the area A of the plane region lying above the x-axis and under the curve $y = 3x - x^2$.



The Fundamental Theorem of Calculus

EXAMPLE Find the area A of the plane region lying above the x-axis and under the curve $y = 3x - x^2$.



$$A = \int_0^3 (3x - x^2)\,dx = \left(\frac{3}{2}x^2 - \frac{1}{3}x^3\right)\Bigg|_0^3$$

$$= \frac{27}{2} - \frac{27}{3} - (0 - 0) = \frac{27}{6} = \frac{9}{2} \text{ square units.}$$

The Fundamental Theorem of Calculus

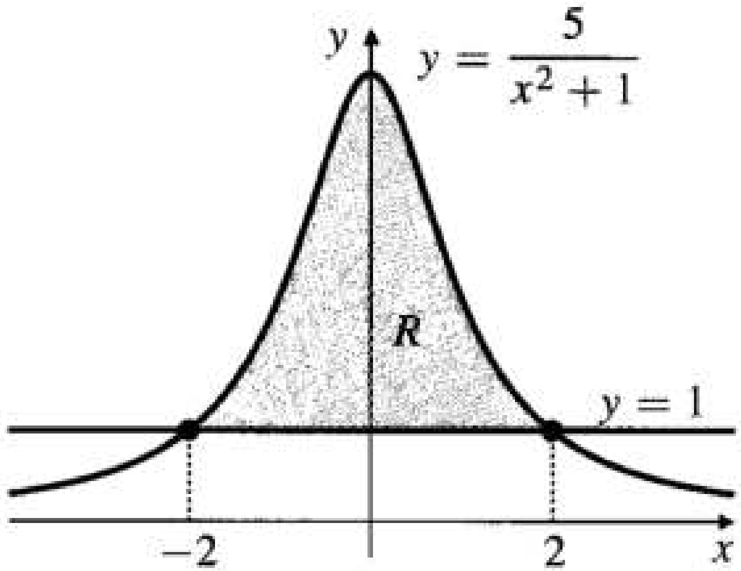
EXAMPLE

Find the area of the region R lying above the line $y = 1$ and below the curve $y = 5/(x^2 + 1)$.

The Fundamental Theorem of Calculus

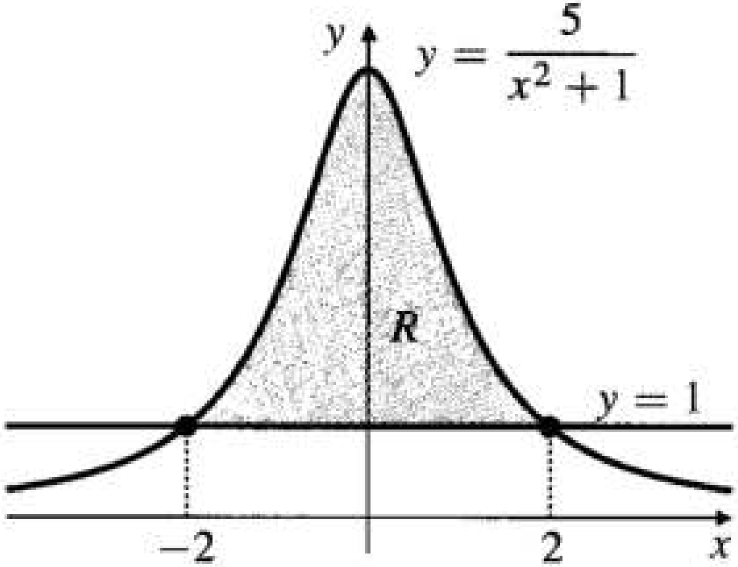
EXAMPLE

Find the area of the region R lying above the line $y = 1$ and below the curve $y = 5/(x^2 + 1)$.



The Fundamental Theorem of Calculus

EXAMPLE Find the area of the region R lying above the line $y = 1$ and below the curve $y = 5/(x^2 + 1)$.



$$A = \int_{-2}^{2} \frac{5}{x^2+1}\,dx - 4 = 2\int_{0}^{2} \frac{5}{x^2+1}\,dx - 4$$

$$= 10\tan^{-1} x \Big|_0^2 - 4 = 10\tan^{-1} 2 - 4 \text{ square units.}$$

The Fundamental Theorem of Calculus

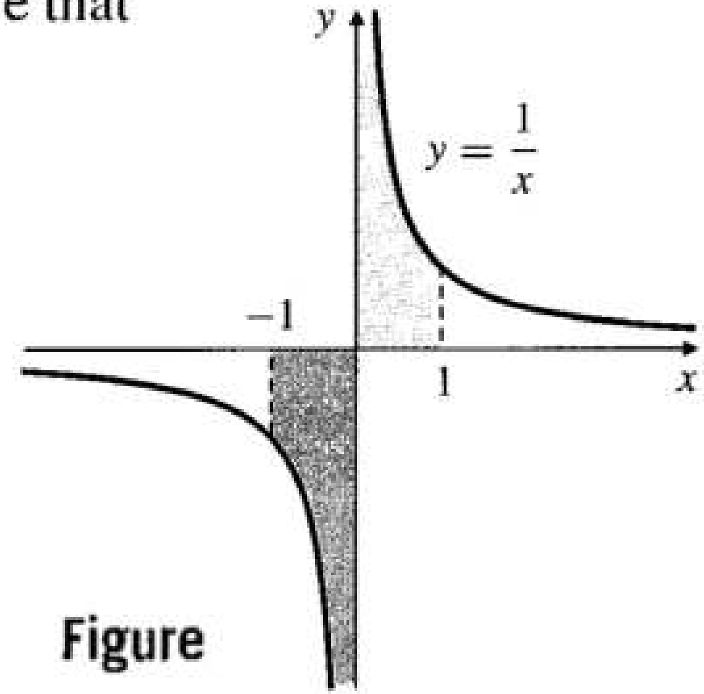
EXAMPLE We know that $\frac{d}{dx} \ln |x| = \frac{1}{x}$ if $x \neq 0$. It is *incorrect*, however, to state that

$$\int_{-1}^{1} \frac{dx}{x} = \ln |x| \Big|_{-1}^{1} = 0 - 0 = 0,$$

even though $1/x$ is an odd function. In fact, $1/x$ is undefined and has no limit at $x = 0$, and it is not integrable on $[-1, 0]$ or $[0, 1]$. Observe that

$$\lim_{c \to 0+} \int_{c}^{1} \frac{1}{x}\, dx = \lim_{c \to 0+} -\ln c = \infty,$$

so both shaded regions in Figure have infinite area.



Figure

The Fundamental Theorem of Calculus

EXAMPLE Find the derivatives of the following functions:

(a) $F(x) = \int_{x}^{3} e^{-t^2}\,dt$, (b) $G(x) = x^2 \int_{-4}^{5x} e^{-t^2}\,dt$, (c) $H(x) = \int_{x^2}^{x^3} e^{-t^2}\,dt$.

The Fundamental Theorem of Calculus

EXAMPLE Find the derivatives of the following functions:

(a) $F(x) = \int_x^3 e^{-t^2}\,dt$, (b) $G(x) = x^2 \int_{-4}^{5x} e^{-t^2}\,dt$, (c) $H(x) = \int_{x^2}^{x^3} e^{-t^2}\,dt$.

(a) Observe that $F(x) = -\int_3^x e^{-t^2}\,dt$. Therefore, by the Fundamental Theorem,

$F'(x) = -e^{-x^2}$.

The Fundamental Theorem of Calculus

EXAMPLE Find the derivatives of the following functions:

(a) $F(x) = \int_x^3 e^{-t^2}\,dt$, (b) $G(x) = x^2 \int_{-4}^{5x} e^{-t^2}\,dt$, (c) $H(x) = \int_{x^2}^{x^3} e^{-t^2}\,dt$.

(a) Observe that $F(x) = -\int_3^x e^{-t^2}\,dt$. Therefore, by the Fundamental Theorem, $F'(x) = -e^{-x^2}$.

(b) By the Product Rule and the Chain Rule,

$$\begin{aligned} G'(x) &= 2x \int_{-4}^{5x} e^{-t^2}\,dt + x^2 \frac{d}{dx} \int_{-4}^{5x} e^{-t^2}\,dt \\ &= 2x \int_{-4}^{5x} e^{-t^2}\,dt + x^2 e^{-(5x)^2}(5) \\ &= 2x \int_{-4}^{5x} e^{-t^2}\,dt + 5x^2 e^{-25x^2}. \end{aligned}$$

The Fundamental Theorem of Calculus

EXAMPLE Find the derivatives of the following functions:

(a) $F(x) = \int_x^3 e^{-t^2}\,dt$, (b) $G(x) = x^2 \int_{-4}^{5x} e^{-t^2}\,dt$, (c) $H(x) = \int_{x^2}^{x^3} e^{-t^2}\,dt$.

(c) Split the integral into a difference of two integrals in each of which the variable x appears only in the upper limit.

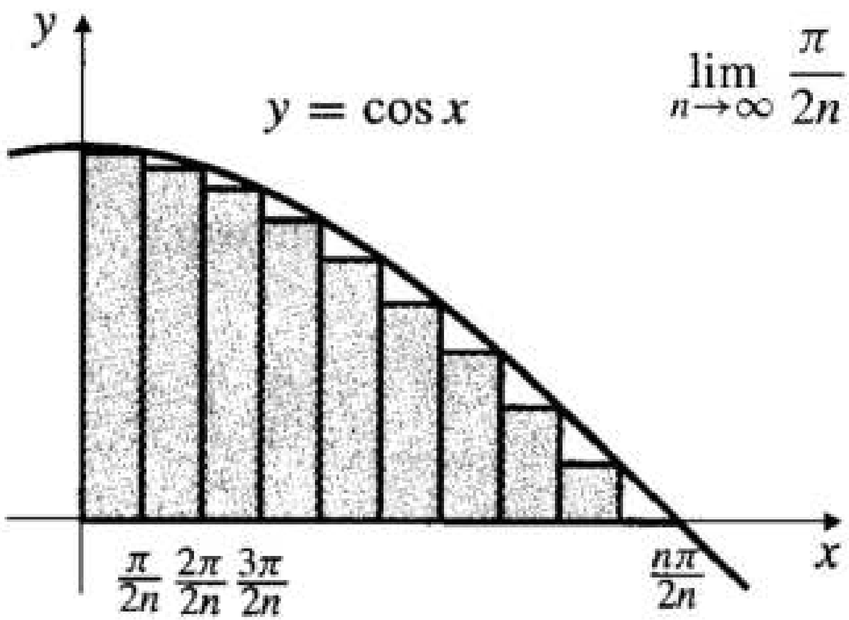
$$H(x) = \int_0^{x^3} e^{-t^2}\,dt - \int_0^{x^2} e^{-t^2}\,dt$$

$$H'(x) = e^{-(x^3)^2}(3x^2) - e^{-(x^2)^2}(2x)$$

$$= 3x^2 e^{-x^6} - 2x\,e^{-x^4}.$$

The Fundamental Theorem of Calculus

EXAMPLE Evaluate $\lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} \cos\left(\frac{j\pi}{2n}\right)$.



$$\lim_{n\to\infty} \frac{\pi}{2n}\sum_{j=1}^{n} \cos\left(\frac{j\pi}{2n}\right) = \int_0^{\pi/2} \cos x\, dx = \sin x\Big|_0^{\pi/2} = 1 - 0 = 1.$$

$$\lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} \cos\left(\frac{j\pi}{2n}\right) = \frac{2}{\pi}.$$

The Method of Substitution

DEFINITION

The **indefinite integral** of $f(x)$ on interval I is

$$\int f(x)\,dx = F(x) + C \qquad \text{on } I,$$

provided $F'(x) = f(x)$ for all x in I.

The Method of Substitution

Some elementary integrals

1. $\displaystyle\int 1\,dx = x + C$

2. $\displaystyle\int x\,dx = \frac{1}{2}x^2 + C$

3. $\displaystyle\int x^2\,dx = \frac{1}{3}x^3 + C$

4. $\displaystyle\int \frac{1}{x^2}\,dx = -\frac{1}{x} + C$

5. $\displaystyle\int \sqrt{x}\,dx = \frac{2}{3}x^{3/2} + C$

6. $\displaystyle\int \frac{1}{\sqrt{x}}\,dx = 2\sqrt{x} + C$

7. $\displaystyle\int x^r\,dx = \frac{1}{r+1}x^{r+1} + C \quad (r \neq -1)$

8. $\displaystyle\int \frac{1}{x}\,dx = \ln|x| + C$

9. $\displaystyle\int \sin ax\,dx = -\frac{1}{a}\cos ax + C$

10. $\displaystyle\int \cos ax\,dx = \frac{1}{a}\sin ax + C$

11. $\displaystyle\int \sec^2 ax\,dx = \frac{1}{a}\tan ax + C$

12. $\displaystyle\int \csc^2 ax\,dx = -\frac{1}{a}\cot ax + C$

13. $\displaystyle\int \sec ax\,\tan ax\,dx = \frac{1}{a}\sec ax + C$

14. $\displaystyle\int \csc ax\cot ax\,dx = -\frac{1}{a}\csc ax + C$

15. $\displaystyle\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\frac{x}{a} + C \quad (a > 0)$

16. $\displaystyle\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$

17. $\displaystyle\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C$

18. $\displaystyle\int b^{ax}\,dx = \frac{1}{a\ln b}b^{ax} + C$

19. $\displaystyle\int \cosh ax\,dx = \frac{1}{a}\sinh ax + C$

20. $\displaystyle\int \sinh ax\,dx = \frac{1}{a}\cosh ax + C$

The Method of Substitution

EXAMPLE **(Combining elementary integrals)**

(a) $$\int (x^4 - 3x^3 + 8x^2 - 6x - 7)\,dx = \frac{x^5}{5} - \frac{3x^4}{4} + \frac{8x^3}{3} - 3x^2 - 7x + C$$

(b) $$\int \left(5x^{3/5} - \frac{3}{2+x^2}\right) dx = \frac{25}{8}x^{8/5} - \frac{3}{\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}} + C$$

(c) $$\int (4\cos 5x - 5\sin 3x)\,dx = \frac{4}{5}\sin 5x + \frac{5}{3}\cos 3x + C$$

(d) $$\int \left(\frac{1}{\pi x} + a^{\pi x}\right) dx = \frac{1}{\pi}\ln|x| + \frac{1}{\pi \ln a}a^{\pi x} + C, \quad (a > 0).$$

The Method of Substitution

EXAMPLE

$$\int \frac{(x+1)^3}{x}\,dx = \int \frac{x^3+3x^2+3x+1}{x}\,dx$$
$$= \int \left(x^2+3x+3+\frac{1}{x}\right)dx$$
$$= \frac{1}{3}x^3+\frac{3}{2}x^2+3x+\ln|x|+C.$$

The Method of Substitution

$$\frac{d}{dx} f\big(g(x)\big) = f'\big(g(x)\big)\, g'(x) \longrightarrow \int f'\big(g(x)\big)\, g'(x)\, dx = f\big(g(x)\big) + C.$$

Let $u = g(x)$. Then $du/dx = g'(x)$, or in differential form, $du = g'(x)\,dx$. Thus,

$$\int f'\underbrace{(g(x))}_{u}\ \underbrace{g'(x)\,dx}_{du} = \int f'(u)\,du = f(u) + C = f\,(g(x)) + C.$$

The Method of Substitution

EXAMPLE **(Examples of substitution)** Find the indefinite integrals:

(a) $\displaystyle\int \frac{x}{x^2+1}\,dx$, (b) $\displaystyle\int \frac{\sin(3\ln x)}{x}\,dx$, and (c) $\displaystyle\int e^x\sqrt{1+e^x}\,dx$.

The Method of Substitution

EXAMPLE **(Examples of substitution)** Find the indefinite integrals:

(a) $\displaystyle\int \frac{x}{x^2+1}\,dx$, (b) $\displaystyle\int \frac{\sin(3\ln x)}{x}\,dx$, and (c) $\displaystyle\int e^x\sqrt{1+e^x}\,dx$.

Solution

(a) $\displaystyle\int \frac{x}{x^2+1}\,dx$ Let $u = x^2 + 1$.

Then $du = 2x\,dx$ and

$x\,dx = \frac{1}{2}\,du$

$$= \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln(x^2+1) + C = \ln\sqrt{x^2+1} + C.$$

The Method of Substitution

EXAMPLE **(Examples of substitution)** Find the indefinite integrals:

(a) $\displaystyle\int \frac{x}{x^2+1}\,dx,$ (b) $\displaystyle\int \frac{\sin(3\ln x)}{x}\,dx,$ and (c) $\displaystyle\int e^x\sqrt{1+e^x}\,dx.$

Solution

(b) $\displaystyle\int \frac{\sin(3\ln x)}{x}\,dx$ Let $u = 3\ln x$.

Then $du = \dfrac{3}{x}\,dx$

$$= \frac{1}{3}\int \sin u\,du = -\frac{1}{3}\cos u + C = -\frac{1}{3}\cos(3\ln x) + C.$$

The Method of Substitution

EXAMPLE **(Examples of substitution)** Find the indefinite integrals:

(a) $\displaystyle\int \frac{x}{x^2+1}\,dx$, (b) $\displaystyle\int \frac{\sin(3\ln x)}{x}\,dx$, and (c) $\displaystyle\int e^x\sqrt{1+e^x}\,dx$.

Solution

(c) $\displaystyle\int e^x\sqrt{1+e^x}\,dx$ Let $v = 1 + e^x$.

Then $dv = e^x\,dx$

$$= \int v^{1/2}\,dv = \frac{2}{3}v^{3/2} + C = \frac{2}{3}(1+e^x)^{3/2} + C.$$

The Method of Substitution

EXAMPLE Evaluate (a) $\displaystyle\int \frac{1}{x^2+4x+5}\,dx$ and (b) $\displaystyle\int \frac{dx}{\sqrt{e^{2x}-1}}$.

The Method of Substitution

EXAMPLE Evaluate (a) $\displaystyle\int \frac{1}{x^2+4x+5}\,dx$ and (b) $\displaystyle\int \frac{dx}{\sqrt{e^{2x}-1}}$.

Solution

(a) $$\int \frac{dx}{x^2+4x+5} = \int \frac{dx}{(x+2)^2+1}$$ Let $t = x+2$. Then $dt = dx$.

$$= \int \frac{dt}{t^2+1}$$

$$= \tan^{-1} t + C = \tan^{-1}(x+2) + C.$$

The Method of Substitution

EXAMPLE Evaluate (a) $\int \frac{1}{x^2+4x+5}\,dx$ and (b) $\int \frac{dx}{\sqrt{e^{2x}-1}}$.

Solution

$$\text{(b)} \quad \int \frac{dx}{\sqrt{e^{2x}-1}} = \int \frac{dx}{e^x\sqrt{1-e^{-2x}}}$$

$$= \int \frac{e^{-x}\,dx}{\sqrt{1-(e^{-x})^2}} \qquad \text{Let } u = e^{-x}. \quad \text{Then } du = -e^{-x}\,dx.$$

$$= -\int \frac{du}{\sqrt{1-u^2}}$$

$$= -\sin^{-1} u + C = -\sin^{-1}\left(e^{-x}\right) + C.$$

The Method of Substitution

THEOREM

Substitution in a definite integral

Suppose that g is a differentiable function on $[a, b]$ that satisfies $g(a) = A$ and $g(b) = B$. Also suppose that f is continuous on the range of g. Then

$$\int_a^b f\big(g(x)\big)\, g'(x)\, dx = \int_A^B f(u)\, du.$$

The Method of Substitution

EXAMPLE Evaluate the integral $I = \displaystyle\int_0^8 \frac{\cos\sqrt{x+1}}{\sqrt{x+1}}\,dx$.

The Method of Substitution

EXAMPLE Evaluate the integral $I = \displaystyle\int_0^8 \frac{\cos\sqrt{x+1}}{\sqrt{x+1}}\,dx$.

Solution **METHOD I.** Let $u = \sqrt{x+1}$. Then $du = \dfrac{dx}{2\sqrt{x+1}}$. If $x = 0$, then $u = 1$; if $x = 8$, then $u = 3$. Thus

$$I = 2\int_1^3 \cos u\,du = 2\sin u\Big|_1^3 = 2\sin 3 - 2\sin 1.$$

The Method of Substitution

EXAMPLE Evaluate the integral $I = \int_0^8 \frac{\cos\sqrt{x+1}}{\sqrt{x+1}}\,dx$.

Solution

METHOD II. We use the same substitution as in Method I, but we do not transform the limits of integration from x values to u values. Hence, we must return to the variable x before substituting in the limits:

$$I = 2\int_{x=0}^{x=8} \cos u\,du = 2\sin u\Big|_{x=0}^{x=8} = 2\sin\sqrt{x+1}\Big|_0^8 = 2\sin 3 - 2\sin 1.$$

The Method of Substitution

EXAMPLE Find the area of the region bounded by $y = \left(2 + \sin \frac{x}{2}\right)^2 \cos \frac{x}{2}$, the x-axis, and the lines $x = 0$ and $x = \pi$.

Solution Because $y \geq 0$ when $0 \leq x \leq \pi$, the required area is

$$A = \int_0^\pi \left(2 + \sin \frac{x}{2}\right)^2 \cos \frac{x}{2}\, dx \qquad \text{Let } v = 2 + \sin \frac{x}{2}.$$

$$\text{Then } dv = \frac{1}{2} \cos \frac{x}{2}\, dx$$

$$= 2 \int_2^3 v^2\, dv = \frac{2}{3} v^3 \Big|_2^3 = \frac{2}{3}(27 - 8) = \frac{38}{3} \text{ square units.}$$

The Method of Substitution

Trigonometric Integrals

Integrals of tangent, cotangent, secant, and cosecant

$$\int \tan x \, dx = \ln|\sec x| + C,$$

$$\int \cot x \, dx = \ln|\sin x| + C = -\ln|\csc x| + C,$$

$$\int \sec x \, dx = \ln|\sec x + \tan x| + C,$$

$$\int \csc x \, dx = -\ln|\csc x + \cot x| + C = \ln|\csc x - \cot x| + C.$$

The Method of Substitution

Trigonometric Integrals

$$\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx \qquad \text{Let } u = \cos x.$$

$$\text{Then } du = -\sin x \, dx.$$

$$= -\int \frac{du}{u} = -\ln |u| + C$$

$$= -\ln |\cos x| + C = \ln \left| \frac{1}{\cos x} \right| + C = \ln |\sec x| + C.$$

Trigonometric Integrals

The integral of $\sec x$ can be evaluated by rewriting it in the form

$$\int \sec x\, dx = \int \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x}\, dx$$

and using the substitution $u = \sec x + \tan x$. The integral of $\csc x$ can be evaluated similarly.

The Method of Substitution

Trigonometric Integrals

We now consider integrals of the form

$$\int \sin^m x \cos^n x \, dx.$$

If either m or n is an odd, positive integer, the integral can be done easily by substitution.

The Method of Substitution

Trigonometric Integrals

We now consider integrals of the form

$$\int \sin^m x \, \cos^n x \, dx.$$

If either m or n is an odd, positive integer, the integral can be done easily by substitution.

EXAMPLE Evaluate: (a) $\int \sin^3 x \, \cos^8 x \, dx$ and (b) $\int \cos^5 ax \, dx$.

The Method of Substitution

Trigonometric Integrals

We now consider integrals of the form

$$\int \sin^m x \cos^n x \, dx.$$

If either m or n is an odd, positive integer, the integral can be done easily by substitution.

EXAMPLE Evaluate: (a) $\int \sin^3 x \cos^8 x \, dx$ and (b) $\int \cos^5 ax \, dx$.

Solution

(a) $$\int \sin^3 x \cos^8 x \, dx = \int (1 - \cos^2 x) \cos^8 x \sin x \, dx \qquad \text{Let } u = \cos x,$$
$$du = -\sin x \, dx.$$
$$= -\int (1 - u^2) u^8 \, du = \int (u^{10} - u^8) \, du$$
$$= \frac{u^{11}}{11} - \frac{u^9}{9} + C = \frac{1}{11} \cos^{11} x - \frac{1}{9} \cos^9 x + C.$$

The Method of Substitution

Trigonometric Integrals

We now consider integrals of the form

$$\int \sin^m x \cos^n x\, dx.$$

If either m or n is an odd, positive integer, the integral can be done easily by substitution.

EXAMPLE Evaluate: (a) $\displaystyle\int \sin^3 x \cos^8 x\, dx$ and (b) $\displaystyle\int \cos^5 ax\, dx$.

Solution (b)

$$\begin{aligned}
\int \cos^5 ax\, dx &= \int (1-\sin^2 ax)^2 \cos ax\, dx \qquad \text{Let } u = \sin ax,\ du = a\cos ax\, dx.\\
&= \frac{1}{a}\int (1-u^2)^2\, du = \frac{1}{a}\int (1-2u^2+u^4)\, du\\
&= \frac{1}{a}\left(u - \frac{2}{3}u^3 + \frac{1}{5}u^5\right) + C\\
&= \frac{1}{a}\left(\sin ax - \frac{2}{3}\sin^3 ax + \frac{1}{5}\sin^5 ax\right) + C.
\end{aligned}$$

The Method of Substitution

Trigonometric Integrals

If the powers of $\sin x$ and $\cos x$ are both even, then we can make use of the *double-angle formulas*

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x) \qquad \text{and} \qquad \sin^2 x = \frac{1}{2}(1 - \cos 2x).$$

The Method of Substitution

Trigonometric Integrals

If the powers of $\sin x$ and $\cos x$ are both even, then we can make use of the *double-angle formulas*

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x) \qquad \text{and} \qquad \sin^2 x = \frac{1}{2}(1 - \cos 2x).$$

EXAMPLE Evaluate $\int \sin^4 x \, dx$.

The Method of Substitution

Trigonometric Integrals

If the powers of $\sin x$ and $\cos x$ are both even, then we can make use of the *double-angle formulas*

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x) \qquad \text{and} \qquad \sin^2 x = \frac{1}{2}(1 - \cos 2x).$$

EXAMPLE Evaluate $\int \sin^4 x\, dx$.

$$\begin{aligned}
\int \sin^4 x\, dx &= \frac{1}{4}\int (1 - \cos 2x)^2\, dx \\
&= \frac{1}{4}\int (1 - 2\cos 2x + \cos^2 2x)\, dx \\
&= \frac{x}{4} - \frac{1}{4}\sin 2x + \frac{1}{8}\int (1 + \cos 4x)\, dx \\
&= \frac{x}{4} - \frac{1}{4}\sin 2x + \frac{x}{8} + \frac{1}{32}\sin 4x + C \\
&= \frac{3}{8}x - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C
\end{aligned}$$

The Method of Substitution

Trigonometric Integrals

$$\int \sec^m x \, \tan^n x \, dx \qquad \text{or} \qquad \int \csc^m x \, \cot^n x \, dx, \quad \text{unless } m \text{ is odd and } n \text{ is even.}$$

The Method of Substitution

Trigonometric Integrals

$$\int \sec^m x \, \tan^n x \, dx \qquad \text{or} \qquad \int \csc^m x \, \cot^n x \, dx, \quad \text{unless } m \text{ is odd and } n \text{ is even.}$$

EXAMPLE **(Integrals involving secants and tangents)** Evaluate the following integrals:

(a) $\displaystyle\int \tan^2 x \, dx$, (b) $\displaystyle\int \sec^4 t \, dt$, and (c) $\displaystyle\int \sec^3 x \, \tan^3 x \, dx$.

The Method of Substitution

Trigonometric Integrals

$$\int \sec^m x \tan^n x \, dx \qquad \text{or} \qquad \int \csc^m x \cot^n x \, dx, \quad \text{unless } m \text{ is odd and } n \text{ is even.}$$

EXAMPLE **(Integrals involving secants and tangents)** Evaluate the following integrals:

(a) $\int \tan^2 x \, dx$, (b) $\int \sec^4 t \, dt$, and (c) $\int \sec^3 x \tan^3 x \, dx$.

Solution

(a) $\int \tan^2 x \, dx = \int (\sec^2 x - 1) \, dx = \tan x - x + C.$

The Method of Substitution

Trigonometric Integrals

$$\int \sec^m x \, \tan^n x \, dx \qquad \text{or} \qquad \int \csc^m x \, \cot^n x \, dx, \quad \text{unless } m \text{ is odd and } n \text{ is even.}$$

EXAMPLE **(Integrals involving secants and tangents)** Evaluate the following integrals:

(a) $\int \tan^2 x \, dx$, (b) $\int \sec^4 t \, dt$, and (c) $\int \sec^3 x \, \tan^3 x \, dx$.

Solution

(a) $\displaystyle\int \tan^2 x \, dx = \int (\sec^2 x - 1) \, dx = \tan x - x + C.$

(b) $\displaystyle\int \sec^4 t \, dt = \int (1 + \tan^2 t) \sec^2 t \, dt$ Let $u = \tan t$,
$du = \sec^2 t \, dt$.

$$= \int (1 + u^2) \, du = u + \frac{1}{3}u^3 + C = \tan t + \frac{1}{3}\tan^3 t + C.$$

The Method of Substitution

Trigonometric Integrals

$$\int \sec^m x \tan^n x \, dx \qquad \text{or} \qquad \int \csc^m x \cot^n x \, dx, \quad \text{unless } m \text{ is odd and } n \text{ is even.}$$

EXAMPLE **(Integrals involving secants and tangents)** Evaluate the following integrals:

(a) $\int \tan^2 x \, dx$, (b) $\int \sec^4 t \, dt$, and (c) $\int \sec^3 x \tan^3 x \, dx$.

Solution

(c) $\int \sec^3 x \tan^3 x \, dx$

$$= \int \sec^2 x \, (\sec^2 x - 1) \sec x \tan x \, dx \qquad \text{Let } u = \sec x,$$

$$du = \sec x \tan x \, dx.$$

$$= \int (u^4 - u^2) \, du = \frac{u^5}{5} - \frac{u^3}{3} + C = \frac{1}{5} \sec^5 x - \frac{1}{3} \sec^3 x + C.$$