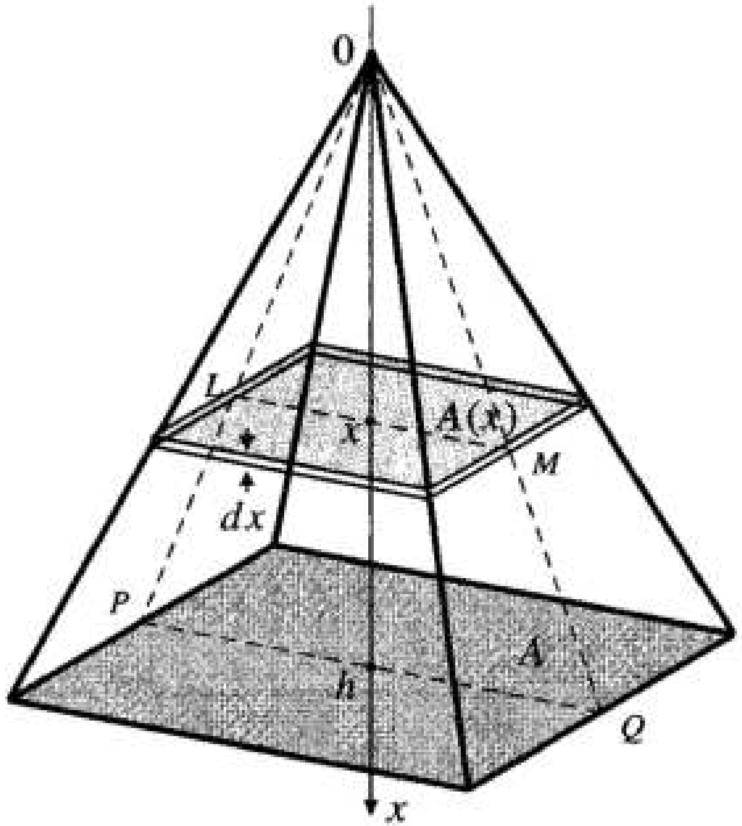
More Volumes by Slicing

EXAMPLE

Verify the formula for the volume of a pyramid with rectangular base of area A and height h.



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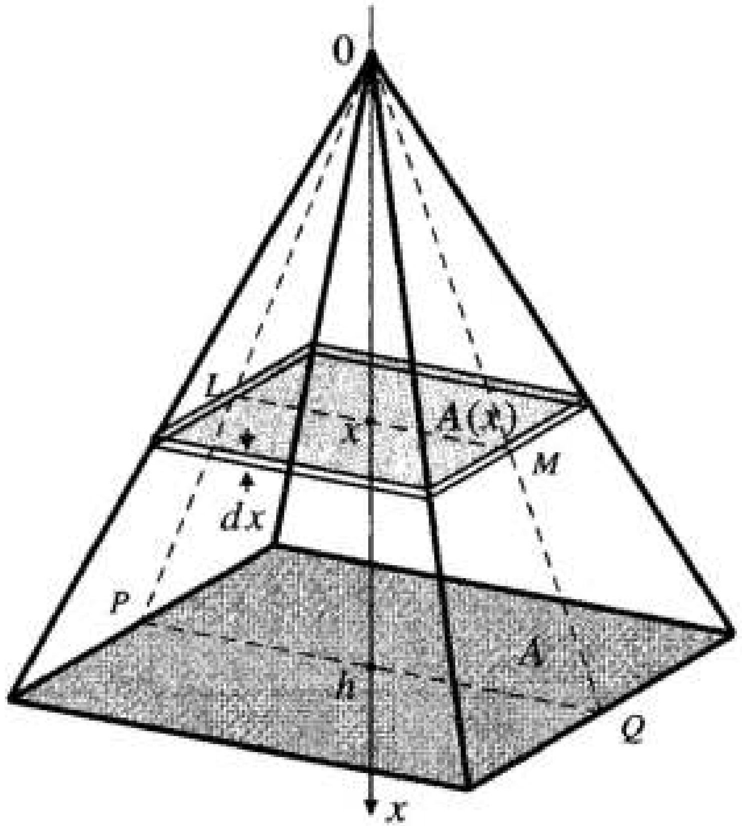

$$A(x) = \left(\frac{x}{h}\right)^2 A.$$

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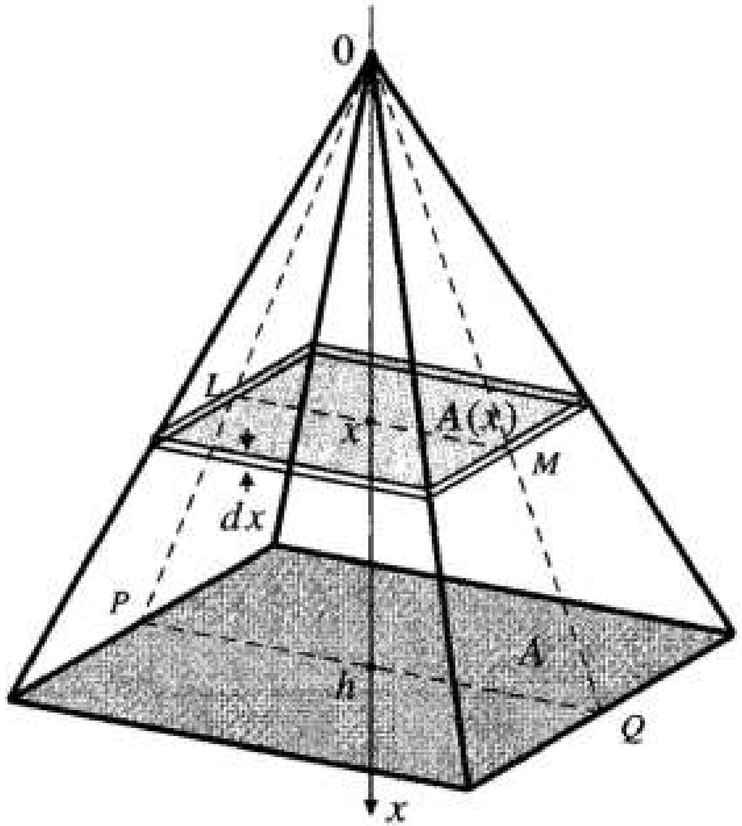
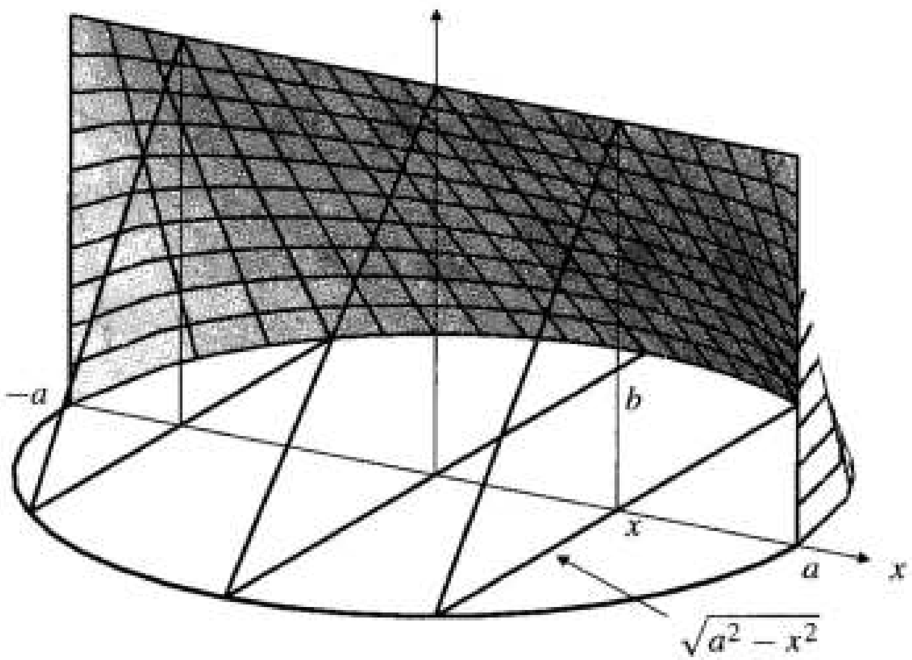


$$A(x) = \left(\frac{x}{h}\right)^2 A.$$

$$V = \int_0^h \left(\frac{x}{h}\right)^2 A\,dx = \frac{A}{h^2}\frac{x^3}{3}\bigg|_0^h = \frac{1}{3}Ah \text{ cubic units.}$$

More Volumes by Slicing

EXAMPLE A tent has a circular base of radius a metres and is supported by a horizontal ridge bar held at height b metres above a diameter of the base by vertical supports at each end of the diameter. The material of the tent is stretched tight so that each cross-section perpendicular to the ridge bar is an isosceles triangle. Find the volume of the tent.



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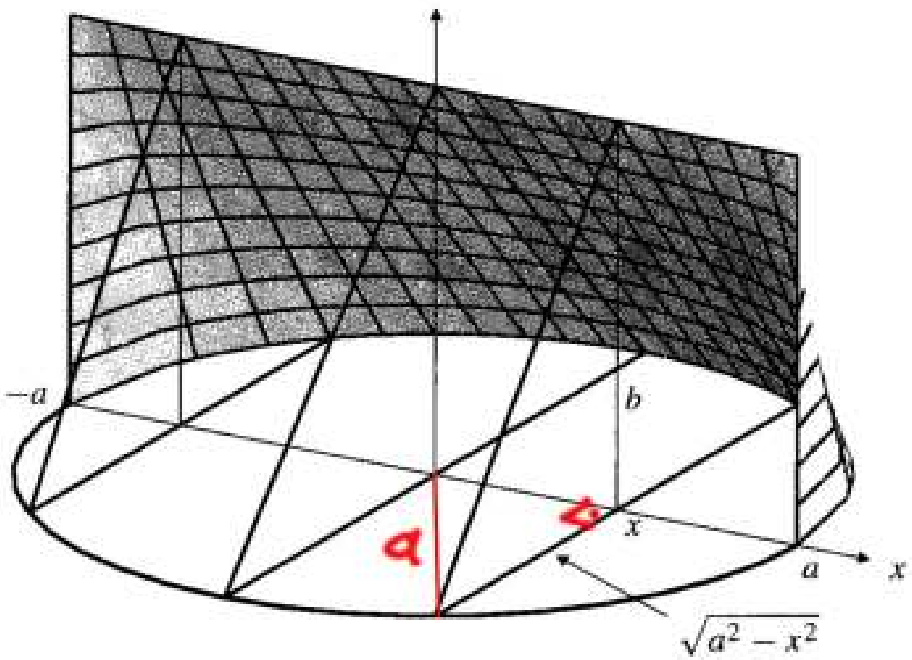

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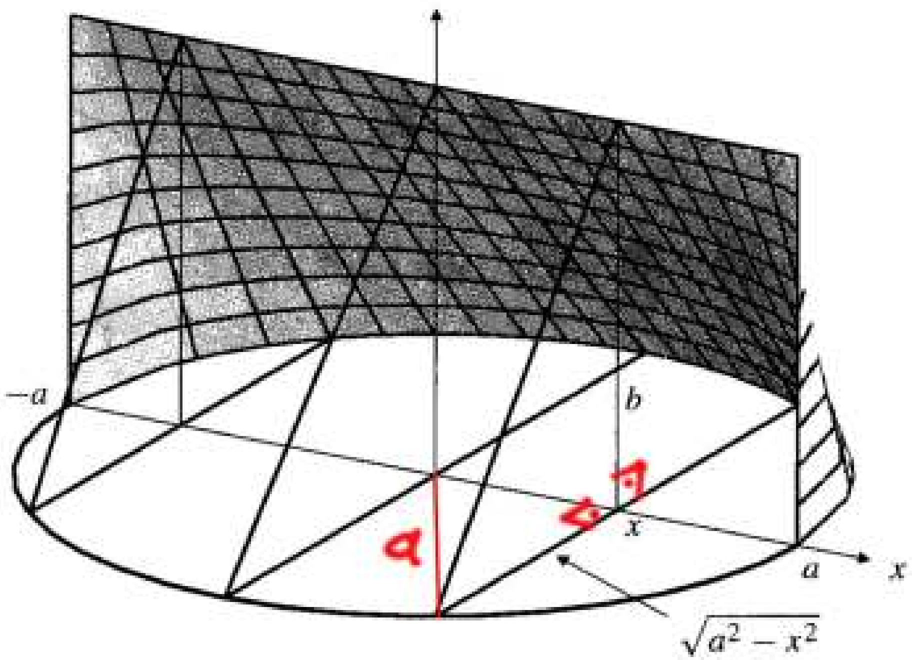

$$A(x) = \frac{1}{2}(2\sqrt{a^2 - x^2})b = b\sqrt{a^2 - x^2}$$

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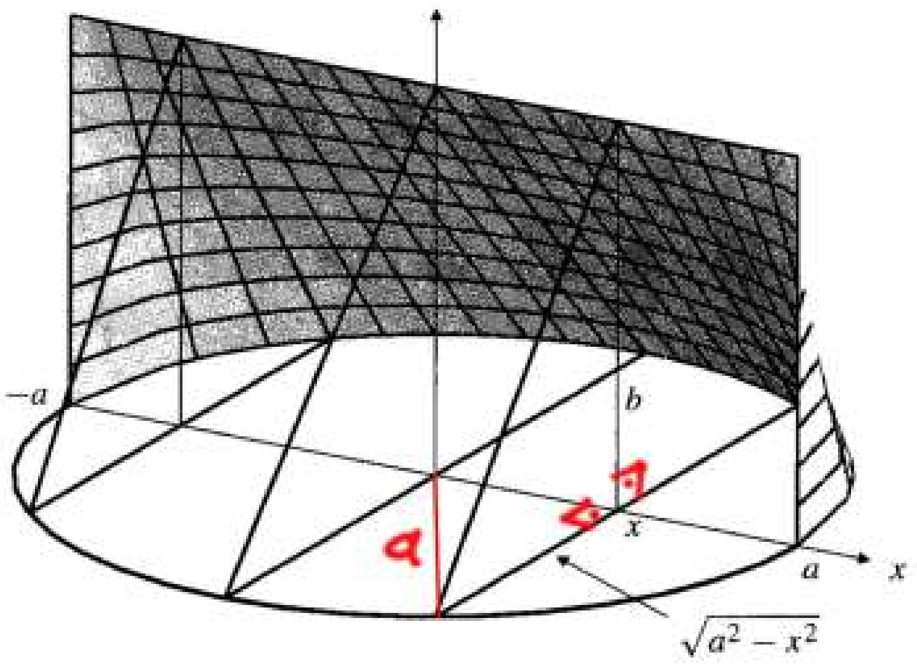

$$V = \int_{-a}^{a} b\sqrt{a^2 - x^2}\,dx = b\int_{-a}^{a} \sqrt{a^2 - x^2}\,dx$$

$$= b\frac{\pi a^2}{2} = \frac{\pi}{2}a^2 b \text{ m}^3.$$

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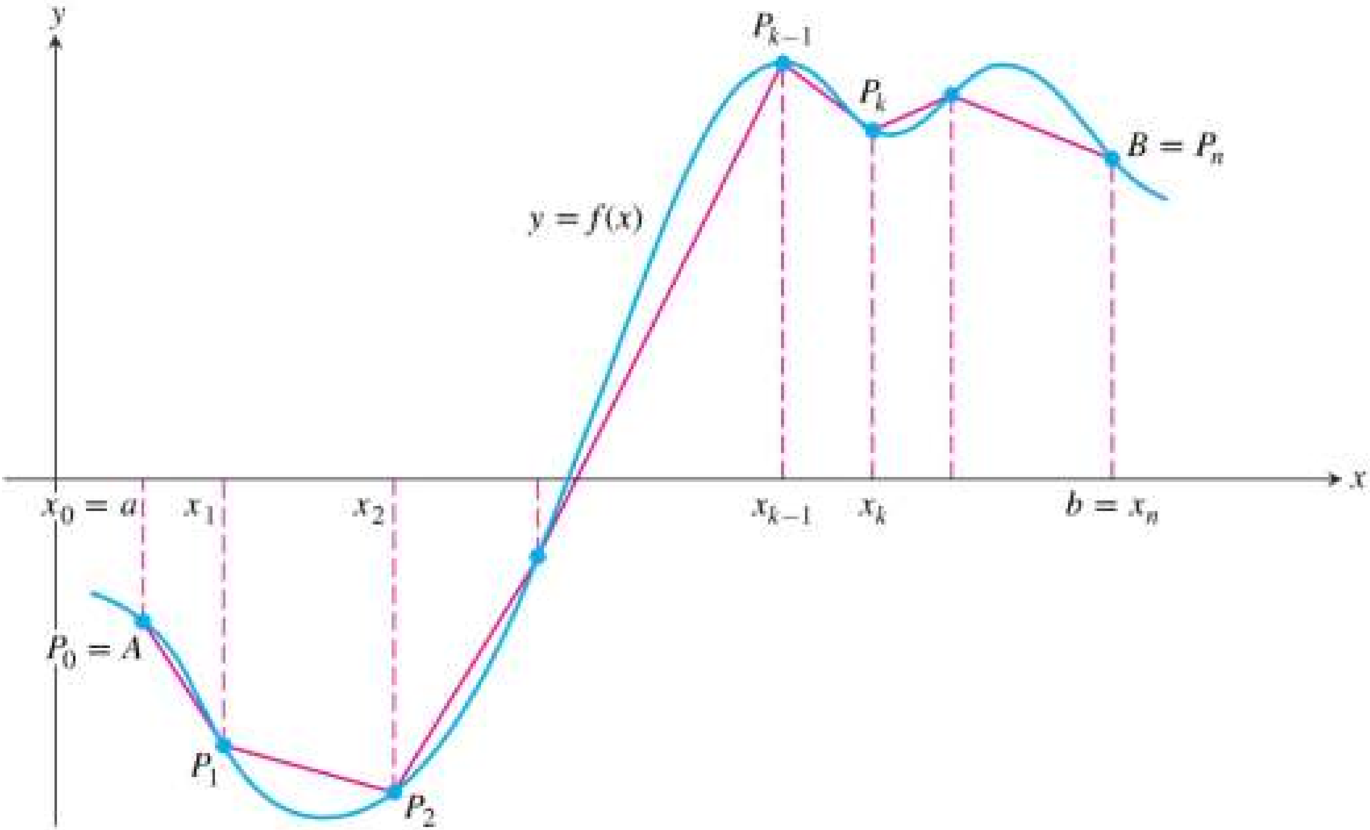
Arc Length and Surface Area



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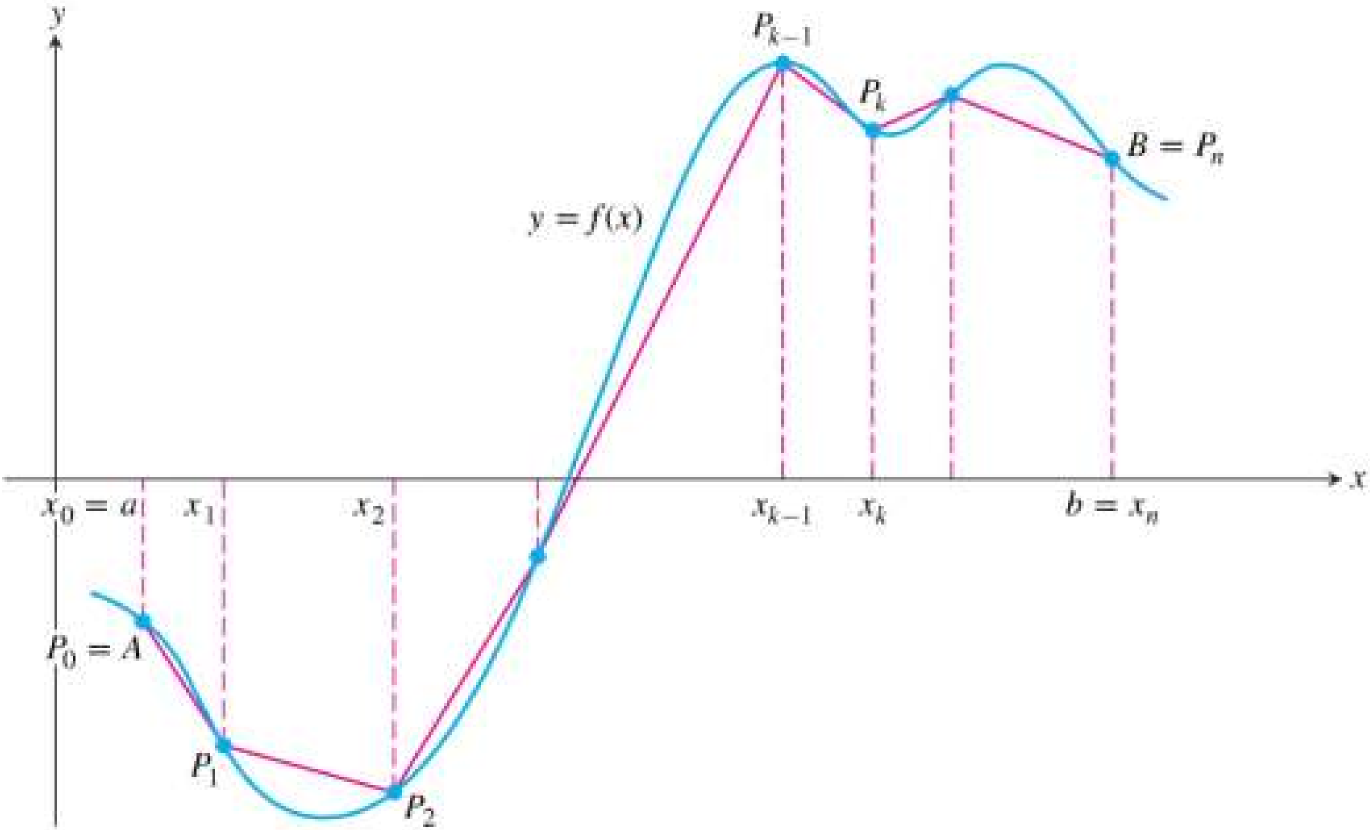
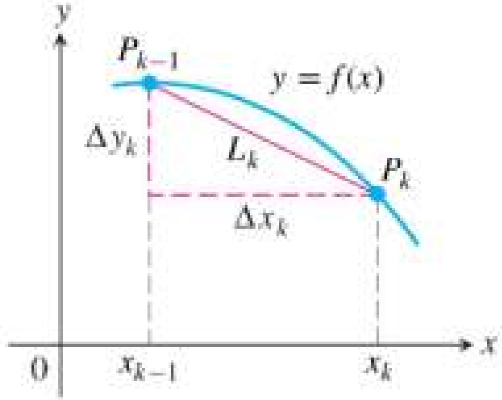
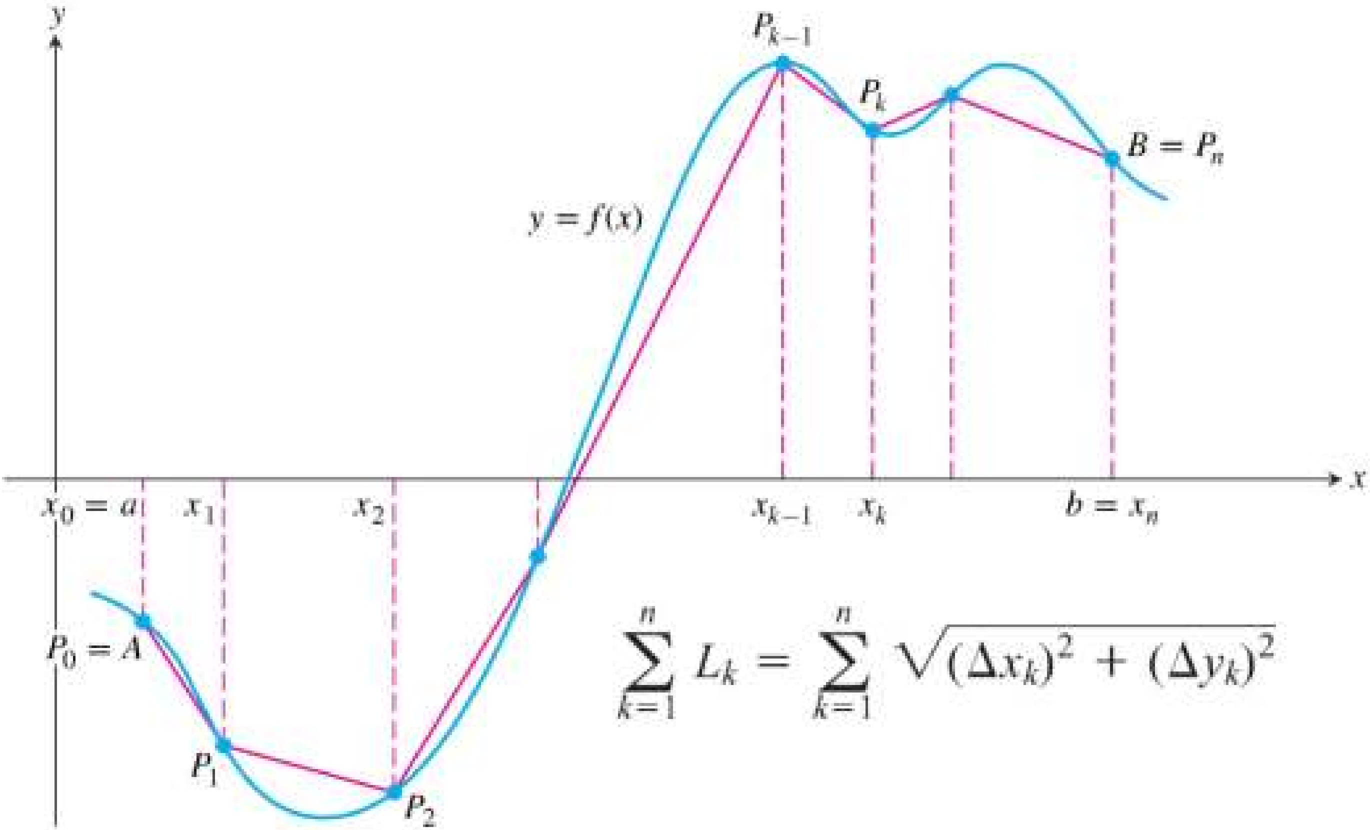


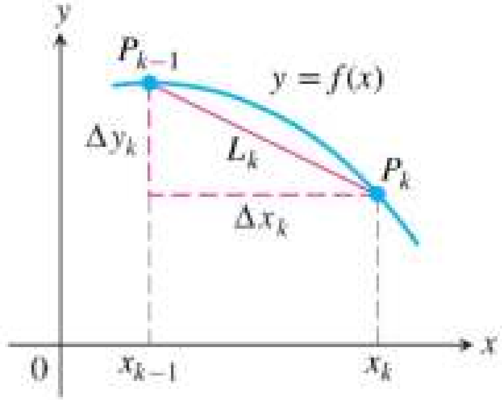


$$L_k = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$$

Arc Length and Surface Area



$$\sum_{k=1}^{n} L_k = \sum_{k=1}^{n} \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$$



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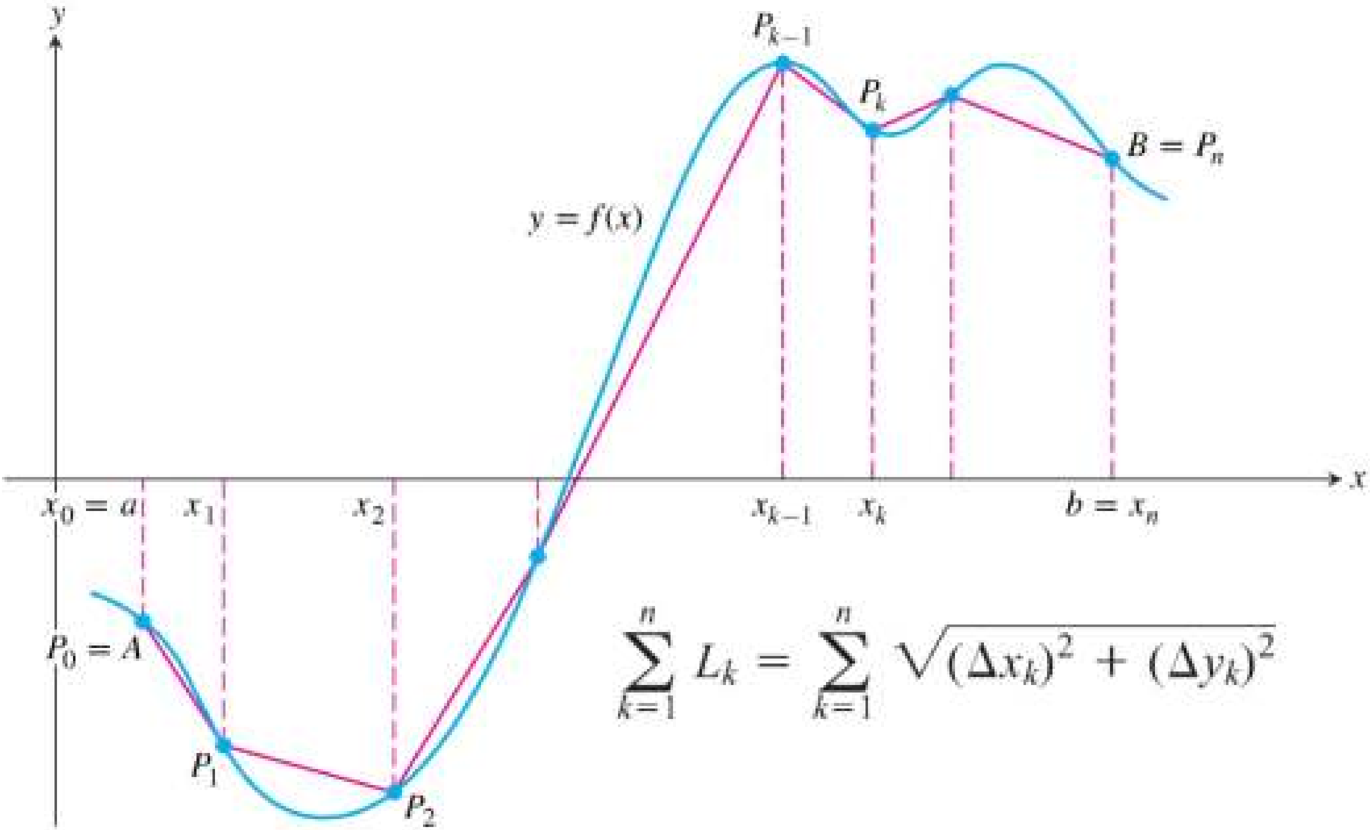

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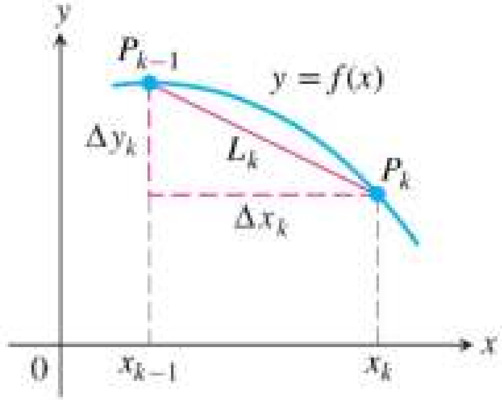

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Now, by the Mean Value Theorem, there is a point c_k, with $x_{k-1} < c_k < x_k$, such that

$$\Delta y_k = f'(c_k)\,\Delta x_k.$$

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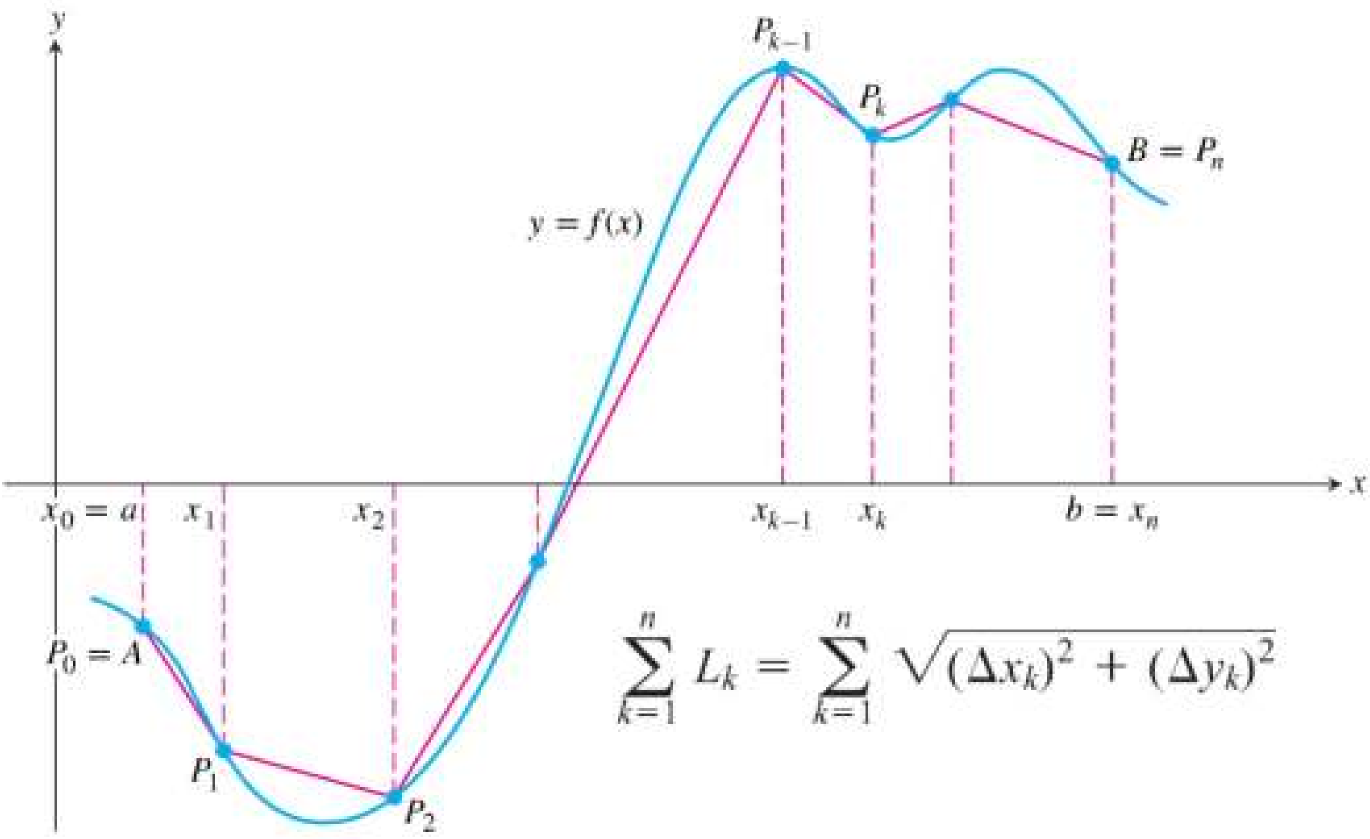

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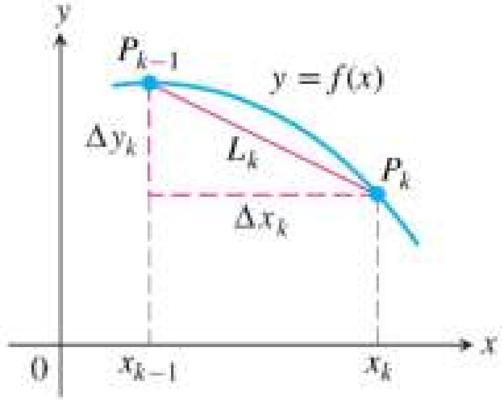

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$$\lim_{n\to\infty} \sum_{k=1}^{n} L_k = \lim_{n\to\infty} \sum_{k=1}^{n} \sqrt{1 + [f'(c_k)]^2}\,\Delta x_k = \int_a^b \sqrt{1 + [f'(x)]^2}\,dx$$

Arc Length and Surface Area

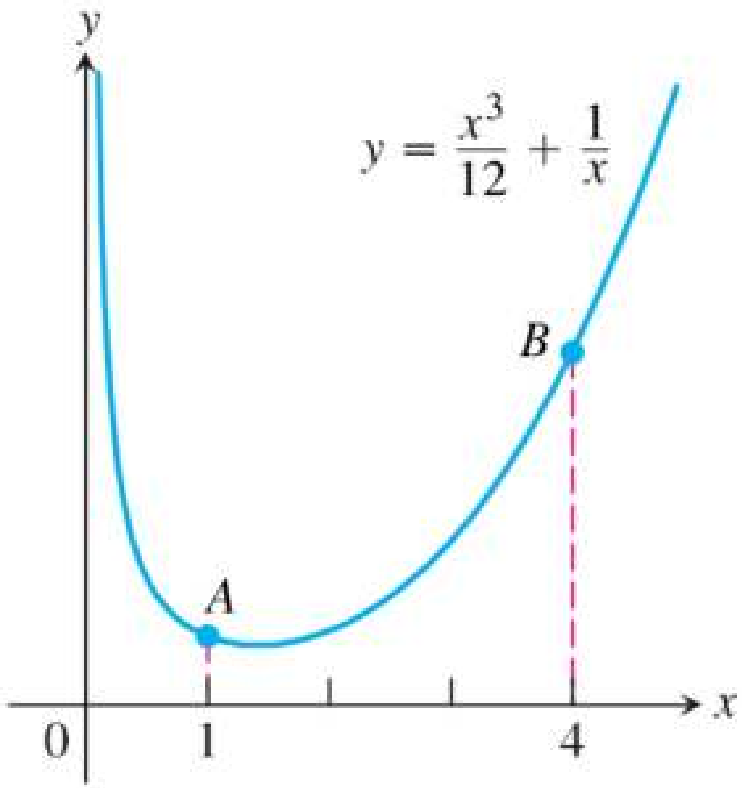
DEFINITION If f' is continuous on $[a, b]$, then the **length** (**arc length**) of the curve $y = f(x)$ from the point $A = (a, f(a))$ to the point $B = (b, f(b))$ is the value of the integral

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

Arc Length and Surface Area

EXAMPLE Find the length of the graph of

$$f(x) = \frac{x^3}{12} + \frac{1}{x}, \qquad 1 \leq x \leq 4.$$



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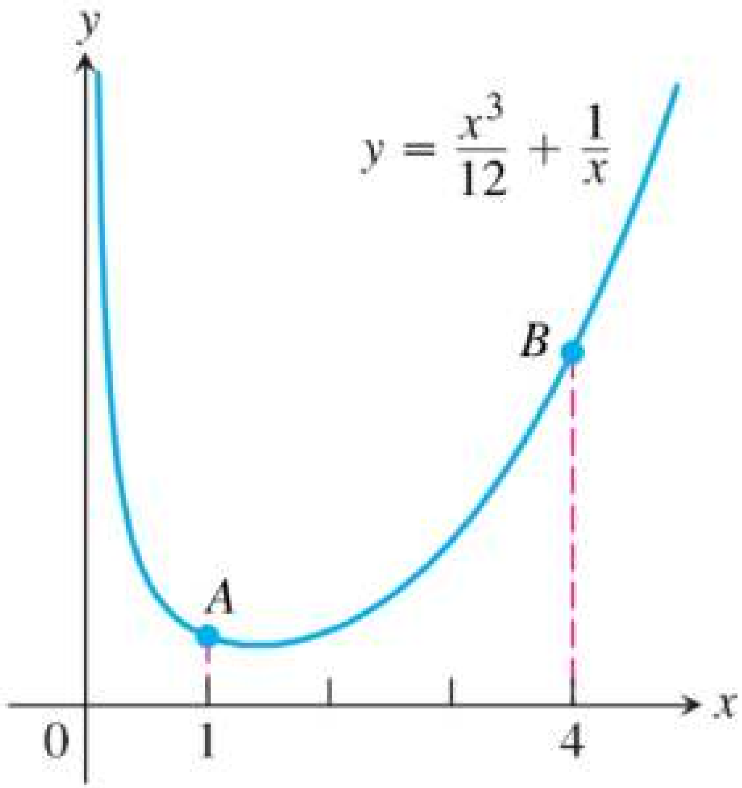

Solution

$$f'(x) = \frac{x^2}{4} - \frac{1}{x^2}$$

$$1 + [f'(x)]^2 = 1 + \left(\frac{x^2}{4} - \frac{1}{x^2}\right)^2 = 1 + \left(\frac{x^4}{16} - \frac{1}{2} + \frac{1}{x^4}\right)$$

$$= \frac{x^4}{16} + \frac{1}{2} + \frac{1}{x^4} = \left(\frac{x^2}{4} + \frac{1}{x^2}\right)^2.$$

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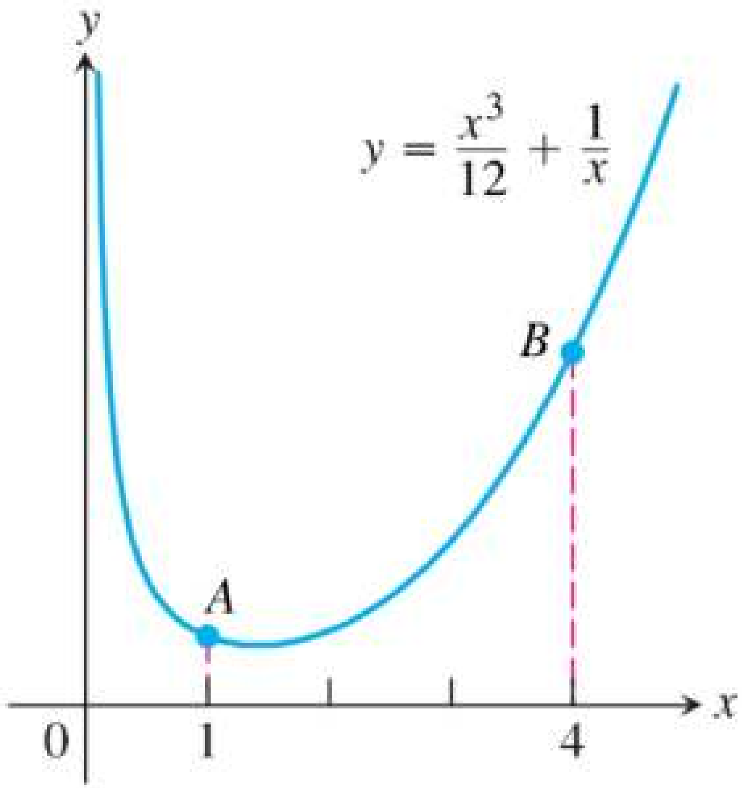

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$$= \frac{x^4}{16} + \frac{1}{2} + \frac{1}{x^4} = \left(\frac{x^2}{4} + \frac{1}{x^2}\right)^2.$$

The length of the graph over [1, 4] is

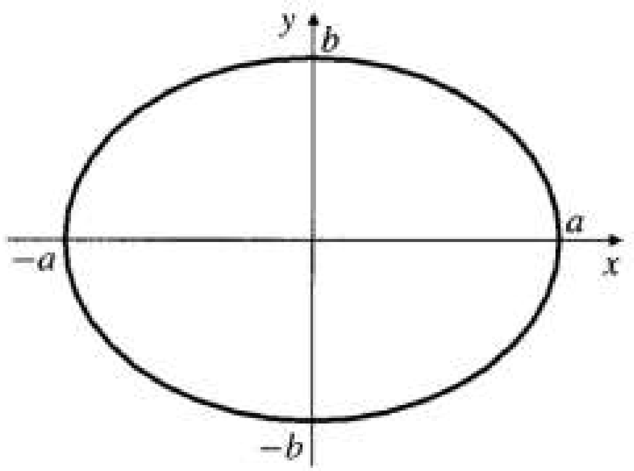
$$L = \int_1^4 \sqrt{1 + [f'(x)]^2}\, dx = \int_1^4 \left(\frac{x^2}{4} + \frac{1}{x^2}\right) dx$$

$$= \left[\frac{x^3}{12} - \frac{1}{x}\right]_1^4 = \left(\frac{64}{12} - \frac{1}{4}\right) - \left(\frac{1}{12} - 1\right) = \frac{72}{12} = 6.$$

EXAMPLE **(The circumference of an ellipse)** Find the circumference of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

where $a \geq b > 0$.



EXAMPLE **(The circumference of an ellipse)** Find the circumference of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \qquad y = b\sqrt{1 - \frac{x^2}{a^2}} = \frac{b}{a}\sqrt{a^2 - x^2}$$

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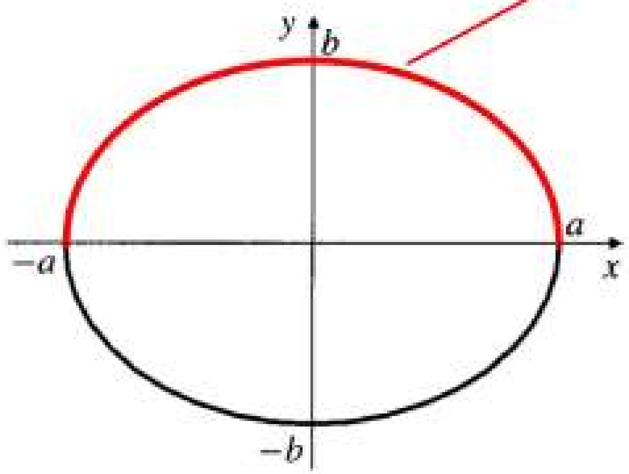

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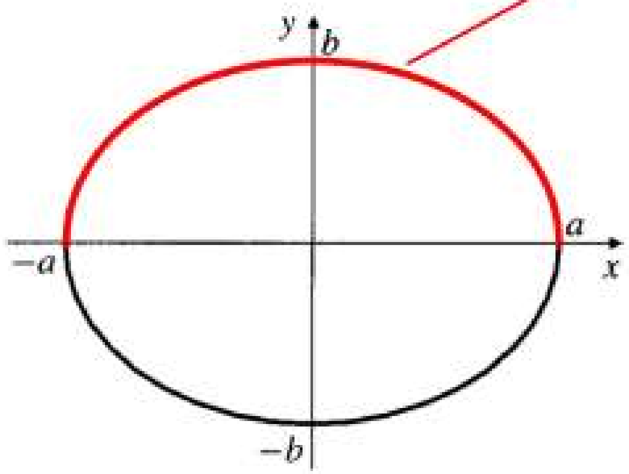

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$$\frac{dy}{dx} = -\frac{b}{a}\frac{x}{\sqrt{a^2 - x^2}},$$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{b^2}{a^2}\frac{x^2}{a^2 - x^2}$$

$$= \frac{a^4 - (a^2 - b^2)x^2}{a^2(a^2 - x^2)}.$$

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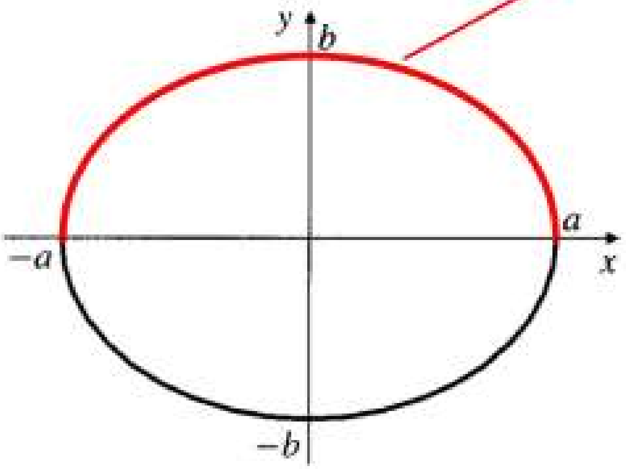


$$y=b\sqrt{1-\frac{x^2}{a^2}}=\frac{b}{a}\sqrt{a^2-x^2}\qquad \frac{dy}{dx}=-\frac{b}{a}\frac{x}{\sqrt{a^2-x^2}},$$

$$1+\left(\frac{dy}{dx}\right)^2=1+\frac{b^2}{a^2}\frac{x^2}{a^2-x^2}$$

$$=\frac{a^4-(a^2-b^2)x^2}{a^2(a^2-x^2)}.$$

$$s=4\int_0^a\frac{\sqrt{a^4-(a^2-b^2)x^2}}{a\sqrt{a^2-x^2}}\,dx$$

Let $x=a\sin t$, $dx=a\cos t\,dt$

$$=4\int_0^{\pi/2}\frac{\sqrt{a^4-(a^2-b^2)a^2\sin^2 t}}{a(a\cos t)}\,a\cos t\,dt$$

$$=4\int_0^{\pi/2}\sqrt{a^2-(a^2-b^2)\sin^2 t}\,dt\quad=4a\int_0^{\pi/2}\sqrt{1-\frac{a^2-b^2}{a^2}\sin^2 t}\,dt$$

$$=4a\int_0^{\pi/2}\sqrt{1-\varepsilon^2\sin^2 t}\,dt\text{ units,}$$

where $\varepsilon=(\sqrt{a^2-b^2})/a$ is the *eccentricity* of the ellipse.

Arc Length and Surface Area

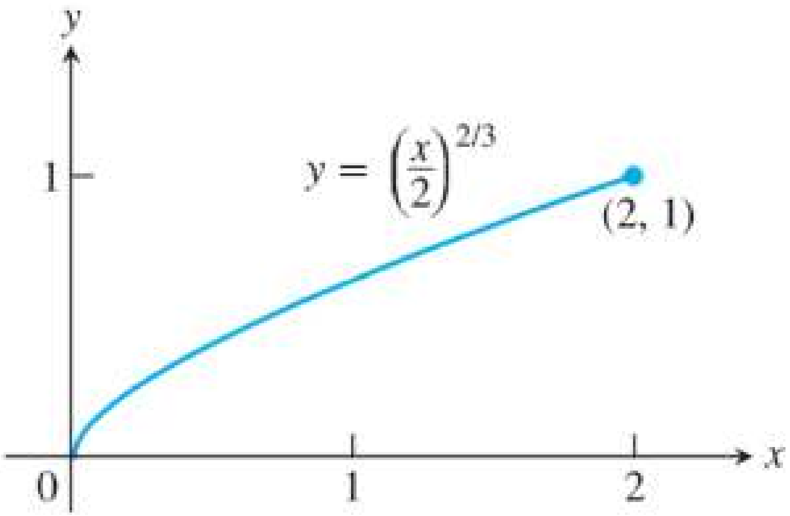
Formula for the Length of $x = g(y)$, $c \leq y \leq d$

If g' is continuous on $[c, d]$, the length of the curve $x = g(y)$ from $A = (g(c), c)$ to $B = (g(d), d)$ is

$$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy.$$

Arc Length and Surface Area

EXAMPLE Find the length of the curve $y = (x/2)^{2/3}$ from $x = 0$ to $x = 2$.



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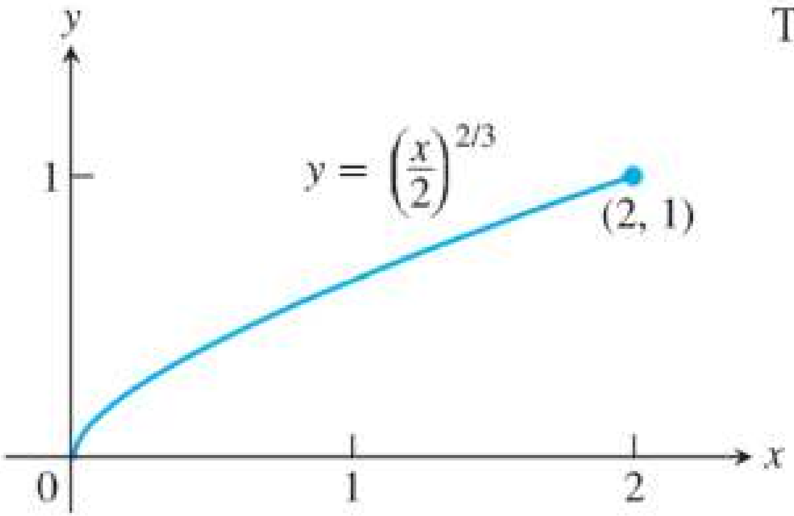

The derivative

$$\frac{dy}{dx} = \frac{2}{3}\left(\frac{x}{2}\right)^{-1/3}\left(\frac{1}{2}\right) = \frac{1}{3}\left(\frac{2}{x}\right)^{1/3}$$

is not defined at $x = 0$, so we cannot find the curve's length.

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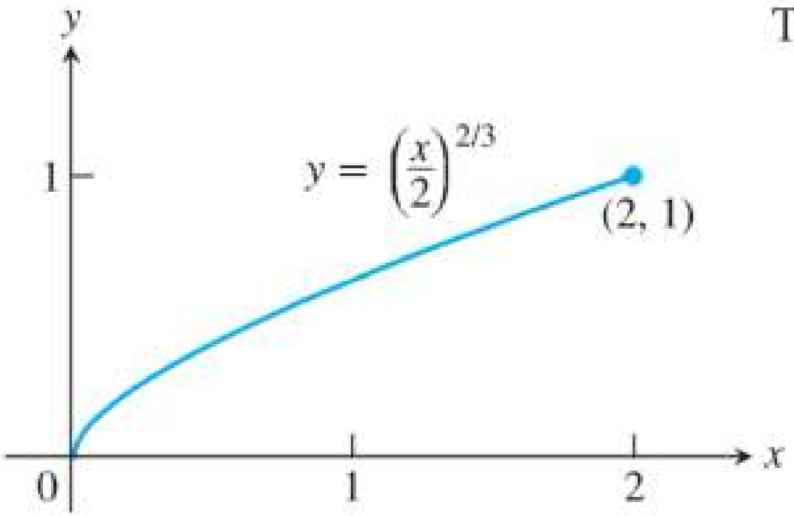

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We therefore rewrite the equation to express x in terms of y:

$$y = \left(\frac{x}{2}\right)^{2/3}$$

$$y^{3/2} = \frac{x}{2} \qquad \text{Raise both sides to the power 3/2.}$$

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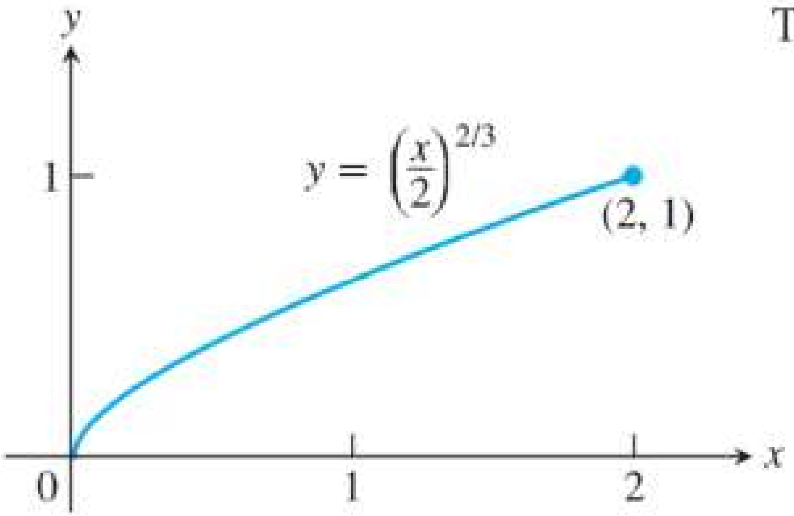

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$$\frac{dx}{dy} = 2\left(\frac{3}{2}\right)y^{1/2} = 3y^{1/2}$$

is continuous on $[0, 1]$.

$$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_0^1 \sqrt{1 + 9y}\, dy$$

$$= \frac{1}{9} \cdot \frac{2}{3}(1 + 9y)^{3/2}\Big]_0^1$$

$$= \frac{2}{27}\left(10\sqrt{10} - 1\right) \approx 2.27.$$

Arc Length and Surface Area

The Differential Formula for Arc Length

If $y = f(x)$ and if f' is continuous on $[a, b]$, then by the Fundamental Theorem of Calculus we can define a new function

$$s(x) = \int_a^x \sqrt{1 + [f'(t)]^2}\, dt.$$

The function s is called the **arc length function** for $y = f(x)$.

Arc Length and Surface Area

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$$\frac{ds}{dx} = \sqrt{1 + [f'(x)]^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

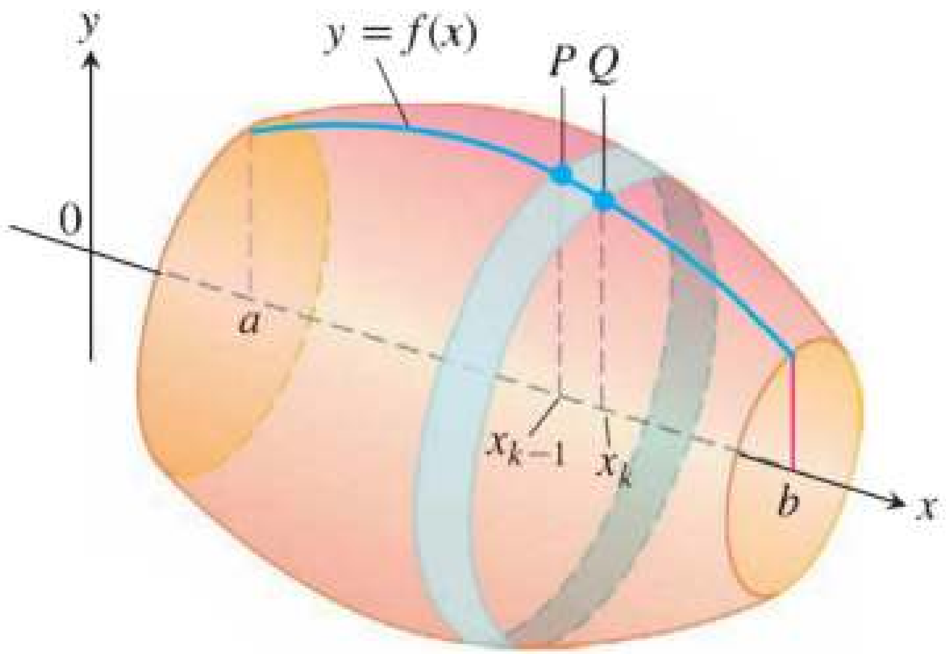
Then the differential of arc length is

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

$$ds = \sqrt{dx^2 + dy^2}$$

Arc Length and Surface Area

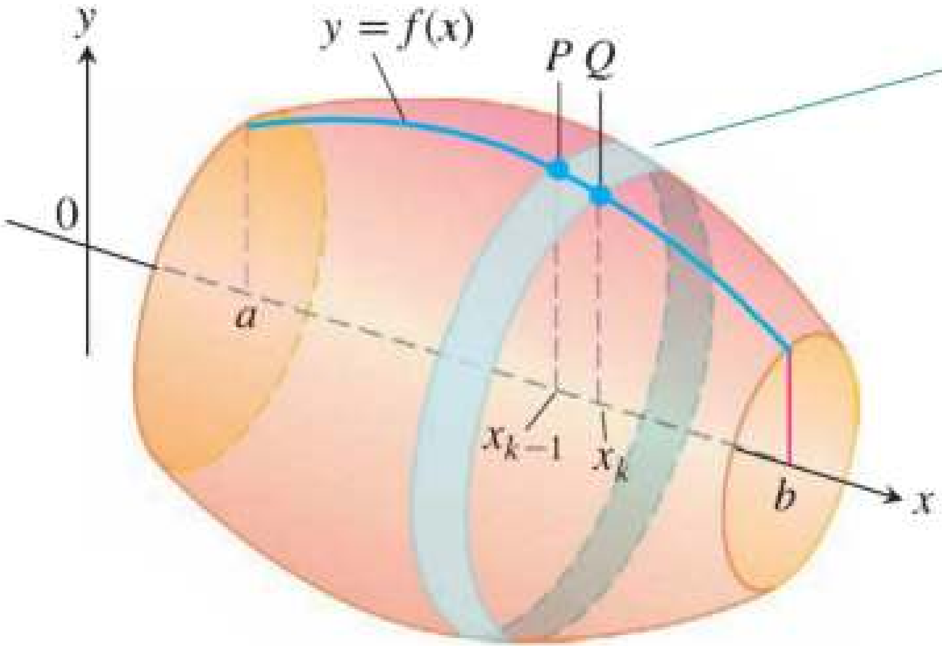
Areas of Surfaces of Revolution

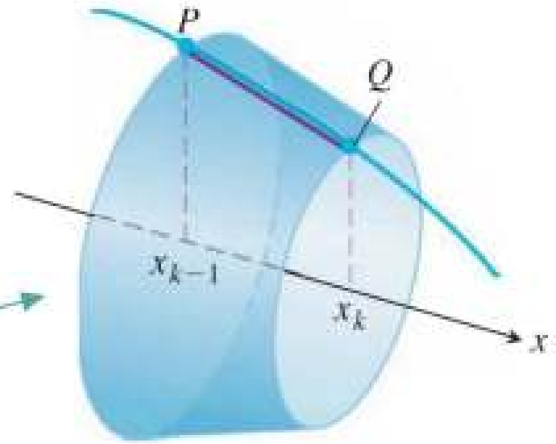


The surface generated by revolving the graph of a nonnegative function $y = f(x)$, $a \leq x \leq b$, about the x-axis. The surface is a union of bands like the one swept out by the arc PQ.

Arc Length and Surface Area

Areas of Surfaces of Revolution





The line segment joining P and Q sweeps out a frustum of a cone.

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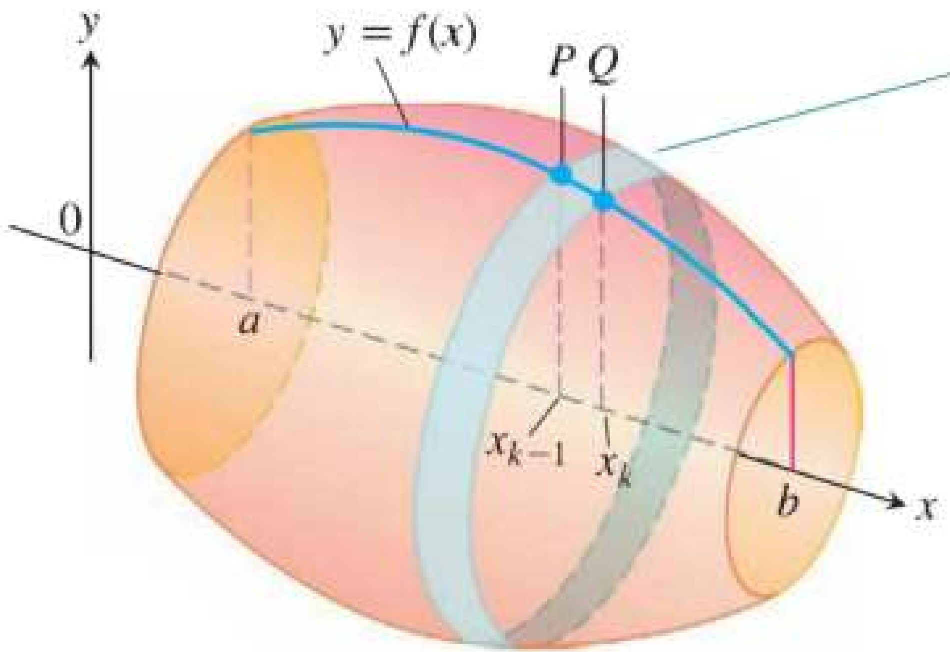

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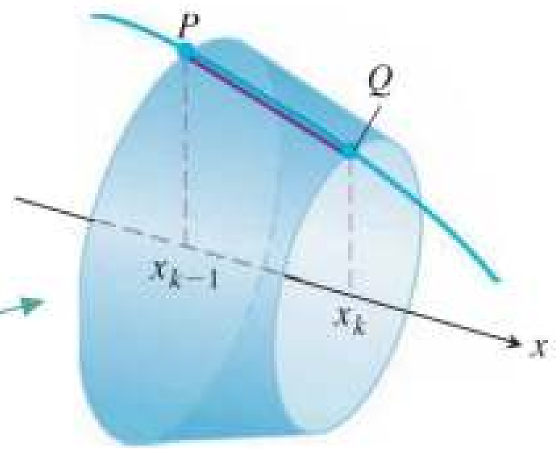

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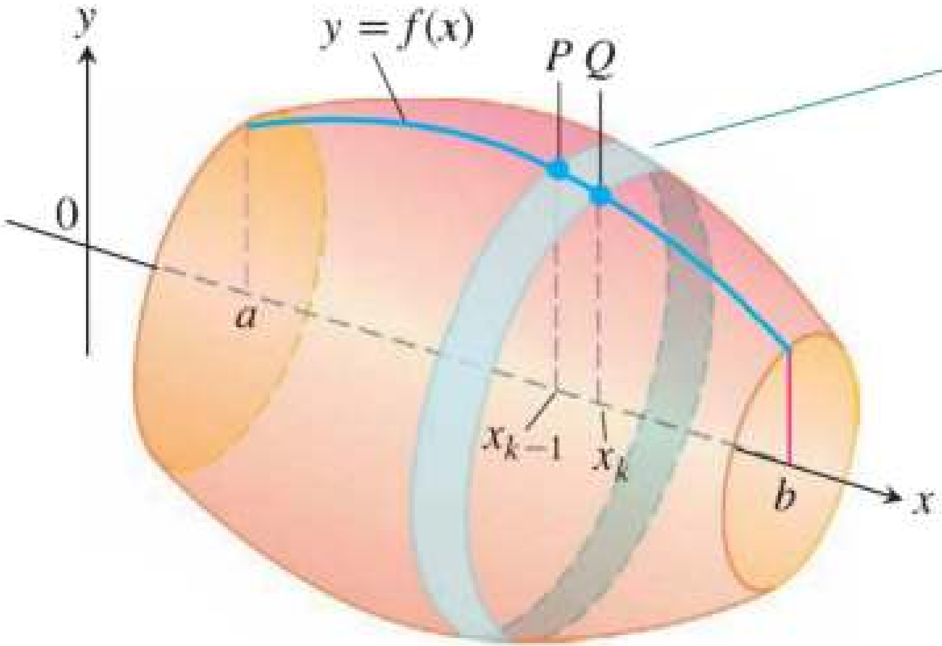
Frustum surface area

$$= 2\pi \cdot \frac{f(x_{k-1}) + f(x_k)}{2} \cdot \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$$

$$= \pi(f(x_{k-1}) + f(x_k))\sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}.$$

Arc Length and Surface Area

Areas of Surfaces of Revolution



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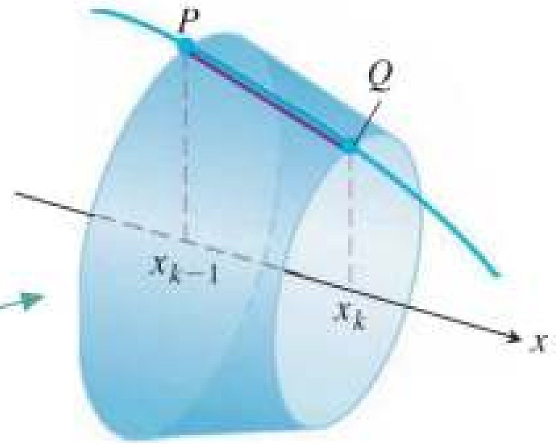

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Frustum surface area

$$= 2\pi \cdot \frac{f(x_{k-1}) + f(x_k)}{2} \cdot \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$$

$$= \pi(f(x_{k-1}) + f(x_k))\sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}.$$

$$\int_a^b 2\pi f(x)\sqrt{1 + (f'(x))^2}\, dx$$

Areas of Surfaces of Revolution

EXAMPLE

(Surface area of a sphere) Find the area of the surface of a sphere of radius a.

Arc Length and Surface Area

Areas of Surfaces of Revolution

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Solution Such a sphere can be generated by rotating the semicircle with equation $y = \sqrt{a^2 - x^2}$, $(-a \le x \le a)$, about the x-axis.

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$$\begin{aligned} S &= 2\pi \int_{-a}^{a} y\sqrt{1 + \left(\frac{x}{y}\right)^2}\, dx \\ &= 4\pi \int_0^a \sqrt{y^2 + x^2}\, dx \\ &= 4\pi \int_0^a \sqrt{a^2}\, dx = 4\pi a x \Big|_0^a = 4\pi a^2 \text{ square units.} \end{aligned}$$

Infinite Sequences and Series

Sequences

Representing Sequences

A sequence is a list of numbers

$$a_1, a_2, a_3, \ldots, a_n, \ldots$$

in a given order. Each of a_1, a_2, a_3 and so on represents a number. These are the **terms** of the sequence. For example, the sequence

$$2, 4, 6, 8, 10, 12, \ldots, 2n, \ldots$$

has first term $a_1 = 2$, second term $a_2 = 4$, and nth term $a_n = 2n$. The integer n is called the **index** of a_n, and indicates where a_n occurs in the list. Order is important. The sequence $2, 4, 6, 8 \ldots$ is not the same as the sequence $4, 2, 6, 8 \ldots$.

Sequences

Representing Sequences

We can think of the sequence

$$a_1, a_2, a_3, \ldots, a_n, \ldots$$

as a function that sends 1 to a_1, 2 to a_2, 3 to a_3, and in general sends the positive integer n to the nth term a_n. More precisely, an **infinite sequence** of numbers is a function whose domain is the set of positive integers.

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The function associated with the sequence

$$2, 4, 6, 8, 10, 12, \ldots, 2n, \ldots$$

sends 1 to $a_1 = 2$, 2 to $a_2 = 4$, and so on. The general behavior of this sequence is described by the formula $a_n = 2n$.

Sequences

Representing Sequences

We can equally well make the domain the integers larger than a given number n_0, and we allow sequences of this type also. For example, the sequence

$$12, 14, 16, 18, 20, 22 \ldots$$

is described by the formula $a_n = 10 + 2n$. It can also be described by the simpler formula $b_n = 2n$, where the index n starts at 6 and increases. To allow such simpler formulas, we let the first index of the sequence be any integer.

Sequences

Representing Sequences

Sequences can be described by writing rules that specify their terms, such as

$$a_n = \sqrt{n}, \qquad b_n = (-1)^{n+1}\frac{1}{n}, \qquad c_n = \frac{n-1}{n}, \qquad d_n = (-1)^{n+1},$$

or by listing terms:

$$\{a_n\} = \left\{\sqrt{1}, \sqrt{2}, \sqrt{3}, \ldots, \sqrt{n}, \ldots\right\}$$

$$\{b_n\} = \left\{1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \ldots, (-1)^{n+1}\frac{1}{n}, \ldots\right\}$$

$$\{c_n\} = \left\{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots, \frac{n-1}{n}, \ldots\right\}$$

$$\{d_n\} = \{1, -1, 1, -1, 1, -1, \ldots, (-1)^{n+1}, \ldots\}.$$

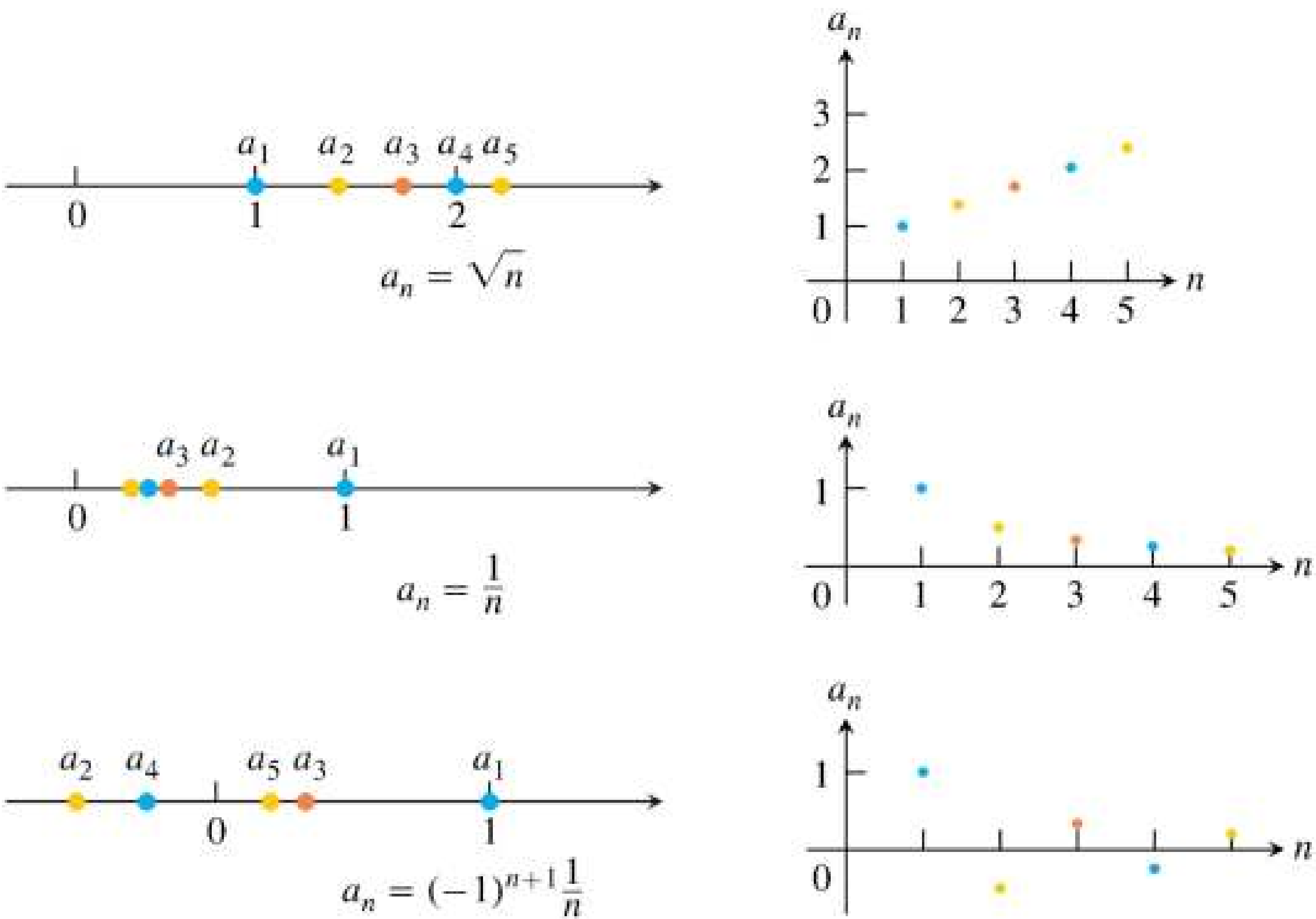
We also sometimes write

$$\{a_n\} = \left\{\sqrt{n}\right\}_{n=1}^{\infty}.$$

Sequences

Representing Sequences

two ways to represent sequences graphically



Sequences

Convergence and Divergence

DEFINITIONS The sequence $\{a_n\}$ **converges** to the number L if for every positive number ϵ there corresponds an integer N such that for all n,

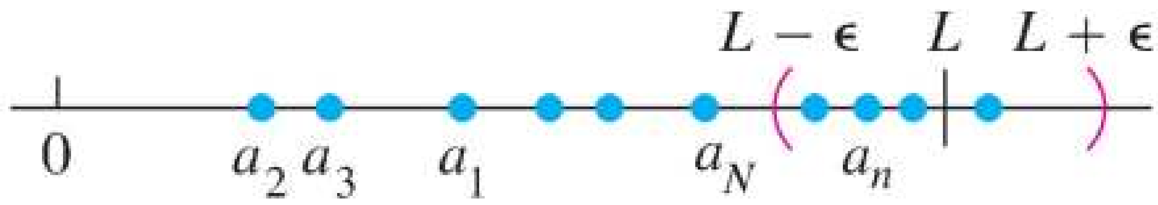
$$n > N \qquad \Rightarrow \qquad |a_n - L| < \epsilon.$$

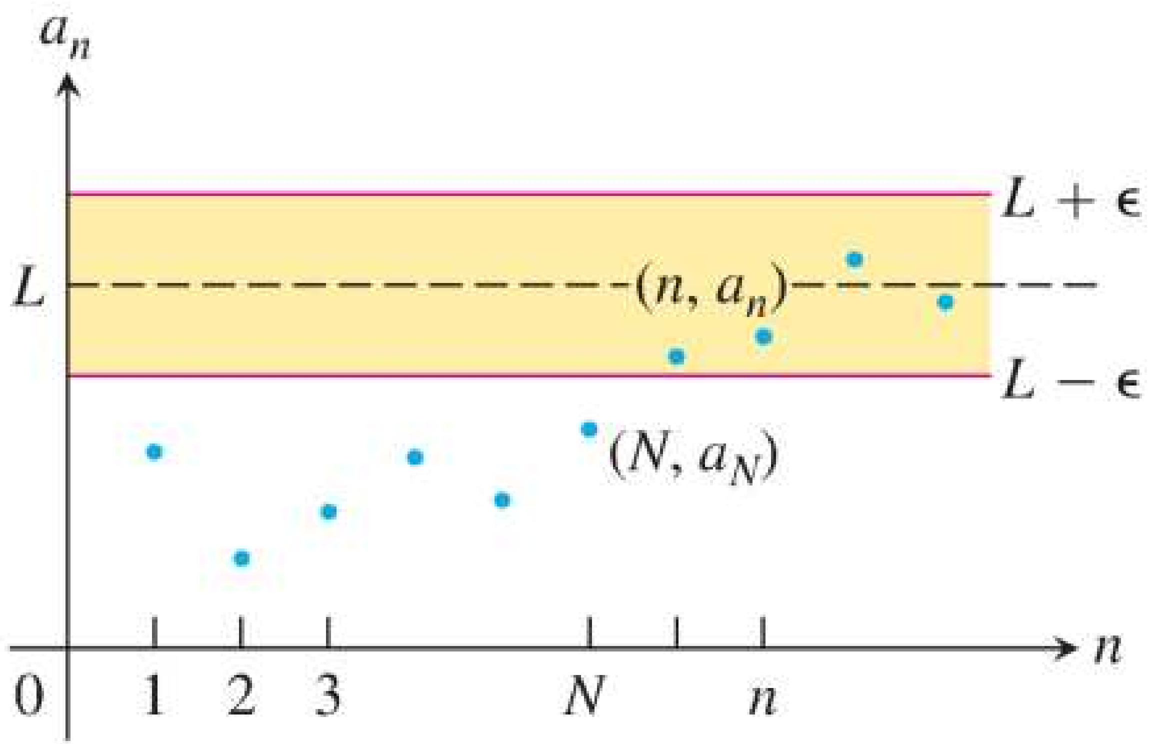
If no such number L exists, we say that $\{a_n\}$ **diverges**.

If $\{a_n\}$ converges to L, we write $\lim_{n\to\infty} a_n = L$, or simply $a_n \to L$, and call L the **limit** of the sequence (Figure 10.2).

Sequences

Convergence and Divergence





Sequences

Convergence and Divergence

EXAMPLE

(a) $\lim_{n \to \infty} \frac{1}{n} = 0$ **(b)** $\lim_{n \to \infty} k = k$ (any constant k)

Sequences

Convergence and Divergence

EXAMPLE

(a) $\lim_{n\to\infty} \frac{1}{n} = 0$ **(b)** $\lim_{n\to\infty} k = k$ (any constant k)

EXAMPLE Show that the sequence $\{1, -1, 1, -1, 1, -1, \ldots, (-1)^{n+1}, \ldots\}$ diverges.

Sequences

Convergence and Divergence

DEFINITION The sequence $\{a_n\}$ **diverges to infinity** if for every number M there is an integer N such that for all n larger than N, $a_n > M$. If this condition holds we write

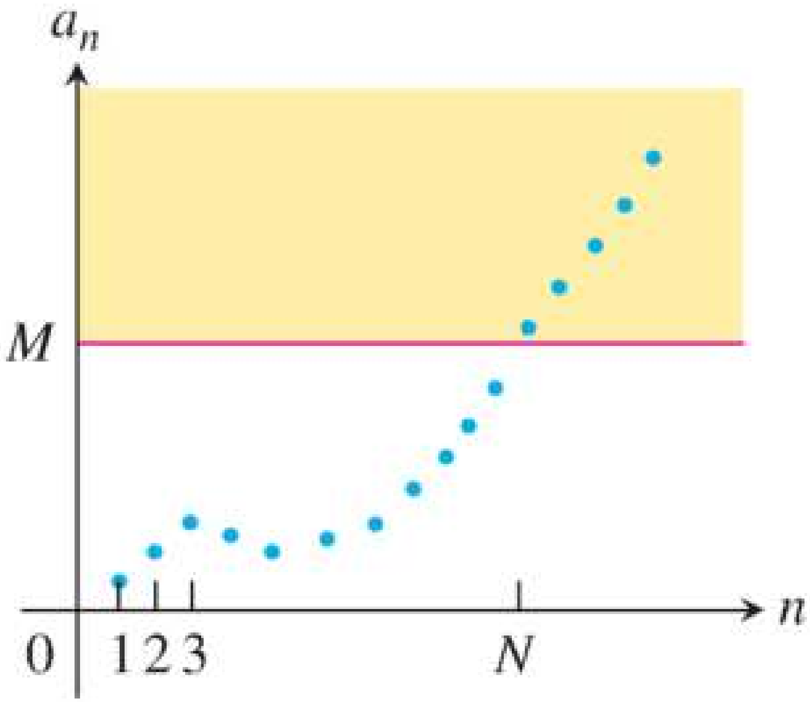
$$\lim_{n \to \infty} a_n = \infty \qquad \text{or} \qquad a_n \to \infty .$$

Similarly if for every number m there is an integer N such that for all $n > N$ we have $a_n < m$, then we say $\{a_n\}$ **diverges to negative infinity** and write

$$\lim_{n \to \infty} a_n = -\infty \qquad \text{or} \qquad a_n \to -\infty .$$

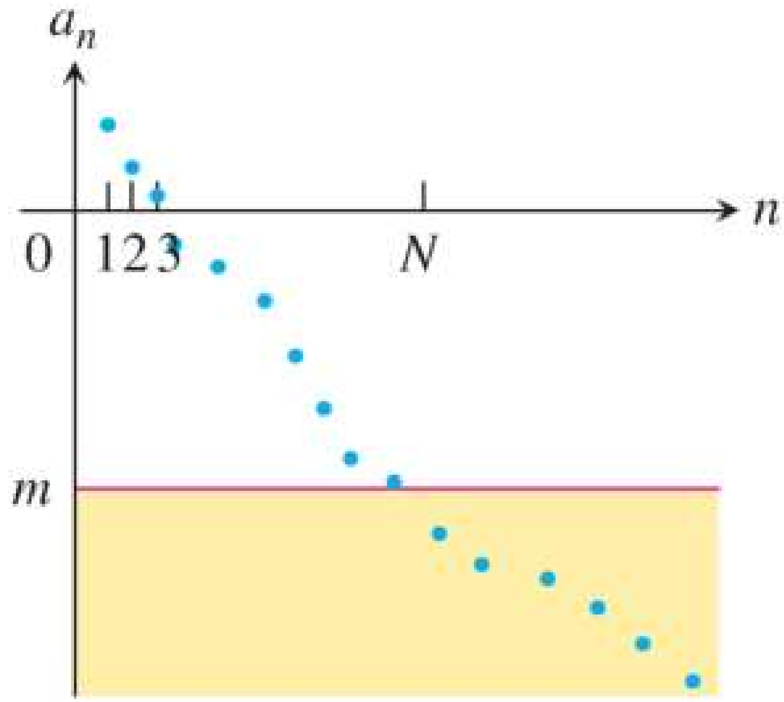
Sequences

Convergence and Divergence



(a)

$a_n \to \infty$



(b)

$a_n \to -\infty$