The Chain Rule

THEOREM The Chain Rule

If $f(u)$ is differentiable at $u = g(x)$, and $g(x)$ is differentiable at x, then the composite function $f \circ g(x) = f(g(x))$ is differentiable at x, and

$$(f \circ g)'(x) = f'(g(x))g'(x).$$

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In terms of Leibniz notation, if $y = f(u)$ where $u = g(x)$, then $y = f(g(x))$ and:

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}, \qquad \text{where } \frac{dy}{du} \text{ is evaluated at } u = g(x).$$

The Chain Rule

EXAMPLE Find the derivative of $y = \sqrt{x^2 + 1}$.

EXAMPLE Find derivatives of the following functions:

(a) $(7x - 3)^{10}$, (b) $f(t) = |t^2 - 1|$, and (c) $\left(3x + \frac{1}{(2x+1)^3}\right)^{1/4}$.

The Chain Rule

Building the Chain Rule into Differentiation Formulas

If u is a differentiable function of x and $y = u^n$, then

$$\frac{d}{dx}u^n = \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = nu^{n-1}\frac{du}{dx}.$$

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$$\frac{d}{dx}\left(\frac{1}{u}\right) = \frac{-1}{u^2}\frac{du}{dx} \qquad \text{(the Reciprocal Rule)}$$

$$\frac{d}{dx}\sqrt{u} = \frac{1}{2\sqrt{u}}\frac{du}{dx} \qquad \text{(the Square Root Rule)}$$

$$\frac{d}{dx}u^r = r\,u^{r-1}\frac{du}{dx} \qquad \text{(the General Power Rule)}$$

$$\frac{d}{dx}|u| = \operatorname{sgn} u\,\frac{du}{dx} = \frac{u}{|u|}\frac{du}{dx} \qquad \text{(the Absolute Value Rule)}$$

Derivatives of Trigonometric Functions

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PROOF

$$\begin{aligned}
\frac{d}{dx}\sin x &= \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h} \\
&= \lim_{h\to 0}\frac{\sin x\cos h+\cos x\sin h-\sin x}{h} \\
&= \lim_{h\to 0}\frac{\sin x(\cos h-1)+\cos x\sin h}{h} \\
&= \lim_{h\to 0}\sin x\cdot\lim_{h\to 0}\frac{\cos h-1}{h}+\lim_{h\to 0}\cos x\cdot\lim_{h\to 0}\frac{\sin h}{h} \\
&= (\sin x)\cdot(0)+(\cos x)\cdot(1)=\cos x.
\end{aligned}$$

Derivatives of Trigonometric Functions

THEOREM **The derivative of the cosine function is the negative of the sine function.**

$$\frac{d}{dx}\cos x = -\sin x$$

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EXAMPLE Evaluate the derivative of $x^2 \sin\sqrt{x}$.

Derivatives of Trigonometric Functions

EXAMPLE

Use two different methods to find the derivative of the function $f(t) = \sin t \cos t$.

Derivatives of Trigonometric Functions

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Solution By the Product Rule:

$$f'(t) = (\cos t)(\cos t) + (\sin t)(-\sin t) = \cos^2 t - \sin^2 t.$$

On the other hand, since $\sin(2t) = 2\sin t \cos t$, we have

$$f'(t) = \frac{d}{dt}\left(\frac{1}{2}\sin(2t)\right) = \left(\frac{1}{2}\right)(2)\cos(2t) = \cos(2t).$$

The two answers are really the same, since $\cos(2t) = \cos^2 t - \sin^2 t$.

Derivatives of Trigonometric Functions

The Derivatives of the Other Trigonometric Functions

$$\frac{d}{dx}\tan x = \sec^2 x \qquad \frac{d}{dx}\sec x = \sec x \tan x$$

$$\frac{d}{dx}\cot x = -\csc^2 x \qquad \frac{d}{dx}\csc x = -\csc x \cot x.$$

Derivatives of Trigonometric Functions

EXAMPLE

Find the tangent and normal lines to the curve $y = \tan(\pi x/4)$ at the point $(1, 1)$.

Derivatives of Trigonometric Functions

EXAMPLE Find the tangent and normal lines to the curve $y = \tan(\pi x/4)$ at the point $(1, 1)$.

Solution The slope of the tangent to $y = \tan(\pi x/4)$ at $(1, 1)$ is:

$$\left.\frac{dy}{dx}\right|_{x=1} = \frac{\pi}{4}\sec^2(\pi x/4)\Big|_{x=1} = \frac{\pi}{4}\sec^2\left(\frac{\pi}{4}\right) = \frac{\pi}{4}\left(\sqrt{2}\right)^2 = \frac{\pi}{2}.$$

The tangent is the line

$$y = 1 + \frac{\pi}{2}(x-1), \qquad \text{or} \qquad y = \frac{\pi x}{2} - \frac{\pi}{2} + 1.$$

The normal has slope $m = -2/\pi$, so its point-slope equation is

$$y = 1 - \frac{2}{\pi}(x-1), \qquad \text{or} \qquad y = -\frac{2x}{\pi} + \frac{2}{\pi} + 1.$$

Higher-Order Derivatives

If the derivative $y' = f'(x)$ of a function $y = f(x)$ is itself differentiable at x, we can calculate *its* derivative, which we call the **second derivative** of f and denote by

$$y'' = f''(x).$$

$$\boxed{y'' = f''(x) = \frac{d^2y}{dx^2} = \frac{d}{dx}\frac{d}{dx}f(x) = \frac{d^2}{dx^2}f(x) = D_x^2 y = D_x^2 f(x)}$$

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the nth derivative of $y = f(x)$ is

$$y^{(n)} = f^{(n)}(x) = \frac{d^n y}{dx^n} = \frac{d^n}{dx^n}f(x) = D_x^n y = D_x^n f(x).$$

Higher-Order Derivatives

EXAMPLE The **velocity** of a moving object is the (instantaneous) rate of change of the position of the object with respect to time; if the object moves along the x-axis and is at position $x = f(t)$ at time t, then its velocity at that time is

$$v = \frac{dx}{dt} = f'(t).$$

Similarly, the **acceleration** of the object is the rate of change of the velocity. Thus, the acceleration is the *second derivative* of the position:

$$a = \frac{dv}{dt} = \frac{d^2x}{dt^2} = f''(t).$$

Higher-Order Derivatives

EXAMPLE If $f(x) = x^n$ (where n is a positive integer), then

$$
\begin{aligned}
f^{(k)}(x) &= n(n-1)(n-2)\cdots(n-(k-1))\,x^{n-k} \\
&= \begin{cases} \dfrac{n!}{(n-k)!}\,x^{n-k} & \text{if } 0 \le k \le n \\ 0 & \text{if } k > n, \end{cases}
\end{aligned}
$$

where $n!$ (called n **factorial**) is defined by:

$$
\begin{aligned}
0! &= 1 \\
1! &= 0! \times 1 = 1 \times 1 = 1 \\
2! &= 1! \times 2 = 1 \times 2 = 2 \\
3! &= 2! \times 3 = 1 \times 2 \times 3 = 6 \\
&\vdots \\
n! &= (n-1)! \times n = 1 \times 2 \times 3 \times \cdots \times (n-1) \times n.
\end{aligned}
$$

Higher-Order Derivatives

EXAMPLE Show that if A, B, and k are constants, then the function $y = A\cos(kt) + B\sin(kt)$ is a solution of the *second-order* **differential equation of simple harmonic motion**

$$\frac{d^2y}{dt^2} + k^2y = 0.$$

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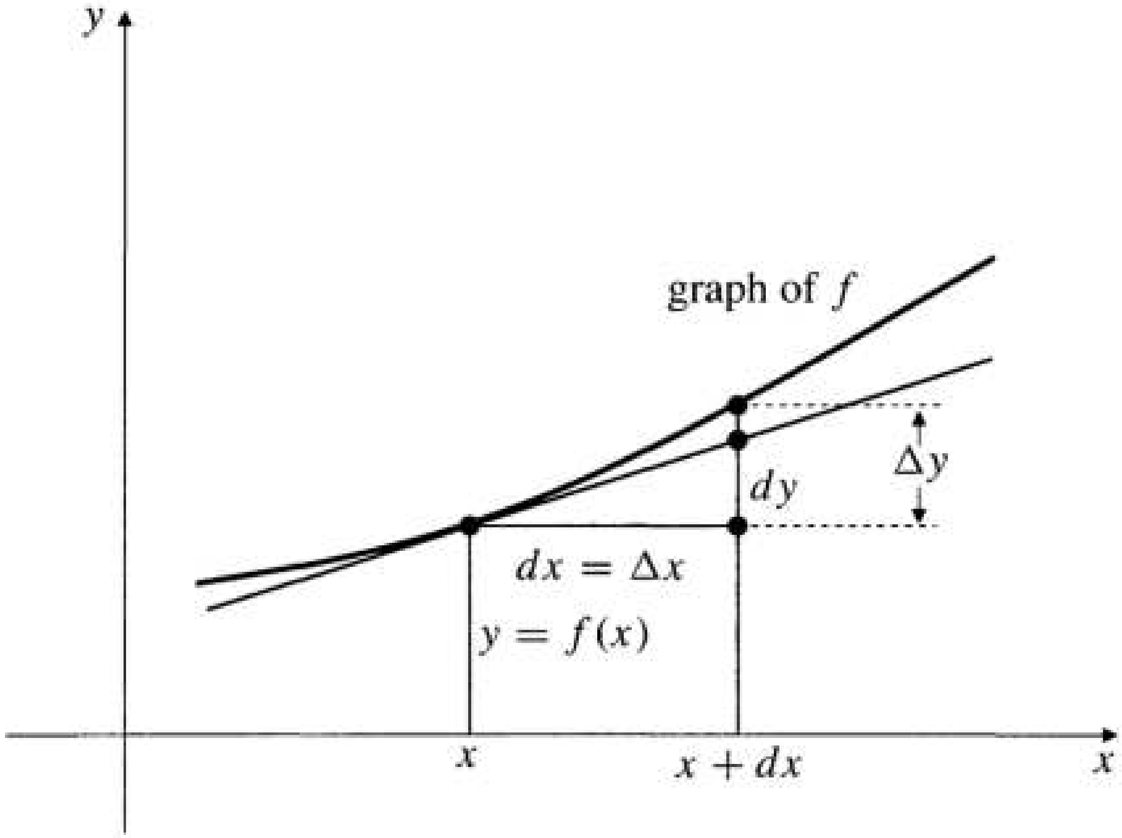
EXAMPLE Find the nth derivative, $y^{(n)}$, of $y = \dfrac{1}{1+x} = (1+x)^{-1}$.

Using Differentials and Derivatives

Approximating Small Changes

$$\Delta y \approx dy = f'(x)\,dx.$$



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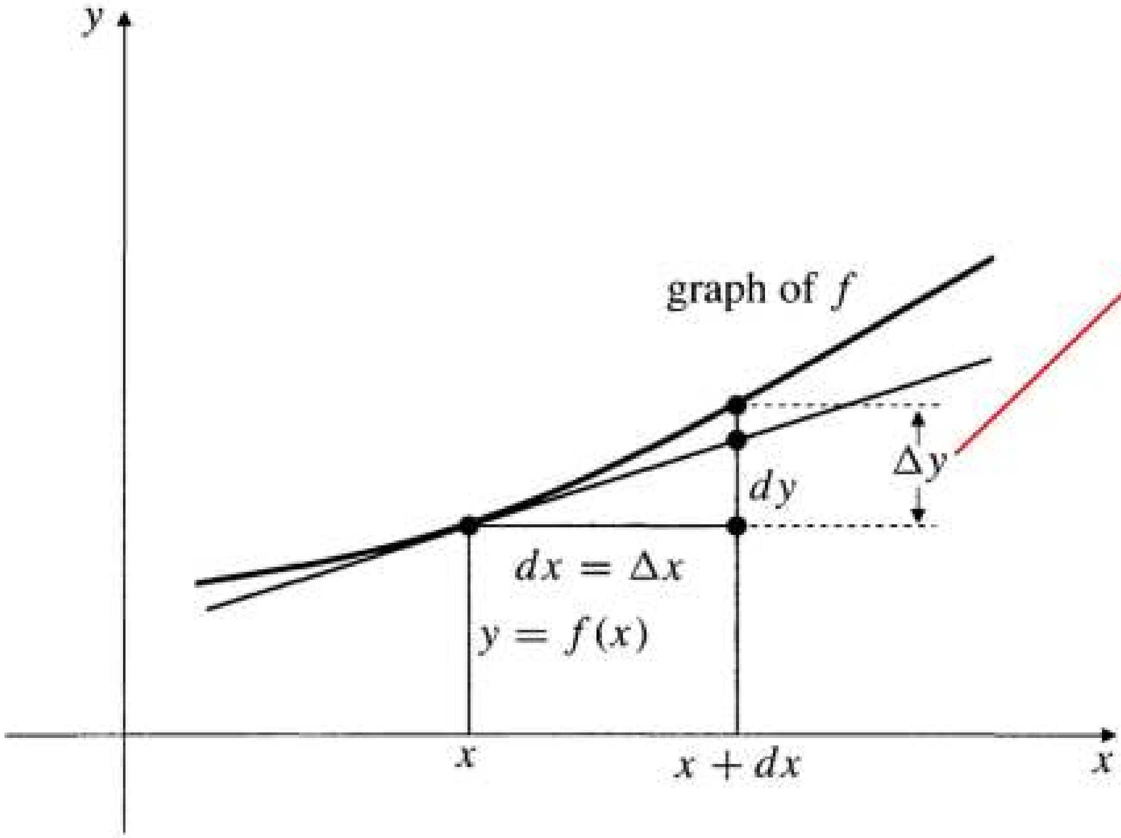

The exact change Δy in y is given by

$$\Delta y = f(x + \Delta x) - f(x)$$

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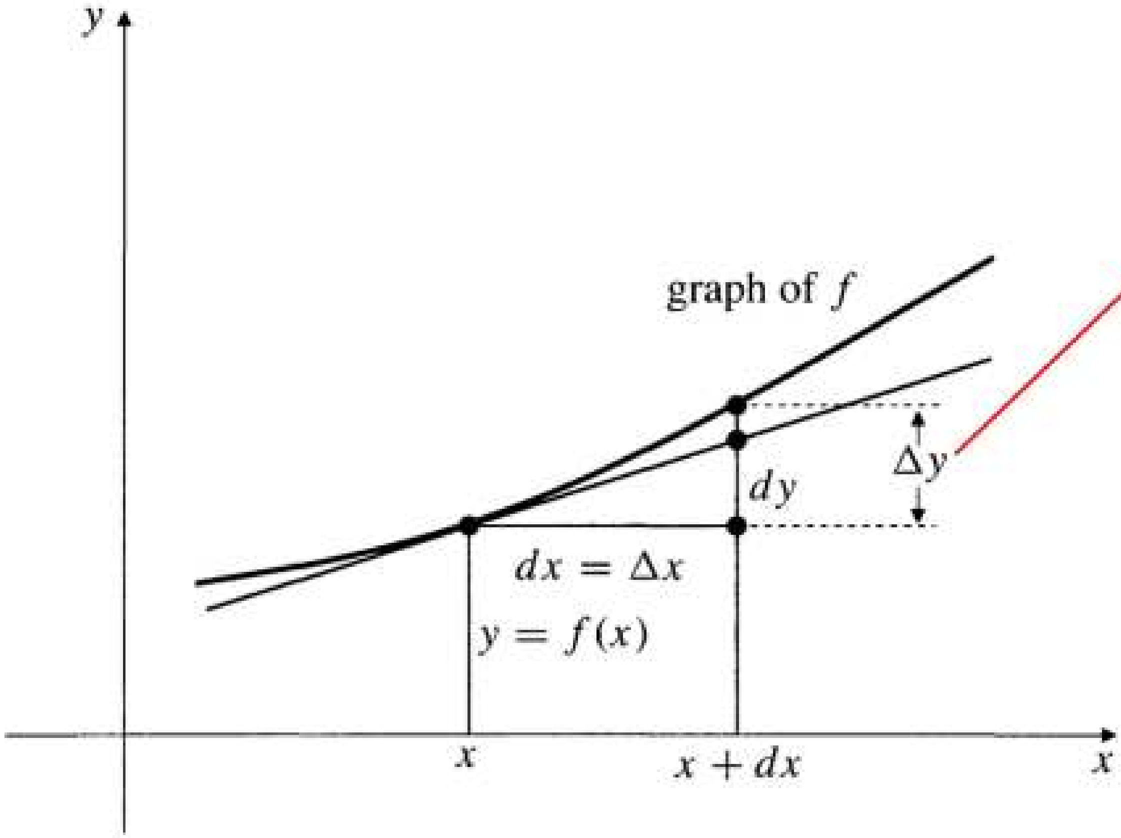

The exact change Δy in y is given by

$$\Delta y = f(x + \Delta x) - f(x)$$

$$\Delta y = \frac{\Delta y}{\Delta x}\,\Delta x \approx \frac{dy}{dx}\,\Delta x = f'(x)\,\Delta x,$$

if the change Δx is small.

Using Differentials and Derivatives

Approximating Small Changes

EXAMPLE Without using a scientific calculator, determine by approximately how much the value of $\sin x$ increases as x increases from $\pi/3$ to $(\pi/3) + 0.006$. To 3 decimal places, what is the value of $\sin\big((\pi/3) + 0.006\big)$?

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Solution If $y = \sin x$, $x = \pi/3 \approx 1.0472$, and $dx = 0.006$, then

$$dy = \cos(x)\,dx = \cos\left(\frac{\pi}{3}\right)\,dx = \frac{1}{2}(0.006) = 0.003.$$

Thus the change in the value of $\sin x$ is approximately 0.003, and

$$\sin\left(\frac{\pi}{3} + 0.006\right) \approx \sin\frac{\pi}{3} + 0.003 = \frac{\sqrt{3}}{2} + 0.003 = 0.869$$

rounded to 3 decimal places.

Using Differentials and Derivatives

Approximating Small Changes

Measuring changes with respect to the size of quantities

$$\text{relative change in } x = \frac{dx}{x}$$

$$\text{percentage change in } x = 100\,\frac{dx}{x}$$

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EXAMPLE

By approximately what percentage does the area of a circle increase if the radius increases by 2%?

Using Differentials and Derivatives

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$$A = \pi r^2 \longrightarrow \Delta A \approx dA = \frac{dA}{dr}\,dr = 2\pi r\,dr \longrightarrow \frac{\Delta A}{A} \approx \frac{dA}{A} = \frac{2\pi r\,dr}{\pi r^2} = 2\,\frac{dr}{r}$$

Using Differentials and Derivatives

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If r increases by 2%, then $dr = \frac{2}{100}r$, so

$$\frac{\Delta A}{A} \approx 2 \times \frac{2}{100} = \frac{4}{100}.$$

Thus, A increases by approximately 4%.

Using Differentials and Derivatives

Average and Instantaneous Rates of Change

The **average rate of change** of a function $f(x)$ with respect to x over the interval from a to $a+h$ is

$$\frac{f(a+h)-f(a)}{h}.$$

The **(instantaneous) rate of change** of f with respect to x at $x=a$ is the derivative

$$f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h},$$

provided the limit exists.

Using Differentials and Derivatives

Average and Instantaneous Rates of Change

EXAMPLE How fast is area A of a circle increasing with respect to its radius when the radius is 5 m?

Solution The rate of change of the area with respect to the radius is

$$\frac{dA}{dr} = \frac{d}{dr}(\pi r^2) = 2\pi\, r.$$

When $r = 5$ m, the area is changing at the rate $2\pi \times 5 = 10\pi$ m^2/m. This means that a small change Δr m in the radius when the radius is 5 m would result in a change of about $10\pi\,\Delta r$ m^2 in the area of the circle.