Integrals of Rational Functions

In this section we are concerned with integrals of the form

$$\int \frac{P(x)}{Q(x)}\,dx,$$

where P and Q are polynomials.

Integrals of Rational Functions

EXAMPLE Evaluate $\displaystyle\int \frac{x^3+3x^2}{x^2+1}\,dx$.

Solution The numerator has degree 3 and the denominator has degree 2 so we need to divide. We use long division:

$$
\begin{array}{rrr|l}
x^3 & +\,3x^2 & & x^2+1 \\
\cline{4-4}
x^3 & & +\,x & x+3 \\
\cline{1-3}
& 3x^2 & -\,x & \\
& 3x^2 & \quad +\,3 & \\
\cline{2-3}
& & -\,x-3 &
\end{array}
$$

$$\frac{x^3+3x^2}{x^2+1} = x+3-\frac{x+3}{x^2+1}.$$

$$
\begin{aligned}
\int \frac{x^3+3x^2}{x^2+1}\,dx &= \int (x+3)\,dx - \int \frac{x}{x^2+1}\,dx - 3\int \frac{dx}{x^2+1} \\
&= \frac{1}{2}x^2+3x-\frac{1}{2}\ln(x^2+1)-3\tan^{-1}x+C.
\end{aligned}
$$

EXAMPLE Evaluate $\int \frac{x}{2x-1}\,dx$.

Solution The numerator and denominator have the same degree, 1, so division is again required. In this case the division can be carried out by manipulation of the integrand:

$$\frac{x}{2x-1} = \frac{1}{2}\,\frac{2x}{2x-1} = \frac{1}{2}\,\frac{2x-1+1}{2x-1} = \frac{1}{2}\left(1 + \frac{1}{2x-1}\right),$$

a process that we call *short division*. We have

$$\int \frac{x}{2x-1}\,dx = \frac{1}{2}\int\left(1 + \frac{1}{2x-1}\right)dx = \frac{x}{2} + \frac{1}{4}\ln|2x-1| + C.$$

The basic problem

Evaluate $\int \frac{P(x)}{Q(x)}\,dx$, where the degree of $P <$ the degree of Q.

Linear and Quadratic Denominators

Suppose that $Q(x)$ has degree 1. Thus, $Q(x) = ax + b$, where $a \neq 0$. Then $P(x)$ must have degree 0 and be a constant c. We have $P(x)/Q(x) = c/(ax+b)$. The substitution $u = ax + b$ leads to

$$\int \frac{c}{ax+b}\,dx = \frac{c}{a}\int \frac{du}{u} = \frac{c}{a}\ln|u| + C,$$

so that for $c = 1$:

The case of a linear denominator

$$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b| + C.$$

Linear and Quadratic Denominators

$$\int \frac{x\,dx}{x^2+a^2}, \qquad \int \frac{x\,dx}{x^2-a^2}, \qquad \int \frac{dx}{x^2+a^2}, \quad \text{and} \quad \int \frac{dx}{x^2-a^2}.$$

Linear and Quadratic Denominators

$$\int \frac{x\,dx}{x^2+a^2}, \qquad \int \frac{x\,dx}{x^2-a^2}, \qquad \int \frac{dx}{x^2+a^2}, \quad \text{and} \qquad \int \frac{dx}{x^2-a^2}.$$

$$\int \frac{x\,dx}{x^2+a^2} = \frac{1}{2}\ln(x^2+a^2) + C,$$

$$\int \frac{x\,dx}{x^2-a^2} = \frac{1}{2}\ln|x^2-a^2| + C,$$

$$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C,$$

Linear and Quadratic Denominators

$$\frac{1}{x^2 - a^2} = \frac{1}{(x-a)(x+a)} = \frac{A}{x-a} + \frac{B}{x+a} = \frac{Ax + Aa + Bx - Ba}{x^2 - a^2},$$

Linear and Quadratic Denominators

$$\frac{1}{x^2-a^2}=\frac{1}{(x-a)(x+a)}=\frac{A}{x-a}+\frac{B}{x+a}=\frac{Ax+Aa+Bx-Ba}{x^2-a^2},$$

$$\begin{aligned} A+B&=0 &\qquad &\text{(the coefficient of } x\text{)},\\ Aa-Ba&=1 &\qquad &\text{(the constant term).}\end{aligned}$$

Linear and Quadratic Denominators

$$\frac{1}{x^2 - a^2} = \frac{1}{(x-a)(x+a)} = \frac{A}{x-a} + \frac{B}{x+a} = \frac{Ax + Aa + Bx - Ba}{x^2 - a^2},$$

$$\begin{aligned} A + B &= 0 \qquad \text{(the coefficient of } x\text{)}, \\ Aa - Ba &= 1 \qquad \text{(the constant term)}. \end{aligned} \longrightarrow \begin{aligned} A &= 1/(2a) \\ B &= -1/(2a) \end{aligned}$$

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$$\begin{aligned} \int \frac{dx}{x^2-a^2} &= \frac{1}{2a}\int\frac{dx}{x-a} - \frac{1}{2a}\int\frac{dx}{x+a} \\ &= \frac{1}{2a}\ln|x-a| - \frac{1}{2a}\ln|x+a| + C \\ &= \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C. \end{aligned}$$

Partial Fractions

Suppose

$$Q(x) = (x - a_1)(x - a_2) \cdots (x - a_n),$$

where $a_i \neq a_j$ if $i \neq j, 1 \leq i, j \leq n$. If $P(x)$ is a polynomial of degree smaller than n, then $P(x)/Q(x)$ has a **partial fraction decomposition** of the form

$$\frac{P(x)}{Q(x)} = \frac{A_1}{x - a_1} + \frac{A_2}{x - a_2} + \cdots + \frac{A_n}{x - a_n}$$

for certain values of the constants $A_1, A_2, \ldots, A_n$.

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METHOD I.

$$\frac{x+4}{x^2-5x+6} = \frac{Ax-3A+Bx-2B}{(x-2)(x-3)}, \longrightarrow \begin{array}{r} A+B=1 \\ -3A-2B=4. \end{array} \longrightarrow A=-6 \text{ and } B=7.$$

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METHOD II.

$$A = \left.\frac{x+4}{x-3}\right|_{x=2} = -6 \qquad \text{and} \qquad B = \left.\frac{x+4}{x-2}\right|_{x=3} = 7.$$

Integrals of Rational Functions

EXAMPLE Evaluate $\int \frac{(x+4)}{x^2-5x+6}\,dx$.

Solution

$$\begin{aligned}\int \frac{(x+4)}{x^2-5x+6}\,dx &= -6\int \frac{1}{x-2}\,dx + 7\int \frac{1}{x-3}\,dx \\ &= -6\ln|x-2| + 7\ln|x-3| + C.\end{aligned}$$

Integrals of Rational Functions

EXAMPLE Evaluate $I = \int \frac{x^3 + 2}{x^3 - x}\, dx$.

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Solution

$$I = \int \frac{x^3 - x + x + 2}{x^3 - x}\, dx = \int \left(1 + \frac{x + 2}{x^3 - x}\right) dx = x + \int \frac{x + 2}{x^3 - x}\, dx.$$

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$$\frac{x+2}{x^3-x} = \frac{x+2}{x(x-1)(x+1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1}$$

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$$\frac{x+2}{x^3-x} = \frac{x+2}{x(x-1)(x+1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1}$$

$$A = \left.\frac{x+2}{(x-1)(x+1)}\right|_{x=0} = -2, \qquad B = \left.\frac{x+2}{x(x+1)}\right|_{x=1} = \frac{3}{2}, \quad \text{and}$$

$$C = \left.\frac{x+2}{x(x-1)}\right|_{x=-1} = \frac{1}{2}.$$

EXAMPLE Evaluate $I = \int \frac{x^3+2}{x^3-x}\,dx$.

Solution

$$
\begin{aligned}
I &= x - 2\int \frac{1}{x}\,dx + \frac{3}{2}\int \frac{1}{x-1}\,dx + \frac{1}{2}\int \frac{1}{x+1}\,dx \\
&= x - 2\ln|x| + \frac{3}{2}\ln|x-1| + \frac{1}{2}\ln|x+1| + C.
\end{aligned}
$$

Integrals of Rational Functions

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Integrals of Rational Functions

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Solution

$$\frac{2+3x+x^2}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1} = \frac{A(x^2+1)+Bx^2+Cx}{x(x^2+1)}.$$

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$$\left.\begin{array}{lll} A + B = 1 & & \text{(coefficient of } x^2\text{)} \\ C = 3 & & \text{(coefficient of } x\text{)} \\ A = 2 & & \text{(constant term).} \end{array}\right. \longrightarrow A = 2,\ B = -1,\ \text{and } C = 3.$$

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$$\left.\begin{array}{llll} A + B & = & 1 & \text{(coefficient of } x^2\text{)} \\ \quad\quad C & = & 3 & \text{(coefficient of } x\text{)} \\ A & = & 2 & \text{(constant term).} \end{array}\right. \longrightarrow A = 2,\ B = -1, \text{ and } C = 3.$$

$$\int \frac{2+3x+x^2}{x(x^2+1)}\,dx = 2\int \frac{1}{x}\,dx - \int \frac{x}{x^2+1}\,dx + 3\int \frac{1}{x^2+1}\,dx$$
$$= 2\ln|x| - \frac{1}{2}\ln(x^2+1) + 3\tan^{-1} x + C.$$

Completing the Square

EXAMPLE Evaluate $I = \int \frac{1}{x^3 + 1}\, dx.$

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Solution Here $Q(x) = x^3 + 1 = (x+1)(x^2 - x + 1)$. The latter factor has no real roots, so it has no real linear subfactors. We have

$$\frac{1}{x^3+1} = \frac{1}{(x+1)(x^2-x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1}$$
$$= \frac{A(x^2-x+1) + B(x^2+x) + C(x+1)}{(x+1)(x^2-x+1)}$$

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$$= \frac{A(x^2-x+1) + B(x^2+x) + C(x+1)}{(x+1)(x^2-x+1)}$$

$$\begin{array}{rcrcrcl} A & + & B & & & = & 0 \quad \text{(coefficient of } x^2\text{)} \\ -A & + & B & + & C & = & 0 \quad \text{(coefficient of } x\text{)} \\ A & & & + & C & = & 1 \quad \text{(constant term).} \end{array} \longrightarrow \begin{array}{l} A = 1/3, \\ B = -1/3, \\ C = 2/3. \end{array}$$

Completing the Square

EXAMPLE Evaluate $I = \int \frac{1}{x^3+1}\,dx.$

Solution

$$I = \frac{1}{3}\int \frac{dx}{x+1} - \frac{1}{3}\int \frac{x-2}{x^2-x+1}\,dx.$$

$$\boxed{x^2-x+1 = \left(x-\frac{1}{2}\right)^2 + \frac{3}{4}}$$

$$= \frac{1}{3}\ln|x+1| - \frac{1}{3}\int \frac{x-\frac{1}{2}-\frac{3}{2}}{\left(x-\frac{1}{2}\right)^2+\frac{3}{4}}\,dx$$

Let $u = x - 1/2$, $du = dx$

$$= \frac{1}{3}\ln|x+1| - \frac{1}{3}\int \frac{u}{u^2+\frac{3}{4}}\,du + \frac{1}{2}\int \frac{1}{u^2+\frac{3}{4}}\,du$$

$$= \frac{1}{3}\ln|x+1| - \frac{1}{6}\ln\left(u^2+\frac{3}{4}\right) + \frac{1}{2}\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2u}{\sqrt{3}}\right) + C$$

$$= \frac{1}{3}\ln|x+1| - \frac{1}{6}\ln(x^2-x+1) + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right) + C.$$

Denominators with Repeated Factors

EXAMPLE Evaluate $\displaystyle\int \frac{1}{x(x-1)^2}\,dx$.

Solution The appropriate partial fraction decomposition here is

$$\begin{aligned}\frac{1}{x(x-1)^2} &= \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2} \\ &= \frac{A(x^2-2x+1) + B(x^2-x) + Cx}{x(x-1)^2}.\end{aligned}$$

Equating coefficients of x^2, x, and 1 in the numerators of both sides, we get

$$\begin{aligned} A + B \quad &= 0 \qquad \text{(coefficient of } x^2\text{)} \\ -2A - B + C &= 0 \qquad \text{(coefficient of } x\text{)} \\ A \quad &= 1 \qquad \text{(constant term).} \end{aligned}$$

Hence $A = 1$, $B = -1$, $C = 1$, and

Denominators with Repeated Factors

EXAMPLE Evaluate $\displaystyle\int \frac{1}{x(x-1)^2}\,dx.$

Solution

$$\begin{aligned}
\int \frac{1}{x(x-1)^2}\,dx &= \int \frac{1}{x}\,dx - \int \frac{1}{x-1}\,dx + \int \frac{1}{(x-1)^2}\,dx \\
&= \ln|x| - \ln|x-1| - \frac{1}{x-1} + C \\
&= \ln\left|\frac{x}{x-1}\right| - \frac{1}{x-1} + C.
\end{aligned}$$

Denominators with Repeated Factors

EXAMPLE Evaluate $I = \int \frac{x^2+2}{4x^5+4x^3+x}\,dx$.

Solution The denominator factors to $x(2x^2+1)^2$, so the appropriate partial fraction decomposition is

$$\begin{aligned}\frac{x^2+2}{x(2x^2+1)^2} &= \frac{A}{x} + \frac{Bx+C}{2x^2+1} + \frac{Dx+E}{(2x^2+1)^2}\\ &= \frac{A(4x^4+4x^2+1)+B(2x^4+x^2)+C(2x^3+x)+Dx^2+Ex}{x(2x^2+1)^2}.\end{aligned}$$

Thus

$$\begin{array}{lll} 4A + 2B = 0 & & \text{(coefficient of } x^4\text{)}\\ 2C = 0 & & \text{(coefficient of } x^3\text{)}\\ 4A + B + D = 1 & & \text{(coefficient of } x^2\text{)}\\ C + E = 0 & & \text{(coefficient of } x\text{)}\\ A = 2 & & \text{(constant term).}\end{array}$$

Solving these equations, we get $A = 2$, $B = -4$, $C = 0$, $D = -3$, and $E = 0$.

Denominators with Repeated Factors

EXAMPLE Evaluate $I = \int \frac{x^2+2}{4x^5+4x^3+x}\,dx.$

Solution

$$I = 2\int \frac{dx}{x} - 4\int \frac{x\,dx}{2x^2+1} - 3\int \frac{x\,dx}{(2x^2+1)^2}$$

$$\boxed{\begin{array}{l} \text{Let } u = 2x^2+1, \\ du = 4x\,dx \end{array}}$$

$$= 2\ln|x| - \int \frac{du}{u} - \frac{3}{4}\int \frac{du}{u^2}$$

$$= 2\ln|x| - \ln|u| + \frac{3}{4u} + C$$

$$= \ln\left(\frac{x^2}{2x^2+1}\right) + \frac{3}{4}\frac{1}{2x^2+1} + C.$$

Inverse Substitutions

$$\int_a^b f(x)\,dx \xrightarrow{\quad x = g(u) \quad} \int_{x=a}^{x=b} f\big(g(u)\big)\,g'(u)\,du.$$

Inverse Substitutions

The Inverse Trigonometric Substitutions

Three very useful inverse substitutions are:

$$x = a\sin\theta, \qquad x = a\tan\theta, \quad \text{and} \qquad x = a\sec\theta.$$

The inverse sine substitution

Integrals involving $\sqrt{a^2 - x^2}$ (where $a > 0$) can frequently be reduced to a simpler form by means of the substitution

$$x = a\sin\theta \quad \text{or, equivalently,} \quad \theta = \sin^{-1}\frac{x}{a}.$$

Observe that $\sqrt{a^2 - x^2}$ makes sense only if $-a \le x \le a$, which corresponds to $-\pi/2 \le \theta \le \pi/2$. Since $\cos\theta \ge 0$ for such θ, we have

$$\sqrt{a^2 - x^2} = \sqrt{a^2(1 - \sin^2\theta)} = \sqrt{a^2\cos^2\theta} = a\cos\theta.$$

The Inverse Trigonometric Substitutions

EXAMPLE Evaluate $\int \frac{1}{(5-x^2)^{3/2}}\,dx$.

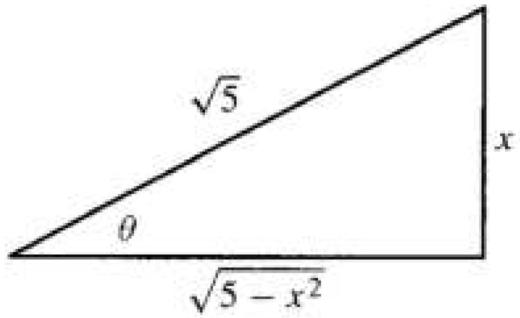
Solution

$$\int \frac{1}{(5-x^2)^{3/2}}\,dx$$

$$\boxed{\begin{aligned}&\text{Let } x = \sqrt{5}\sin\theta,\\ &dx = \sqrt{5}\cos\theta\,d\theta\end{aligned}}$$

$$= \int \frac{\sqrt{5}\cos\theta\,d\theta}{5^{3/2}\cos^3\theta}$$

$$= \frac{1}{5}\int \sec^2\theta\,d\theta = \frac{1}{5}\tan\theta + C = \frac{1}{5}\frac{x}{\sqrt{5-x^2}} + C$$



Inverse Substitutions

The Inverse Trigonometric Substitutions

The inverse tangent substitution

Integrals involving $\sqrt{a^2 + x^2}$ or $\dfrac{1}{x^2 + a^2}$ (where $a > 0$) are often simplified by the substitution

$$x = a \tan \theta \quad \text{or, equivalently,} \quad \theta = \tan^{-1} \frac{x}{a}.$$

Since x can take any real value, we have $-\pi/2 < \theta < \pi/2$, so $\sec \theta > 0$ and

$$\sqrt{a^2 + x^2} = a\sqrt{1 + \tan^2 \theta} = a \sec \theta.$$

Inverse Substitutions

EXAMPLE Evaluate $\displaystyle\int \frac{1}{(1+9x^2)^2}\,dx$.

Solution

$$\int \frac{1}{(1+9x^2)^2}\,dx$$

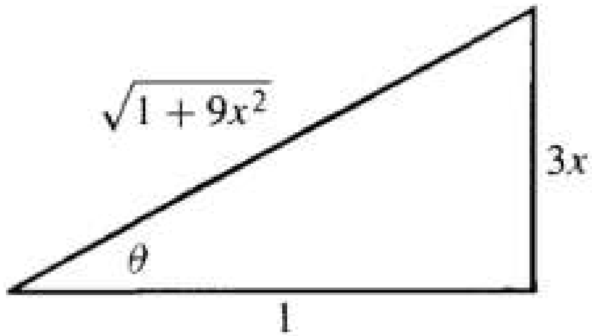
$$\boxed{\begin{aligned} &\text{Let } 3x = \tan\theta,\\ &3\,dx = \sec^2\theta\,d\theta,\\ &1+9x^2 = \sec^2\theta \end{aligned}}$$

$$= \frac{1}{3}\int \frac{\sec^2\theta\,d\theta}{\sec^4\theta}$$

$$= \frac{1}{3}\int \cos^2\theta\,d\theta = \frac{1}{6}(\theta + \sin\theta\,\cos\theta) + C$$

$$= \frac{1}{6}\tan^{-1}(3x) + \frac{1}{6}\frac{3x}{\sqrt{1+9x^2}}\frac{1}{\sqrt{1+9x^2}} + C$$

$$= \frac{1}{6}\tan^{-1}(3x) + \frac{1}{2}\frac{x}{1+9x^2} + C$$



Inverse Substitutions

The Inverse Trigonometric Substitutions

The inverse secant substitution

Integrals involving $\sqrt{x^2 - a^2}$ (where $a > 0$) can frequently be simplified by using the substitution

$$x = a \sec\theta \quad \text{or, equivalently,} \quad \theta = \sec^{-1}\frac{x}{a}.$$

$$\sqrt{x^2 - a^2} = a\sqrt{\sec^2\theta - 1} = a\sqrt{\tan^2\theta} = a|\tan\theta|,$$

Observe that $\sqrt{x^2 - a^2}$ makes sense for $x \geq a$ and for $x \leq -a$.

If $x \geq a$, then $0 \leq \theta = \sec^{-1}\frac{x}{a} = \arccos\frac{a}{x} < \frac{\pi}{2}$, and $\tan\theta \geq 0$.

If $x \leq -a$, then $\frac{\pi}{2} < \theta = \sec^{-1}\frac{x}{a} = \arccos\frac{a}{x} \leq \pi$, and $\tan\theta \leq 0$.

In the first case $\sqrt{x^2 - a^2} = a\tan\theta$; in the second case $\sqrt{x^2 - a^2} = -a\tan\theta$.

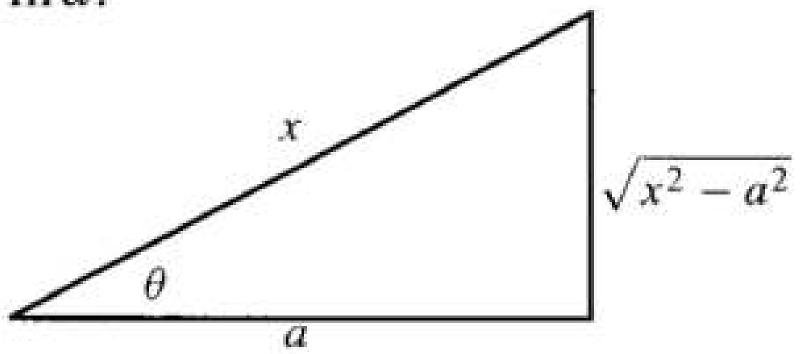
The Inverse Trigonometric Substitutions

EXAMPLE Find $I = \displaystyle\int \frac{dx}{\sqrt{x^2 - a^2}}$, where $a > 0$.

Solution For the moment, assume that $x \geq a$. If $x = a \sec\theta$, then $dx = a \sec\theta \, \tan\theta \, d\theta$ and $\sqrt{x^2 - a^2} = a \tan\theta$.

$$\begin{aligned} I &= \int \sec\theta \, d\theta = \ln|\sec\theta + \tan\theta| + C \\ &= \ln\left|\frac{x}{a} + \frac{\sqrt{x^2 - a^2}}{a}\right| + C = \ln|x + \sqrt{x^2 - a^2}| + C_1, \end{aligned}$$

where $C_1 = C - \ln a$.



Inverse Substitutions

The Inverse Trigonometric Substitutions

EXAMPLE Find $I = \int \frac{dx}{\sqrt{x^2 - a^2}}$, where $a > 0$.

Solution If $x \le -a$, let $u = -x$ so that $u \ge a$ and $du = -dx$. We have

$$I = -\int \frac{du}{\sqrt{u^2 - a^2}} = -\ln|u + \sqrt{u^2 - a^2}| + C_1$$

$$= \ln\left|\frac{1}{-x + \sqrt{x^2 - a^2}} \frac{x + \sqrt{x^2 - a^2}}{x + \sqrt{x^2 - a^2}}\right| + C_1$$

$$= \ln\left|\frac{x + \sqrt{x^2 - a^2}}{-a^2}\right| + C_1 = \ln|x + \sqrt{x^2 - a^2}| + C_2,$$

where $C_2 = C_1 - 2\ln a$. Thus, in either case, we have

$$I = \ln|x + \sqrt{x^2 - a^2}| + C.$$

Inverse Substitutions

The Inverse Trigonometric Substitutions

EXAMPLE Evaluate $\displaystyle\int \frac{1}{\sqrt{2x - x^2}}\,dx$

Solution

(a) $$\int \frac{1}{\sqrt{2x - x^2}}\,dx = \int \frac{dx}{\sqrt{1 - (1 - 2x + x^2)}}$$

$$= \int \frac{dx}{\sqrt{1 - (x - 1)^2}} \qquad \text{Let } u = x - 1,\ du = dx$$

$$= \int \frac{du}{\sqrt{1 - u^2}}$$

$$= \sin^{-1} u + C = \sin^{-1}(x - 1) + C.$$

Other Inverse Substitutions

EXAMPLE

$$\int \frac{1}{1+\sqrt{2x}}\,dx \qquad \text{Let } 2x = u^2, \quad 2\,dx = 2u\,du$$

$$= \int \frac{u}{1+u}\,du$$

$$= \int \frac{1+u-1}{1+u}\,du$$

$$= \int \left(1 - \frac{1}{1+u}\right) du \qquad \text{Let } v = 1+u, \quad dv = du$$

$$= u - \int \frac{dv}{v} = u - \ln|v| + C$$

$$= \sqrt{2x} - \ln\left(1+\sqrt{2x}\right) + C$$

Other Inverse Substitutions

EXAMPLE

$$\int_{-1/3}^{2} \frac{x}{\sqrt[3]{3x+2}}\,dx \qquad \text{Let } 3x+2=u^3,\quad 3\,dx = 3u^2\,du$$

$$= \int_{1}^{2} \frac{u^3-2}{3u}\,u^2\,du$$

$$= \frac{1}{3}\int_{1}^{2} (u^4-2u)\,du = \frac{1}{3}\left(\frac{u^5}{5}-u^2\right)\Bigg|_{1}^{2} = \frac{16}{15}.$$

Other Inverse Substitutions

EXAMPLE Evaluate $\int \frac{1}{x^{1/2}(1+x^{1/3})}\,dx$.

Solution

$$\int \frac{dx}{x^{1/2}(1+x^{1/3})} \qquad \begin{aligned} &\text{Let } x = u^6, \\ &dx = 6u^5\,du \end{aligned}$$

$$= 6\int \frac{u^5\,du}{u^3(1+u^2)} = 6\int \frac{u^2}{1+u^2}\,du = 6\int \left(1 - \frac{1}{1+u^2}\right) du$$
$$= 6\,(u - \tan^{-1} u) + C = 6\,(x^{1/6} - \tan^{-1} x^{1/6}) + C.$$

Inverse Substitutions

The tan(θ/2) Substitution

$$x = \tan\frac{\theta}{2} \longrightarrow$$

$$\cos^2\frac{\theta}{2} = \frac{1}{\sec^2\frac{\theta}{2}} = \frac{1}{1+\tan^2\frac{\theta}{2}} = \frac{1}{1+x^2},$$

$$\boxed{\begin{aligned}\cos\theta &= 2\cos^2\frac{\theta}{2} - 1 = \frac{2}{1+x^2} - 1 = \frac{1-x^2}{1+x^2}\\ \sin\theta &= 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = 2\tan\frac{\theta}{2}\cos^2\frac{\theta}{2} = \frac{2x}{1+x^2}.\end{aligned}}$$

$$dx = \frac{1}{2}\sec^2\frac{\theta}{2}\,d\theta$$

$$\boxed{d\theta = 2\cos^2\frac{\theta}{2}\,dx = \frac{2\,dx}{1+x^2}.}$$

Inverse Substitutions

The tan(θ/2) Substitution

In summary:

If $x = \tan(\theta/2)$, then

$$\cos\theta = \frac{1-x^2}{1+x^2}, \qquad \sin\theta = \frac{2x}{1+x^2}, \quad \text{and} \qquad d\theta = \frac{2\,dx}{1+x^2}.$$

Inverse Substitutions

The tan(θ/2) Substitution

EXAMPLE

$$\int \frac{1}{2+\cos\theta}\,d\theta$$

$$\boxed{\begin{aligned}&\text{Let } x = \tan(\theta/2), \text{ so}\\ &\cos\theta = \frac{1-x^2}{1+x^2},\\ &d\theta = \frac{2\,dx}{1+x^2}\end{aligned}}$$

$$= \int \frac{\dfrac{2\,dx}{1+x^2}}{2+\dfrac{1-x^2}{1+x^2}} = 2\int \frac{1}{3+x^2}\,dx$$

$$= \frac{2}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}} + C$$

$$= \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{1}{\sqrt{3}}\tan\frac{\theta}{2}\right) + C.$$