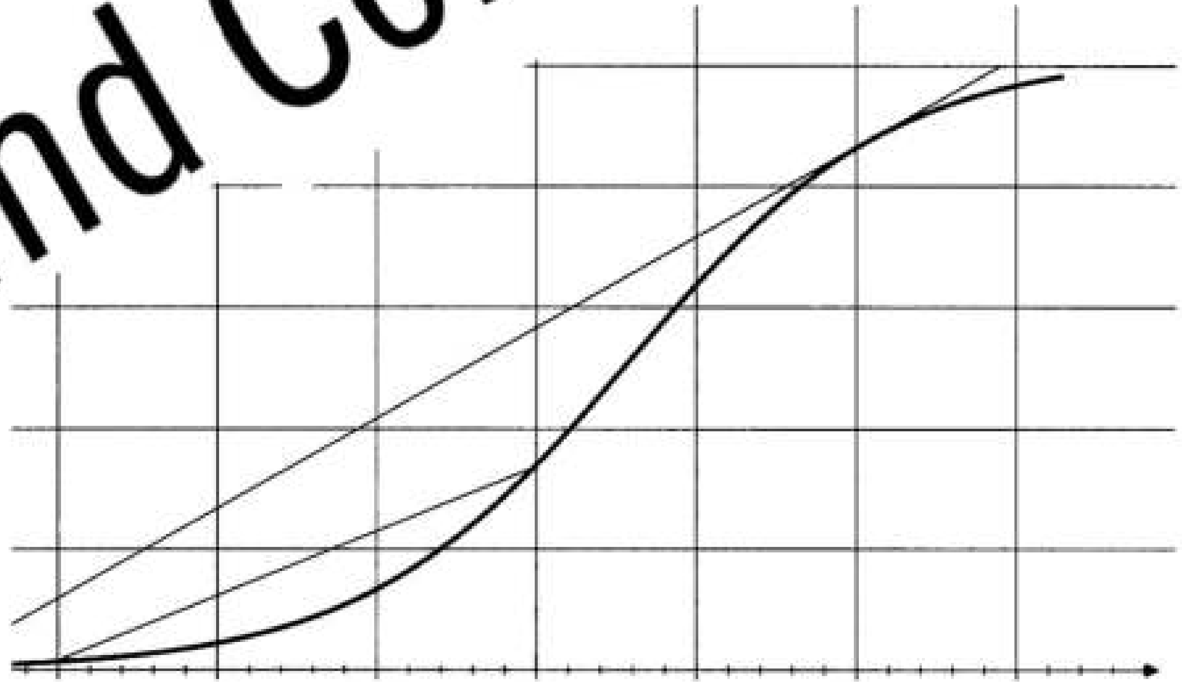
Limits and Continuity



Average Velocity and Instantaneous Velocity

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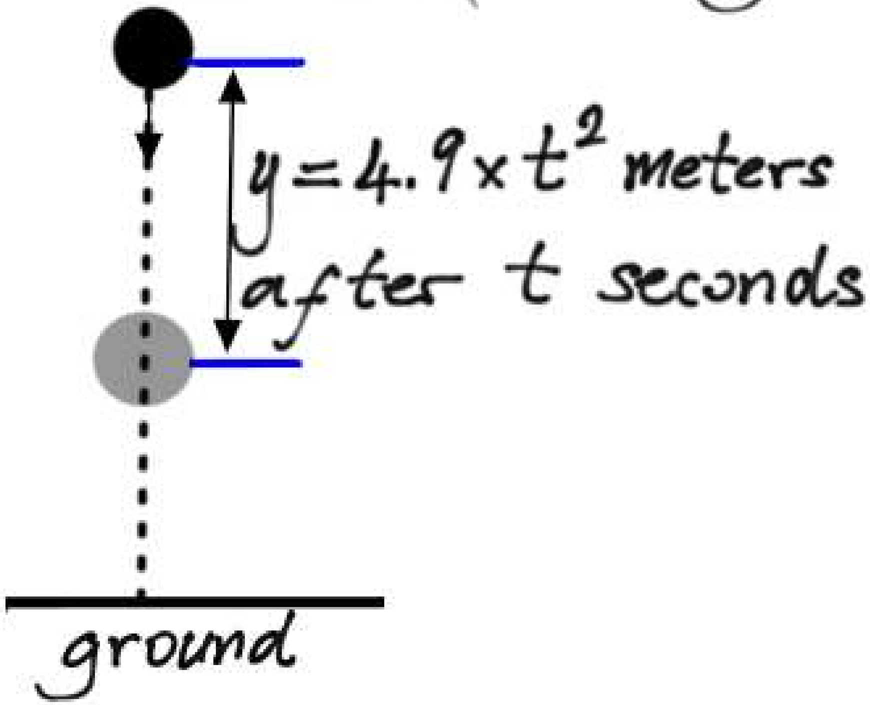
EXAMPLE (Average velocity of a falling rock)



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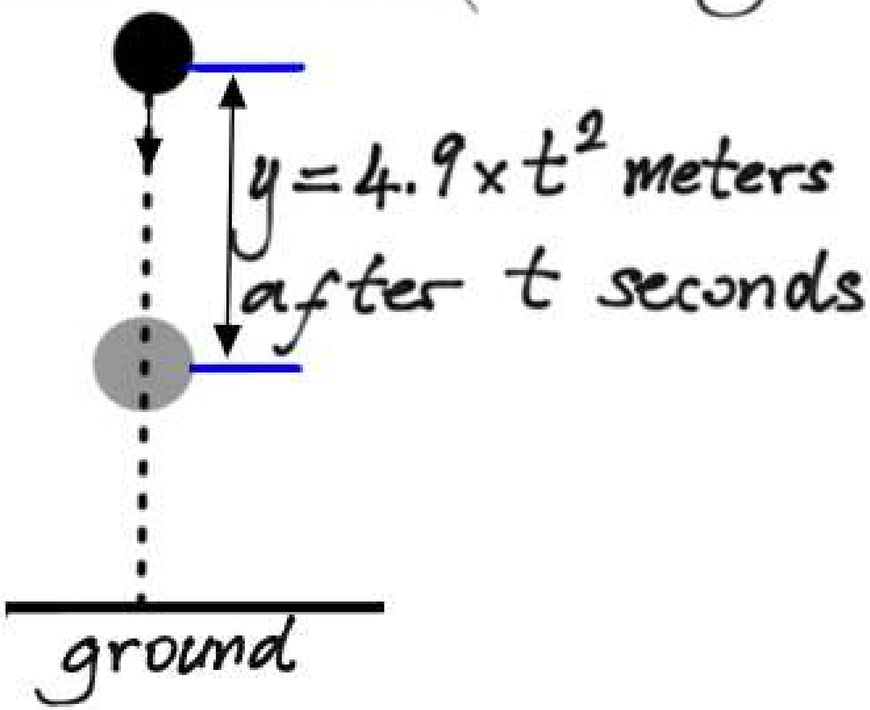

What is the average velocity during the first 2 s?

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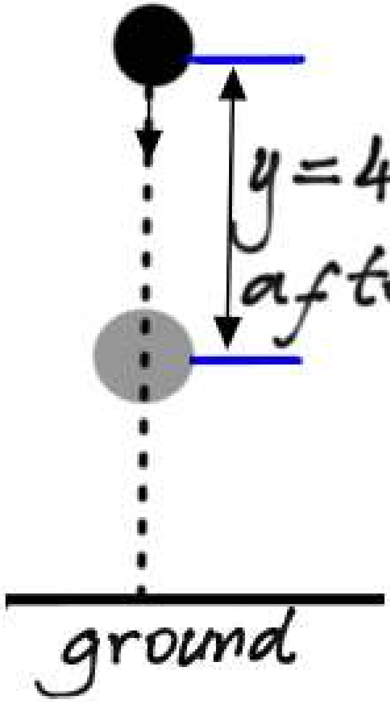

$y = 4.9 \times t^2$ meters after t seconds

What is the average velocity during the first 2 s?

It is

$$\frac{\Delta y}{\Delta t} = \frac{4.9 \times 2^2 - 4.9 \times 0^2}{2-0} = 9.8 \text{ m/s}$$

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Average velocity over $[2, 2+h]$

h	$\Delta y/\Delta t$
1	24.5000
0.1	20.0900
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$$\frac{\Delta y}{\Delta t} = \frac{4.9(2+h)^2 - 4.9t^2}{h}$$

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average velocities get closer and closer to 19.6 as the lengths of time intervals get closer and closer to 0.

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$$\frac{\Delta y}{\Delta t} = \frac{4.9(2+h)^2 - 4.9t^2}{h}$$

notationally

$$\lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t} =$$

$$\lim_{h \to 0} \frac{4.9(2+h)^2 - 4.9}{h} = 19.6$$

Average Velocity and Instantaneous Velocity

EXAMPLE How fast is the rock falling t seconds after it is dropped?

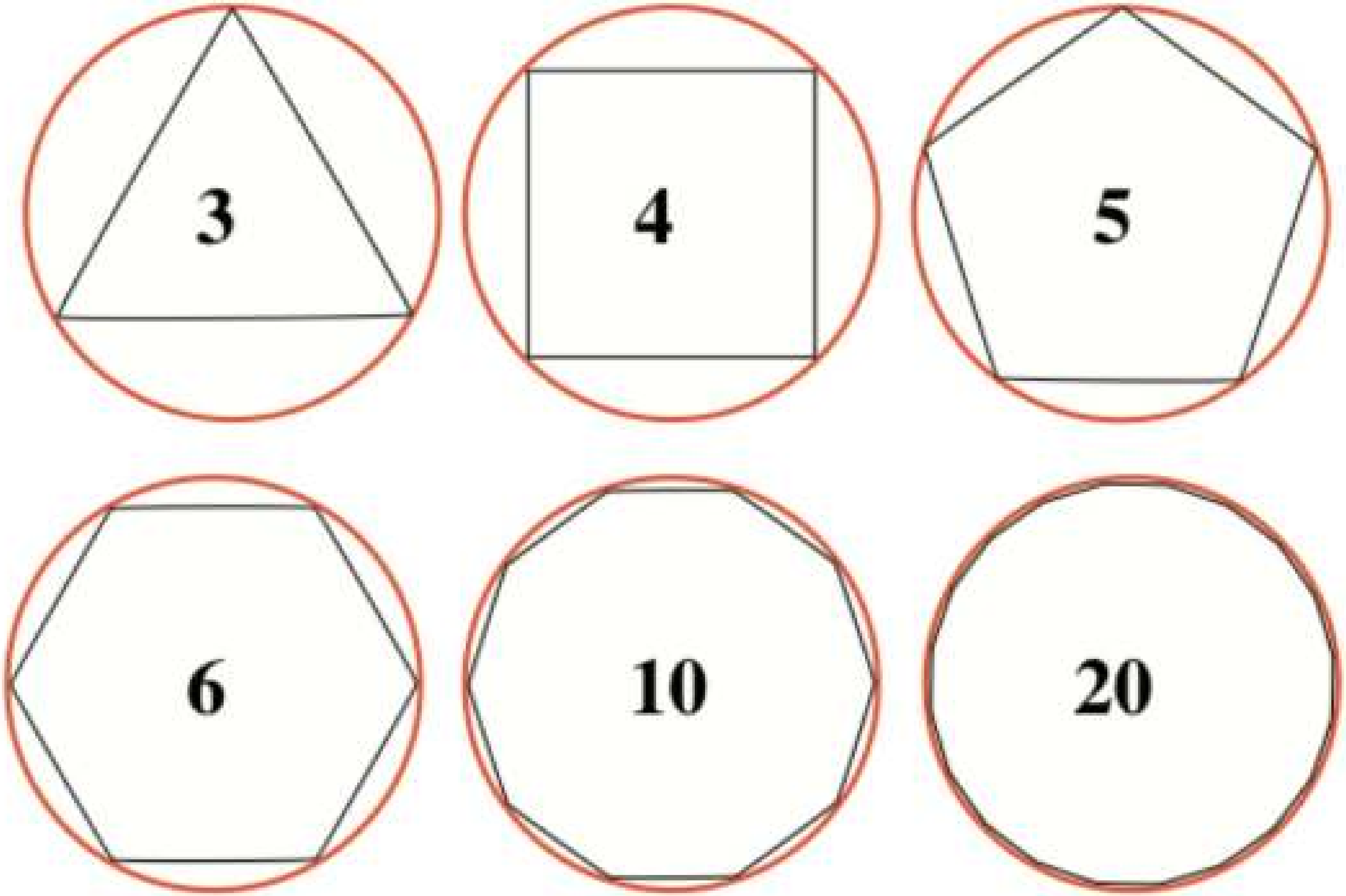
the average velocity over the time interval $[t, t+h] =$

$$\frac{\Delta y}{\Delta t} = \frac{4.9(t+h)^2 - 4.9t^2}{h} = 9.8t + 4.9h \xrightarrow{\text{as } h \to 0} 9.8t$$

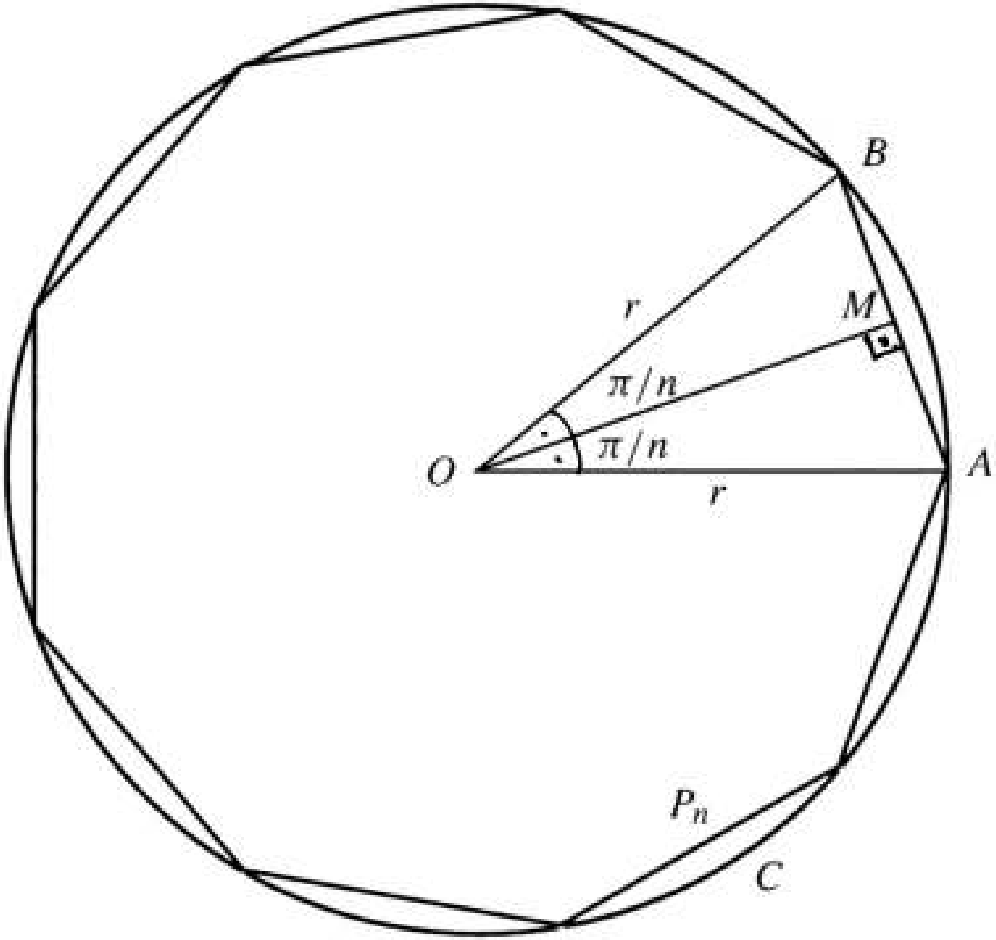
t seconds after the rock is dropped, its velocity is $9.8t$ m/s.

The Area of a Circle

Approximating a circle with polygons



The Area of a Circle



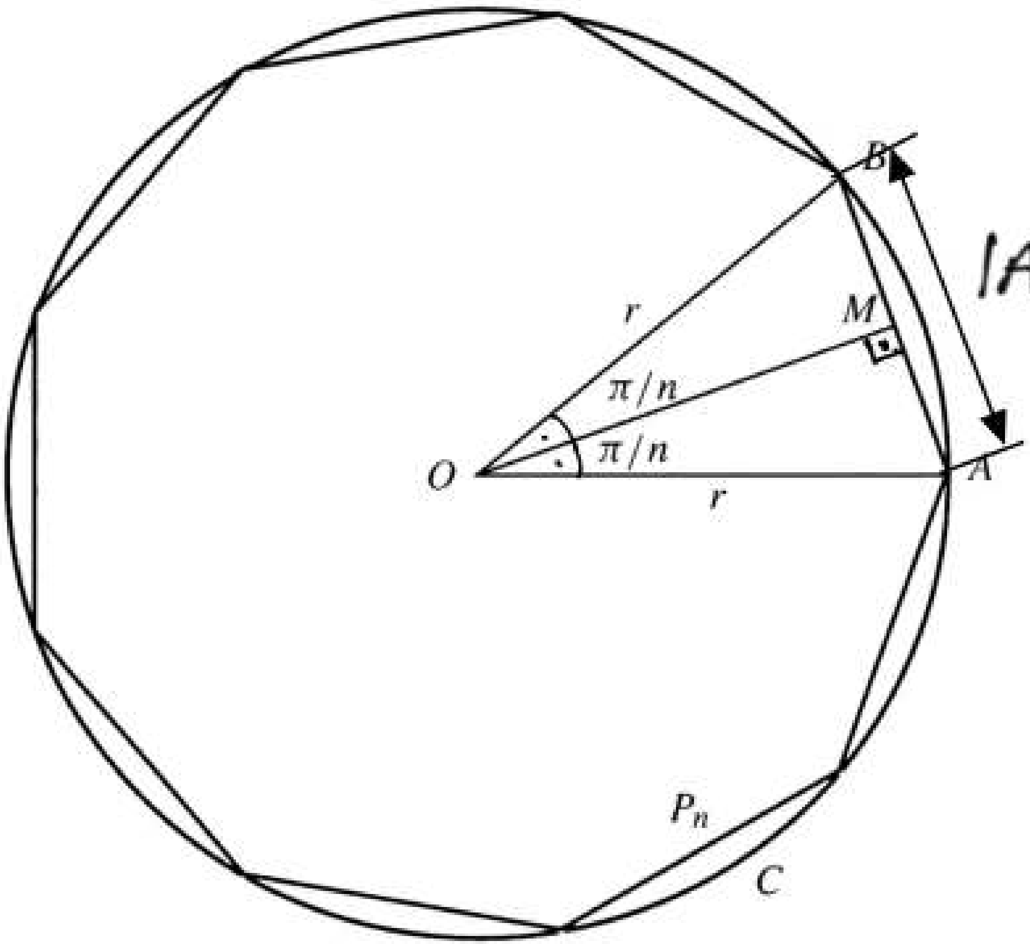
P_n : the perimeter of the polygon

A_n : the area of the polygon

C : the circumference of the circle

A : the area of the circle

The Area of a Circle



$$|AB| = 2|MB| = 2r\sin\left(\frac{\pi}{n}\right)$$

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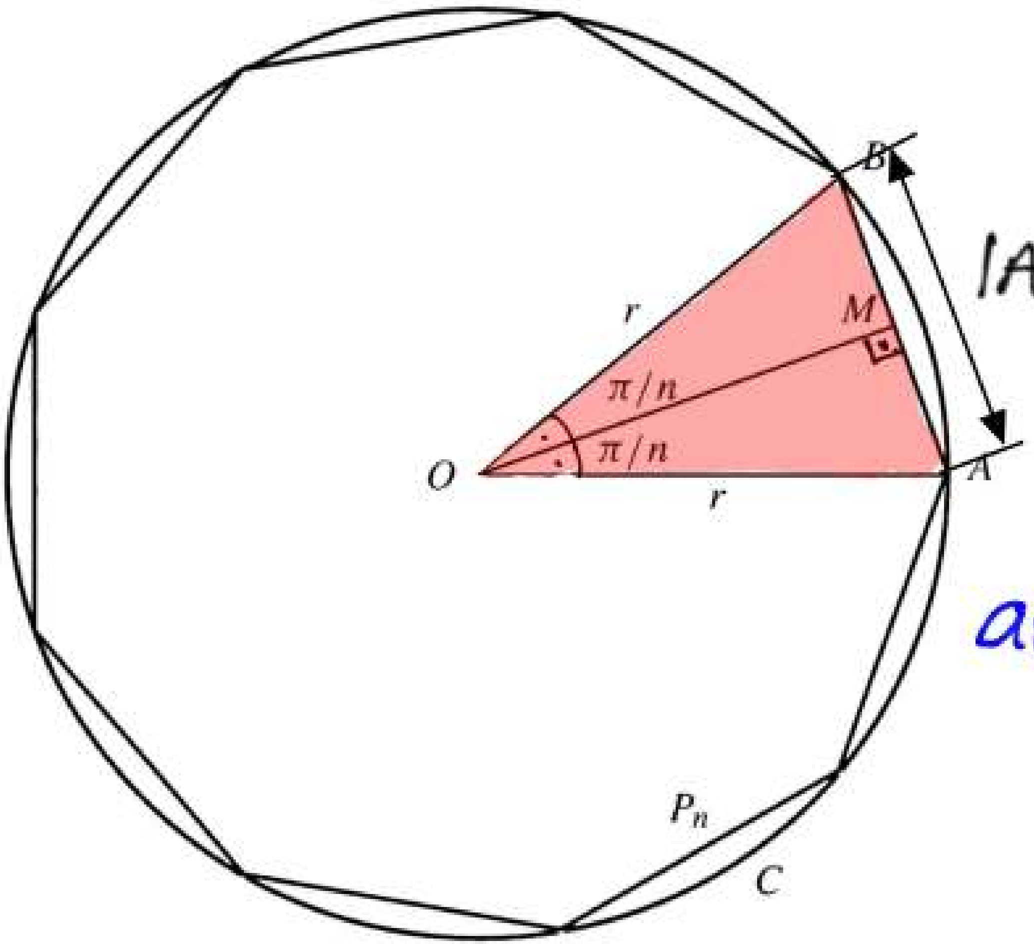

$$|AB| = 2|MB| = 2r\sin\left(\frac{\pi}{n}\right)$$

$$\text{area } \overset{\triangle}{AOB} = \frac{1}{2}|AB||OM|$$

$$= \frac{1}{2}\left[2r\sin\left(\frac{\pi}{n}\right)\right]\left[r\cos\left(\frac{\pi}{n}\right)\right]$$

$$= r^2\sin\left(\frac{\pi}{n}\right)\cos\left(\frac{\pi}{n}\right)$$

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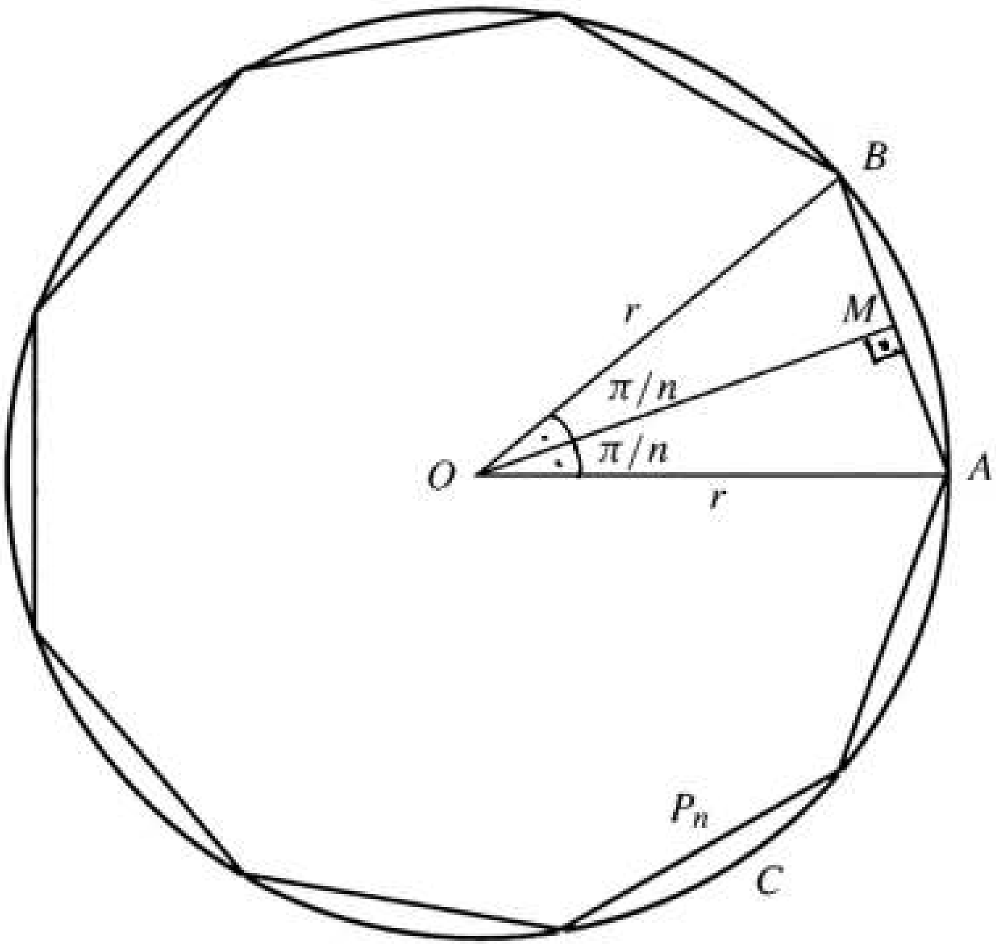

$$P_n = 2rn\sin\left(\frac{\pi}{n}\right)$$

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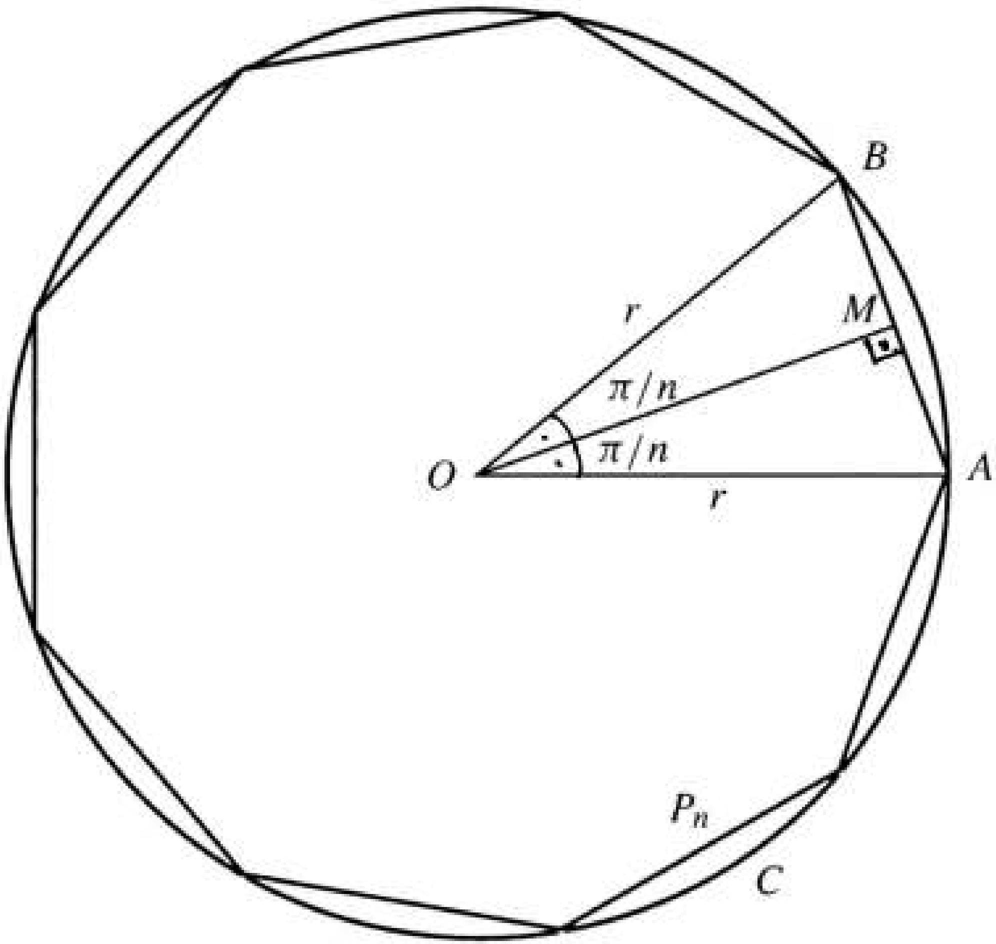

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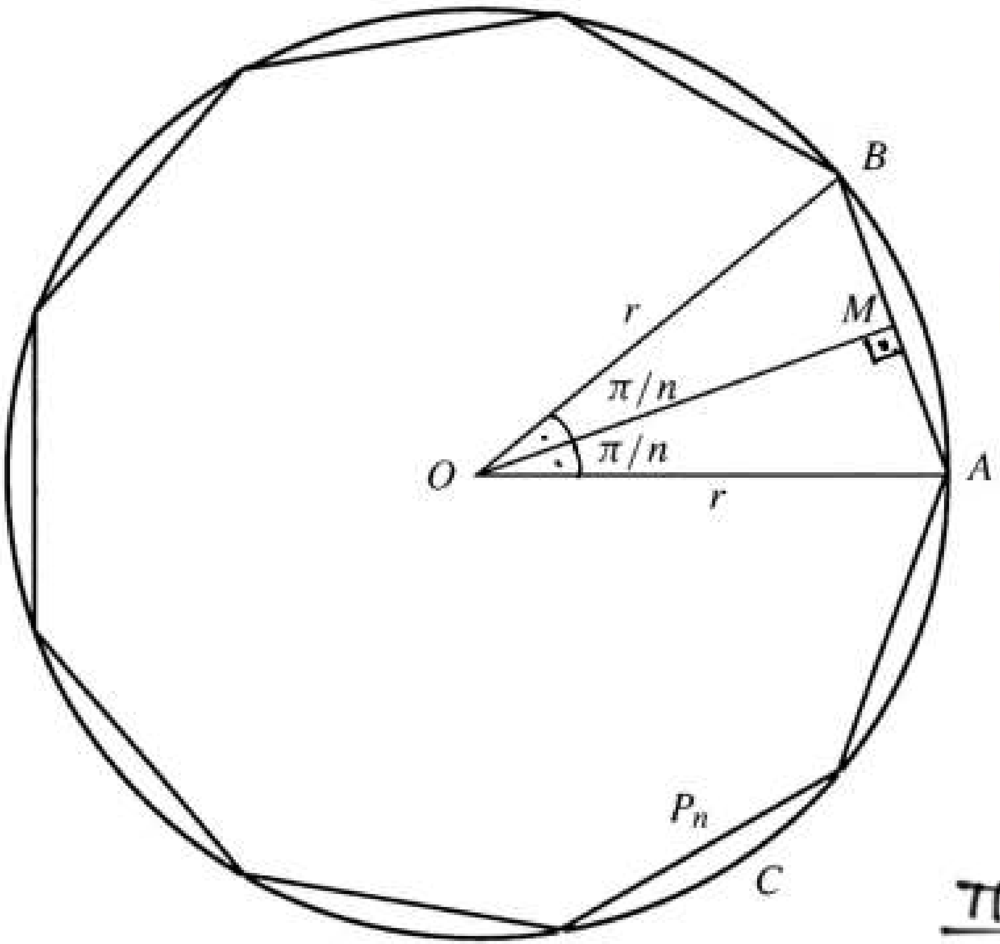

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As n grows large,

$\frac{\pi}{n} \longrightarrow 0$; so $\cos\left(\frac{\pi}{n}\right) = \frac{|OM|}{|OA|} \longrightarrow 1.$

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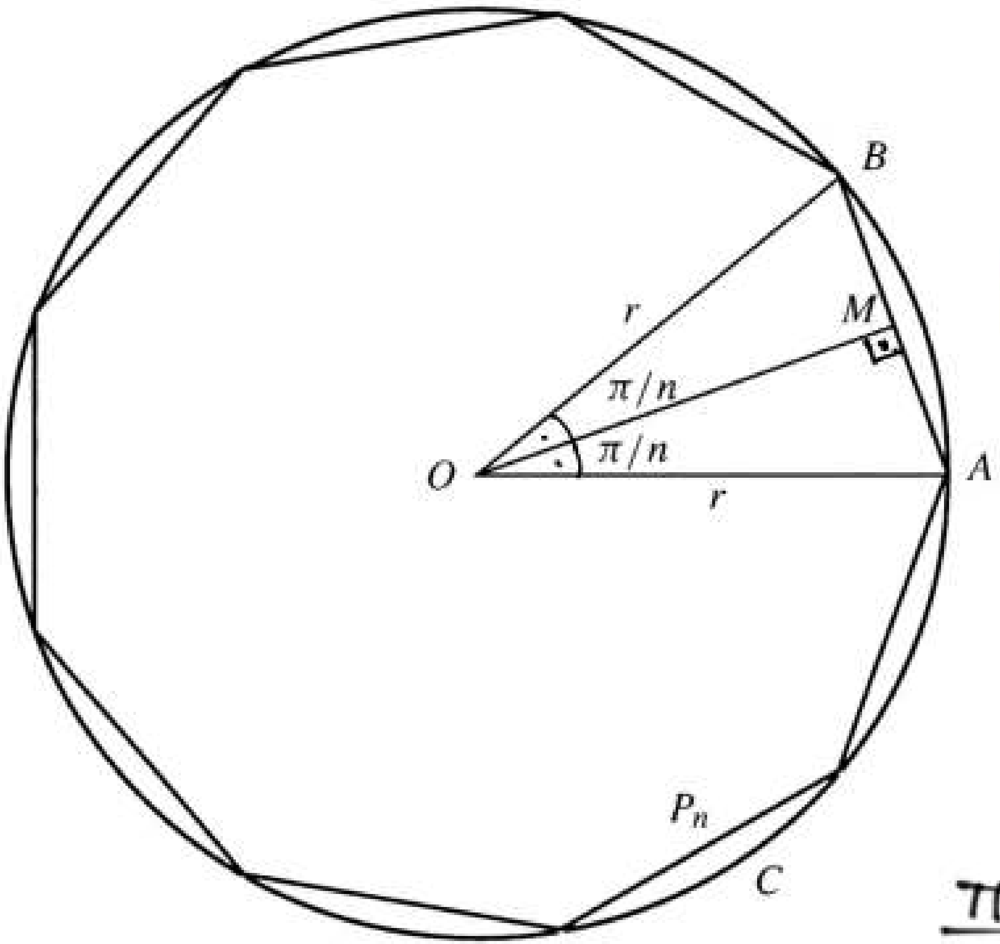

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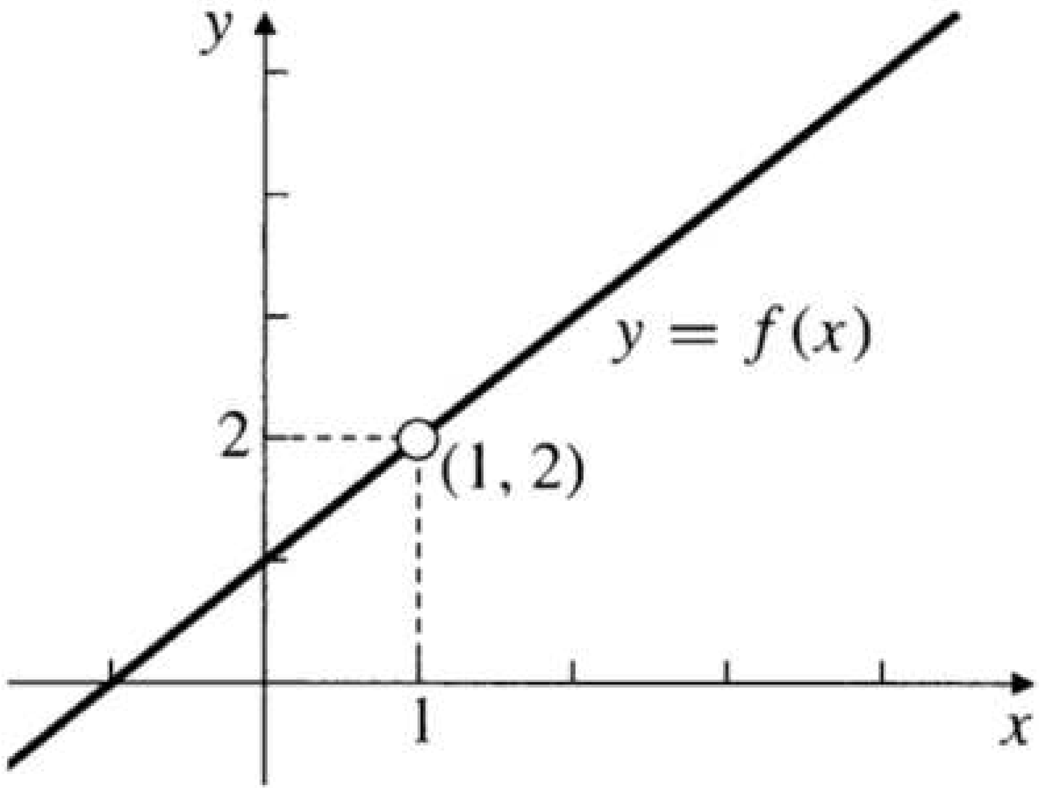
Also, as n grows large, $P_n \to C = 2\pi r$; so $A_n \to \pi r^2$.

Limits of Functions

Limits of Functions

EXAMPLE Describe the behaviour of the function $f(x) = \dfrac{x^2 - 1}{x - 1}$ near $x = 1$.



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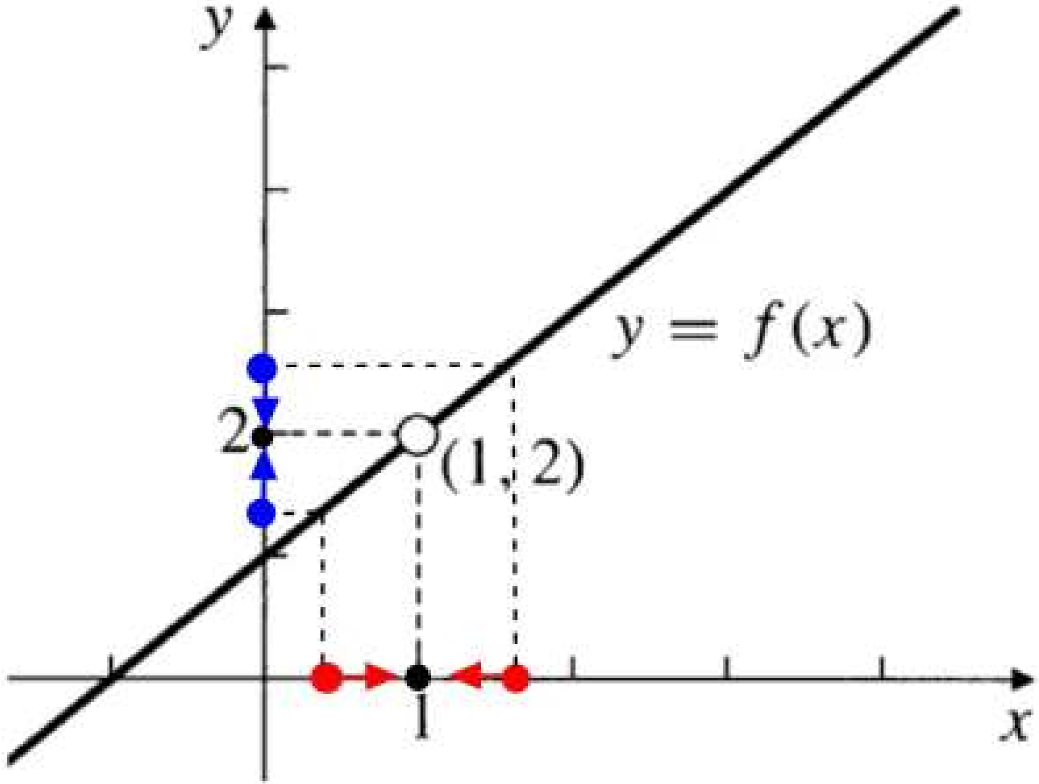

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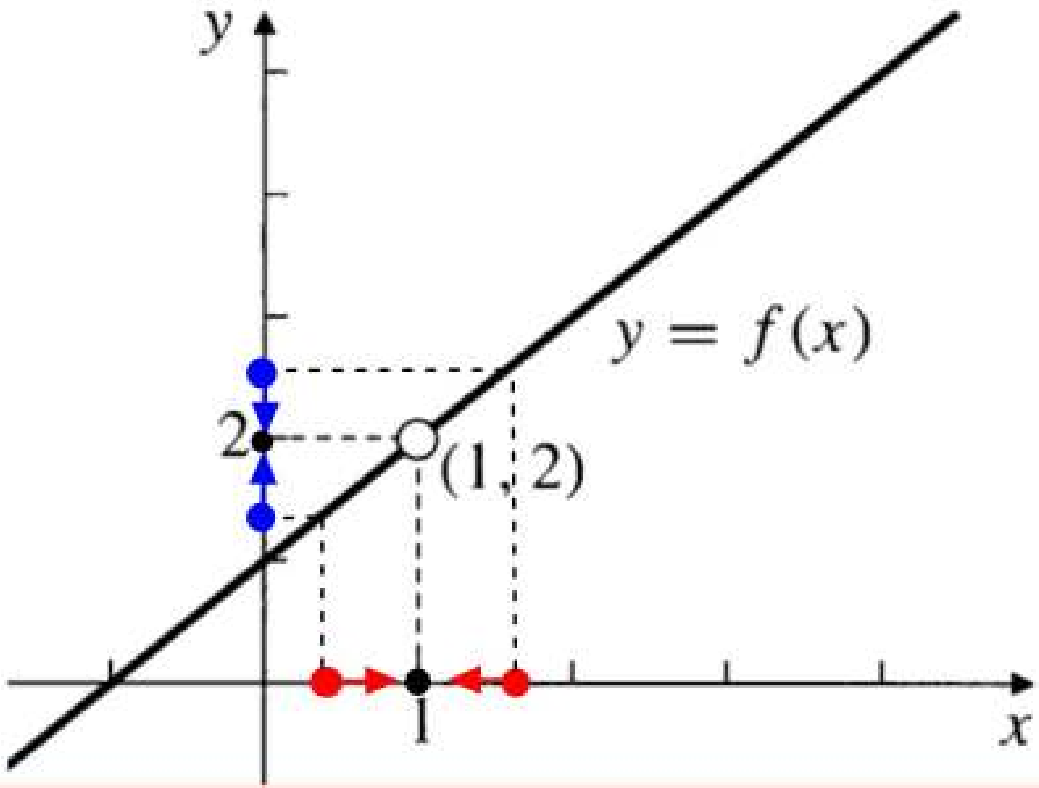

$$\lim_{x \to 1} f(x) = 2 \qquad \text{or} \qquad \lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2$$

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$\lim_{x \to 0} g(x) = 2.71828\ldots = e$

as we shall see later.

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$= e$

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round-off error

it should have been $(1.0000000001)^{10\,000\,000\,000}$

Limits of Functions

An informal definition of limit

If $f(x)$ is defined for all x near a, except possibly at a itself, and if we can ensure that $f(x)$ is as close as we want to L by taking x close enough to a, but not equal to a, we say that the function f approaches the **limit** L as x approaches a, and we write

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EXAMPLE Let $g(x) = \begin{cases} x & \text{if } x \neq 2 \\ 1 & \text{if } x = 2. \end{cases}$ Then

$$\lim_{x \to 2} g(x) = \lim_{x \to 2} x = 2, \qquad \text{although} \quad g(2) = 1.$$

Limits of Functions

One-Sided Limits

Informal definition of left and right limits

If $f(x)$ is defined on some interval (b, a) extending to the left of $x = a$, and if we can ensure that $f(x)$ is as close as we want to L by taking x to the left of a and close enough to a, then we say $f(x)$ has **left limit** L at $x = a$, and we write

$$\lim_{x \to a-} f(x) = L.$$

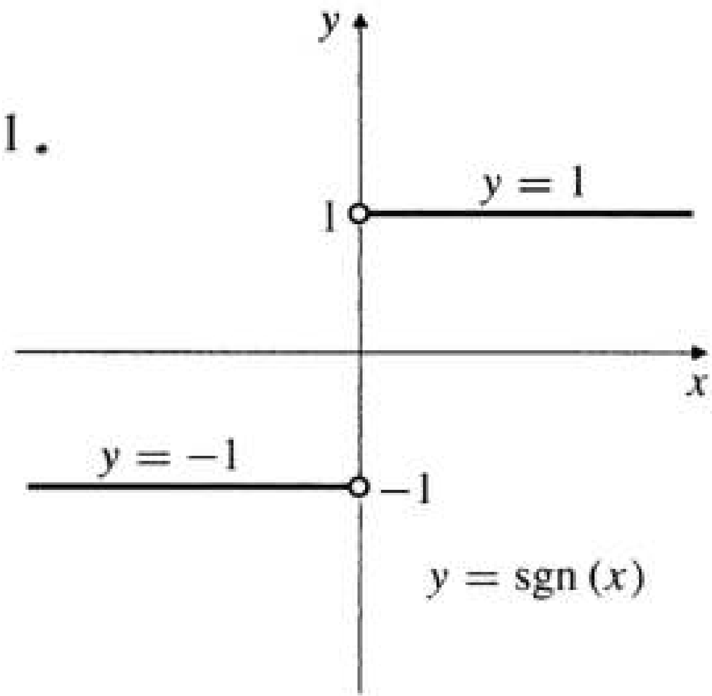
If $f(x)$ is defined on some interval (a, b) extending to the right of $x = a$, and if we can ensure that $f(x)$ is as close as we want to L by taking x to the right of a and close enough to a, then we say $f(x)$ has **right limit** L at $x = a$, and we write

$$\lim_{x \to a+} f(x) = L.$$

One-Sided Limits

EXAMPLE The signum function $\operatorname{sgn}(x) = x/|x|$ has left limit -1 and right limit 1 at $x = 0$:

$$\lim_{x \to 0-} \operatorname{sgn}(x) = -1 \qquad \text{and} \qquad \lim_{x \to 0+} \operatorname{sgn}(x) = 1 .$$



One-Sided Limits

THEOREM **Relationship between one-sided and two-sided limits**

A function $f(x)$ has limit L at $x = a$ if and only if it has both left and right limits there and these one-sided limits are both equal to L:

$$\lim_{x \to a} f(x) = L \quad \Longleftrightarrow \quad \lim_{x \to a-} f(x) = \lim_{x \to a+} f(x) = L.$$

Limits of Functions

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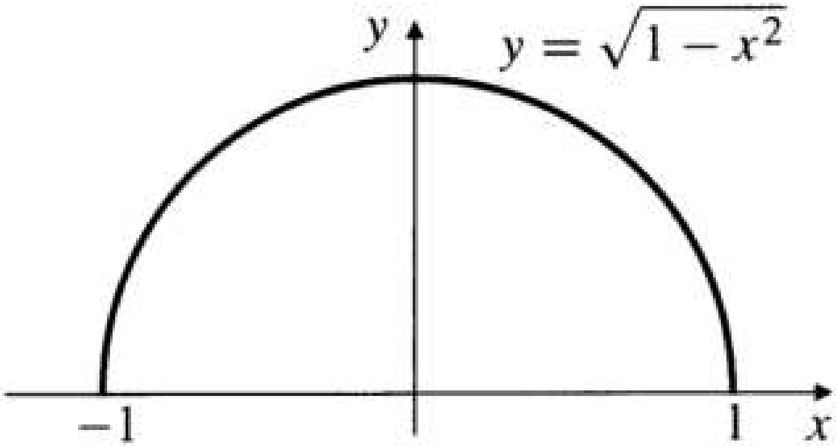
EXAMPLE If $f(x) = \dfrac{|x-2|}{x^2+x-6}$, find: $\displaystyle\lim_{x \to 2+} f(x)$, $\displaystyle\lim_{x \to 2-} f(x)$, and $\displaystyle\lim_{x \to 2} f(x)$.

One-Sided Limits

EXAMPLE What one-sided limits does $g(x) = \sqrt{1 - x^2}$ have at $x = -1$ and $x = 1$?



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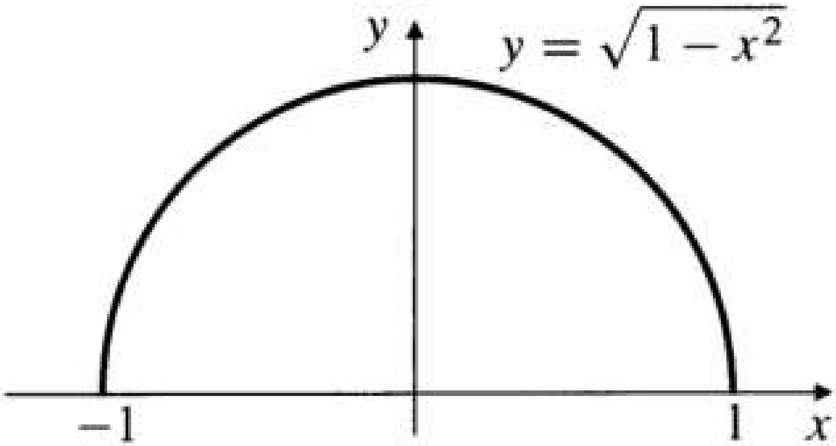

$$\lim_{x \to -1^+} g(x) = 0$$

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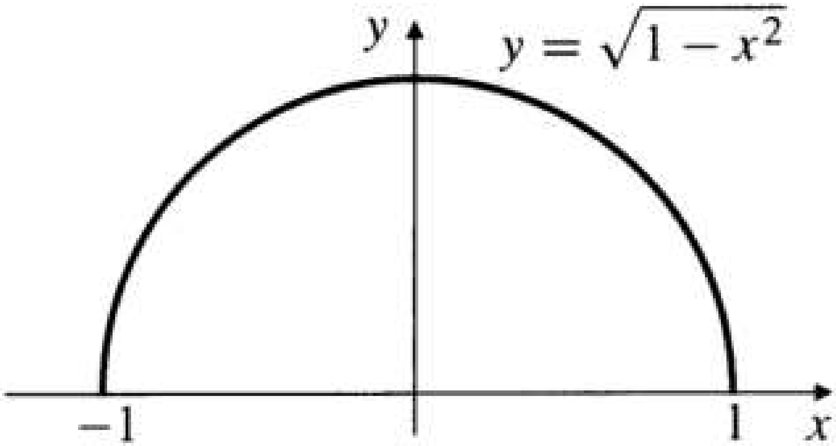


$$\lim_{x \to -1^+} g(x) = 0$$

$$\lim_{x \to 1^-} g(x) = 0.$$

$g(x)$ has no left limit or limit at $x=-1$ and no right limit or limit at $x=1$.

Limits of Functions

Rules for Calculating Limits

If $\lim_{x \to a} f(x) = L$, $\lim_{x \to a} g(x) = M$, and k is a constant, then

1. **Limit of a sum:** $\lim\limits_{x \to a} [f(x) + g(x)] = L + M$
2. **Limit of a difference:** $\lim\limits_{x \to a} [f(x) - g(x)] = L - M$
3. **Limit of a product:** $\lim\limits_{x \to a} f(x)g(x) = LM$
4. **Limit of a multiple:** $\lim\limits_{x \to a} kf(x) = kL$
5. **Limit of a quotient:** $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}$, if $M \neq 0$.

If m is an integer and n is a positive integer, then

6. **Limit of a power:** $\lim\limits_{x \to a} \left[f(x)\right]^{m/n} = L^{m/n}$, provided $L > 0$ if n is even, and $L \neq 0$ if $m < 0$.

If $f(x) \leq g(x)$ on an interval containing a in its interior, then

7. **Order is preserved:** $L \leq M$

Limits of Functions

Rules for Calculating Limits

EXAMPLE Evaluate:

$$\text{(a)} \lim_{x\to-2} \frac{x^2+x-2}{x^2+5x+6}, \quad \text{(b)} \lim_{x\to a} \frac{\frac{1}{x}-\frac{1}{a}}{x-a}, \quad \text{and} \quad \text{(c)} \lim_{x\to 4} \frac{\sqrt{x}-2}{x^2-16}.$$

EXAMPLE Find:

$$\text{(a)} \lim_{x\to a} \frac{x^2+x+4}{x^3-2x^2+7} \quad \text{and} \quad \text{(b)} \lim_{x\to 2} \sqrt{2x+1}.$$

Limits of Functions

THEOREM

Limits of Polynomials and Rational Functions

1. If $P(x)$ is a polynomial and a is any real number, then

$$\lim_{x \to a} P(x) = P(a).$$

2. If $P(x)$ and $Q(x)$ are polynomials and $Q(a) \neq 0$, then

$$\lim_{x \to a} \frac{P(x)}{Q(x)} = \frac{P(a)}{Q(a)}.$$

Limits of Functions

EXAMPLE Find the following limits:

(a) $$\lim_{h \to 0} \frac{\sqrt{1+h}-1}{h}$$

(b) $$\lim_{t \to 0} \frac{1}{t\sqrt{1+t}} - \frac{1}{t}$$

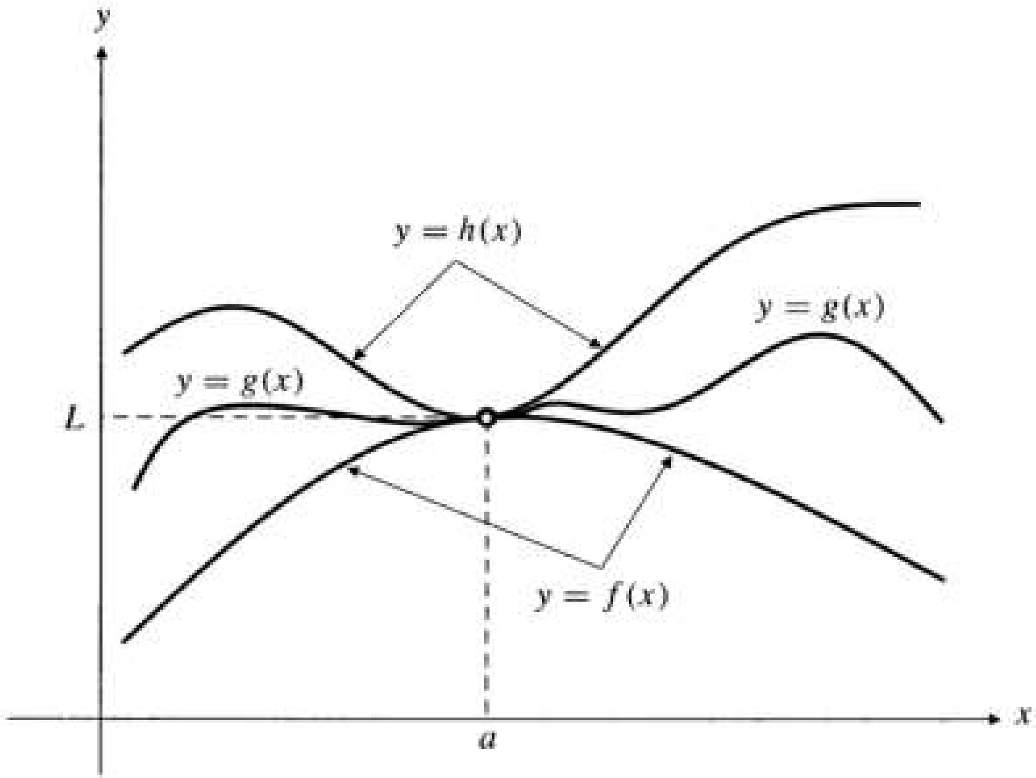
Limits of Functions

THEOREM The sequeeze (or sandwich) theorem

Suppose that $f(x) \leq g(x) \leq h(x)$ holds for all x in some open interval containing a, except possibly at $x = a$ itself. Suppose also that

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L.$$

Then $\lim_{x \to a} g(x) = L$ also. Similar statements hold for left and right limits.



Limits of Functions

EXAMPLE Given that $3 - x^2 \leq u(x) \leq 3 + x^2$ for all $x \neq 0$, find $\lim_{x \to 0} u(x)$.

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EXAMPLE Show that $\lim\limits_{x \to 0} x^2 \sin \frac{1}{x} = 0$.

Limits of Functions

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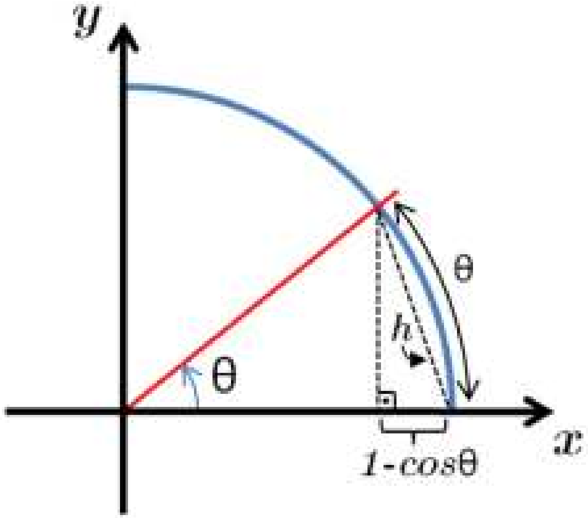

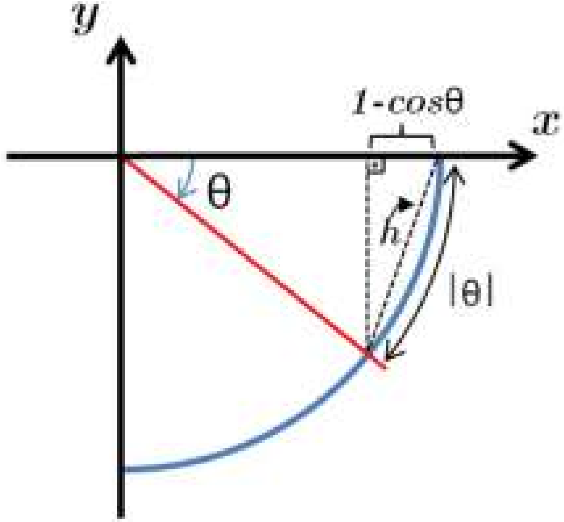

Limits of Functions

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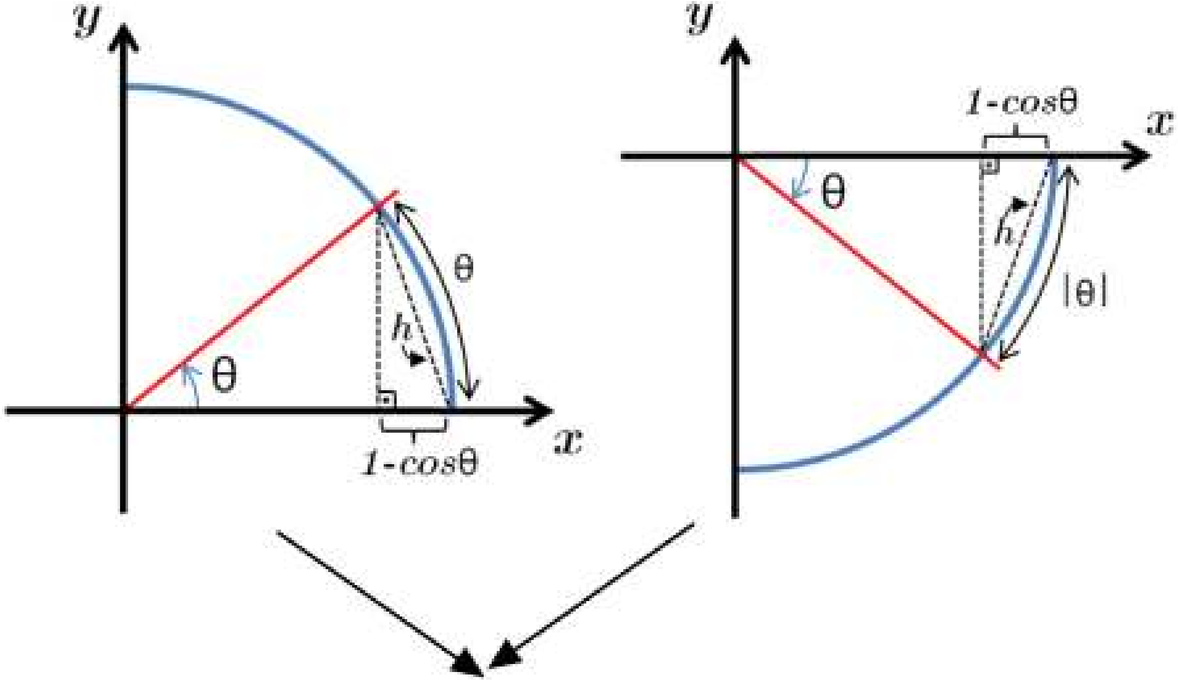
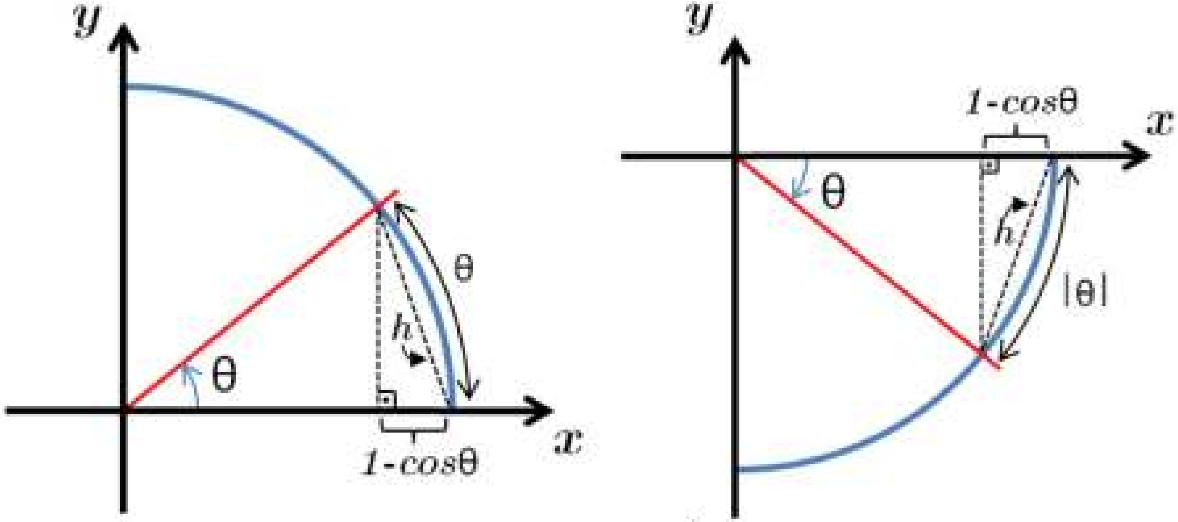


$$0 \leq 1-\cos\theta \leq h \leq |\theta|$$

Limits of Functions

EXAMPLE



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by sequeeze theorem

$$\lim_{\theta \to 0} 1 - \cos\theta = 0$$

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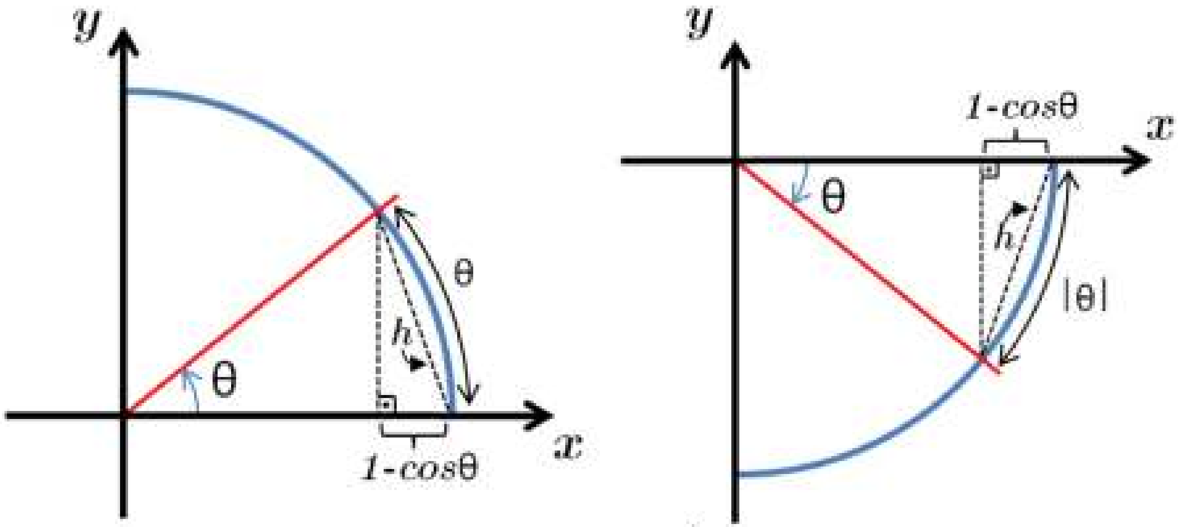

$$0 \leq 1 - \cos\theta \leq h \leq |\theta|$$

by sequeeze theorem

$$\lim_{\theta \to 0} 1 - \cos\theta = 0 \longrightarrow \boxed{\lim_{\theta \to 0} \cos\theta = 1}$$

Limits of Functions

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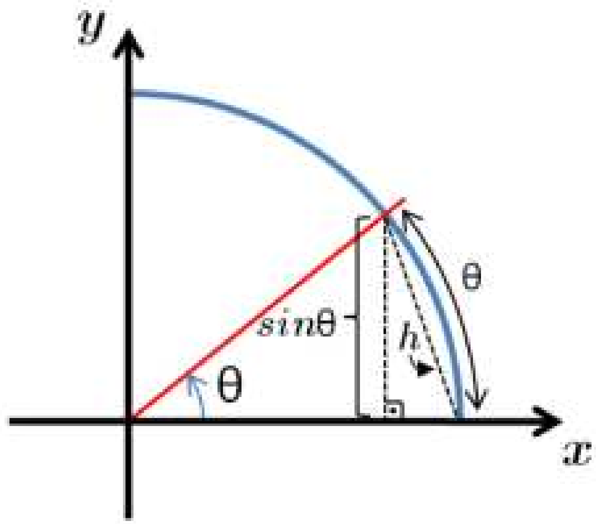
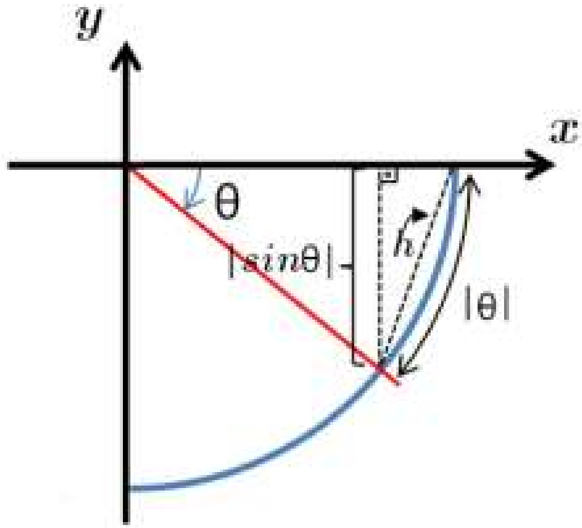
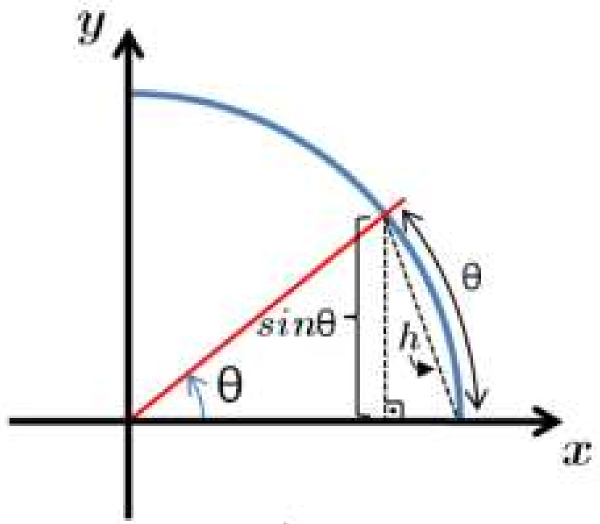
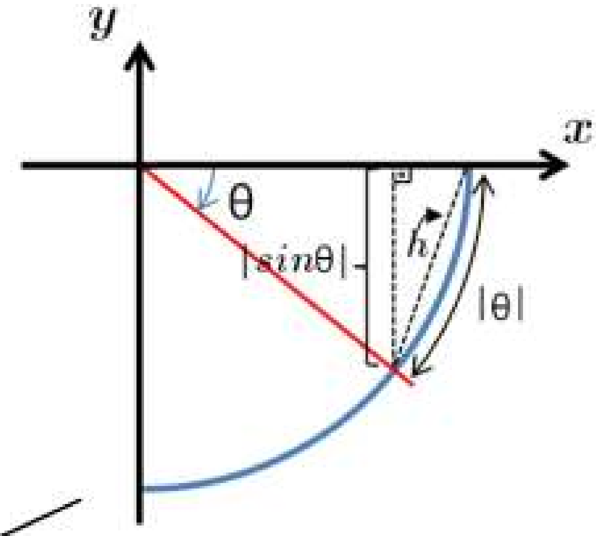




Limits of Functions

EXAMPLE

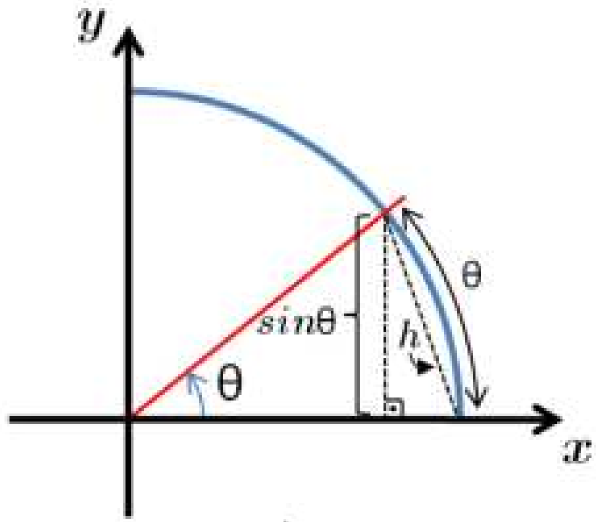


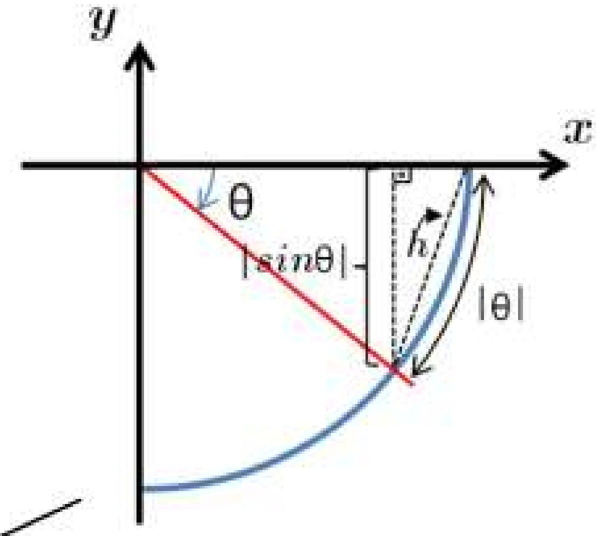


$$|\sin\theta| \leq |\theta|$$

Limits of Functions

EXAMPLE



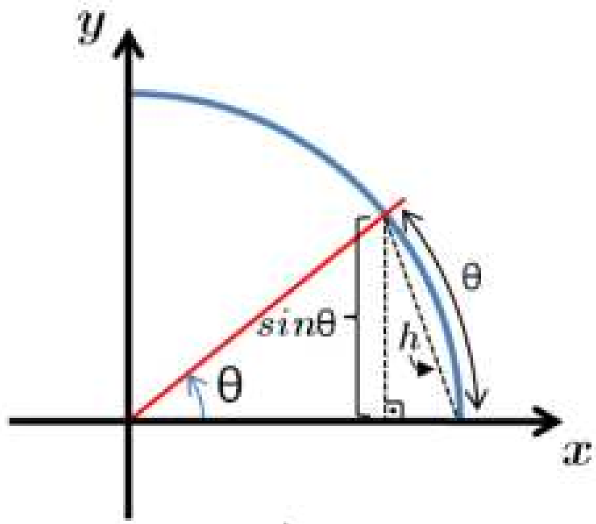


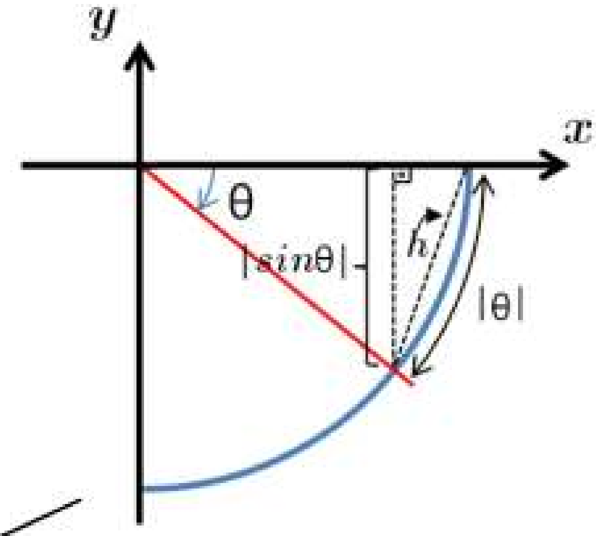
$$|\sin\theta| \leq |\theta|$$

$$-|\theta| \leq \sin\theta \leq |\theta|$$

Limits of Functions

EXAMPLE





$$|\sin\theta| \le |\theta|$$

$$-|\theta| \le \sin\theta \le |\theta|$$

by sequeeze theorem

$$\boxed{\lim_{\theta \to 0} \sin\theta = 0}$$