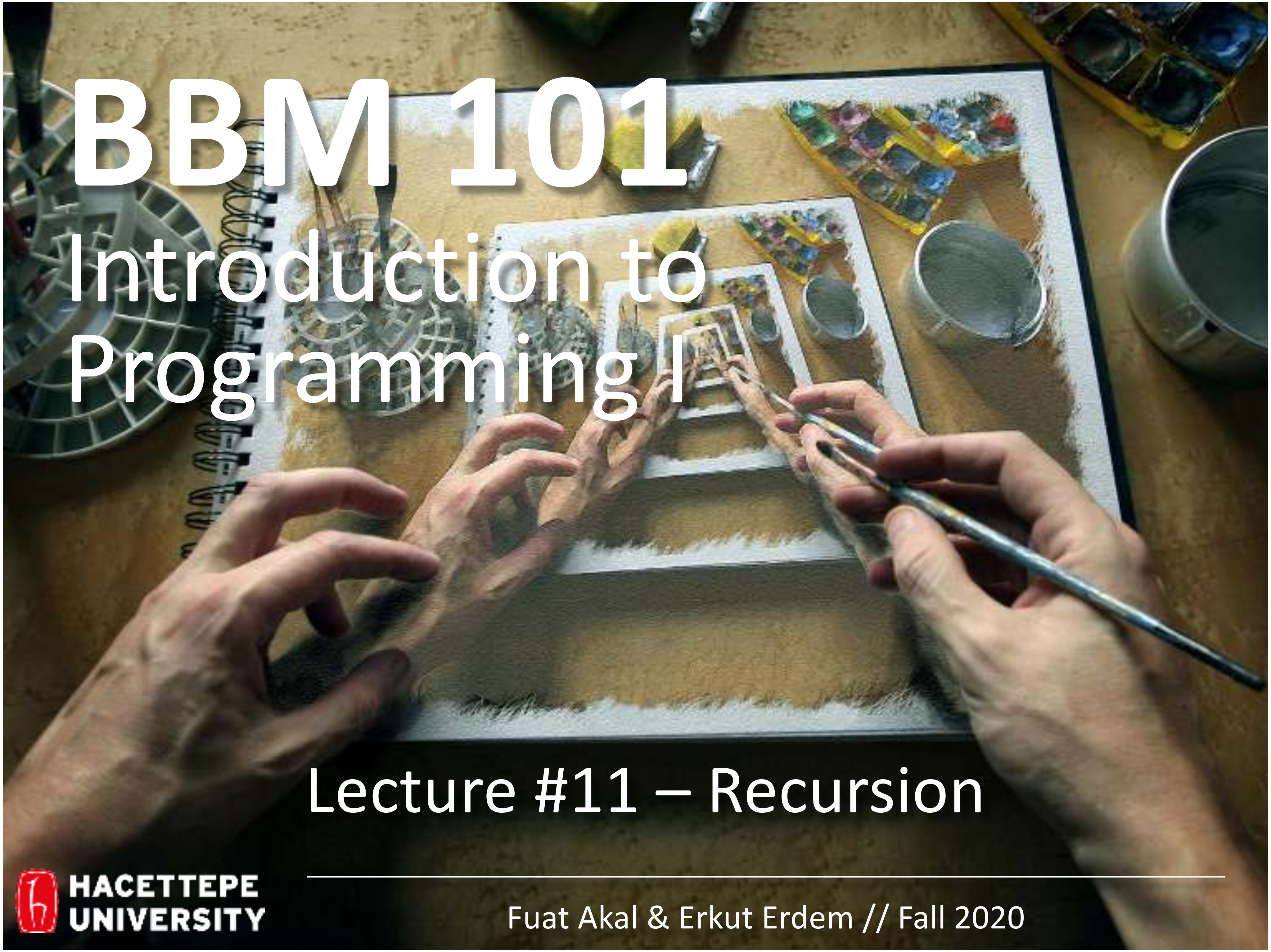


BBM 101

Introduction to Programming I

Lecture #11 – Recursion

HACETTEPE UNIVERSITY

Fuat Akal & Erkut Erdem // Fall 2020

Last time… **Testing, debugging, exceptions**

Exceptions


```
try:
      .....
      .....
except:
      .....
finally:
      .....
```

Debugging

Lecture Overview

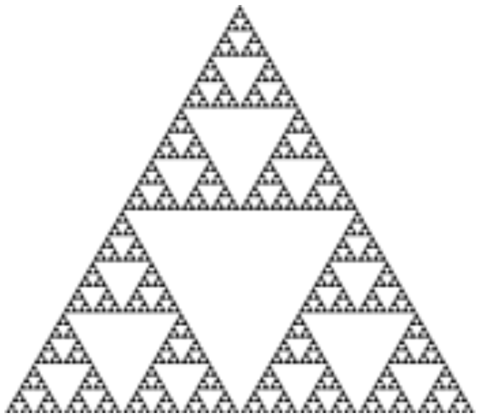
- Notion of state in computation
- Recursion as a programming concept
- Mutual recursion
- Recursion tree
- Pitfalls of recursion

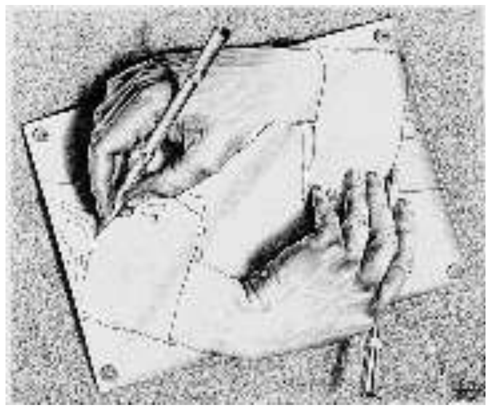
Disclaimer: Much of the material and slides for this lecture were borrowed from
—E. Grimson, J. Guttag and C. Terman in MITx 6.00.1x,
—J. DeNero in CS 61A (Berkeley),
—T. Cortina in 15110 Principles of Computing (CMU)
—R. Sedgewick, K. Wayne and R. Dondero (Princeton)

Recursion

- **Recursion** is a programming concept whereby a function invokes itself.

- Recursion is typically used to solve problems that are decomposable into subproblems which are just like the original problem, but a step closer to being solved.





Drawing Hands, by M. C. Escher (lithograph, 1948)

Computation

- All **computation** consists of chugging along from **state** to state to state...

- There is a set of **rules** that tells us, given the current state, which state to go to next.

Arithmetic as Rewrite Rules

- Expression evaluation
  - We stop when we reach a number

```
2 + 3 + 4
5 + 4
9
```

Functions as New Rules

```
def square(n):
    return n * n
```

When we see: `square(`*`something`*`)`

Rewrite it as: *`something * something`*

Functions as Rewrite Rules

```
def square(n):
    return n * n

square(3)
3 * 3
9
```

Piecewise Functions

$$\mathbf{f}(n) = \begin{cases} 1 & \text{if } n = 1 \\ n - 1 & \text{if } n > 1 \end{cases}$$

f(4)

4 - 1

3

In Python

```
def f(n):
    if n == 1:
        return 1
    else:
        return n - 1
```

This is just math, right?

- Difference between mathematical functions and computation functions.
  - Computation functions must be ***effective***.

- For example, we can define the square-root function as

  $\sqrt{x} = y$ such that $y \geq 0$ and $y^2 = x$

- This defines a valid mathematical function, but it doesn't tell us **how to compute** the square root of a given number.

Fancier Functions

```
def f(n):
    return n + (n - 1)
```

Find **f(4)**

Fancier Functions

```
def f(n):
    return n + (n - 1)

def g(n):
    return n + f(n - 1)
```

Find `g(4)`

Fancier Functions

```
def f(n):
    return n + (n - 1)

def g(n):
    return n + f(n - 1)

def h(n):
    return n + h(n - 1)
```

Find **`h(4)`**

Recursion

```
def h(n):
    return n + h(n - 1)
```

- **`h`** is a ***recursive*** function,
  because it is defined in terms of itself.

Definition

Recursion

- See: "Recursion".

Recursion

```
def h(n):
    return n + h(n - 1)
```

```
h(4)
4 + h(3)
4 + 3 + h(2)
4 + 3 + 2 + h(1)
4 + 3 + 2 + 1 + h(0)
4 + 3 + 2 + 1 + 0 + h(-1)
4 + 3 + 2 + 1 + 0 + -1 + h(-2)
```

…

Evaluating **h** leads to an infinite loop!

What you are thinking?

"Ok, recursion is bad.
What's the big deal?"

Recursion

```
def f(n):
    if n == 1:
        return 1
    else:
        return f(n - 1)
```

Find **`f(1)`**

Find **`f(2)`**

Find **`f(3)`**

Find **`f(100)`**

Recursion

```
def f(n):
    if n == 1:
        return 1
    else:
        return f(n - 1)

f(3)
f(3 - 1)
f(2)
f(2 - 1)
f(1)
1
```

Terminology

```
def f(n):
    if n == 1:
        return 1
    else:
        return f(n - 1)
```

base case

recursive case

"Useful" recursive functions have:

- at least one *recursive case*
- at least one *base case*
  so that the computation terminates

Recursion

```
def f(n):
    if n == 1:
        return 1
    else:
        return f(n + 1)
```

Find **`f(5)`**

We have a base case and a recursive case. What's wrong?

The recursive case should call the function on a *simpler input*, bringing us closer and closer to the base case.

Recursion

```
def f(n):
    if n == 0:
        return 0
    else:
        return 1 + f(n - 1)
```

Find f(0)

Find f(1)

Find f(2)

Find f(100)

Recursion

```
def f(n):
    if n == 0:
        return 0
    else:
        return 1 + f(n - 1)

f(3)
1 + f(2)
1 + 1 + f(1)
1 + 1 + 1 + f(0)
1 + 1 + 1 + 0
3
```

Iterative algorithms

- Looping constructs (e.g. while or for loops) lead naturally to **iterative** algorithms

- Can conceptualize as capturing computation in a set of “state variables” which update on each iteration through the loop

Iterative multiplication by successive additions

- Imagine we want to perform multiplication by successive additions:
  - To multiply a by b, add a to itself b times
- State variables:
  - i – iteration number; starts at b
  - result – current value of computation; starts at 0
- Update rules
  - i←i -1; stop when 0
  - result ← result + a

Multiplication by successive additions

```
def iterMul(a, b):
    result = 0
    while b > 0:
        result += a
        b -= 1
    return result
```

Recursive version

- An alternative is to think of this computation as:

$$\texttt{a * b} = \underbrace{\texttt{a + a + ... + a}}_{\texttt{b}\text{ copies}}$$

$$= \texttt{a + }\underbrace{\texttt{a + ... + a}}_{\texttt{b-1}\text{ copies}}$$

$$= \texttt{a + a * (b - 1)}$$

Recursion

- This is an instance of a **recursive** algorithm
  - Reduce a problem to a simpler (or smaller) version of the same problem, plus some simple computations
    [**Recursive step]**
  - Keep reducing until reach a simple case that can be solved directly
    [**Base case]**

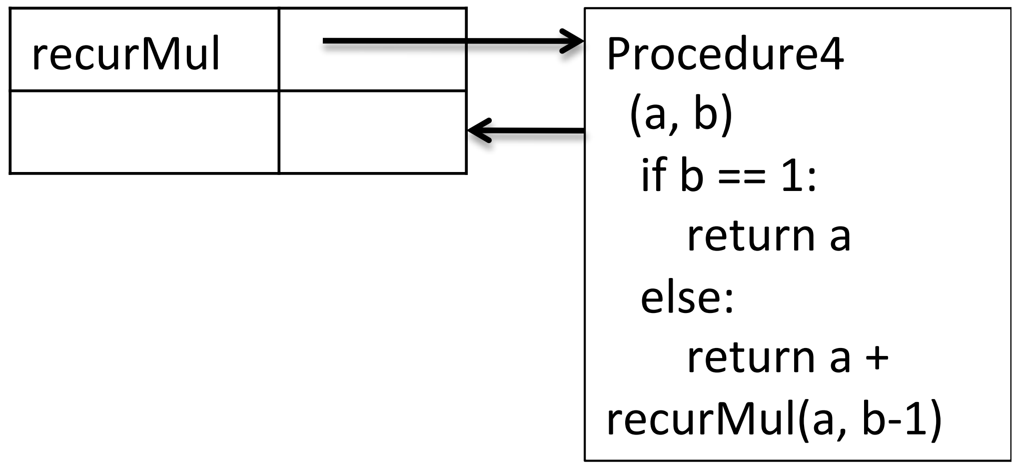
- **`a*b=a; if b=1`**
  (Base case)
- **`a * b = a + a * (b-1); otherwise`**
  (Recursive case)

Recursive Multiplication

```
def recurMul(a, b):
    if b == 1:
        return a
    else:
        return a + recurMul(a, b-1)
```

Let’s try it out

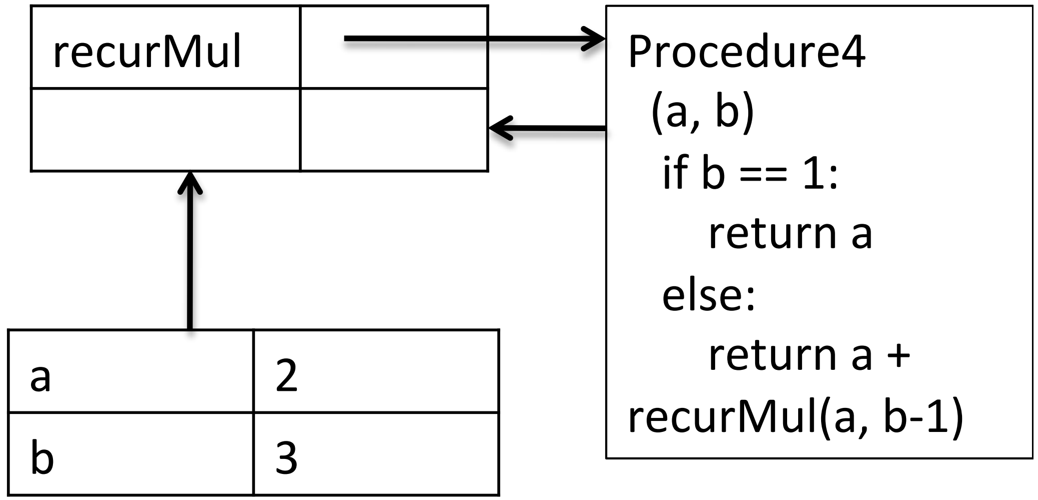
```
def recurMul(a, b):
      if b == 1:
         return a
      else:
         return a +
         recurMul(a, b-1)
```



Let’s try it out

```
def recurMul(a,b):
      if b == 1:
         return a
      else:
         return a +
         recurMul(a, b-1)

recurMul(2,3)
```



Let’s try it out

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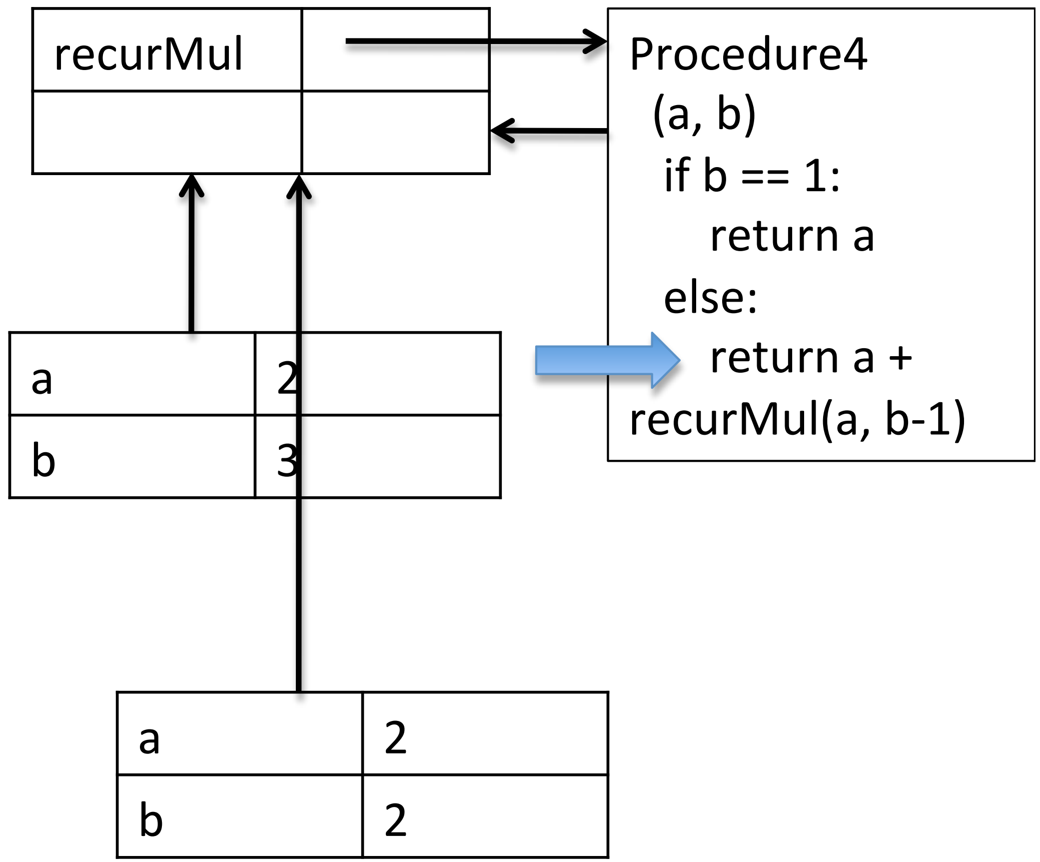

Let's try it out

```
def recurMul(a,b):
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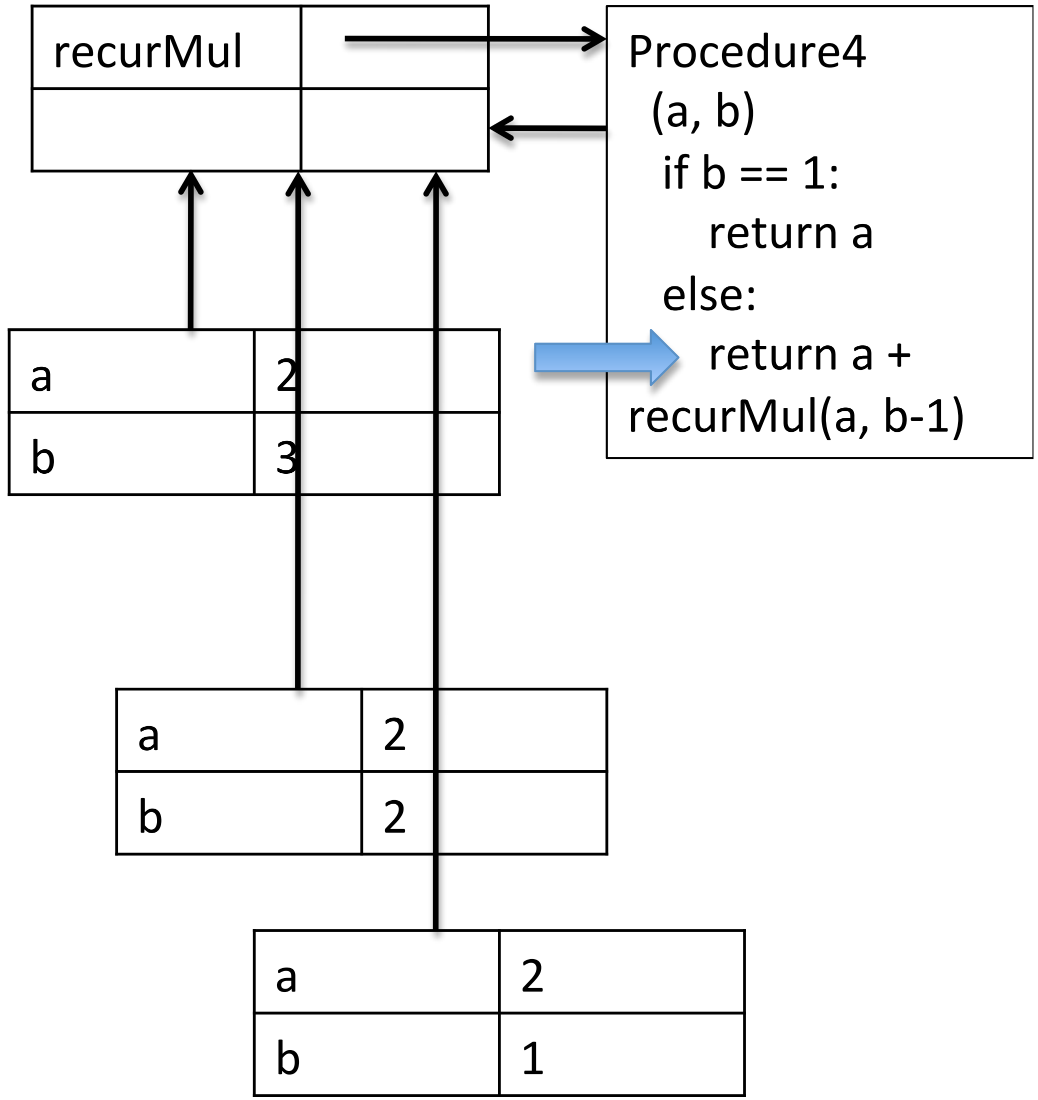

Let's try it out

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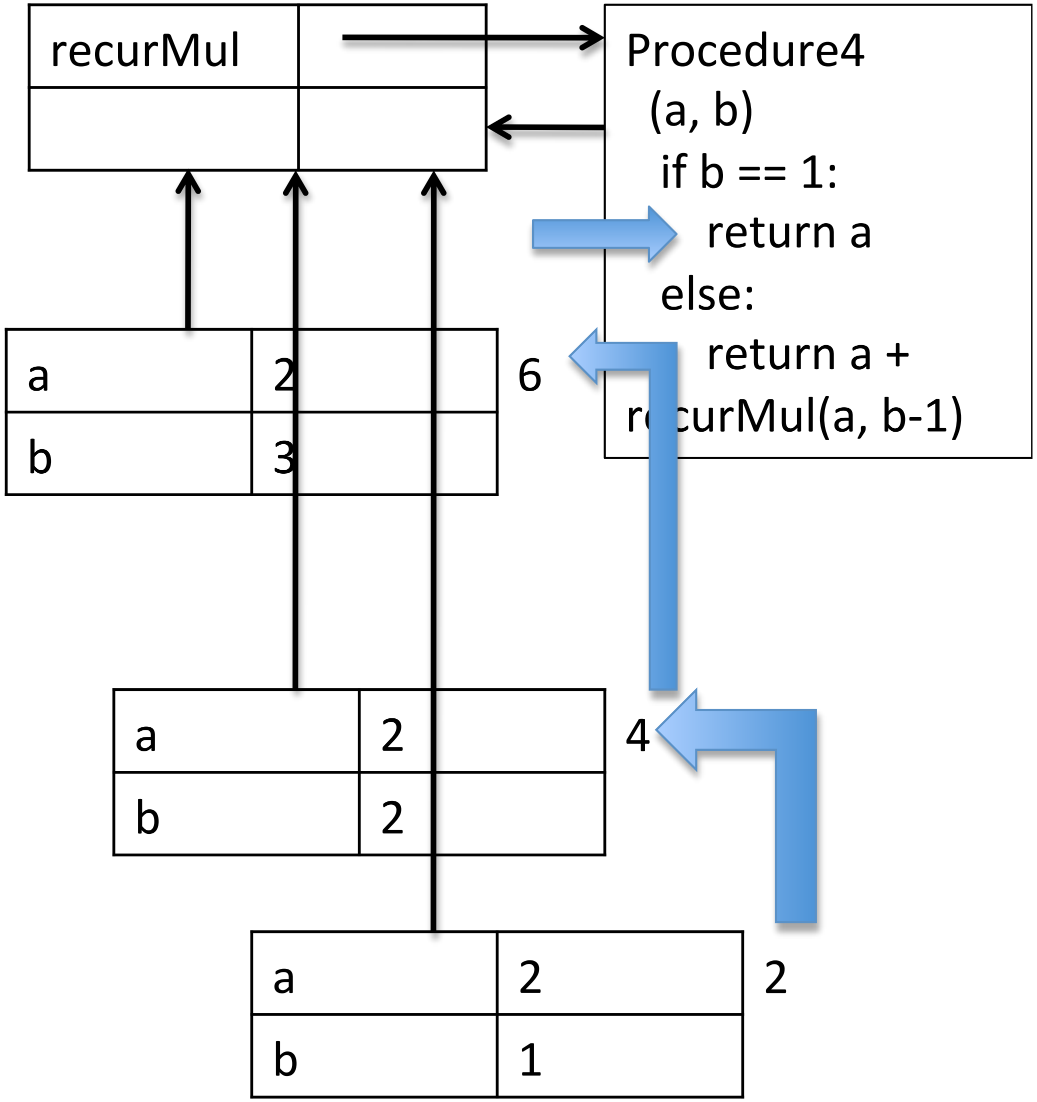

The Anatomy of a Recursive Function

- The def statement header is similar to other functions
- Conditional statements check for base cases
- Base cases are evaluated without recursive calls
- Recursive cases are evaluated with recursive calls

```
def recurMul(a, b):
    if b == 1:
        return a
    else:
        return a + recurMul(a, b-1)
```

Inductive Reasoning

- How do we know that our recursive code will work?
- **`iterMul`** terminates because b is initially positive, and decrease by **`1`** each time around loop; thus must eventually become less than **`1`**
- **`recurMul`** called with **`b = 1`** has no recursive call and stops
- **`recurMul`** called with **`b > 1`** makes a recursive call with a smaller version of **`b`**; must eventually reach call with **`b = 1`**

Mathematical Induction

- To prove a statement indexed on integers is true for all values of n:
  - Prove it is true when n is smallest value (e.g. $n = 0$ or $n = 1$)
  - Then prove that if it is true for an arbitrary value of n, one can show that it must be true for $n+1$

Example

- 0+1+2+3+...+n=(n(n+1))/2
- Proof
  - If n = 0, then LHS is 0 and RHS is 0*1/2 = 0, so true
  - Assume true for some k, then need to show that
    - 0 + 1 + 2 + ... + k + (k+1) = ((k+1)(k+2))/2
    - LHS is k(k+1)/2 + (k+1) by assumption that property holds for problem of size k
    - This becomes, by algebra, ((k+1)(k+2))/2
  - Hence expression holds for all n >= 0

What does this have to do with code?

- Same logic applies

```
def recurMul(a, b):
    if b == 1:
       return a
    else:
       return a + recurMul(a, b-1)
```

- Base case, we can show that **`recurMul`** must return correct answer

- For recursive case, we can assume that **`recurMul`** correctly returns an answer for problems of size smaller than **`b`**, then by the addition step, it must also return a correct answer for problem of size **`b`**

- Thus by induction, code correctly returns answer

Sum digits of a number

```
def split(n):
    """Split positive n into all but its last digit and its last digit."""
    return n // 10, n % 10

def sum_digits(n):
    """Return the sum of the digits of positive integer n."""
    if n < 10:
        return n
    else:
        all_but_last, last = split(n)
        return sum_digits(all_but_last) + last
```

```
>>> 123 // 10
12
>>> 123 % 10
3
```

Verify the correctness of this recursive definition.

Some Observations

- Each recursive call to a function creates its own environment, with local scoping of variables

- Bindings for variable in each frame distinct, and not changed by recursive call

- Flow of control will pass back to earlier frame once function call returns value

The "classic" Recursive Problem

- Factorial

$$n! = n * (n-1) * \ldots * 1$$

$$n! = \begin{cases} 1 & \text{if } n = 0 \\ n * (n-1)! & \text{otherwise} \end{cases}$$

Recursion in Environment Diagrams

```
def fact(n):
    if n == 0:
        return 1
    else:
        return n * fact(n-1)

fact(3)
```

Recursion in Environment Diagrams

```
def fact(n):
    if n == 0:
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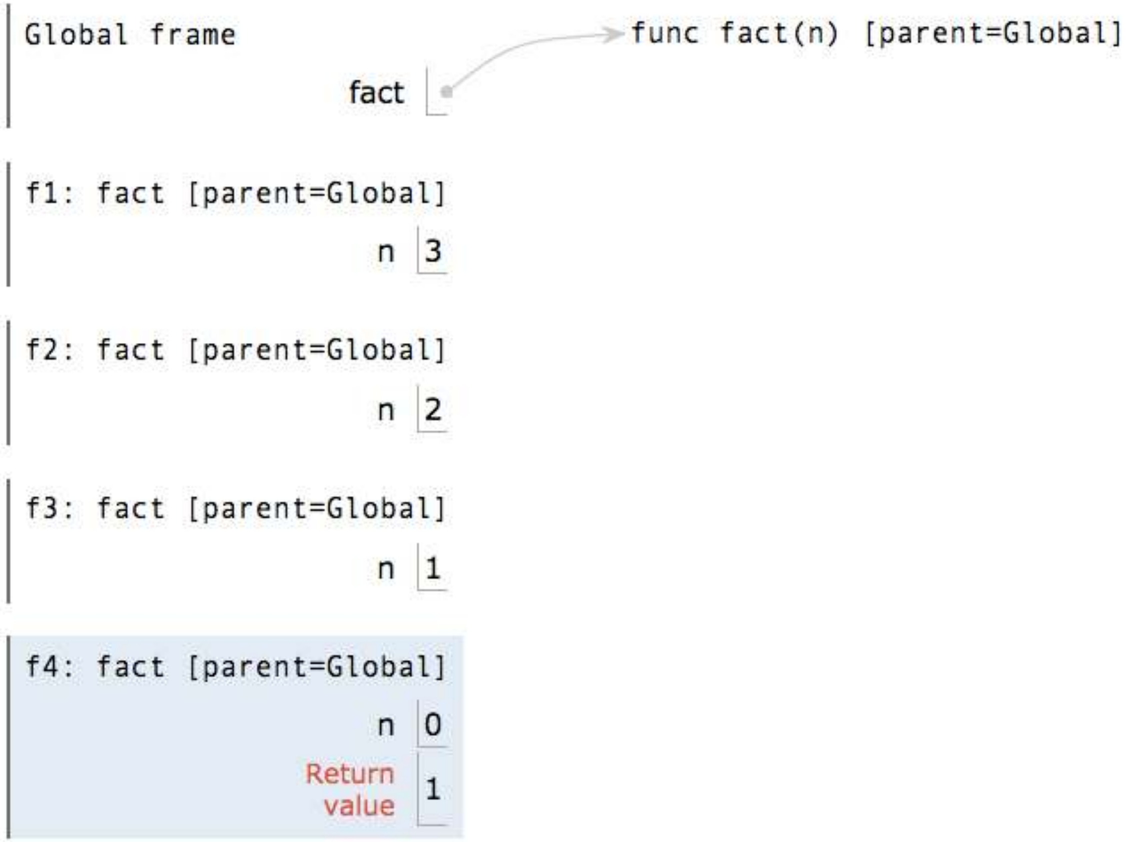
fact(3)
```

(Demo)



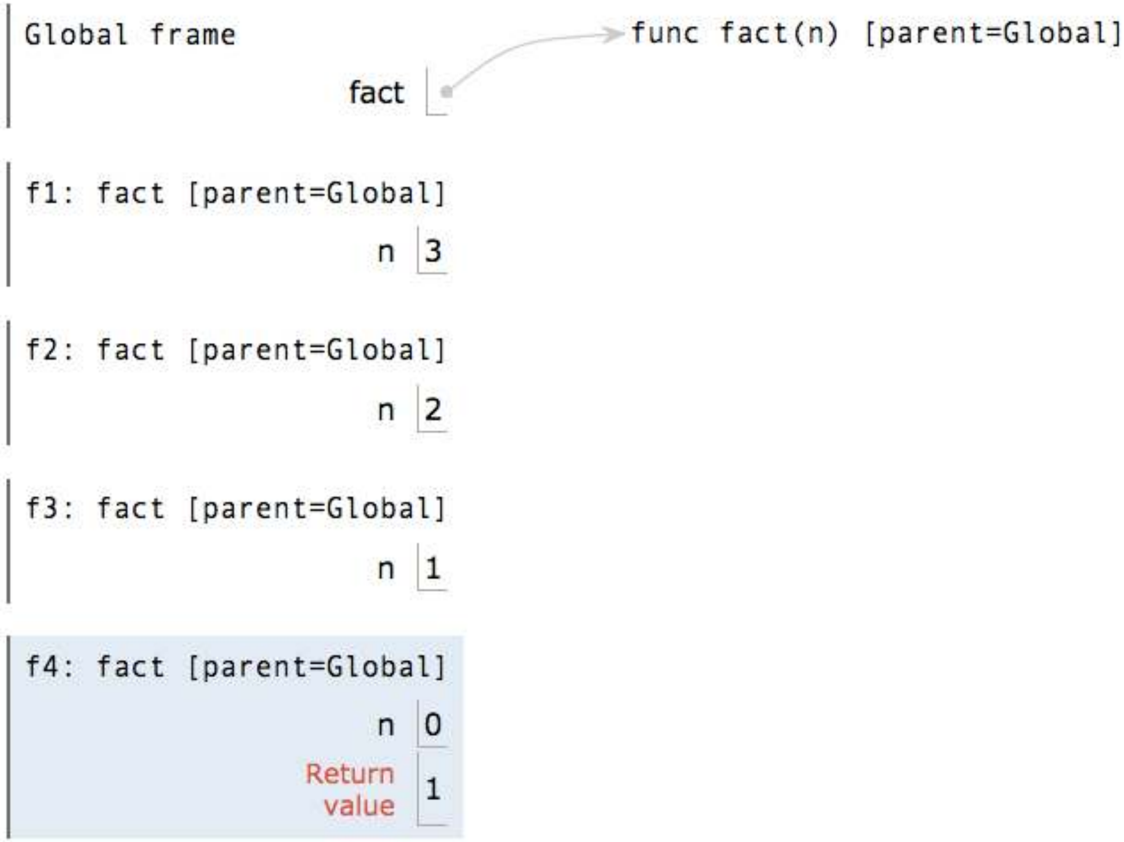
Recursion in Environment Diagrams

```
def fact(n):
    if n == 0:
        return 1
    else:
        return n * fact(n-1)

fact(3)
```

- The same function **fact** is called multiple times

(Demo)



Recursion in Environment Diagrams

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def fact(n):
    if n == 0:
        return 1
    else:
        return n * fact(n-1)

fact(3)
```
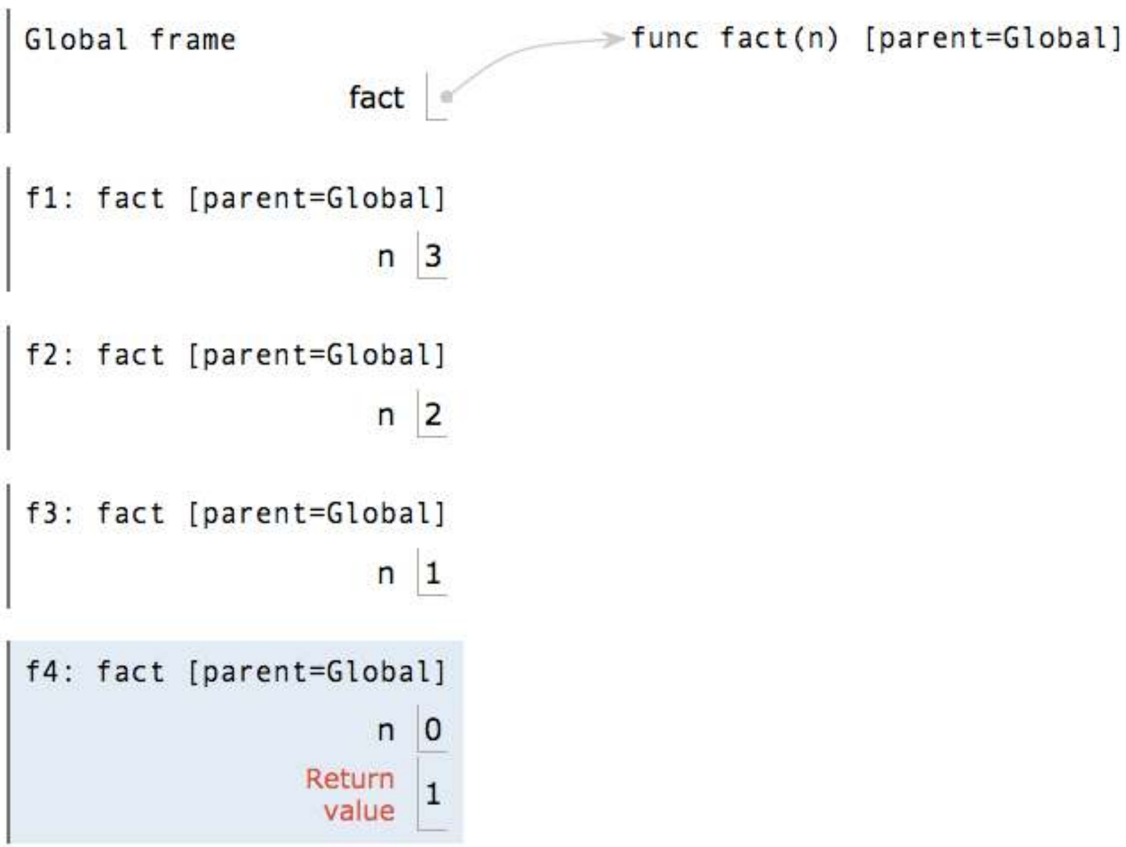

(Demo)

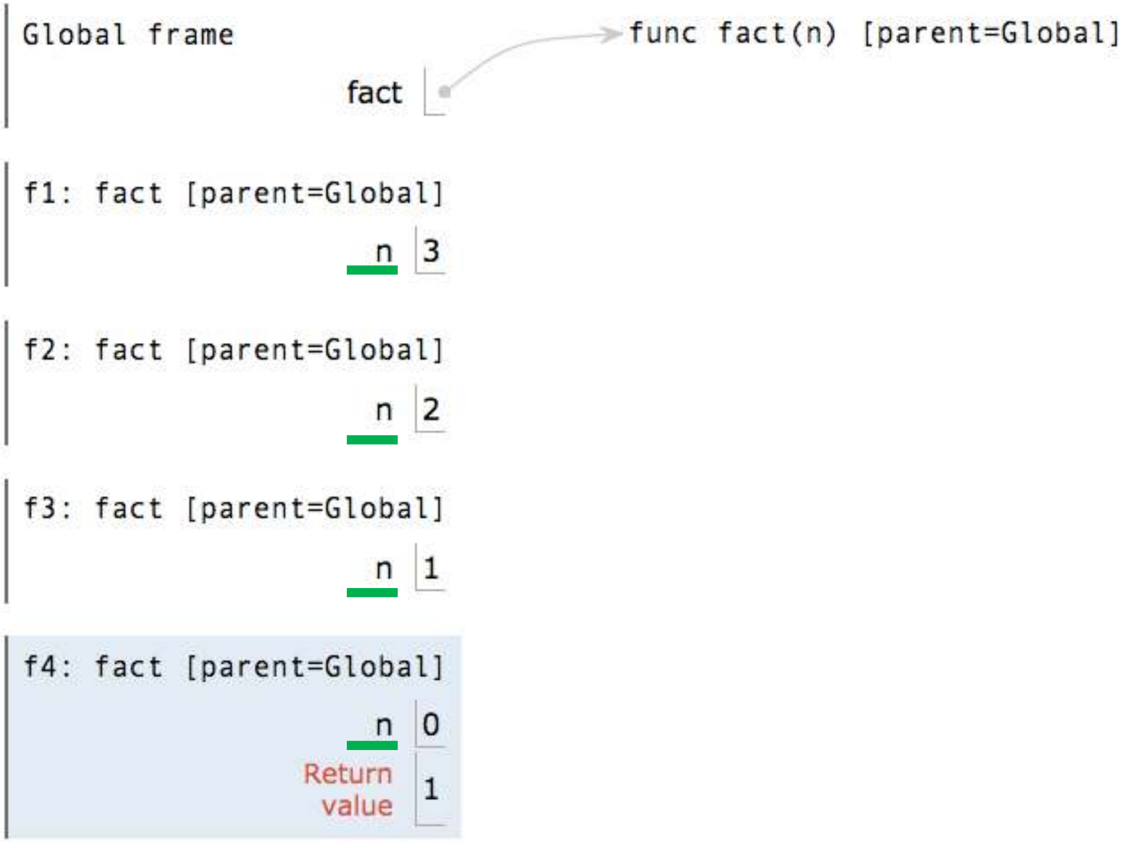


- The same function **fact** is called multiple times
- Different frames keep track of the different arguments in each call

Recursion in Environment Diagrams

```
def fact(n):
    if n == 0:
        return 1
    else:
        return n * fact(n-1)

fact(3)
```

(Demo)

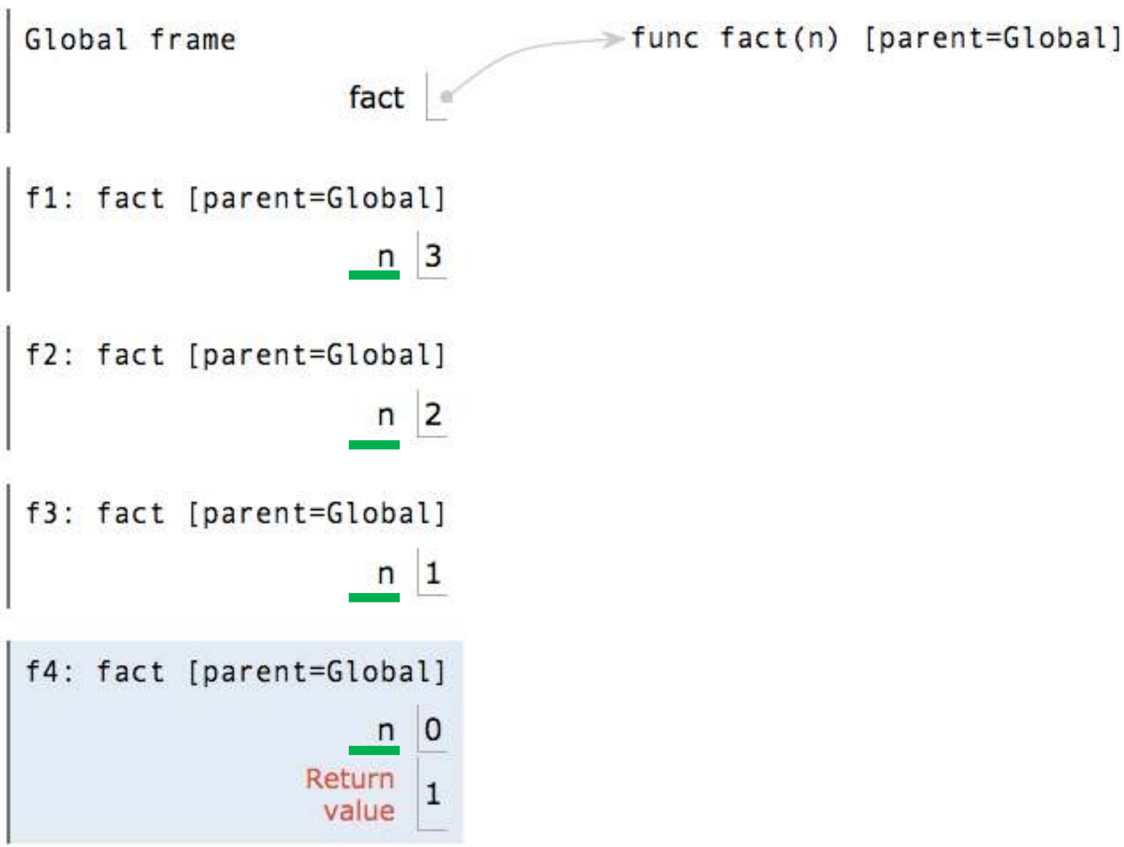


- The same function **fact** is called multiple times
- Different frames keep track of the different arguments in each call
- What **n** evaluates to depends upon the current environment

Recursion in Environment Diagrams

```
def fact(n):
    if n == 0:
        return 1
    else:
        return n * fact(n-1)

fact(3)
```

(Demo)



- The same function **fact** is called multiple times
- Different frames keep track of different arguments in each call
- What **n** evaluates to depends upon the current environment
- Each call to **fact** solves a simpler problem than the last: smaller **n**

Iteration vs. Recursion

$$4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$$

Iteration vs. Recursion

$$4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$$

Using while:

```
def fact_iter(n):
    total, k = 1, 1
    while k <= n:
        total, k = total*k, k+1
    return total
```

Math: $$n! = \prod_{k=1}^{n} k$$

Names: n, total, k, fact_iter

Iteration vs. Recursion

$$4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$$

Using while:

```
def fact_iter(n):
    total, k = 1, 1
    while k <= n:
        total, k = total*k, k+1
    return total
```

Using recursion:

```
def fact(n):
    if n == 0:
        return 1
    else:
        return n * fact(n-1)
```

Math:

$$n! = \prod_{k=1}^{n} k$$

$$n! = \begin{cases} 1 & \text{if } n = 0 \\ n \cdot (n-1)! & \text{otherwise} \end{cases}$$

Names:

`n, total, k, fact_iter`

`n, fact`

Recursion on Non-numerics

- How could we check whether a string of characters is a palindrome, i.e., reads the same forwards and backwards
  - **`"Able was I ere I saw Elba"`**
    attributed to Napolean
  - **`"Are we not drawn onward, we few, drawn onward to new era?"`**
  - **`"Ey Edip Adana'da pide ye"`**

How to solve this recursively?

- First, convert the string to just characters, by stripping out punctuation, and converting upper case to lower case

- Then
  - a string of length 0 or 1 is a palindrome **[Base case]**
  - If the first character matches the last character, then is a palindrome if middle section is a palindrome **[Recursive case]**

Example

- **"Able was I ere I saw Elba" →**
  "ablewasiereisawelba"

- **isPalindrome("ablewasiereisawelba")**
  is same as
  "a"=="a" and isPalindrome("blewasiereisawleb")

Palindrome or not?

```
def toChars(s):
    s = s.lower()
    ans = ''
    for c in s:
        if c in 'abcdefghijklmnopqrstuvwxyz':
            ans = ans + c
    return ans
```

Palindrome or not?

```
def isPal(s):
    if len(s) <= 1:
        return True
    else:
        return s[0] == s[-1] and isPal(s[1:-1])

def isPalindrome(s):
    return isPal(toChars(s))
```

Divide and Conquer

- This is an example of a “divide and conquer” algorithm
  - Solve a hard problem by breaking it into a set of sub-problems such that:
  - Sub-problems are easier to solve than the original
  - Solutions of the sub-problems can be combined to solve the original

Global Variables

- Suppose we wanted to count the number of times **`fact`** calls itself recursively

- Can do this using a global variable

- So far, all functions communicate with their environment through their parameters and return values

- But, (though a bit dangerous), can declare a variable to be global – means name is defined at the outermost scope of the program, rather than scope of function in which appears

Example

```
def factMetered(n):
    global numCalls
    numCalls += 1
    if n == 0:
       return 1
    else:
       return n * factMetered(n-1)

def testFac(n):
    for i in range(n+1):
        global numCalls
        numCalls = 0
        print('fac of '+str(i)+' = '+str(factMetered(i)))
        print('fac called ' + str(numCalls) + ' times')

testFac(4)
```

Global Variables

- Use with care!!
- Destroy locality of code
- Since can be modified or read in a wide range of places, can be easy to break locality and introduce bugs!!

Mutual Recursion

- **Mutual recursion** is a form of **recursion** where two functions or data types are **defined** in terms of each other.

Mutual Recursion Example

```
def  even(n):
     if n == 0:
          return True
     else:
          return odd(n - 1)

def  odd(n):
     if n == 0:
          return False
     else:
          return even(n - 1)


even(4)
```

The Luhn Algorithm

- A simple checksum formula used to validate a variety of identification numbers, such as credit card numbers, IMEI numbers, etc.

The Luhn Algorithm

- From Wikipedia: http://en.wikipedia.org/wiki/Luhn_algorithm
- **First:** From the rightmost digit, which is the check digit, moving left, double the value of every second digit; if product of this doubling operation is greater than 9 (e.g., 7 * 2 = 14), then sum the digits of the products (e.g., 10: 1 + 0 = 1, 14: 1 + 4 = 5)
- **Second:** Take the sum of all the digits

1	3	8	7	4	3	
2	3	1+6=7	7	8	3	= 30

- The Luhn sum of a valid credit card number is a multiple of 10

The Luhn Algorithm

```
def luhn_sum(n):
    """Return the digit sum of n computed by the Luhn algorithm"""
    if n < 10:
        return n
    else:
        all_but_last, last = split(n)
        return luhn_sum_double(all_but_last) + last

def luhn_sum_double(n):
    """Return the Luhn sum of n, doubling the last digit."""
    all_but_last, last = split(n)
    luhn_digit = sum_digits(2 * last)
    if n < 10:
        return luhn_digit
    else:
        return luhn_sum(all_but_last) + luhn_digit
```

Tree Recursion

- Tree-shaped processes arise whenever executing the body of a recursive function makes more than one recursive call.

Tree Recursion

- Fibonacci numbers
- Leonardo of Pisa (aka Fibonacci) modeled the following challenge
  - Newborn pair of rabbits (one female, one male) are put in a pen
  - Rabbits mate at age of one month
  - Rabbits have a one month gestation period
  - Assume rabbits never die, that female always produces one new pair (one male, one female) every month from its second month on.
  - How many female rabbits are there at the end of one year?

Fibonacci

- After one month (call it 0) – 1 female
- After second month – still 1 female (now pregnant)
- After third month – two females, one pregnant, one not
- In general, females(n) = females(n-1) + females(n-2)
  - Every female alive at month n-2 will produce one female in month n;
  - These can be added those alive in month n-1 to get total alive in month n

Month	Females
0	1
1	1
2	2
3	3
4	5
5	8
6	13

Fibonacci

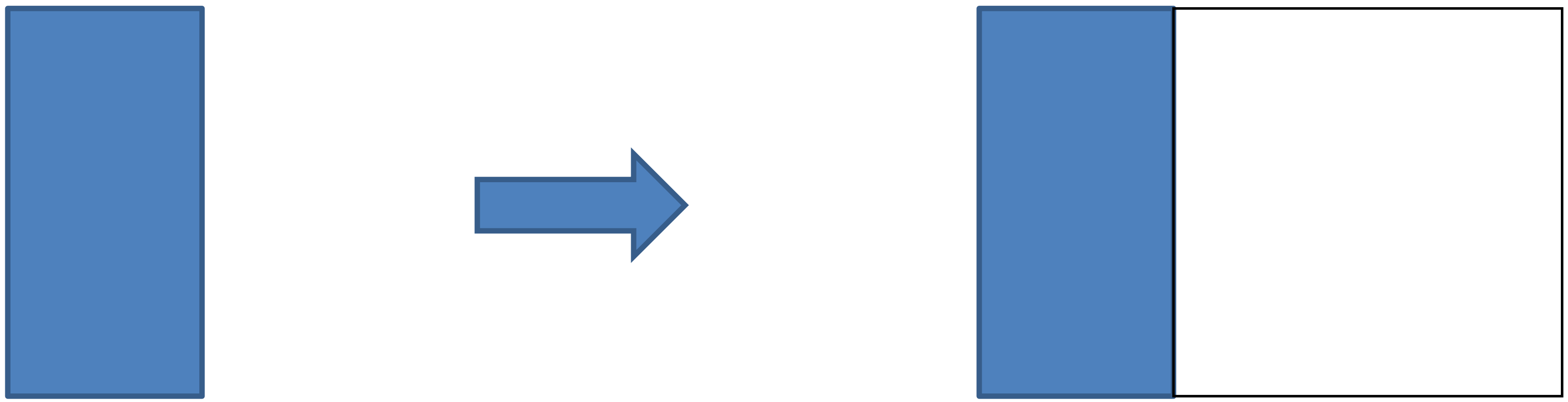
- Base cases:
  – Females(0) = 1
  – Females(1) = 1
- Recursive case
  – Females(n) = Females(n-1) + Females(n-2)

Fibonacci

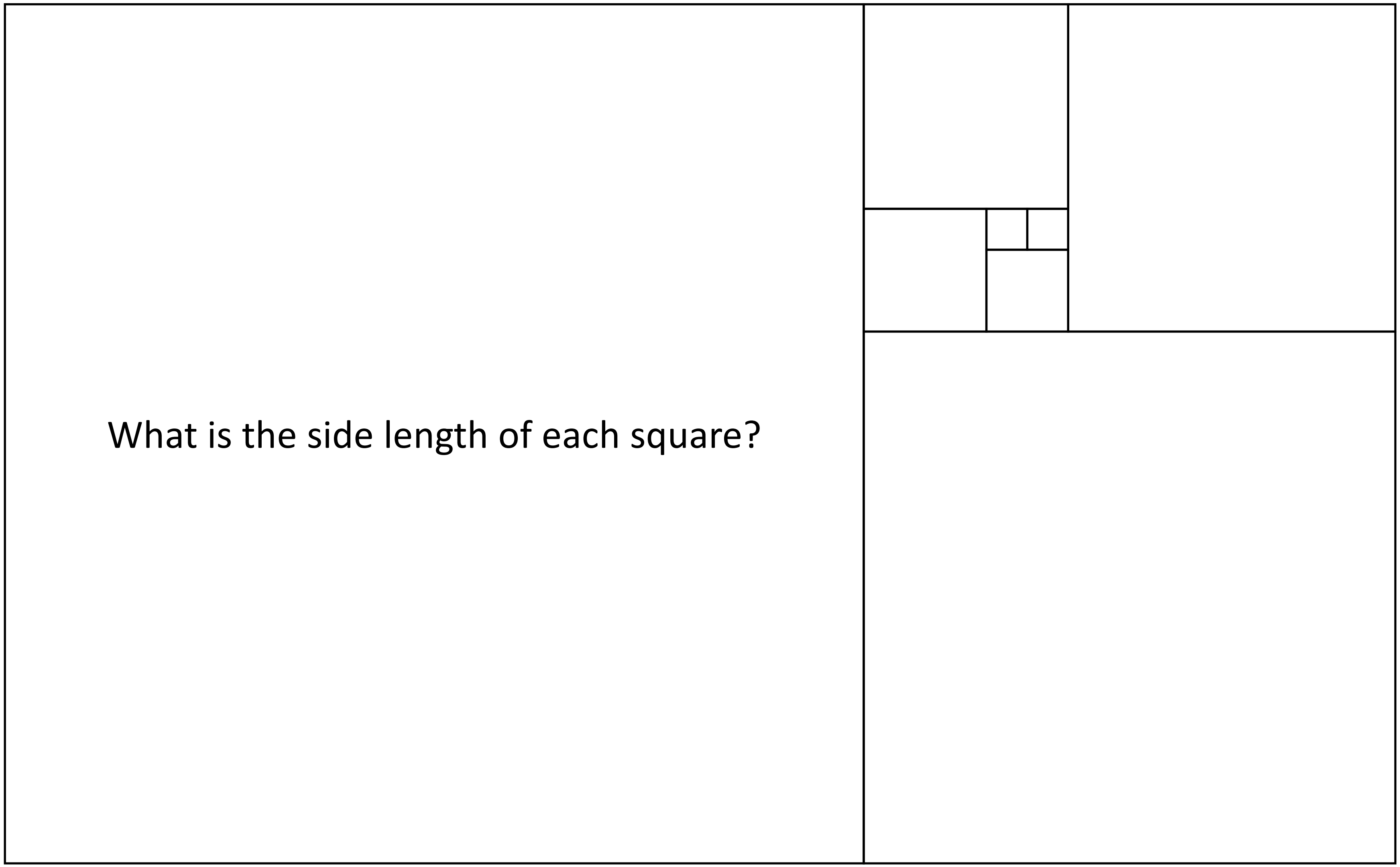
```
def fib(n):
    """assumes n an int >= 0
    returns Fibonacci of n"""
    assert type(n) == int and n >= 0
    if n == 0:
        return 1
    elif n == 1:
        return 1
    else:
        return fib(n-2) + fib(n-1)
```

Tiling Squares

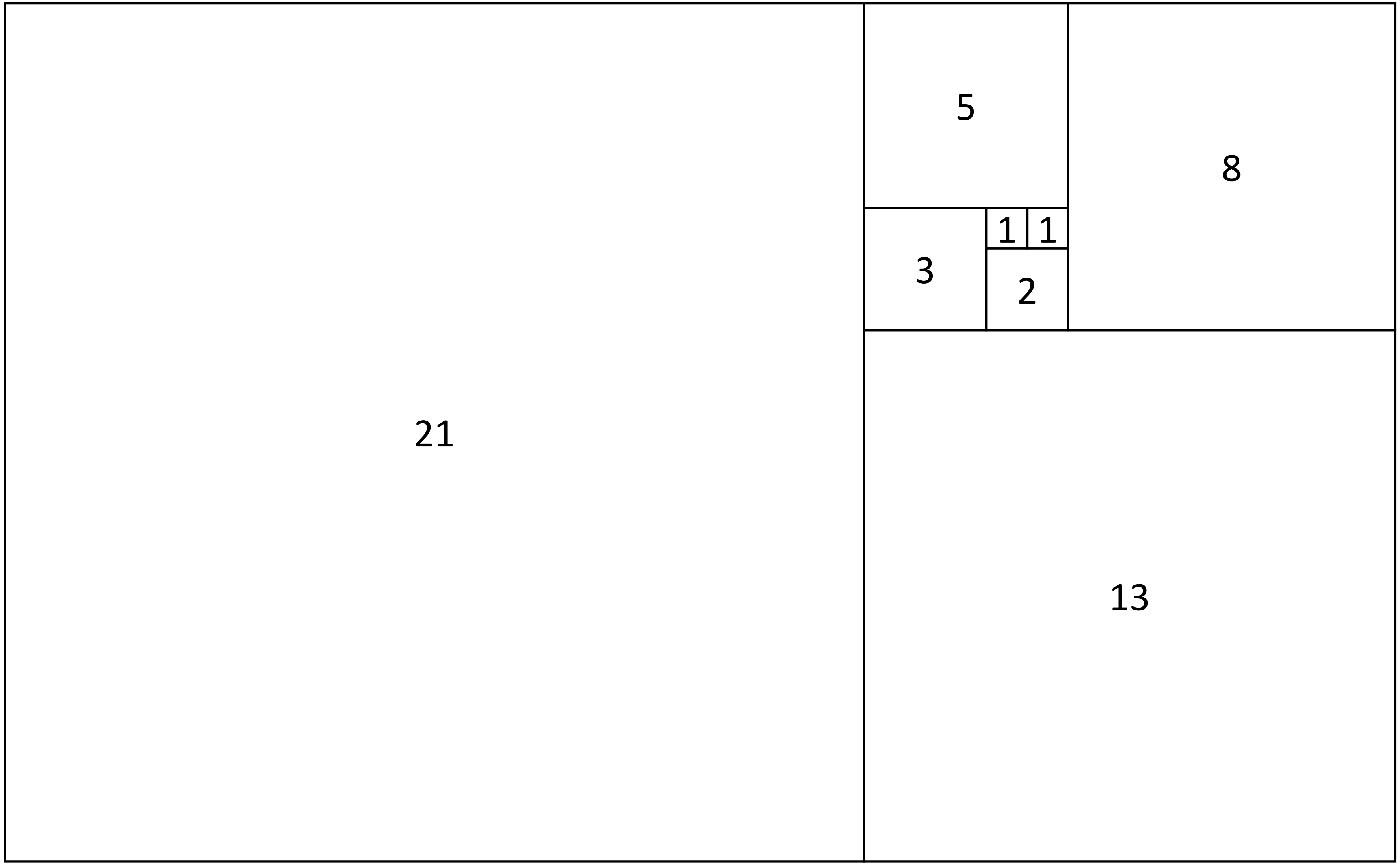
Rewrite rule: Add square to long side.



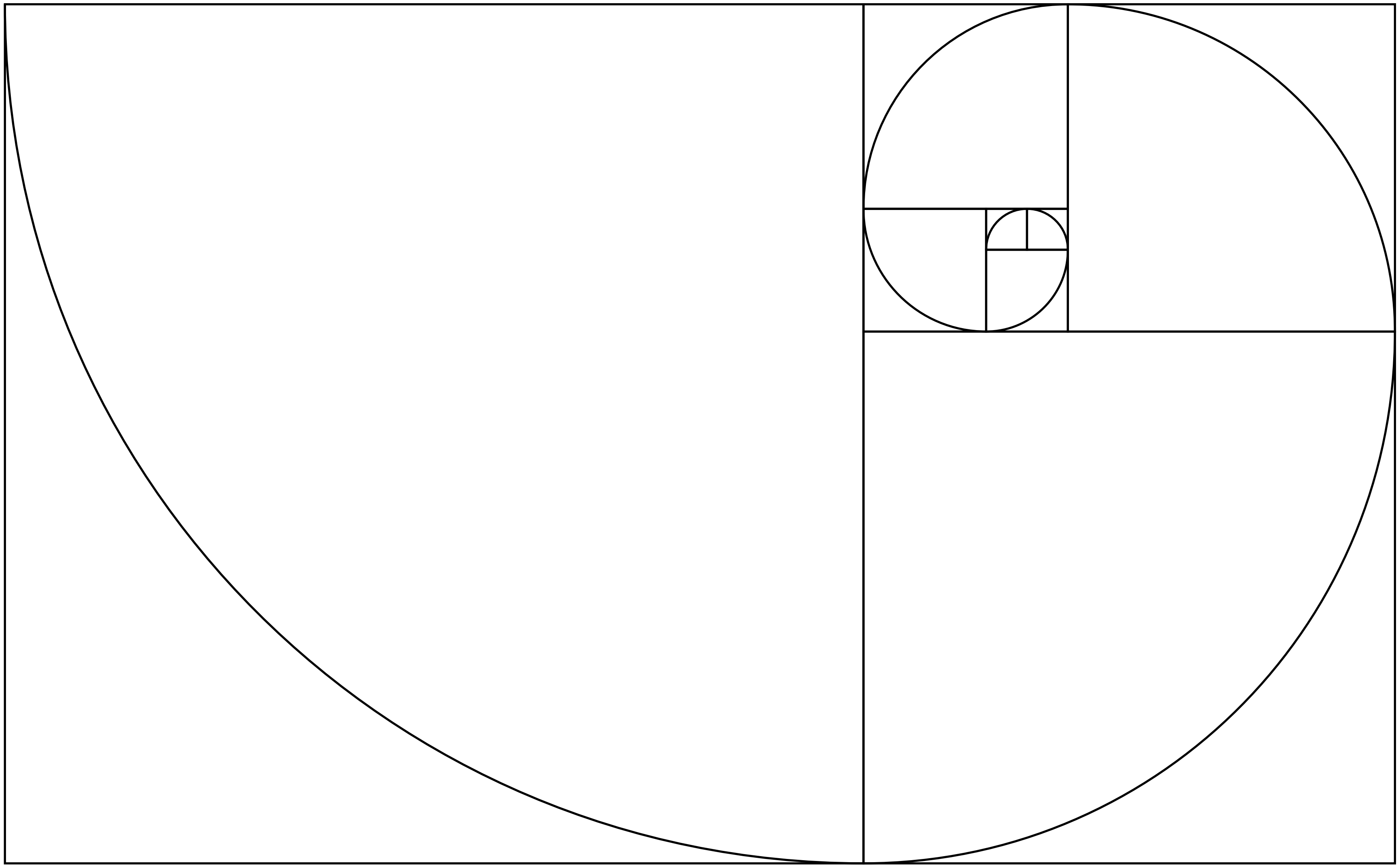
Tiling Squares



Tiling Squares



Spiral



Fibonacci

$1 \div 1 = 1$

$2 \div 1 = 2$

$3 \div 2 = 1.5$

$5 \div 3 = 1.666...$

$8 \div 5 = 1.6$

$13 \div 8 = 1.625$

$21 \div 13 = 1.615...$

$34 \div 21 = 1.619...$

Limit

What is the limit of $$\frac{\texttt{fib(}\mathit{n}\texttt{)}}{\texttt{fib(}\mathit{n}\texttt{ - 1)}}$$

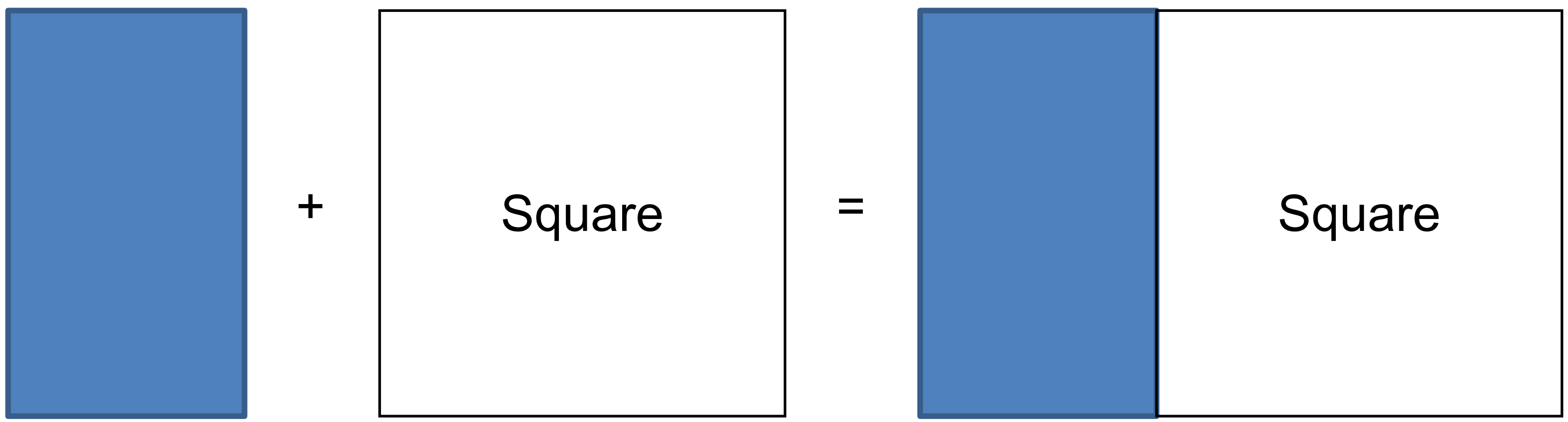
as **n** approaches infinity?

1.6180339887498948482...

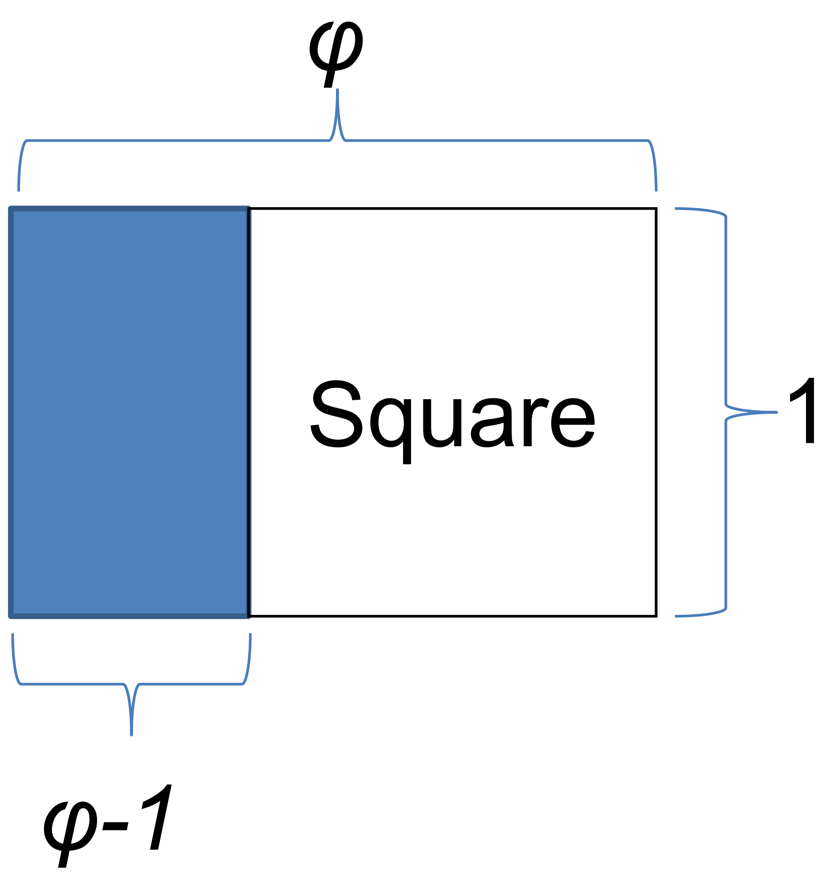
What's that called?

The Golden Ratio

The proportions of a rectangle that,
when a square is added to it
results in a rectangle
with the same proportions.



The Golden Ratio



$$\frac{\varphi}{1} = \frac{1}{\varphi - 1}$$

$$\varphi^2 - \varphi - 1 = 0$$

$$\varphi = \frac{1 + \sqrt{5}}{2}$$

$$= 1.618...$$

Fibonacci

$$\texttt{fib}(n) = \begin{cases} 1 & n = 1, 2 \\ \texttt{fib}(n-1) + \texttt{fib}(n-2) & n > 2 \end{cases}$$

$$\texttt{fib}(n) = \frac{\varphi^n - (1 - \varphi)^n}{\sqrt{5}}$$

Recursion Tree

- The computational process of **`fib`** evolves into a tree structure

Recursion Tree

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```
fib(5)
```

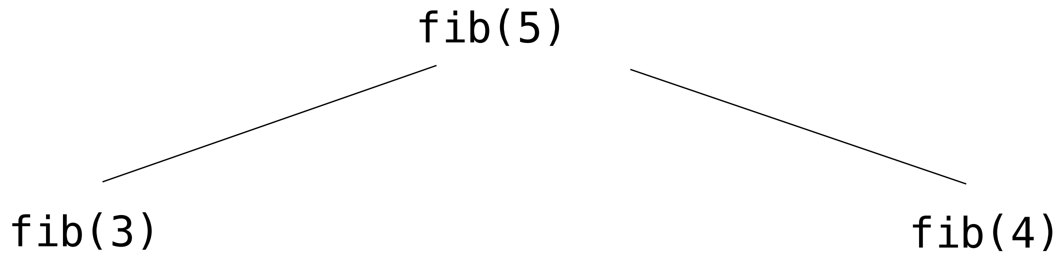
Recursion Tree

- The computational process of **`fib`** evolves into a tree structure

```
            fib(5)
           /
     fib(3)
```

Recursion Tree

- The computational process of **`fib`** evolves into a tree structure



Recursion Tree

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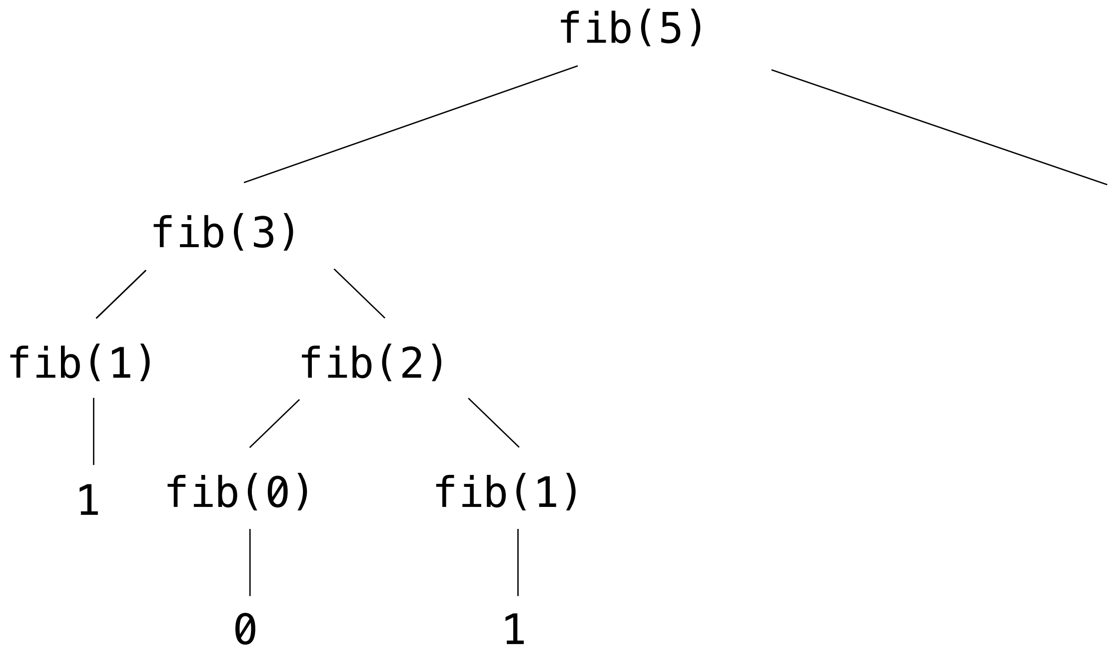

Recursion Tree

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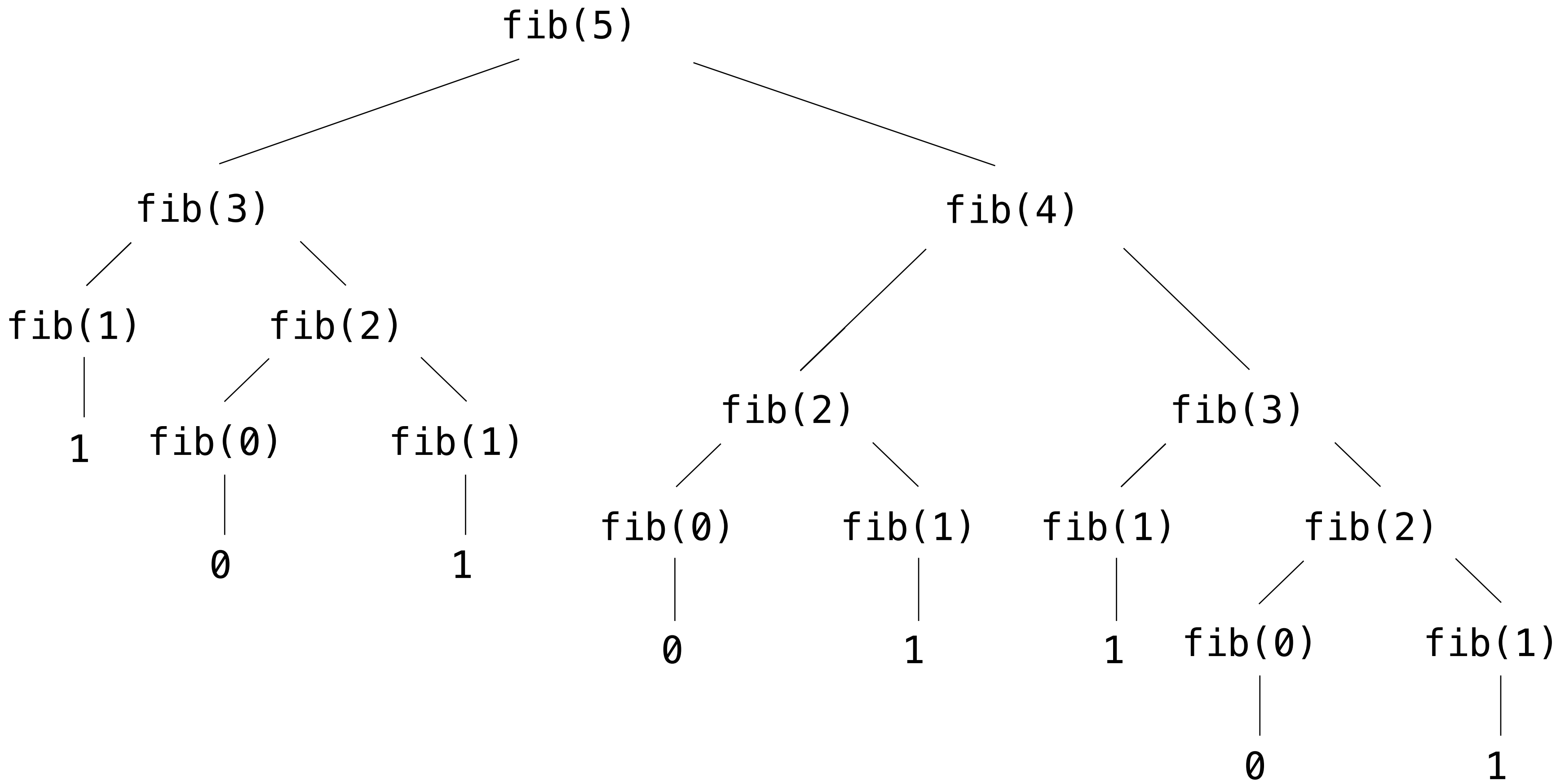

Recursion Tree

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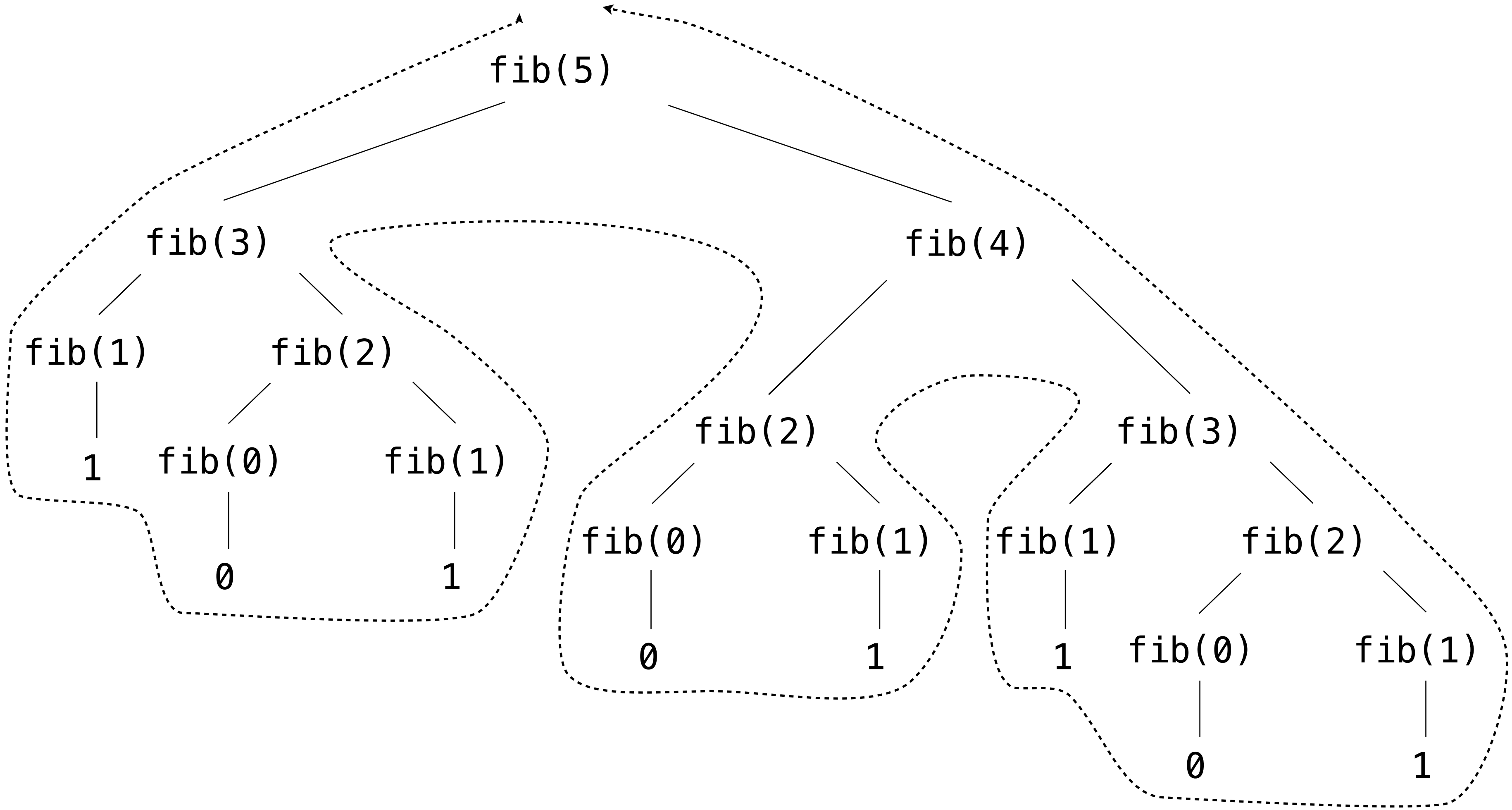
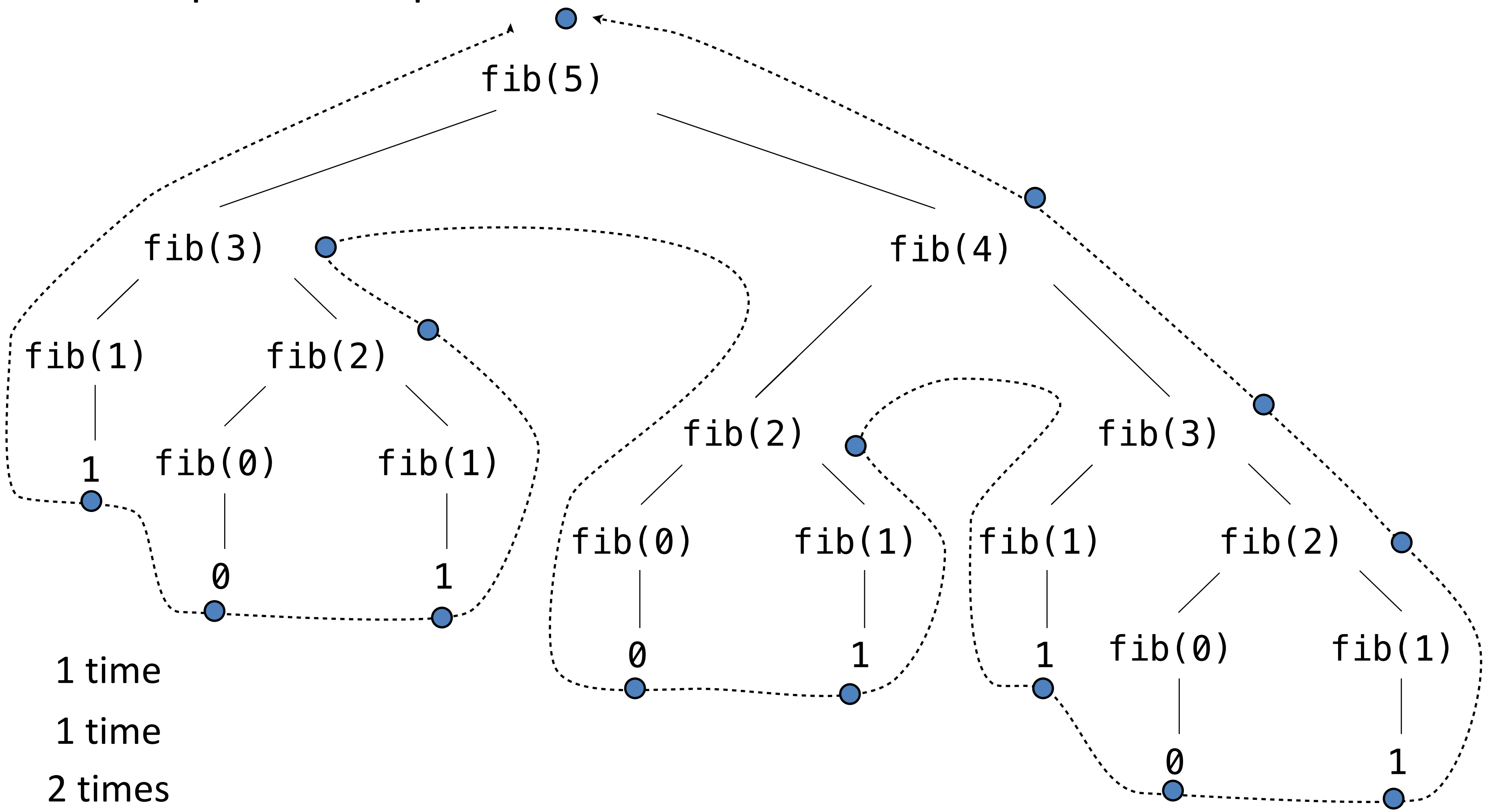


Recursion Tree

- The computational process of **`fib`** evolves into a tree structure



fib(5)	1 time
fib(4)	1 time
fib(3)	2 times
fib(2)	3 times

Pitfalls of Recursion

- With recursion, you can compose compact and elegant programs that fail spectacularly at runtime.
  - Missing base case
  - No guarentee of convergence
  - Excessive space requirements
  - Excessive recomputation

Missing base case

```
def H(n):
    return H(n-1) + 1.0/n;
```

- This recursive function is supposed to compute Harmonic numbers, but is missing a base case.
- If you call this function, it will repeatedly call itself and never return.

No guarantee of convergence

```
def H(n):
    if n == 1:
        return 1.0
    return H(n) + 1.0/n
```

- This recursive function will go into an infinite recursive loop if it is invoked with an argument n having any value other than 1.

- Another common problem is to include within a recursive function a recursive call to solve a subproblem that is not smaller.

Excessive space requirements

- Python needs to keep track of each recursive call to implement the function abstraction as expected.

- If a function calls itself recursively an excessive number of times before returning, the space required by Python for this task may be prohibitive.

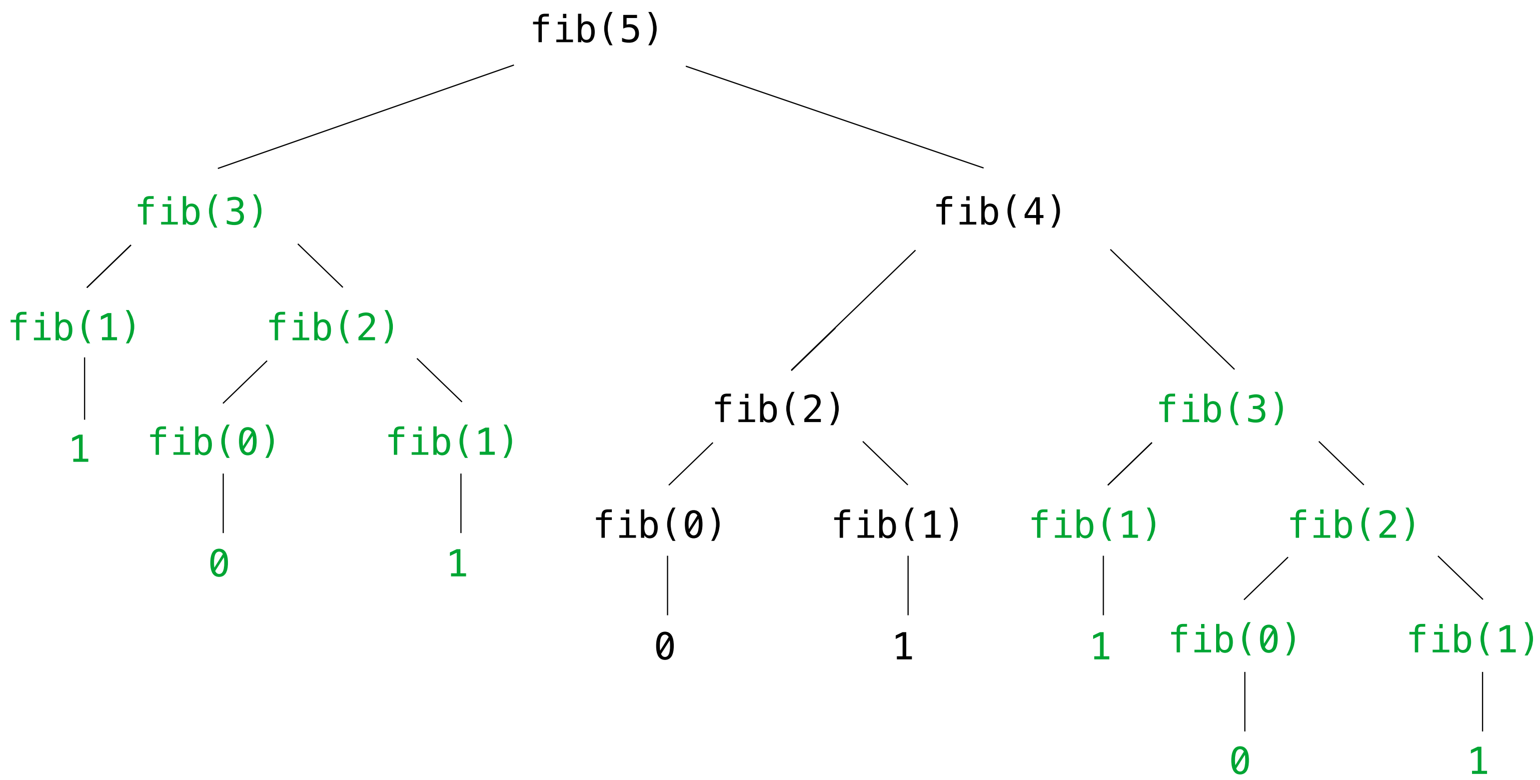
```
def H(n):
    if n == 0:
       return 0.0
    return H(n-1) + 1.0/n
```

- This recursive function correctly computes the n^{th} harmonic number.

- However, we cannot use it for large *n* because the recursive depth is proportional to *n*, and this creates a **StackOverflowError**.

Excessive recomputation

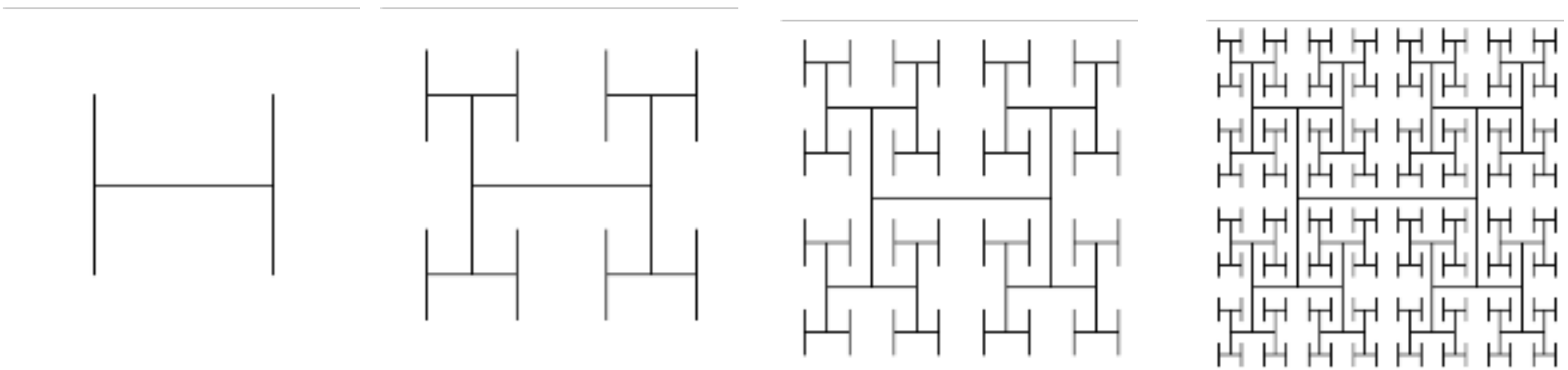
- A simple recursive program might require exponential time (unnecessarily), due to excessive recomputation.
- For example, **`fib`** is called on the same argument multiple times



Recursive Graphics

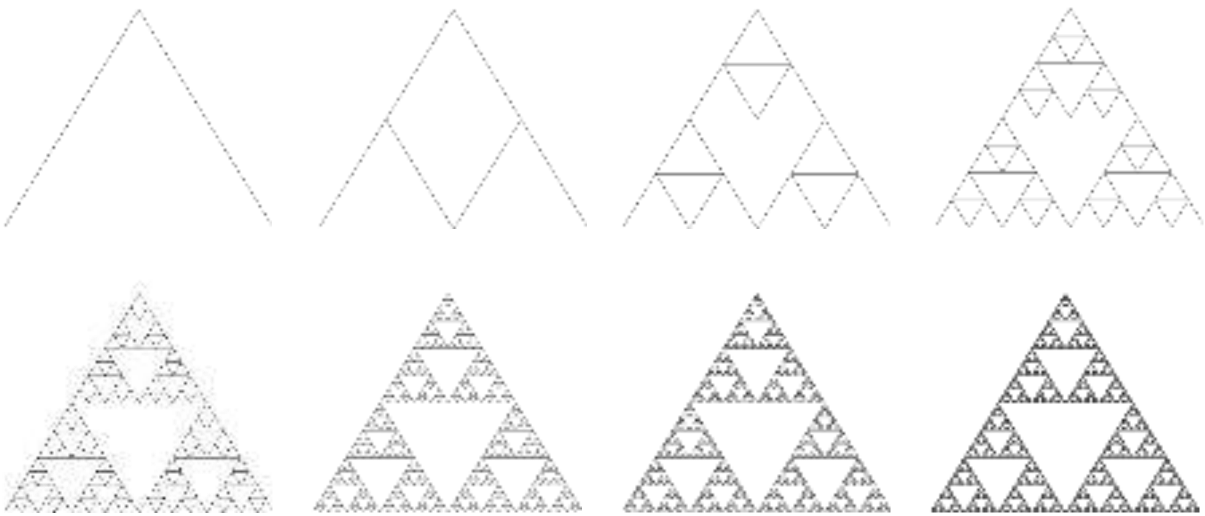
- Simple recursive drawing schemes can lead to pictures that are remarkably intricate – **Fractals**

- For example, an *H-tree of order n* is defined as follows:
  - The base case is null for $n = 0$.
  - The reduction step is to draw, within the unit square three lines in the shape of the letter H four H-trees of order n-1.
  - One connected to each tip of the H with the additional provisos that the H-trees of order n-1 are centered in the four quadrants of the square, halved in size.



More recursive graphics

- Sierpinski triangles



- Recursive trees

Next time… **Understanding Data**

Data science is the study of data.

Data scientist is part **mathematician**, part **statistician**, part **computer scientist** and part **trend-spotter**.

Machine Learning

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