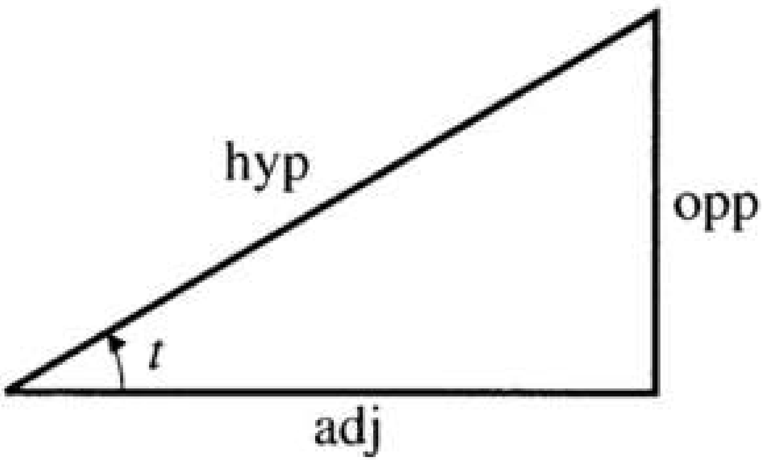
The Trigonometric Functions



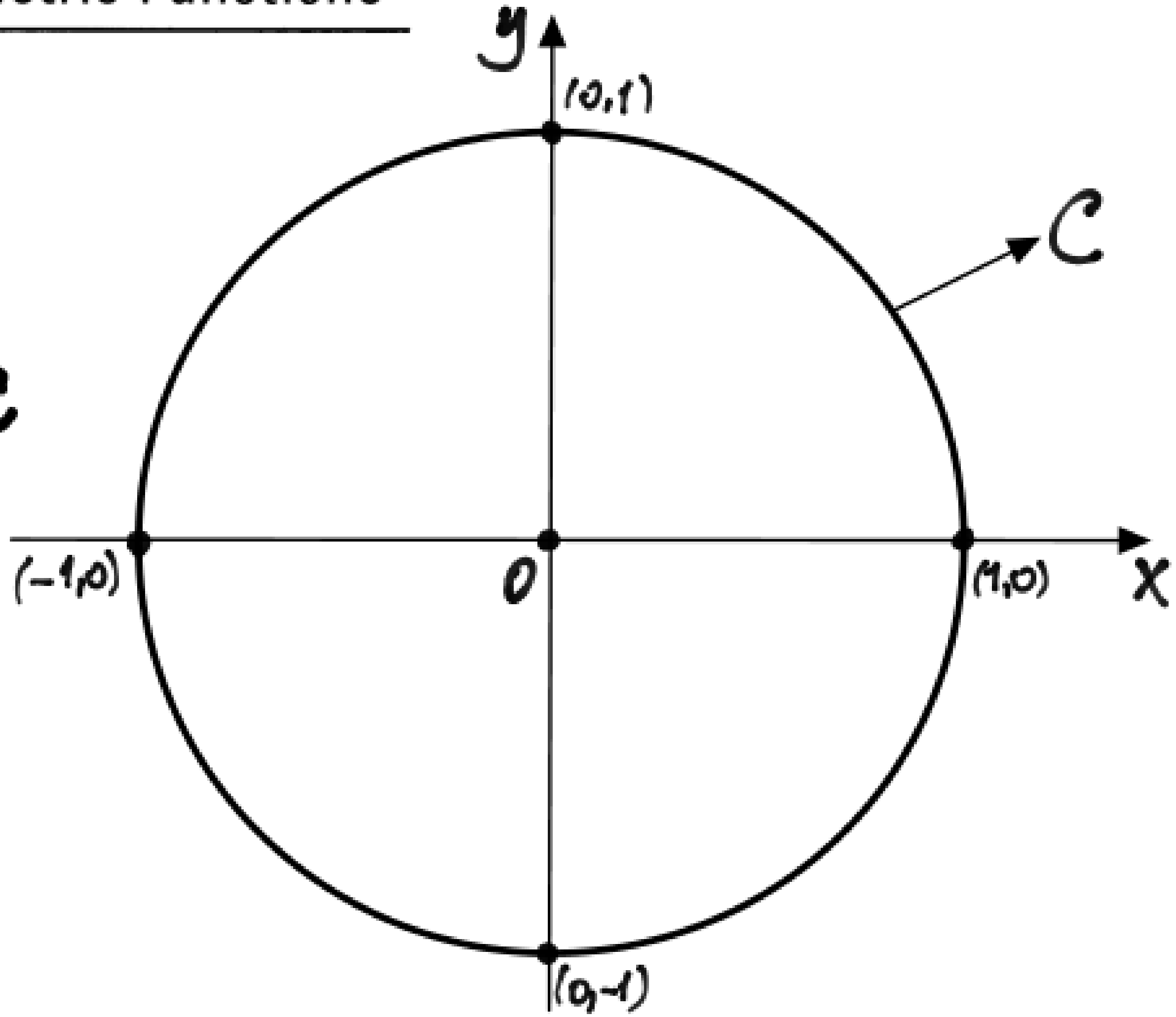
$$\cos t = \frac{\text{adj}}{\text{hyp}}$$

$$\sin t = \frac{\text{opp}}{\text{hyp}}$$

The Trigonometric Functions

The unit circle **C**

$x^2+y^2=1$



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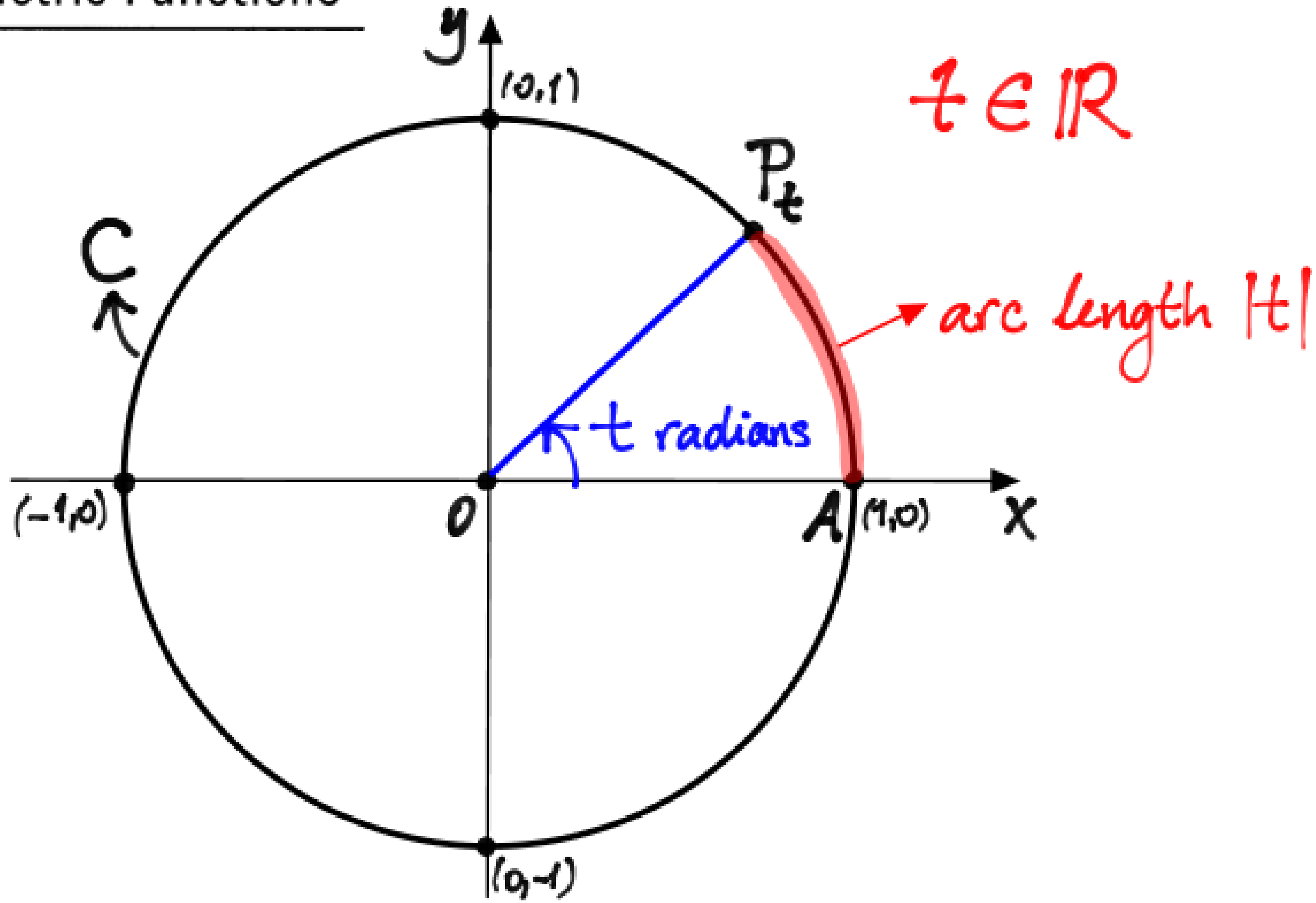

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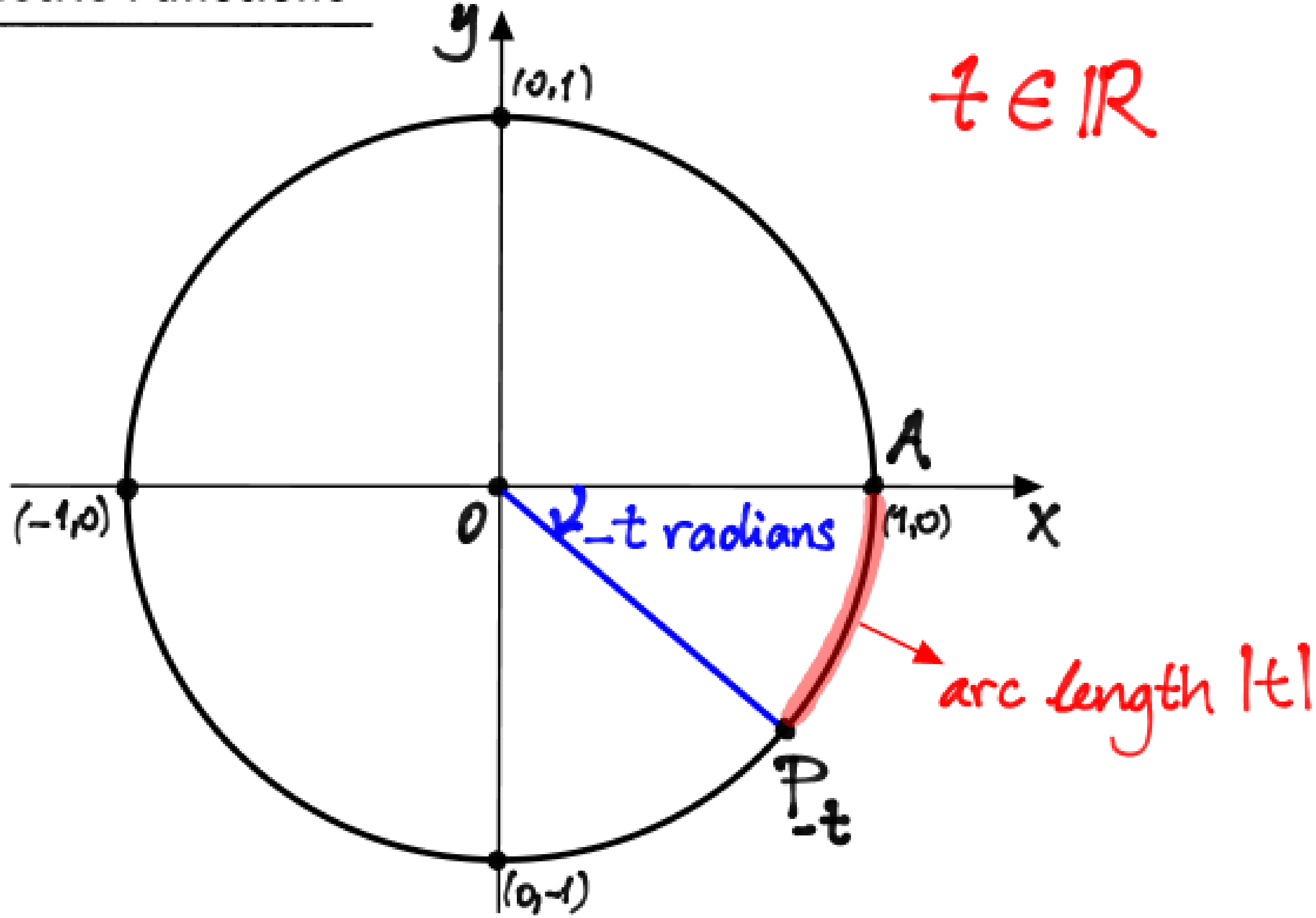
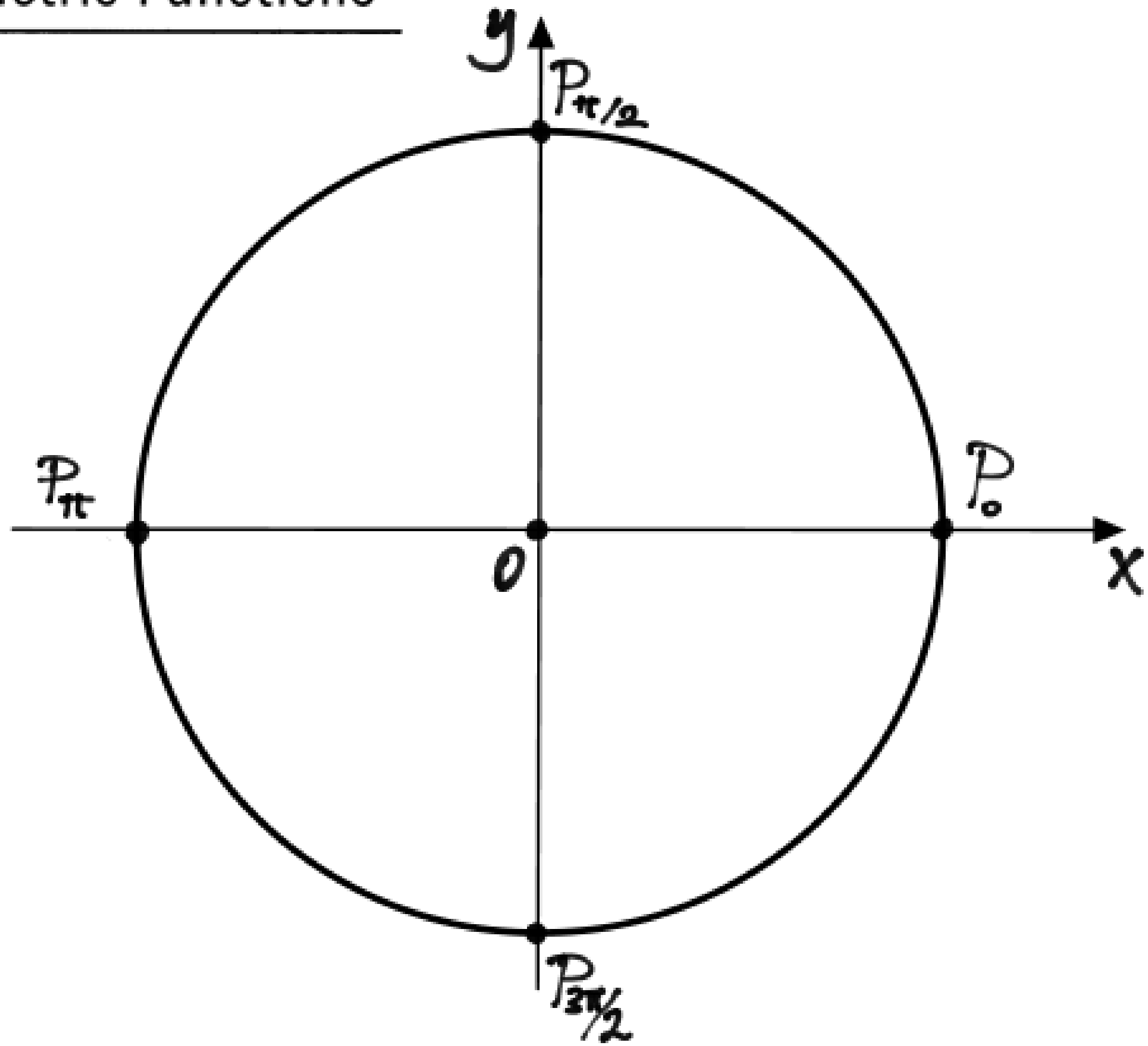


$t \in \mathbb{R}$

The Trigonometric Functions



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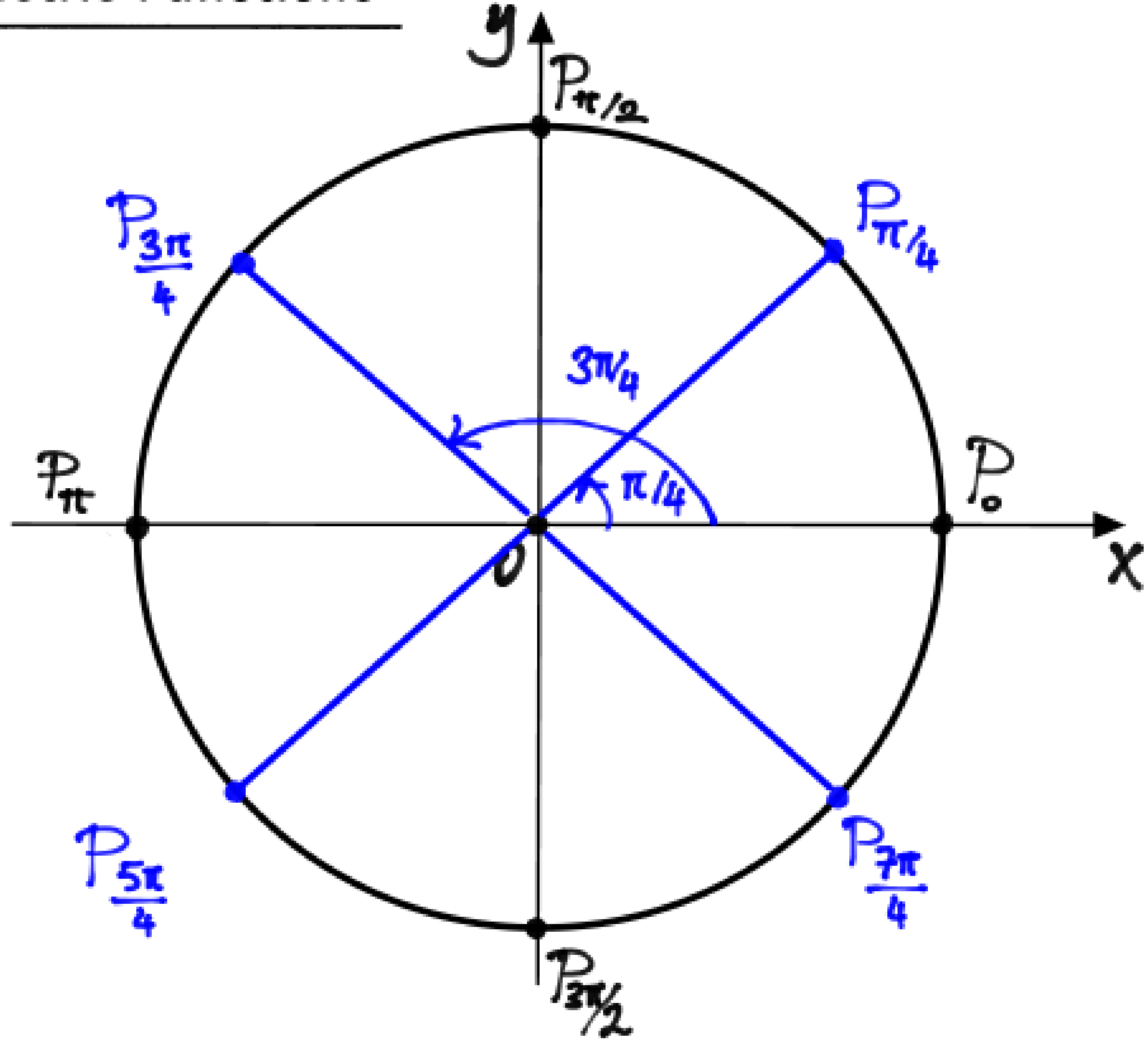

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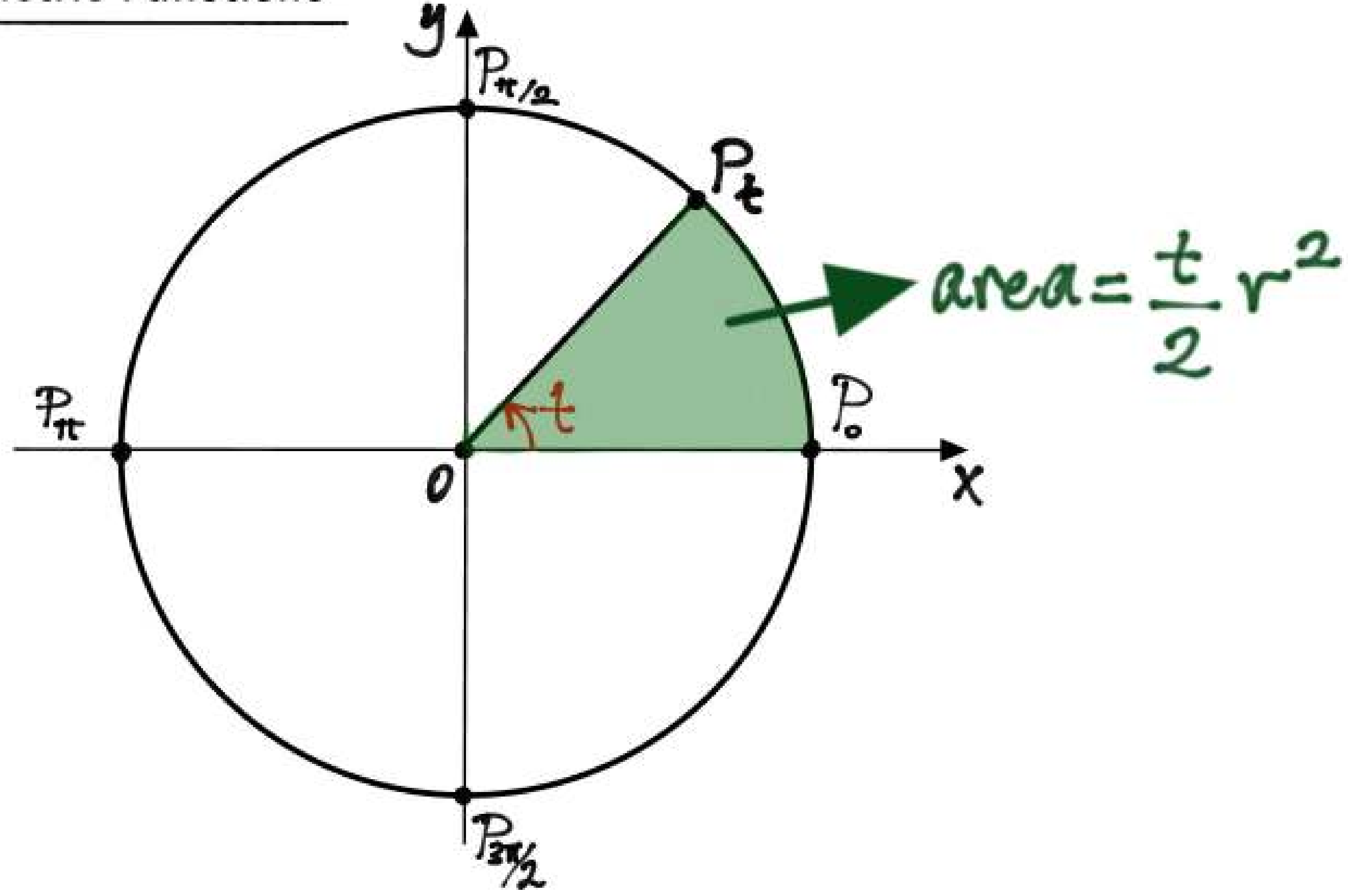
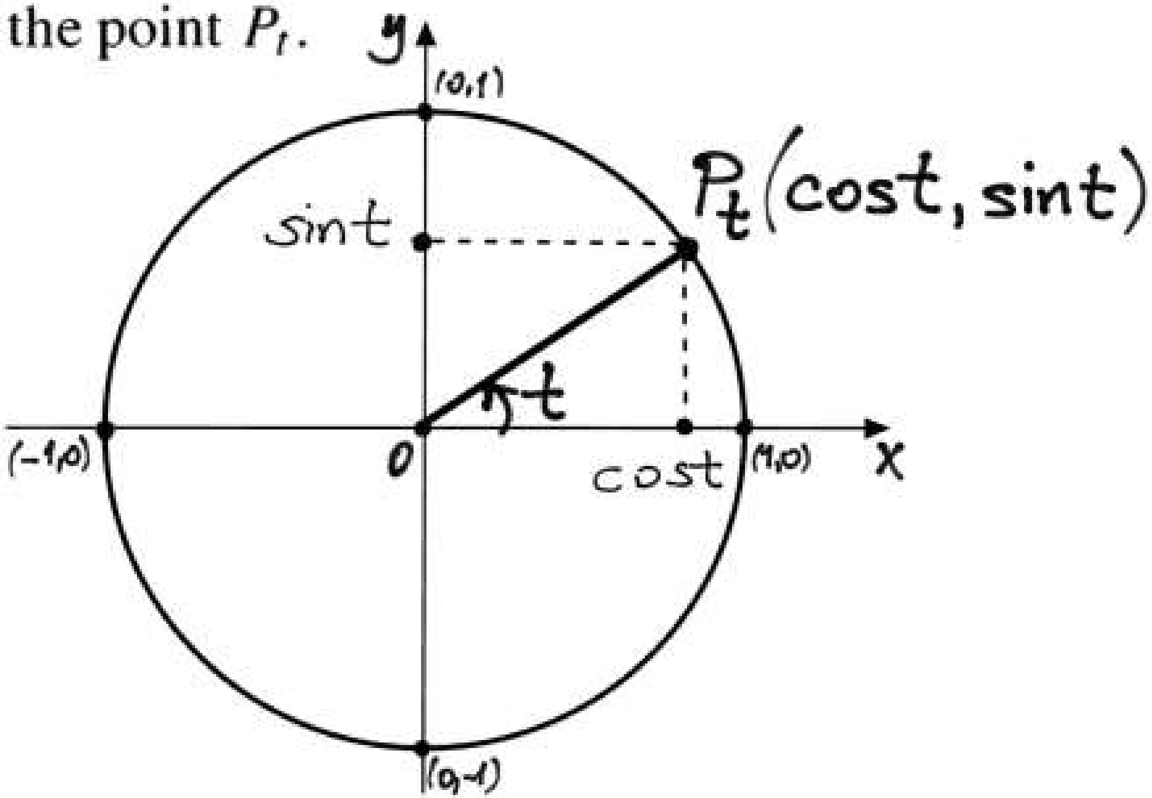


The Trigonometric Functions

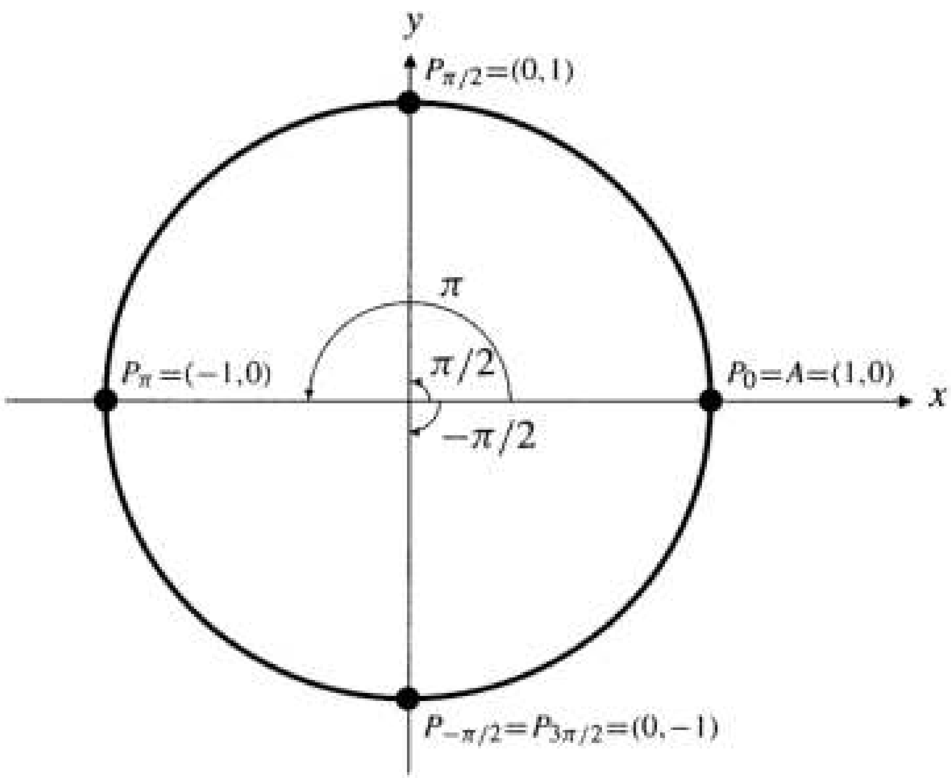
Cosine and sine

For any real t, the **cosine** of t (abbreviated $\cos t$) and the **sine** of t (abbreviated $\sin t$) are the x- and y-coordinates of the point P_t.

$$\cos t = \text{the } x\text{-coordinate of } P_t$$
$$\sin t = \text{the } y\text{-coordinate of } P_t$$



The Trigonometric Functions



EXAMPLE

Examining the coordinates of $P_0 = A$, $P_{\pi/2}$, P_π, and $P_{-\pi/2} = P_{3\pi/2}$ we obtain the following values:

$$\cos 0 = 1 \quad \cos\frac{\pi}{2} = 0 \quad \cos\pi = -1 \quad \cos\left(-\frac{\pi}{2}\right) = \cos\frac{3\pi}{2} = 0$$

$$\sin 0 = 0 \quad \sin\frac{\pi}{2} = 1 \quad \sin\pi = 0 \quad \sin\left(-\frac{\pi}{2}\right) = \sin\frac{3\pi}{2} = -1$$

The Trigonometric Functions

The range of cosine and sine. For every real number t,

$$-1 \leq \cos t \leq 1 \qquad \text{and} \qquad -1 \leq \sin t \leq 1.$$

The Pythagorean identity. The coordinates $x = \cos t$ and $y = \sin t$ of P_t must satisfy the equation of the circle. Therefore, for every real number t,

$$\cos^2 t + \sin^2 t = 1.$$

(Note that $\cos^2 t$ means $(\cos t)^2$, not $\cos(\cos t)$. This is an unfortunate notation, but it is used everywhere in technical literature, so you have to get used to it!)

Periodicity. Since C has circumference 2π, adding 2π to t causes the point P_t to go one extra complete revolution around C and end up in the same place: $P_{t+2\pi} = P_t$. Thus, for every t,

$$\cos(t + 2\pi) = \cos t \qquad \text{and} \qquad \sin(t + 2\pi) = \sin t.$$

The Trigonometric Functions

Cosine is an even function. Sine is an odd function. Since the circle $x^2 + y^2 = 1$ is symmetric about the x-axis, the points P_{-t} and P_t have the same x-coordinates and opposite y-coordinates

$$\cos(-t) = \cos t \quad \text{and} \quad \sin(-t) = -\sin t.$$

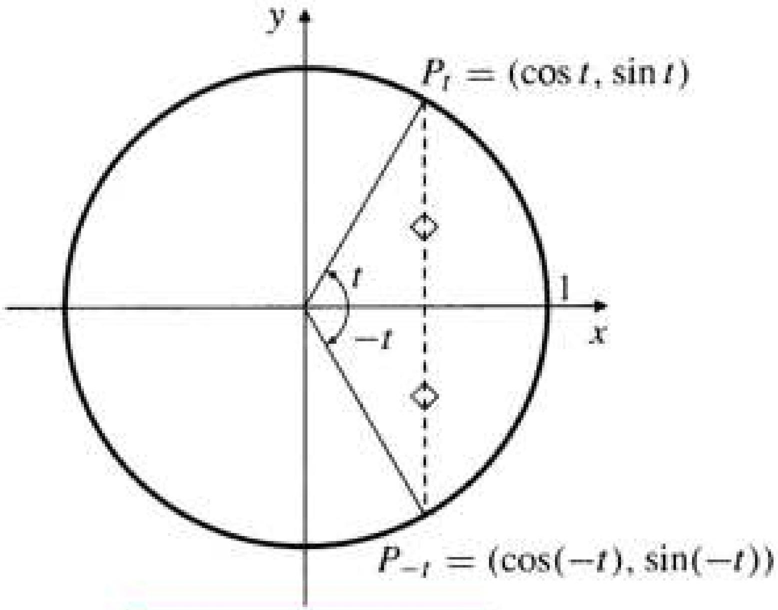
Complementary angle identities. Two angles are complementary if their sum is $\pi/2$ (or 90°). The points $P_{(\pi/2)-t}$ and P_t are reflections of each other in the line $y = x$ so the x-coordinate of one is the y-coordinate of the other and vice versa. Thus,

$$\cos\left(\frac{\pi}{2} - t\right) = \sin t \quad \text{and} \quad \sin\left(\frac{\pi}{2} - t\right) = \cos t.$$

Supplementary angle identities. Two angles are supplementary if their sum is π (or 180°). Since the circle is symmetric about the y-axis, $P_{\pi-t}$ and P_t have the same y-coordinates and opposite x-coordinates. Thus,

$$\cos(\pi - t) = -\cos t \quad \text{and} \quad \sin(\pi - t) = \sin t.$$

The Trigonometric Functions



$$\cos(-t) = \cos t$$
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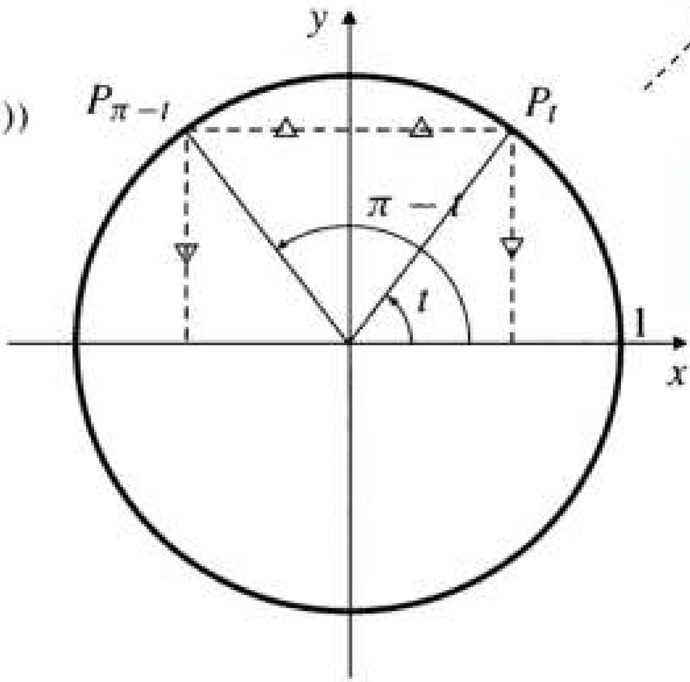
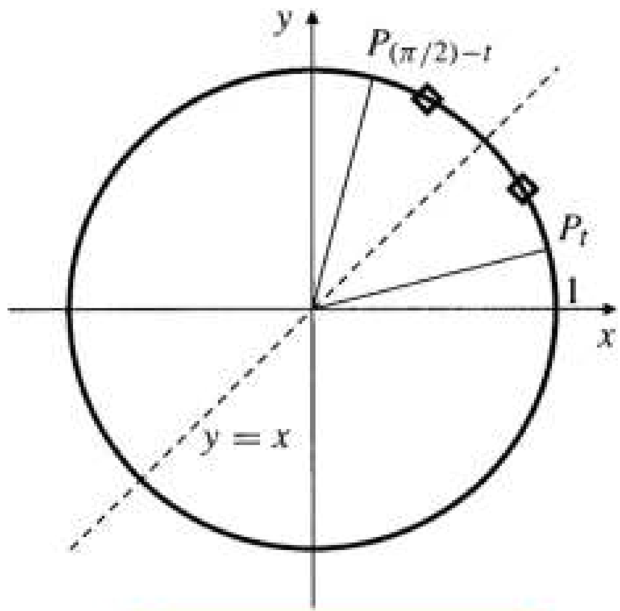


$$\cos(\pi - t) = -\cos t$$
$$\sin(\pi - t) = \sin t$$



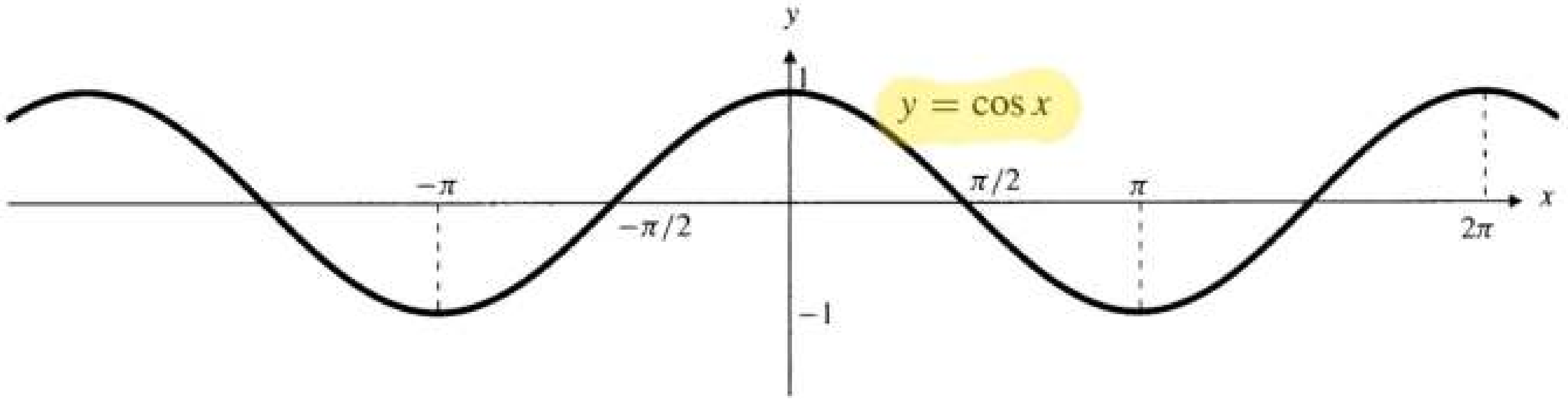
$$\cos((\pi/2) - t) = \sin t$$
$$\sin((\pi/2) - t) = \cos t$$

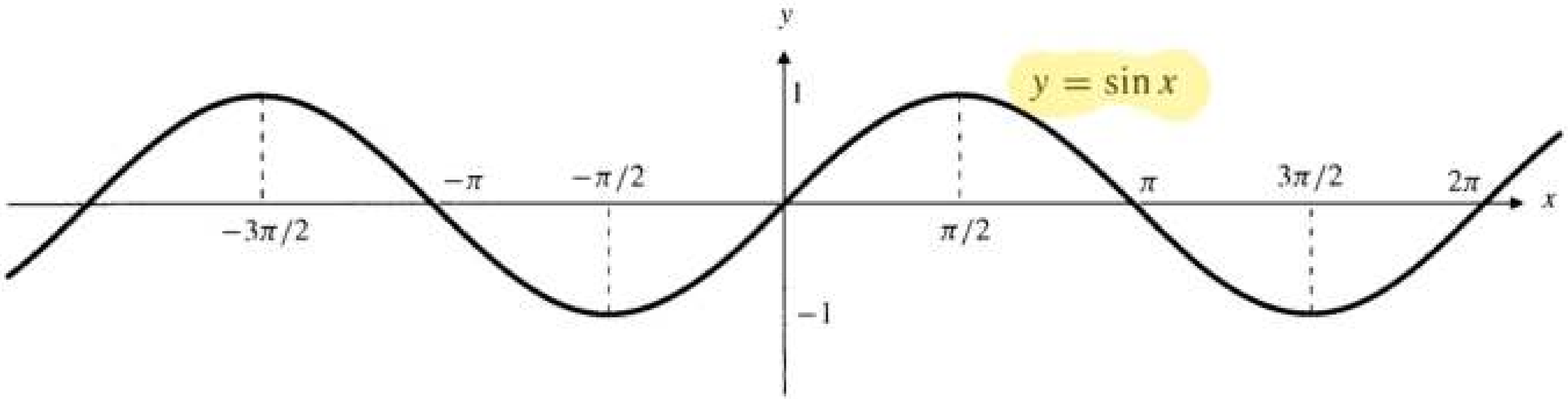
Some Special Angles

Cosines and sines of special angles

Degrees	$0°$	$30°$	$45°$	$60°$	$90°$	$120°$	$135°$	$150°$	$180°$
Radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
Cosine	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$	-1
Sine	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0

The Trigonometric Functions





The Addition Formulas

$$\cos(s+t) = \cos s \cos t - \sin s \sin t$$

$$\sin(s+t) = \sin s \cos t + \cos s \sin t$$

$$\cos(s-t) = \cos s \cos t + \sin s \sin t$$

$$\sin(s-t) = \sin s \cos t - \cos s \sin t$$

The Addition Formulas

$$\cos(s + t) = \cos s \cos t - \sin s \sin t$$
$$\sin(s + t) = \sin s \cos t + \cos s \sin t$$
$$\cos(s - t) = \cos s \cos t + \sin s \sin t$$
$$\sin(s - t) = \sin s \cos t - \cos s \sin t$$

EXAMPLE Find the value of $\cos(\pi/12) = \cos 15°$.

The double-angle formulas:

$$\sin 2t = 2\sin t\cos t \qquad \text{and}$$

$$\begin{aligned}\cos 2t &= \cos^2 t - \sin^2 t\\ &= 2\cos^2 t - 1 \qquad\qquad (\text{using } \sin^2 t + \cos^2 t = 1)\\ &= 1 - 2\sin^2 t\end{aligned}$$

Solving the last two formulas for $\cos^2 t$ and $\sin^2 t$, we obtain

$$\cos^2 t = \frac{1+\cos 2t}{2} \qquad \text{and} \qquad \sin^2 t = \frac{1-\cos 2t}{2},$$

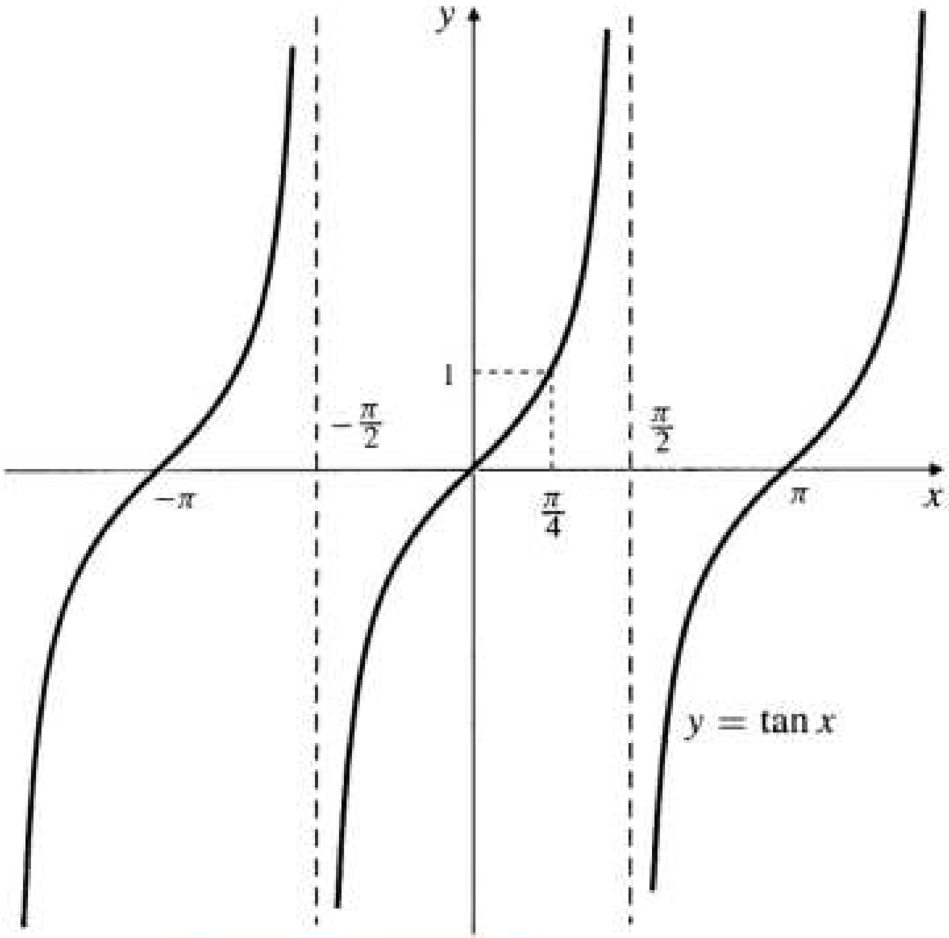
Other Trigonometric Functions

Tangent, cotangent, secant, and cosecant

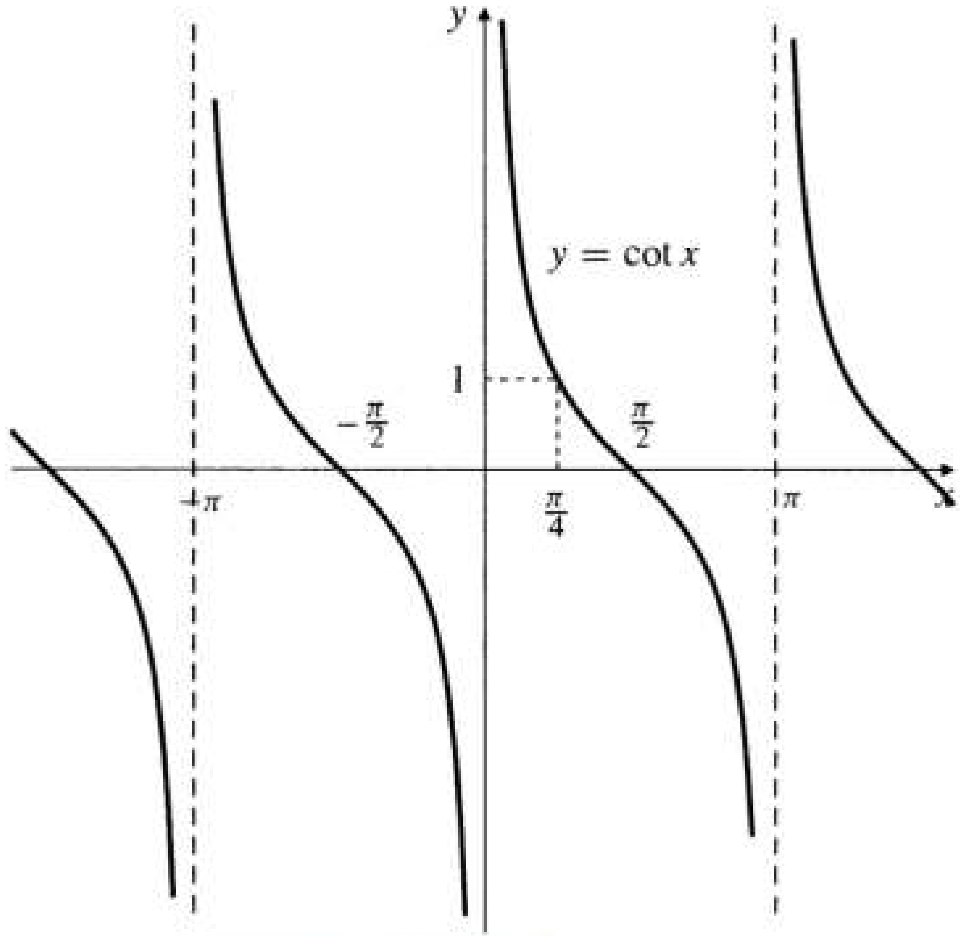
$$\tan t = \frac{\sin t}{\cos t} \qquad \sec t = \frac{1}{\cos t}$$

$$\cot t = \frac{\cos t}{\sin t} = \frac{1}{\tan t} \qquad \csc t = \frac{1}{\sin t}$$

Other Trigonometric Functions

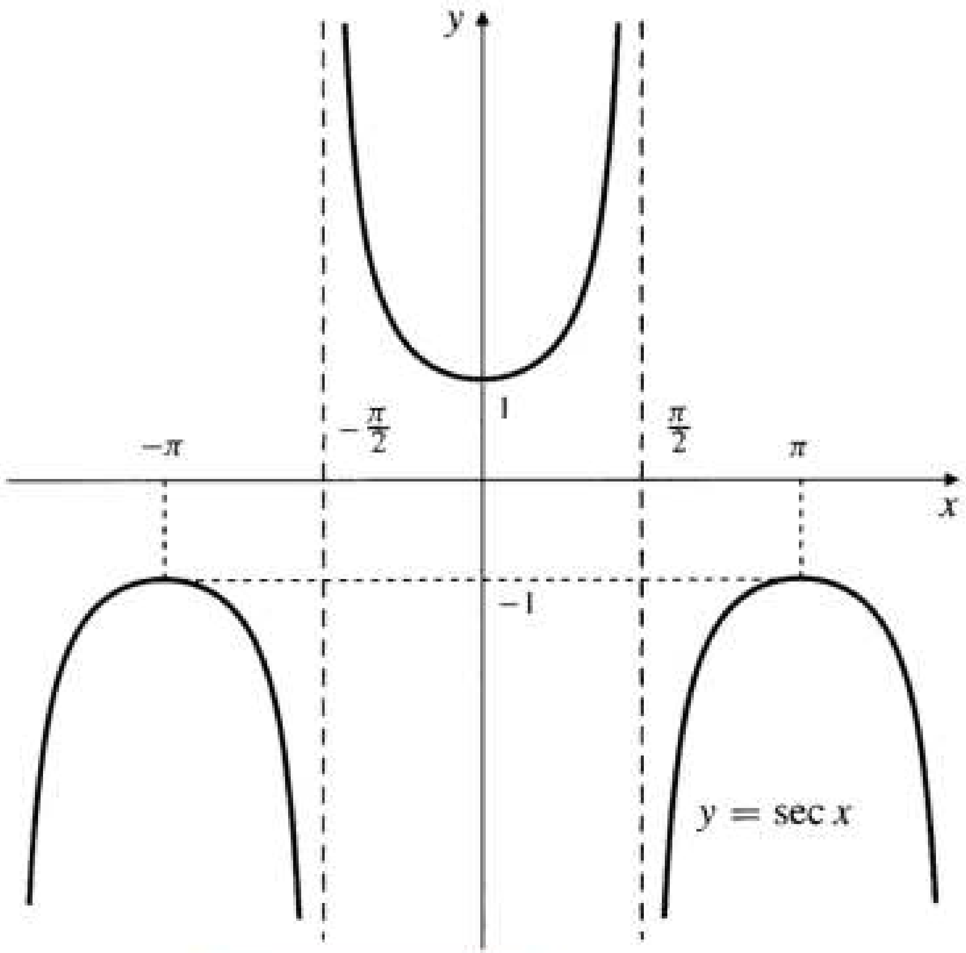


The graph of $\tan x$

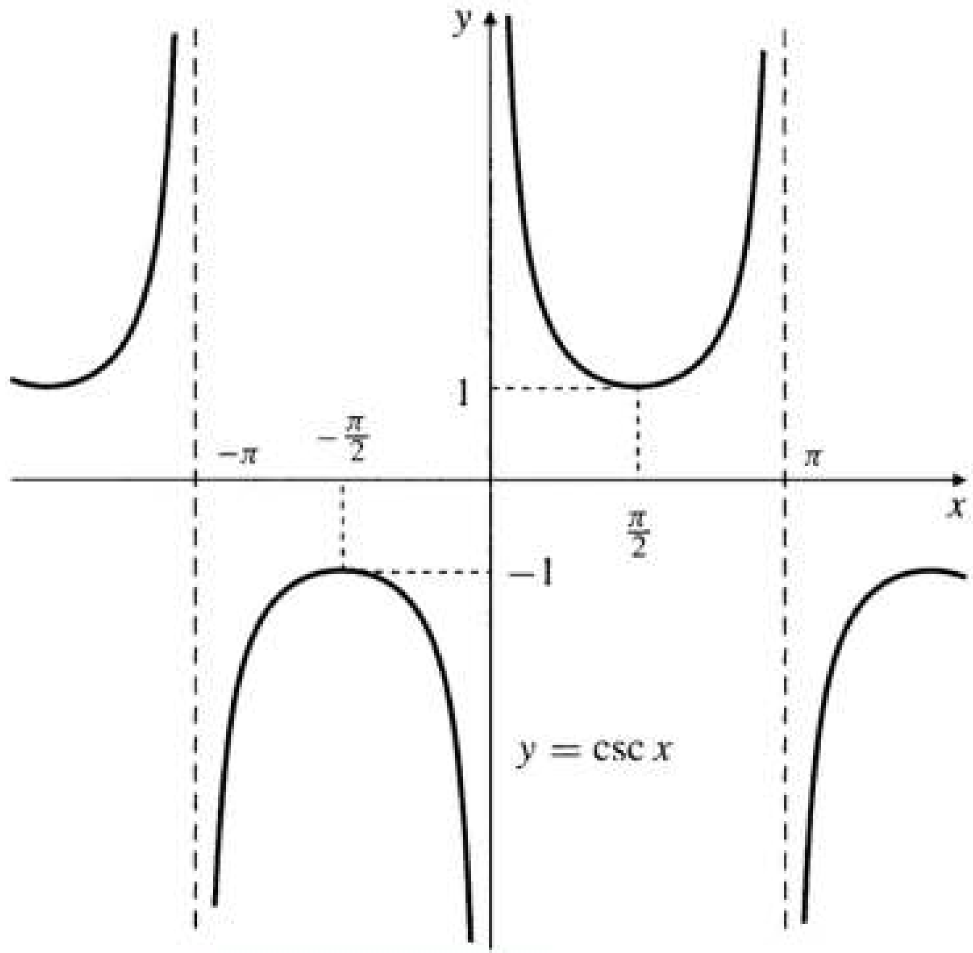


The graph of $\cot x$

Other Trigonometric Functions

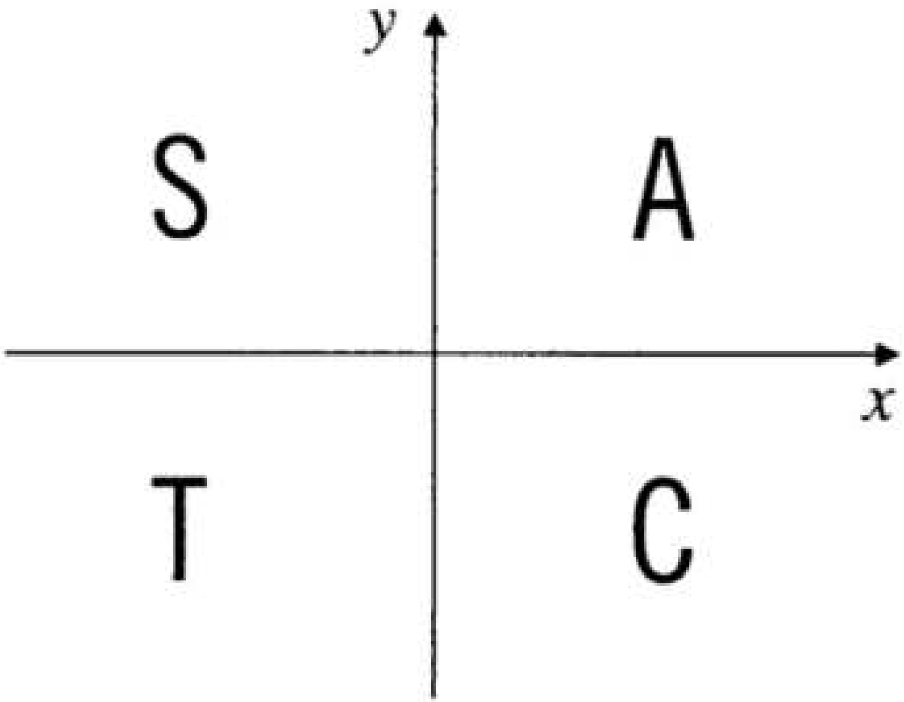


The graph of $\sec x$



The graph of $\csc x$

The Trigonometric Functions



The CAST rule for signs of trigonometric functions

EXAMPLE Find the sine and tangent of the angle θ in $\left[\pi, \frac{3\pi}{2}\right]$ for which we have $\cos\theta = -\frac{1}{3}$.

Other Trigonometric Identities

$$\tan(x+\pi) = \tan x$$

$$1 + \tan^2 x = \sec^2 x$$

$$1 + \cot^2 x = \csc^2 x$$

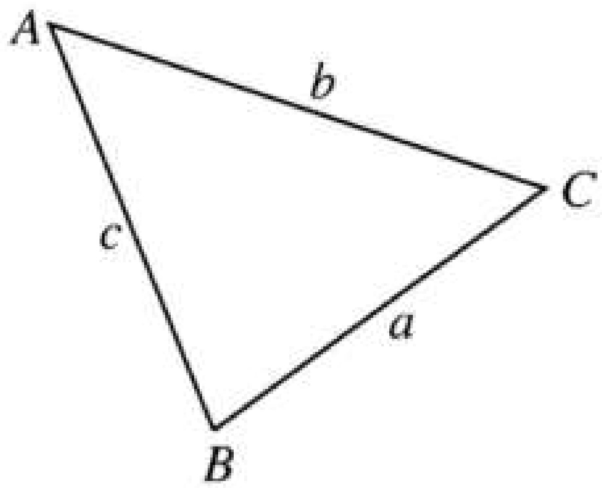
$$\tan(s \mp t) = \frac{\tan s \mp \tan t}{1 \pm \tan s . \tan t}$$

The Trigonometric Functions

THEOREM

Sine Law: $$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

Cosine Law:
$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$b^2 = a^2 + c^2 - 2ac\cos B$$
$$c^2 = a^2 + b^2 - 2ab\cos C$$



Inverse Functions

A function f is **one-to-one** if $f(x_1) \neq f(x_2)$ whenever x_1 and x_2 belong to the domain of f and $x_1 \neq x_2$ or, equivalently, if

$$f(x_1) = f(x_2) \quad \Longrightarrow \quad x_1 = x_2.$$

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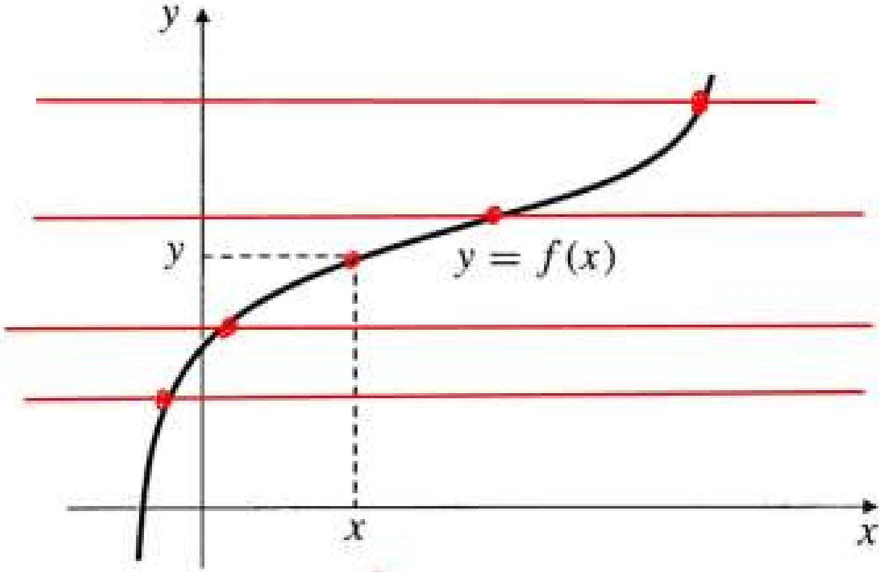
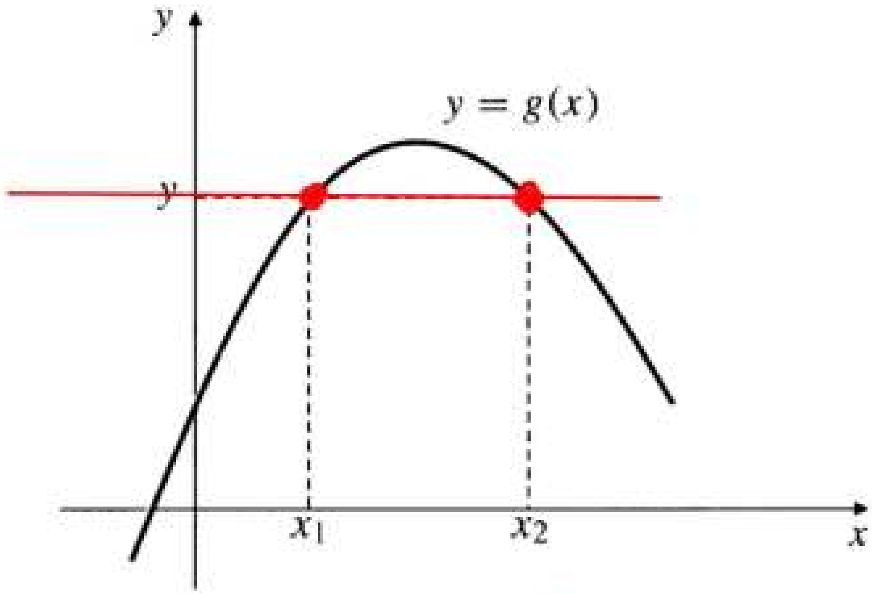


one-to-one



not one-to one

Inverse Functions

If f is one-to-one, then it has an **inverse function** f^{-1}. The value of $f^{-1}(x)$ is the unique number y in the domain of f for which $f(y) = x$. Thus,

$$y = f^{-1}(x) \quad \Longleftrightarrow \quad x = f(y).$$

Inverse Functions

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$$y = f^{-1}(x) \iff x = f(y).$$

EXAMPLE Show that $f(x) = 2x - 1$ is one-to-one, and find its inverse $f^{-1}(x)$.

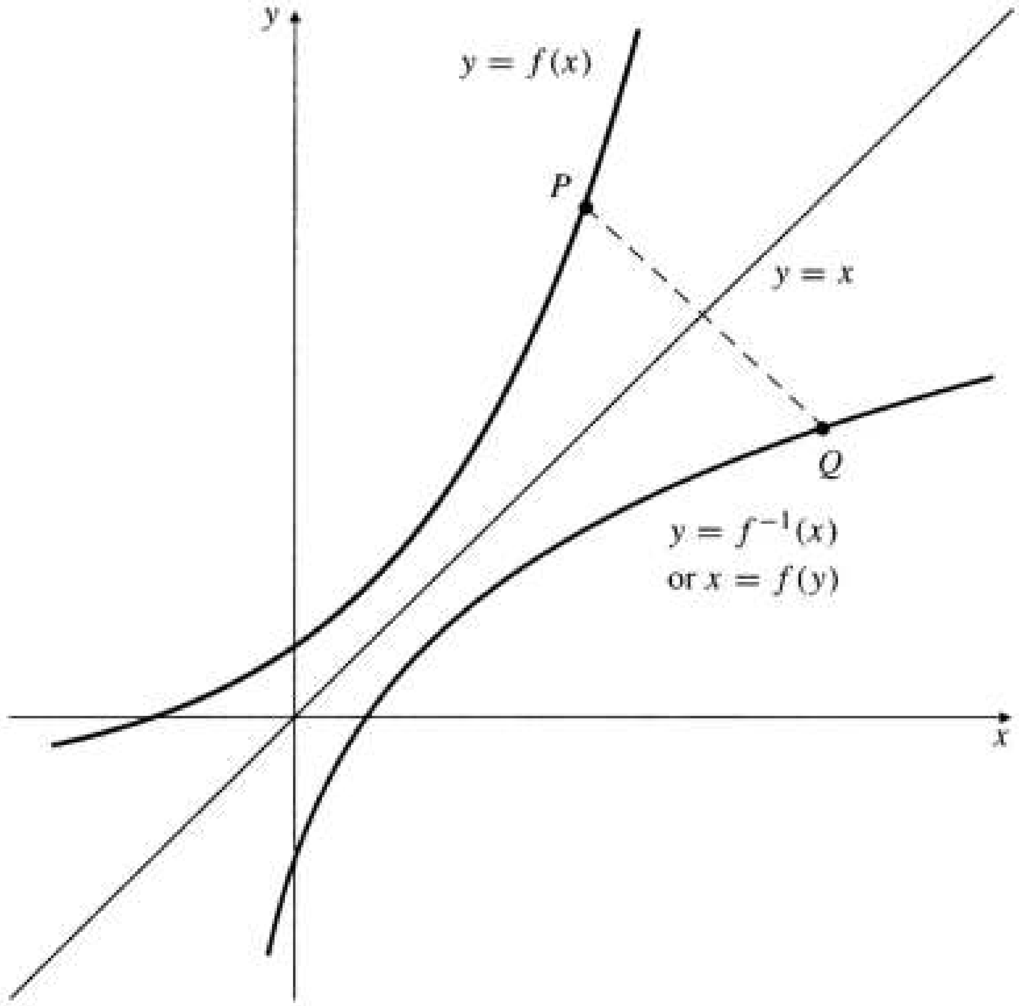
Inverse Functions

Properties of inverse functions

1. $y = f^{-1}(x) \iff x = f(y)$.
2. The domain of f^{-1} is the range of f.
3. The range of f^{-1} is the domain of f.
4. $f^{-1}\big(f(x)\big) = x$ for all x in the domain of f. $\longrightarrow$ $f^{-1} \circ f = I_{\mathcal{D}(f)}$,
5. $f\big(f^{-1}(x)\big) = x$ for all x in the domain of f^{-1}. $\longrightarrow$ $f \circ f^{-1} = I_{\mathcal{D}(f^{-1})}$
6. $(f^{-1})^{-1}(x) = f(x)$ for all x in the domain of f.

Inverse Functions

7. The graph of f^{-1} is the reflection of the graph of f in the line $x = y$.



Inverse Functions

EXAMPLE. Show that the function

$$f(x) = \frac{1-2x}{1+x}$$

is one-to-one, and calculate the inverse function f^{-1}. Specify the domains and ranges of both f and f^{-1}.

Exponential and Logarithmic Functions

An exponential function is a function of the form

$$f(x) = a^x,$$

where a is a positive constant which is not 1.

But how can we raise a number to an irrational number?

Exponential and Logarithmic Functions

If x is a positive integer, then

$$a^x = \underbrace{a \ldots a}_{x \text{ factors}}$$

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Exponential and Logarithmic Functions

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If $x = p/q$ is a rational number where p and $q > 0$ are integers, then

$$a^x = a^{p/q} = \sqrt[q]{a^p} = \left(\sqrt[q]{a}\right)^p.$$

Irrational powers can be calculated by approximating the irrational exponent by rational powers.

For a concrete example, lets consider the number 2 raised to the power $\sqrt{3}$, i.e., $2^{\sqrt{3}}$.

Exponential and Logarithmic Functions

$$
\begin{array}{rcl}
1.7 < \sqrt{3} < 1.8 & \Longrightarrow & 2^{1.7} < 2^{\sqrt{3}} < 2^{1.8} \\
1.73 < \sqrt{3} < 1.74 & \Longrightarrow & 2^{1.73} < 2^{\sqrt{3}} < 2^{1.74} \\
1.732 < \sqrt{3} < 1.733 & \Longrightarrow & 2^{1.732} < 2^{\sqrt{3}} < 2^{1.733} \\
1.7320 < \sqrt{3} < 1.7321 & \Longrightarrow & 2^{1.7320} < 2^{\sqrt{3}} < 2^{1.7321} \\
1.73205 < \sqrt{3} < 1.73206 & \Longrightarrow & 2^{1.73205} < 2^{\sqrt{3}} < 2^{1.73206} \\
\vdots & & \vdots
\end{array}
$$

Exponential and Logarithmic Functions

$$
\begin{aligned}
1.7 < \sqrt{3} < 1.8 &\implies \boxed{2^{1.7}} < 2^{\sqrt{3}} < \boxed{2^{1.8}} \\
1.73 < \sqrt{3} < 1.74 &\implies \boxed{2^{1.73}} < 2^{\sqrt{3}} < \boxed{2^{1.74}} \\
1.732 < \sqrt{3} < 1.733 &\implies \boxed{2^{1.732}} < 2^{\sqrt{3}} < \boxed{2^{1.733}} \\
1.7320 < \sqrt{3} < 1.7321 &\implies \boxed{2^{1.7320}} < 2^{\sqrt{3}} < \boxed{2^{1.7321}} \\
1.73205 < \sqrt{3} < 1.73206 &\implies \boxed{2^{1.73205}} < 2^{\sqrt{3}} < \boxed{2^{1.73206}} \\
\vdots \qquad & \qquad\quad \rightarrow \ \vdots \ \leftarrow
\end{aligned}
$$

Exponential and Logarithmic Functions

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\begin{aligned}
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1.7320 < \sqrt{3} < 1.7321 &\implies 2^{1.7320} < 2^{\sqrt{3}} < 2^{1.7321} \\
1.73205 < \sqrt{3} < 1.73206 &\implies 2^{1.73205} < 2^{\sqrt{3}} < 2^{1.73206} \\
\vdots \qquad & \qquad\qquad\quad \rightarrow \vdots \leftarrow
\end{aligned}
$$

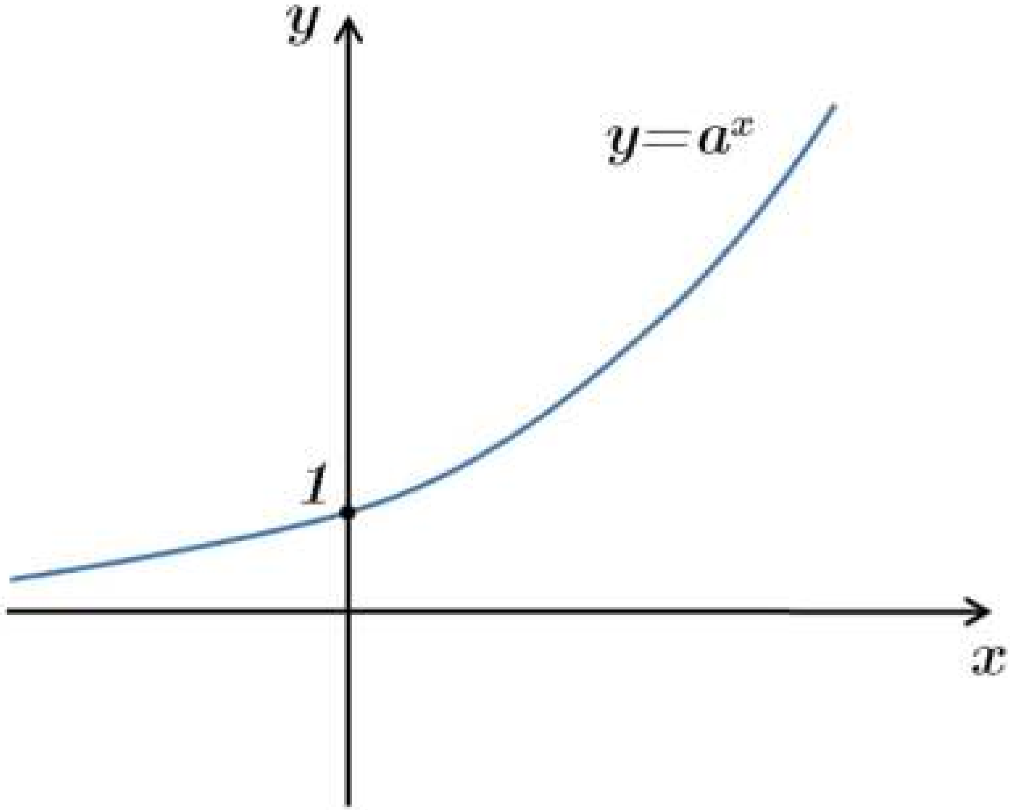
there is exactly one number that is greater than all of the numbers

$$2^{1.7},\ 2^{1.73},\ 2^{1.732},\ 2^{1.73205}, \ldots$$

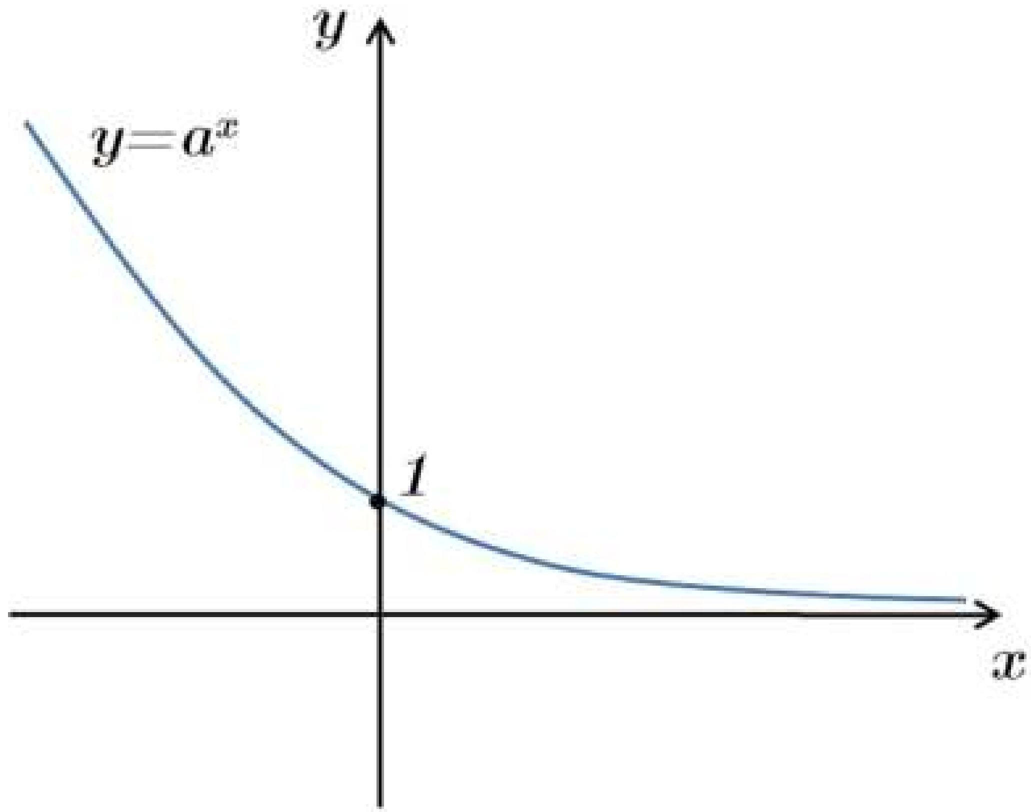
and less than all the numbers

$$2^{1.8},\ 2^{1.74},\ 2^{1.733},\ 2^{1.7321},\ 2^{1.73206}, \ldots$$

Exponential and Logarithmic Functions



$a>1$



$0<a<1$

Exponential and Logarithmic Functions

Laws of exponents

If $a > 0$ and $b > 0$, and x and y are any real numbers, then

(i) $a^0 = 1$

(ii) $a^{x+y} = a^x a^y$

(iii) $a^{-x} = \dfrac{1}{a^x}$

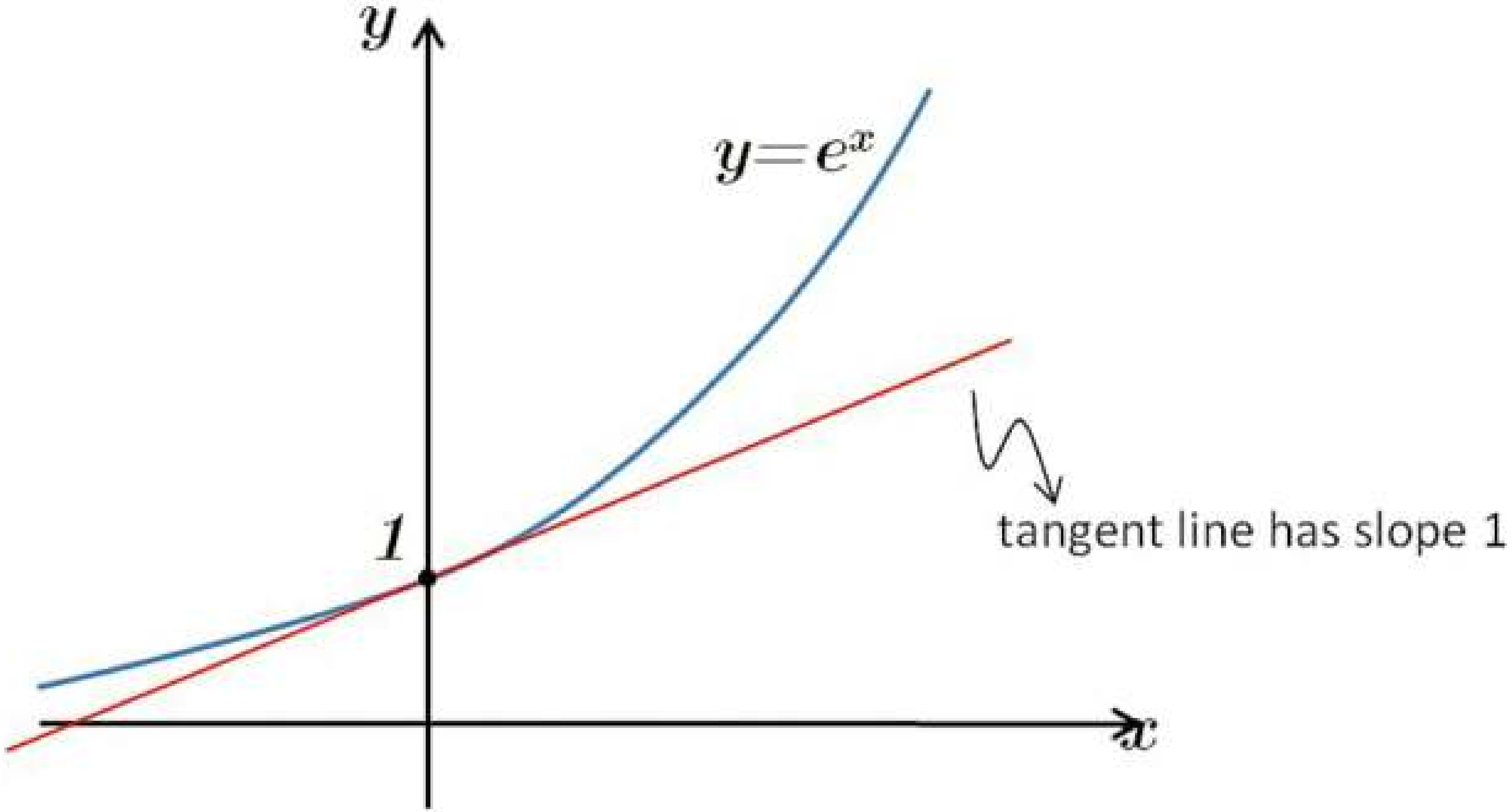
(iv) $a^{x-y} = \dfrac{a^x}{a^y}$

(v) $(a^x)^y = a^{xy}$

(vi) $(ab)^x = a^x b^x$

Exponential and Logarithmic Functions

The Natural Exponential Function



Exponential and Logarithmic Functions

Logarithms

If $a > 0$ and $a \neq 1$, the function $\log_a x$, called **the logarithm of** x **to the base** a, is the inverse of the one-to-one function a^x:

$$y = \log_a x \quad \Longleftrightarrow \quad x = a^y, \qquad (a > 0, \quad a \neq 1).$$

a^x is a function $\mathbb{R} \longrightarrow (0, \infty)$

$\log_a x$ is a function $(0, \infty) \longrightarrow \mathbb{R}$

Exponential and Logarithmic Functions

Logarithms

cancellation identities

$\log_a (a^x) = x$ for all real x and $a^{\log_a x} = x$ for all $x > 0$.

Exponential and Logarithmic Functions

Logarithms

cancellation identities

$\log_a (a^x) = x$ for all real x and $a^{\log_a x} = x$ for all $x > 0$.

Laws of logarithms

If $x > 0$, $y > 0$, $a > 0$, $b > 0$, $a \neq 1$, and $b \neq 1$, then

(i) $\log_a 1 = 0$

(ii) $\log_a (xy) = \log_a x + \log_a y$

(iii) $\log_a \left(\frac{1}{x}\right) = -\log_a x$

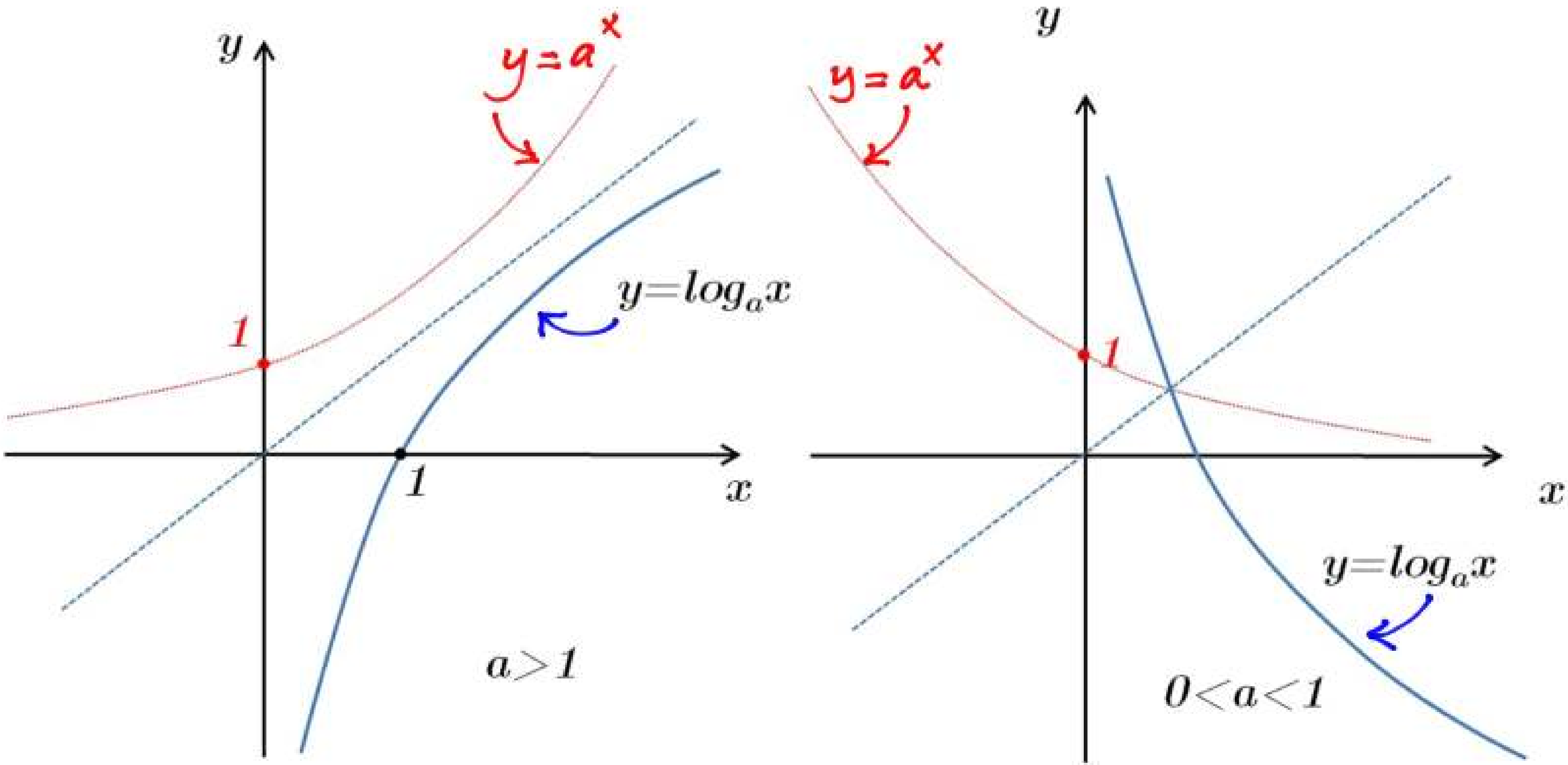
(iv) $\log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y$

(v) $\log_a (x^y) = y \log_a x$

(vi) $\log_a x = \dfrac{\log_b x}{\log_b a}$

Exponential and Logarithmic Functions

Logarithms



Exponential and Logarithmic Functions

Logarithms

EXAMPLE Solve the equation $3^{x-1} = 2^x$.