Sketching the Graph of a Function

Asymptotes

The graph of $y = f(x)$ has a **vertical asymptote** at $x = a$ if

$$\text{either} \quad \lim_{x \to a-} f(x) = \pm\infty \qquad \text{or} \quad \lim_{x \to a+} f(x) = \pm\infty, \qquad \text{or both.}$$

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EXAMPLE

Find the vertical asymptotes of $f(x) = \dfrac{1}{x^2 - x}$. How does the graph approach these asymptotes?

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Solution The denominator $x^2 - x = x(x - 1)$ approaches 0 as x approaches 0 or 1, so f has vertical asymptotes at $x = 0$ and $x = 1$.

$$\lim_{x \to 0-} \frac{1}{x^2 - x} = \infty, \qquad \lim_{x \to 1-} \frac{1}{x^2 - x} = -\infty,$$

$$\lim_{x \to 0+} \frac{1}{x^2 - x} = -\infty, \qquad \lim_{x \to 1+} \frac{1}{x^2 - x} = \infty.$$

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$$\lim_{x \to 0-} \frac{1}{x^2 - x} = \infty, \qquad \lim_{x \to 1-} \frac{1}{x^2 - x} = -\infty,$$

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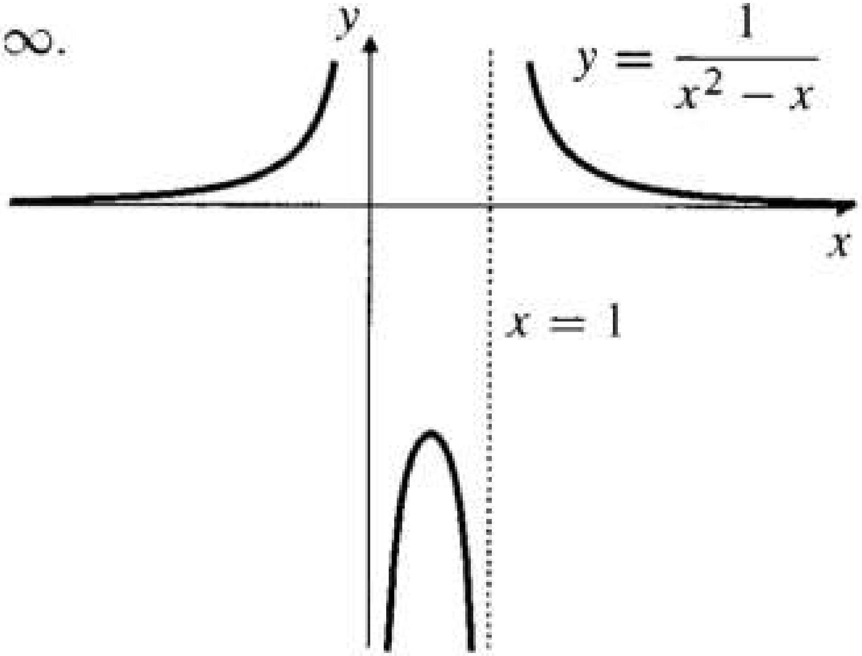

Sketching the Graph of a Function

Asymptotes

The graph $y=f(x)$ has a horizontal asymptote at $y=b$ if either

$$\lim_{x \to \infty} f(x) = b \quad \text{or} \quad \lim_{x \to -\infty} f(x) = b.$$

Sketching the Graph of a Function

Asymptotes

EXAMPLE

Find the horizontal asymptotes of

(a) $f(x) = \dfrac{1}{x^2 - x}$ and (b) $g(x) = \dfrac{x^4 + x^2}{x^4 + 1}$.

Sketching the Graph of a Function

Asymptotes

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Solution

(a) The function f has horizontal asymptote $y = 0$ since

$$\lim_{x \to \pm\infty} \frac{1}{x^2 - x} = \lim_{x \to \pm\infty} \frac{1/x^2}{1 - (1/x)} = \frac{0}{1} = 0.$$

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(b) The function g has horizontal asymptote $y = 1$ since

$$\lim_{x \to \pm\infty} \frac{x^4 + x^2}{x^4 + 1} = \lim_{x \to \pm\infty} \frac{1 + (1/x^2)}{1 + (1/x^4)} = \frac{1}{1} = 1.$$

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Asymptotes

EXAMPLE Find the horizontal asymptotes of

(a) $f(x) = \dfrac{1}{x^2 - x}$ and (b) $g(x) = \dfrac{x^4 + x^2}{x^4 + 1}$.

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Observe that the graph of g crosses its asymptote twice.

Sketching the Graph of a Function

Asymptotes

The straight line $y = ax + b$ (where $a \neq 0$) is an **oblique asymptote** of the graph of $y = f(x)$ if

$$\text{either} \quad \lim_{x \to -\infty} \big(f(x) - (ax + b)\big) = 0 \quad \text{or} \quad \lim_{x \to \infty} \big(f(x) - (ax + b)\big) = 0,$$

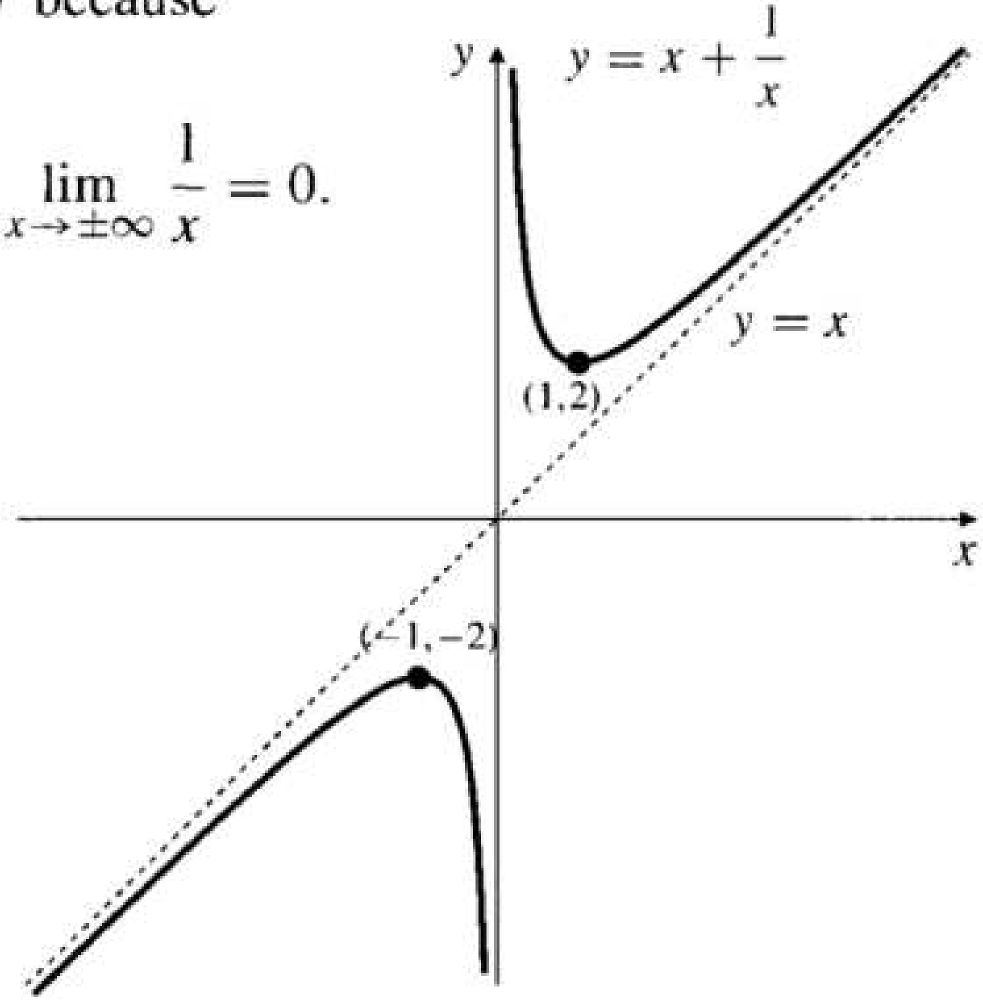
or both.

Sketching the Graph of a Function

Asymptotes

EXAMPLE Consider the function $f(x) = \frac{x^2+1}{x} = x + \frac{1}{x}$, whose graph is shown. The straight line $y = x$ is a *two-sided* oblique asymptote of the graph of f because

$$\lim_{x\to\pm\infty}\big(f(x) - x\big) = \lim_{x\to\pm\infty}\frac{1}{x} = 0.$$



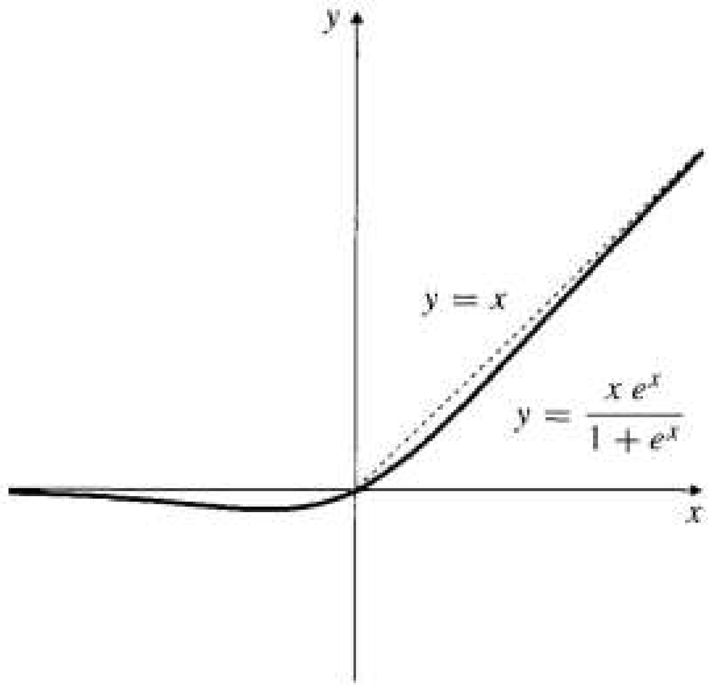
Sketching the Graph of a Function

Asymptotes

EXAMPLE The graph of $y = \dfrac{x\,e^x}{1+e^x}$ has a horizontal asymptote $y = 0$ at the left and an oblique asymptote $y = x$ at the right:

$$\lim_{x\to-\infty} \frac{x\,e^x}{1+e^x} = \frac{0}{1} = 0 \qquad \text{and}$$

$$\lim_{x\to\infty} \left(\frac{x\,e^x}{1+e^x} - x\right) = \lim_{x\to\infty} \frac{x(e^x - 1 - e^x)}{1+e^x} = \lim_{x\to\infty} \frac{-x}{1+e^x} = 0.$$



Sketching the Graph of a Function

Asymptotes

Asymptotes of a rational function

Suppose that $f(x) = \dfrac{P_m(x)}{Q_n(x)}$, where P_m and Q_n are polynomials of degree m and n, respectively. Suppose also that P_m and Q_n have no common linear factors. Then

(a) The graph of f has a vertical asymptote at every position x such that $Q_n(x) = 0$.

(b) The graph of f has a two-sided horizontal asymptote $y = 0$ if $m < n$.

(c) The graph of f has a two-sided horizontal asymptote $y = L$, $(L \neq 0)$ if $m = n$. L is the quotient of the coefficients of the highest degree terms in P_m and Q_n.

(d) The graph of f has a two-sided oblique asymptote if $m = n + 1$. This asymptote can be found by dividing Q_n into P_m to obtain a linear quotient, $ax + b$, and remainder, R, a polynomial of degree at most $n - 1$. That is,

$$f(x) = ax + b + \frac{R(x)}{Q_n(x)}.$$

The oblique asymptote is $y = ax + b$.

(e) The graph of f has no horizontal or oblique asymptotes if $m > n + 1$.

Sketching the Graph of a Function

Asymptotes

EXAMPLE Find the oblique asymptote of $y = \dfrac{x^3}{x^2 + x + 1}$.

Sketching the Graph of a Function

Asymptotes

EXAMPLE Find the oblique asymptote of $y = \dfrac{x^3}{x^2+x+1}$.

Solution We can either obtain the quotient by long division:

$$\begin{array}{r|l} x^3 \qquad\qquad & x^2 + x + 1 \\ \hline \underline{x^3 + x^2 + x} & x - 1 \\ -x^2 - x \qquad \\ \underline{-x^2 - x - 1} \\ 1 \end{array}$$

$$\frac{x^3}{x^2+x+1} = x - 1 + \frac{1}{x^2+x+1}$$

Sketching the Graph of a Function

Asymptotes

EXAMPLE Find the oblique asymptote of $y = \dfrac{x^3}{x^2 + x + 1}$.

Solution We can either obtain the quotient by long division:

$$
\begin{array}{l|l}
x^3 & x^2 + x + 1 \\ \cline{2-2}
\underline{x^3 + x^2 + x} & x \;\; - 1 \\
\quad - x^2 - x & \\
\quad \underline{- x^2 - x - 1} & \\
\qquad\qquad\quad 1 &
\end{array}
$$

$$\frac{x^3}{x^2 + x + 1} = \boxed{x - 1} + \frac{1}{x^2 + x + 1}$$

$y = x - 1$ is the oblique asymptote.

Sketching the Graph of a Function

Checklist for curve sketching

Sketching the Graph of a Function

Checklist for curve sketching

1. Calculate $f'(x)$ and $f''(x)$, and express the results in factored form.

Sketching the Graph of a Function

Checklist for curve sketching

1. Calculate $f'(x)$ and $f''(x)$, and express the results in factored form.

2. Examine $f(x)$ to determine its domain and the following items:
   (a) Any vertical asymptotes. (Look for zeros of denominators.)
   (b) Any horizontal or oblique asymptotes. (Consider $\lim_{x\to\pm\infty} f(x)$.)
   (c) Any obvious symmetry. (Is f even or odd?)
   (d) Any easily calculated intercepts (points with coordinates $(x, 0)$ or $(0, y)$) or endpoints or other "obvious" points.

Sketching the Graph of a Function

Checklist for curve sketching

3. Examine $f'(x)$ for the following:
   (a) Any critical points.
   (b) Any points where f' is not defined. (These will include singular points, endpoints of the domain of f, and vertical asymptotes.)
   (c) Intervals on which f' is positive or negative. It's a good idea to convey this information in the form of a chart such as those used in the examples. Conclusions about where f is increasing and decreasing and classification of some critical and singular points as local maxima and minima can also be indicated on the chart.

Sketching the Graph of a Function

Checklist for curve sketching

4. Examine $f''(x)$ for the following:
   (a) Points where $f''(x) = 0$.
   (b) Points where $f''(x)$ is undefined. (These will include singular points, endpoints, vertical asymptotes, and possibly other points as well, where f' is defined but f'' isn't.)
   (c) Intervals where f'' is positive or negative and where f is therefore concave up or down. Use a chart.
   (d) Any inflection points.

Sketching the Graph of a Function

EXAMPLE Sketch the graph of $y = \dfrac{x^2 + 2x + 4}{2x}$.

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		CP		ASY		CP	
x		-2		0		2	
y'	$+$	0	$-$	undef	$-$	0	$+$
y''	$-$		$-$	undef	$+$		$+$
y	$\nearrow$	max	$\searrow$	undef	$\searrow$	min	$\nearrow$
	$\frown$		$\frown$		$\smile$		$\smile$

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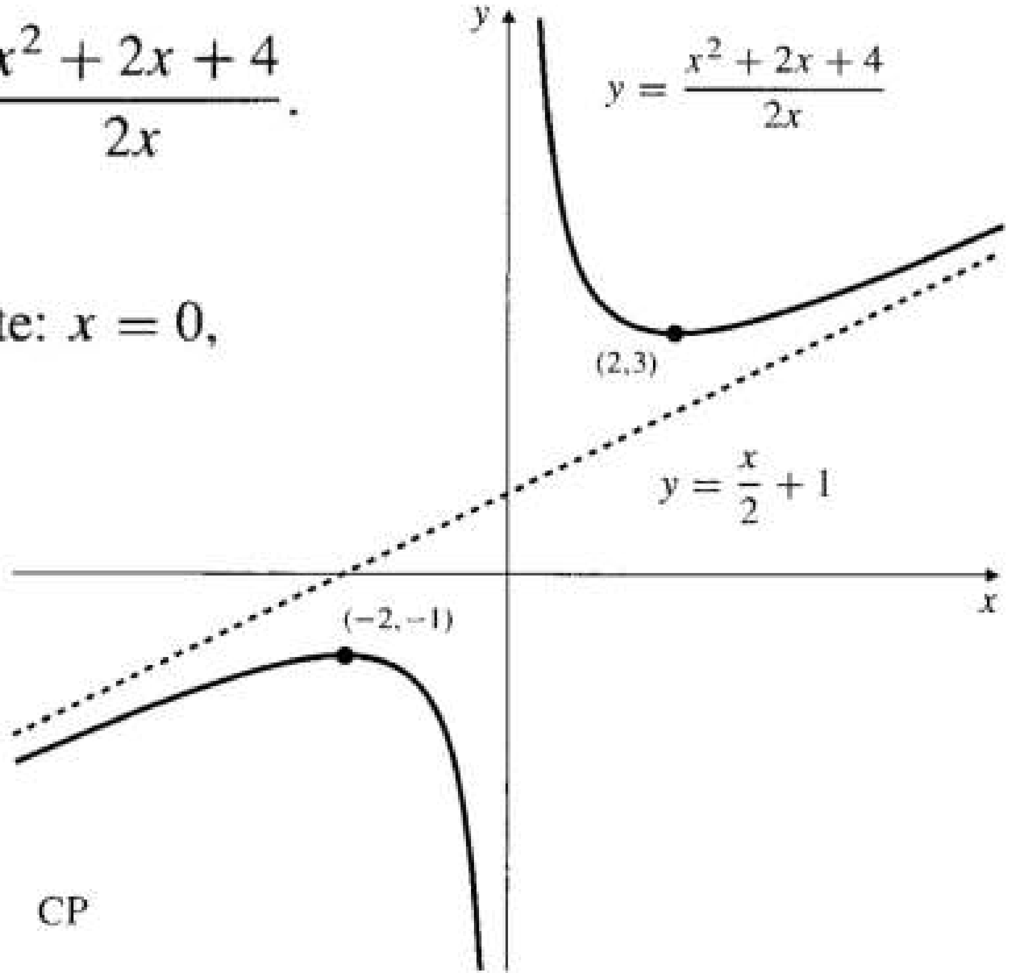
x		-2		0		2	
y'	+	0	−	undef	−	0	+
y''	−		−	undef	+		+
y	↗	max	↘	undef	↘	min	↗
	⌒		⌒		⌣		⌣



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f''	$+$	undef	$-$		$-$	undef	$+$
f	$\nearrow$	undef	$\nearrow$	max	$\searrow$	undef	$\searrow$
	$\smile$		$\frown$		$\frown$		$\smile$

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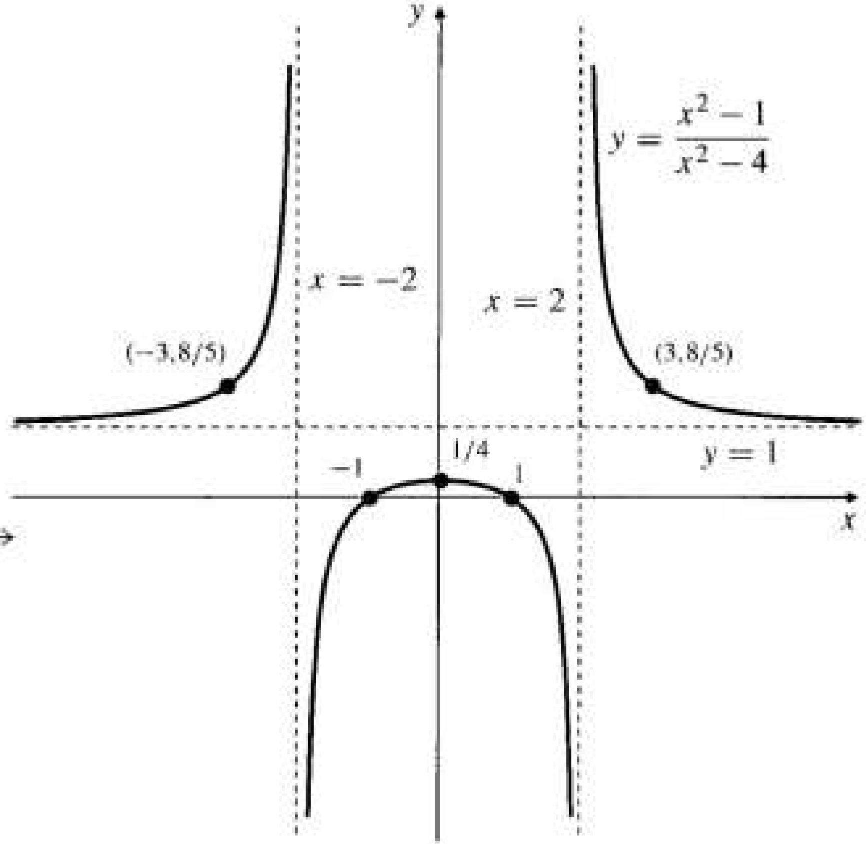
f	$\nearrow$	undef	$\nearrow$	max	$\searrow$	undef	$\searrow$
	$\smile$		$\frown$		$\frown$		$\smile$



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Critical points: $x = \pm 1$; points $(\pm 1, \pm 1/\sqrt{e}) \approx (\pm 1, \pm 0.61)$.

$y'' = 0$ at $x = 0$ and $x = \pm\sqrt{3}$;
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				CP				CP			
x		$-\sqrt{3}$		-1		0		1		$\sqrt{3}$	
y'	$-$		$-$	0	$+$		$+$	0	$-$		$-$
y''	$-$	0	$+$		$+$	0	$-$		$-$	0	$+$
y	$\searrow$		$\searrow$	min	$\nearrow$		$\nearrow$	max	$\searrow$		$\searrow$
	$\frown$	infl	$\smile$		$\smile$	infl	$\frown$		$\frown$	infl	$\smile$

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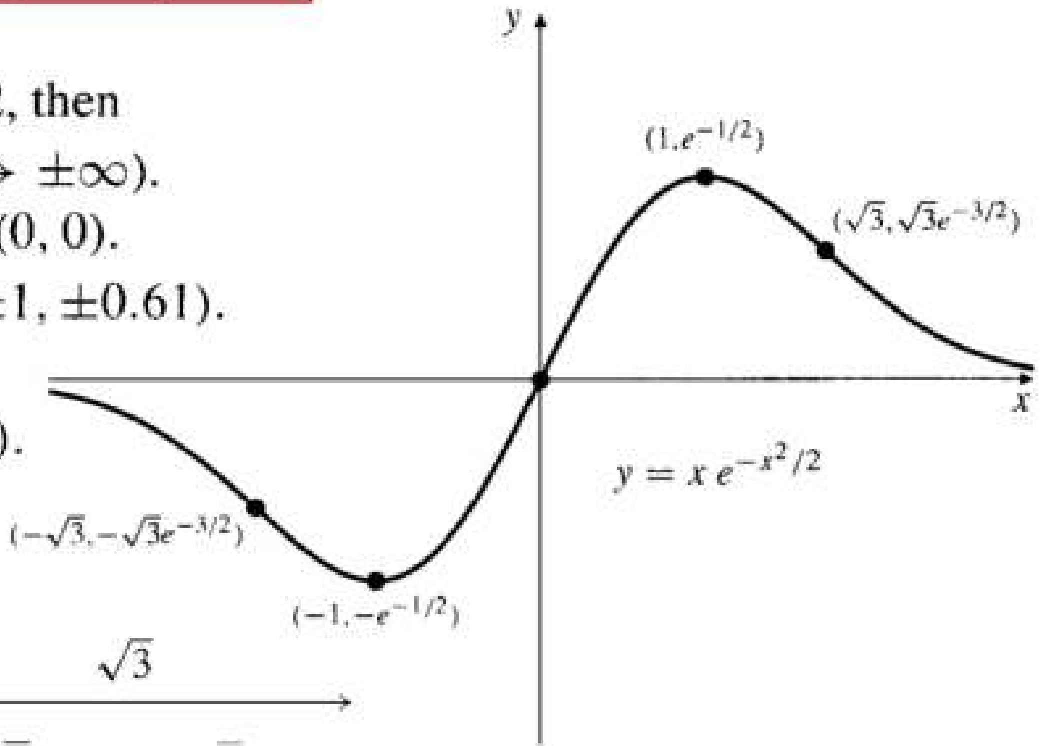

				CP				CP			
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y'	$-$		$-$	0	$+$		$+$	0	$-$		$-$
y''	$-$	0	$+$		$+$	0	$-$		$-$	0	$+$
y	$\searrow$		$\searrow$	min	$\nearrow$		$\nearrow$	max	$\searrow$		$\searrow$
	$\frown$	infl	$\smile$		$\smile$	infl	$\frown$		$\frown$	infl	$\smile$

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Solution $f'(x) = \dfrac{4}{3}\dfrac{x}{(x^2-1)^{1/3}}$, $\qquad f''(x) = \dfrac{4}{9}\dfrac{x^2-3}{(x^2-1)^{4/3}}$.

Domain: all x.
Asymptotes: none. ($f(x)$ grows like $x^{4/3}$ as $x \to \pm\infty$.)
Symmetry: about the y-axis (f is an even function).
Intercepts: $(\pm 1, 0)$, $(0, 1)$.

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Critical points: $x = 0$; singular points: $x = \pm 1$.

$f''(x) = 0$ at $x = \pm\sqrt{3}$; points $(\pm\sqrt{3}, 2^{2/3}) \approx (\pm 1.73, 1.59)$;
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				SP		CP		SP			
x		$-\sqrt{3}$		-1		0		1		$\sqrt{3}$	
f'	$-$		$-$	undef	$+$	0	$-$	undef	$+$		$+$
f''	$+$	0	$-$	undef	$-$		$-$	undef	$-$	0	$+$
f	↘		↘	min	↗	max	↘	min	↗		↗
	⌣	infl	⌢		⌢		⌢		⌢	infl	⌣

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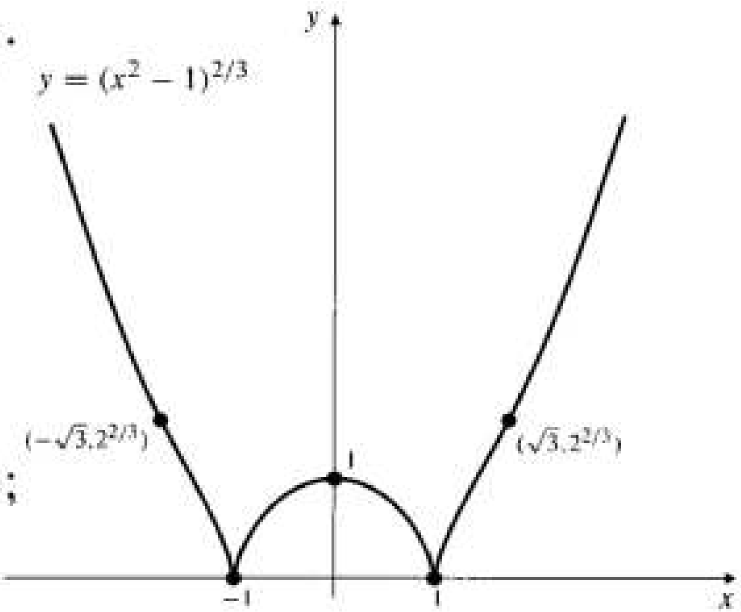

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f	$\searrow$		$\searrow$	min	$\nearrow$	max	$\searrow$	min	$\nearrow$		$\nearrow$
	$\smile$	infl	$\frown$		$\frown$		$\frown$		$\frown$	infl	$\smile$

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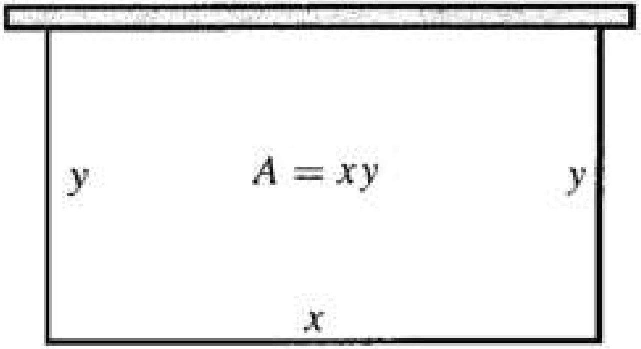

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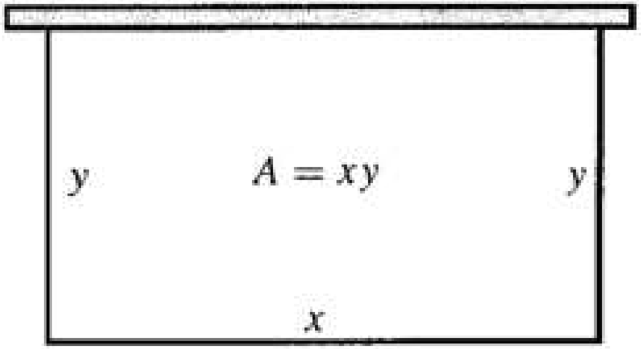

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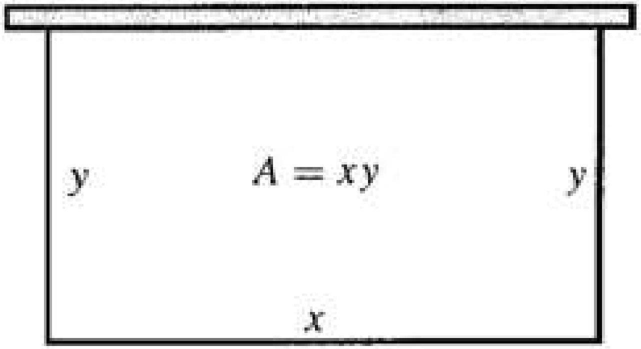

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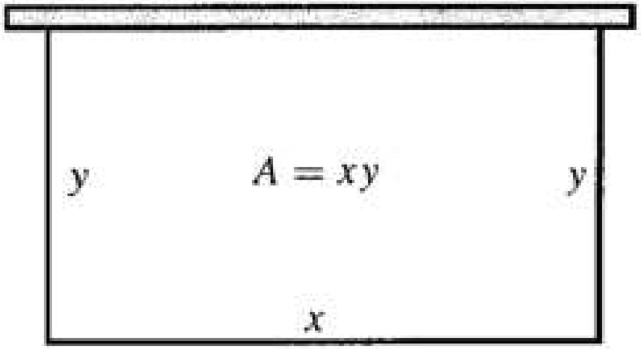

$A' = 100 - 4y = 0 \iff y = 25$

The only critical point of A is $y = 25$. $\Rightarrow$ Largest area $= A(25)$

$= 1250\ m^2$.

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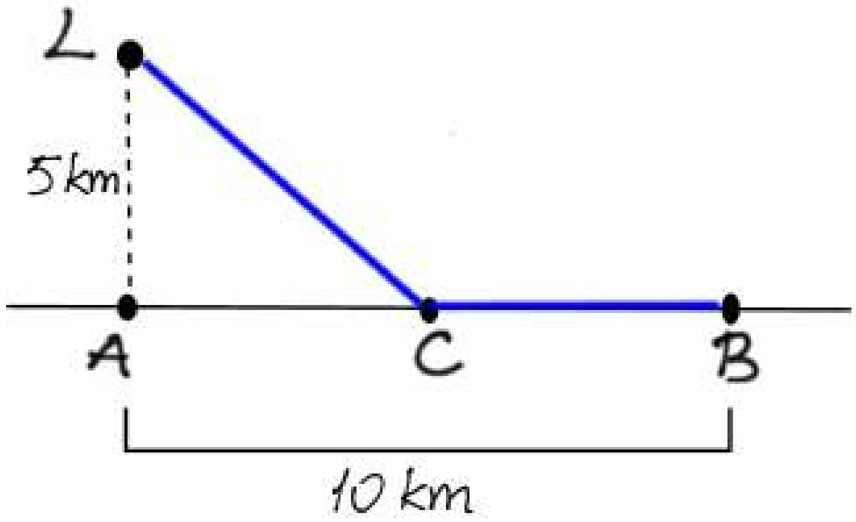

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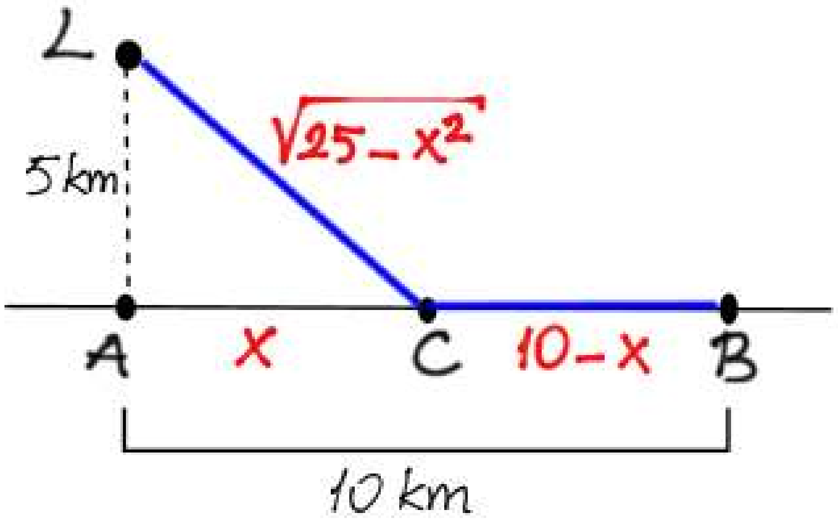

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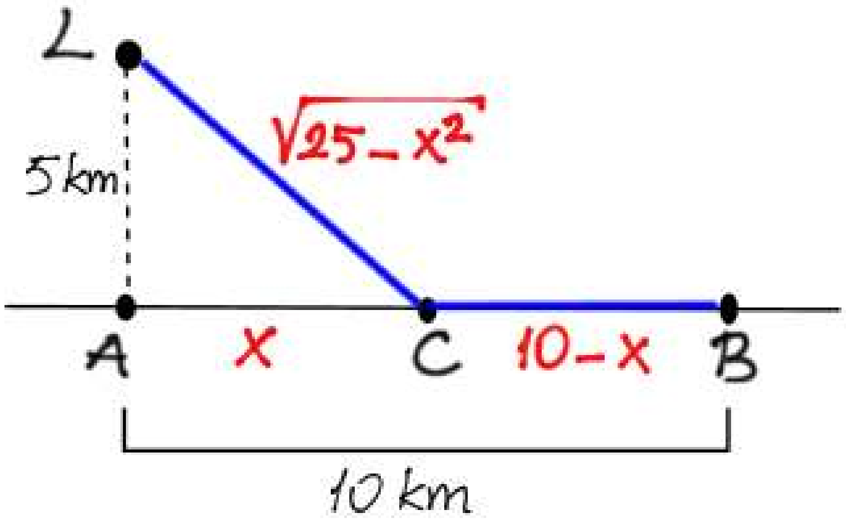

$$T = T(x) = 5{,}000\sqrt{25 + x^2} + 3{,}000(10 - x), \quad (0 \le x \le 10).$$

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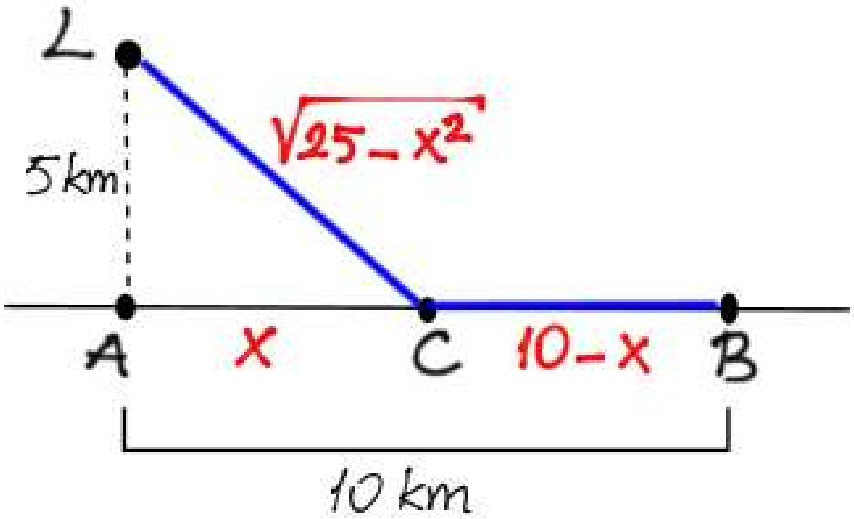

$$0 = \frac{dT}{dx} = \frac{5{,}000x}{\sqrt{25+x^2}} - 3{,}000.$$

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$$25x^2 = 9(25+x^2)$$

$$16x^2 = 225$$

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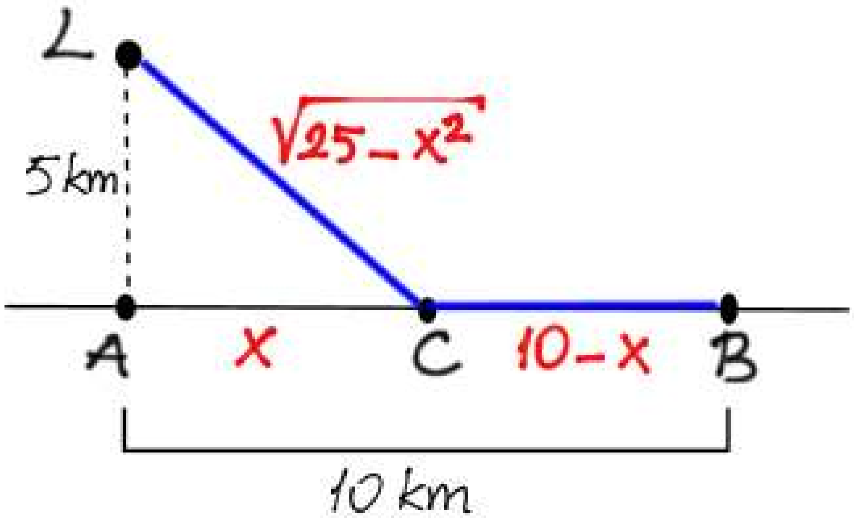

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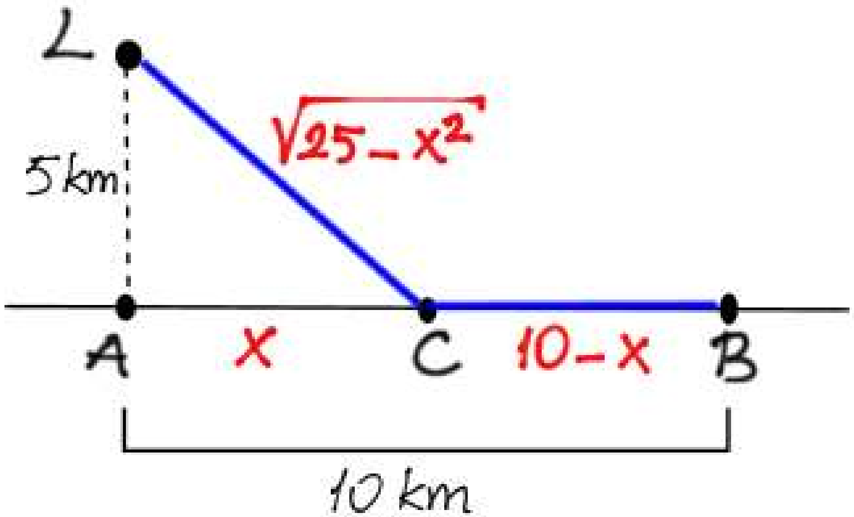

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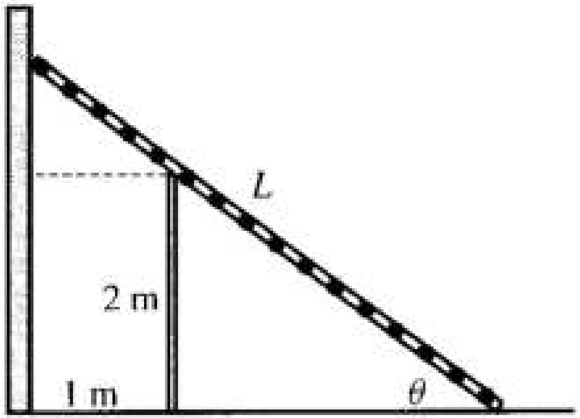

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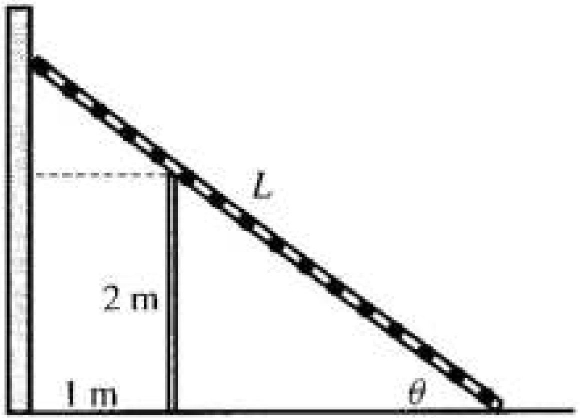

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$$L = L(\theta) = \frac{1}{\cos\theta} + \frac{2}{\sin\theta},$$

where $0 < \theta < \pi/2$.

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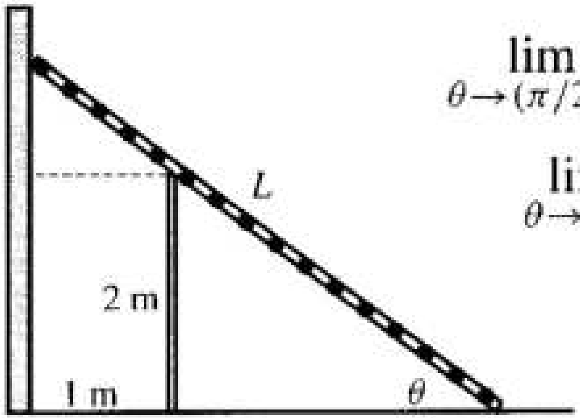

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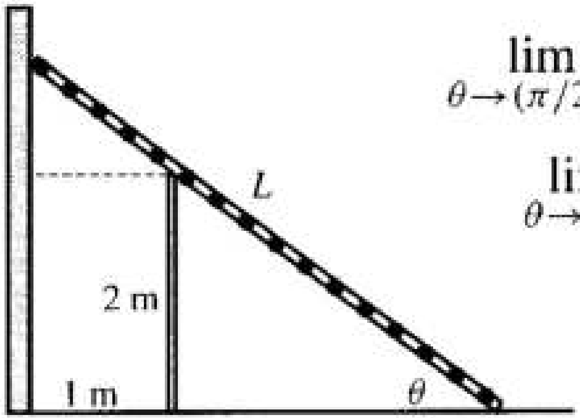

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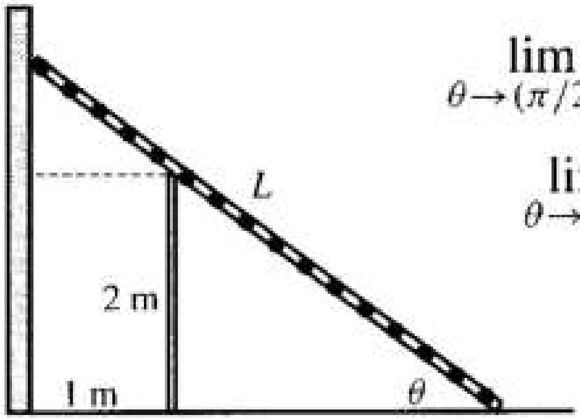

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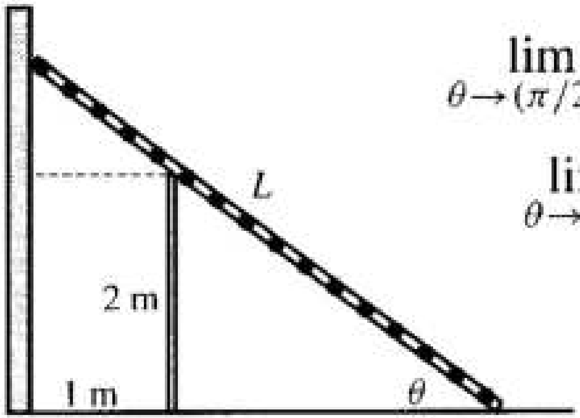

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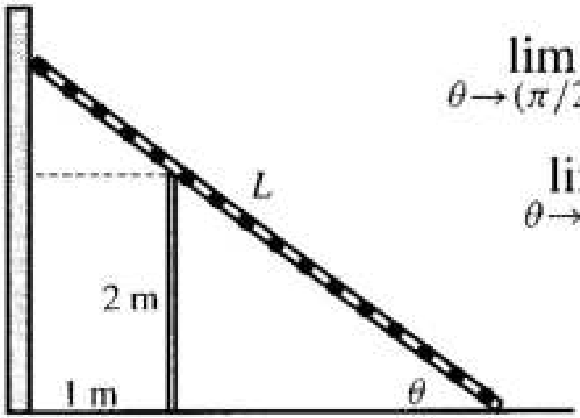

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Therefore the minimal value of $L(\theta)$ is

$$\frac{1}{\cos\theta} + \frac{2}{\sin\theta} = (1+2^{2/3})^{1/2} + 2\frac{(1+2^{2/3})^{1/2}}{2^{1/3}}$$

$$= \left(1+2^{2/3}\right)^{3/2} \approx 4.16.$$