Areas as Limits of Sums

EXAMPLE

Find the area of the region bounded by the parabola $y = x^2$ and the straight lines $y = 0$, $x = 0$, and $x = b$, where $b > 0$.

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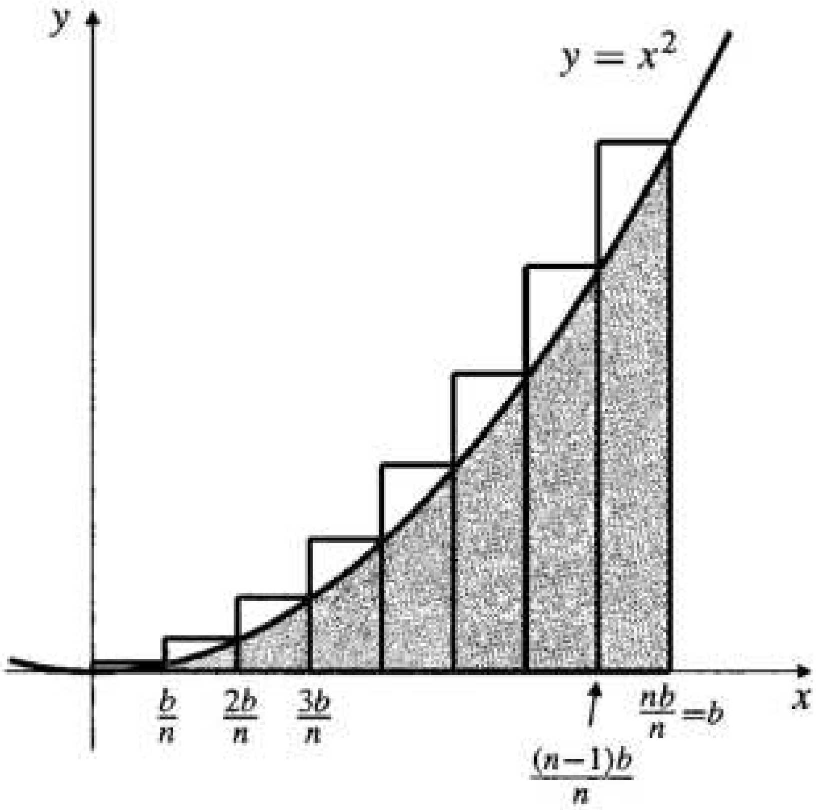

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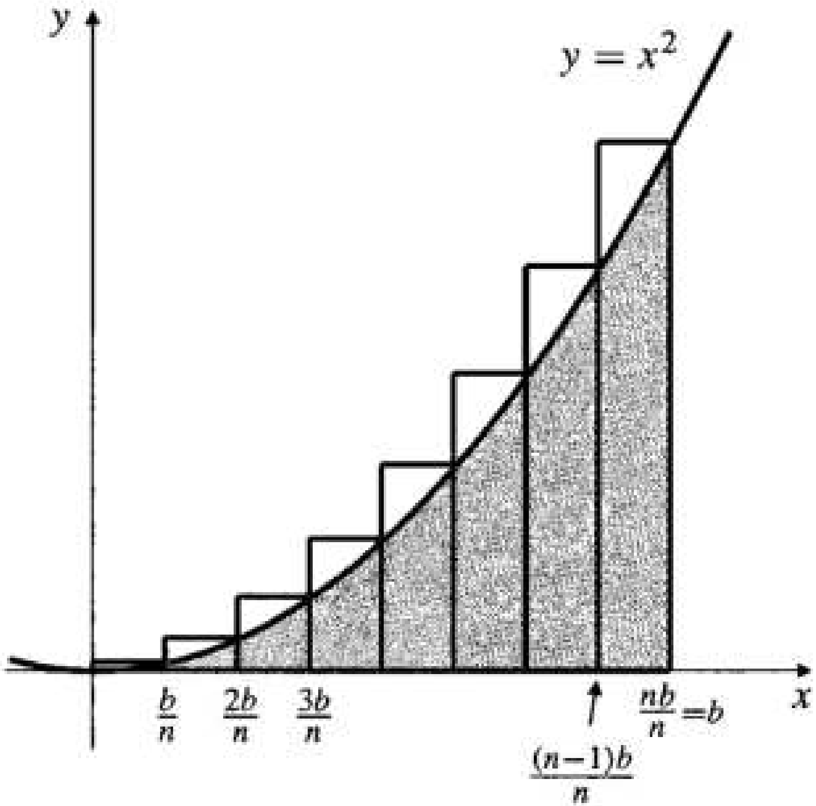

$$S_n = \sum_{i=1}^{n} \left(\frac{ib}{n}\right)^2 \frac{b}{n}$$

$$= \frac{b^3}{n^3} \sum_{i=1}^{n} i^2$$

$$= \frac{b^3}{n^3} \frac{n(n+1)(2n+1)}{6}$$

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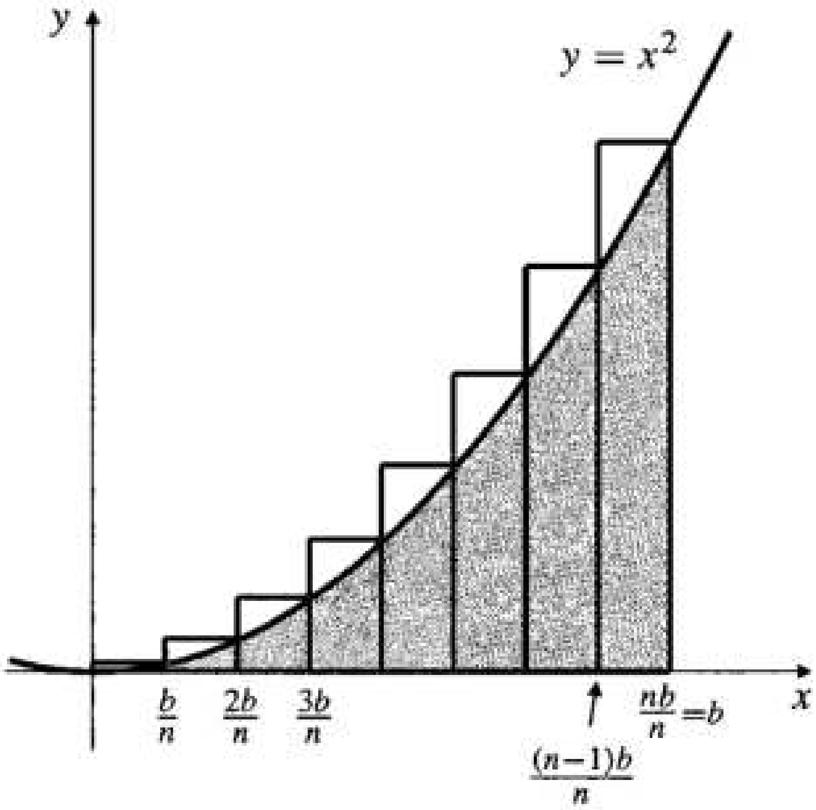

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Hence, the required area is

$$A = \lim_{n\to\infty} S_n = \lim_{n\to\infty} b^3 \frac{(n+1)(2n+1)}{6n^2} = \frac{b^3}{3} \text{ square units.}$$

The Definite Integral

Partitions

Let P be a finite set of points arranged in order between a and b on the real line, say

$$P = \{x_0, x_1, x_2, x_3, \ldots, x_{n-1}, x_n\},$$

where $a = x_0 < x_1 < x_2 < x_3 < \cdots < x_{n-1} < x_n = b$. Such a set P is called a **partition** of $[a, b]$.

The Definite Integral

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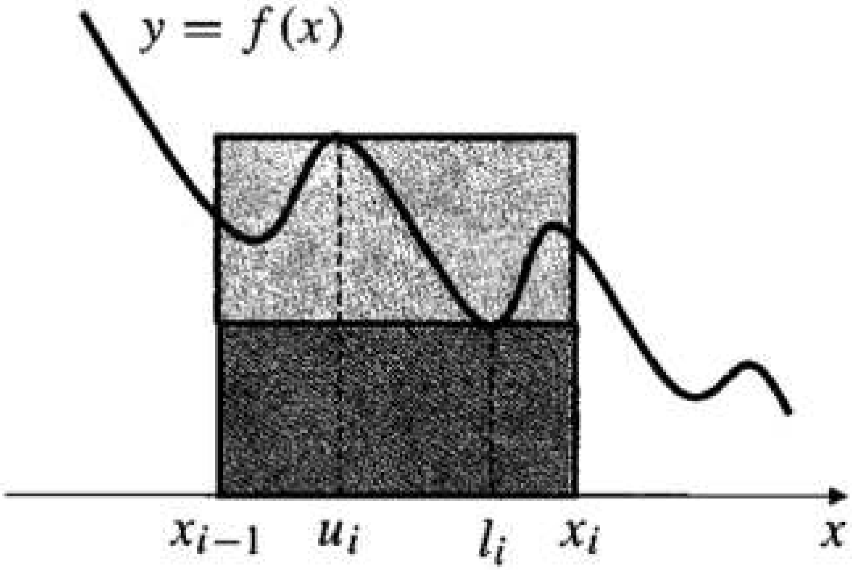
$$\Delta x_i = x_i - x_{i-1}, \qquad (\text{for } 1 \le i \le n)$$

and we call the greatest of these numbers Δx_i the **norm** of the partition P and denote it $\|P\|$:

$$\|P\| = \max_{1 \le i \le n} \Delta x_i.$$

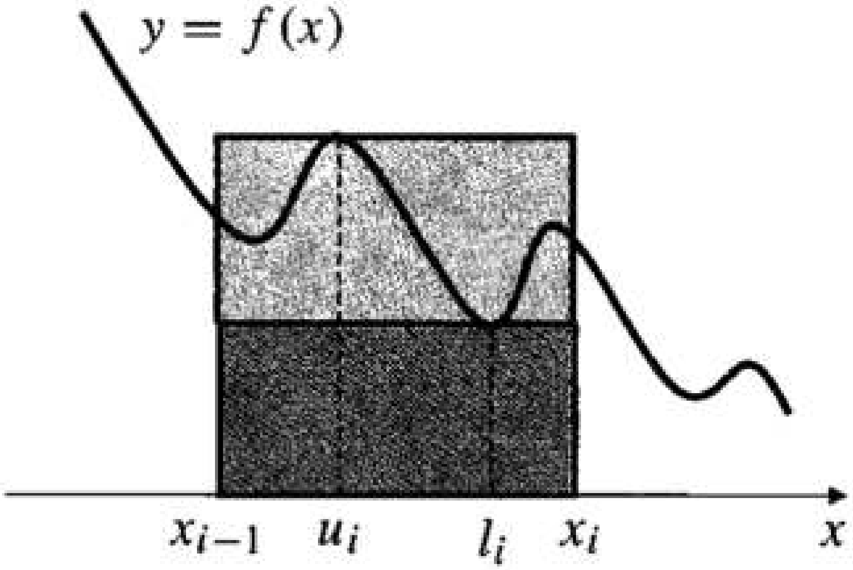
The Definite Integral

Riemann Sums



The Definite Integral

Riemann Sums

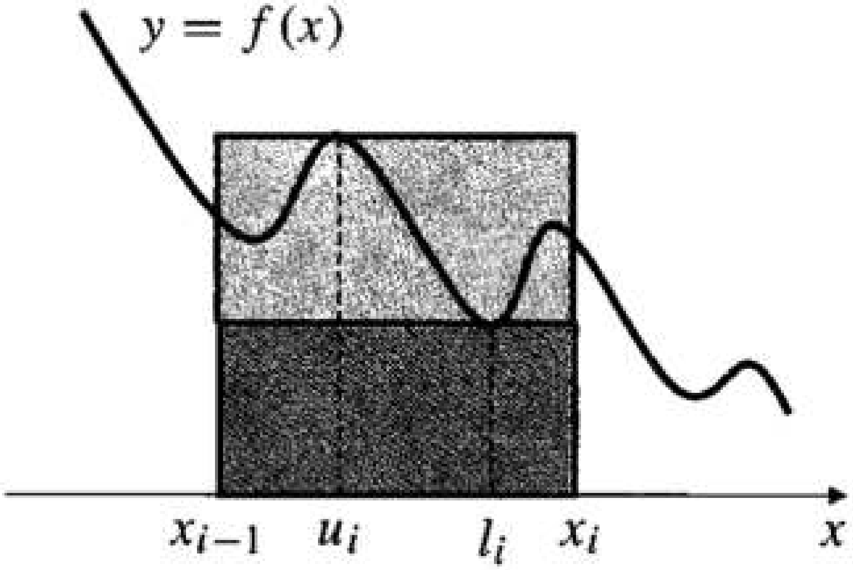


If A_i is that part of the area under $y = f(x)$ and above the x-axis that lies in the vertical strip between $x = x_{i-1}$ and $x = x_i$, then

$$f(l_i)\,\Delta x_i \le A_i \le f(u_i)\,\Delta x_i.$$

The Definite Integral

Riemann Sums



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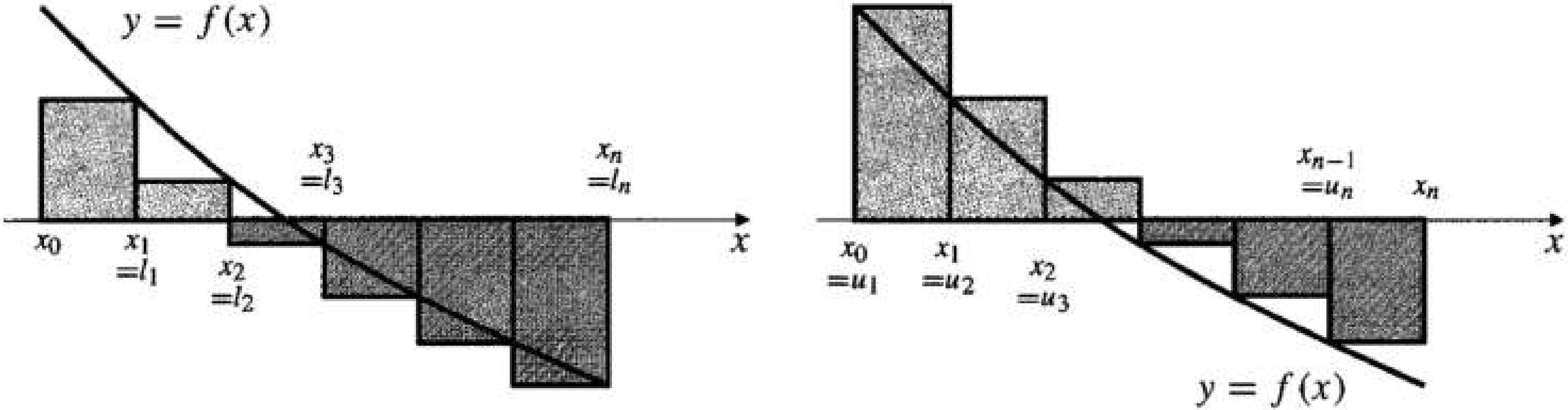
The **lower (Riemann) sum, $L(f, P)$**, and the **upper (Riemann) sum, $U(f, P)$**, for the function f and the partition P are defined by:

$$L(f, P) = f(l_1)\,\Delta x_1 + f(l_2)\,\Delta x_2 + \cdots + f(l_n)\,\Delta x_n = \sum_{i=1} f(l_i)\,\Delta x_i ,$$

$$U(f, P) = f(u_1)\,\Delta x_1 + f(u_2)\,\Delta x_2 + \cdots + f(u_n)\,\Delta x_n = \sum_{i=1}^{n} f(u_i)\Delta x_i .$$

The Definite Integral

Riemann Sums



EXAMPLE A lower Riemann sum and an upper Riemann sum for a decreasing function f.

The Definite Integral

Riemann Sums

EXAMPLE Calculate the lower and upper Riemann sums for the function $f(x) = x^2$ on the interval $[0, a]$ (where $a > 0$), corresponding to the partition P_n of $[0, a]$ into n subintervals of equal length.

The Definite Integral

Riemann Sums

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Solution Each subinterval of P_n has length $\Delta x = a/n$, and the division points are given by $x_i = ia/n$ for $i = 0, 1, 2, \ldots, n$. Since x^2 is increasing on $[0, a]$, its minimum and maximum values over the ith subinterval $[x_{i-1}, x_i]$ occur at $l_i = x_{i-1}$ and $u_i = x_i$, respectively.

The Definite Integral

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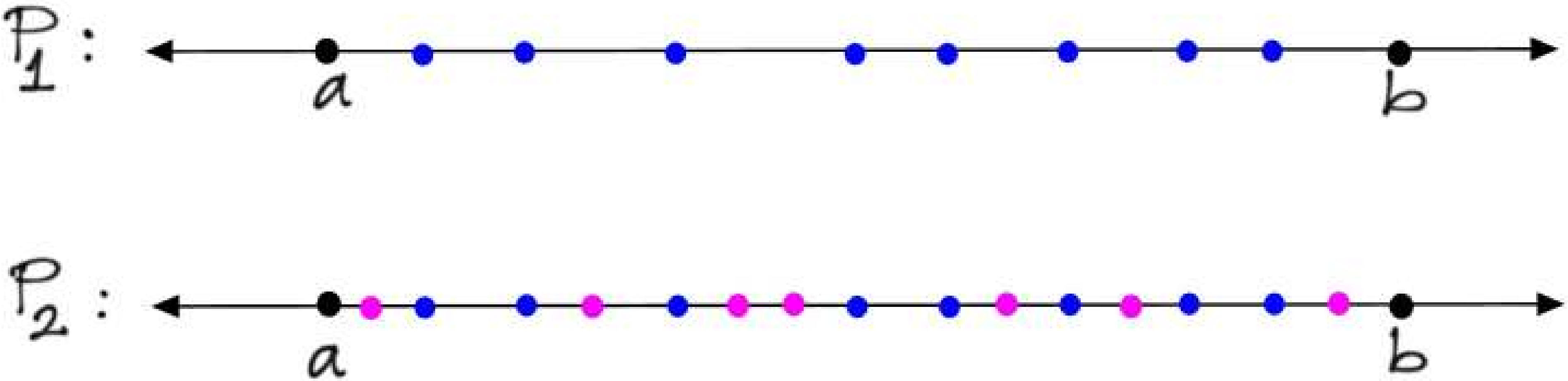
$$\begin{aligned} L(f, P_n) &= \sum_{i=1}^{n} (x_{i-1})^2 \Delta x = \frac{a^3}{n^3} \sum_{i=1}^{n} (i-1)^2 \\ &= \frac{a^3}{n^3} \sum_{j=0}^{n-1} j^2 = \frac{a^3}{n^3} \frac{(n-1)n(2(n-1)+1)}{6} = \frac{(n-1)(2n-1)a^3}{6n^2}, \end{aligned}$$

$$\begin{aligned} U(f, P_n) &= \sum_{i=1}^{n} (x_i)^2 \Delta x \\ &= \frac{a^3}{n^3} \sum_{i=1}^{n} i^2 = \frac{a^3}{n^3} \frac{n(n+1)(2n+1)}{6} = \frac{(n+1)(2n+1)a^3}{6n^2}. \end{aligned}$$

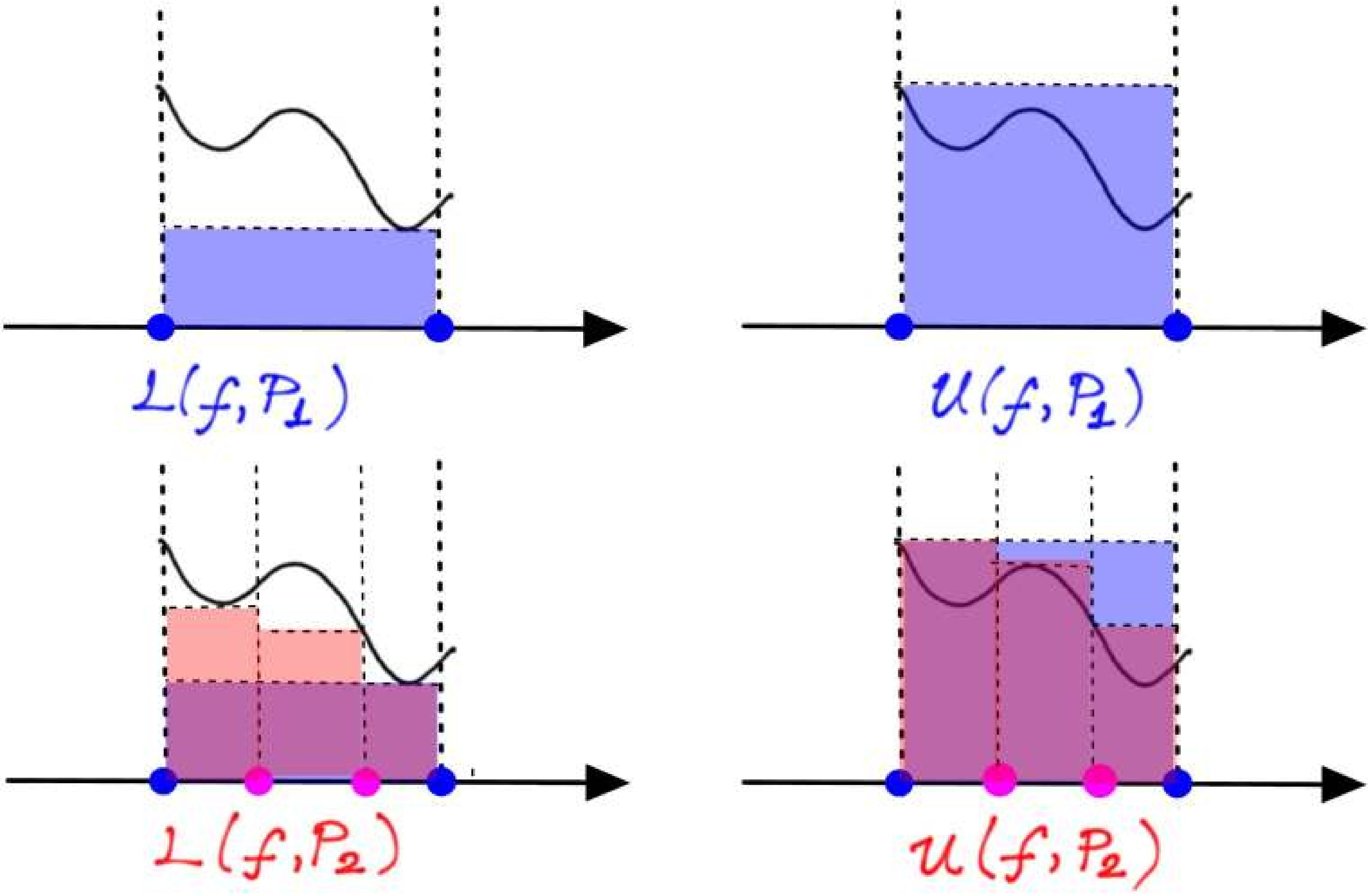
The Definite Integral

If P_1 and P_2 are two partitions of $[a, b]$ such that every point of P_1 also belongs to P_2, then we say that P_2 is a **refinement** of P_1.



P_2 is a refinement of P_1

The Definite Integral



The Definite Integral

If P_2 is a refinement of P_1, then we have

$$L(f, P_1) \leq L(f, P_2) \leq U(f, P_2) \leq U(f, P_1).$$

The Definite Integral

If P_2 is a refinement of P_1, then we have

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Also, given any two partitions P_1 and P_2, we can have a common refinement P by combining these partitions.

The Definite Integral

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$$L(f,P_1) \leq L(f,P) \leq U(f,P) \leq U(f,P_2)$$

Each lower sum is $\leq$ each upper sum!

Therefore, there is at least one number I such that

$$\mathcal{L}(f, P) \leq I \leq \mathcal{U}(f, P)$$

for any partition P.

(By the completeness of reals!)

The Definite Integral

DEFINITION

The definite integral

Suppose there is exactly one number I such that for every partition P of $[a, b]$ we have

$$L(f, P) \leq I \leq U(f, P).$$

Then we say that the function f is **integrable** on $[a, b]$, and we call I the **definite integral** of f on $[a, b]$. The definite integral is denoted by the symbol

$$I = \int_a^b f(x)\, dx.$$

The Definite Integral

The definite integral of $f(x)$ over $[a, b]$ is a *number*; it is not a function of x.

$$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt.$$

The Definite Integral

$$\underbrace{\int_a^b f(x)\,dx}_{\text{Integral of } f \text{ from } a \text{ to } b}$$

The Definite Integral

Integral sign

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The Definite Integral

Upper limit of integration

Integral sign

$$\underbrace{\int_a^b f(x)\,dx}_{\text{Integral of } f \text{ from } a \text{ to } b}$$

Lower limit of integration

The Definite Integral

The function is the integrand.

Upper limit of integration

x is the variable of integration.

Integral sign

$$\underbrace{\int_a^b f(x)\,dx}_{\text{Integral of } f \text{ from } a \text{ to } b}$$

Lower limit of integration

The Definite Integral

EXAMPLE

Show that $f(x) = x^2$ is integrable over the interval $[0, a]$, where $a > 0$, and evaluate $\int_0^a x^2\,dx$.

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Solution

$$\lim_{n\to\infty} L(f, P_n) = \lim_{n\to\infty} \frac{(n-1)(2n-1)a^3}{6n^2} = \frac{a^3}{3},$$

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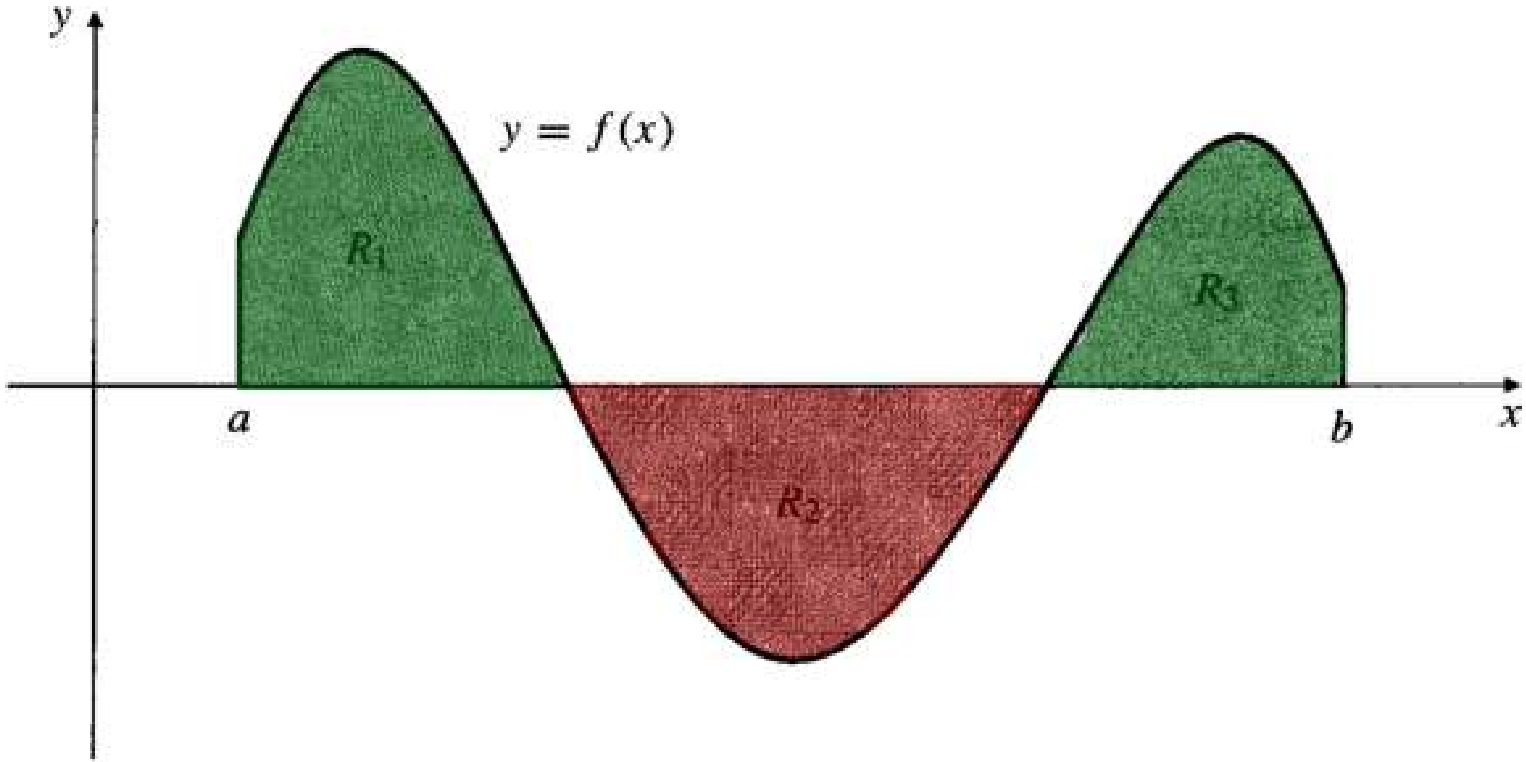
If $L(f, P_n) \le I \le U(f, P_n)$, we must have $I = a^3/3$. Thus, $f(x) = x^2$ is integrable over $[0, a]$, and

$$\int_0^a f(x)\,dx = \int_0^a x^2\,dx = \frac{a^3}{3}.$$

The Definite Integral



$$\int_a^b f(x)\,dx \text{ equals area } R_1 - \text{area } R_2 + \text{area } R_3$$

The Definite Integral

General Riemann Sums

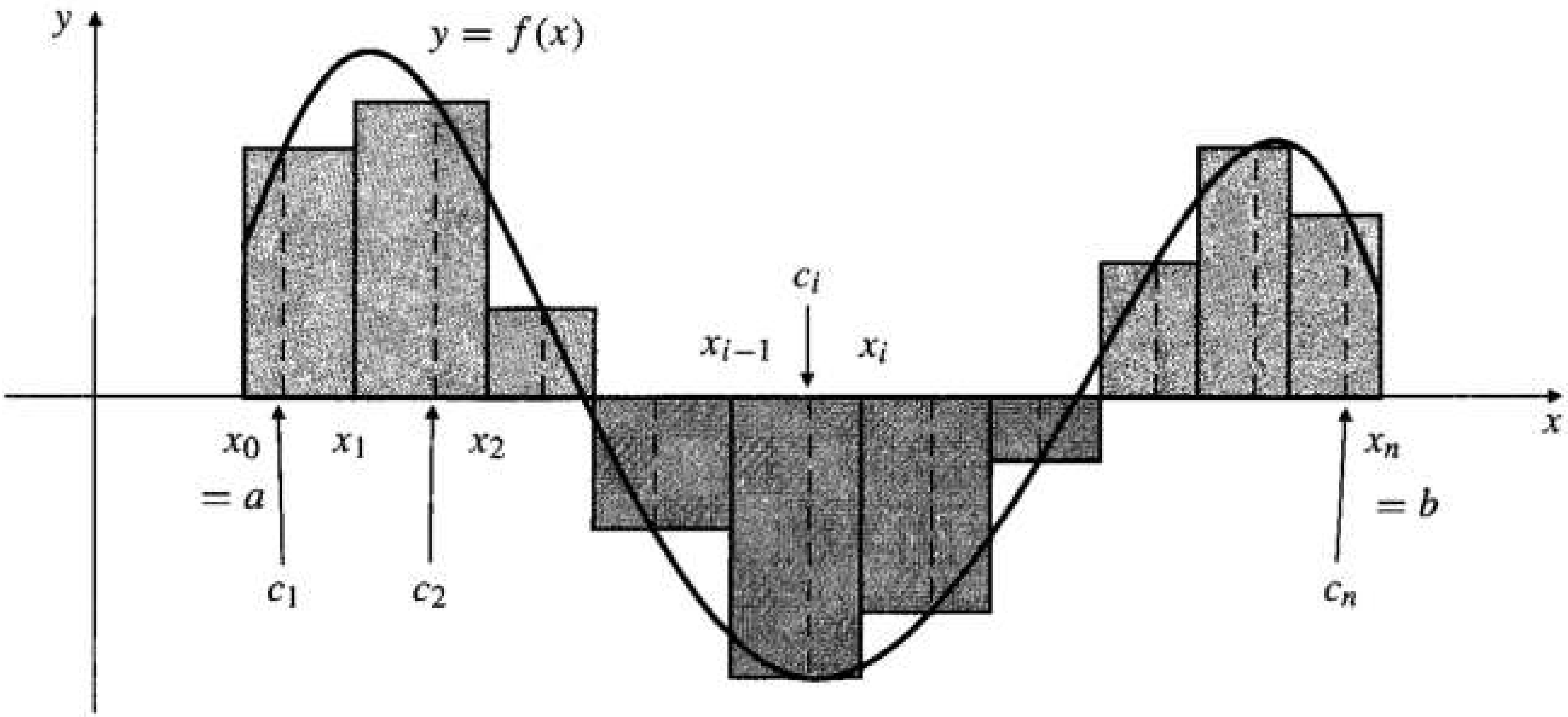
Let $P = \{x_0, x_1, x_2, \ldots, x_n\}$, where $a = x_0 < x_1 < x_2 < \cdots < x_n = b$, be a partition of $[a, b]$ having norm $\|P\| = \max_{1 \le i \le n} \Delta x_i$. In each subinterval $[x_{i-1}, x_i]$ of P pick a point c_i (called a *tag*). Let $c = (c_1, c_2, \ldots, c_n)$ denote the set of these tags. The sum

$$\begin{aligned} R(f, P, c) &= \sum_{i=1}^{n} f(c_i)\,\Delta x_i \\ &= f(c_1)\,\Delta x_1 + f(c_2)\,\Delta x_2 + f(c_3)\,\Delta x_3 + \cdots + f(c_n)\,\Delta x_n \end{aligned}$$

is called the **Riemann sum** of f on $[a, b]$ corresponding to partition P and tags c.

The Definite Integral

General Riemann Sums



The Definite Integral

General Riemann Sums

$$L(f, P) \le R(f, P, c) \le U(f, P).$$

if f is integrable
on $[a, b]$

$$\lim_{\substack{n(P)\to\infty \\ \|P\|\to 0}} R(f, P, c) = \int_a^b f(x)\,dx.$$

The Definite Integral

THEOREM

If f is continuous on $[a, b]$, then f is integrable on $[a, b]$.

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EXAMPLE Express the limit $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n}\left(1 + \frac{2i-1}{n}\right)^{1/3}$ as a definite integral.

Properties of the Definite Integral

Let f and g be integrable on an interval containing the points a, b, and c. Then

(a) An integral over an interval of zero length is zero.

$$\int_a^a f(x)\,dx = 0.$$

(b) Reversing the limits of integration changes the sign of the integral.

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx.$$

(c) An integral depends linearly on the integrand. If A and B are constants, then

$$\int_a^b \big(Af(x) + Bg(x)\big)\,dx = A\int_a^b f(x)\,dx + B\int_a^b g(x)\,dx.$$

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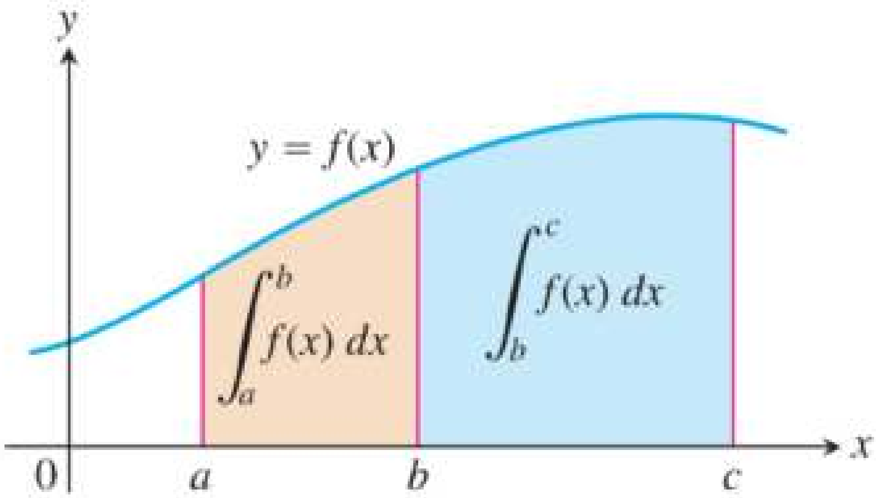
Properties of the Definite Integral

(d) An integral depends additively on the interval of integration.

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx.$$



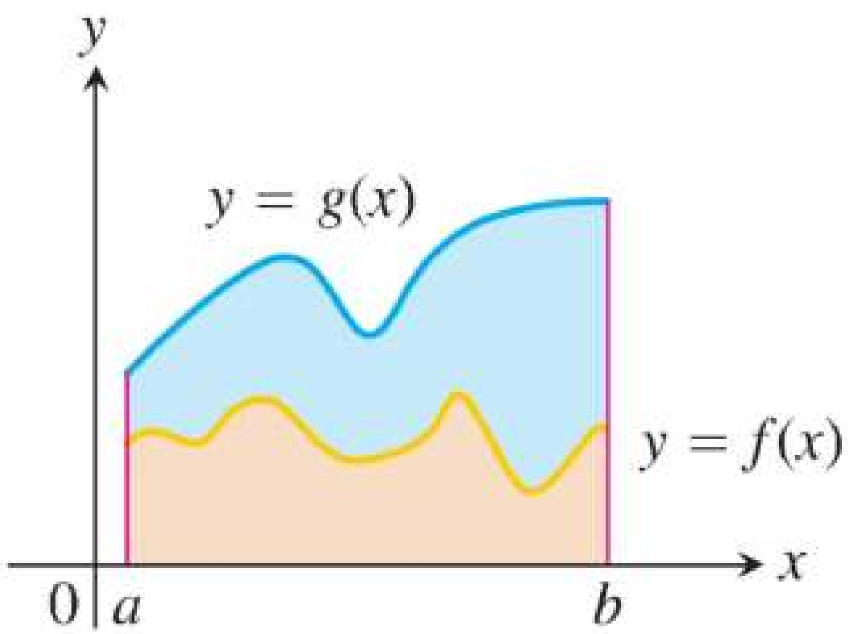
Additivity for definite integrals:

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$

Properties of the Definite Integral

(e) If $a \leq b$ and $f(x) \leq g(x)$ for $a \leq x \leq b$, then

$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx.$$



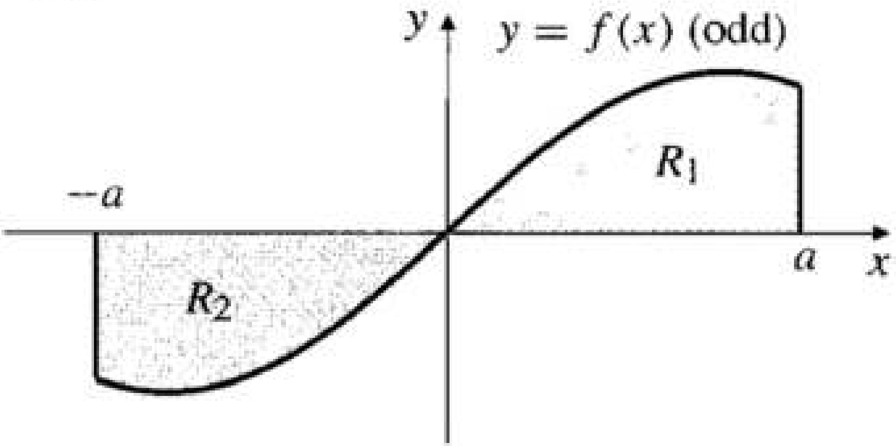
Properties of the Definite Integral

(f) The **triangle inequality** for sums extends to definite integrals. If $a \leq b$, then

$$\left| \int_a^b f(x)\,dx \right| \leq \int_a^b |f(x)|\,dx.$$

(g) The integral of an odd function over an interval symmetric about zero is zero. If f is an odd function (i.e., $f(-x) = -f(x)$), then

$$\int_{-a}^{a} f(x)\,dx = 0.$$

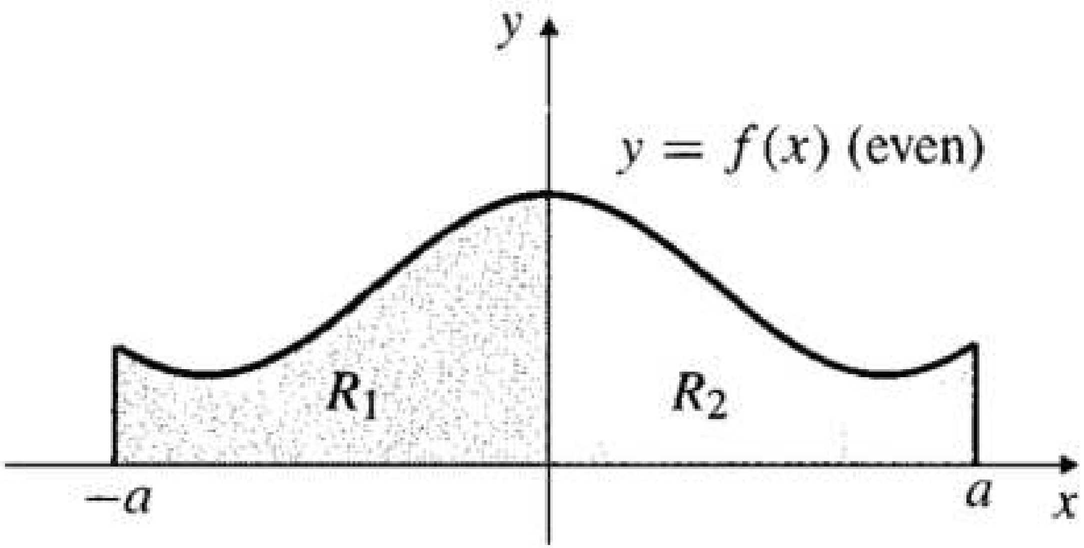


area R_1 − area $R_2 = 0$

Properties of the Definite Integral

(h) The integral of an even function over an interval symmetric about zero is twice the integral over the positive half of the interval. If f is an even function (i.e., $f(-x) = f(x)$), then

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx.$$



area R_1 + area R_2 = 2 area R_2

Properties of the Definite Integral

EXAMPLE Evaluate

$$\text{(a)} \int_{-2}^{2} (2+5x)\,dx, \quad \text{(b)} \int_{0}^{3} (2+x)\,dx, \quad \text{and} \quad \text{(c)} \int_{-3}^{3} \sqrt{9-x^2}\,dx.$$

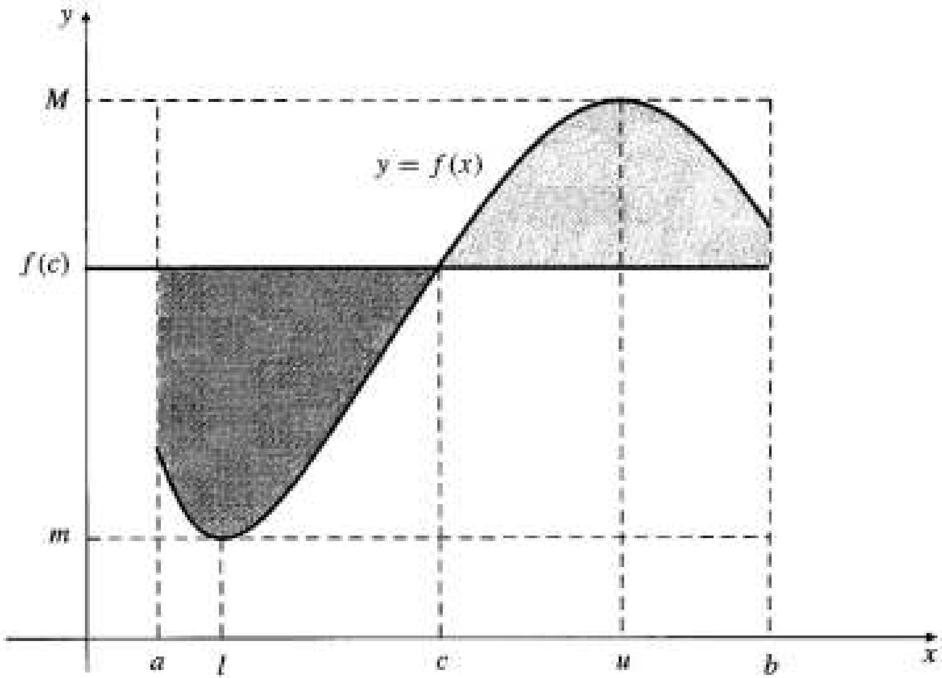
Properties of the Definite Integral

A Mean-Value Theorem for Integrals

THEOREM **The Mean-Value Theorem for integrals**

If f is continuous on $[a, b]$, then there exists a point c in $[a, b]$ such that

$$\int_a^b f(x)\,dx = (b-a)f(c).$$



Properties of the Definite Integral

DEFINITION

Average value of a function

If f is integrable on $[a, b]$, then the **average value** or **mean value** of f on $[a, b]$, denoted by $\bar{f}$, is

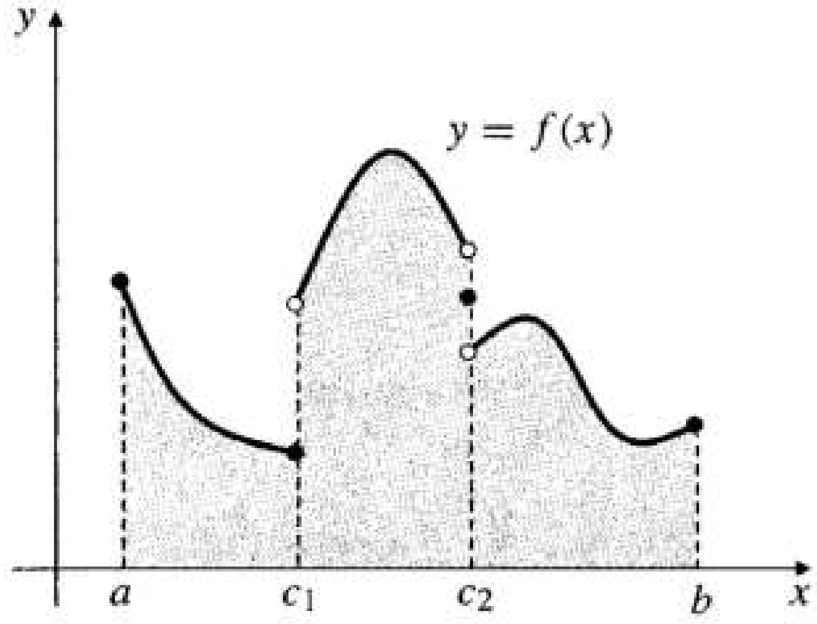
$$\bar{f} = \frac{1}{b-a} \int_a^b f(x)\, dx.$$

EXAMPLE Find the average value of $f(x) = 2x$ on the interval $[1, 5]$.

Piecewise continuous functions

Let $c_0 < c_1 < c_2 < \cdots < c_n$ be a finite set of points on the real line. A function f defined on $[c_0, c_n]$ except possibly at some of the points c_i, $(0 \le i \le n)$, is called **piecewise continuous** on that interval if for each i $(1 \le i \le n)$ there exists a function F_i continuous on the *closed* interval $[c_{i-1}, c_i]$ such that

$$f(x) = F_i(x) \qquad \text{on the } open \text{ interval} \quad (c_{i-1}, c_i).$$



In this case,

Properties of the Definite Integral

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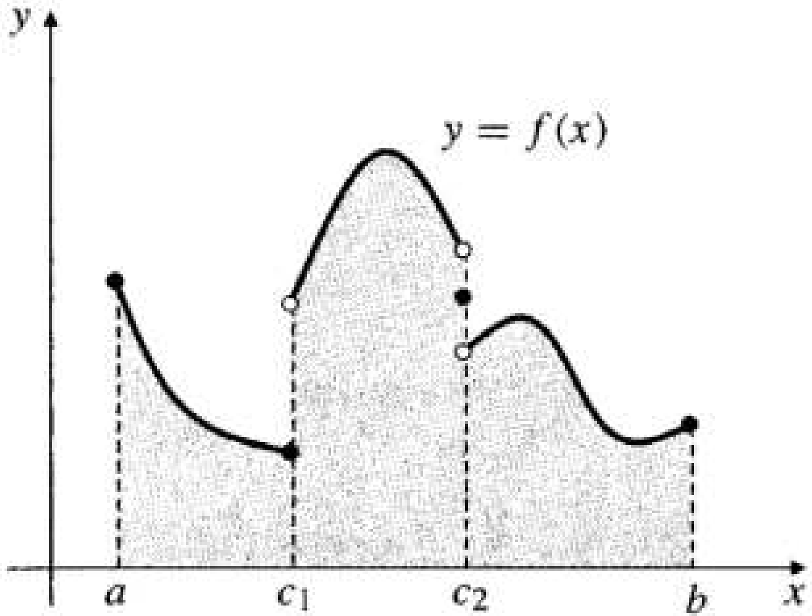
$$f(x) = F_i(x) \qquad \text{on the } open \text{ interval} \quad (c_{i-1}, c_i).$$

In this case,

$$\boxed{\int_{c_0}^{c_n} f(x)\,dx = \sum_{i=1}^{n} \int_{c_{i-1}}^{c_i} F_i(x)\,dx.}$$



Properties of the Definite Integral

EXAMPLE Find $\int_0^3 f(x)\,dx$, where $f(x) = \begin{cases} \sqrt{1-x^2} & \text{if } 0 \le x \le 1 \\ 2 & \text{if } 1 < x \le 2 \\ x-2 & \text{if } 2 < x \le 3. \end{cases}$