The Mean-Value Theorem

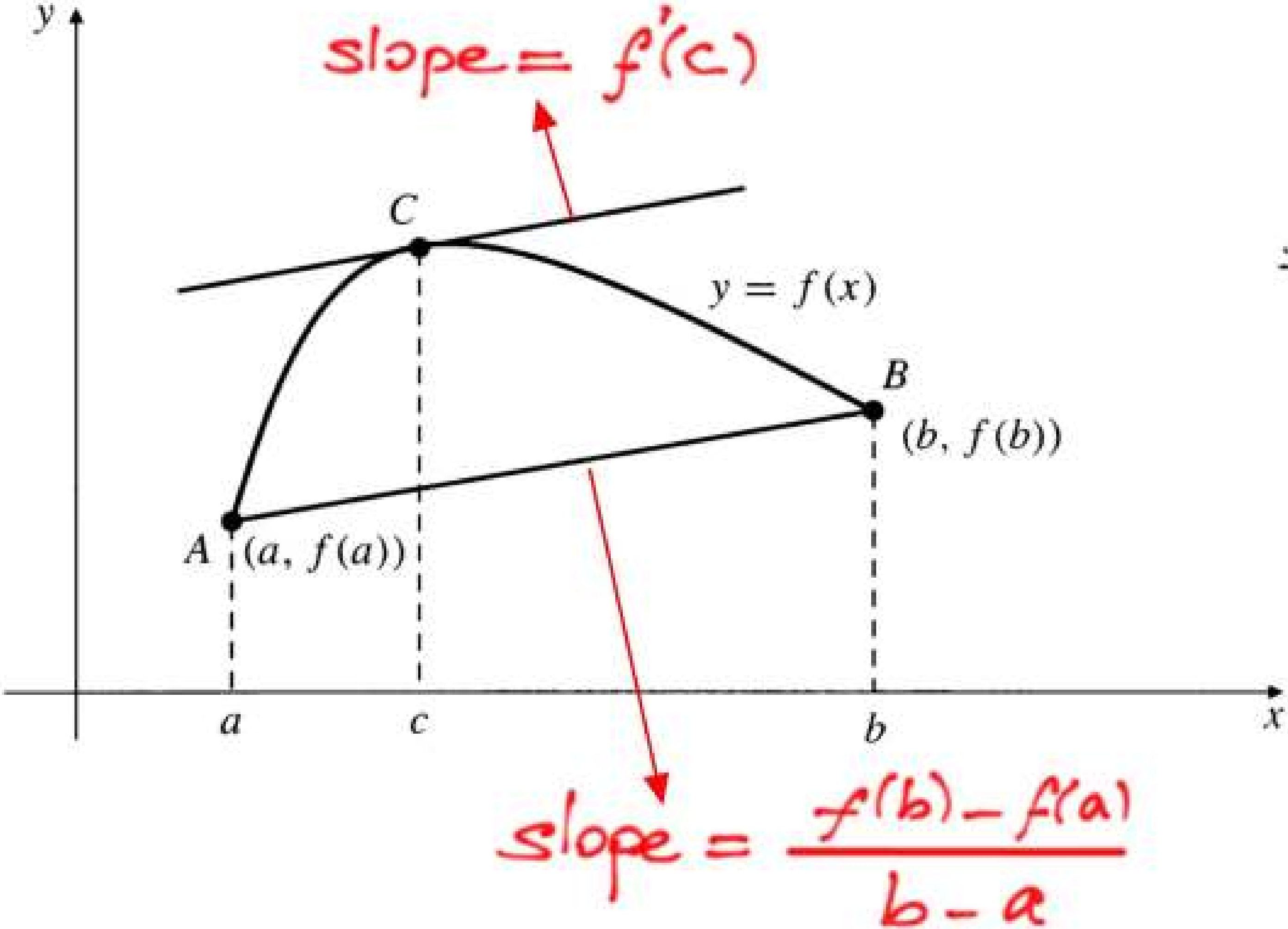
THEOREM **The Mean-Value Theorem**

Suppose that the function f is continuous on the closed, finite interval $[a, b]$ and that it is differentiable on the open interval (a, b). Then there exists a point c in the open interval (a, b) such that

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

This says that the slope of the chord line joining the points $(a, f(a))$ and $(b, f(b))$ is equal to the slope of the tangent line to the curve $y = f(x)$ at the point $(c, f(c))$, so the two lines are parallel.

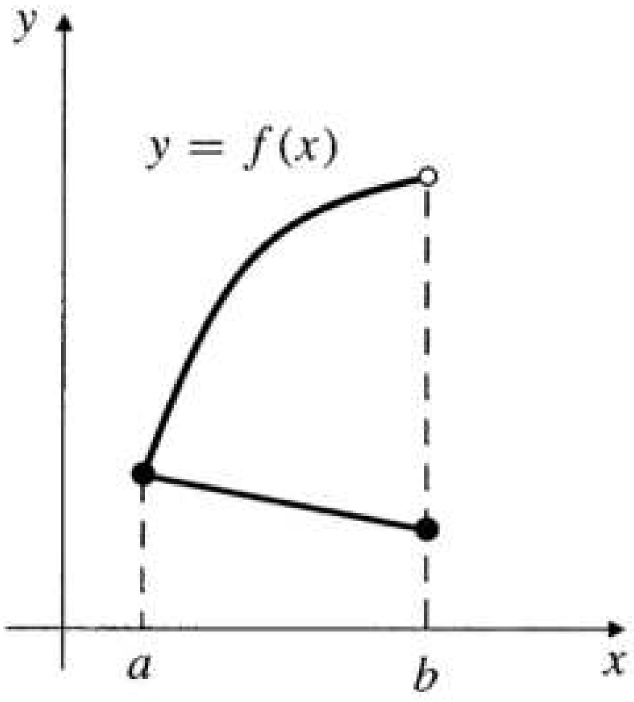
The Mean-Value Theorem



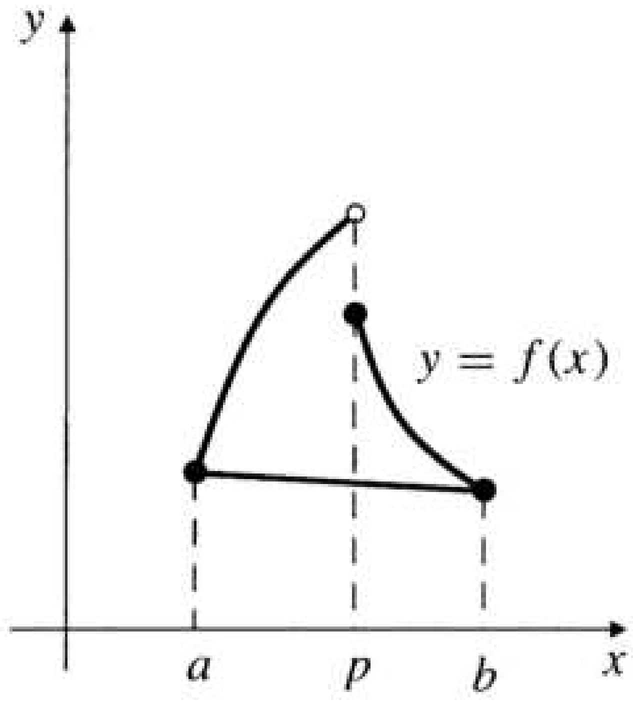
$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

The Mean-Value Theorem

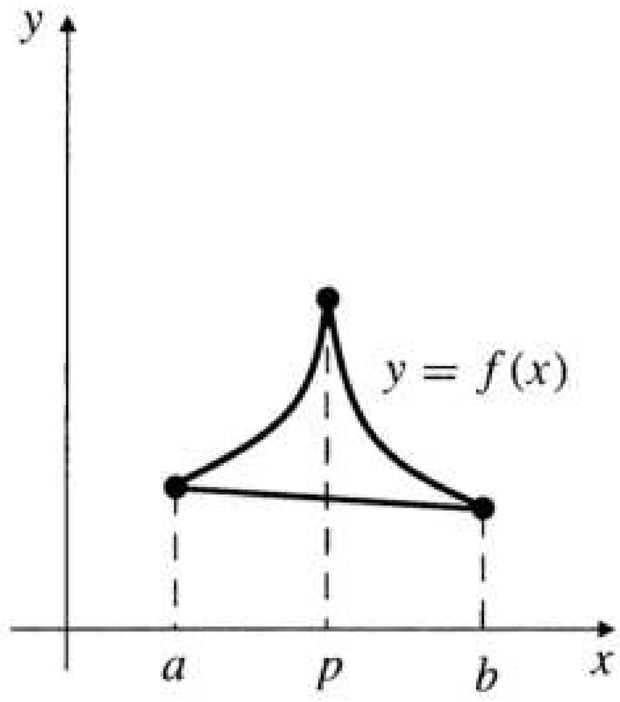
Functions that fail to satisfy the hypotheses of the Mean-Value Theorem and for which the conclusion is false:



f is discontinuous at endpoint b



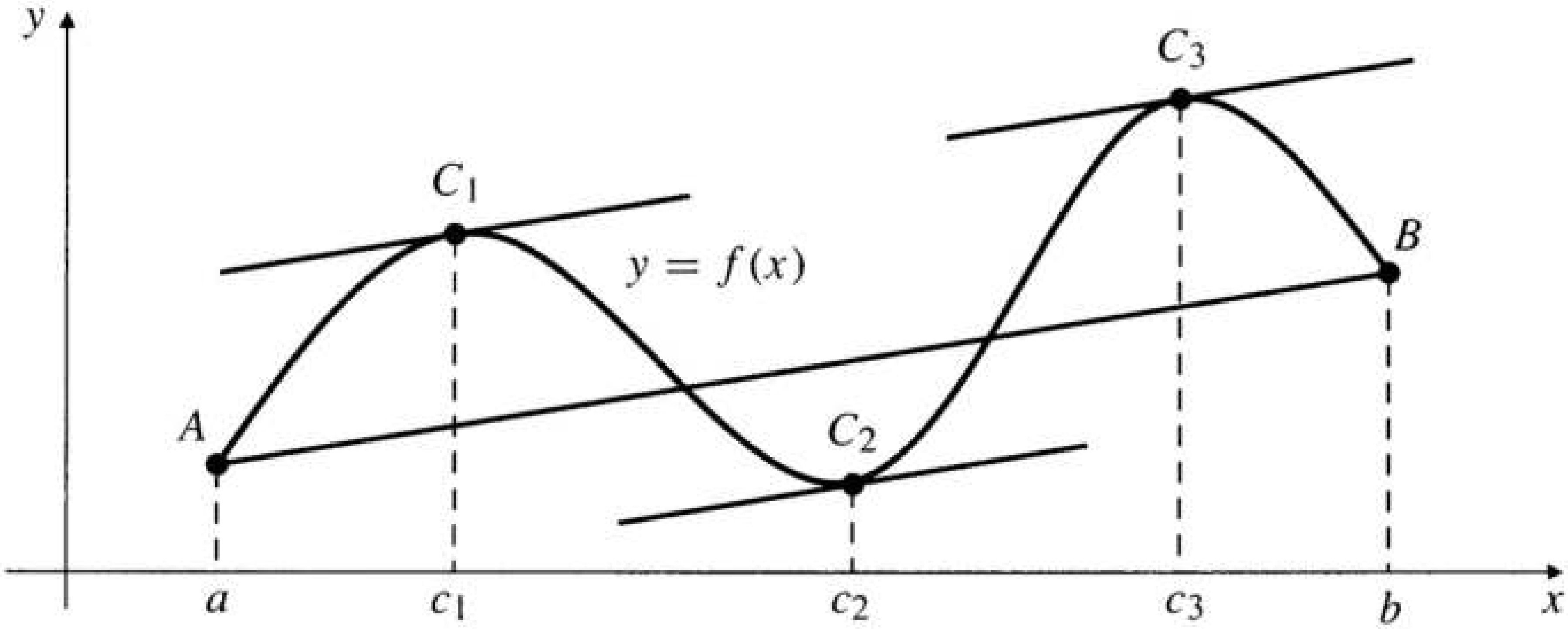
f is discontinuous at p



f is not differentiable at p

The Mean-Value Theorem

The Mean-Value Theorem gives no indication of how many points C there may be on the curve between A and B where the tangent is parallel to AB.



The Mean-Value Theorem

EXAMPLE Show that $\sin x < x$ for all $x > 0$.

EXAMPLE Show that $\sqrt{1+x} < 1 + \frac{x}{2}$ for $x > 0$ and for $-1 \le x < 0$.

The Mean-Value Theorem

Mathematical Consequences

COROLLARY If $f'(x) = 0$ at each point x of an open interval (a, b), then $f(x) = C$ for all $x \in (a, b)$, where C is a constant.

The Mean-Value Theorem

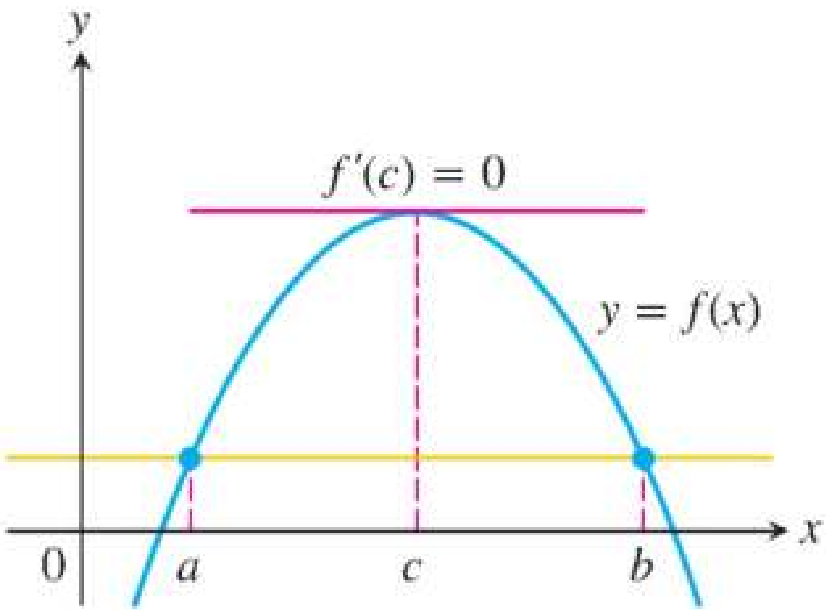
Mathematical Consequences

COROLLARY If $f'(x) = 0$ at each point x of an open interval (a, b), then $f(x) = C$ for all $x \in (a, b)$, where C is a constant.

COROLLARY If $f'(x) = g'(x)$ at each point x in an open interval (a, b), then there exists a constant C such that $f(x) = g(x) + C$ for all $x \in (a, b)$. That is, $f - g$ is a constant function on (a, b).

The Mean-Value Theorem

THEOREM —Rolle's Theorem Suppose that $y = f(x)$ is continuous at every point of the closed interval $[a, b]$ and differentiable at every point of its interior (a, b). If $f(a) = f(b)$, then there is at least one number c in (a, b) at which $f'(c) = 0$.



The Mean-Value Theorem

EXAMPLE show that the equation

$$x^3 + 3x + 1 = 0$$

has exactly one solution.

The Mean-Value Theorem

THEOREM **The Generalized Mean-Value Theorem**

If functions f and g are both continuous on $[a, b]$ and differentiable on (a, b), and if $g'(x) \neq 0$ for every x in (a, b), then there exists a number c in (a, b) such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$

The Mean-Value Theorem

THEOREM **The Generalized Mean-Value Theorem**

If functions f and g are both continuous on $[a, b]$ and differentiable on (a, b), and if $g'(x) \neq 0$ for every x in (a, b), then there exists a number c in (a, b) such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$

PROOF Note that $g(b) \neq g(a)$; otherwise, there would be some number in (a, b) where $g' = 0$. Hence, neither denominator above can be zero. Apply the Mean-Value Theorem to

$$h(x) = \big(f(b) - f(a)\big)\big(g(x) - g(a)\big) - \big(g(b) - g(a)\big)\big(f(x) - f(a)\big).$$

Since $h(a) = h(b) = 0$, there exists c in (a, b) such that $h'(c) = 0$. Thus,

$$\big(f(b) - f(a)\big)g'(c) - \big(g(b) - g(a)\big)f'(c) = 0,$$

and the result follows on division by the g factors.

The Mean-Value Theorem

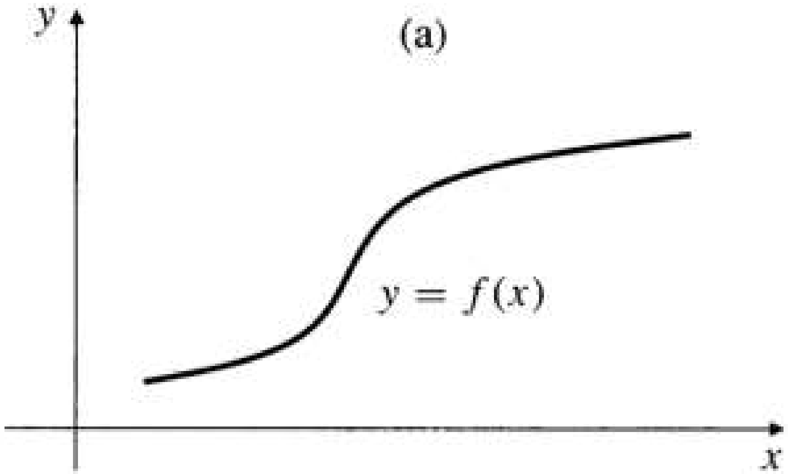
Increasing and Decreasing Functions

DEFINITION

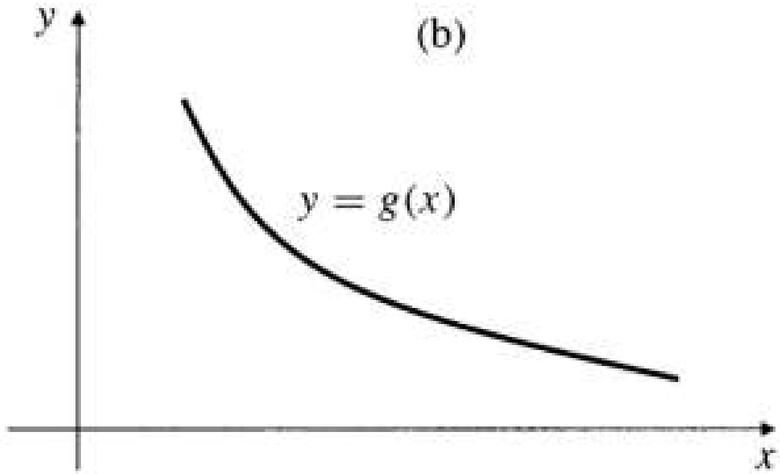
Suppose that the function f is defined on an interval I and that x_1 and x_2 are two points of I.

(a) If $f(x_2) > f(x_1)$ whenever $x_2 > x_1$, we say f is **increasing** on I.
(b) If $f(x_2) < f(x_1)$ whenever $x_2 > x_1$, we say f is **decreasing** on I.
(c) If $f(x_2) \geq f(x_1)$ whenever $x_2 > x_1$, we say f is **nondecreasing** on I.
(d) If $f(x_2) \leq f(x_1)$ whenever $x_2 > x_1$, we say f is **nonincreasing** on I.

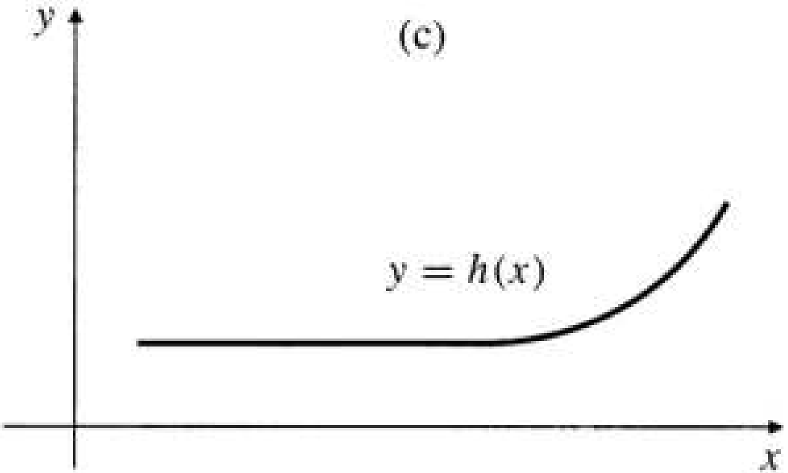
The Mean-Value Theorem



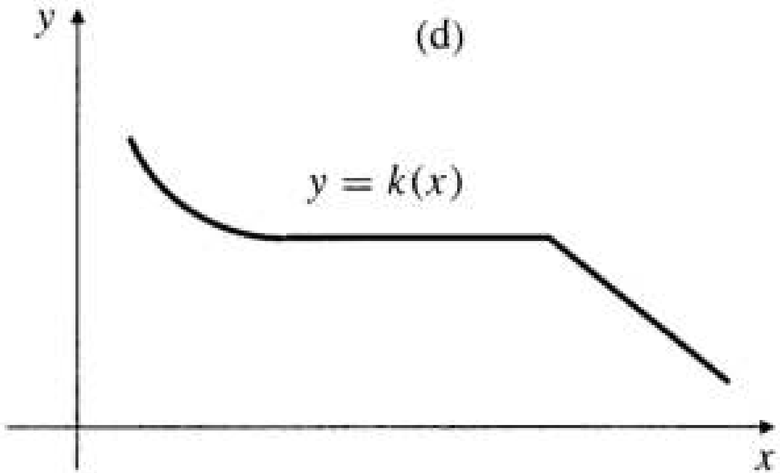
Function f is increasing



Function g is decreasing



Function h is nondecreasing



Function k is nonincreasing

The Mean-Value Theorem

Increasing and Decreasing Functions

THEOREM

Let J be an open interval, and let I be an interval consisting of all the points in J and possibly one or both of the endpoints of J. Suppose that f is continuous on I and differentiable on J.

(a) If $f'(x) > 0$ for all x in J, then f is increasing on I.

(b) If $f'(x) < 0$ for all x in J, then f is decreasing on I.

(c) If $f'(x) \geq 0$ for all x in J, then f is nondecreasing on I.

(d) If $f'(x) \leq 0$ for all x in J, then f is nonincreasing on I.

The Mean-Value Theorem

Increasing and Decreasing Functions

EXAMPLE On what intervals is the function $f(x) = x^3 - 12x + 1$ increasing? On what intervals is it decreasing?

The Mean-Value Theorem

Increasing and Decreasing Functions

EXAMPLE On what intervals is the function $f(x) = x^3 - 12x + 1$ increasing? On what intervals is it decreasing?

EXAMPLE Show that $f(x) = x^3$ is increasing on any interval.

Implicit Differentiation

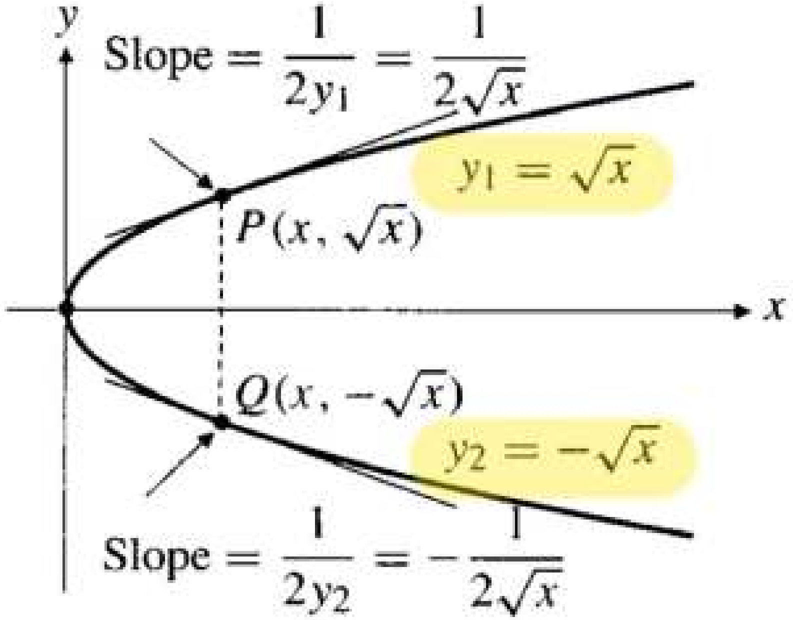
Curves are generally the graphs of *equations* in two variables. Such equations can be written in the form

$$F(x, y) = 0,$$

where $F(x, y)$ denotes an expression involving the two variables x and y. For example,

Implicit Differentiation

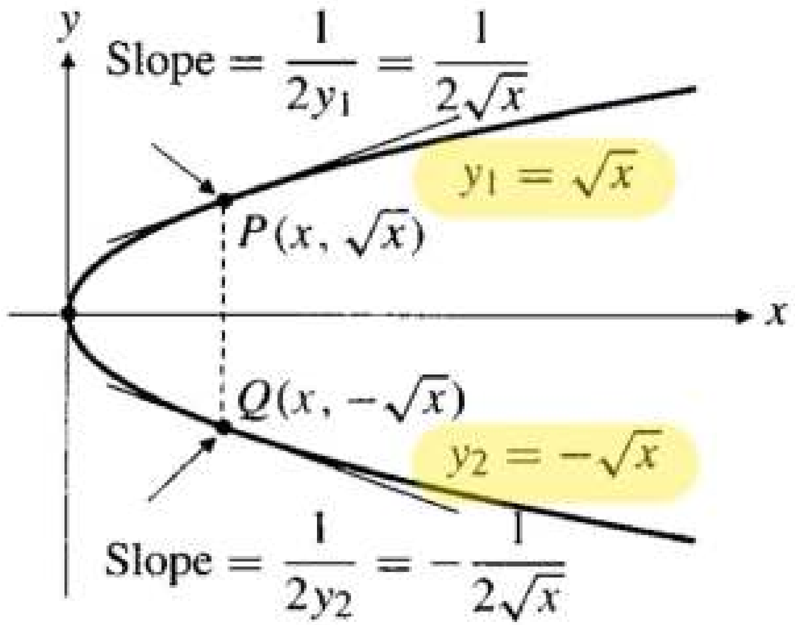
EXAMPLE Find dy/dx if $y^2 = x$.



The equation $y^2 = x$ defines two differentiable functions of x on the interval $x \geq 0$

Implicit Differentiation

EXAMPLE Find dy/dx if $y^2 = x$.



The equation $y^2 = x$ defines two differentiable functions of x on the interval $x \geq 0$

differentiating implicitly

$$\frac{d}{dx}(y^2) = \frac{d}{dx}(x)$$

$$2y\frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{2y}.$$

Implicit Differentiation

EXAMPLE Find the slope of circle $x^2 + y^2 = 25$ at the point $(3, -4)$.

Implicit Differentiation

EXAMPLE Find the slope of circle $x^2 + y^2 = 25$ at the point $(3, -4)$.

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = \frac{d}{dx}(25)$$

$$2x + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{x}{y}.$$

Implicit Differentiation

EXAMPLE Find the slope of circle $x^2 + y^2 = 25$ at the point $(3, -4)$.

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = \frac{d}{dx}(25)$$

$$2x + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{x}{y}.$$

The slope at $(3, -4)$ is $-\left.\frac{x}{y}\right|_{(3,-4)} = -\frac{3}{-4} = \frac{3}{4}$.

Implicit Differentiation

EXAMPLE Find $\frac{dy}{dx}$ if $y \sin x = x^3 + \cos y$.

Implicit Differentiation

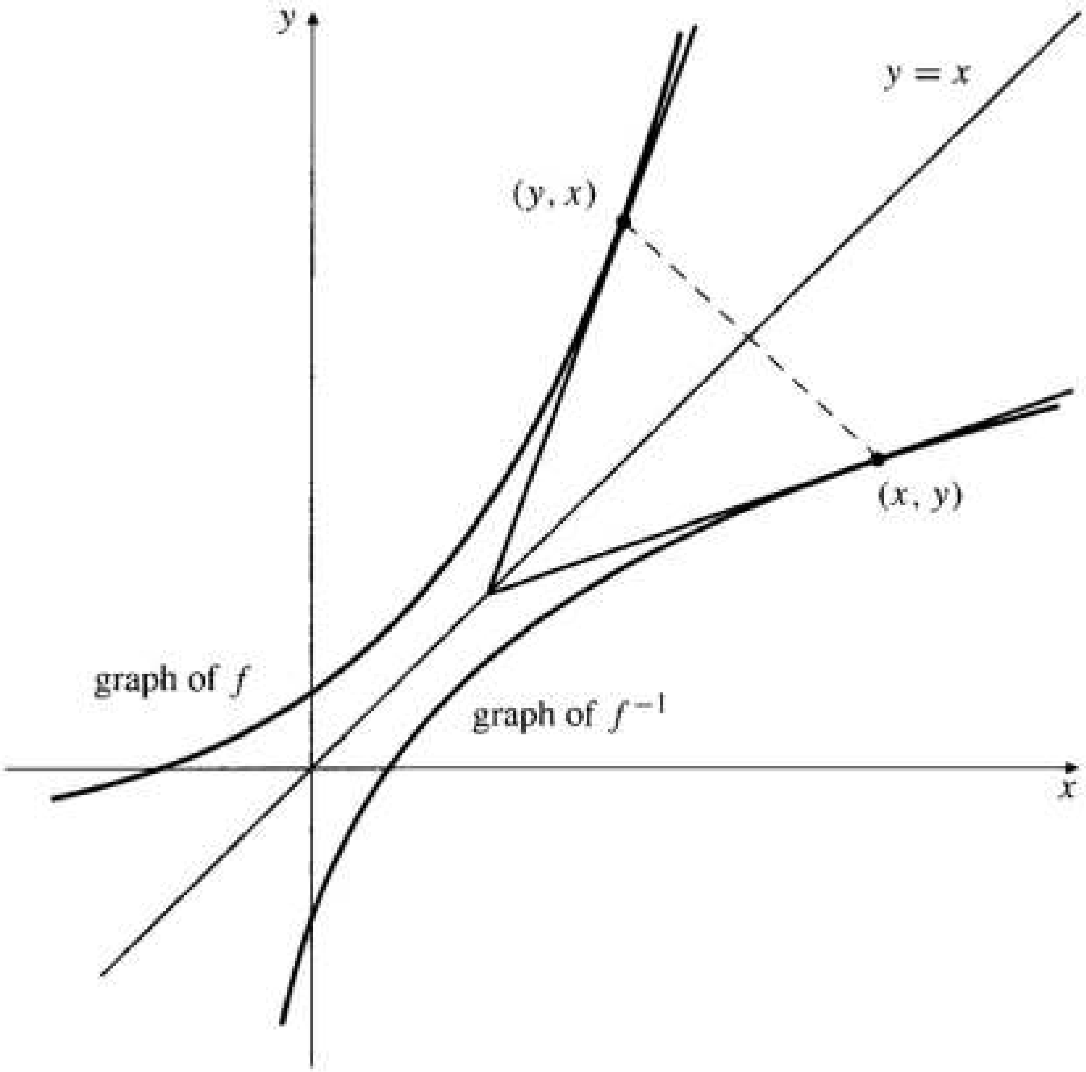
Derivatives of Inverse Functions

Suppose that the function f is differentiable on an interval (a, b) and that either $f'(x) > 0$ for $a < x < b$, so that f is increasing on (a, b), or $f'(x) < 0$ for $a < x < b$, so that f is decreasing on (a, b). In either case f is one-to-one on (a, b) and has an inverse, f^{-1}, defined by

$$y = f^{-1}(x) \quad \Longleftrightarrow \quad x = f(y), \qquad (a < y < b).$$

Implicit Differentiation

Derivatives of Inverse Functions



Implicit Differentiation

Derivatives of Inverse Functions

Let $y = f^{-1}(x)$. We want to find dy/dx. Solve the equation $y = f^{-1}(x)$ for $x = f(y)$ and differentiate implicitly with respect to x to obtain

$$1 = f'(y)\,\frac{dy}{dx}, \qquad \text{so} \qquad \frac{dy}{dx} = \frac{1}{f'(y)} = \frac{1}{f'\left(f^{-1}(x)\right)}.$$

Implicit Differentiation

Derivatives of Inverse Functions

EXAMPLE

Show that $f(x) = x^3 + x$ is one-to-one on the whole real line, and, noting that $f(2) = 10$, find $\left(f^{-1}\right)'(10)$.

Implicit Differentiation

Derivatives of Logarithmic and Exponential Functions

$$\frac{d}{dx}(\log_a x) = \frac{1}{\ln a \cdot x}$$

In particular,

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

Implicit Differentiation

Derivatives of Logarithmic and Exponential Functions

$$y = a^x$$

Implicit Differentiation

Derivatives of Logarithmic and Exponential Functions

$$y = a^x \Rightarrow x = \log_a y$$

Implicit Differentiation

Derivatives of Logarithmic and Exponential Functions

$$y = a^x \Rightarrow x = \log_a y$$

$$\Rightarrow 1 = \frac{1}{\ln a \cdot y} \cdot y'$$

Implicit Differentiation

Derivatives of Logarithmic and Exponential Functions

$$y = a^x \Rightarrow x = \log_a y$$

$$\Rightarrow 1 = \frac{1}{\ln a . y} . y'$$

$$\Rightarrow \boxed{y' = \ln a . y = \ln a . a^x}$$

Implicit Differentiation

Derivatives of Logarithmic and Exponential Functions

EXAMPLE

Find the derivatives of (a) $\ln|\cos x|$ and (b) $\ln\left(x + \sqrt{x^2 + 1}\right)$. Simplify your answers as much as possible.

Implicit Differentiation

Derivatives of Logarithmic and Exponential Functions

EXAMPLE Find the derivatives of (a) $\ln|\cos x|$ and (b) $\ln\left(x+\sqrt{x^2+1}\right)$. Simplify your answers as much as possible.

Solution

(a)

$$\frac{d}{dx}\ln|\cos x| = \frac{1}{\cos x}(-\sin x) = -\tan x.$$

Implicit Differentiation

Derivatives of Logarithmic and Exponential Functions

EXAMPLE Find the derivatives of (a) $\ln|\cos x|$ and (b) $\ln\left(x+\sqrt{x^2+1}\right)$. Simplify your answers as much as possible.

Solution

(a)

$$\frac{d}{dx}\ln|\cos x| = \frac{1}{\cos x}(-\sin x) = -\tan x.$$

(b)

$$\begin{aligned}\frac{d}{dx}\ln\left(x+\sqrt{x^2+1}\right) &= \frac{1}{x+\sqrt{x^2+1}}\left(1+\frac{2x}{2\sqrt{x^2+1}}\right)\\ &= \frac{1}{x+\sqrt{x^2+1}}\,\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}}\\ &= \frac{1}{\sqrt{x^2+1}}.\end{aligned}$$

Implicit Differentiation

Logarithmic Differentiation

EXAMPLE Find y' if $y = x^x$ for $x > 0$.

Implicit Differentiation

Logarithmic Differentiation

EXAMPLE Find y' if $y = x^x$ for $x > 0$.

EXAMPLE Find dy/dt if $y = (\sin t)^{\ln t}$, where $0 < t < \pi$.

Implicit Differentiation

Logarithmic Differentiation

EXAMPLE Differentiate $y = [(x+1)(x+2)(x+3)]/(x+4)$.

Implicit Differentiation

Logarithmic Differentiation

EXAMPLE Differentiate $y = [(x + 1)(x + 2)(x + 3)]/(x + 4)$.

Solution $\ln|y| = \ln|x + 1| + \ln|x + 2| + \ln|x + 3| - \ln|x + 4|$.

Implicit Differentiation

Logarithmic Differentiation

EXAMPLE Differentiate $y = [(x+1)(x+2)(x+3)]/(x+4)$.

Solution $\ln|y| = \ln|x+1| + \ln|x+2| + \ln|x+3| - \ln|x+4|$. Thus,

$$\frac{1}{y}y' = \frac{1}{x+1} + \frac{1}{x+2} + \frac{1}{x+3} - \frac{1}{x+4}$$

Implicit Differentiation

Logarithmic Differentiation

EXAMPLE Differentiate $y = [(x+1)(x+2)(x+3)]/(x+4)$.

Solution $\ln|y| = \ln|x+1| + \ln|x+2| + \ln|x+3| - \ln|x+4|$. Thus,

$$\frac{1}{y}y' = \frac{1}{x+1} + \frac{1}{x+2} + \frac{1}{x+3} - \frac{1}{x+4}$$

$$y' = \frac{(x+1)(x+2)(x+3)}{x+4}\left(\frac{1}{x+1} + \frac{1}{x+2} + \frac{1}{x+3} - \frac{1}{x+4}\right)$$

Implicit Differentiation

Logarithmic Differentiation

EXAMPLE Differentiate $y = [(x+1)(x+2)(x+3)]/(x+4)$.

Solution $\ln|y| = \ln|x+1| + \ln|x+2| + \ln|x+3| - \ln|x+4|$. Thus,

$$
\begin{aligned}
\frac{1}{y}y' &= \frac{1}{x+1} + \frac{1}{x+2} + \frac{1}{x+3} - \frac{1}{x+4} \\
y' &= \frac{(x+1)(x+2)(x+3)}{x+4}\left(\frac{1}{x+1} + \frac{1}{x+2} + \frac{1}{x+3} - \frac{1}{x+4}\right) \\
&= \frac{(x+2)(x+3)}{x+4} + \frac{(x+1)(x+3)}{x+4} + \frac{(x+1)(x+2)}{x+4} \\
&\qquad - \frac{(x+1)(x+2)(x+3)}{(x+4)^2}.
\end{aligned}
$$

Implicit Differentiation

Logarithmic Differentiation

EXAMPLE Find $\left.\dfrac{du}{dx}\right|_{x=1}$ if $u = \sqrt{(x+1)(x^2+1)(x^3+1)}$.

Implicit Differentiation

Logarithmic Differentiation

EXAMPLE Find $\left.\dfrac{du}{dx}\right|_{x=1}$ if $u = \sqrt{(x+1)(x^2+1)(x^3+1)}$.

Solution

$$\ln u = \frac{1}{2}\Big(\ln(x+1) + \ln(x^2+1) + \ln(x^3+1)\Big)$$

Implicit Differentiation

Logarithmic Differentiation

EXAMPLE Find $\left.\dfrac{du}{dx}\right|_{x=1}$ if $u = \sqrt{(x+1)(x^2+1)(x^3+1)}$.

Solution

$$\ln u = \frac{1}{2}\Big(\ln(x+1) + \ln(x^2+1) + \ln(x^3+1)\Big)$$

$$\frac{1}{u}\frac{du}{dx} = \frac{1}{2}\left(\frac{1}{x+1} + \frac{2x}{x^2+1} + \frac{3x^2}{x^3+1}\right).$$

Implicit Differentiation

Logarithmic Differentiation

EXAMPLE Find $\left.\dfrac{du}{dx}\right|_{x=1}$ if $u = \sqrt{(x+1)(x^2+1)(x^3+1)}$.

Solution

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$$\frac{1}{u}\frac{du}{dx} = \frac{1}{2}\left(\frac{1}{x+1} + \frac{2x}{x^2+1} + \frac{3x^2}{x^3+1}\right).$$

At $x = 1$ we have $u = \sqrt{8} = 2\sqrt{2}$. Hence,

$$\left.\frac{du}{dx}\right|_{x=1} = \sqrt{2}\left(\frac{1}{2} + 1 + \frac{3}{2}\right) = 3\sqrt{2}.$$

Implicit Differentiation

Derivatives of Inverse Trigonometric Functions

If $y = \sin^{-1} x$, then $x = \sin y$ and $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$. Differentiating with respect to x, we obtain

$$1 = (\cos y)\frac{dy}{dx}.$$

Implicit Differentiation

Derivatives of Inverse Trigonometric Functions

If $y = \sin^{-1} x$, then $x = \sin y$ and $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$. Differentiating with respect to x, we obtain

$$1 = (\cos y)\frac{dy}{dx}.$$

Since $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$, we know that $\cos y \geq 0$. Therefore,

$$\cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2},$$

and $dy/dx = 1/\cos y = 1/\sqrt{1 - x^2}$;

Implicit Differentiation

Derivatives of Inverse Trigonometric Functions

If $y = \sin^{-1} x$, then $x = \sin y$ and $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$. Differentiating with respect to x, we obtain

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Since $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$, we know that $\cos y \geq 0$. Therefore,

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and $dy/dx = 1/\cos y = 1/\sqrt{1 - x^2}$;

$$\frac{d}{dx}\sin^{-1} x = \frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1 - x^2}}.$$

Implicit Differentiation

Derivatives of Inverse Trigonometric Functions

$$\frac{d}{dx}\cos^{-1}x = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}$$

$$\frac{d}{dx}\sec^{-1}x = \frac{1}{|x|\sqrt{x^2-1}}$$

Hyperbolic Functions

$$\frac{d}{dx}\cosh x = \sinh x$$

$$\frac{d}{dx}\sinh x = \cosh x$$

$$\frac{d}{dx}\tanh x = \operatorname{sech}^2 x$$

$$\frac{d}{dx}\coth x = -\operatorname{csch}^2 x$$

$$\frac{d}{dx}\operatorname{sech} x = -\operatorname{sech} x \, \tanh x$$

$$\frac{d}{dx}\operatorname{csch} x = -\operatorname{csch} x \, \coth x$$