The Derivative

DEFINITION

The **derivative** of a function f is another function f' defined by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

at all points x for which the limit exists (i.e., is a finite real number). If $f'(x)$ exists, we say that f is **differentiable** at x.

Values of x in $\mathcal{D}(f)$ where f is not differentiable and that are not endpoints of $\mathcal{D}(f)$ are **singular points** of f.

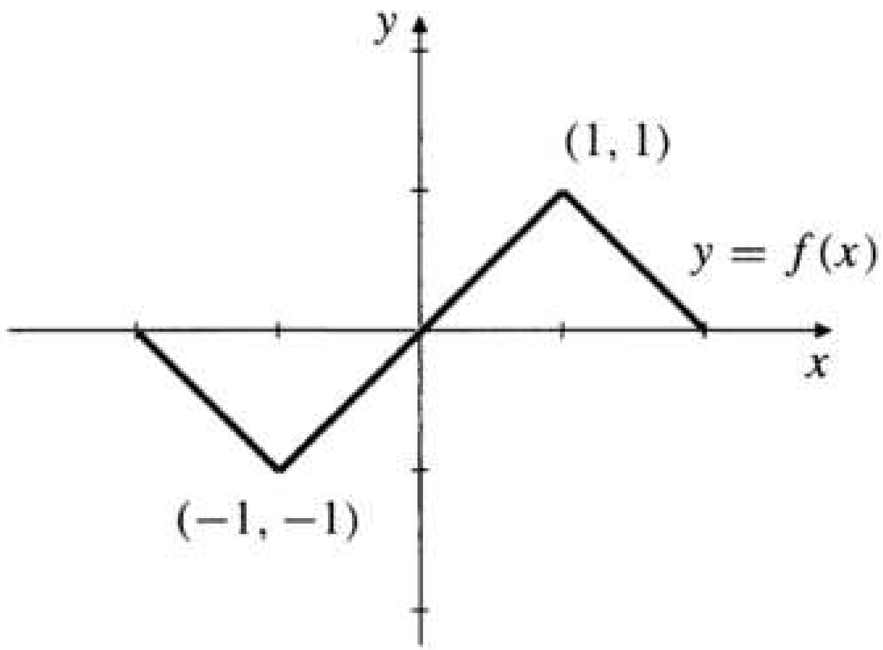
The Derivative

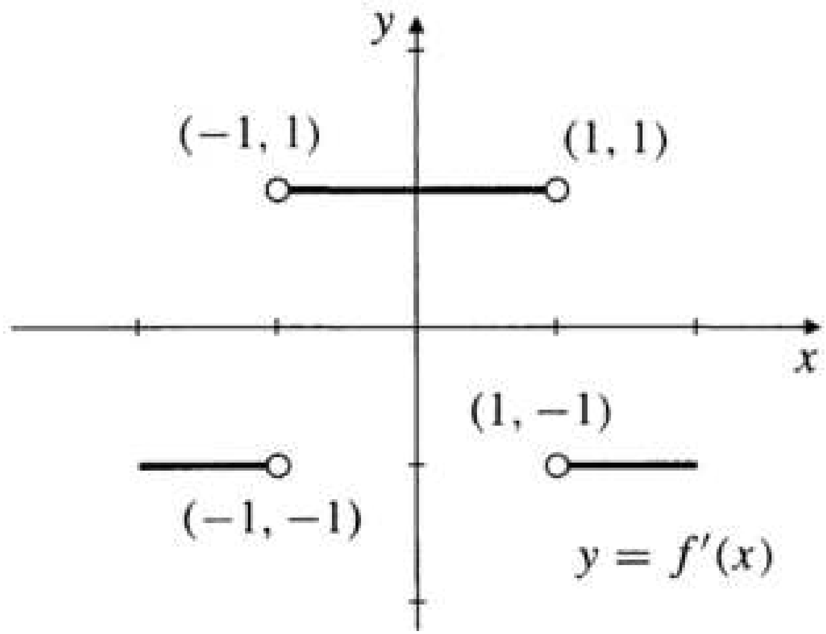
Remark The value of the derivative of f at a particular point x_0 can be expressed as a limit in either of two ways:

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}.$$

In the second limit $x_0 + h$ is replaced by x, so that $h = x - x_0$ and $h \to 0$ is equivalent to $x \to x_0$.

The Derivative





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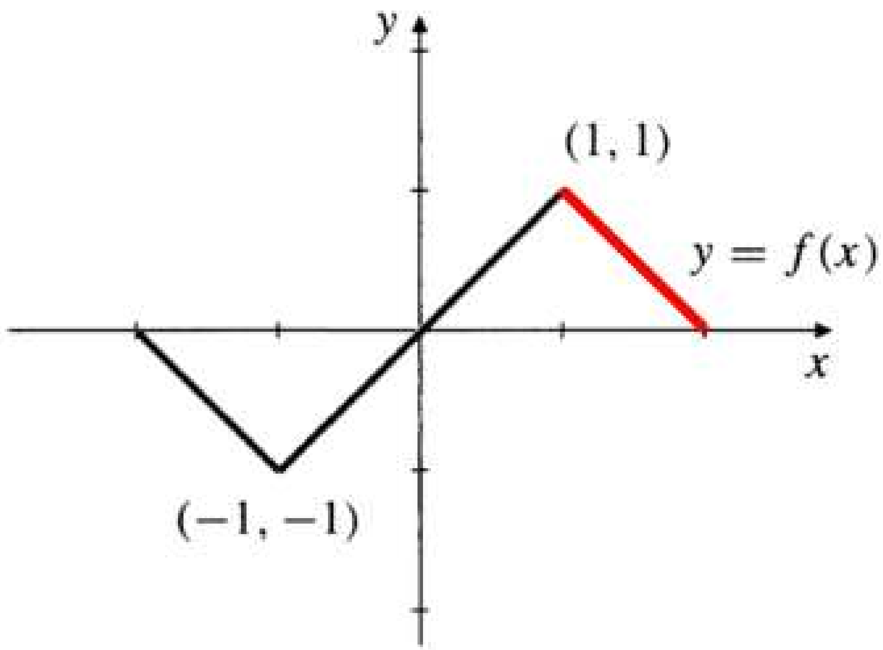

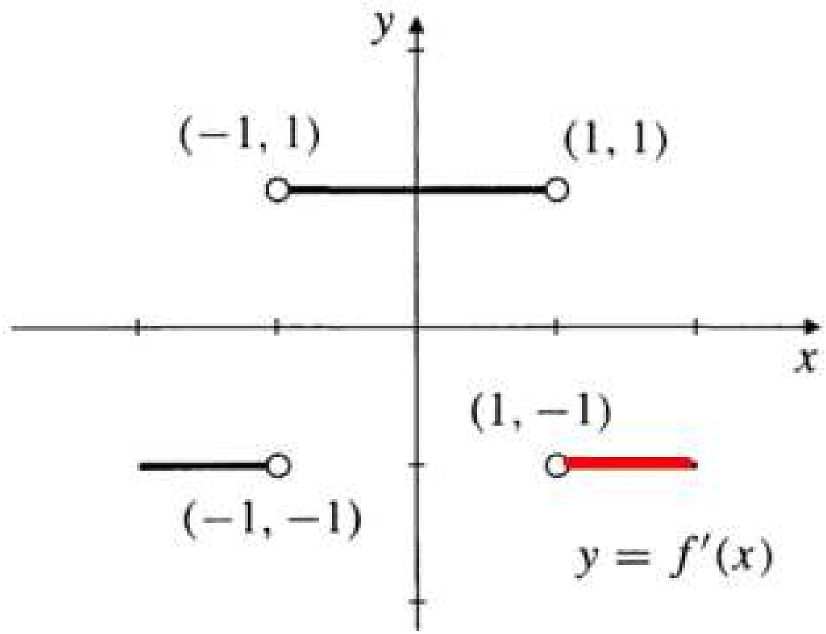

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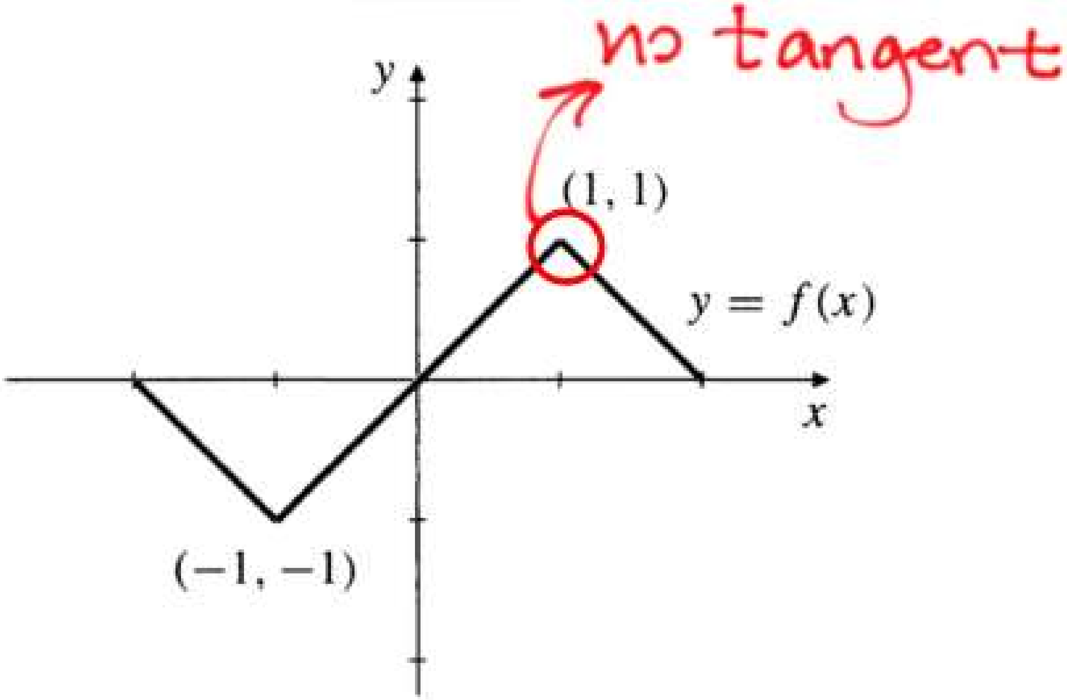

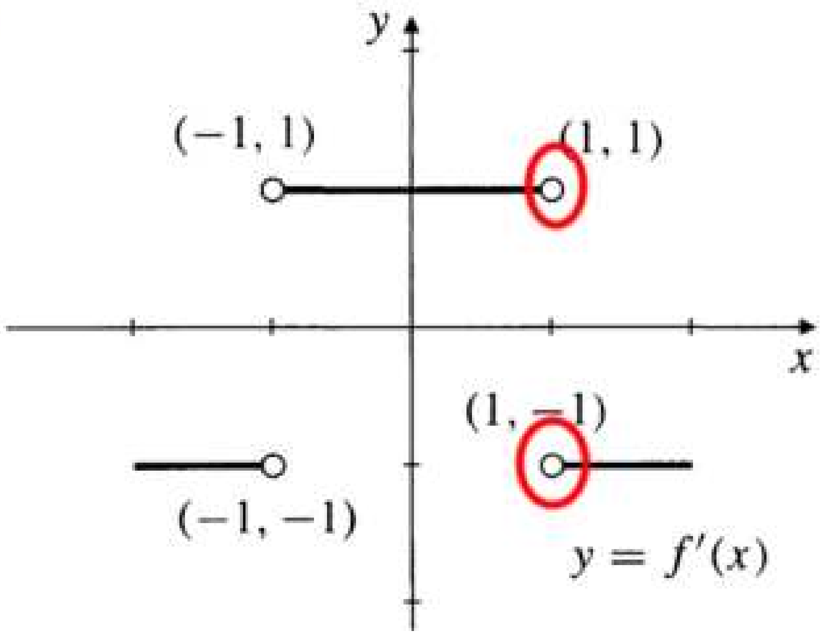

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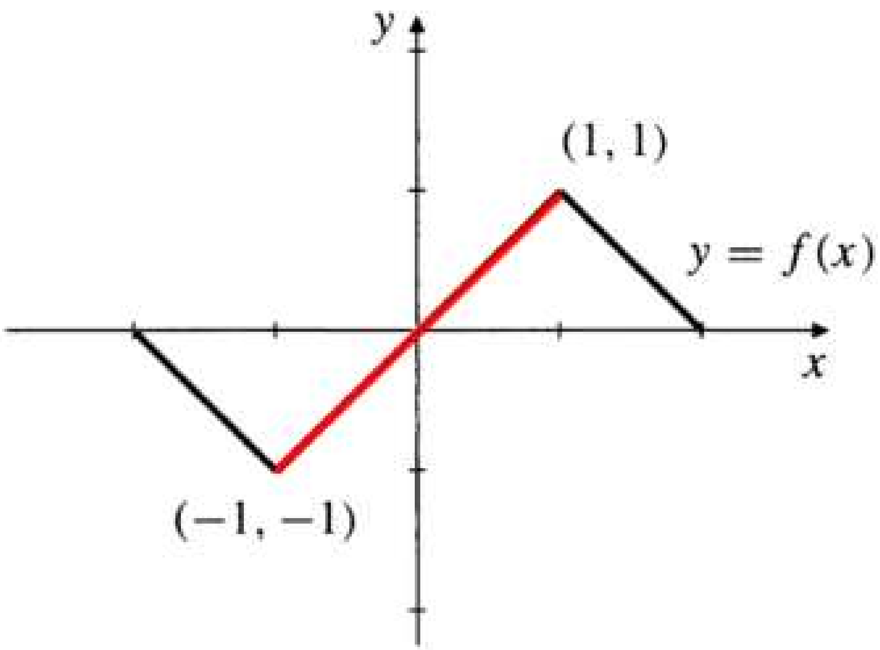

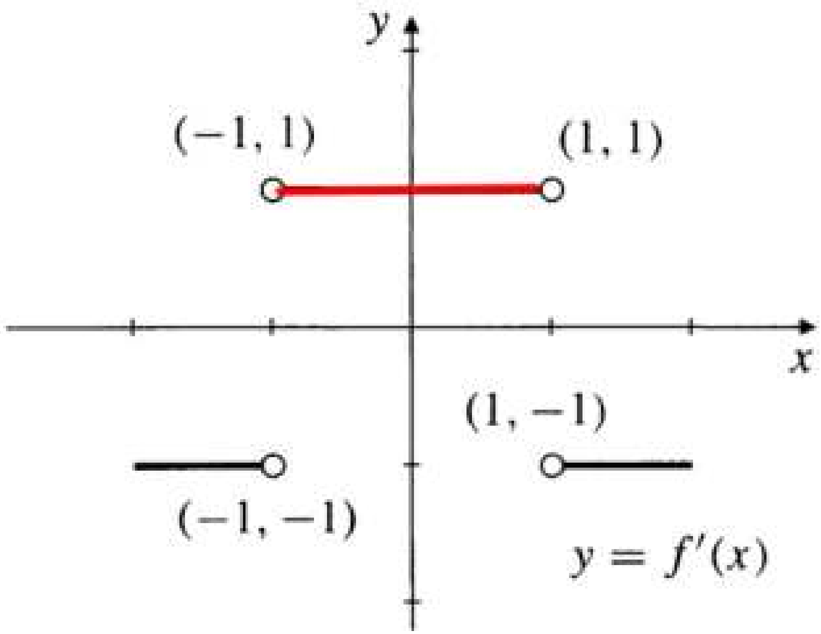

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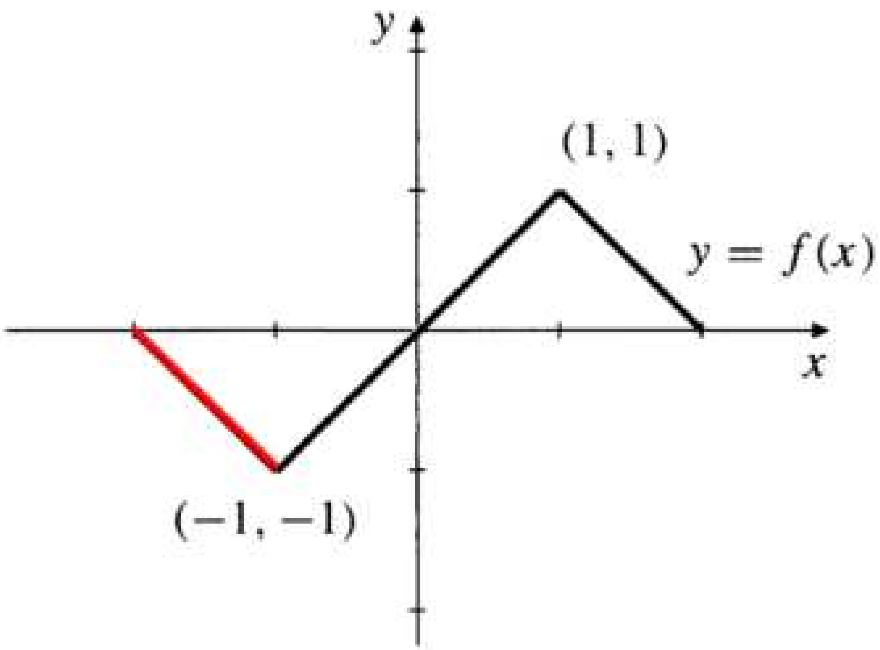

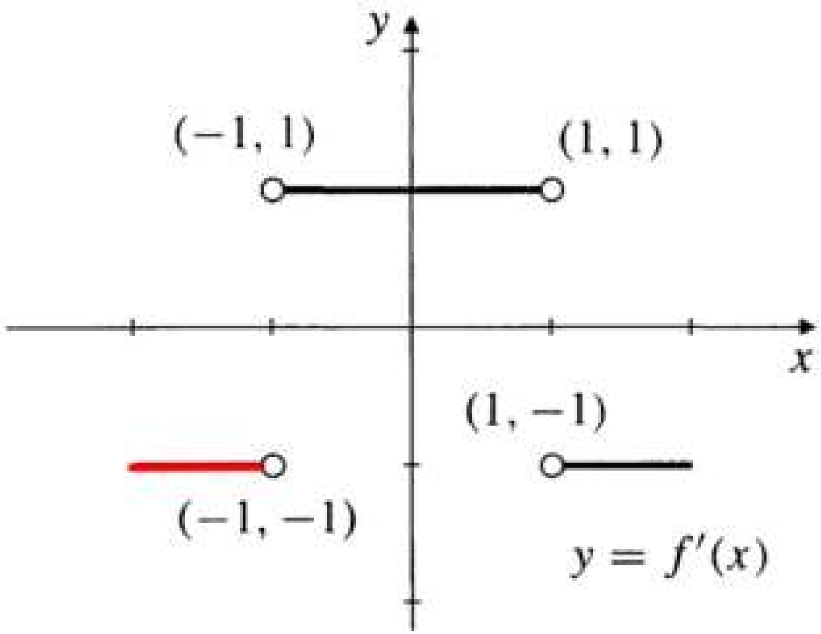

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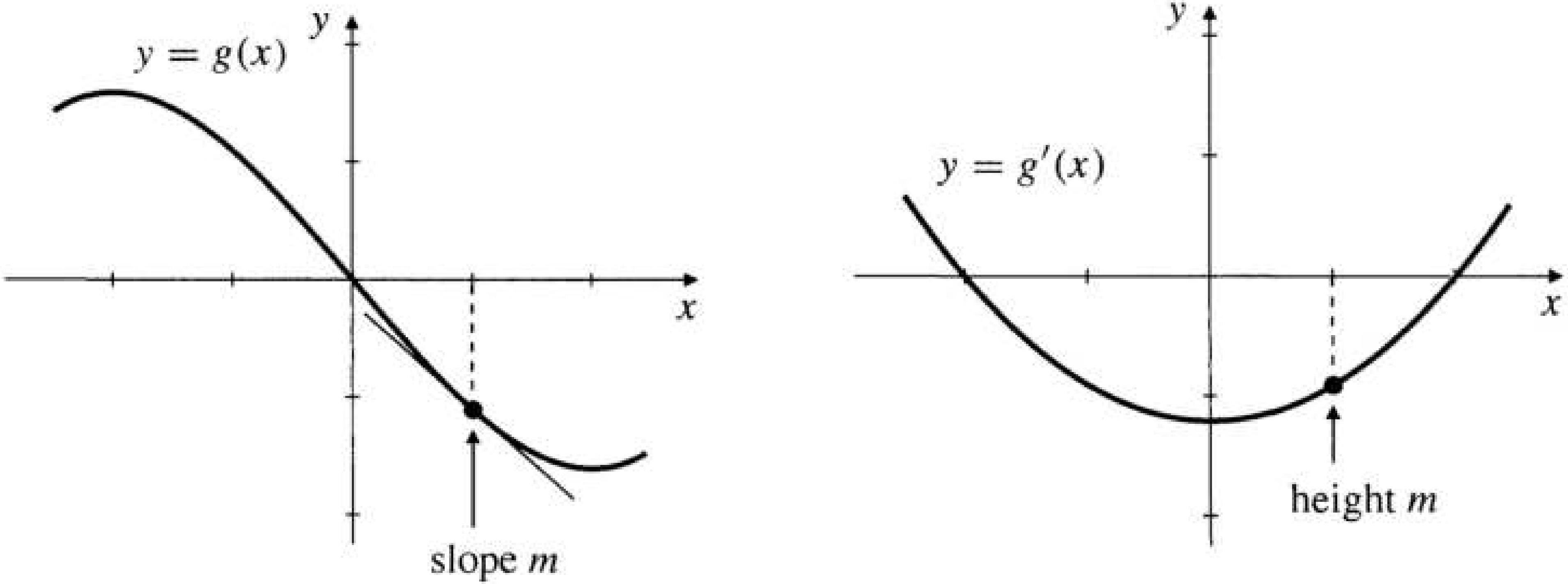

The Derivative

$$f'_+(a) = \lim_{h \to 0+} \frac{f(a+h) - f(a)}{h}$$

right derivative

$$f'_-(b) = \lim_{h \to 0-} \frac{f(b+h) - f(b)}{h}$$

left derivative

The Derivative

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right derivative

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left derivative

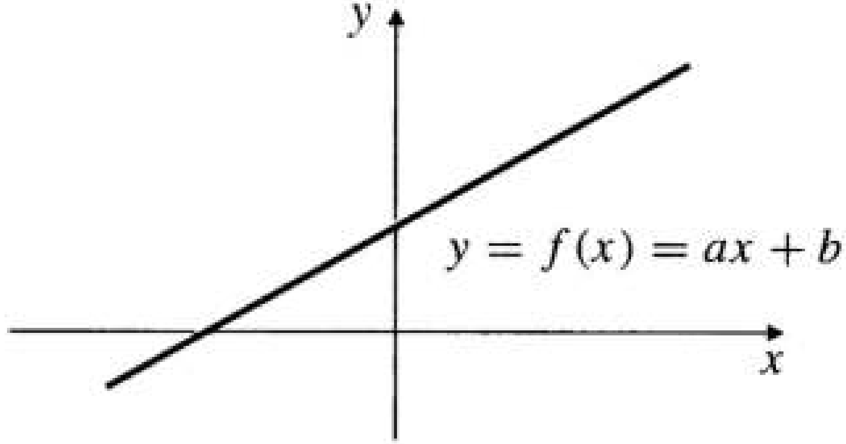
We now say that f is **differentiable** on $[a, b]$ if $f'(x)$ exists for all x in (a, b) and $f'_+(a)$ and $f'_-(b)$ both exist.

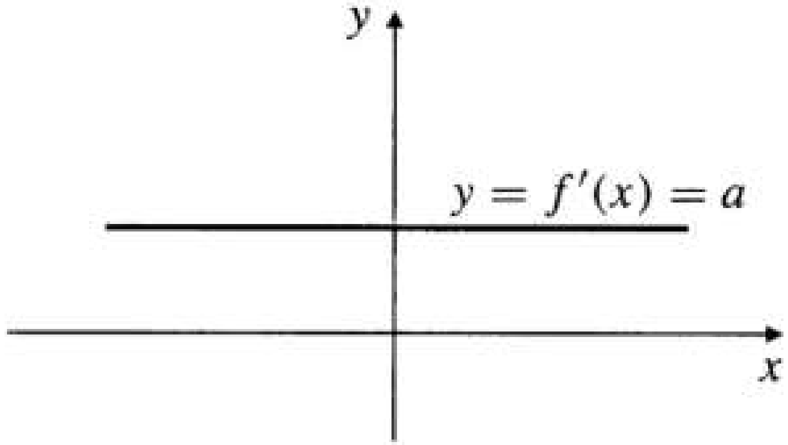
The Derivative

EXAMPLE **(The derivative of a linear function)** Show that if $f(x) = ax + b$, then $f'(x) = a$.

Solution

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$
$$= \lim_{h\to 0} \frac{a(x+h) + b - (ax+b)}{h}$$
$$= \lim_{h\to 0} \frac{ah}{h} = a.$$





The Derivative

EXAMPLE Use the definition of the derivative to calculate the derivatives of the functions:

$$\text{(a) } f(x) = x^2, \quad \text{(b) } g(x) = \frac{1}{x}, \quad \text{and} \quad \text{(c) } k(x) = \sqrt{x}.$$

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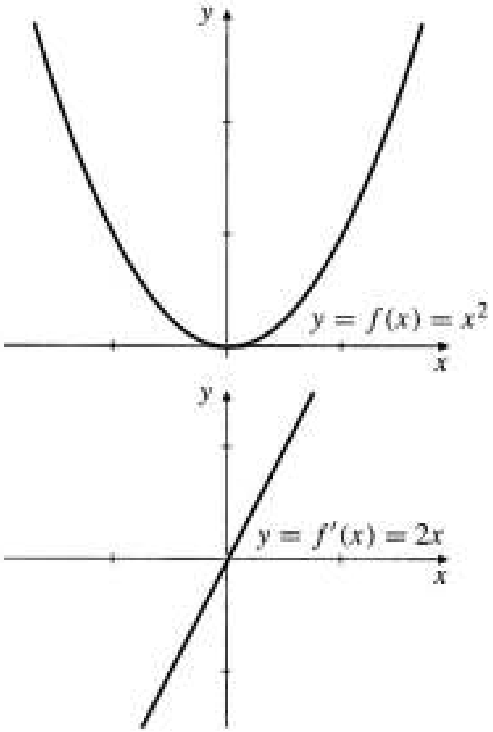

Figure 2.12 The derivative of $f(x) = x^2$ is $f'(x) = 2x$

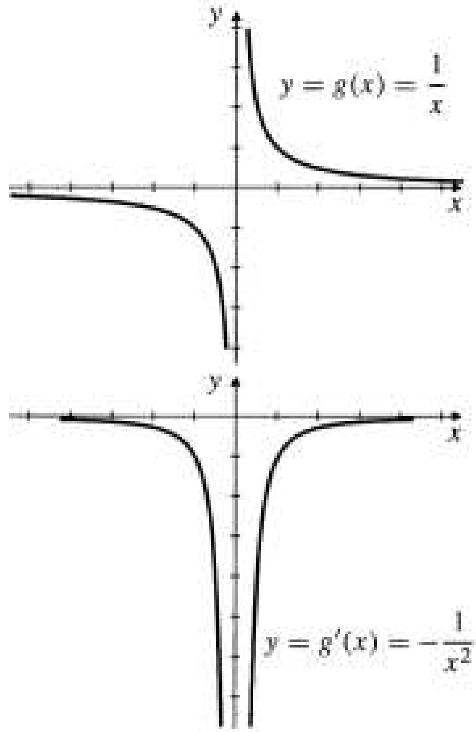

Figure 2.13 The derivative of $g(x) = 1/x$ is $g'(x) = -1/x^2$

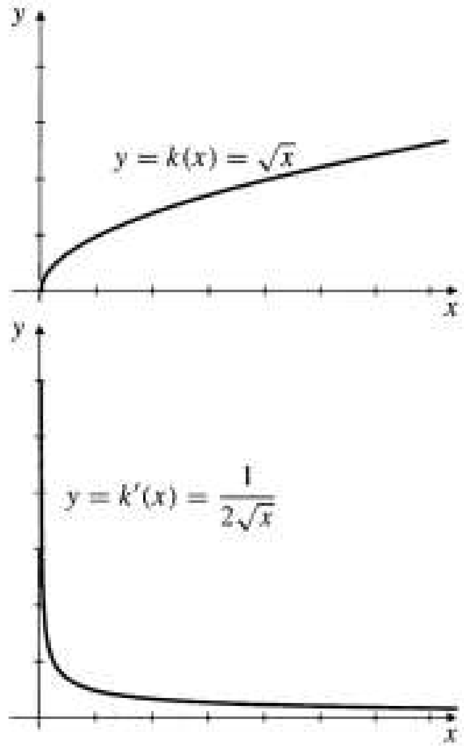

Figure 2.14 The derivative of $k(x) = \sqrt{x}$ is $k'(x) = 1/(2\sqrt{x})$

The Derivative

General Power Rule:

$$\text{If } f(x) = x^r, \quad \text{then} \quad f'(x) = r\,x^{r-1}.$$

This formula is valid for *all values of* r *and* x *for which* x^{r-1} *makes sense as a real number.*

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EXAMPLE **(Differentiating powers)**

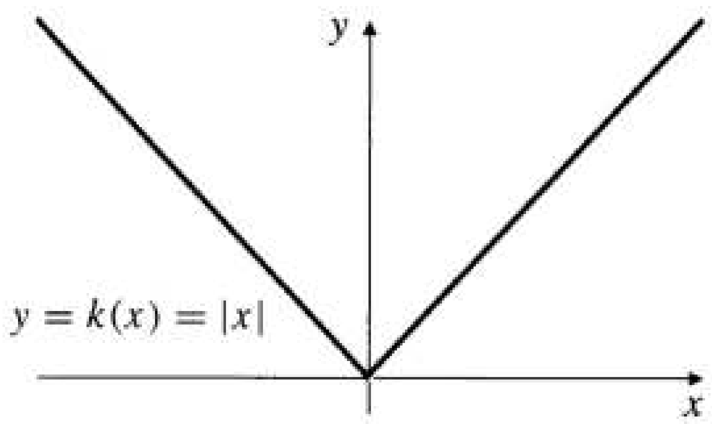
If $f(x) = x^{5/3}$, then $f'(x) = \frac{5}{3}x^{(5/3)-1} = \frac{5}{3}x^{2/3}$ for all real x.

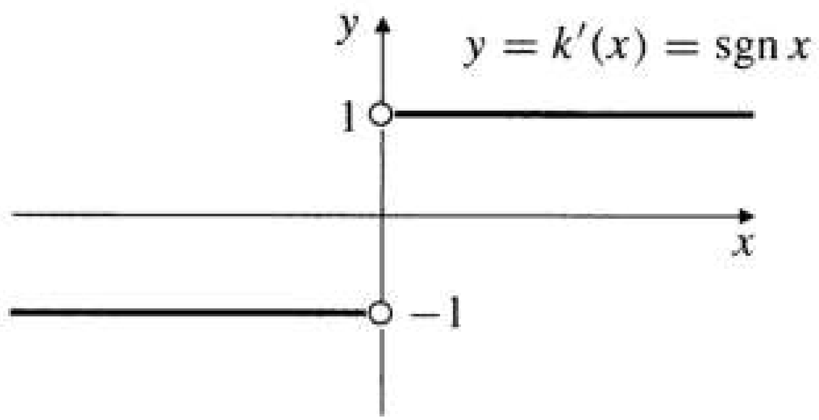
If $g(t) = \frac{1}{\sqrt{t}} = t^{-1/2}$, then $g'(t) = -\frac{1}{2}t^{-(1/2)-1} = -\frac{1}{2}t^{-3/2}$ for $t > 0$.

The Derivative

EXAMPLE **(Differentiating the absolute value function)** Verify that:

$$\text{If } f(x) = |x|, \quad \text{then} \quad f'(x) = \frac{x}{|x|} = \operatorname{sgn} x.$$





The Derivative

Leibniz Notation

$$y = f(x)$$

$$D_x y = y' = \frac{dy}{dx} = \frac{d}{dx} f(x) = f'(x) = D_x f(x) = Df(x)$$

The Derivative

Leibniz Notation

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$$D_x y = y' = \frac{dy}{dx} = \frac{d}{dx} f(x) = f'(x) = D_x f(x) = Df(x)$$

$$\frac{d}{dx} x^2 = 2x \quad \text{(the derivative with respect to } x \text{ of } x^2 \text{ is } 2x\text{)}$$

$$\frac{d}{dx} \sqrt{x} = \frac{1}{2\sqrt{x}}$$

$$\frac{d}{dt} t^{100} = 100\, t^{99}$$

if $y = u^3$, then $\dfrac{dy}{du} = 3u^2$.

The Derivative

Leibniz Notation

The value of the derivative of a function at a particular number x_0 in its domain can also be expressed in several ways:

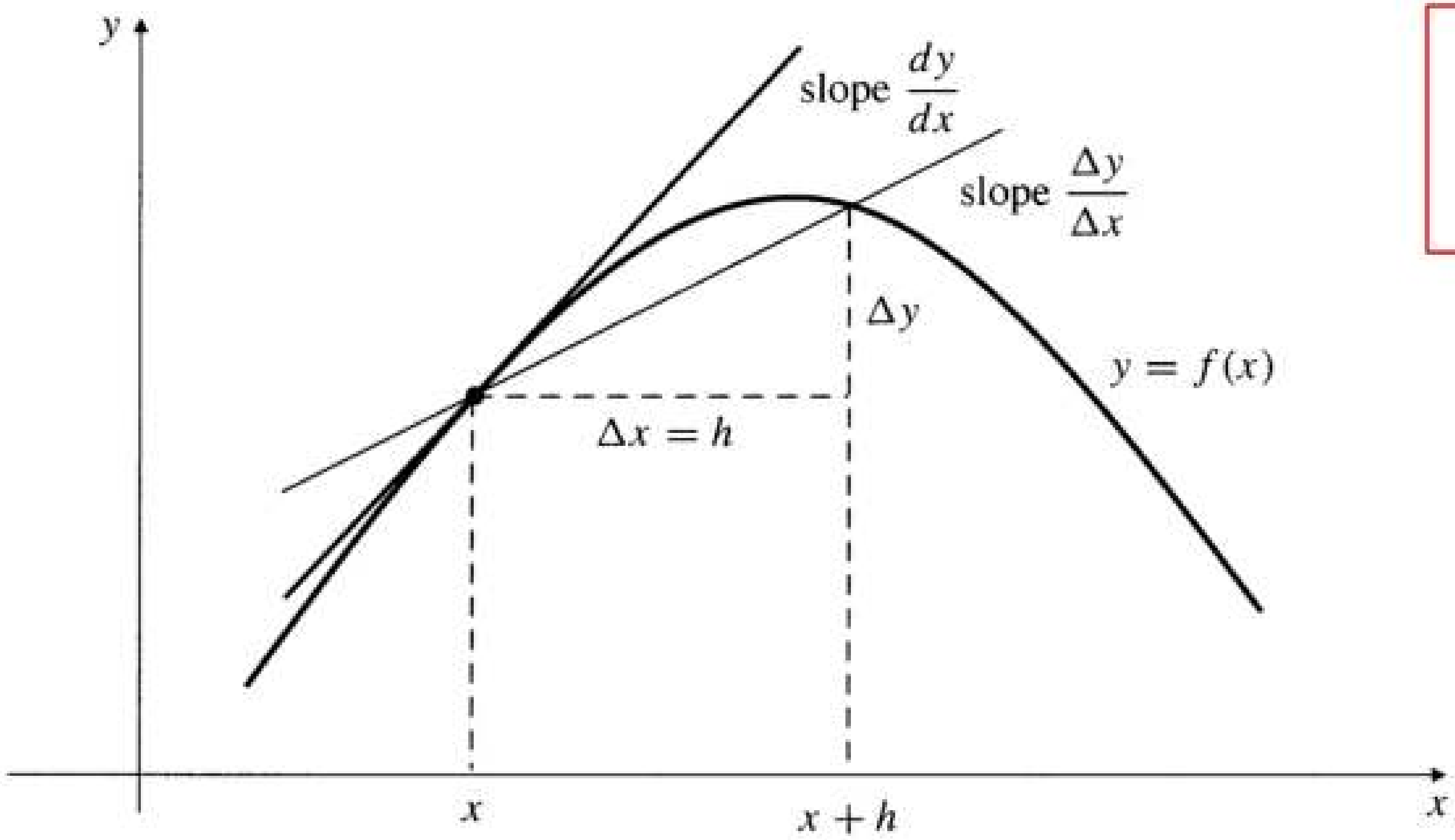
$$D_x y\Big|_{x=x_0} = y'\Big|_{x=x_0} = \frac{dy}{dx}\Big|_{x=x_0} = \frac{d}{dx}f(x)\Big|_{x=x_0} = f'(x_0) = D_x f(x_0).$$

The symbol $\Big|_{x=x_0}$ is called an **evaluation symbol**. It signifies that the expression preceding it should be evaluated at $x = x_0$. Thus,

$$\frac{d}{dx}x^4\Big|_{x=-1} = 4x^3\Big|_{x=-1} = 4(-1)^3 = -4.$$

The Derivative

Leibniz Notation



$$\boxed{\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}}$$

The Derivative

Differentials

$$dy = \frac{dy}{dx}\,dx = f'(x)\,dx$$

The Derivative

Differentials

regarding dx and dy as quantities

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as a new dependent variable (called the differential of y)

as a new independent variable (called the differential of x)

The Derivative

Differentials

regarding dx and dy as <u>quantities</u>

$$dy = \frac{dy}{dx}\,dx = f'(x)\,dx$$

as a new dependent variable (called the differential of y)

as a new independent variable (called the differential of x)

$y = x^2 \Rightarrow dy = 2x\,dx$ (equivalently, $\frac{dy}{dx} = 2x$)

$f(x) = 1/x \Rightarrow df(x) = -\frac{1}{x^2}\,dx$

Differentiation Rules

THEOREM Differentiability implies continuity

If f is differentiable at x, then f is continuous at x.

Differentiation Rules

THEOREM **Differentiation rules for sums, differences, and constant multiples**

If functions f and g are differentiable at x, and if C is a constant, then the functions $f+g$, $f-g$, and Cf are all differentiable at x and

$$
\begin{aligned}
(f+g)'(x) &= f'(x) + g'(x), \\
(f-g)'(x) &= f'(x) - g'(x), \\
(Cf)'(x) &= Cf'(x).
\end{aligned}
$$

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$$\begin{aligned}(f+g)'(x) &= f'(x)+g'(x),\\ (f-g)'(x) &= f'(x)-g'(x),\\ (Cf)'(x) &= Cf'(x).\end{aligned}$$

PROOF

$$\begin{aligned}(f+g)'(x) &= \lim_{h\to 0}\frac{(f+g)(x+h)-(f+g)(x)}{h}\\ &= \lim_{h\to 0}\frac{(f(x+h)+g(x+h))-(f(x)+g(x))}{h}\\ &= \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}+\frac{g(x+h)-g(x)}{h}\right)\\ &= f'(x)+g'(x),\end{aligned}$$

Differentiation Rules

EXAMPLE

Calculate the derivatives of the functions:

(a) $2x^3 - 5x^2 + 4x + 7$, (b) $f(x) = 5\sqrt{x} + \dfrac{3}{x} - 18$, (c) $y = \dfrac{1}{7}t^4 - 3t^{7/3}$.

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(a) $2x^3 - 5x^2 + 4x + 7$, (b) $f(x) = 5\sqrt{x} + \dfrac{3}{x} - 18$, (c) $y = \dfrac{1}{7}t^4 - 3t^{7/3}$.

EXAMPLE Find an equation of the tangent to the curve $y = \dfrac{3x^3 - 4}{x}$ at the point on the curve where $x = -2$.

Differentiation Rules

THEOREM **The Product Rule**

If functions f and g are differentiable at x, then their product fg is also differentiable at x, and

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x).$$

Differentiation Rules

THEOREM The Product Rule

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$$(fg)'(x) = f'(x)g(x) + f(x)g'(x).$$

PROOF

$$\begin{aligned}
(fg)'(x) &= \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\
&= \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h} \\
&= \lim_{h \to 0} \left(\frac{f(x+h) - f(x)}{h} g(x+h) + f(x) \frac{g(x+h) - g(x)}{h} \right) \\
&= f'(x)g(x) + f(x)g'(x).
\end{aligned}$$

Differentiation Rules

EXAMPLE

Find the derivative of $(x^2 + 1)(x^3 + 4)$ using and without using the Product Rule.

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Solution Using the Product Rule with $f(x) = x^2+1$ and $g(x) = x^3+4$, we calculate

$$\frac{d}{dx}\big((x^2 + 1)(x^3 + 4)\big) = 2x(x^3 + 4) + (x^2 + 1)(3x^2) = 5x^4 + 3x^2 + 8x.$$

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On the other hand, we can calculate the derivative

$$\frac{d}{dx}\big((x^2 + 1)(x^3 + 4)\big) = \frac{d}{dx}(x^5 + x^3 + 4x^2 + 4) = 5x^4 + 3x^2 + 8x.$$

Differentiation Rules

The Product Rule can be extended to products of any number of factors, for instance:

$$\begin{aligned}(fgh)'(x) &= f'(x)(gh)(x) + f(x)(gh)'(x) \\ &= f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x).\end{aligned}$$

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In general,

$$(f_1 f_2 f_3 \cdots f_n)' = f_1' f_2 f_3 \cdots f_n + f_1 f_2' f_3 \cdots f_n + \cdots + f_1 f_2 f_3 \cdots f_n'.$$

Differentiation Rules

THEOREM **The Reciprocal Rule**

If f is differentiable at x and $f(x) \neq 0$, then $1/f$ is differentiable at x, and

$$\left(\frac{1}{f}\right)'(x) = \frac{-f'(x)}{(f(x))^2}.$$

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PROOF

$$\begin{aligned}
\frac{d}{dx}\frac{1}{f(x)} &= \lim_{h\to 0} \frac{\dfrac{1}{f(x+h)} - \dfrac{1}{f(x)}}{h} \\
&= \lim_{h\to 0} \frac{f(x) - f(x+h)}{hf(x+h)f(x)} \\
&= \lim_{h\to 0} \left(\frac{-1}{f(x+h)f(x)}\right) \frac{f(x+h) - f(x)}{h} = \frac{-1}{(f(x))^2} f'(x)
\end{aligned}$$

Differentiation Rules

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(a) $\dfrac{1}{x^2+1}$ and (b) $f(t) = \dfrac{1}{t + \dfrac{1}{t}}$.

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Solution Using the Reciprocal Rule:

(a) $\dfrac{d}{dx}\left(\dfrac{1}{x^2+1}\right) = \dfrac{-2x}{(x^2+1)^2}$.

(b) $f'(t) = \dfrac{-1}{\left(t + \dfrac{1}{t}\right)^2}\left(1 - \dfrac{1}{t^2}\right) = \dfrac{-t^2}{(t^2+1)^2}\,\dfrac{t^2-1}{t^2} = \dfrac{1-t^2}{(t^2+1)^2}$.

Differentiation Rules

THEOREM **The Quotient Rule**

If f and g are differentiable at x, and if $g(x) \neq 0$, then the quotient f/g is differentiable at x and

$$\left(\frac{f}{g}\right)'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}.$$

PROOF

$$\begin{aligned}\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{d}{dx}\left(f(x)\frac{1}{g(x)}\right) &= f'(x)\frac{1}{g(x)} + f(x)\left(-\frac{g'(x)}{(g(x))^2}\right)\\ &= \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}.\end{aligned}$$

Differentiation Rules

EXAMPLE Find the derivatives of

(a) $y = \dfrac{1 - x^2}{1 + x^2}$, (b) $\dfrac{\sqrt{t}}{3 - 5t}$, and (c) $f(\theta) = \dfrac{a + b\theta}{m + n\theta}$.

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EXAMPLE Find equations of any lines that pass through the point $(-1, 0)$ and are tangent to the curve $y = (x - 1)/(x + 1)$.

The Chain Rule

THEOREM The Chain Rule

If $f(u)$ is differentiable at $u = g(x)$, and $g(x)$ is differentiable at x, then the composite function $f \circ g(x) = f(g(x))$ is differentiable at x, and

$$(f \circ g)'(x) = f'(g(x))g'(x).$$

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In terms of Leibniz notation, if $y = f(u)$ where $u = g(x)$, then $y = f(g(x))$ and:

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}, \qquad \text{where } \frac{dy}{du} \text{ is evaluated at } u = g(x).$$

The Chain Rule

EXAMPLE Find the derivative of $y = \sqrt{x^2 + 1}$.

EXAMPLE Find derivatives of the following functions:

(a) $(7x - 3)^{10}$, (b) $f(t) = |t^2 - 1|$, and (c) $\left(3x + \dfrac{1}{(2x+1)^3}\right)^{1/4}$.

The Chain Rule

Building the Chain Rule into Differentiation Formulas

If u is a differentiable function of x and $y = u^n$, then

$$\frac{d}{dx}u^n = \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = nu^{n-1}\frac{du}{dx}.$$

The Chain Rule

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If u is a differentiable function of x and $y = u^n$, then

$$\frac{d}{dx}u^n = \frac{dy}{dx} = \frac{dy}{du}\,\frac{du}{dx} = nu^{n-1}\,\frac{du}{dx}.$$

$$\frac{d}{dx}\left(\frac{1}{u}\right) = \frac{-1}{u^2}\,\frac{du}{dx} \qquad \text{(the Reciprocal Rule)}$$

$$\frac{d}{dx}\sqrt{u} = \frac{1}{2\sqrt{u}}\,\frac{du}{dx} \qquad \text{(the Square Root Rule)}$$

$$\frac{d}{dx}u^r = r\,u^{r-1}\,\frac{du}{dx} \qquad \text{(the General Power Rule)}$$

$$\frac{d}{dx}|u| = \operatorname{sgn} u\,\frac{du}{dx} = \frac{u}{|u|}\,\frac{du}{dx} \qquad \text{(the Absolute Value Rule)}$$