The Formal Definition of Limit

A formal definition of limit

We say that $f(x)$ **approaches the limit** L as x **approaches** a, and we write

$$\lim_{x \to a} f(x) = L,$$

if the following condition is satisfied:
for every number $\epsilon > 0$ there exists a number $\delta > 0$, possibly depending on ϵ, such that if $0 < |x - a| < \delta$, then x belongs to the domain of f and

$$|f(x) - L| < \epsilon.$$

$\forall \varepsilon > 0$, $\exists \delta > 0$ such that

$$0 < |x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon.$$

The Formal Definition of Limit

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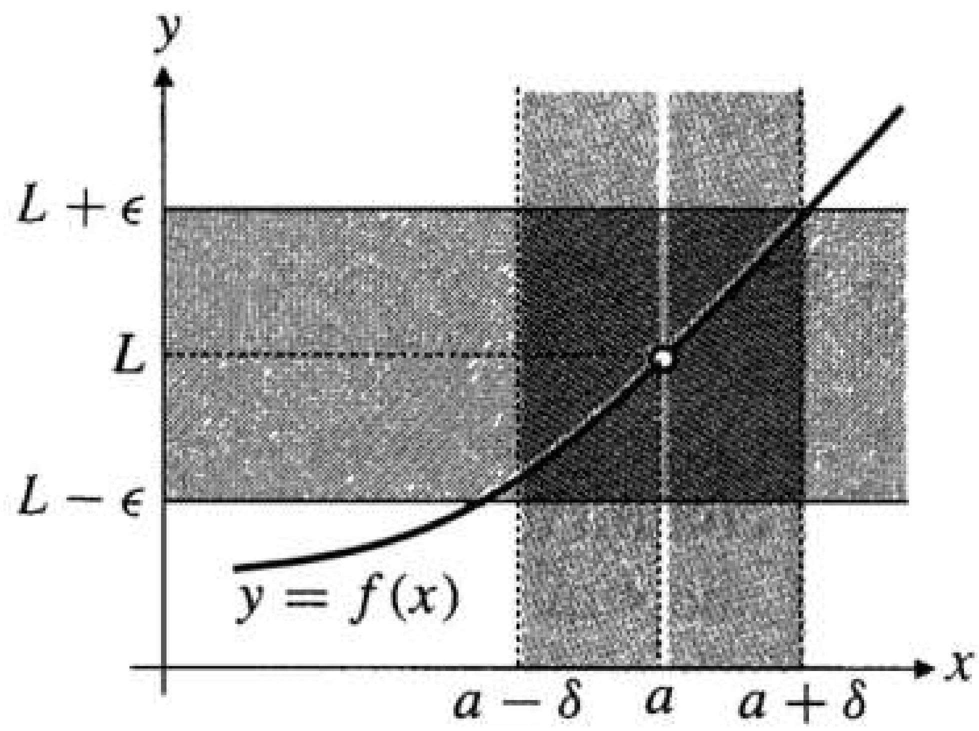


The Formal Definition of Limit

EXAMPLE

(Two important limits) Verify: (a) $\lim_{x \to a} x = a$ and (b) $\lim_{x \to a} k = k$ (k = constant).

The Formal Definition of Limit

EXAMPLE **(Two important limits)** Verify: (a) $\lim_{x \to a} x = a$ and (b) $\lim_{x \to a} k = k$ (k = constant).

Solution

(a) Let $\epsilon > 0$ be given. We must find $\delta > 0$ so that

$$0 < |x - a| < \delta \qquad \text{implies} \qquad |x - a| < \epsilon.$$

Clearly, we can take $\delta = \epsilon$ and the implication above will be true. This proves that $\lim_{x \to a} x = a$.

The Formal Definition of Limit

EXAMPLE **(Two important limits)** Verify: (a) $\lim\limits_{x \to a} x = a$ and (b) $\lim\limits_{x \to a} k = k$ (k = constant).

Solution

(b) Let $\epsilon > 0$ be given. We must find $\delta > 0$ so that

$$0 < |x - a| < \delta \qquad \text{implies} \qquad |k - k| < \epsilon.$$

Since $k - k = 0$, we can use any positive number for δ and the implication above will be true. This proves that $\lim_{x \to a} k = k$.

The Formal Definition of Limit

EXAMPLE Verify that $\lim_{x \to 2} x^2 = 4$.

Other Kinds of Limits

Right limits

We say that $f(x)$ has **right limit** L at a, and we write

$$\lim_{x\to a+} f(x) = L,$$

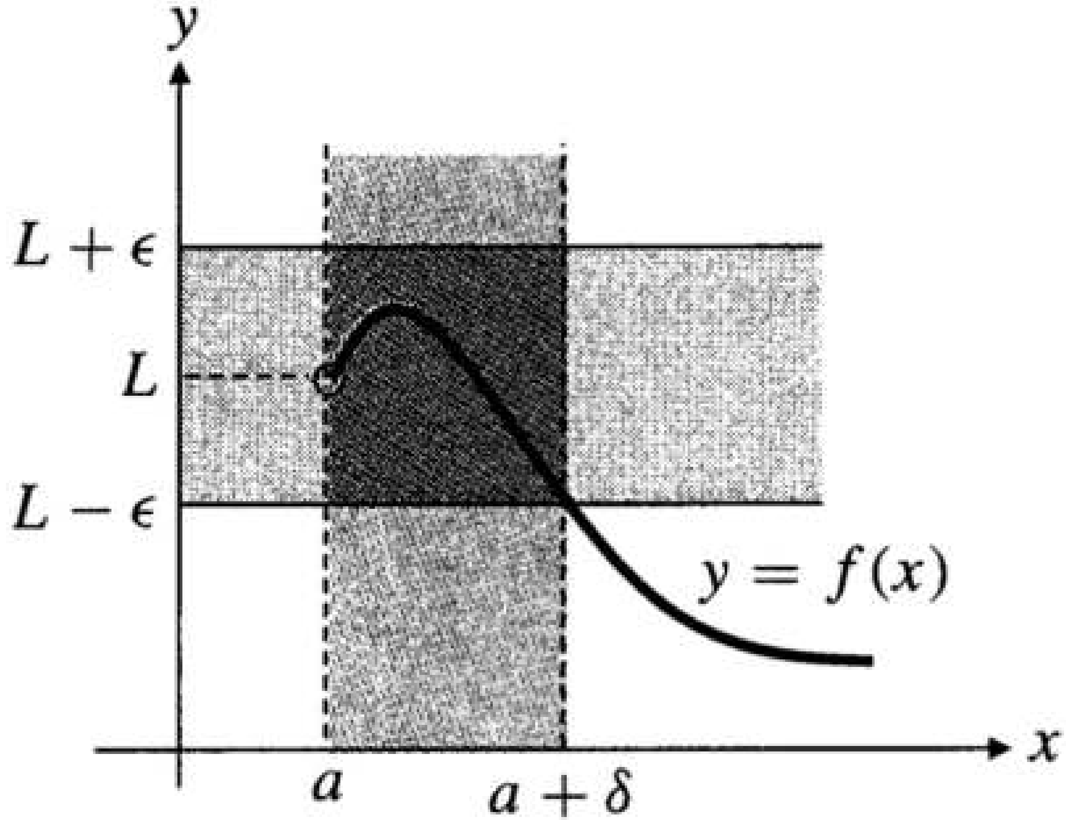
if the following condition is satisfied:
for every number $\epsilon > 0$ there exists a number $\delta > 0$, possibly depending on ϵ, such that if $a < x < a + \delta$, then x belongs to the domain of f and

$$|f(x) - L| < \epsilon.$$

$\forall \varepsilon > 0$, $\exists \delta > 0$ such that

$$a < x < a + \delta \implies |f(x) - L| < \varepsilon.$$

Other Kinds of Limits



If $a < x < a + \delta$, then $|f(x) - L| < \epsilon$

Other Kinds of Limits

EXAMPLE Show that $\lim_{x \to 0+} \sqrt{x} = 0$.

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The Formal Definition of Limit

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EXAMPLE Show that $\lim_{x \to 0+} \sqrt{x} = 0$.

Solution Let $\epsilon > 0$ be given. If $x > 0$, then $|\sqrt{x} - 0| = \sqrt{x}$.

Other Kinds of Limits

EXAMPLE Show that $\lim_{x \to 0+} \sqrt{x} = 0$.

Solution Let $\epsilon > 0$ be given. If $x > 0$, then $|\sqrt{x} - 0| = \sqrt{x}$. We can ensure that $\sqrt{x} < \epsilon$ by requiring $x < \epsilon^2$. Thus we can take $\delta = \epsilon^2$ and the condition of the definition will be satisfied:

$$0 < x < \delta = \epsilon^2 \quad \text{implies} \quad |\sqrt{x} - 0| < \epsilon.$$

The Formal Definition of Limit

Other Kinds of Limits

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$$0 < x < \delta = \epsilon^2 \qquad \text{implies} \qquad |\sqrt{x} - 0| < \epsilon.$$

Therefore, $\lim_{x \to 0+} \sqrt{x} = 0$.

Other Kinds of Limits

Limit at infinity

We say that $f(x)$ **approaches the limit** L **as** x **approaches infinity**, and we write

$$\lim_{x\to\infty} f(x) = L,$$

if the following condition is satisfied:
for every number $\epsilon > 0$ there exists a number R, possibly depending on ϵ, such that if $x > R$, then x belongs to the domain of f and

$$|f(x) - L| < \epsilon.$$

$\forall \varepsilon > 0$, $\exists R$ such that
$x > R \Longrightarrow |f(x) - L| < \varepsilon$.

The Formal Definition of Limit

Other Kinds of Limits

EXAMPLE Show that $\lim\limits_{x \to \infty} \dfrac{1}{x} = 0$.

Solution Let ϵ be a given positive number.

The Formal Definition of Limit

Other Kinds of Limits

EXAMPLE Show that $\lim_{x\to\infty} \frac{1}{x} = 0.$

Solution Let ϵ be a given positive number. For $x > 0$ we have

$$\left|\frac{1}{x} - 0\right| = \frac{1}{|x|} = \frac{1}{x} < \epsilon \qquad \text{provided} \qquad x > \frac{1}{\epsilon}.$$

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Therefore, the condition of the definition is satisfied with $R = 1/\epsilon$. We have shown that

$$\lim_{x\to\infty} 1/x = 0.$$

Other Kinds of Limits

Infinite limits

We say that $f(x)$ approaches infinity as x approaches a and write

$$\lim_{x \to a} f(x) = \infty,$$

if for every positive number B we can find a positive number δ, possibly depending on B, such that if $0 < |x - a| < \delta$, then x belongs to the domain of f and $f(x) > B$.

$\forall B > 0$, $\exists \delta > 0$ such that

$0 < |x - a| < \delta \Longrightarrow f(x) > B$

The Formal Definition of Limit

Other Kinds of Limits

EXAMPLE Verify that $\lim_{x \to 0} \frac{1}{x^2} = \infty$.

The Formal Definition of Limit

Other Kinds of Limits

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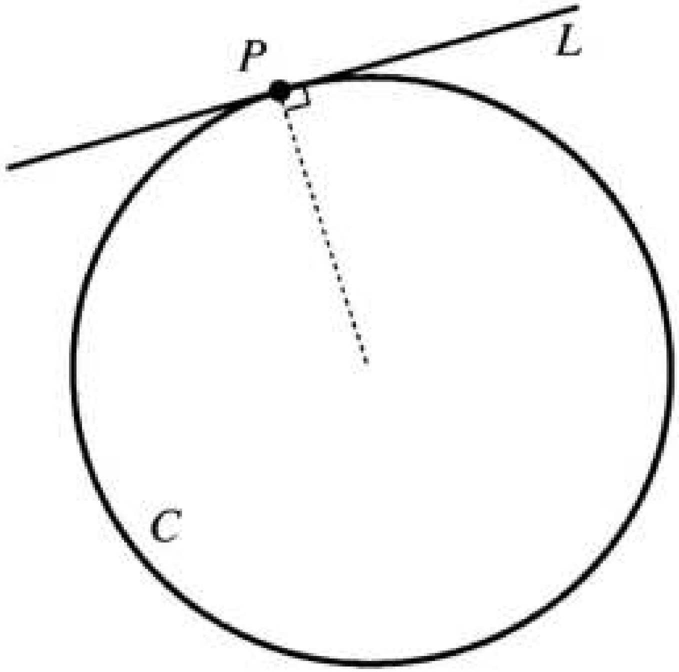
$$0 < |x| < \delta \quad \Rightarrow \quad x^2 < \delta^2 = \frac{1}{B} \quad \Rightarrow \quad \frac{1}{x^2} > B.$$

Therefore $\lim_{x \to 0} 1/x^2 = \infty$.

Differentiation

Tangent Lines and Their Slopes

A tangent line to a circle has the following properties

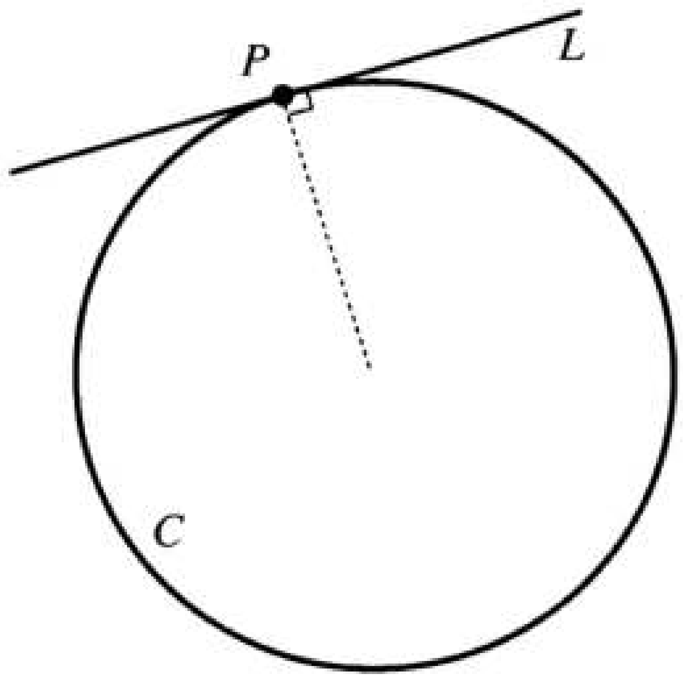


L is tangent to C at P

Tangent Lines and Their Slopes

A tangent line to a circle has the following properties

(i) It meets the circle at only one point.



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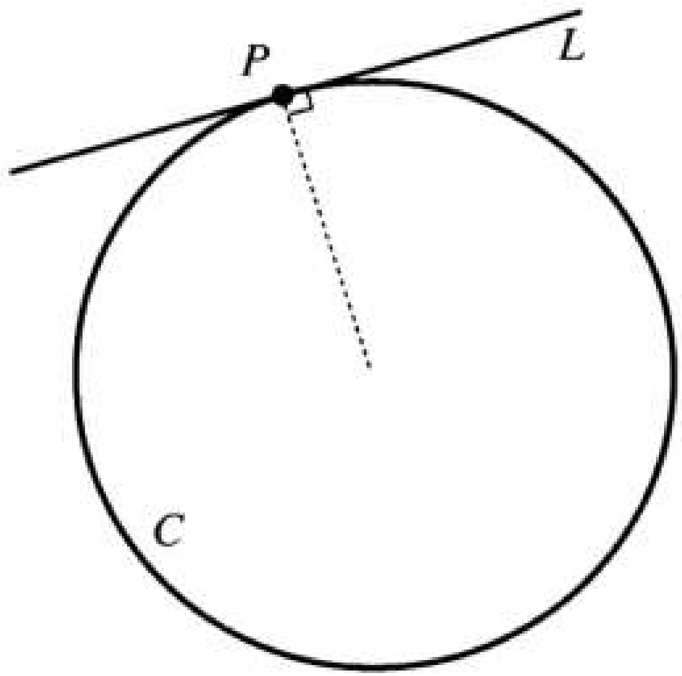
Tangent Lines and Their Slopes

A tangent line to a circle has the following properties

(i) It meets the circle at only one point.

(ii) The circle lies on only one side of the line.



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Tangent Lines and Their Slopes

A tangent line to a circle has the following properties

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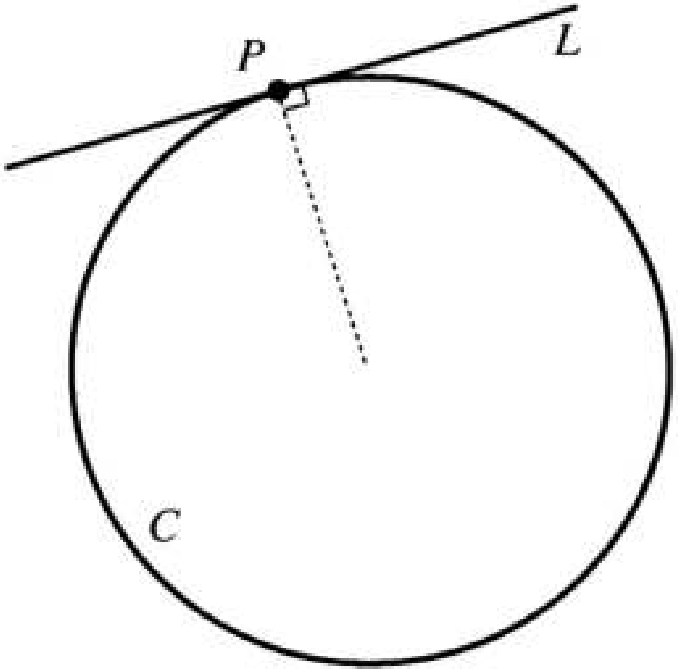
(iii) The tangent is perpendicular to the line joining the centre of the circle to the point of contact.



L is tangent to C at P

Tangent Lines and Their Slopes

What about tangents to a general curve?

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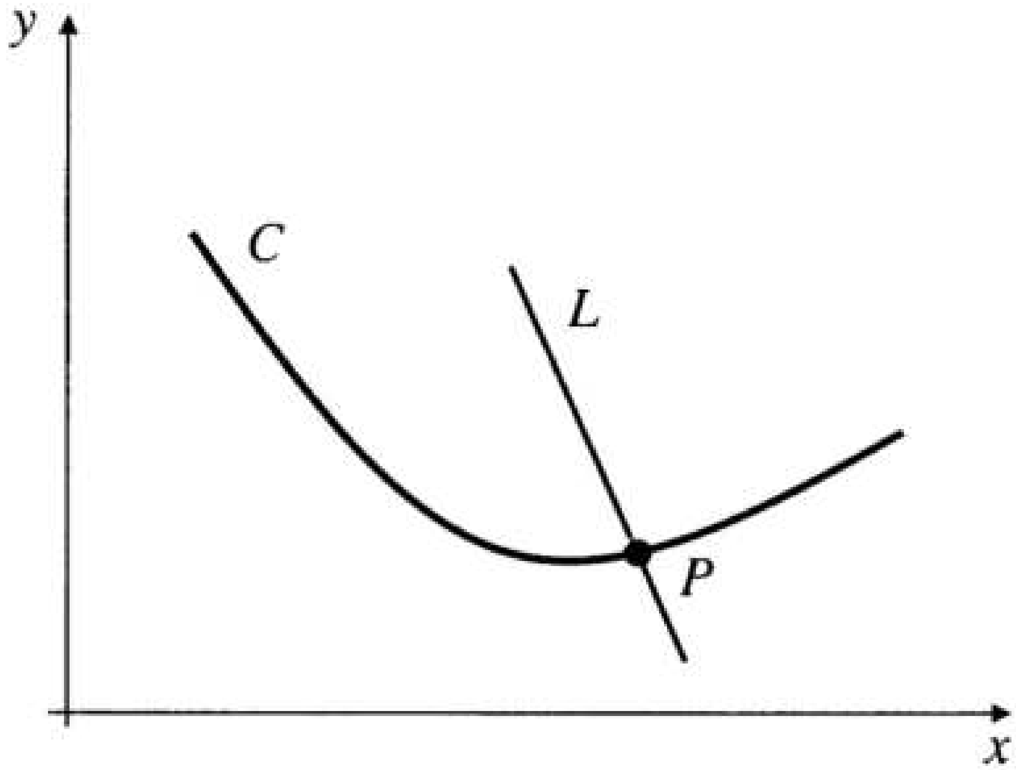

L meets C only at P but is not tangent to C

Tangent Lines and Their Slopes

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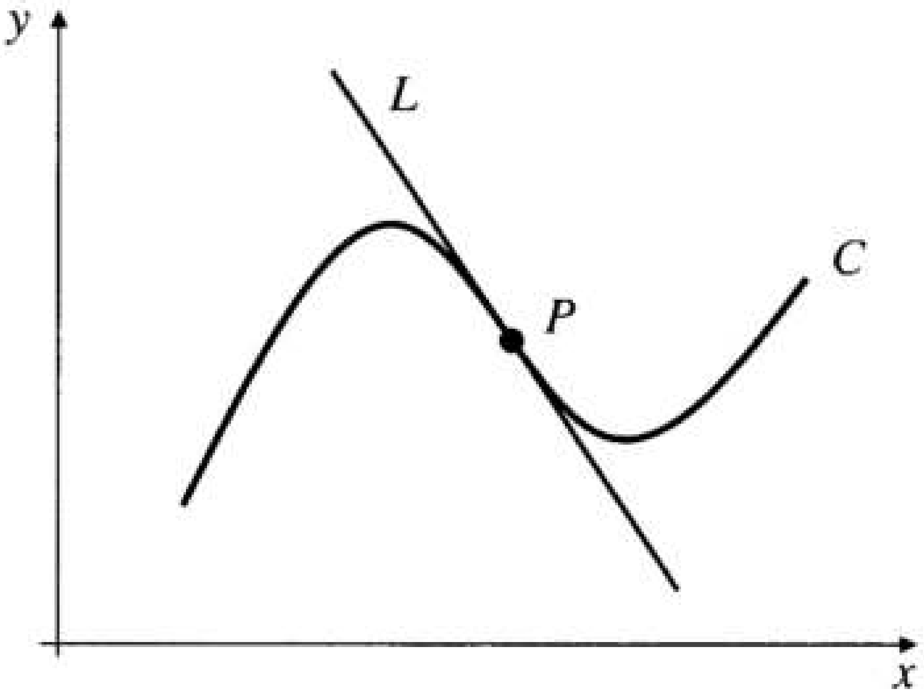

L is tangent to C at P but crosses C at P

Tangent Lines and Their Slopes

What about tangents to a general curve?

(i) It meets the circle at only one point.

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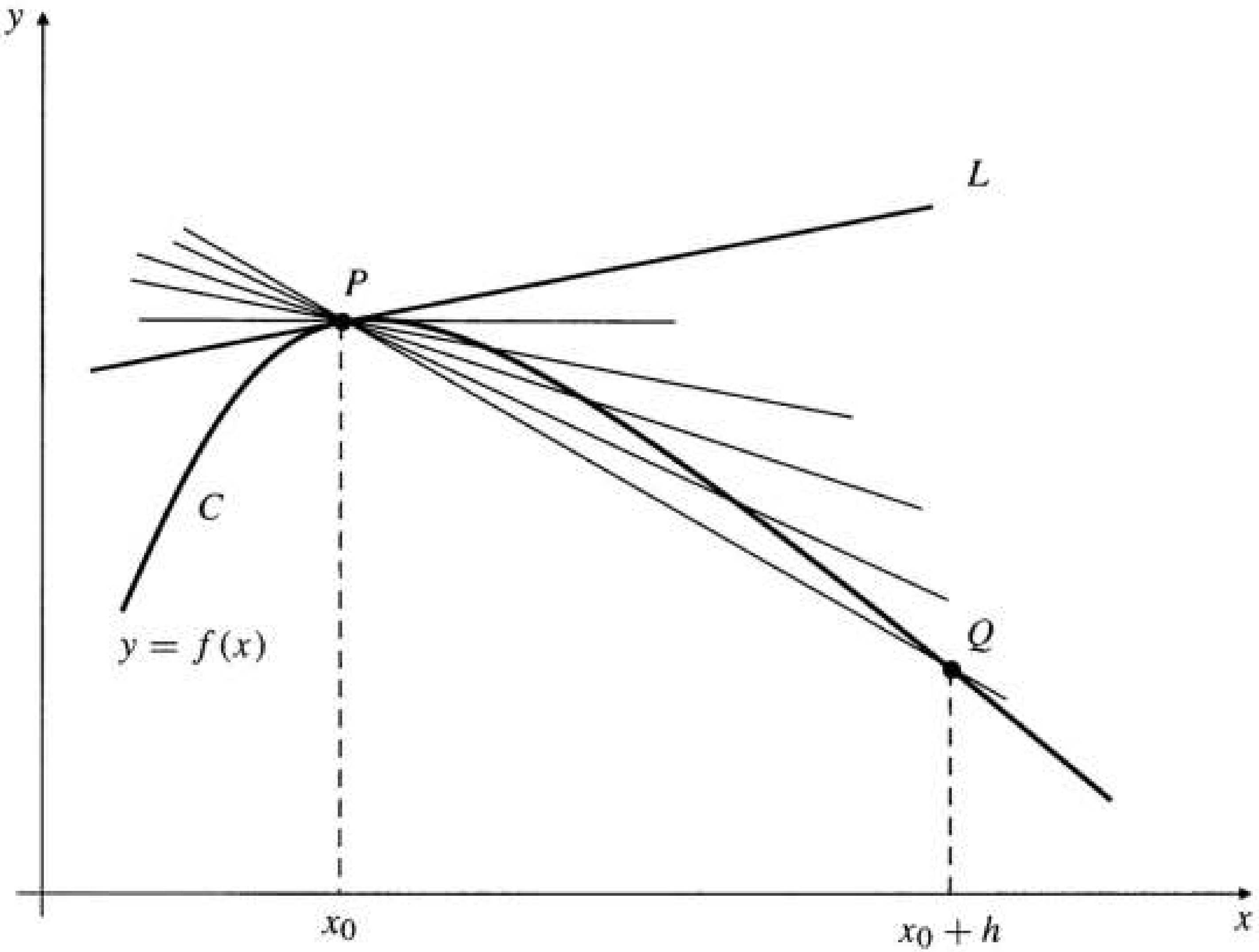
(iii) The tangent is perpendicular to the line joining the centre of the circle to the point of contact.

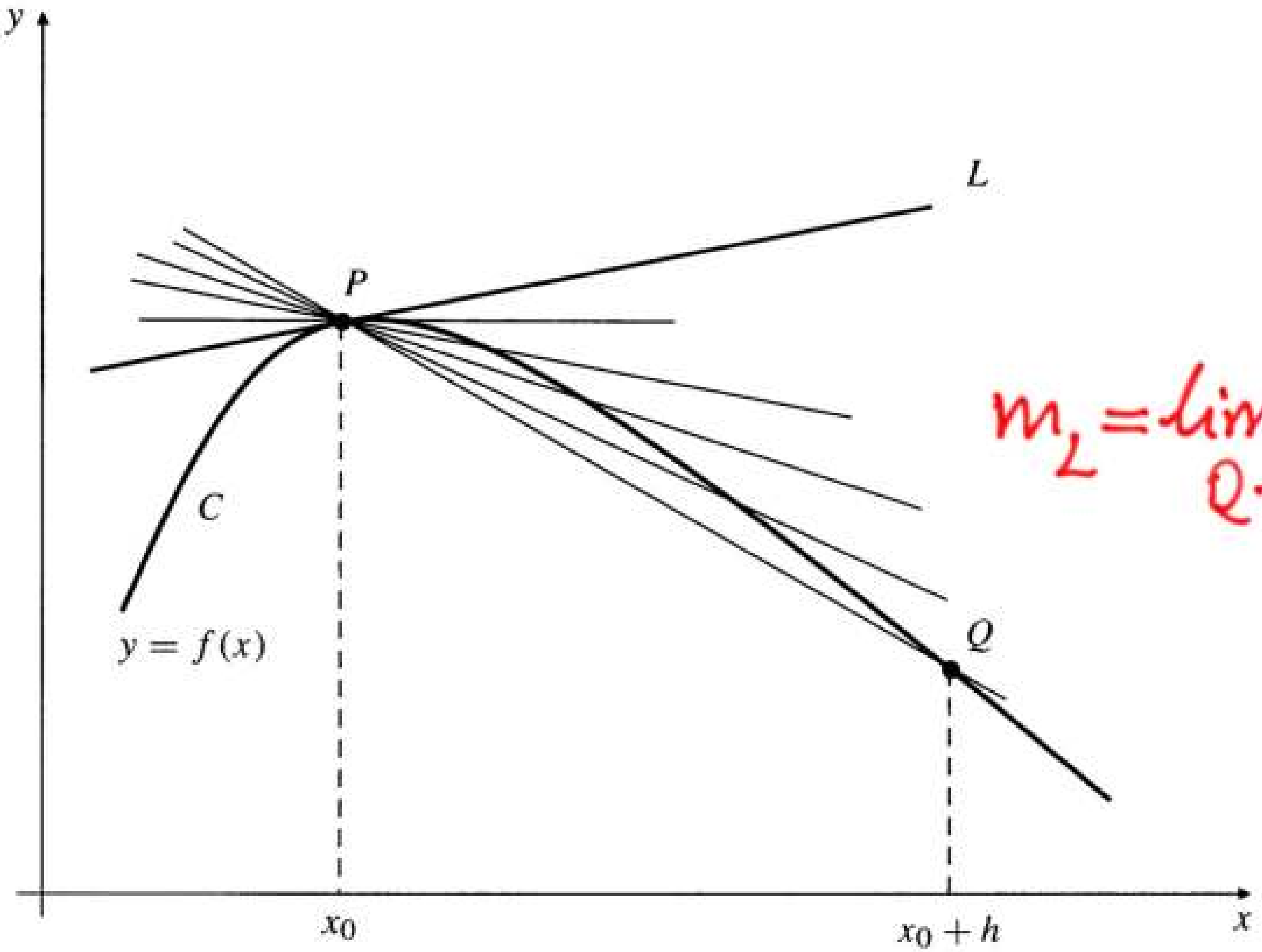
many curves do NOT have centre!

Tangent Lines and Their Slopes



Secant lines PQ approach tangent line L as Q approaches P along the curve C

Tangent Lines and Their Slopes



$$m_L = \lim_{Q \to P} m_{PQ}$$

Secant lines PQ approach tangent line L as Q approaches P along the curve C

Tangent Lines and Their Slopes

Nonvertical tangent lines

Suppose that the function f is continuous at $x = x_0$ and that

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} = m$$

exists. Then the straight line having slope m and passing through the point $P = (x_0, f(x_0))$ is called the **tangent line** (or simply the **tangent**) to the graph of $y = f(x)$ at P. An equation of this tangent is

$$y = m(x - x_0) + y_0.$$

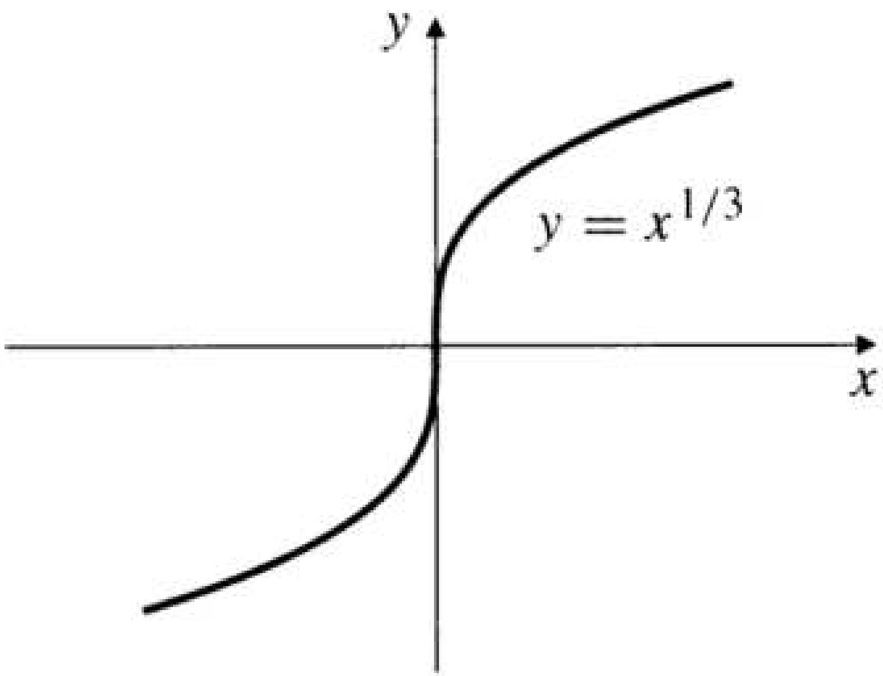
Tangent Lines and Their Slopes

EXAMPLE

Find an equation of the tangent line to the curve $y = x^2$ at the point $(1, 1)$.

Tangent Lines and Their Slopes

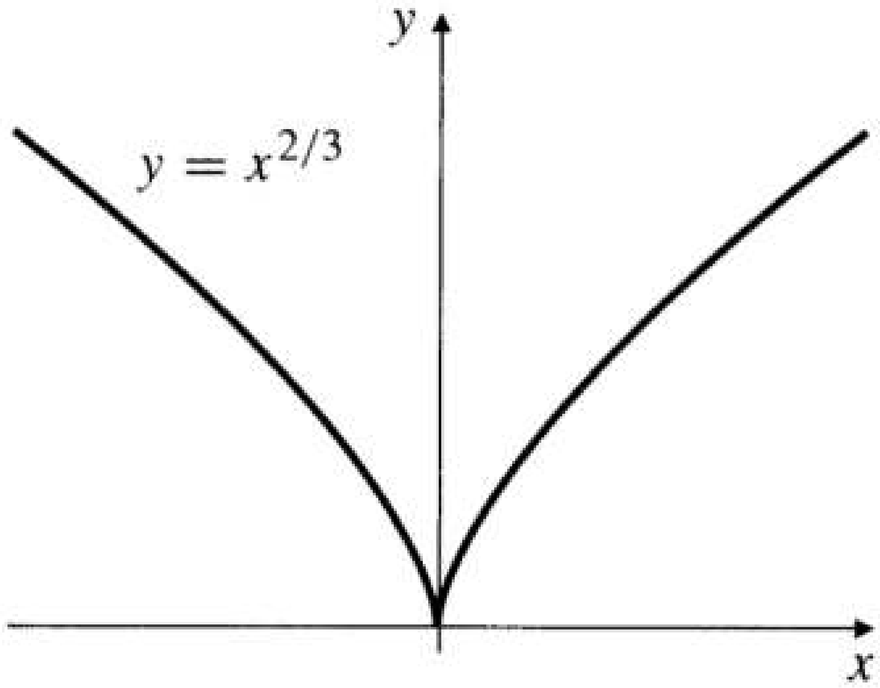
EXAMPLE $f(x) = \sqrt[3]{x} = x^{1/3}$



The y-axis is tangent to $y = x^{1/3}$ at the origin

Tangent Lines and Their Slopes

EXAMPLE $f(x) = x^{2/3}$



This graph has no tangent at the origin

Tangent Lines and Their Slopes

Vertical tangents

If f is continuous at $P = (x_0, y_0)$, where $y_0 = f(x_0)$, and if either

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} = \infty \quad \text{or} \quad \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} = -\infty,$$

then the vertical line $x = x_0$ is tangent to the graph $y = f(x)$ at P. If the limit of the Newton quotient fails to exist in any other way than by being ∞ or $-\infty$, the graph $y = f(x)$ has no tangent line at P.

Tangent Lines and Their Slopes

EXAMPLE Does the graph of $y = |x|$ have a tangent line at $x = 0$?

Tangent Lines and Their Slopes

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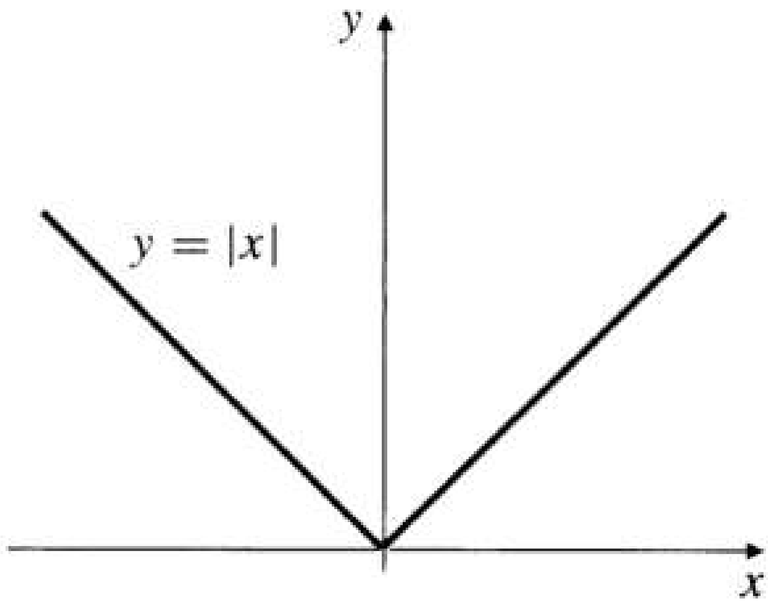

$y = |x|$ has no tangent at the origin

Tangent Lines and Their Slopes

DEFINITION

The slope of a curve

The **slope** of a curve C at a point P is the slope of the tangent line to C at P if such a tangent line exists. In particular, the slope of the graph of $y = f(x)$ at the point x_0 is

$$\lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h}.$$

Tangent Lines and Their Slopes

Normals

If a curve C has a tangent line L at point P, then the straight line N through P perpendicular to L is called the **normal** to C at P. If L is horizontal, then N is vertical; if L is vertical, then N is horizontal. If L is neither horizontal nor vertical, then, the slope of N is the negative reciprocal of the slope of L, that is,

$$\text{slope of the normal} = \frac{-1}{\text{slope of the tangent}}.$$

EXAMPLE Find an equation of the normal to $y = x^2$ at $(1, 1)$.

The Derivative

DEFINITION

The **derivative** of a function f is another function f' defined by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

at all points x for which the limit exists (i.e., is a finite real number). If $f'(x)$ exists, we say that f is **differentiable** at x.

Values of x in $\mathcal{D}(f)$ where f is not differentiable and that are not endpoints of $\mathcal{D}(f)$ are **singular points** of f.

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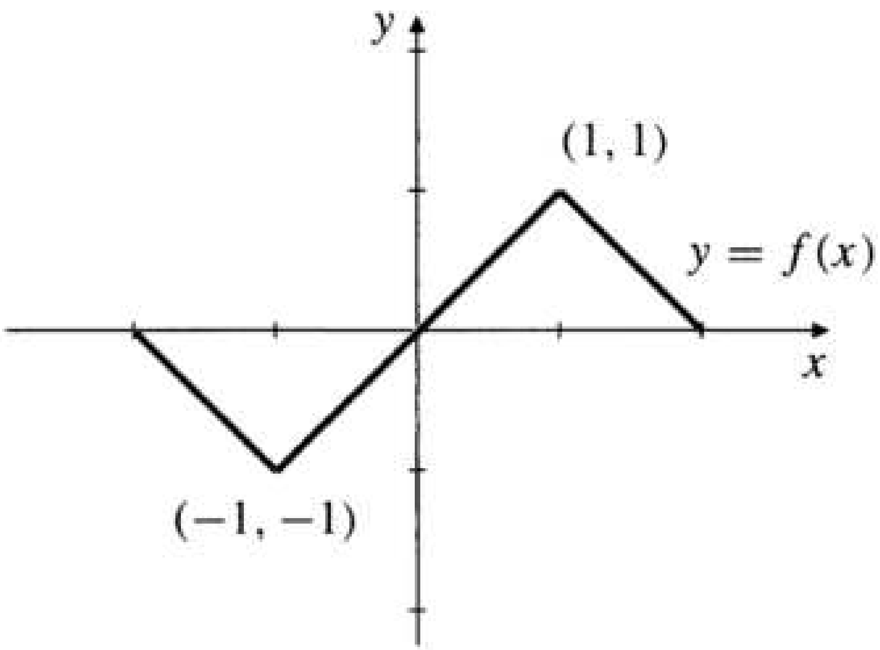
The Derivative

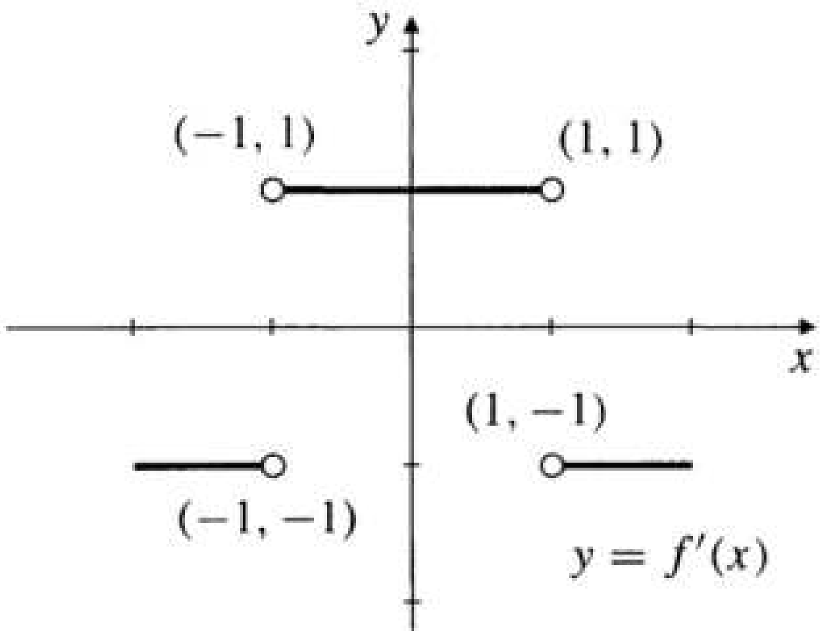
Remark The value of the derivative of f at a particular point x_0 can be expressed as a limit in either of two ways:

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h} = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}.$$

In the second limit $x_0 + h$ is replaced by x, so that $h = x - x_0$ and $h \to 0$ is equivalent to $x \to x_0$.

The Derivative





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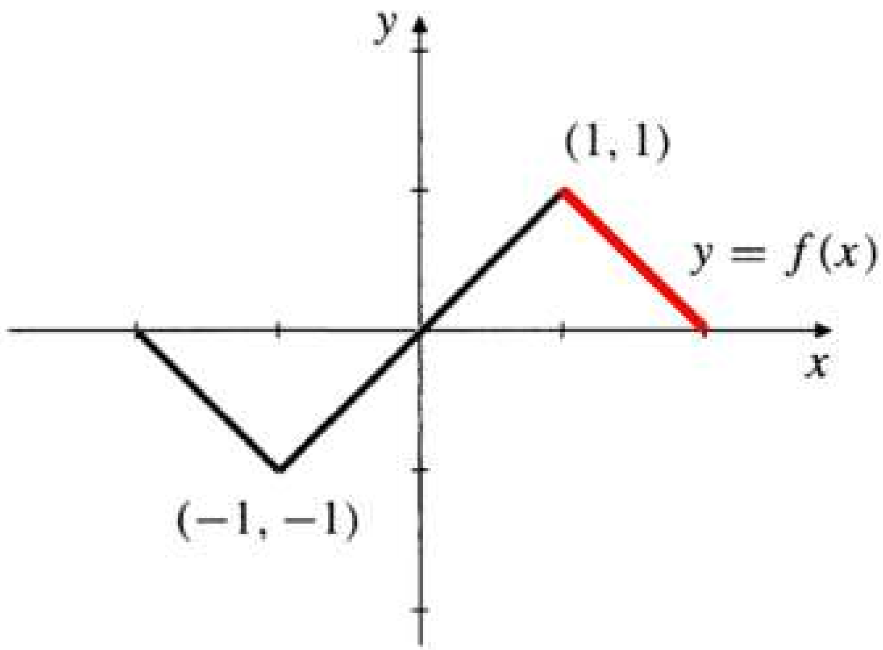

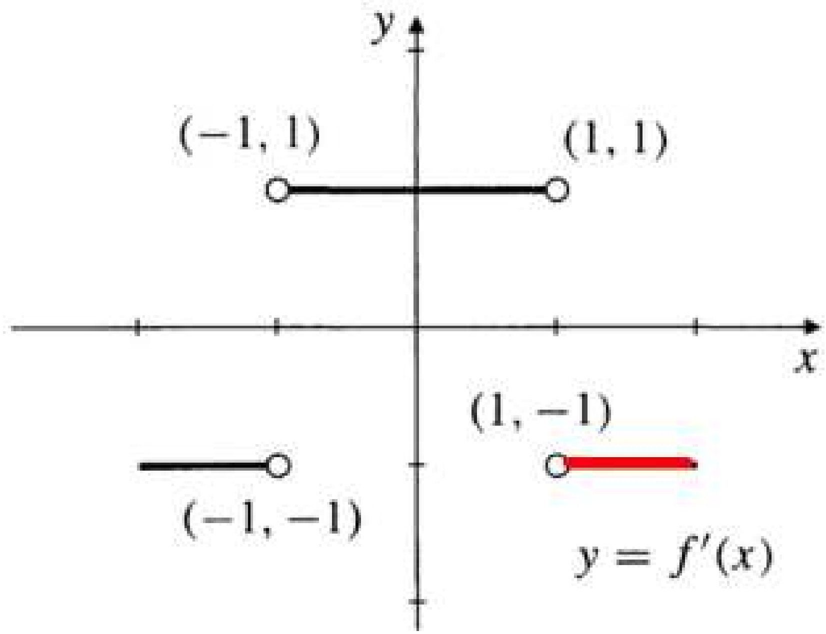

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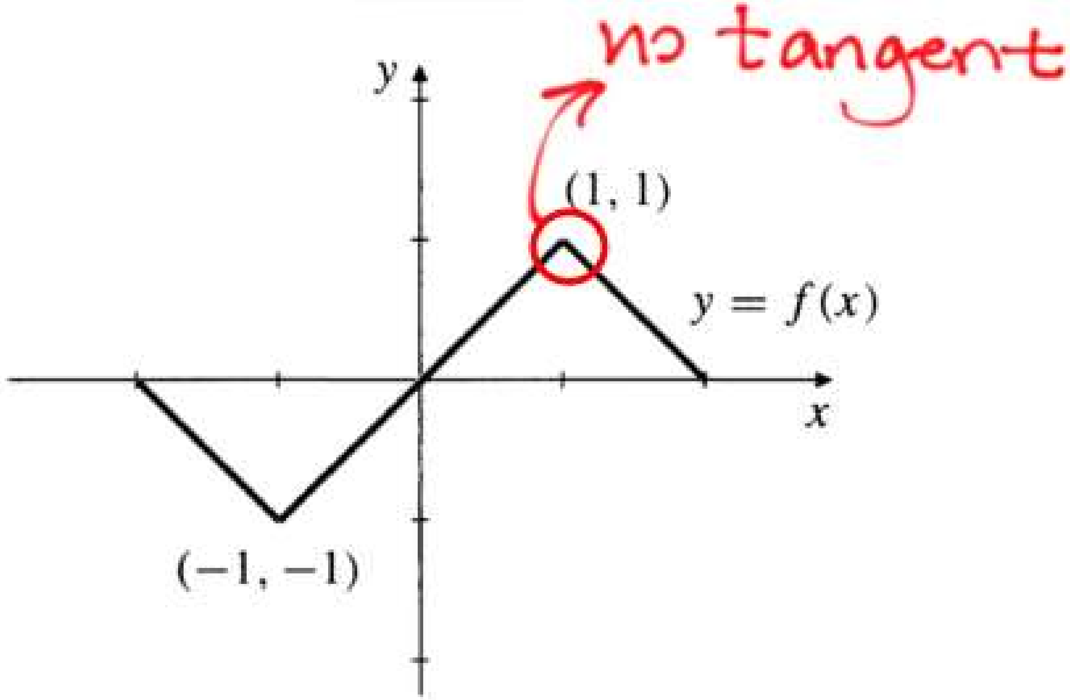

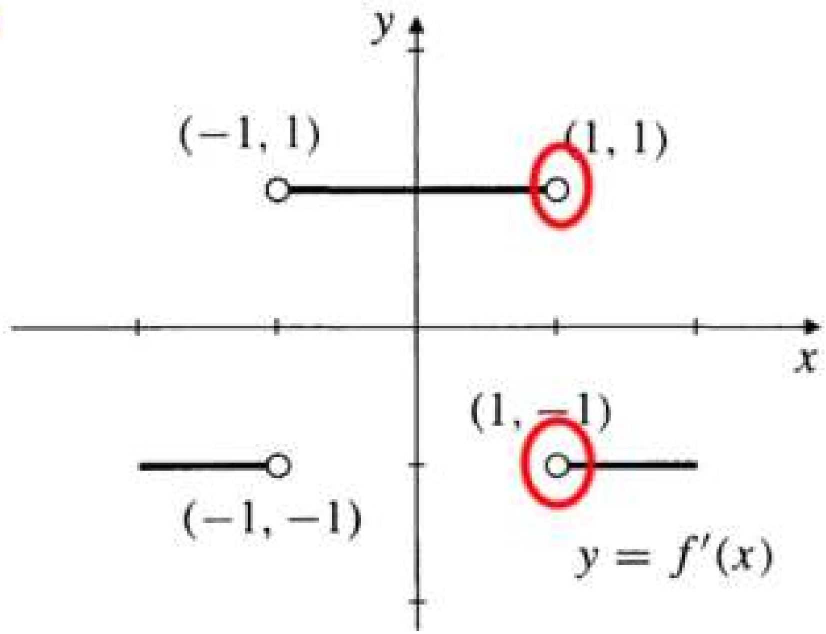

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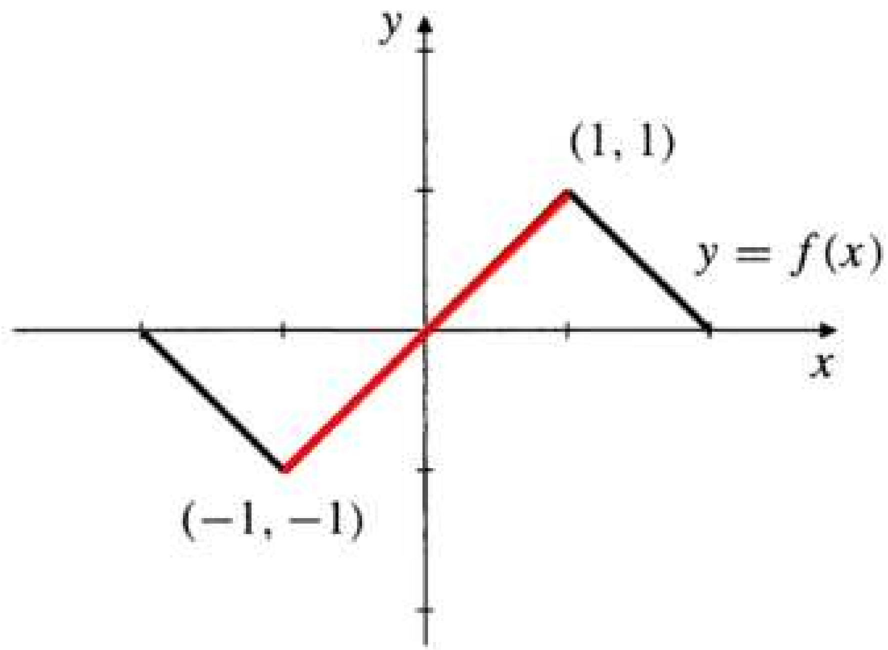

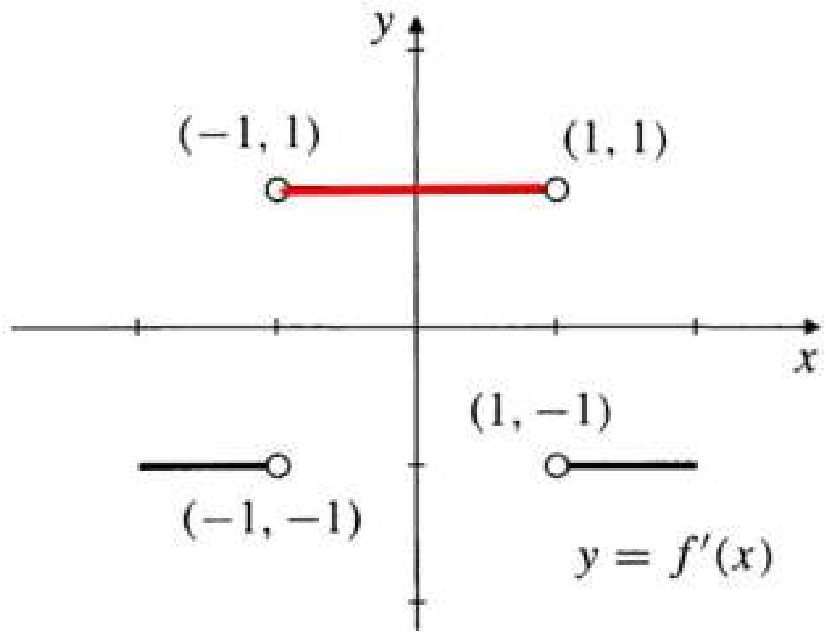

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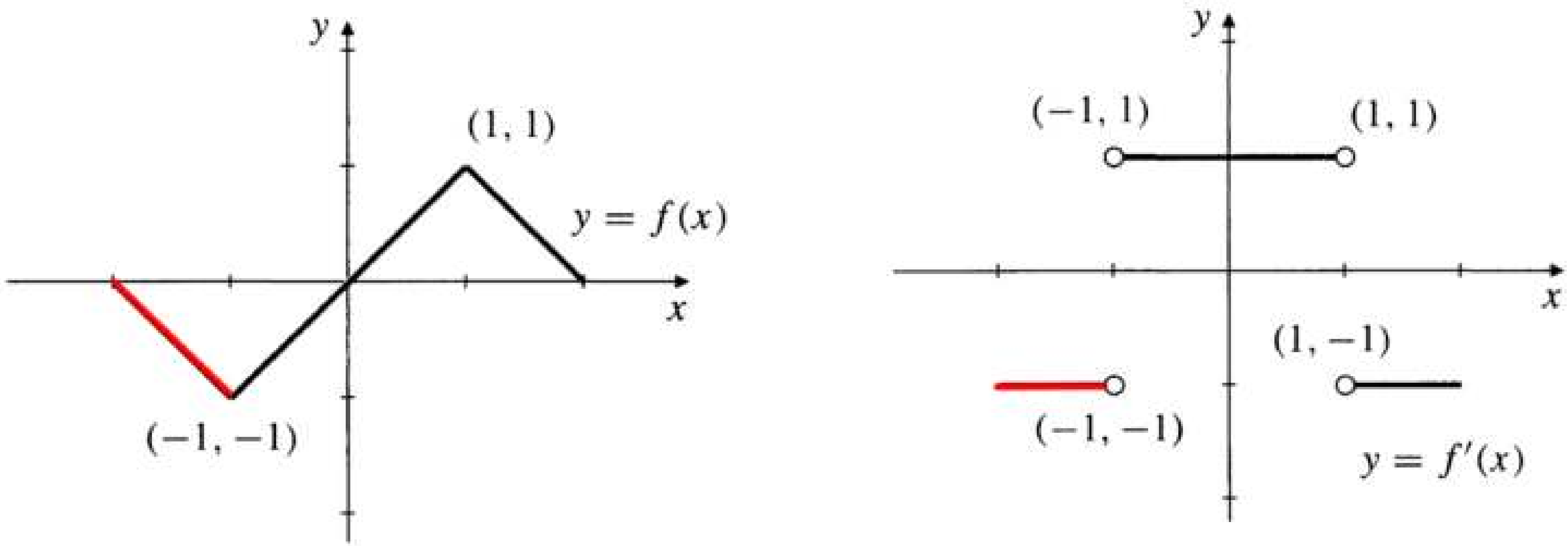

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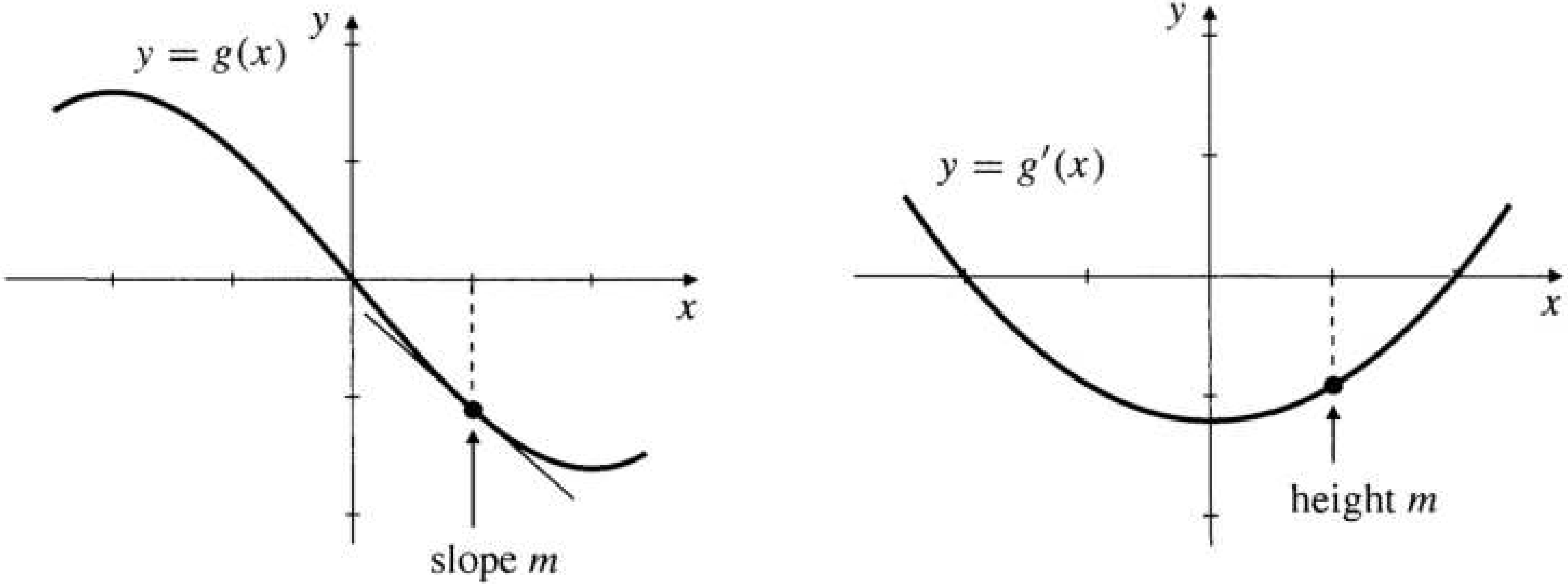

The Derivative

$$f'_{+}(a) = \lim_{h \to 0+} \frac{f(a+h) - f(a)}{h}$$ **right derivative**

$$f'_{-}(b) = \lim_{h \to 0-} \frac{f(b+h) - f(b)}{h}$$ **left derivative**