Implicit Differentiation

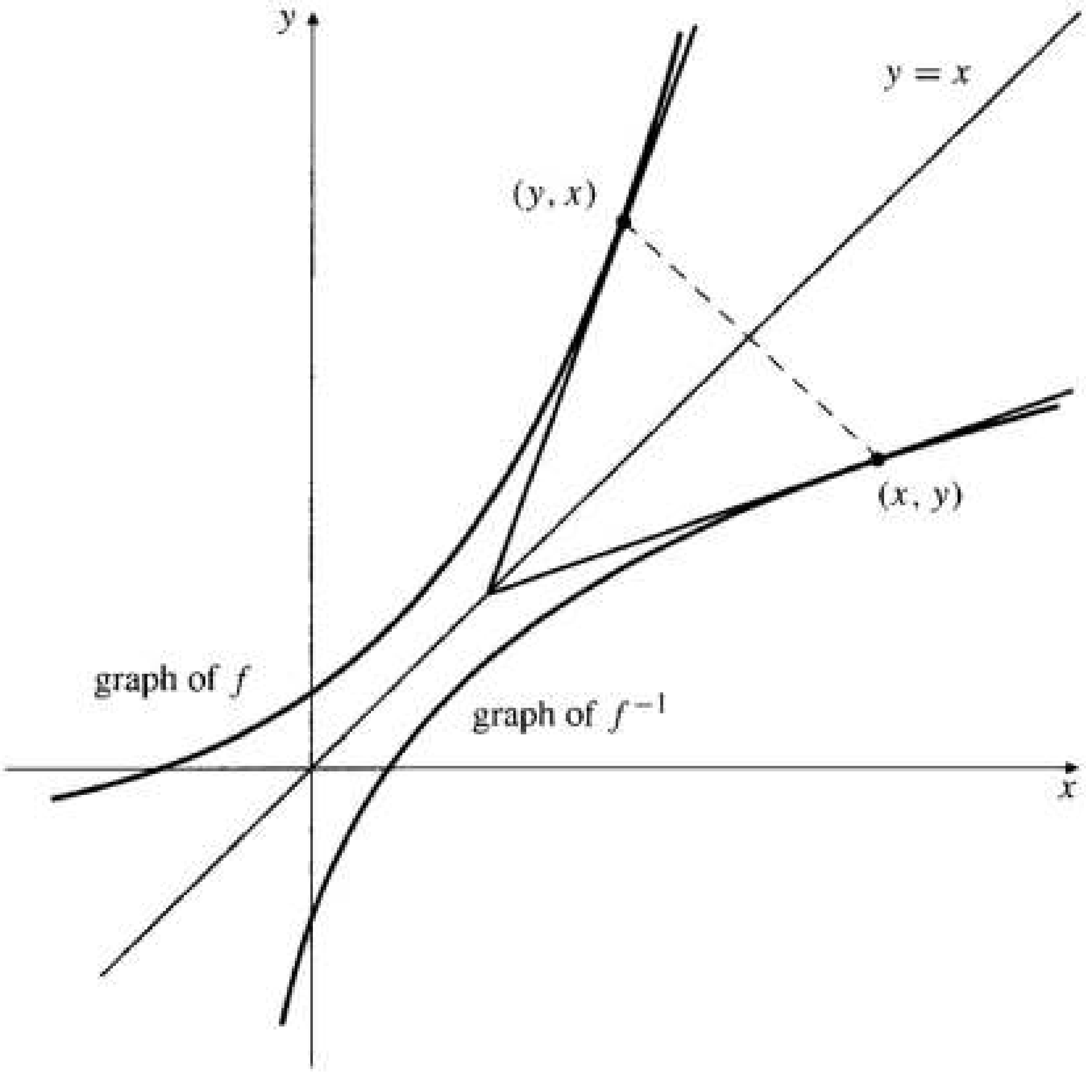
Derivatives of Inverse Functions

Suppose that the function f is differentiable on an interval (a, b) and that either $f'(x) > 0$ for $a < x < b$, so that f is increasing on (a, b), or $f'(x) < 0$ for $a < x < b$, so that f is decreasing on (a, b). In either case f is one-to-one on (a, b) and has an inverse, f^{-1}, defined by

$$y = f^{-1}(x) \quad \Longleftrightarrow \quad x = f(y), \qquad (a < y < b).$$

Implicit Differentiation

Derivatives of Inverse Functions



Implicit Differentiation

Derivatives of Inverse Functions

Let $y = f^{-1}(x)$. We want to find dy/dx. Solve the equation $y = f^{-1}(x)$ for $x = f(y)$ and differentiate implicitly with respect to x to obtain

$$1 = f'(y)\,\frac{dy}{dx}, \qquad \text{so} \qquad \boxed{\frac{dy}{dx} = \frac{1}{f'(y)} = \frac{1}{f'\left(f^{-1}(x)\right)}}.$$

Implicit Differentiation

Derivatives of Inverse Functions

EXAMPLE

Show that $f(x) = x^3 + x$ is one-to-one on the whole real line, and, noting that $f(2) = 10$, find $\left(f^{-1}\right)'(10)$.

Implicit Differentiation

Derivatives of Logarithmic and Exponential Functions

$$\frac{d}{dx}(\log_a x) = \frac{1}{\ln a . x}$$

In particular,

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

Implicit Differentiation

Derivatives of Logarithmic and Exponential Functions

$$y = a^x$$

Implicit Differentiation

Derivatives of Logarithmic and Exponential Functions

$$y = a^x \Rightarrow x = \log_a y$$

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$$\Rightarrow 1 = \frac{1}{\ln a \cdot y} \cdot y'$$

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Derivatives of Logarithmic and Exponential Functions

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$$\Rightarrow 1 = \frac{1}{\ln a \cdot y} \cdot y'$$

$$\Rightarrow \boxed{y' = \ln a \cdot y = \ln a \cdot a^x}$$

Implicit Differentiation

Derivatives of Logarithmic and Exponential Functions

EXAMPLE

Find the derivatives of (a) $\ln|\cos x|$ and (b) $\ln\left(x + \sqrt{x^2 + 1}\right)$. Simplify your answers as much as possible.

Implicit Differentiation

Derivatives of Logarithmic and Exponential Functions

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Solution

(a)

$$\frac{d}{dx}\ln|\cos x| = \frac{1}{\cos x}(-\sin x) = -\tan x.$$

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Derivatives of Logarithmic and Exponential Functions

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Solution

(a)

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(b)

$$\frac{d}{dx}\ln\left(x+\sqrt{x^2+1}\right) = \frac{1}{x+\sqrt{x^2+1}}\left(1+\frac{2x}{2\sqrt{x^2+1}}\right)$$

$$= \frac{1}{x+\sqrt{x^2+1}}\,\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}}$$

$$= \frac{1}{\sqrt{x^2+1}}.$$

Implicit Differentiation

Logarithmic Differentiation

EXAMPLE Find y' if $y = x^x$ for $x > 0$.

Implicit Differentiation

Logarithmic Differentiation

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EXAMPLE Find dy/dt if $y = (\sin t)^{\ln t}$, where $0 < t < \pi$.

Implicit Differentiation

Logarithmic Differentiation

EXAMPLE Differentiate $y = [(x+1)(x+2)(x+3)]/(x+4)$.

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Logarithmic Differentiation

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Solution $\ln|y| = \ln|x+1| + \ln|x+2| + \ln|x+3| - \ln|x+4|$.

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$$y' = \frac{(x+1)(x+2)(x+3)}{x+4}\left(\frac{1}{x+1} + \frac{1}{x+2} + \frac{1}{x+3} - \frac{1}{x+4}\right)$$

Implicit Differentiation

Logarithmic Differentiation

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$$
\begin{aligned}
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&= \frac{(x+2)(x+3)}{x+4} + \frac{(x+1)(x+3)}{x+4} + \frac{(x+1)(x+2)}{x+4} \\
&\qquad - \frac{(x+1)(x+2)(x+3)}{(x+4)^2}.
\end{aligned}
$$

Implicit Differentiation

Logarithmic Differentiation

EXAMPLE Find $\left.\dfrac{du}{dx}\right|_{x=1}$ if $u = \sqrt{(x+1)(x^2+1)(x^3+1)}$.

Implicit Differentiation

Logarithmic Differentiation

EXAMPLE Find $\left.\dfrac{du}{dx}\right|_{x=1}$ if $u = \sqrt{(x+1)(x^2+1)(x^3+1)}$.

Solution

$$\ln u = \frac{1}{2}\left(\ln(x+1) + \ln(x^2+1) + \ln(x^3+1)\right)$$

Implicit Differentiation

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At $x = 1$ we have $u = \sqrt{8} = 2\sqrt{2}$. Hence,

$$\left.\frac{du}{dx}\right|_{x=1} = \sqrt{2}\left(\frac{1}{2} + 1 + \frac{3}{2}\right) = 3\sqrt{2}.$$

Implicit Differentiation

Derivatives of Inverse Trigonometric Functions

If $y = \sin^{-1} x$, then $x = \sin y$ and $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$. Differentiating with respect to x, we obtain

$$1 = (\cos y)\frac{dy}{dx}.$$

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Since $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$, we know that $\cos y \geq 0$. Therefore,

$$\cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2},$$

and $dy/dx = 1/\cos y = 1/\sqrt{1 - x^2}$;

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$$\frac{d}{dx}\sin^{-1} x = \frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1 - x^2}}.$$

Implicit Differentiation

Derivatives of Inverse Trigonometric Functions

$$\frac{d}{dx}\cos^{-1}x = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}$$

$$\frac{d}{dx}\sec^{-1}x = \frac{1}{|x|\sqrt{x^2-1}}$$

Hyperbolic Functions

$$\frac{d}{dx}\cosh x = \sinh x$$

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$$\frac{d}{dx}\tanh x = \operatorname{sech}^2 x$$

$$\frac{d}{dx}\coth x = -\operatorname{csch}^2 x$$

$$\frac{d}{dx}\operatorname{sech} x = -\operatorname{sech} x \tanh x$$

$$\frac{d}{dx}\operatorname{csch} x = -\operatorname{csch} x \coth x$$

More Applications of Differentiation

1. Related rates problems
2. Graphing problems
3. Optimization problems
4. Root finding methods
5. Approximation problems
6. Evaluation of limits

Related Rates

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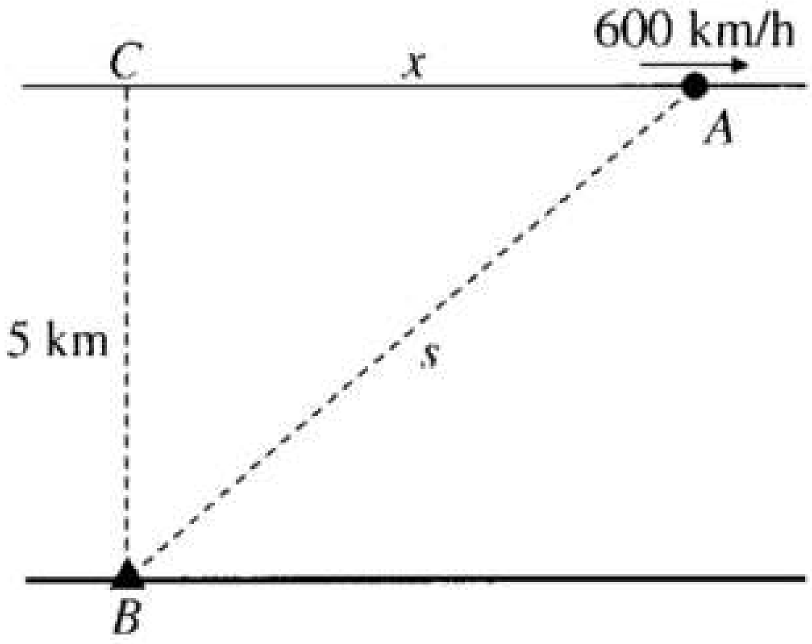

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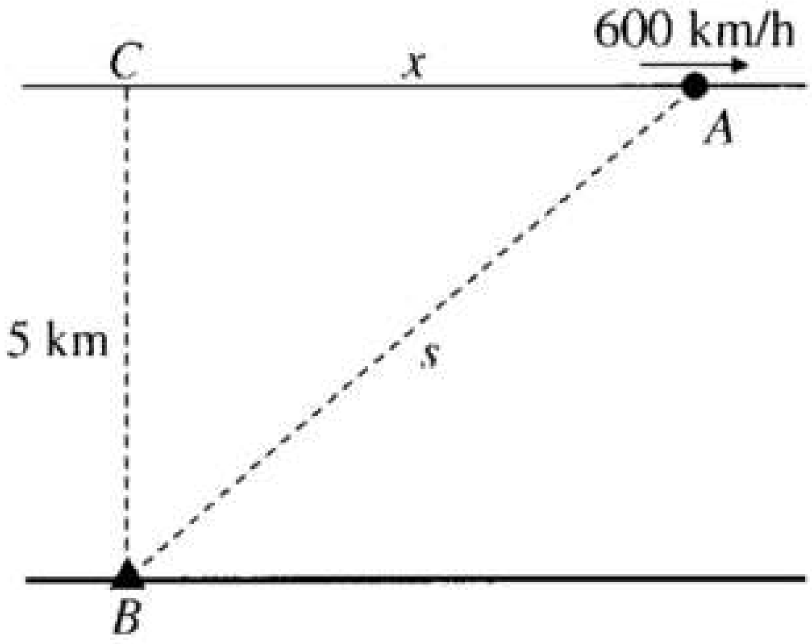

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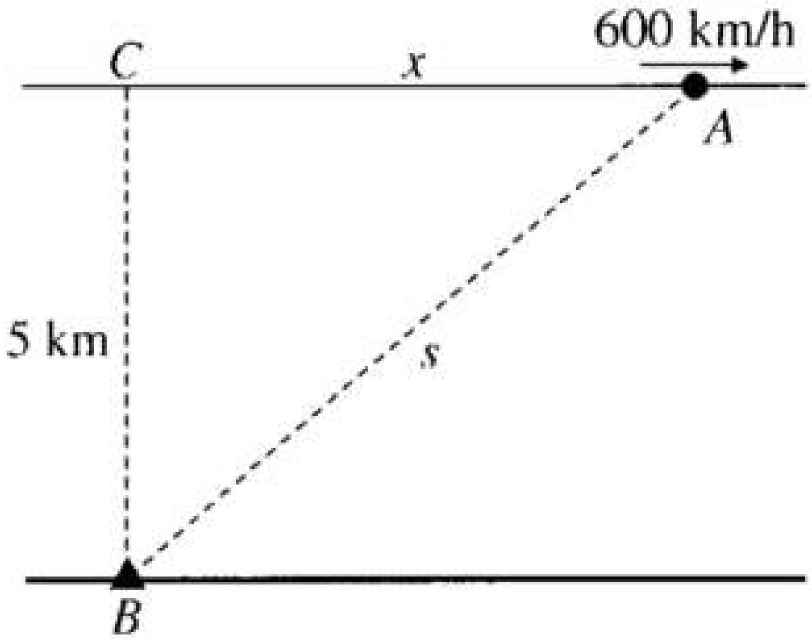

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$$s^2 = x^2 + 5^2$$

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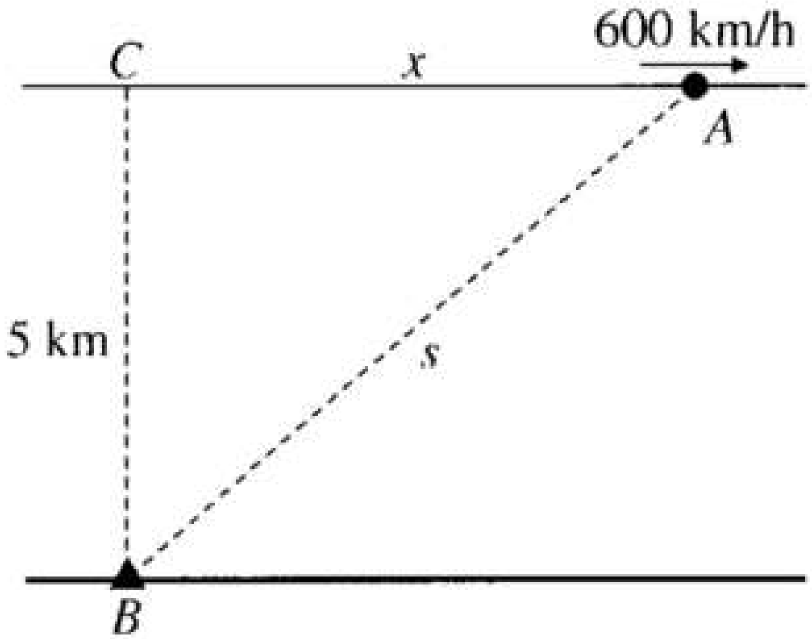

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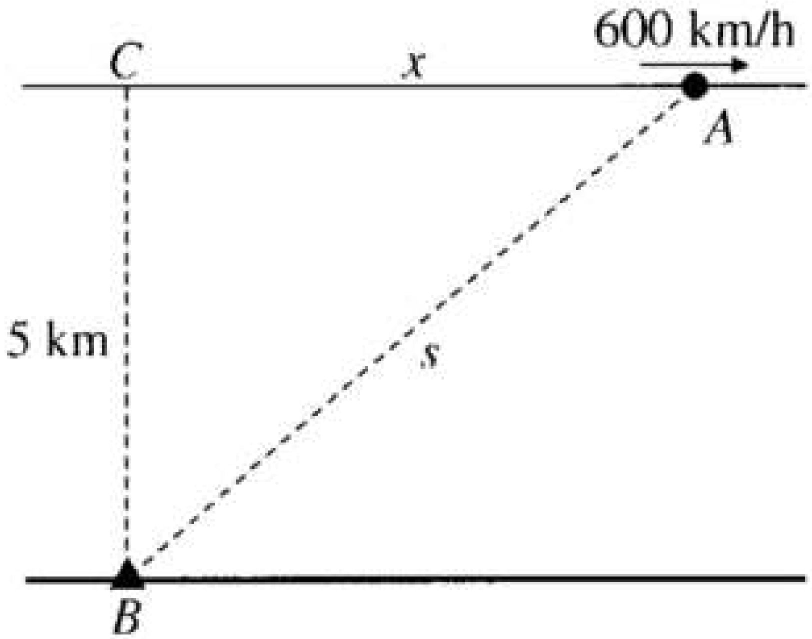

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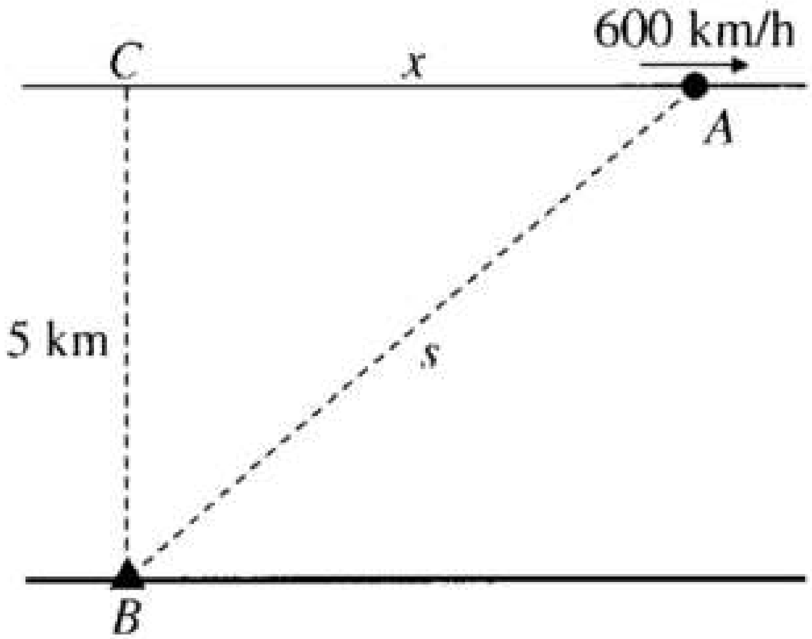

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$$\frac{ds}{dt} = \frac{x}{s}\,\frac{dx}{dt} = \frac{10}{5\sqrt{5}}(600) = \frac{1,200}{\sqrt{5}} \approx 536.7 \text{ km/h}.$$

Related Rates

EXAMPLE How fast is the area of a rectangle changing if one side is 10 cm long and is increasing at a rate of 2 cm/s and the other side is 8 cm long and is decreasing at a rate of 3 cm/s?

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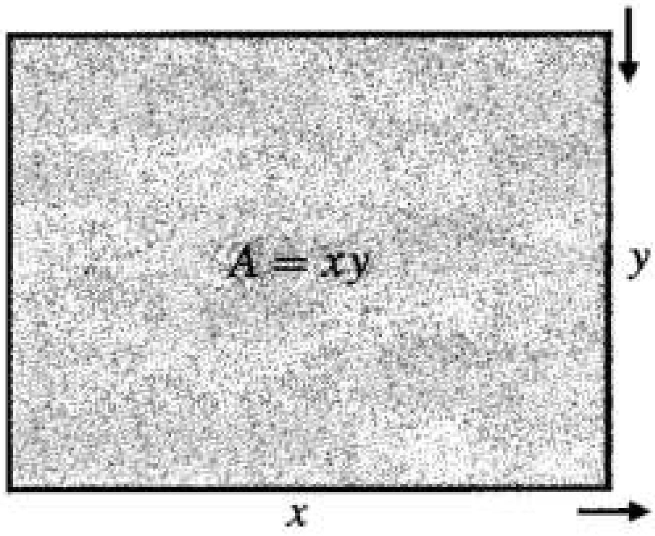

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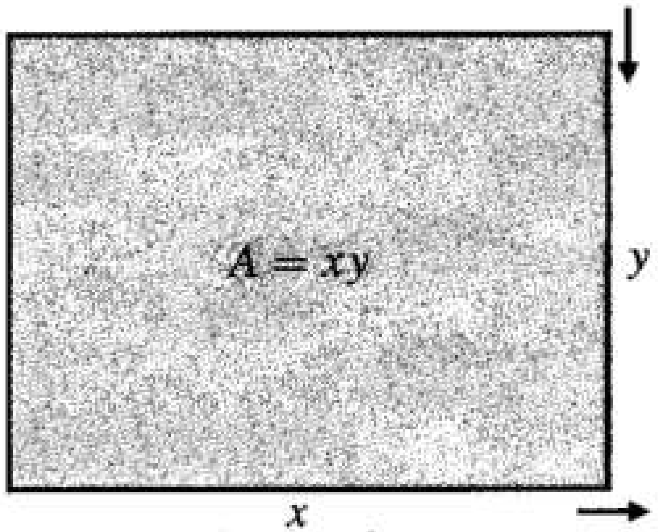

Given: $\dfrac{dx}{dt} = 2$ cm/s

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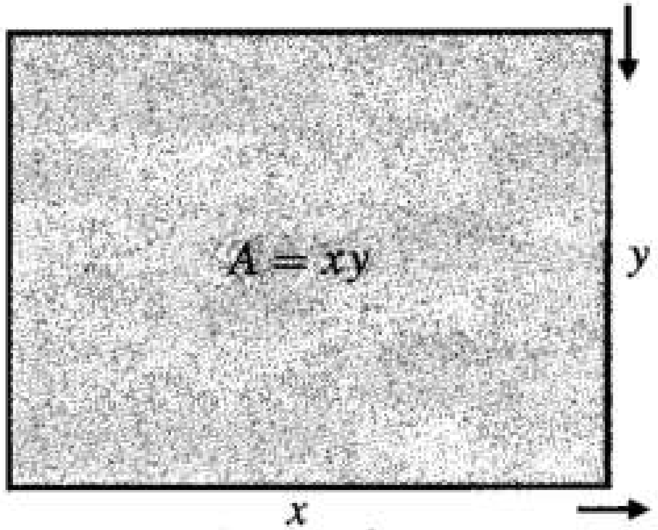

Solution

$$A = xy \Rightarrow$$

$$\frac{dA}{dt} = \frac{dx}{dt} \cdot y + x \cdot \frac{dy}{dt}$$

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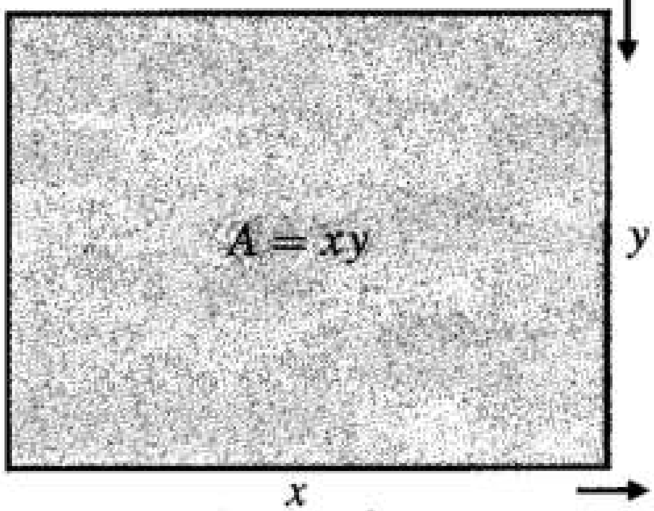

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$$\left.\frac{dA}{dt}\right|_{\substack{x=10\text{ cm}\\ y=8\text{ cm}}} = 2(8) + 10(-3)$$

$$= -14 \text{ cm}^2/\text{s}$$

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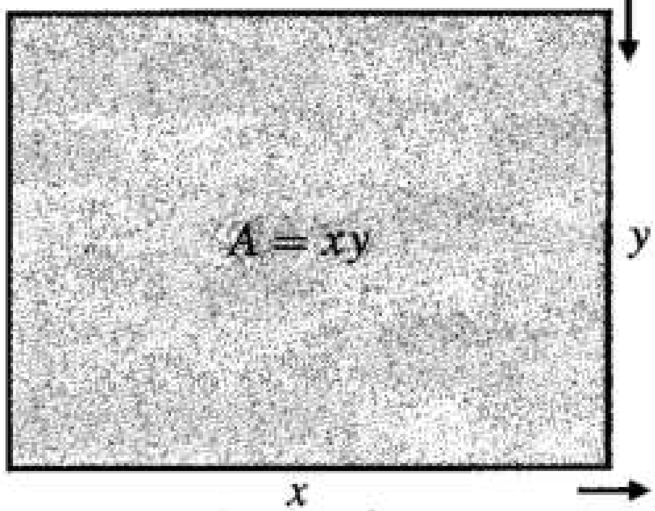

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decreasing

Related Rates

Procedures for Related-Rates Problems

1. Read the problem very carefully. Try to understand the relationships between the variable quantities. What is given? What is to be found?
2. Make a sketch if appropriate.
3. Define any symbols you want to use that are not defined in the statement of the problem. Express given and required quantities and rates in terms of these symbols.
4. From a careful reading of the problem or consideration of the sketch, identify one or more equations linking the variable quantities.
5. Differentiate the equation(s) implicitly with respect to time, regarding all variable quantities as functions of time.
6. Substitute any given values for the quantities and their rates, then solve the resulting equation(s) for the unknown quantities and rates.
7. Make a concluding statement answering the question asked.

Related Rates

EXAMPLE A leaky water tank is in the shape of an inverted right circular cone with depth 5 m and top radius 2 m. When the water in the tank is 4 m deep, it is leaking out at a rate of $1/12$ m^3/min. How fast is the water level in the tank dropping at that time?

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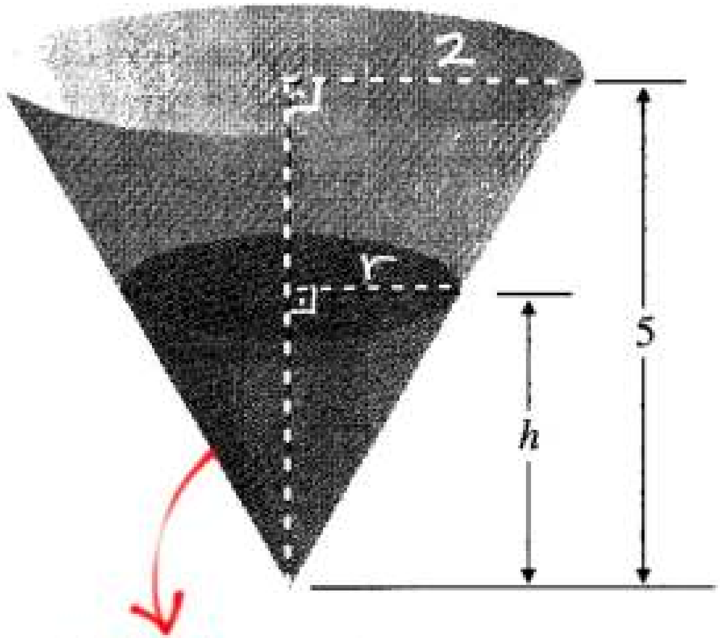

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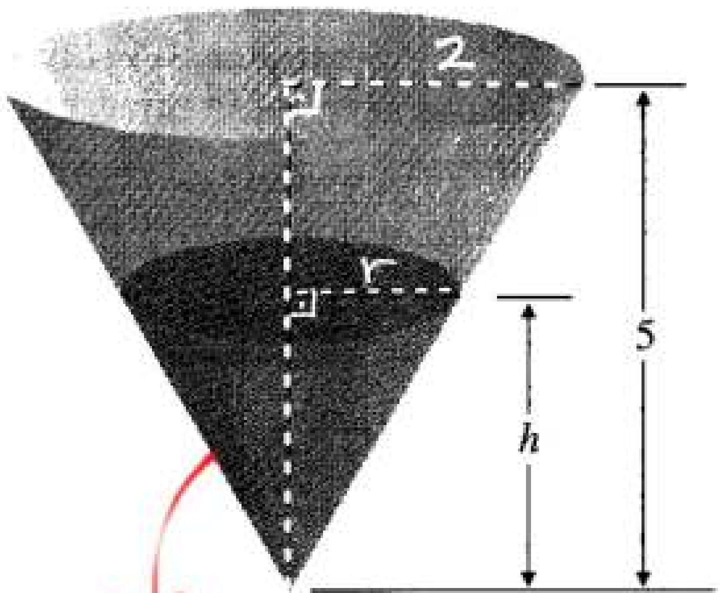

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<u>Given</u>: $\left.\frac{dV}{dt}\right|_{h=4} = -\frac{1}{12}$ m^3/min.

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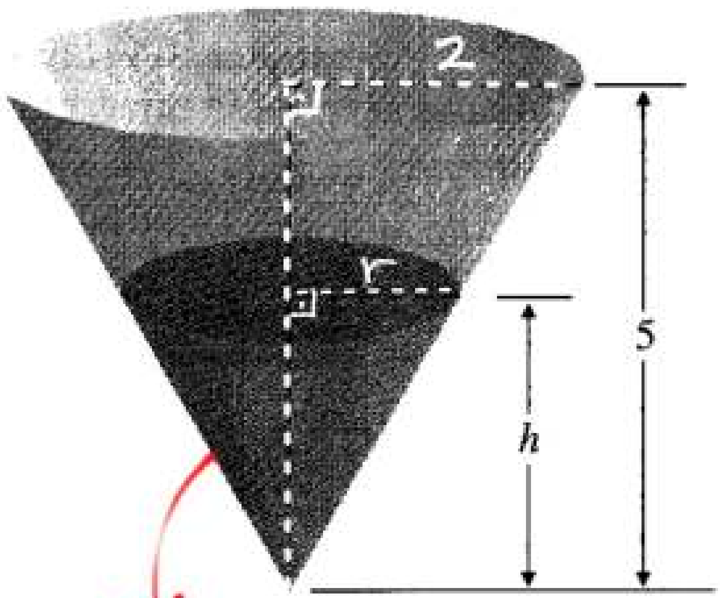

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$$\frac{r}{h} = \frac{2}{5}, \qquad \text{so} \quad r = \frac{2h}{5} \qquad \text{and}$$

$$V = \frac{1}{3}\pi\left(\frac{2h}{5}\right)^2 h = \frac{4\pi}{75}h^3.$$

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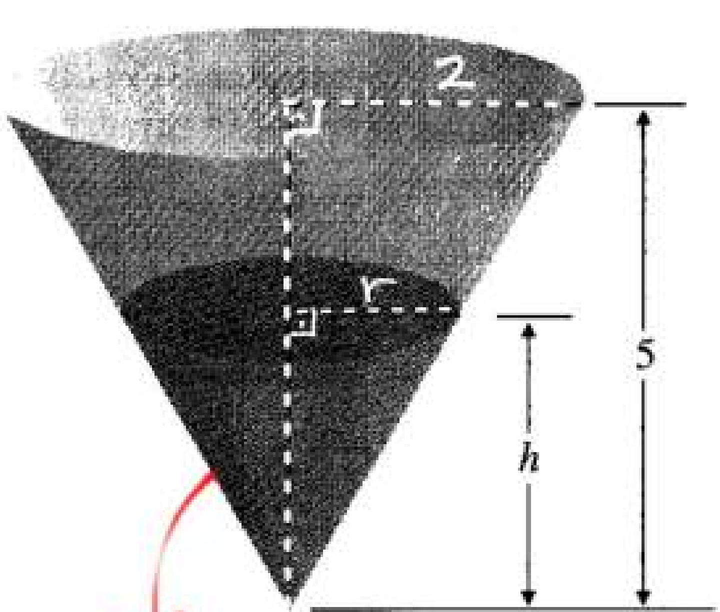

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$$V = \frac{1}{3}\pi\left(\frac{2h}{5}\right)^2 h = \frac{4\pi}{75}h^3. \longrightarrow$$

$$\frac{dV}{dt} = \frac{4\pi}{25}h^2\frac{dh}{dt} \longrightarrow \frac{-1}{12} = \frac{4\pi}{25}(4^2)\frac{dh}{dt} \longrightarrow \frac{dh}{dt} = -\frac{25}{768\pi}.$$