Power Series

DEFINITIONS A **power series about $x = 0$** is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots. \tag{1}$$

A **power series about $x = a$** is a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n + \cdots \tag{2}$$

in which the **center** a and the **coefficients** $c_0, c_1, c_2, \ldots, c_n, \ldots$ are constants.

Power Series

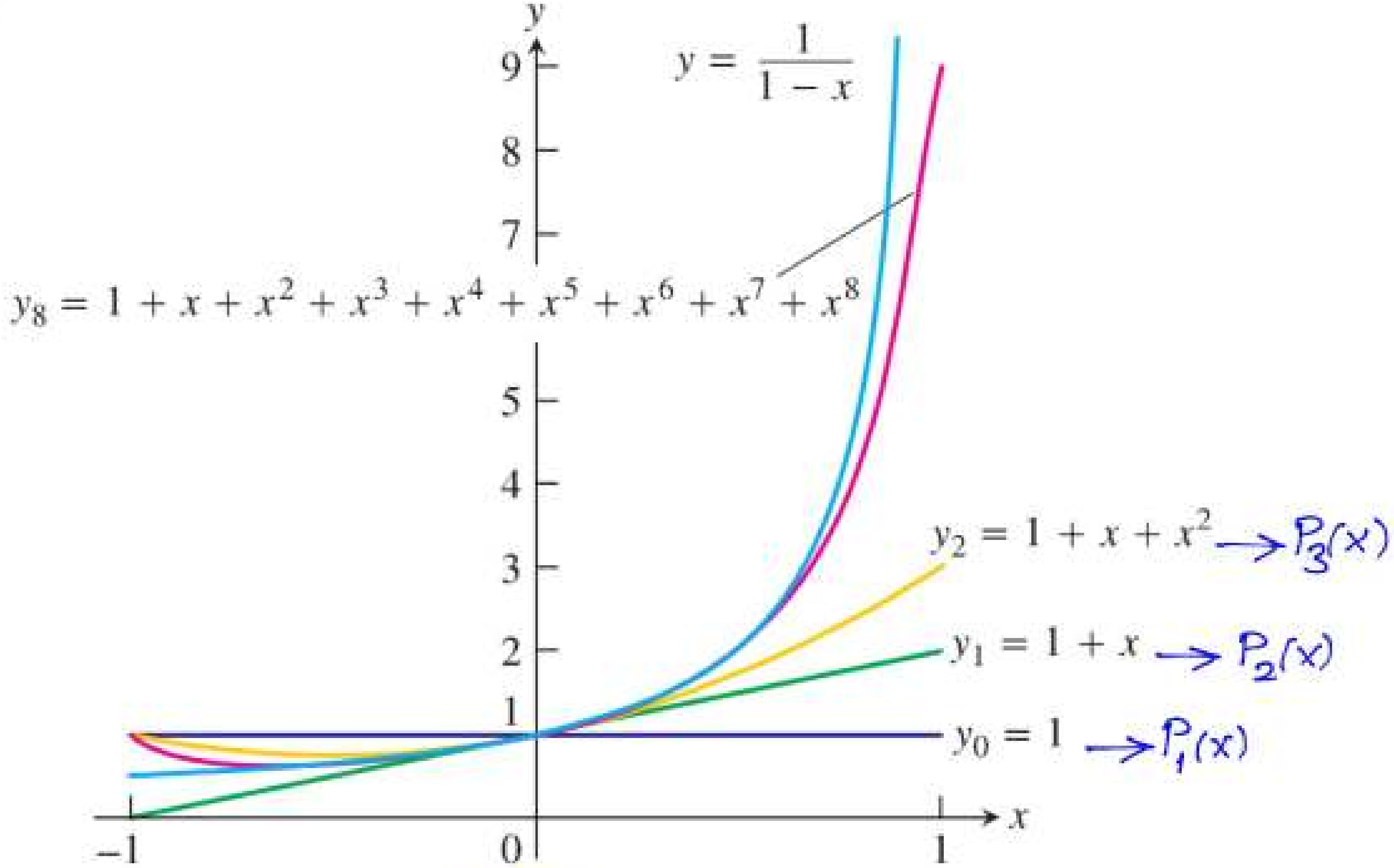
EXAMPLE Taking all the coefficients to be 1 in Equation (1) gives the geometric power series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots + x^n + \cdots.$$

This is the geometric series with first term 1 and ratio x. It converges to $1/(1 - x)$ for $|x| < 1$. We express this fact by writing

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots, \qquad -1 < x < 1. \tag{3}$$

Power Series



$$\frac{1}{1-x} = \underbrace{\underbrace{\underbrace{\overbrace{1 + x + x^2 + \cdots + x^n}^{P_n(x)}}_{}}_{}}_{} + \cdots, \qquad -1 < x < 1.$$

$$\underbrace{\underbrace{\underbrace{1}_{P_1(x)} + x}_{P_2(x)} + x^2}_{P_3(x)}$$

Power Series

EXAMPLE The power series

$$1 - \frac{1}{2}(x-2) + \frac{1}{4}(x-2)^2 + \cdots + \left(-\frac{1}{2}\right)^n (x-2)^n + \cdots$$

matches Equation (2) with $a = 2, c_0 = 1, c_1 = -1/2, c_2 = 1/4, \ldots, c_n = (-1/2)^n$. This is a geometric series with first term 1 and ratio $r = -\frac{x-2}{2}$. The series converges for $\left|\frac{x-2}{2}\right| < 1$ or $0 < x < 4$. The sum is

$$\frac{1}{1-r} = \frac{1}{1 + \frac{x-2}{2}} = \frac{2}{x},$$

so

$$\frac{2}{x} = 1 - \frac{(x-2)}{2} + \frac{(x-2)^2}{4} - \cdots + \left(-\frac{1}{2}\right)^n (x-2)^n + \cdots, \qquad 0 < x < 4.$$

Power Series

EXAMPLE For what values of x do the following power series converge?

(a) $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$

Power Series

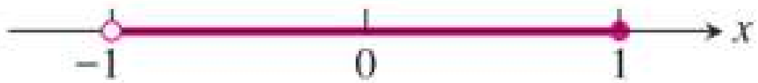
EXAMPLE For what values of x do the following power series converge?

(a) $$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$$

Solution Apply the Ratio Test to the series $\sum|u_n|$, where u_n is the nth term of the power series in question.

(a) $$\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{x^{n+1}}{n+1}\cdot\frac{n}{x}\right| = \frac{n}{n+1}|x| \to |x|.$$

The series converges absolutely for $|x| < 1$. It diverges if $|x| > 1$ because the nth term does not converge to zero. At $x = 1$, we get the alternating harmonic series $1 - 1/2 + 1/3 - 1/4 + \cdots$, which converges. At $x = -1$, we get $-1 - 1/2 - 1/3 - 1/4 - \cdots$, the negative of the harmonic series; it diverges. Series (a) converges for $-1 < x \le 1$ and diverges elsewhere.



Power Series

EXAMPLE For what values of x do the following power series converge?

(b) $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n-1}}{2n-1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$

Solution Apply the Ratio Test to the series $\sum |u_n|$, where u_n is the nth term of the power series in question.

Power Series

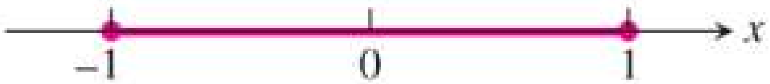
EXAMPLE For what values of x do the following power series converge?

(b) $$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{2n-1}}{2n-1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$$

Solution Apply the Ratio Test to the series $\sum|u_n|$, where u_n is the nth term of the power series in question.

(b) $$\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{x^{2n+1}}{2n+1}\cdot\frac{2n-1}{x^{2n-1}}\right| = \frac{2n-1}{2n+1}x^2 \to x^2.$$

The series converges absolutely for $x^2 < 1$. It diverges for $x^2 > 1$ because the nth term does not converge to zero. At $x = 1$ the series becomes $1 - 1/3 + 1/5 - 1/7 + \cdots$, which converges by the Alternating Series Theorem. It also converges at $x = -1$ because it is again an alternating series that satisfies the conditions for convergence. The value at $x = -1$ is the negative of the value at $x = 1$. Series (b) converges for $-1 \le x \le 1$ and diverges elsewhere.



Power Series

EXAMPLE For what values of x do the following power series converge?

(c) $$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \cdots$$

Solution Apply the Ratio Test to the series $\sum |u_n|$, where u_n is the nth term of the power series in question.

Power Series

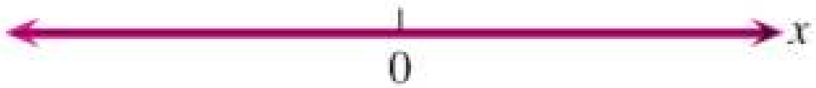
EXAMPLE For what values of x do the following power series converge?

$$\text{(c)}\quad \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \quad \cdots$$

Solution Apply the Ratio Test to the series $\sum |u_n|$, where u_n is the nth term of the power series in question.

$$\text{(c)}\quad \left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n}\right| = \frac{|x|}{n+1} \to 0 \text{ for every } x.$$

The series converges absolutely for all x.



Power Series

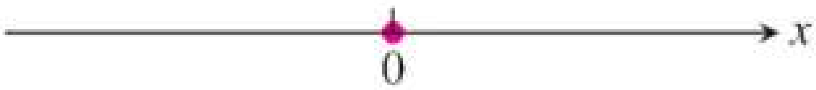
EXAMPLE For what values of x do the following power series converge?

(d) $\sum_{n=0}^{\infty} n!x^n = 1 + x + 2!x^2 + 3!x^3 + \cdots$

Solution Apply the Ratio Test to the series $\sum|u_n|$, where u_n is the nth term of the power series in question.

(d) $$\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{(n+1)!x^{n+1}}{n!x^n}\right| = (n+1)|x| \to \infty \text{ unless } x = 0.$$

The series diverges for all values of x except $x = 0$.



Power Series

THEOREM —The Convergence Theorem for Power Series If the power series $\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$ converges at $x = c \neq 0$, then it converges absolutely for all x with $|x| < |c|$. If the series diverges at $x = d$, then it diverges for all x with $|x| > |d|$.

Power Series

The Radius of Convergence of a Power Series

COROLLARY The convergence of the series $\sum c_n(x - a)^n$ is described by one of the following three cases:

1. There is a positive number R such that the series diverges for x with $|x - a| > R$ but converges absolutely for x with $|x - a| < R$. The series may or may not converge at either of the endpoints $x = a - R$ and $x = a + R$.
2. The series converges absolutely for every x $(R = \infty)$.
3. The series converges at $x = a$ and diverges elsewhere $(R = 0)$.

Power Series

The Radius of Convergence of a Power Series

COROLLARY The convergence of the series $\sum c_n(x-a)^n$ is described by one of the following three cases:

1. There is a positive number R such that the series diverges for x with $|x-a|>R$ but converges absolutely for x with $|x-a|<R$. The series may or may not converge at either of the endpoints $x=a-R$ and $x=a+R$.
2. The series converges absolutely for every x $(R=\infty)$.
3. The series converges at $x=a$ and diverges elsewhere $(R=0)$.

R is called the **radius of convergence** of the power series, and the interval of radius R centered at $x=a$ is called the **interval of convergence**. If the series converges for all values of x, we say its radius of convergence is infinite. If it converges only at $x=a$, we say its radius of convergence is zero.

Power Series

The Radius of Convergence of a Power Series

EXAMPLE Determine the centre, radius, and interval of convergence of

$$\sum_{n=0}^{\infty} \frac{(2x+5)^n}{(n^2+1)3^n}.$$

Power Series

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EXAMPLE Determine the centre, radius, and interval of convergence of

$$\sum_{n=0}^{\infty} \frac{(2x+5)^n}{(n^2+1)3^n}. = \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n \frac{1}{n^2+1} \left(x+\frac{5}{2}\right)^n$$

Power Series

The Radius of Convergence of a Power Series

EXAMPLE Determine the centre, radius, and interval of convergence of

$$\sum_{n=0}^{\infty} \frac{(2x+5)^n}{(n^2+1)3^n}. = \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n \frac{1}{n^2+1} \left(x+\frac{5}{2}\right)^n$$

SOLUTION. Let $a_n = \frac{(2x+5)^n}{(n^2+1)3^n}$. Then

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n^2+1)3^n}{(n^2+2n+2)3^{n+1}} \cdot |2x+5| \longrightarrow \frac{1}{3}|2x+5| < 1 \Rightarrow -3 < 2x+5 < 3$$

$\Rightarrow -4 < x < -1 \Rightarrow R = \frac{-1-(-4)}{2} = \frac{3}{2}$ and the series converges absolutely on $(-4,-1)$; diverges if $x > -1$ or $x < -4$.

$x = -1 \Rightarrow \sum_{n=0}^{\infty} \frac{1}{n^2+1}$ converges

$x = -4 \Rightarrow \sum_{n=0}^{\infty} \frac{(-1)^n}{n^2+1}$ converges

$\rightarrow$ the radius of convergence is $[-4,-1]$

Power Series

Operations on Power Series

THEOREM —The Term-by-Term Differentiation Theorem If $\sum c_n(x-a)^n$ has radius of convergence $R > 0$, it defines a function

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n \qquad \text{on the interval} \qquad a - R < x < a + R.$$

This function f has derivatives of all orders inside the interval, and we obtain the derivatives by differentiating the original series term by term:

$$f'(x) = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1},$$

$$f''(x) = \sum_{n=2}^{\infty} n(n-1)c_n(x-a)^{n-2},$$

and so on. Each of these derived series converges at every point of the interval $a - R < x < a + R$.

Power Series

Operations on Power Series

THEOREM —The Term-by-Term Integration Theorem Suppose that

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

converges for $a - R < x < a + R$ $(R > 0)$. Then

$$\sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$$

converges for $a - R < x < a + R$ and

$$\int f(x)\,dx = \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1} + C$$

for $a - R < x < a + R$.

Power Series

Operations on Power Series

EXAMPLE Find power series representations for the functions

$$\text{(a)}\quad \frac{1}{(1-x)^2}, \qquad \text{(b)}\quad \frac{1}{(1-x)^3}, \qquad \text{and} \qquad \text{(c)}\quad \ln(1+x)$$

by starting with the geometric series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \qquad (-1 < x < 1)$$

and using differentiation, integration, and substitution. Where is each series valid?

Power Series

Operations on Power Series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \qquad (-1 < x < 1)$$

Solution

(a) Differentiate the geometric series term by term to obtain

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1} = 1 + 2x + 3x^2 + 4x^3 + \cdots \qquad (-1 < x < 1).$$

Power Series

Operations on Power Series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \qquad (-1 < x < 1)$$

Solution

(a) Differentiate the geometric series term by term to obtain

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1} = 1 + 2x + 3x^2 + 4x^3 + \cdots \qquad (-1 < x < 1).$$

(b) Differentiate again to get, for $-1 < x < 1$,

$$\frac{2}{(1-x)^3} = \sum_{n=2}^{\infty} n(n-1)\, x^{n-2} = (1 \times 2) + (2 \times 3)x + (3 \times 4)x^2 + \cdots.$$

Now divide by 2:

$$\frac{1}{(1-x)^3} = \sum_{n=2}^{\infty} \frac{n(n-1)}{2} x^{n-2} = 1 + 3x + 6x^2 + 10x^3 + \cdots \quad (-1 < x < 1).$$

Power Series

Operations on Power Series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \qquad (-1 < x < 1)$$

Solution

(c) Substitute $-t$ in place of x in the original geometric series:

$$\frac{1}{1+t} = \sum_{n=0}^{\infty} (-1)^n t^n = 1 - t + t^2 - t^3 + t^4 - \cdots \qquad (-1 < t < 1).$$

Integrate from 0 to x, where $|x| < 1$, to get

$$\begin{aligned} \ln(1+x) &= \int_0^x \frac{1}{1+t}\,dt = t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \cdots \Big]_0^x \\ &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \end{aligned}$$

or

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}, \qquad -1 < x < 1.$$

Power Series

Operations on Power Series

EXAMPLE

Use the geometric series of the previous example to find a power series representation for $\tan^{-1} x$.

Power Series

Operations on Power Series

EXAMPLE Use the geometric series of the previous example to find a power series representation for $\tan^{-1} x$.

Solution Substitute $-t^2$ for x in the geometric series. Since $0 \le t^2 < 1$ whenever $-1 < t < 1$, we obtain

$$\frac{1}{1+t^2} = 1 - t^2 + t^4 - t^6 + t^8 - \cdots \qquad (-1 < t < 1).$$

Now integrate from 0 to x, where $|x| < 1$:

$$\begin{aligned}
\tan^{-1} x = \int_0^x \frac{dt}{1+t^2} &= \int_0^x (1 - t^2 + t^4 - t^6 + t^8 - \cdots)\, dt \\
&= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \qquad (-1 < x < 1).
\end{aligned}$$

Power Series

Operations on Power Series

EXAMPLE Find a series representation of $f(x) = 1/(2+x)$ in powers of $x - 1$. What is the interval of convergence of this series?

Solution Let $t = x - 1$ so that $x = t + 1$. We have

$$
\begin{aligned}
\frac{1}{2+x} = \frac{1}{3+t} &= \frac{1}{3}\,\frac{1}{1+\dfrac{t}{3}} \\
&= \frac{1}{3}\left(1 - \frac{t}{3} + \frac{t^2}{3^2} - \frac{t^3}{3^3} + \cdots\right) && (-1 < t/3 < 1) \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{t^n}{3^{n+1}} && (-3 < t < 3) \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{(x-1)^n}{3^{n+1}} && (-2 < x < 4).
\end{aligned}
$$

Taylor and Maclaurin Series

THEOREM

Suppose the series

$$f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \cdots$$

converges to $f(x)$ for $c - R < x < c + R$, where $R > 0$. Then

$$a_k = \frac{f^{(k)}(c)}{k!} \qquad \text{for } k = 0, 1, 2, 3, \ldots.$$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots.$$

Taylor and Maclaurin Series

$$f'(x) = \sum_{n=1}^{\infty} na_n(x-c)^{n-1} = a_1 + 2a_2(x-c) + 3a_3(x-c)^2 + \cdots$$

$$f''(x) = \sum_{n=2}^{\infty} n(n-1)a_n(x-c)^{n-2} = 2a_2 + 6a_3(x-c) + 12a_4(x-c)^2 + \cdots$$

$$\vdots$$

$$f^{(k)}(x) = \sum_{n=k}^{\infty} n(n-1)(n-2)\cdots(n-k+1)a_n(x-c)^{n-k}$$

$$= k!a_k + \frac{(k+1)!}{1!}a_{k+1}(x-c) + \frac{(k+2)!}{2!}a_{k+2}(x-c)^2 + \cdots.$$

Taylor and Maclaurin Series

Taylor and Maclaurin series

If $f(x)$ has derivatives of all orders at $x = c$ (i.e., if $f^{(k)}(c)$ exists for $k = 0, 1, 2, 3, \ldots$), then the series

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k$$

$$= f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f^{(3)}(c)}{3!}(x-c)^3 + \cdots$$

is called the **Taylor series of f about c** (or the **Taylor series of f in powers of $x - c$**). If $c = 0$, the term **Maclaurin series** is usually used in place of Taylor series.

Taylor and Maclaurin Series

Analytic functions

A function f is **analytic at c** if f has a Taylor series at c and that series converges to $f(x)$ in an open interval containing c. If f is analytic at each point of an open interval, then we say it is analytic on that interval.

Taylor and Maclaurin Series

Maclaurin Series for Some Elementary Functions

EXAMPLE Find the Taylor series for e^x about $x = c$. Where does the series converge to e^x? Where is e^x analytic? What is the Maclaurin series for e^x?

<u>SOLUTION</u>. Let $f(x) = e^x$. The Taylor series for e^x about $x = c$ is

$$\sum_{n=0}^{\infty} \frac{e^c}{n!}(x-c)^n = e^c + e^c(x-c) + \frac{e^c}{2!}(x-c)^2 + \cdots .$$

Because

$$\left| \frac{\frac{e^c}{(n+1)!}(x-c)^{n+1}}{\frac{e^c}{n!}(x-c)^n} \right| = \frac{1}{n+1}|x-c| \to 0 < 1,$$

this series converges everywhere, i.e., $R = \infty$.

Taylor and Maclaurin Series

Maclaurin Series for Some Elementary Functions

EXAMPLE Find the Taylor series for e^x about $x = c$. Where does the series converge to e^x? Where is e^x analytic? What is the Maclaurin series for e^x?

SOLUTION. Suppose the sum is $g(x)$:

$$g(x) = \sum_{n=0}^{\infty} \frac{e^c}{n!}(x-c)^n = e^c + e^c(x-c) + \frac{e^c}{2!}(x-c)^2 + \frac{e^c}{3!}(x-c)^3 + \cdots.$$

Then $g'(x) = e^c + \frac{e^c}{2!}2(x-c) + \frac{e^c}{3!}3(x-c)^2 + \cdots = g(x).$

It follows that $g(x) = Ke^x$ for some real number K.

But $e^c = g(c) = Ke^c$ implies $K = 1$, i.e.,

$$g(x) = e^x.$$

Taylor and Maclaurin Series

Maclaurin Series for Some Elementary Functions

EXAMPLE Find the Taylor series for e^x about $x = c$. Where does the series converge to e^x? Where is e^x analytic? What is the Maclaurin series for e^x?

SOLUTION.

The Taylor series for e^x about $x=c$ converges to e^x for every real number x:

$$e^x = \sum_{n=0}^{\infty} \frac{e^c}{n!}(x-c)^n.$$

Setting $c=0$, we obtain the Maclaurin series for e^x:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad (\text{for } x).$$

Taylor and Maclaurin Series

Maclaurin Series for Some Elementary Functions

EXAMPLE Find the Maclaurin series for (a) $\sin x$ and (b) $\cos x$. Where does each series converge?

SOLUTION. Let $f(x)=\sin x$. Then $f(0)=0$ and

$f'(x)=\cos x \to f'(0)=1$, $f''(x)=-\sin x \to f''(0)=0$,

$f^{(3)}(x)=-\cos x \to f^{(3)}(0)=-1$, $f^{(4)}(x)=\sin x \to f^{(4)}(0)=0, \ldots$

Thus the Maclaurin series for $\sin x$ is

$$g(x)=0+x+0-\frac{x^3}{3!}+0+\frac{x^5}{5!}+0-\cdots$$

$$=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1}.$$

Maclaurin Series for Some Elementary Functions

EXAMPLE Find the Maclaurin series for (a) $\sin x$ and (b) $\cos x$. Where does each series converge?

SOLUTION. The series $g(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$ converges everywhere because

$$\left| \frac{\frac{(-1)^{n+1}}{(2n+3)!} x^{2n+3}}{\frac{(-1)^n}{(2n+1)!} x^{2n+1}} \right| = \frac{x^2}{(2n+3)(2n+2)} \longrightarrow 0 < 1 .$$

It follows that the radius of converges R is ∞.

Taylor and Maclaurin Series

Maclaurin Series for Some Elementary Functions

EXAMPLE Find the Maclaurin series for (a) $\sin x$ and (b) $\cos x$. Where does each series converge?

SOLUTION. $g'(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$

$$g''(x) = -x + \frac{x^3}{3!} - \frac{x^5}{5!} + \frac{x^7}{7!} - \cdots = -g(x).$$

Thus $g(x)$ satisfies the diff. eq. $g''(x) + g(x) = 0$, whose general solution is $g(x) = A\cos x + B\sin x$. Observe, from the series, that $g(0) = 0$ and $g'(0) = 1$. Thus $A = 0$ and $B = 1$. This gives that

$$g(x) = \sin x \quad \text{and} \quad g'(x) = \cos x$$

for all x.

Taylor and Maclaurin Series

Maclaurin Series for Some Elementary Functions

EXAMPLE Find the Maclaurin series for (a) $\sin x$ and (b) $\cos x$. Where does each series converge?

SOLUTION.

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \qquad (\text{for all } x)$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \qquad (\text{for all } x)$$

Taylor and Maclaurin Series

Some Maclaurin series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \qquad (-1 < x < 1)$$

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1} = 1 + 2x + 3x^2 + 4x^3 + \cdots \qquad (-1 < x < 1)$$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \qquad (-1 < x \le 1)$$

$$\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \qquad (-1 \le x \le 1)$$

Taylor and Maclaurin Series

Other Maclaurin and Taylor Series

- $$e^{-x} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots \quad \text{(for all } x\text{).}$$

- $$\sinh x = \frac{e^x - e^{-x}}{2} = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots \quad \text{(for all } x\text{)}$$

- $$\cosh x = \frac{e^x + e^{-x}}{2} = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots \quad \text{(for all } x\text{).}$$

Taylor and Maclaurin Series

Other Maclaurin and Taylor Series

EXAMPLE Obtain Maclaurin series for the following functions:

(a) $e^{-x^2/3}$, (b) $\dfrac{\sin(x^2)}{x}$, (c) $\sin^2 x$.

Solution

(a) We substitute $-x^2/3$ for x in the Maclaurin series for e^x:

$$e^{-x^2/3} = 1 - \frac{x^2}{3} + \frac{1}{2!}\left(\frac{x^2}{3}\right)^2 - \frac{1}{3!}\left(\frac{x^2}{3}\right)^3 + \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{1}{3^n n!} x^{2n} \qquad \text{(for all real } x\text{)}.$$

Other Maclaurin and Taylor Series

EXAMPLE

Obtain Maclaurin series for the following functions:

(a) $e^{-x^2/3}$, (b) $\dfrac{\sin(x^2)}{x}$, (c) $\sin^2 x$.

Solution

(b) For all $x \neq 0$ we have

$$\frac{\sin(x^2)}{x} = \frac{1}{x}\left(x^2 - \frac{(x^2)^3}{3!} + \frac{(x^2)^5}{5!} - \cdots\right)$$
$$= x - \frac{x^5}{3!} + \frac{x^9}{5!} - \cdots = \sum_{n=0}^{\infty}(-1)^n \frac{x^{4n+1}}{(2n+1)!}.$$

Note that $f(x) = (\sin(x^2))/x$ is not defined at $x = 0$ but does have a limit (namely 0) as x approaches 0. If we define $f(0) = 0$ (the continuous extension of $f(x)$ to $x = 0$), then the series converges to $f(x)$ for all x.

Other Maclaurin and Taylor Series

EXAMPLE Obtain Maclaurin series for the following functions:

(a) $e^{-x^2/3}$, (b) $\dfrac{\sin(x^2)}{x}$, (c) $\sin^2 x$.

Solution

(c) We use a trigonometric identity to express $\sin^2 x$ in terms of $\cos 2x$ and then use the Maclaurin series for $\cos x$ with x replaced by $2x$.

$$\begin{aligned}\sin^2 x &= \frac{1-\cos 2x}{2} = \frac{1}{2} - \frac{1}{2}\left(1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \cdots\right)\\ &= \frac{1}{2}\left(\frac{(2x)^2}{2!} - \frac{(2x)^4}{4!} + \frac{(2x)^6}{6!} - \cdots\right)\\ &= \sum_{n=0}^{\infty}(-1)^n \frac{2^{2n+1}}{(2n+2)!}x^{2n+2} \qquad \text{(for all real } x\text{)}.\end{aligned}$$

Taylor and Maclaurin Series

Other Maclaurin and Taylor Series

EXAMPLE Find the Taylor series for $\ln x$ in powers of $x - 2$. Where does the series converge to $\ln x$?

Solution

$$\ln x = \ln(2 + (x-2)) = \ln\left[2\left(1 + \overset{t}{\frac{x-2}{2}}\right)\right] = \ln 2 + \ln(1+t).$$

We use the known Maclaurin series for $\ln(1+t)$:

$$\begin{aligned}
\ln x &= \ln 2 + \ln(1+t) \\
&= \ln 2 + t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} - \cdots \\
&= \ln 2 + \frac{x-2}{2} - \frac{(x-2)^2}{2 \times 2^2} + \frac{(x-2)^3}{3 \times 2^3} - \frac{(x-2)^4}{4 \times 2^4} + \cdots \\
&= \ln 2 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n\,2^n}(x-2)^n.
\end{aligned}$$

Since the series for $\ln(1+t)$ is valid for $-1 < t \le 1$, this series for $\ln x$ is valid for $-1 < (x-2)/2 \le 1$, that is, for $0 < x \le 4$.