Finding Roots of Equations

Fixed-Point Iteration

Fixed-Point Iteration

Looking for solutions of equations of the form

$$f(x) = x,$$

called fixed points of f.

Finding Roots of Equations

Fixed-Point Iteration

Looking for solutions of equations of the form

$$f(x) = x,$$

called fixed points of f.

Starting from a number x_0, we generate a sequence $x_1, x_2, \ldots$ of numbers by the rule

$$x_{n+1} = f(x_n).$$

Finding Roots of Equations

Fixed-Point Iteration

Looking for solutions of equations of the form

$$f(x) = x,$$

called fixed points of f.

Starting from a number x_0, we generate a sequence $x_1, x_2, \dots$ of numbers by the rule

$$x_{n+1} = f(x_n).$$

In certain circumstances, $\lim x_n$ exists and converges to a fixed point.

Fixed-Point Iteration

THEOREM **A fixed-point theorem**

Suppose that f is defined on an interval $I = [a, b]$ and satisfies the following two conditions:

(i) $f(x)$ belongs to I whenever x belongs to I and

(ii) there exists a constant K with $0 < K < 1$ such that for every u and v in I,

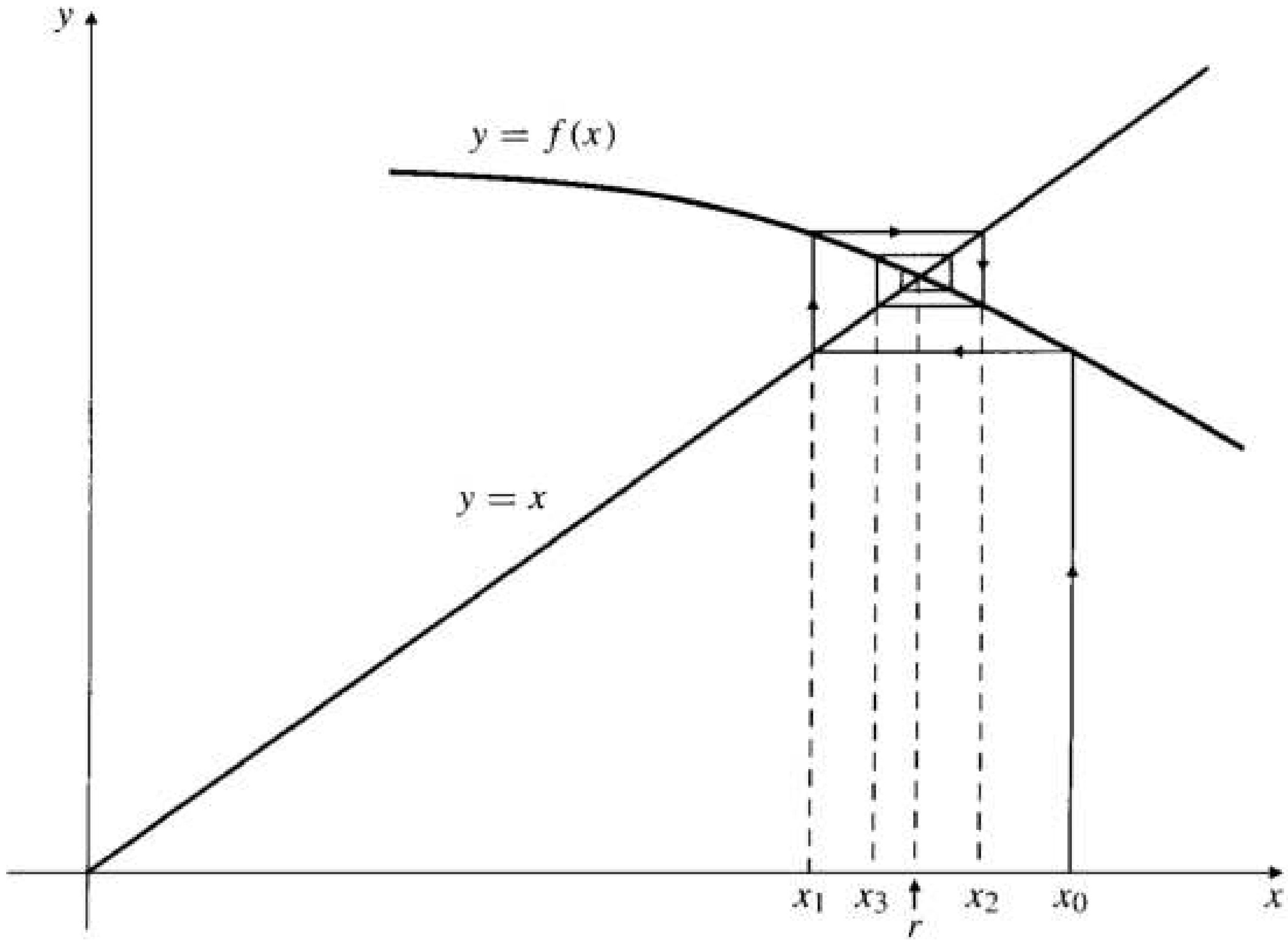
$$|f(u) - f(v)| \leq K|u - v|.$$

Then f has a unique fixed point r in I, that is, $f(r) = r$, and starting with any number x_0 in I, the iterates

$$x_1 = f(x_0), \quad x_2 = f(x_1), \quad \ldots$$

converge to r.

Fixed-Point Iteration



Finding Roots of Equations

Fixed-Point Iteration

EXAMPLE Find a root of the equation $\cos x = 5x$.

Finding Roots of Equations

Fixed-Point Iteration

EXAMPLE Find a root of the equation $\cos x = 5x$.

Solution This equation is of the form $f(x) = x$, where $f(x) = \frac{1}{5}\cos x$. Since $\cos x$ is close to 1 for x near 0, we see that $\frac{1}{5}\cos x$ will be close to $\frac{1}{5}$ when $x = \frac{1}{5}$. This suggests that a reasonable first guess at the fixed point is $x_0 = \frac{1}{5} = 0.2$.

Finding Roots of Equations

Fixed-Point Iteration

EXAMPLE Find a root of the equation $\cos x = 5x$.

Solution This equation is of the form $f(x) = x$, where $f(x) = \frac{1}{5}\cos x$. Since $\cos x$ is close to 1 for x near 0, we see that $\frac{1}{5}\cos x$ will be close to $\frac{1}{5}$ when $x = \frac{1}{5}$. This suggests that a reasonable first guess at the fixed point is $x_0 = \frac{1}{5} = 0.2$.

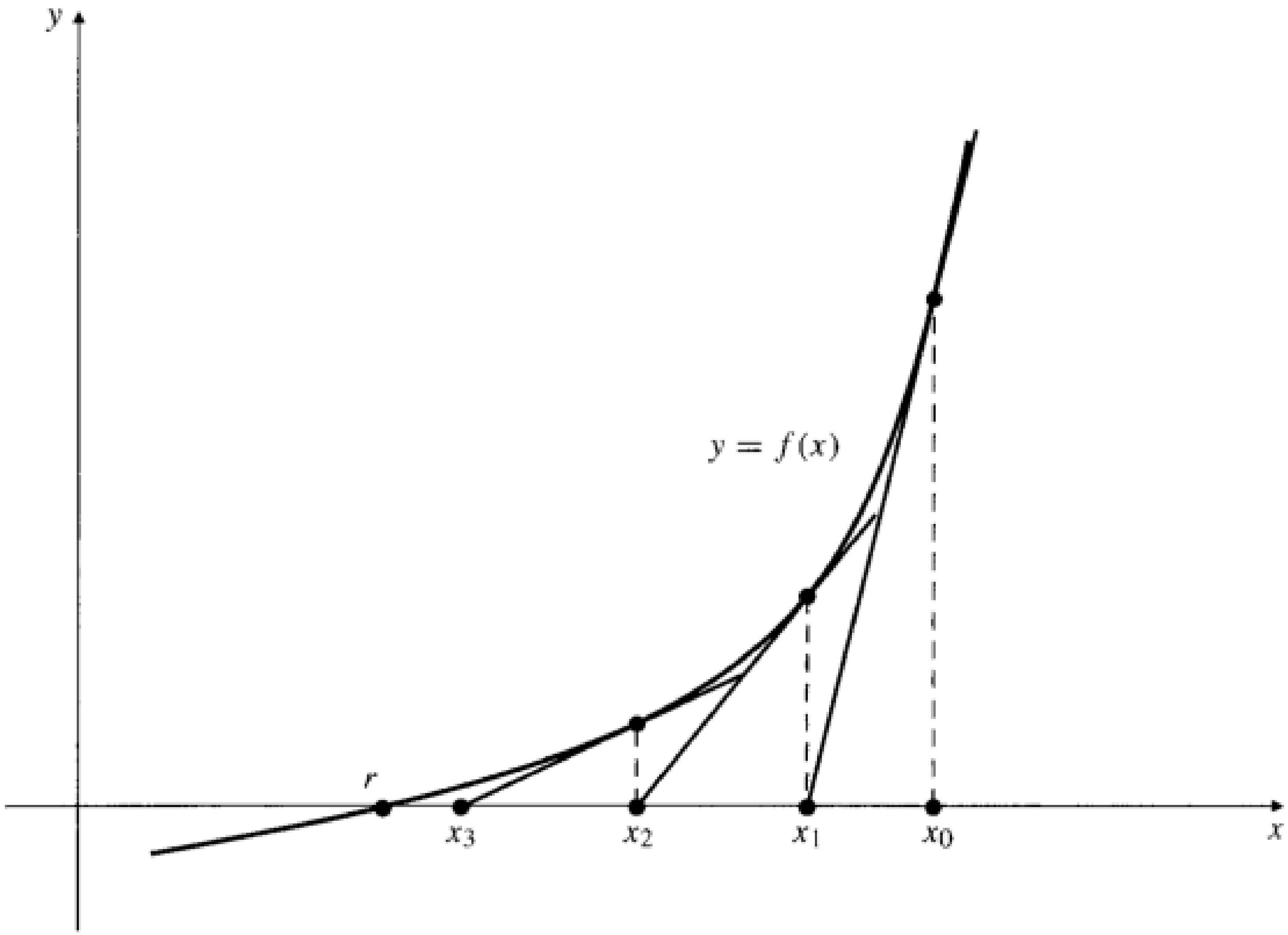
$$x_1 = \frac{1}{5}\cos x_0, \quad x_2 = \frac{1}{5}\cos x_1, \quad x_3 = \frac{1}{5}\cos x_2, \ldots$$

n	x_n
0	0.2
1	0.196 013 32
2	0.196 170 16
3	0.196 164 05
4	0.196 164 29
5	0.196 164 28
6	0.196 164 28

The root is 0.196 164 28 to 8 decimal places.

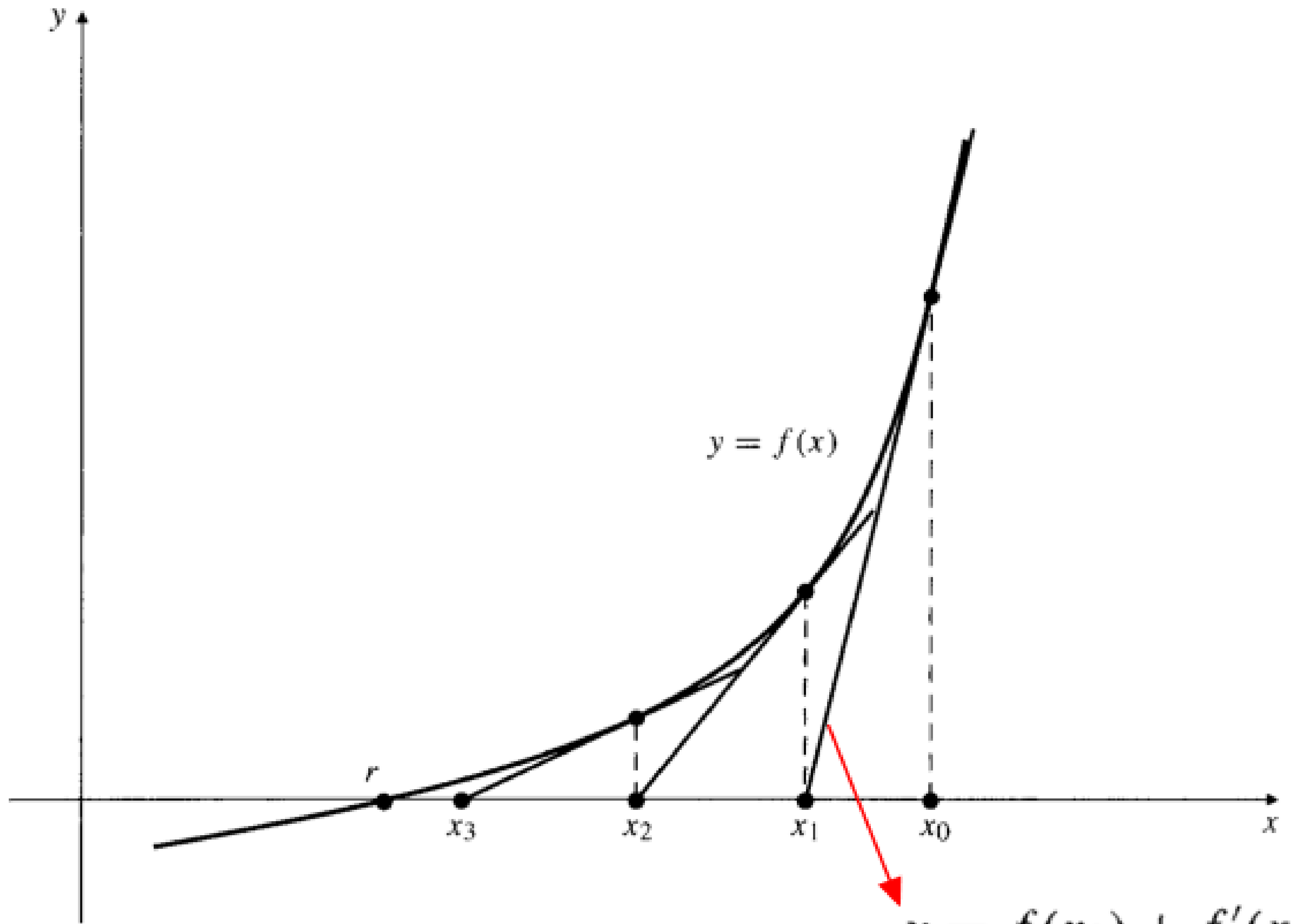
Finding Roots of Equations

Newton's Method



Finding Roots of Equations

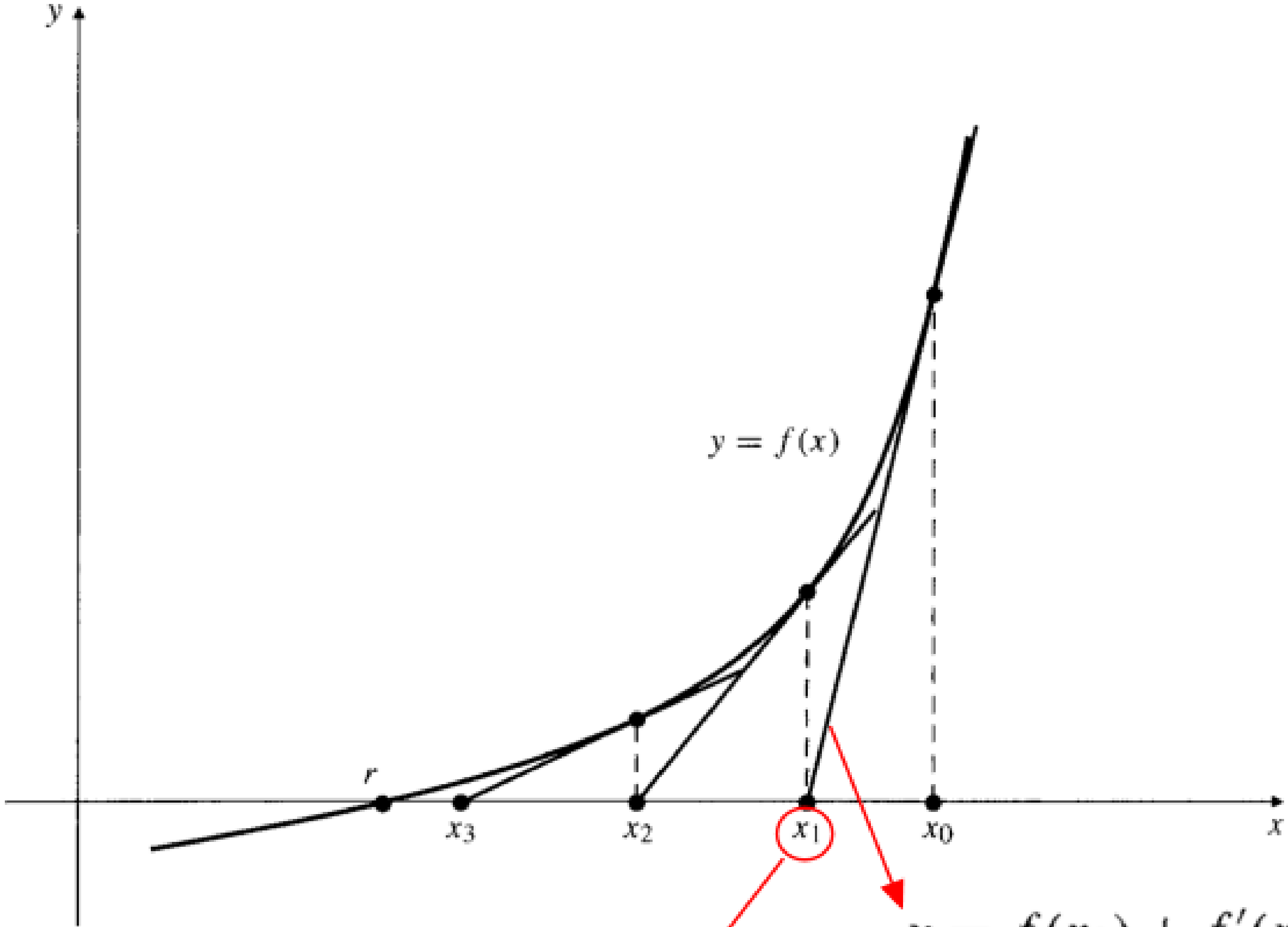
Newton's Method



$$y = f(x_0) + f'(x_0)(x - x_0)$$

Finding Roots of Equations

Newton's Method

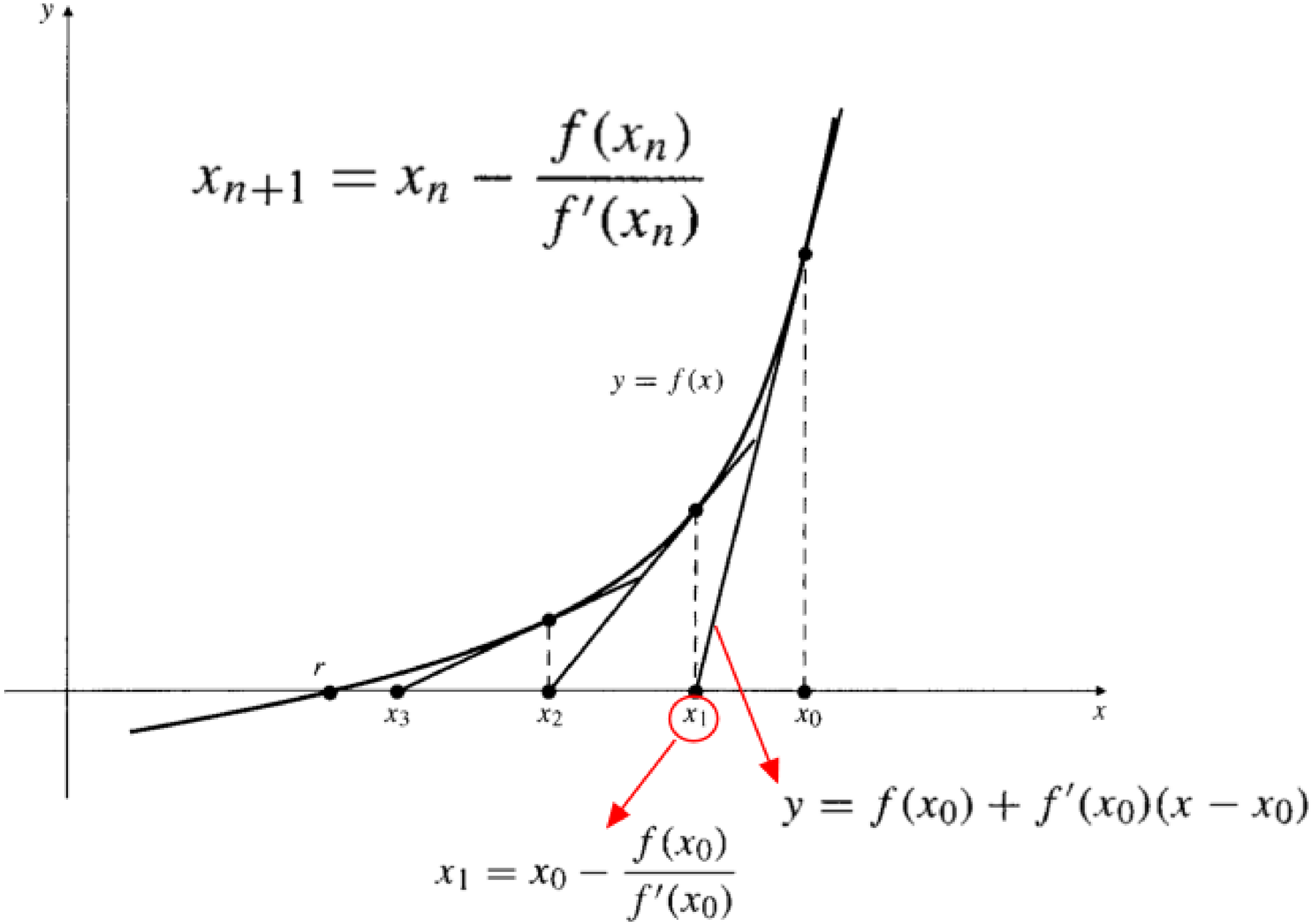


$$y = f(x_0) + f'(x_0)(x - x_0)$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Finding Roots of Equations

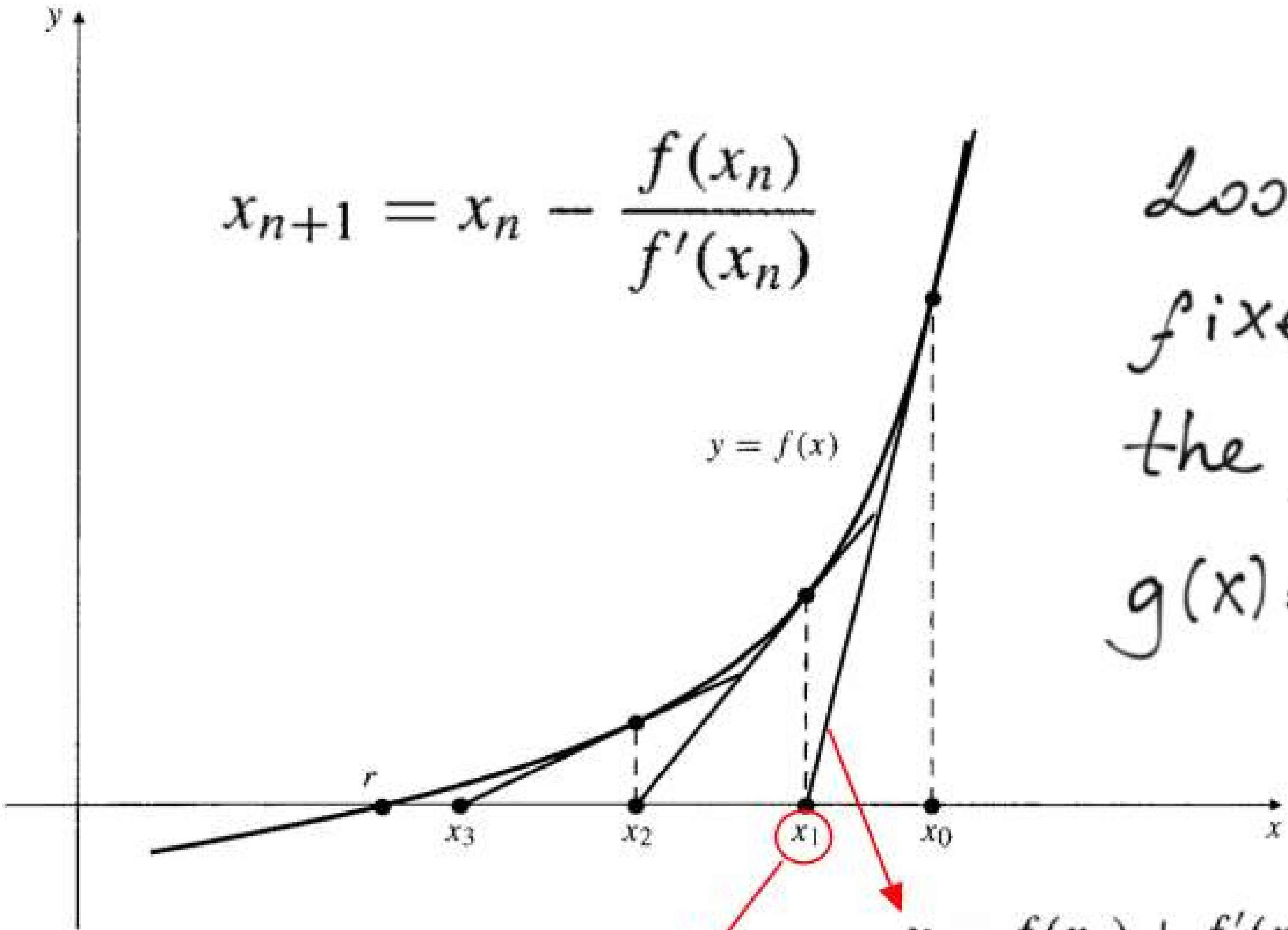
Newton's Method



Finding Roots of Equations

Newton's Method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$



$$y = f(x_0) + f'(x_0)(x - x_0)$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Looking for a fixed point of the function $g(x) = x - \dfrac{f(x)}{f'(x)}$.

Finding Roots of Equations

Newton's Method

EXAMPLE

Use Newton's Method to find the only real root of the equation $x^3 - x - 1 = 0$ correct to 10 decimal places.

Newton's Method

EXAMPLE Use Newton's Method to find the only real root of the equation $x^3 - x - 1 = 0$ correct to 10 decimal places.

Solution We have $f(x) = x^3 - x - 1$ and $f'(x) = 3x^2 - 1$. Since f is continuous and since $f(1) = -1$ and $f(2) = 5$, the equation has a root in the interval $[1, 2]$.

$$x_{n+1} = x_n - \frac{x_n^3 - x_n - 1}{3x_n^2 - 1} = \frac{2x_n^3 + 1}{3x_n^2 - 1},$$

n	x_n	$f(x_n)$
0	1.5	0.875 000 000 000 ···
1	1.347 826 086 96 ···	0.100 682 173 091 ···
2	1.325 200 398 95 ···	0.002 058 361 917 ···
3	1.324 718 174 00 ···	0.000 000 924 378 ···
4	1.324 717 957 24 ···	0.000 000 000 000 ···
5	1.324 717 957 24 ···	

Indeterminate Forms

Type	Example
$[0/0]$	$\lim\limits_{x\to 0} \dfrac{\sin x}{x}$
$[\infty/\infty]$	$\lim\limits_{x\to 0} \dfrac{\ln(1/x^2)}{\cot(x^2)}$
$[0\cdot\infty]$	$\lim\limits_{x\to 0+} x \ln \dfrac{1}{x}$
$[\infty-\infty]$	$\lim\limits_{x\to(\pi/2)-} \left(\tan x - \dfrac{1}{\pi - 2x}\right)$
$[0^0]$	$\lim\limits_{x\to 0+} x^x$
$[\infty^0]$	$\lim\limits_{x\to(\pi/2)-} (\tan x)^{\cos x}$
$[1^\infty]$	$\lim\limits_{x\to\infty} \left(1+\dfrac{1}{x}\right)^x$

Indeterminate Forms

THEOREM **The first l'Hôpital Rule**

Suppose the functions f and g are differentiable on the interval (a, b), and $g'(x) \neq 0$ there. Suppose also that

(i) $\lim_{x \to a+} f(x) = \lim_{x \to a+} g(x) = 0$ and

(ii) $\displaystyle \lim_{x \to a+} \frac{f'(x)}{g'(x)} = L$ (where L is finite or ∞ or $-\infty$).

Then

$$\lim_{x \to a+} \frac{f(x)}{g(x)} = L.$$

Similar results hold if every occurrence of $\lim_{x \to a+}$ is replaced by $\lim_{x \to b-}$ or even $\lim_{x \to c}$ where $a < c < b$. The cases $a = -\infty$ and $b = \infty$ are also allowed.

Indeterminate Forms

EXAMPLE Evaluate $\lim_{x \to 1} \frac{\ln x}{x^2 - 1}$.

Solution We have

$$\lim_{x \to 1} \frac{\ln x}{x^2 - 1} \qquad \left[\frac{0}{0}\right]$$

$$= \lim_{x \to 1} \frac{1/x}{2x} = \lim_{x \to 1} \frac{1}{2x^2} = \frac{1}{2}.$$

Indeterminate Forms

EXAMPLE Evaluate $\lim\limits_{x\to 0} \dfrac{2\sin x - \sin(2x)}{2e^x - 2 - 2x - x^2}$.

Solution We have (using l'Hôpital's Rule three times)

$$
\begin{aligned}
&\lim_{x\to 0} \frac{2\sin x - \sin(2x)}{2e^x - 2 - 2x - x^2} && \left[\frac{0}{0}\right] \\
&= \lim_{x\to 0} \frac{2\cos x - 2\cos(2x)}{2e^x - 2 - 2x} && \text{cancel the 2s} \\
&= \lim_{x\to 0} \frac{\cos x - \cos(2x)}{e^x - 1 - x} && \text{still } \left[\frac{0}{0}\right] \\
&= \lim_{x\to 0} \frac{-\sin x + 2\sin(2x)}{e^x - 1} && \text{still } \left[\frac{0}{0}\right] \\
&= \lim_{x\to 0} \frac{-\cos x + 4\cos(2x)}{e^x} = \frac{-1+4}{1} = 3.
\end{aligned}
$$

Indeterminate Forms

EXAMPLE Evaluate (a) $\lim_{x\to(\pi/2)-} \dfrac{2x-\pi}{\cos^2 x}$ and (b) $\lim_{x\to 1+} \dfrac{x}{\ln x}$.

Indeterminate Forms

EXAMPLE Evaluate (a) $\lim_{x \to (\pi/2)-} \frac{2x - \pi}{\cos^2 x}$ and (b) $\lim_{x \to 1+} \frac{x}{\ln x}$.

Solution

(a)
$$\lim_{x \to (\pi/2)-} \frac{2x - \pi}{\cos^2 x} \qquad \left[\frac{0}{0}\right]$$
$$= \lim_{x \to (\pi/2)-} \frac{2}{-2 \sin x \cos x} = -\infty$$

Indeterminate Forms

EXAMPLE Evaluate (a) $\displaystyle\lim_{x\to(\pi/2)-}\frac{2x-\pi}{\cos^2 x}$ and (b) $\displaystyle\lim_{x\to 1+}\frac{x}{\ln x}$.

Solution

(a) $$\lim_{x\to(\pi/2)-}\frac{2x-\pi}{\cos^2 x} \qquad \left[\frac{0}{0}\right]$$
$$= \lim_{x\to(\pi/2)-}\frac{2}{-2\sin x\cos x} = -\infty$$

(b) l'Hôpital's Rule cannot be used to evaluate $\lim_{x\to 1+} x/(\ln x)$ because this is not an indeterminate form. The denominator approaches 0 as $x \to 1+$, but the numerator does not approach 0. Since $\ln x > 0$ for $x > 1$, we have, directly,

$$\lim_{x\to 1+}\frac{x}{\ln x} = \infty.$$

(Had we tried to apply l'Hôpital's Rule, we would have been led to the erroneous answer $\lim_{x\to 1+}(1/(1/x)) = 1$.)

Indeterminate Forms

EXAMPLE Evaluate $\lim_{x\to 0+}\left(\frac{1}{x}-\frac{1}{\sin x}\right)$.

Solution The indeterminate form here is of type $[\infty-\infty]$ to which l'Hôpital's Rule cannot be applied. However, it becomes $[0/0]$ after we combine the fractions into one fraction.

$$\begin{aligned}
&\lim_{x\to 0+}\left(\frac{1}{x}-\frac{1}{\sin x}\right) && [\infty-\infty]\\
&=\lim_{x\to 0+}\frac{\sin x - x}{x\sin x} && \left[\frac{0}{0}\right]\\
&=\lim_{x\to 0+}\frac{\cos x - 1}{\sin x + x\cos x} && \left[\frac{0}{0}\right]\\
&=\lim_{x\to 0+}\frac{-\sin x}{2\cos x - x\sin x}=\frac{-0}{2}=0.
\end{aligned}$$

Indeterminate Forms

THEOREM **The second l'Hôpital Rule**

Suppose that f and g are differentiable on the interval (a, b) and that $g'(x) \neq 0$ there. Suppose also that

(i) $\lim_{x \to a+} g(x) = \pm\infty$ and

(ii) $\lim_{x \to a+} \dfrac{f'(x)}{g'(x)} = L$ (where L is finite, or ∞ or $-\infty$).

Then

$$\lim_{x \to a+} \frac{f(x)}{g(x)} = L.$$

Again, similar results hold for $\lim_{x \to b-}$ and for $\lim_{x \to c}$, and the cases $a = -\infty$ and $b = \infty$ are allowed.

Indeterminate Forms

EXAMPLE Evaluate (a) $\lim_{x\to\infty} \dfrac{x^2}{e^x}$ and (b) $\lim_{x\to 0+} x^a \ln x$, where $a > 0$.

Indeterminate Forms

EXAMPLE Evaluate (a) $\lim_{x \to \infty} \dfrac{x^2}{e^x}$ and (b) $\lim_{x \to 0+} x^a \ln x$, where $a > 0$.

Solution

(a)
$$\begin{aligned} &\lim_{x \to \infty} \frac{x^2}{e^x} && \left[\frac{\infty}{\infty}\right] \\ &= \lim_{x \to \infty} \frac{2x}{e^x} && \text{still } \left[\frac{\infty}{\infty}\right] \\ &= \lim_{x \to \infty} \frac{2}{e^x} = 0. \end{aligned}$$

Similarly, one can show that $\lim_{x \to \infty} x^n / e^x = 0$ for any positive integer n by repeated applications of l'Hôpital's Rule.

Indeterminate Forms

EXAMPLE Evaluate (a) $\lim_{x\to\infty} \dfrac{x^2}{e^x}$ and (b) $\lim_{x\to 0+} x^a \ln x$, where $a > 0$.

Solution

(a)
$$\begin{aligned}
&\lim_{x\to\infty} \frac{x^2}{e^x} \qquad \left[\frac{\infty}{\infty}\right] \\
&= \lim_{x\to\infty} \frac{2x}{e^x} \qquad \text{still } \left[\frac{\infty}{\infty}\right] \\
&= \lim_{x\to\infty} \frac{2}{e^x} = 0.
\end{aligned}$$

Similarly, one can show that $\lim_{x\to\infty} x^n/e^x = 0$ for any positive integer n by repeated applications of l'Hôpital's Rule.

(b)
$$\begin{aligned}
&\lim_{x\to 0+} x^a \ln x \quad (a > 0) \qquad [0 \cdot (-\infty)] \\
&= \lim_{x\to 0+} \frac{\ln x}{x^{-a}} \qquad \left[\frac{-\infty}{\infty}\right] \\
&= \lim_{x\to 0+} \frac{1/x}{-ax^{-a-1}} = \lim_{x\to 0+} \frac{x^a}{-a} = 0.
\end{aligned}$$

Indeterminate Forms

EXAMPLE Evaluate $\lim_{x\to 0+} x^x$.

Solution This indeterminate form is of type $[0^0]$. Let $y = x^x$. Then

$$\lim_{x\to 0+} \ln y = \lim_{x\to 0+} x \ln x = 0,$$

Hence $\lim_{x\to 0} x^x = \lim_{x\to 0+} y = e^0 = 1$.

$$\boxed{\lim y = e^{\ln(\lim y)} = e^{\lim(\ln y)}}$$

Indeterminate Forms

EXAMPLE Evaluate $\lim_{x\to\infty}\left(1+\sin\frac{3}{x}\right)^x$.

Solution This indeterminate form is of type 1^∞. Let $y=\left(1+\sin\frac{3}{x}\right)^x$. Then, taking ln of both sides,

$$\lim_{x\to\infty}\ln y=\lim_{x\to\infty}x\ln\left(1+\sin\frac{3}{x}\right)\qquad[\infty\cdot 0]$$

$$=\lim_{x\to\infty}\frac{\ln\left(1+\sin\frac{3}{x}\right)}{\frac{1}{x}}\qquad\left[\frac{0}{0}\right]$$

$$=\lim_{x\to\infty}\frac{\frac{1}{1+\sin\frac{3}{x}}\left(\cos\frac{3}{x}\right)\left(-\frac{3}{x^2}\right)}{-\frac{1}{x^2}}=\lim_{x\to\infty}\frac{3\cos\frac{3}{x}}{1+\sin\frac{3}{x}}=3.$$

Hence $\lim_{x\to\infty}\left(1+\sin\frac{3}{x}\right)^x=e^3$.

Extreme Values

Absolute extreme values

Function f has an **absolute maximum value** $f(x_0)$ at the point x_0 in its domain if $f(x) \le f(x_0)$ holds for every x in the domain of f.
Similarly, f has an **absolute minimum value** $f(x_1)$ at the point x_1 in its domain if $f(x) \ge f(x_1)$ holds for every x in the domain of f.

Extreme Values

Absolute extreme values

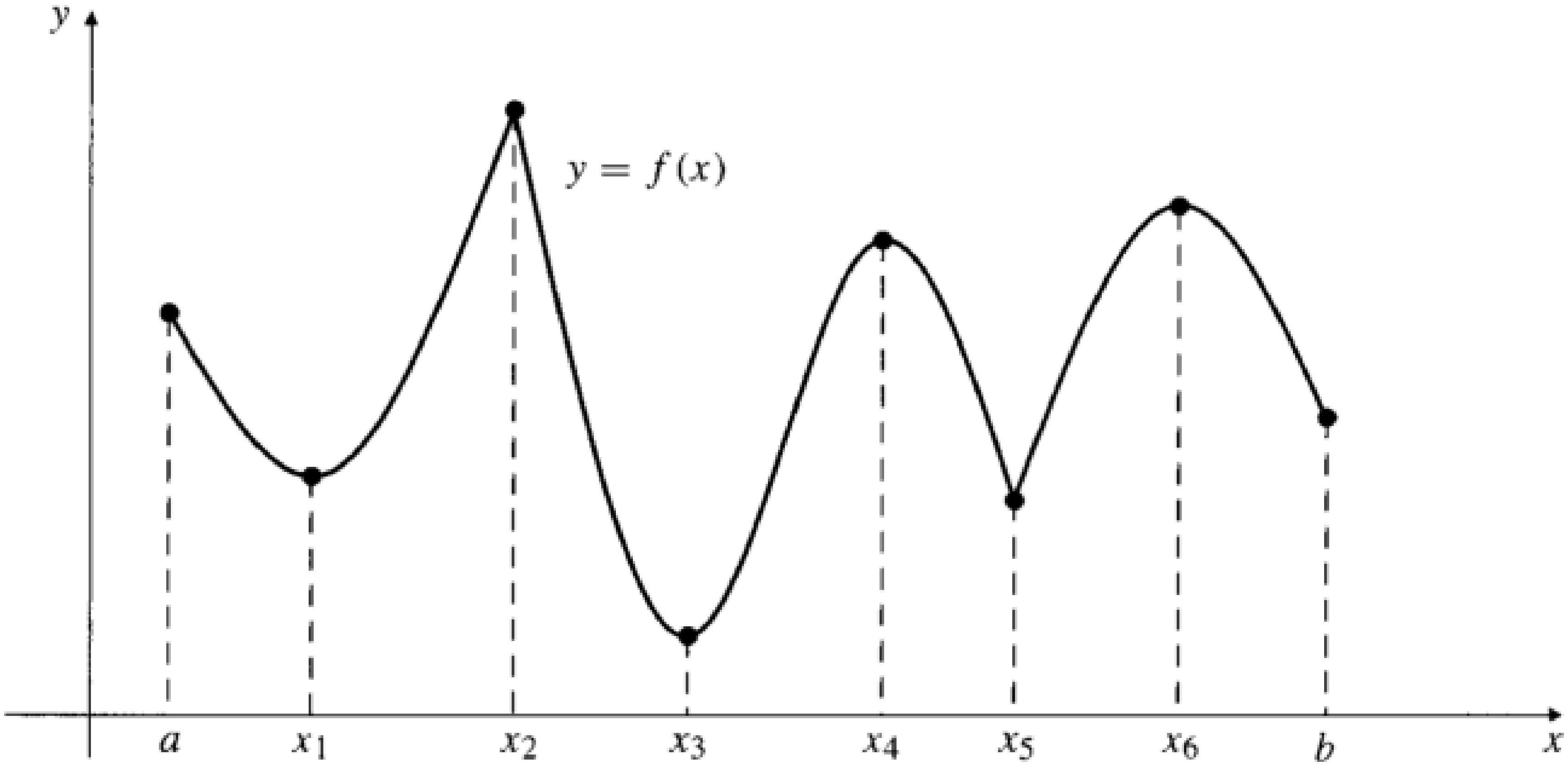
Function f has an **absolute maximum value** $f(x_0)$ at the point x_0 in its domain if $f(x) \leq f(x_0)$ holds for every x in the domain of f.
Similarly, f has an **absolute minimum value** $f(x_1)$ at the point x_1 in its domain if $f(x) \geq f(x_1)$ holds for every x in the domain of f.

THEOREM

Existence of extreme values

If the domain of the function f is a *closed, finite interval* or a union of finitely many such intervals, and if f is *continuous* on that domain, then f must have an absolute maximum value and an absolute minimum value.

Extreme Values



Extreme Values

Local extreme values

Function f has a **local maximum value (loc max)** $f(x_0)$ at the point x_0 in its domain provided there exists a number $h > 0$ such that $f(x) \leq f(x_0)$ whenever x is in the domain of f and $|x - x_0| < h$.

Similarly, f has a **local minimum value (loc min)** $f(x_1)$ at the point x_1 in its domain provided there exists a number $h > 0$ such that $f(x) \geq f(x_1)$ whenever x is in the domain of f and $|x - x_1| < h$.

Extreme Values

Critical Points, Singular Points, and Endpoints

Figure 4.17 suggests that a function $f(x)$ can have local extreme values only at points x of three special types:

(i) **critical points** of f (points x in $\mathcal{D}(f)$ where $f'(x) = 0$),

(ii) **singular points** of f (points x in $\mathcal{D}(f)$ where $f'(x)$ is not defined), and

(iii) **endpoints** of the domain of f (points in $\mathcal{D}(f)$ that do not belong to any open interval contained in $\mathcal{D}(f)$).

Extreme Values

Critical Points, Singular Points, and Endpoints

Figure 4.17 suggests that a function $f(x)$ can have local extreme values only at points x of three special types:

(i) **critical points** of f (points x in $\mathcal{D}(f)$ where $f'(x) = 0$),

(ii) **singular points** of f (points x in $\mathcal{D}(f)$ where $f'(x)$ is not defined), and

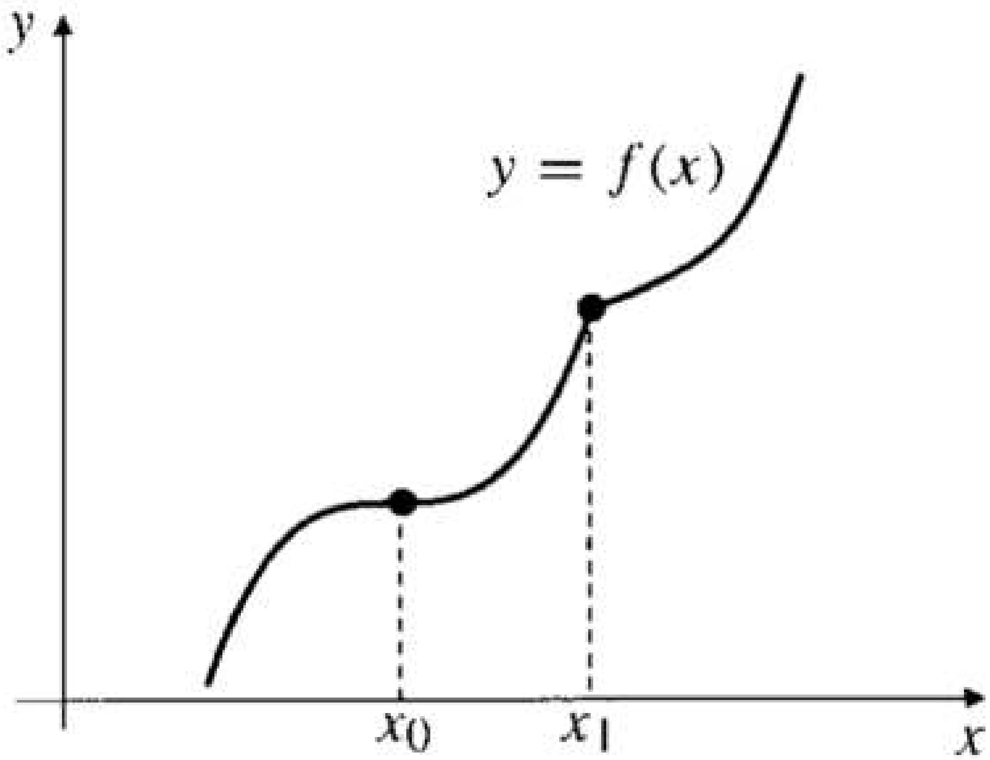
(iii) **endpoints** of the domain of f (points in $\mathcal{D}(f)$ that do not belong to any open interval contained in $\mathcal{D}(f)$).

THEOREM

Locating extreme values

If the function f is defined on an interval I and has a local maximum (or local minimum) value at point $x = x_0$ in I, then x_0 must be either a critical point of f, a singular point of f, or an endpoint of I.

Extreme Values



A function need not have extreme values at a critical point or a singular point

Extreme Values

Finding Absolute Extreme Values

EXAMPLE

Find the maximum and minimum values of the function $g(x) = x^3 - 3x^2 - 9x + 2$ on the interval $-2 \leq x \leq 2$.

Extreme Values

Finding Absolute Extreme Values

EXAMPLE Find the maximum and minimum values of the function $g(x) = x^3 - 3x^2 - 9x + 2$ on the interval $-2 \le x \le 2$.

Solution Since g is a polynomial, it can have no singular points. For critical points, we calculate

$$\begin{aligned} g'(x) = 3x^2 - 6x - 9 &= 3(x^2 - 2x - 3) \\ &= 3(x+1)(x-3) \\ &= 0 \qquad \text{if} \quad x = -1 \text{ or } x = 3. \end{aligned}$$

Extreme Values

Finding Absolute Extreme Values

EXAMPLE Find the maximum and minimum values of the function $g(x) = x^3 - 3x^2 - 9x + 2$ on the interval $-2 \leq x \leq 2$.

Solution Since g is a polynomial, it can have no singular points. For critical points, we calculate

$$\begin{aligned} g'(x) = 3x^2 - 6x - 9 &= 3(x^2 - 2x - 3) \\ &= 3(x+1)(x-3) \\ &= 0 \qquad \text{if} \quad x = -1 \text{ or } \boxed{x = 3.} \end{aligned}$$

not in the domain!

Extreme Values

Finding Absolute Extreme Values

EXAMPLE Find the maximum and minimum values of the function $g(x) = x^3 - 3x^2 - 9x + 2$ on the interval $-2 \le x \le 2$.

Solution Since g is a polynomial, it can have no singular points. For critical points, we calculate

$$\begin{aligned} g'(x) = 3x^2 - 6x - 9 &= 3(x^2 - 2x - 3) \\ &= 3(x+1)(x-3) \\ &= 0 \qquad \text{if} \quad x = -1 \text{ or } \boxed{x = 3.} \end{aligned}$$

not in the domain!

$$g(-2) = 0, \qquad g(-1) = 7, \qquad g(2) = -20.$$

Extreme Values

Finding Absolute Extreme Values

EXAMPLE Find the maximum and minimum values of the function $g(x) = x^3 - 3x^2 - 9x + 2$ on the interval $-2 \leq x \leq 2$.

Solution Since g is a polynomial, it can have no singular points. For critical points, we calculate

$$\begin{aligned} g'(x) = 3x^2 - 6x - 9 &= 3(x^2 - 2x - 3) \\ &= 3(x+1)(x-3) \\ &= 0 \qquad \text{if} \quad x = -1 \text{ or } \boxed{x = 3.} \end{aligned}$$

not in the domain!

$$g(-2) = 0, \qquad \boxed{g(-1) = 7}, \qquad \boxed{g(2) = -20.}$$

maximum value

minimum value

Extreme Values

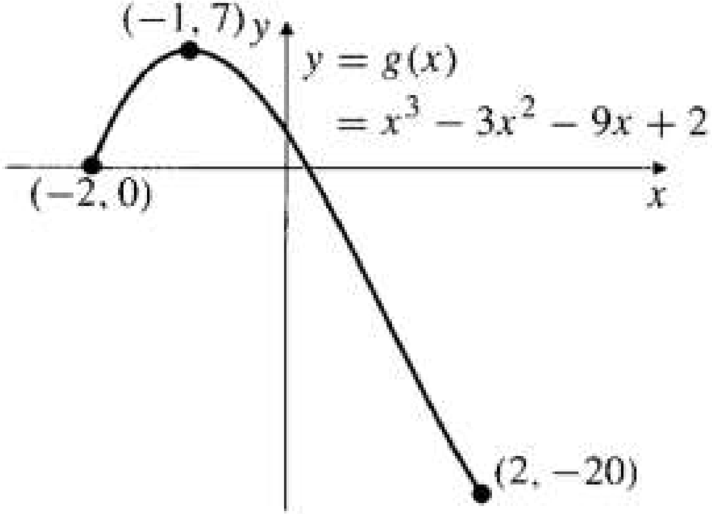
Finding Absolute Extreme Values

EXAMPLE Find the maximum and minimum values of the function $g(x) = x^3 - 3x^2 - 9x + 2$ on the interval $-2 \le x \le 2$.

Solution Since g is a polynomial, it can have no singular points. For critical points, we calculate

$$
\begin{aligned}
g'(x) = 3x^2 - 6x - 9 &= 3(x^2 - 2x - 3) \\
&= 3(x+1)(x-3) \\
&= 0 \qquad \text{if} \quad x = -1 \text{ or } \boxed{x = 3.}
\end{aligned}
$$

not in the domain!



$$g(-2) = 0, \qquad \boxed{g(-1) = 7}, \qquad \boxed{g(2) = -20.}$$

maximum value   minimum value

Extreme Values

Finding Absolute Extreme Values

EXAMPLE

Find the maximum and minimum values of $h(x) = 3x^{2/3} - 2x$ on the interval $[-1, 1]$.

Extreme Values

Finding Absolute Extreme Values

EXAMPLE Find the maximum and minimum values of $h(x) = 3x^{2/3} - 2x$ on the interval $[-1, 1]$.

Solution The derivative of h is

$$h'(x) = 3\left(\frac{2}{3}\right)x^{-1/3} - 2 = 2(x^{-1/3} - 1).$$

Extreme Values

Finding Absolute Extreme Values

EXAMPLE Find the maximum and minimum values of $h(x) = 3x^{2/3} - 2x$ on the interval $[-1, 1]$.

Solution The derivative of h is

$$h'(x) = 3\left(\frac{2}{3}\right)x^{-1/3} - 2 = 2(x^{-1/3} - 1).$$

$x = 0$ is a singular point of h. Also, h has a critical point at $x = 1$

$$h(-1) = 5, \qquad h(0) = 0, \qquad h(1) = 1.$$

Extreme Values

Finding Absolute Extreme Values

EXAMPLE Find the maximum and minimum values of $h(x) = 3x^{2/3} - 2x$ on the interval $[-1, 1]$.

Solution The derivative of h is

$$h'(x) = 3\left(\frac{2}{3}\right)x^{-1/3} - 2 = 2(x^{-1/3} - 1).$$

$x = 0$ is a singular point of h. Also, h has a critical point at $x = 1$

$\boxed{h(-1) = 5,}$ $\boxed{h(0) = 0,}$ $h(1) = 1.$

max. min.

Extreme Values

THEOREM

The First Derivative Test

PART I. Testing interior critical points and singular points.

Suppose that f is continuous at x_0, and x_0 is not an endpoint of the domain of f.

(a) If there exists an open interval (a, b) containing x_0 such that $f'(x) > 0$ on (a, x_0) and $f'(x) < 0$ on (x_0, b), then f has a local maximum value at x_0.

(b) If there exists an open interval (a, b) containing x_0 such that $f'(x) < 0$ on (a, x_0) and $f'(x) > 0$ on (x_0, b), then f has a local minimum value at x_0.

PART II. Testing endpoints of the domain.

Suppose a is a left endpoint of the domain of f and f is right continuous at a.

(c) If $f'(x) > 0$ on some interval (a, b), then f has a local minimum value at a.

(d) If $f'(x) < 0$ on some interval (a, b), then f has a local maximum value at a.

Suppose b is a right endpoint of the domain of f and f is left continuous at b.

(e) If $f'(x) > 0$ on some interval (a, b), then f has a local maximum value at b.

(f) If $f'(x) < 0$ on some interval (a, b), then f has a local minimum value at b.

Extreme Values

EXAMPLE

Find the local and absolute extreme values of $f(x) = x^4 - 2x^2 - 3$ on the interval $[-2, 2]$. Sketch the graph of f.