Improper Integrals

$$I = \int_a^b f(x)\,dx,$$

(i) We may have $a = -\infty$ or $b = \infty$ or both.

(ii) f may be unbounded as x approaches a or b or both.

Improper Integrals

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(i) We may have $a = -\infty$ or $b = \infty$ or both.

(ii) f may be unbounded as x approaches a or b or both.

Integrals satisfying (i) are called **improper integrals of type I**;

integrals satisfying (ii) are called **improper integrals of type II**.

Improper Integrals

Improper Integrals of Type I

EXAMPLE Find the area of the region A lying under the curve $y = 1/x^2$ and above the x-axis to the right of $x = 1$.

Solution We would like to calculate the area with an integral

$$A = \int_1^\infty \frac{dx}{x^2},$$

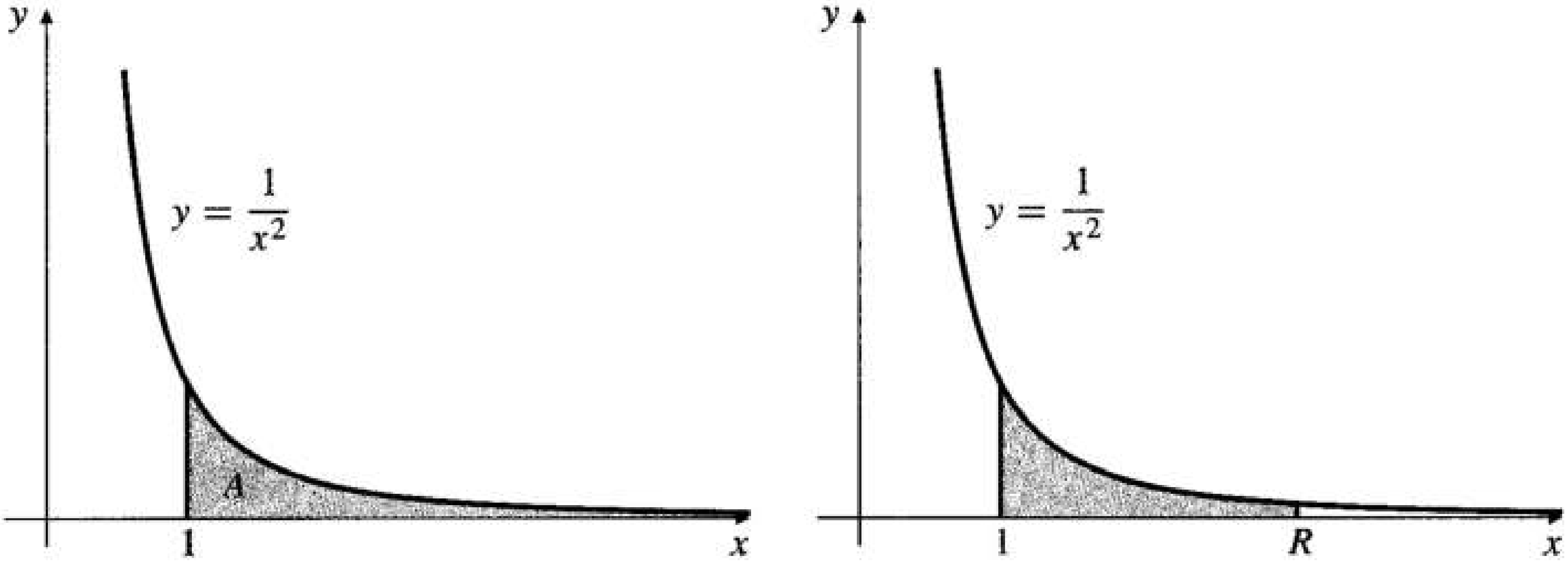
which is improper of type I, since its interval of integration is infinite.

Improper Integrals

Improper Integrals of Type I

EXAMPLE Find the area of the region A lying under the curve $y = 1/x^2$ and above the x-axis to the right of $x = 1$.

Solution



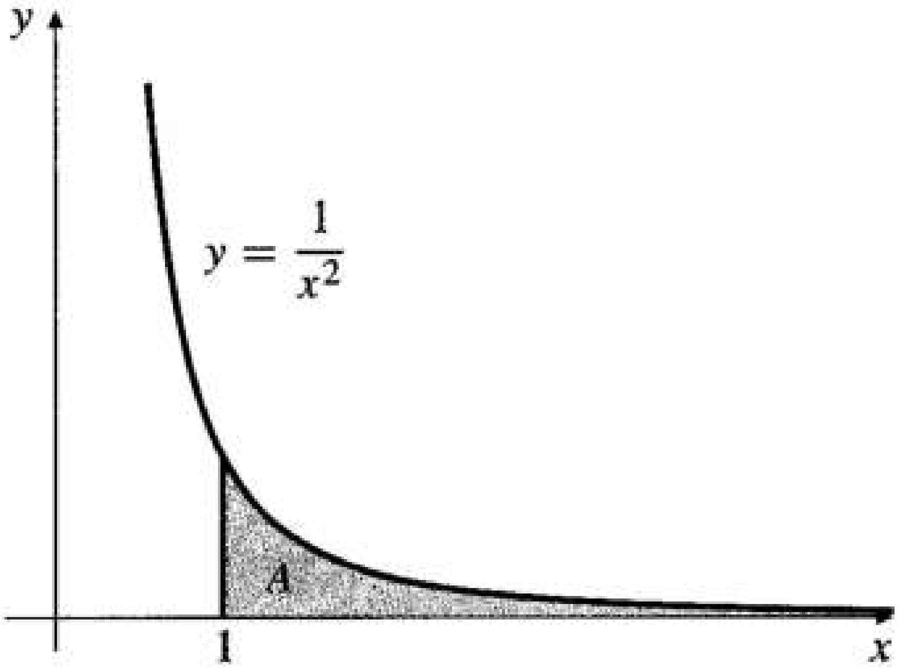
Improper Integrals

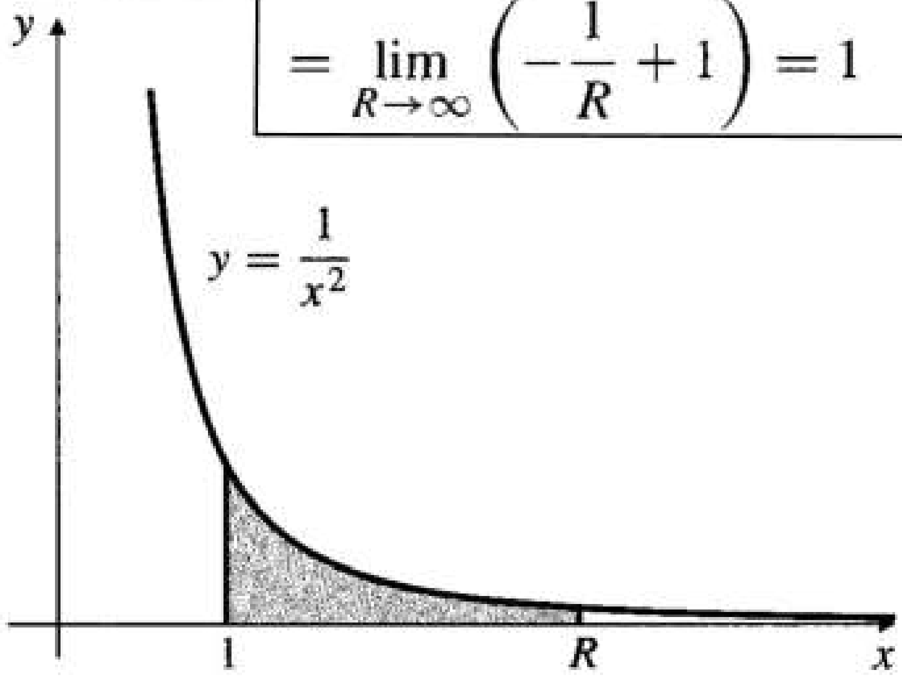
Improper Integrals of Type I

EXAMPLE Find the area of the region A lying under the curve $y = 1/x^2$ and above the x-axis to the right of $x = 1$.

Solution

$$A = \int_1^\infty \frac{dx}{x^2} = \lim_{R\to\infty} \int_1^R \frac{dx}{x^2} = \lim_{R\to\infty} \left(-\frac{1}{x}\right)\bigg|_1^R$$
$$= \lim_{R\to\infty} \left(-\frac{1}{R} + 1\right) = 1$$

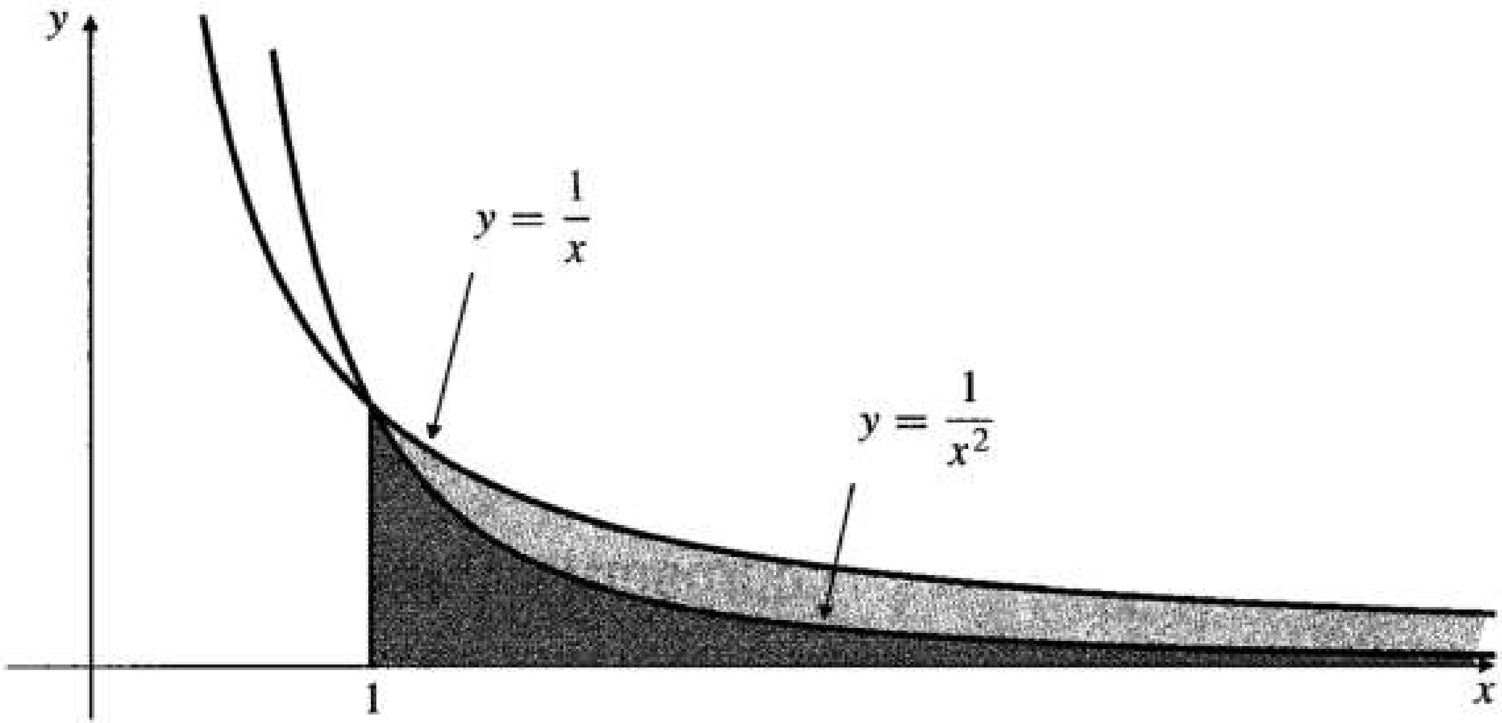




Improper Integrals

Improper Integrals of Type I

EXAMPLE 2 Find the area of the region under $y = 1/x$, above $y = 0$, and to the right of $x = 1$.



Improper Integrals

Improper Integrals of Type I

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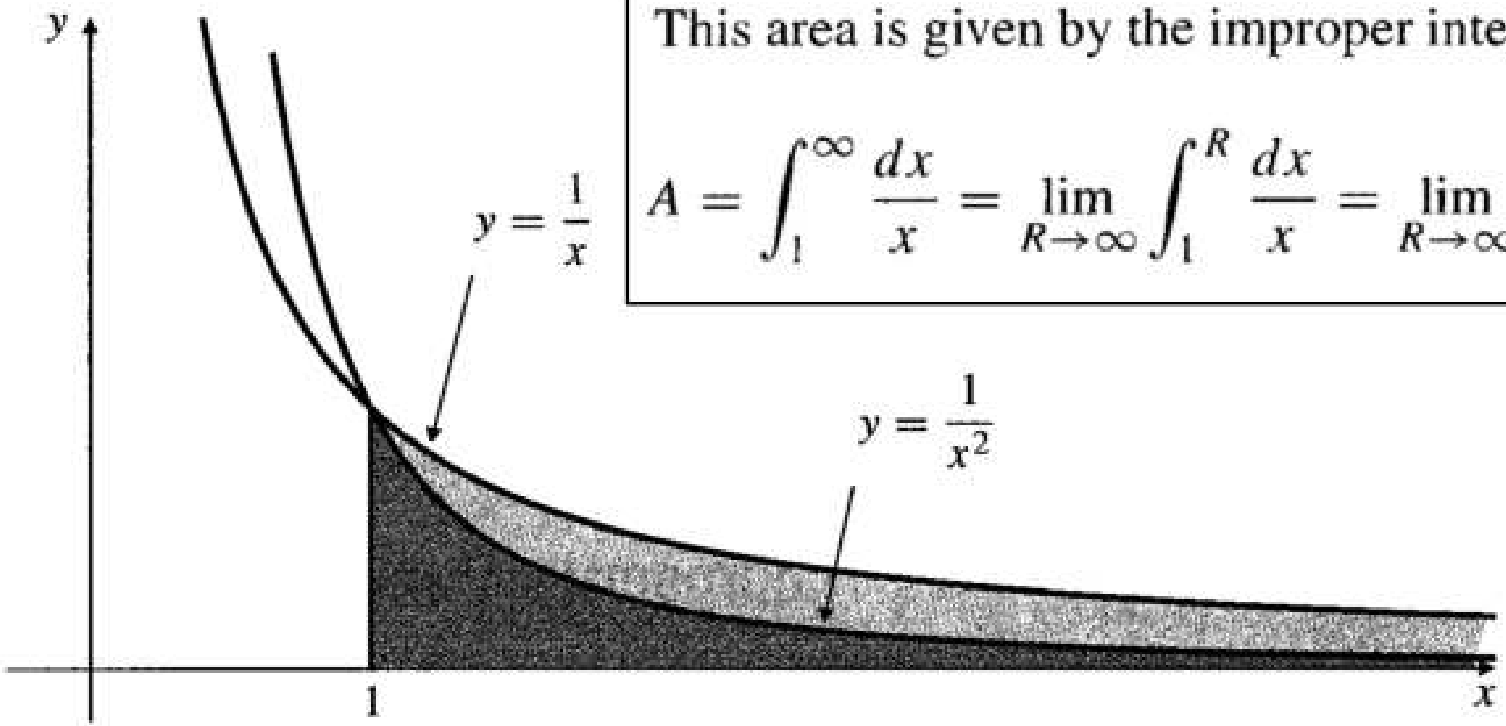


This area is given by the improper integral

$$A = \int_1^\infty \frac{dx}{x} = \lim_{R\to\infty} \int_1^R \frac{dx}{x} = \lim_{R\to\infty} \ln x \Big|_1^R = \lim_{R\to\infty} \ln R = \infty.$$

Improper Integrals

Improper Integrals of Type I

If f is continuous on $[a, \infty)$, we define the improper integral of f over $[a, \infty)$ as a limit of proper integrals:

$$\int_a^\infty f(x)\,dx = \lim_{R\to\infty} \int_a^R f(x)\,dx.$$

Similarly, if f is continuous on $(-\infty, b]$, then we define

$$\int_{-\infty}^b f(x)\,dx = \lim_{R\to-\infty} \int_R^b f(x)\,dx.$$

In either case, if the limit exists (is a finite number), we say that the improper integral **converges**; if the limit does not exist, we say that the improper integral **diverges**. If the limit is ∞ (or $-\infty$), we say the improper integral **diverges to infinity** (or **diverges to negative infinity**).

Improper Integrals

Improper Integrals of Type I

The integral $\int_{-\infty}^{\infty} f(x)\,dx$ is, for f continuous on the real line, improper of type I at both endpoints. We break it into two separate integrals:

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{\infty} f(x)\,dx.$$

The integral on the left converges if and only if *both* integrals on the right converge.

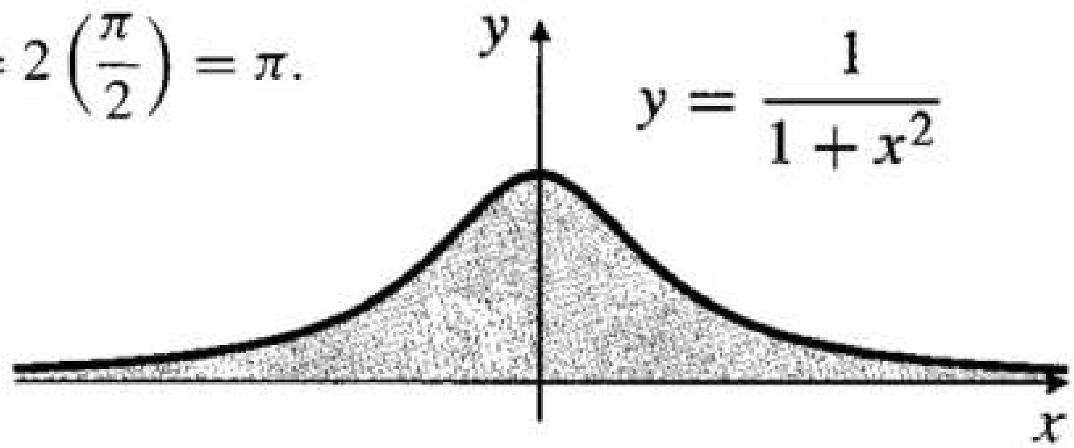
Improper Integrals

Improper Integrals of Type I

EXAMPLE Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx.$

Solution By the (even) symmetry of the integrand, we have

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{dx}{1+x^2} &= \int_{-\infty}^{0} \frac{dx}{1+x^2} + \int_{0}^{\infty} \frac{dx}{1+x^2} \\
&= 2 \lim_{R\to\infty} \int_0^R \frac{dx}{1+x^2} \\
&= 2 \lim_{R\to\infty} \tan^{-1} R = 2\left(\frac{\pi}{2}\right) = \pi.
\end{aligned}$$

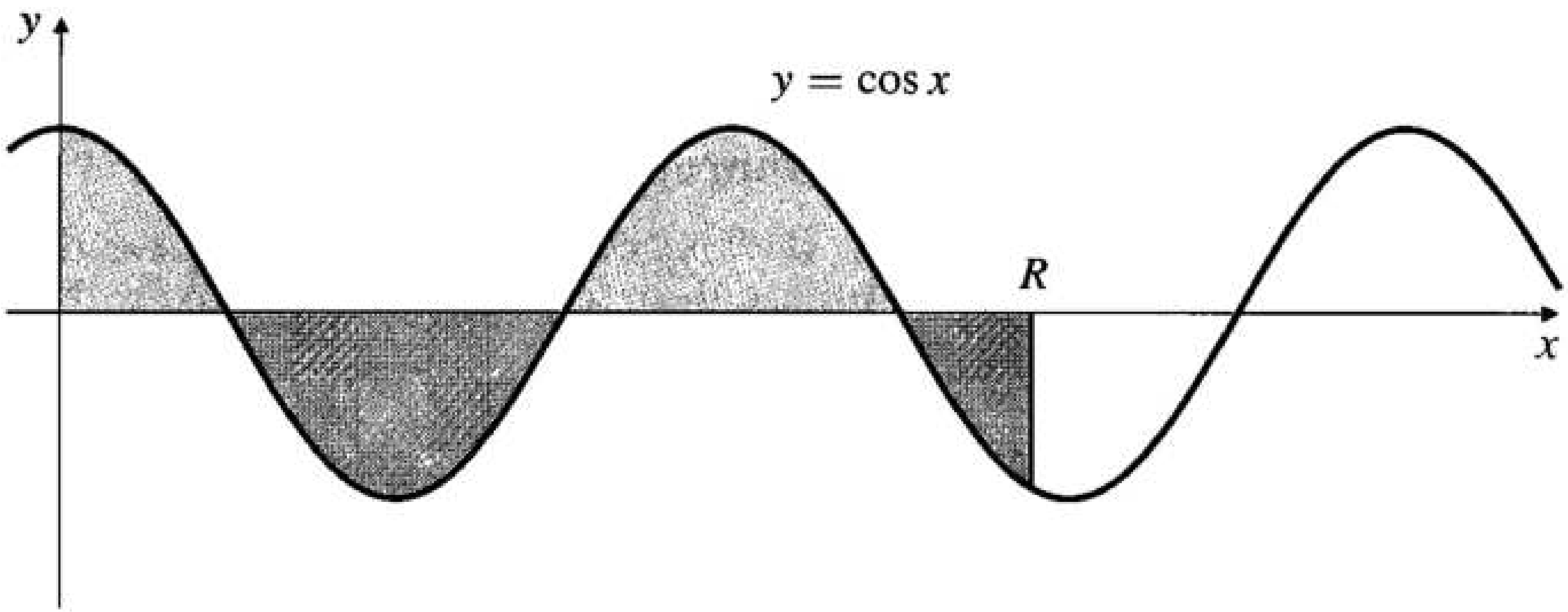


Improper Integrals

Improper Integrals of Type I

EXAMPLE $$\int_0^\infty \cos x\,dx = \lim_{R\to\infty}\int_0^R \cos x\,dx = \lim_{R\to\infty}\sin R.$$

This limit does not exist (and it is not ∞ or $-\infty$), so all we can say is that the given integral diverges.



Improper Integrals

Improper Integrals of Type II

If f is continuous on the interval $(a, b]$ and is possibly unbounded near a, we define the improper integral

$$\int_a^b f(x)\,dx = \lim_{c \to a+} \int_c^b f(x)\,dx.$$

Similarly, if f is continuous on $[a, b)$ and is possibly unbounded near b, we define

$$\int_a^b f(x)\,dx = \lim_{c \to b-} \int_a^c f(x)\,dx.$$

These improper integrals may converge, diverge, diverge to infinity, or diverge to negative infinity.

Improper Integrals

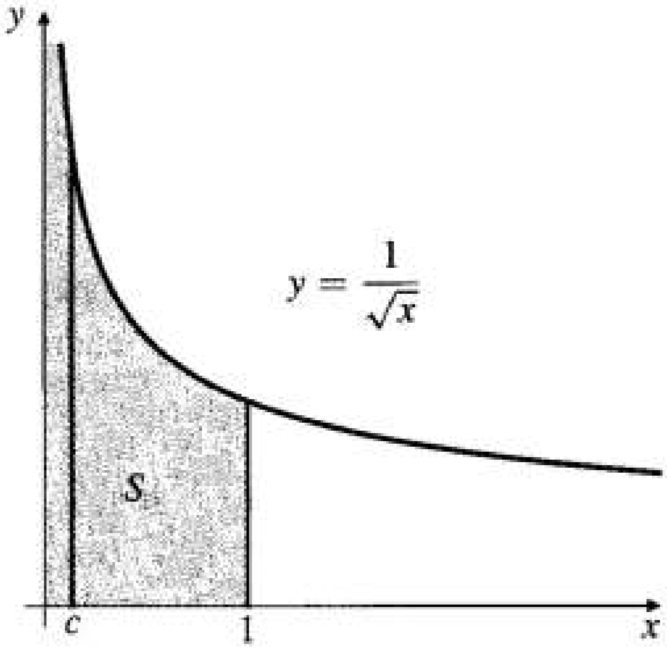
Improper Integrals of Type II

EXAMPLE Find the area of the region S lying under $y = 1/\sqrt{x}$, above the x-axis, between $x = 0$ and $x = 1$.

Solution The area A is given by

$$A = \int_0^1 \frac{1}{\sqrt{x}}\,dx,$$

which is an improper integral of type II since the integrand is unbounded near $x = 0$.



$$A = \lim_{c \to 0+} \int_c^1 x^{-1/2}\,dx$$

$$= \lim_{c \to 0+} 2x^{1/2}\Big|_c^1 = \lim_{c \to 0+} (2 - 2\sqrt{c}) = 2.$$

Improper Integrals

Improper Integrals of Type II

Be alert on the singularities!

$$\int_{-1}^{1} \frac{\ln|x|\,dx}{\sqrt{1-x}} = \int_{-1}^{0} \frac{\ln|x|\,dx}{\sqrt{1-x}} + \int_{0}^{1/2} \frac{\ln|x|\,dx}{\sqrt{1-x}} + \int_{1/2}^{1} \frac{\ln|x|\,dx}{\sqrt{1-x}}.$$

Each integral on the right is improper because of a singularity at one endpoint.

Improper Integrals

Improper Integrals of Type II

EXAMPLE Evaluate each of the following integrals or show that it diverges:

$$\text{(a)} \quad \int_0^1 \frac{1}{x}\,dx, \qquad \text{(b)} \quad \int_0^2 \frac{1}{\sqrt{2x - x^2}}\,dx, \qquad \text{and} \qquad \text{(c)} \quad \int_0^1 \ln x\,dx.$$

Solution

(a) $\displaystyle \int_0^1 \frac{1}{x}\,dx = \lim_{c \to 0+} \int_c^1 \frac{1}{x}\,dx = \lim_{c \to 0+} (\ln 1 - \ln c) = \infty.$

This integral diverges to infinity.

Improper Integrals

Improper Integrals of Type II

EXAMPLE Evaluate each of the following integrals or show that it diverges:

(a) $\displaystyle\int_0^1 \frac{1}{x}\,dx$, (b) $\displaystyle\int_0^2 \frac{1}{\sqrt{2x-x^2}}\,dx$, and (c) $\displaystyle\int_0^1 \ln x\,dx$.

Solution

(b)
$$\int_0^2 \frac{1}{\sqrt{2x-x^2}}\,dx = \int_0^2 \frac{1}{\sqrt{1-(x-1)^2}}\,dx \qquad \text{Let } u = x-1,\ du = dx$$
$$= \int_{-1}^1 \frac{1}{\sqrt{1-u^2}}\,du$$
$$= 2\int_0^1 \frac{1}{\sqrt{1-u^2}}\,du \qquad \text{(by symmetry)}$$
$$= 2\lim_{c\to 1-}\int_0^c \frac{1}{\sqrt{1-u^2}}\,du$$
$$= 2\lim_{c\to 1-} \sin^{-1} u\Big|_0^c = 2\lim_{c\to 1-}\sin^{-1} c = \pi.$$

Improper Integrals

Improper Integrals of Type II

EXAMPLE Evaluate each of the following integrals or show that it diverges:

$$\text{(a)} \quad \int_0^1 \frac{1}{x}\,dx, \qquad \text{(b)} \quad \int_0^2 \frac{1}{\sqrt{2x - x^2}}\,dx, \qquad \text{and} \qquad \text{(c)} \quad \int_0^1 \ln x\,dx.$$

Solution

$$\begin{aligned}
\text{(c)} \quad \int_0^1 \ln x\,dx &= \lim_{c\to 0+} \int_c^1 \ln x\,dx \\
&= \lim_{c\to 0+} (x \ln x - x)\Big|_c^1 \\
&= \lim_{c\to 0+} (0 - 1 - c\ln c + c) \\
&= -1 + 0 - \lim_{c\to 0+} \frac{\ln c}{1/c} \qquad \left[\frac{-\infty}{\infty}\right] \\
&= -1 - \lim_{c\to 0+} \frac{1/c}{-(1/c^2)} \qquad \text{(by l'Hôpital's Rule)} \\
&= -1 - \lim_{c\to 0+} (-c) = -1 + 0 = -1.
\end{aligned}$$

Improper Integrals

THEOREM *p*-integrals

If $0 < a < \infty$, then

(a) $$\int_a^\infty x^{-p}\,dx \quad \begin{cases} \text{converges to } \dfrac{a^{1-p}}{p-1} & \text{if } p > 1 \\ \text{diverges to } \infty & \text{if } p \le 1 \end{cases}$$

(b) $$\int_0^a x^{-p}\,dx \quad \begin{cases} \text{converges to } \dfrac{a^{1-p}}{1-p} & \text{if } p < 1 \\ \text{diverges to } \infty & \text{if } p \ge 1. \end{cases}$$

Improper Integrals

THEOREM **A comparison theorem for integrals**

Let $-\infty \le a < b \le \infty$, and suppose that functions f and g are continuous on the interval (a, b) and satisfy $0 \le f(x) \le g(x)$. If $\int_a^b g(x)\,dx$ converges, then so does $\int_a^b f(x)\,dx$, and

$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx.$$

Equivalently, if $\int_a^b f(x)\,dx$ diverges to ∞, then so does $\int_a^b g(x)\,dx$.

Improper Integrals

EXAMPLE Show that $\int_1^\infty e^{-x^2}\,dx$ converges, and find an upper bound for its value.

Solution

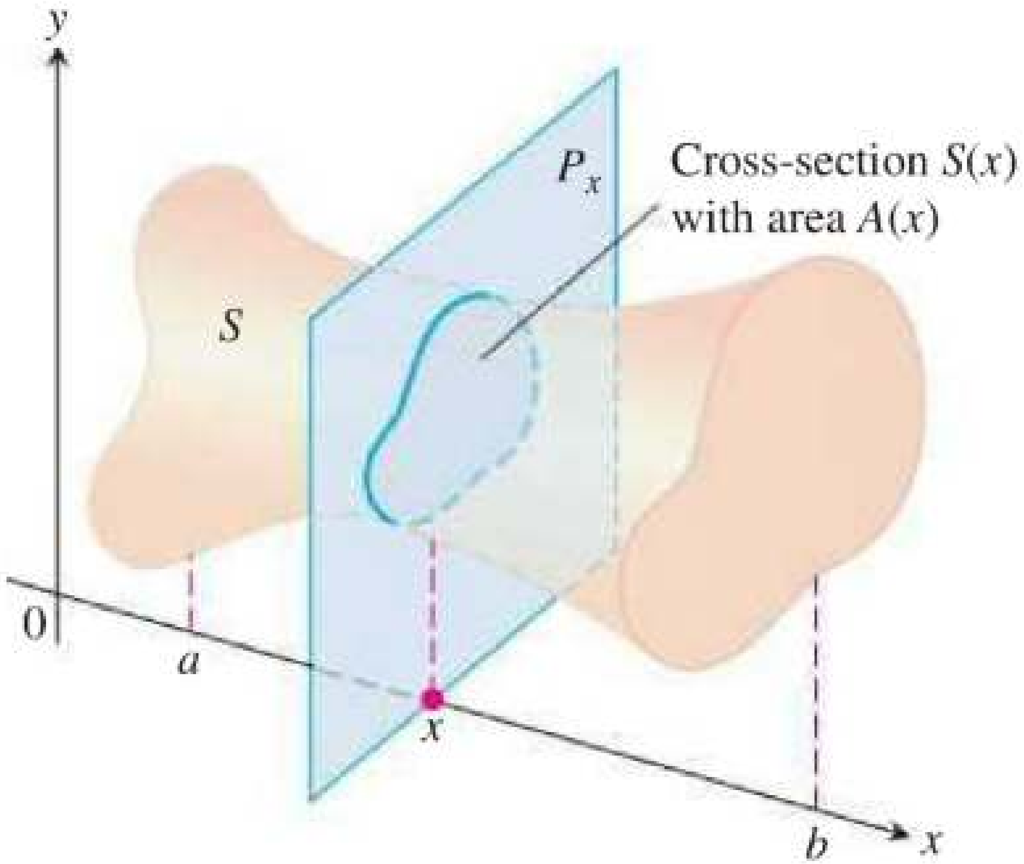
On $[1, \infty)$ we have $x^2 \geq x$, so $-x^2 \leq -x$ and $0 < e^{-x^2} \leq e^{-x}$. Thus,

$$
\begin{aligned}
0 < \int_1^\infty e^{-x^2}\,dx \leq \int_1^\infty e^{-x}\,dx &= \lim_{R\to\infty} \left.\frac{e^{-x}}{-1}\right|_1^R \\
&= \lim_{R\to\infty}\left(\frac{1}{e} - \frac{1}{e^R}\right) = \frac{1}{e}.
\end{aligned}
$$

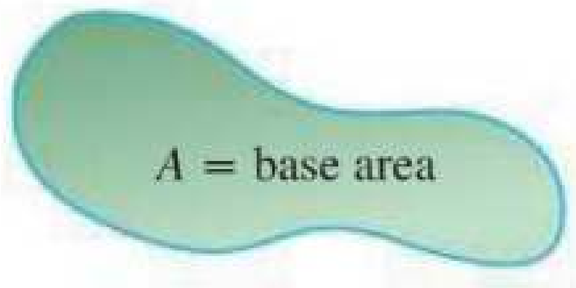
Hence, $\int_0^\infty e^{-x^2}\,dx$ converges and its value is not greater than $1 + (1/e)$.

Applications of Integration

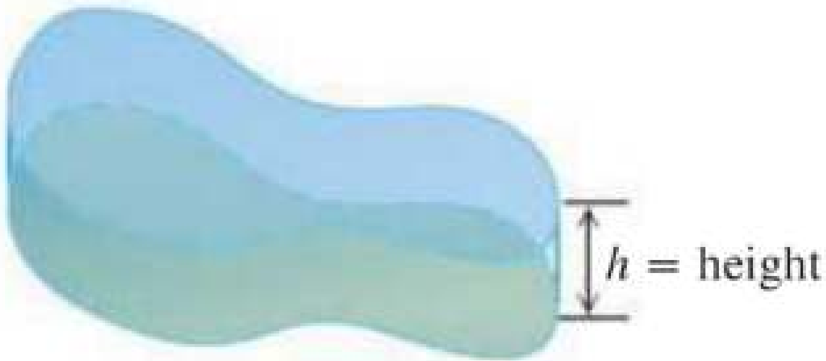
Volumes by Slicing—Solids of Revolution



Volumes by Slicing—Solids of Revolution

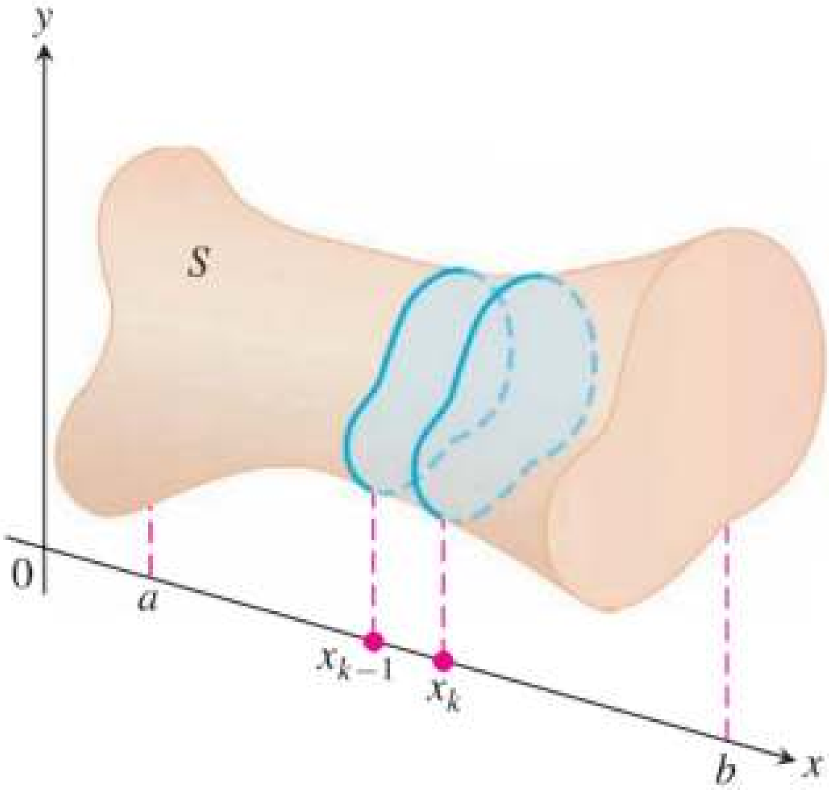


Plane region whose
area we know

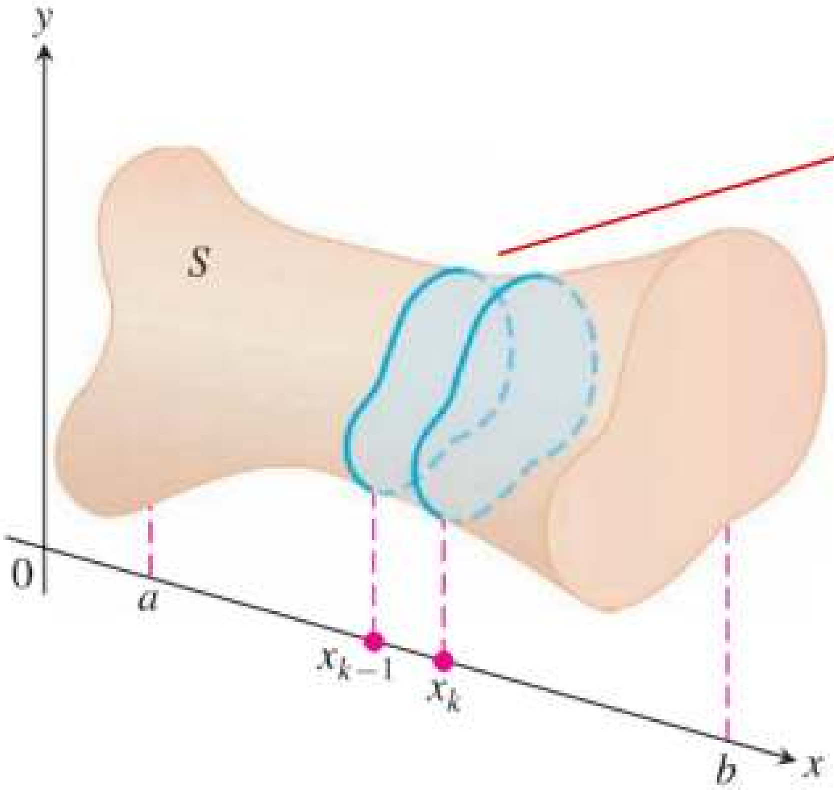


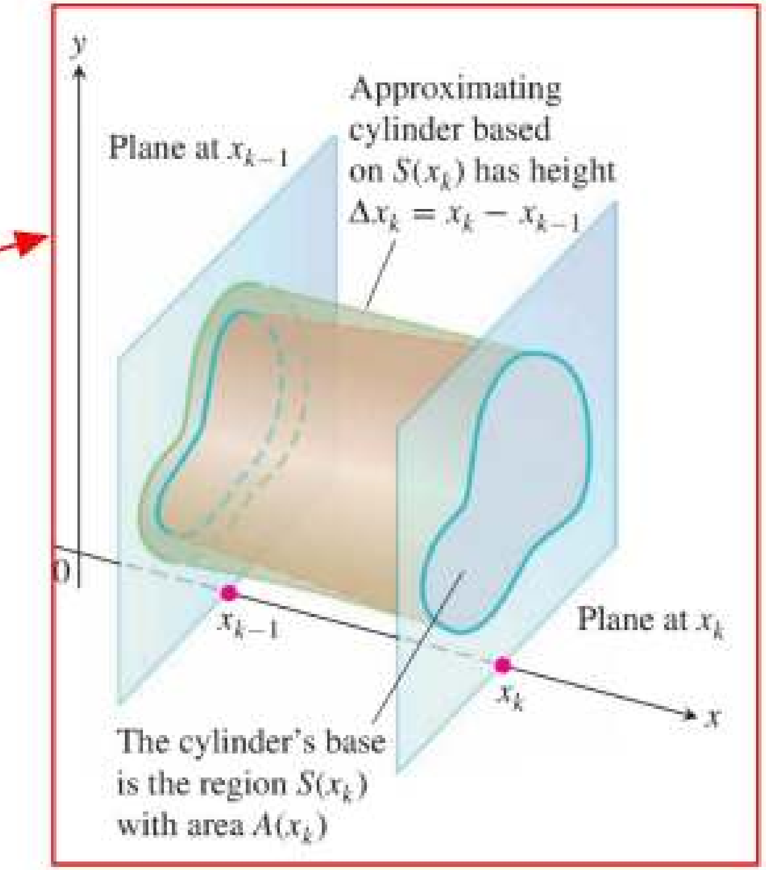
Cylindrical solid based on region
Volume = base area × height = Ah

Volumes by Slicing—Solids of Revolution



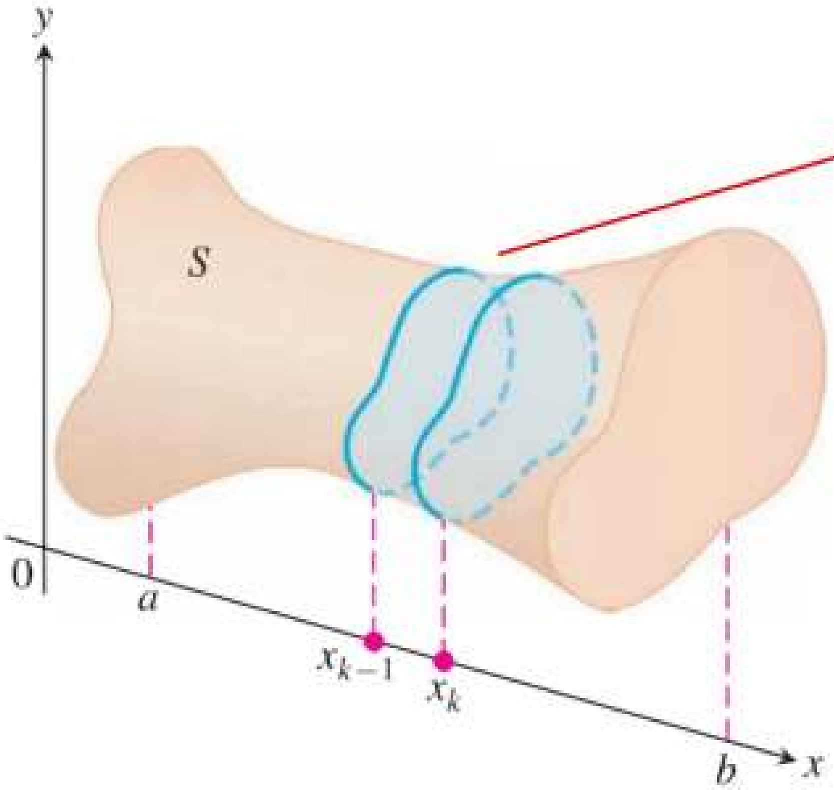
Volumes by Slicing—Solids of Revolution

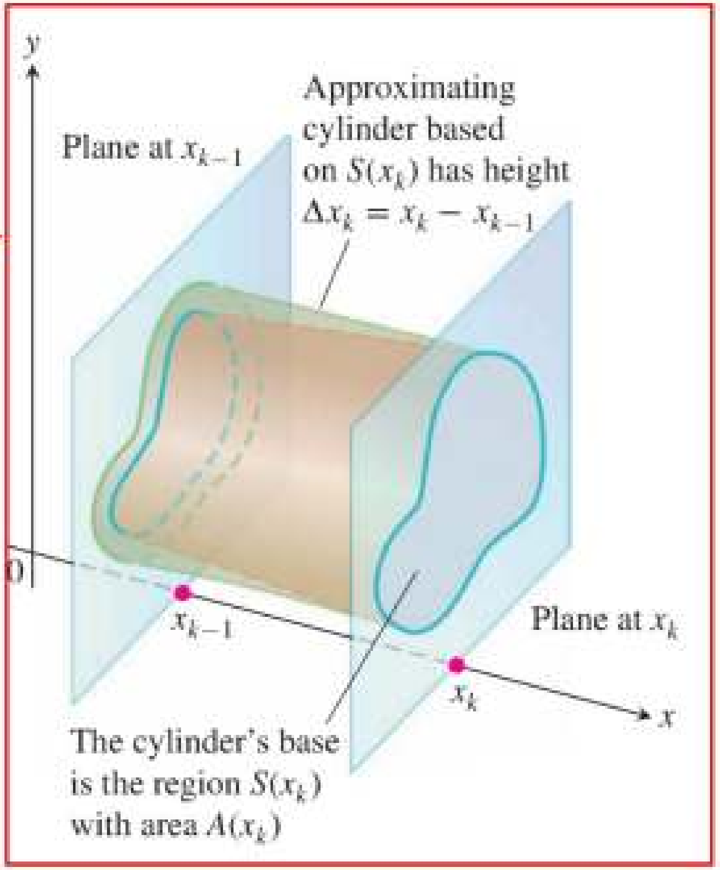




Volume of the kth slab $\approx V_k = A(x_k)\,\Delta x_k$.

Volumes by Slicing—Solids of Revolution





Volume of the kth slab $\approx V_k = A(x_k)\,\Delta x_k$.

$$V \approx \sum_{k=1}^{n} V_k = \sum_{k=1}^{n} A(x_k)\,\Delta x_k$$

$$\lim_{n\to\infty} \sum_{k=1}^{n} A(x_k)\,\Delta x_k = \int_a^b A(x)\,dx$$

Volumes by Slicing—Solids of Revolution

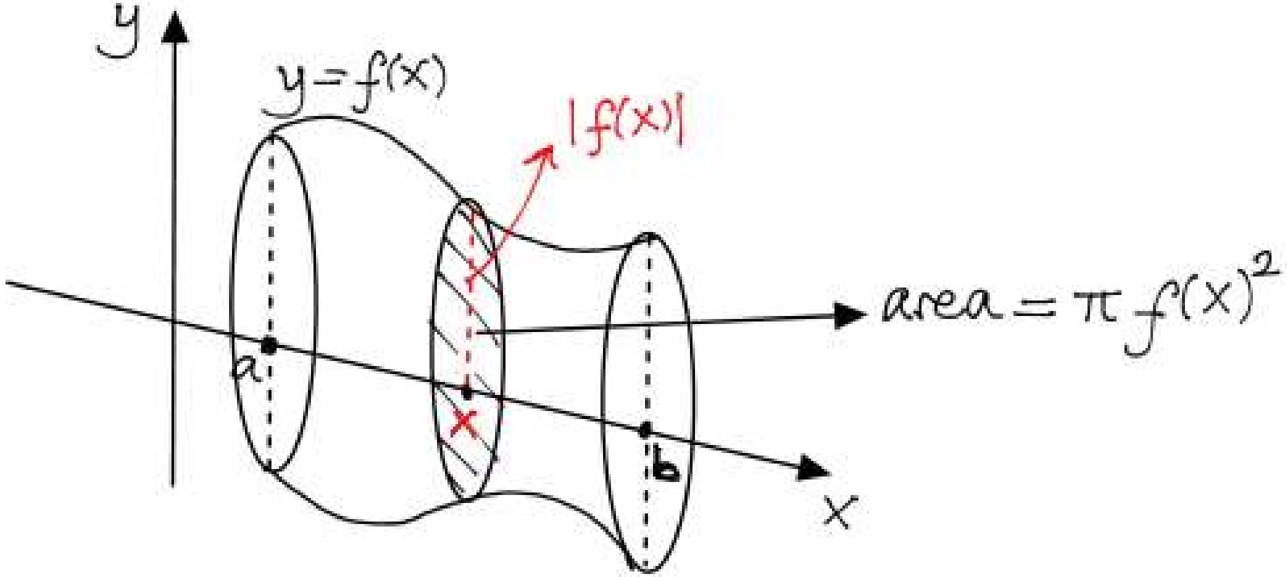
DEFINITION The **volume** of a solid of integrable cross-sectional area $A(x)$ from $x = a$ to $x = b$ is the integral of A from a to b,

$$V = \int_a^b A(x)\,dx.$$

Solids of Revolution

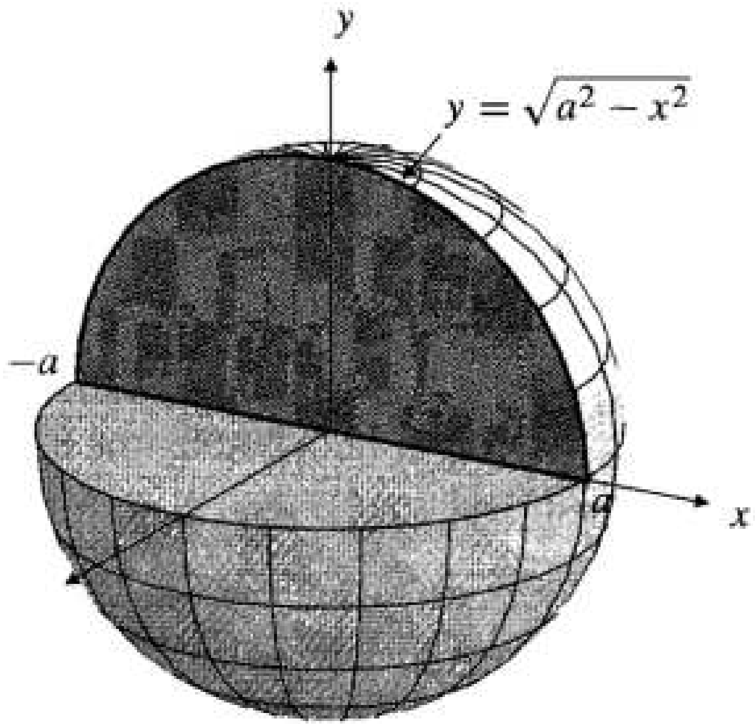
If the region R bounded by $y = f(x)$, $y = 0$, $x = a$, and $x = b$ is rotated about the x-axis, then the cross-section of the solid generated in the plane perpendicular to the x-axis at x is a circular disk of radius $|f(x)|$. The area of this cross-section is $A(x) = \pi\left(f(x)\right)^2$, so the volume of the solid of revolution is

$$V = \pi \int_a^b (f(x))^2 \, dx.$$



Solids of Revolution

EXAMPLE **(The volume of a ball)** Find the volume of a solid ball having radius a.



$$V = \pi \int_{-a}^{a} (\sqrt{a^2 - x^2})^2\, dx = 2\pi \int_0^a (a^2 - x^2)\, dx$$

$$= 2\pi \left(a^2 x - \frac{x^3}{3} \right)\Bigg|_0^a = 2\pi \left(a^3 - \frac{1}{3}a^3 \right) = \frac{4}{3}\pi a^3 \text{ cubic units.}$$

Volumes by Slicing—Solids of Revolution

Solids of Revolution

EXAMPLE A ring-shaped solid is generated by rotating the finite plane region R bounded by the curve $y = x^2$ and the line $y = 1$ about the line $y = 2$. Find its volume.

Solids of Revolution

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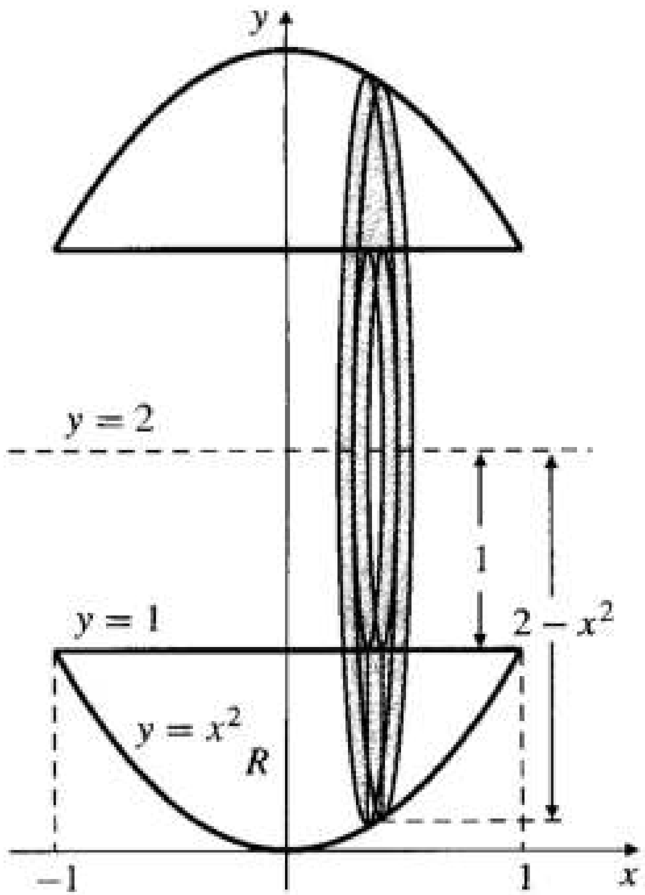

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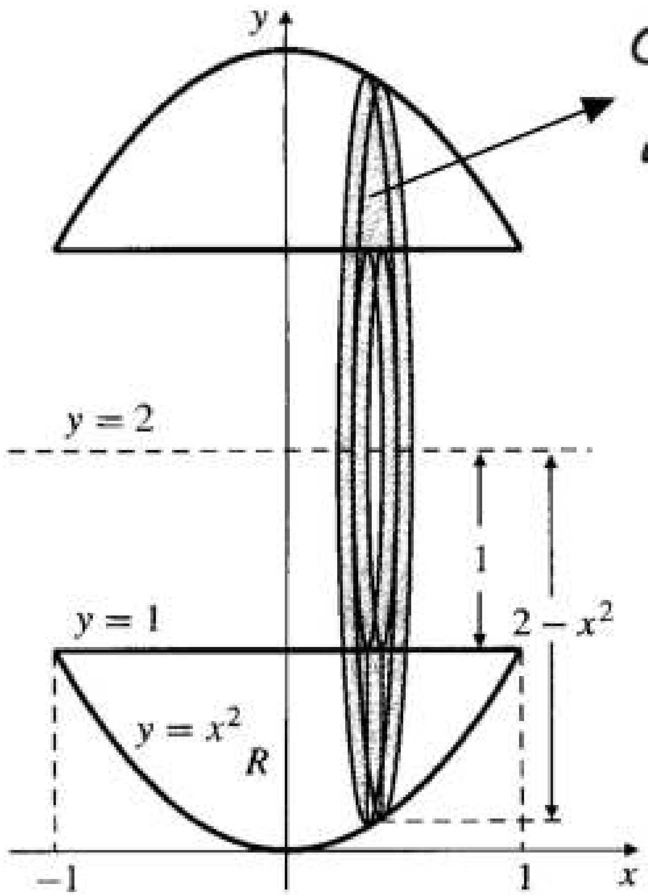

cross-sectional area $= \pi(2 - x^2)^2 - \pi(1)^2 = \pi(3 - 4x^2 + x^4)$

Volumes by Slicing—Solids of Revolution

Solids of Revolution

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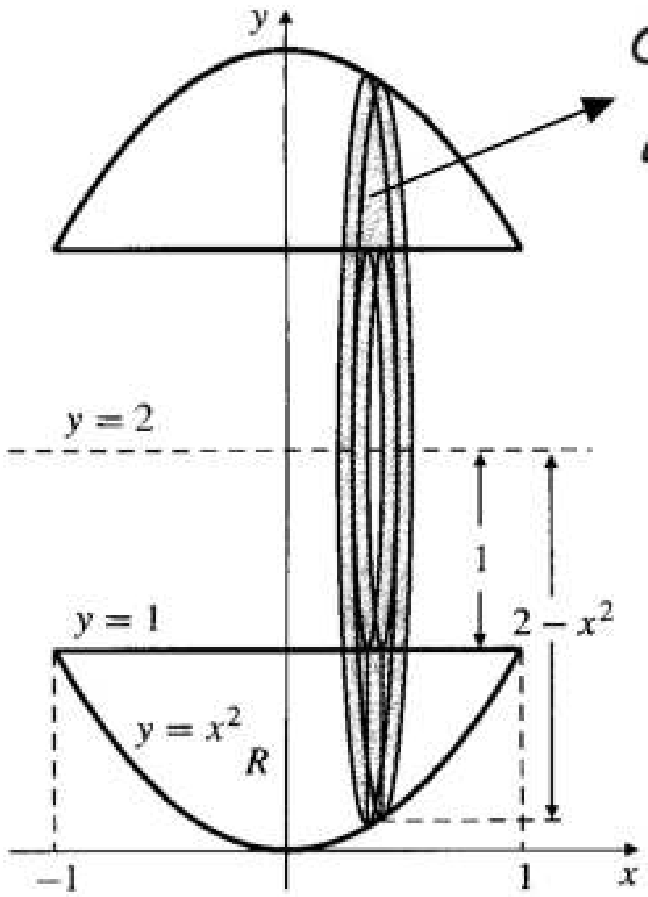

cross-sectional area $= \pi(2 - x^2)^2 - \pi(1)^2 = \pi(3 - 4x^2 + x^4)$

$$V = \pi \int_{-1}^{1} (3 - 4x^2 + x^4)\, dx = 2\pi \int_{0}^{1} (3 - 4x^2 + x^4)\, dx$$

$$= 2\pi \left(3x - \frac{4x^3}{3} + \frac{x^5}{5}\right)\Bigg|_0^1 = 2\pi \left(3 - \frac{4}{3} + \frac{1}{5}\right) = \frac{56\pi}{15} \text{ cubic units.}$$

Volumes by Slicing—Solids of Revolution

Solids of Revolution

EXAMPLE

Find the volume of the solid generated by rotating the region to the right of the y-axis and to the left of the curve $x = 2y - y^2$ about the y-axis.

Volumes by Slicing—Solids of Revolution

Solids of Revolution

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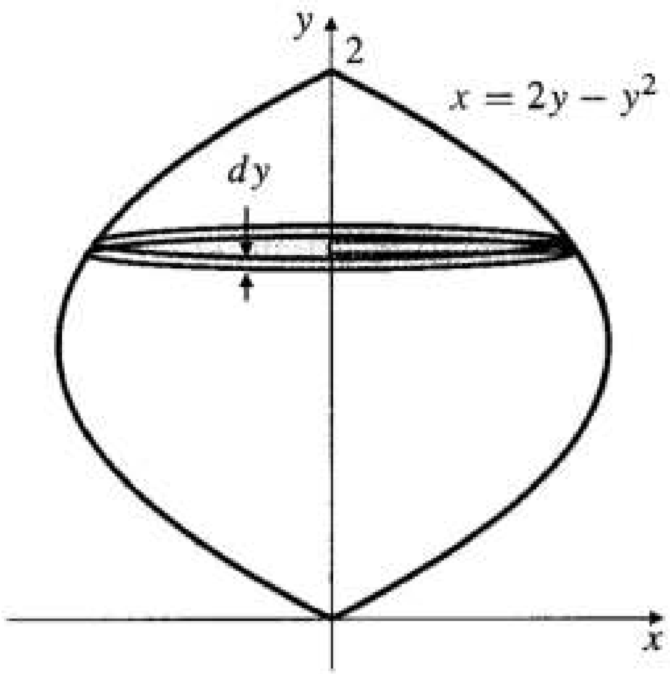

Volumes by Slicing—Solids of Revolution

Solids of Revolution

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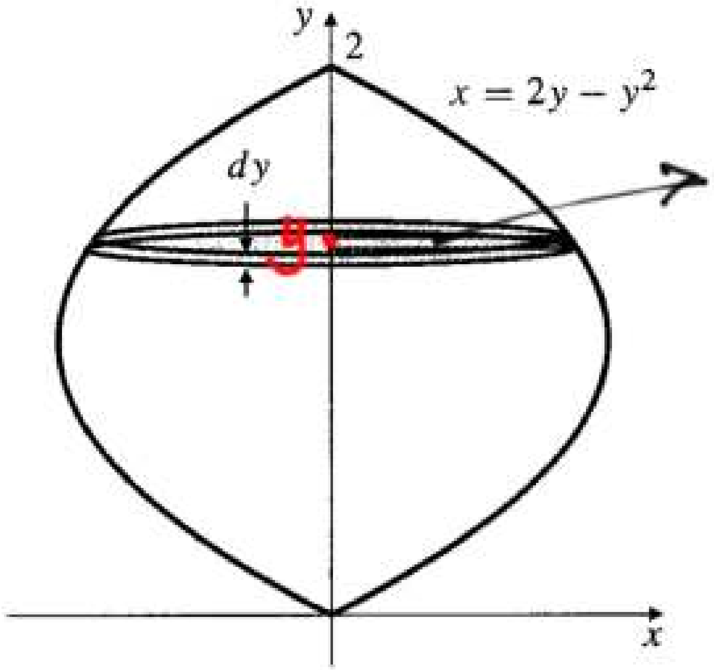

cross-sectional area $= \pi(2y - y^2)^2$

$= \pi(4y^2 - 4y^3 + y^4)$

Volumes by Slicing—Solids of Revolution

Solids of Revolution

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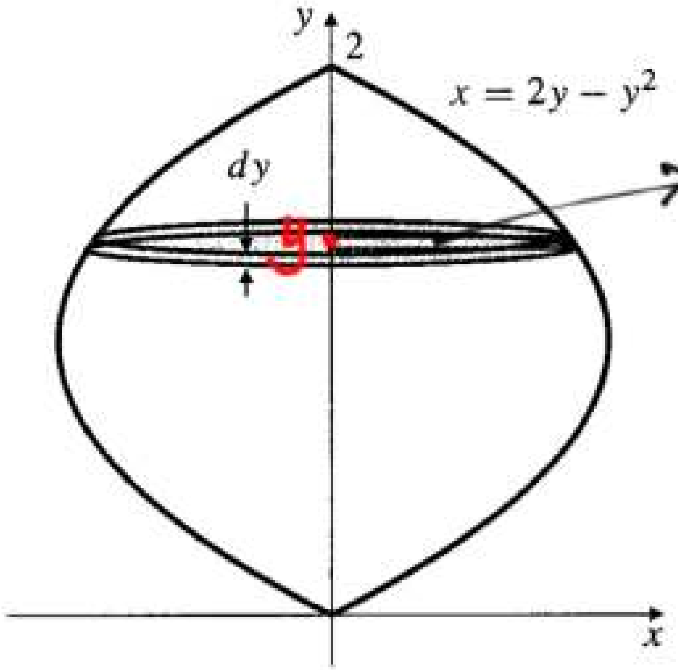


cross-sectional area $= \pi(2y - y^2)^2$

$$= \pi(4y^2 - 4y^3 + y^4)$$

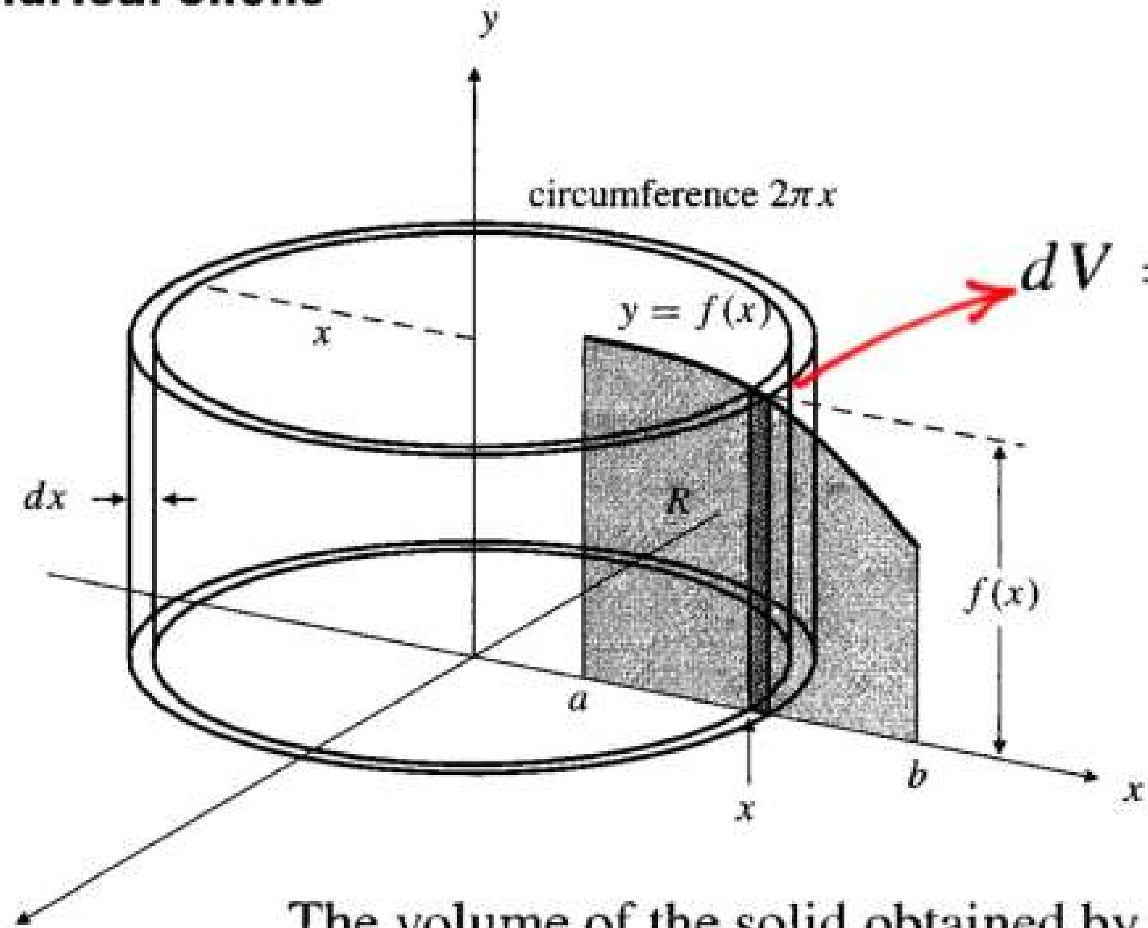
$$V = \pi \int_0^2 (4y^2 - 4y^3 + y^4)\, dy$$

$$= \pi \left(\frac{4y^3}{3} - y^4 + \frac{y^5}{5} \right) \Bigg|_0^2$$

$$= \pi \left(\frac{32}{3} - 16 + \frac{32}{5} \right) = \frac{16\pi}{15} \text{ cubic units.}$$

Volumes by Slicing—Solids of Revolution

Cylindrical Shells



$$dV = 2\pi x\ f(x)\, dx.$$

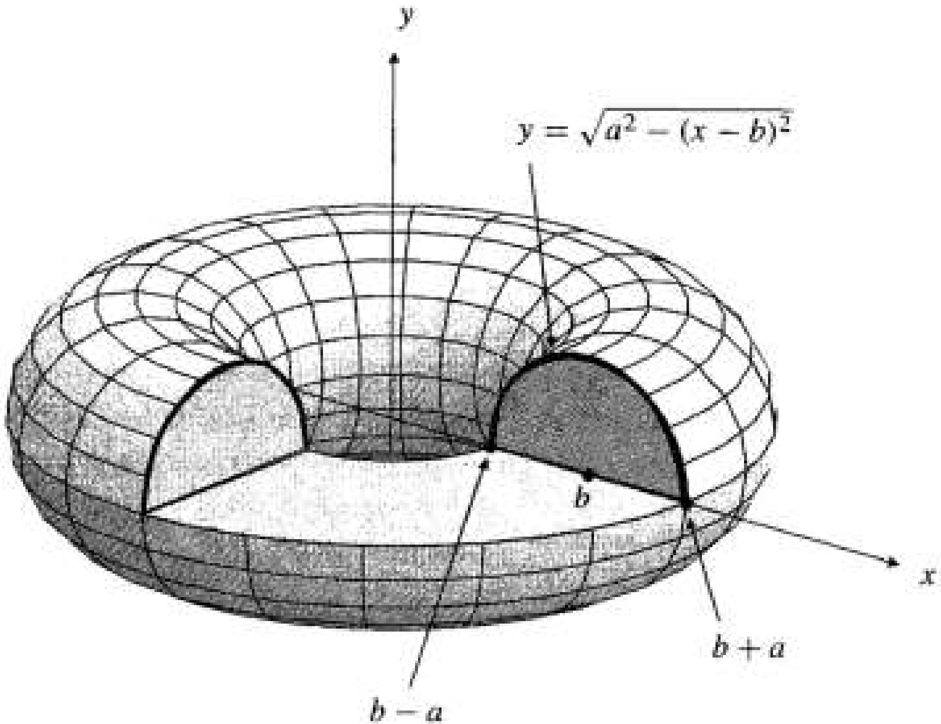
The volume of the solid obtained by rotating the plane region $0 \le y \le f(x)$, $0 \le a < x < b$ about the y-axis is

$$V = 2\pi \int_a^b x\ f(x)\, dx.$$

Volumes by Slicing—Solids of Revolution

Cylindrical Shells

EXAMPLE **(The volume of a torus)** A disk of radius a has centre at the point $(b, 0)$, where $b > a > 0$. The disk is rotated about the y-axis to generate a **torus** (a doughnut-shaped solid). Find its volume.



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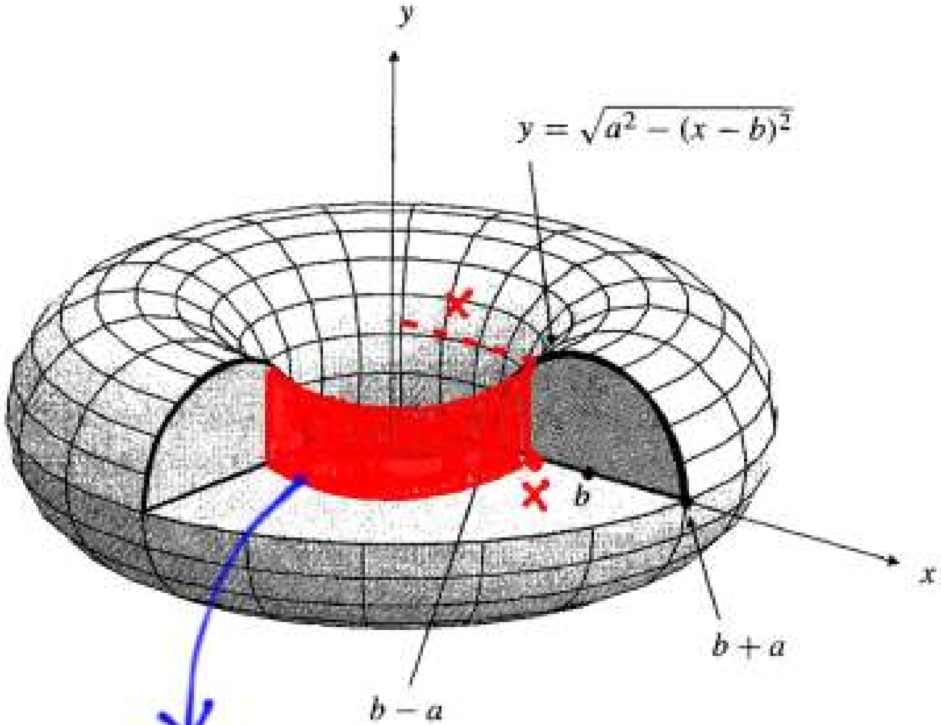
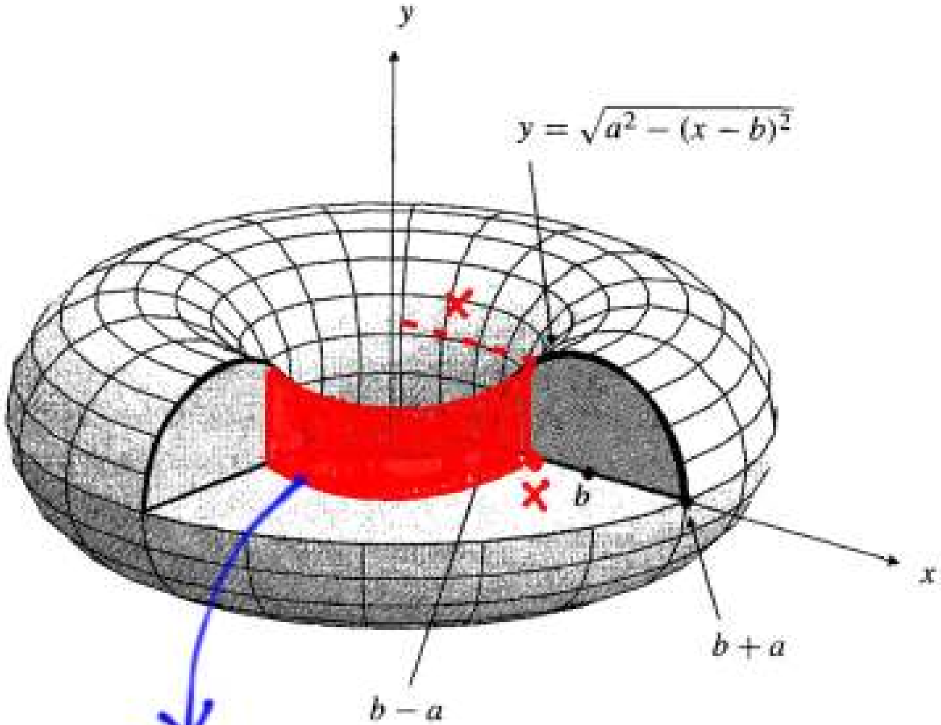


shell at x $(b-a<x<b+a)$
has radius x and height $\sqrt{a^2-(x-b)^2}$

Volumes by Slicing—Solids of Revolution

Cylindrical Shells

EXAMPLE **(The volume of a torus)** A disk of radius a has centre at the point $(b, 0)$, where $b > a > 0$. The disk is rotated about the y-axis to generate a **torus** (a doughnut-shaped solid). Find its volume.



$$V = 2 \times 2\pi \int_{b-a}^{b+a} x\sqrt{a^2 - (x-b)^2}\, dx \qquad \text{Let } u = x - b,\ du = dx$$

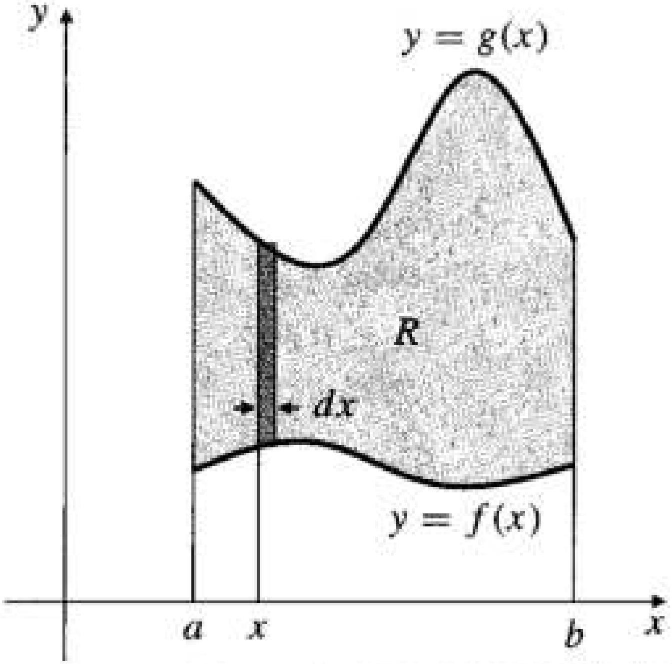
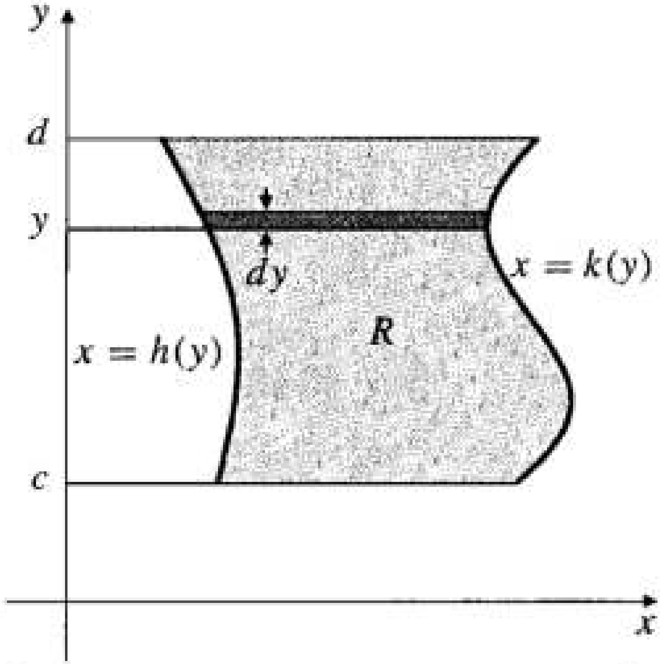
$$= 4\pi \int_{-a}^{a} (u + b)\sqrt{a^2 - u^2}\, du$$

$$= 4\pi \int_{-a}^{a} u\sqrt{a^2 - u^2}\, du + 4\pi b \int_{-a}^{a} \sqrt{a^2 - u^2}\, du$$

$$= 0 + 4\pi b \frac{\pi a^2}{2} = 2\pi^2 a^2 b \text{ cubic units.}$$

shell at x $(b-a<x<b+a)$
has radius x and height $\sqrt{a^2-(x-b)^2}$

Volumes by Slicing—Solids of Revolution

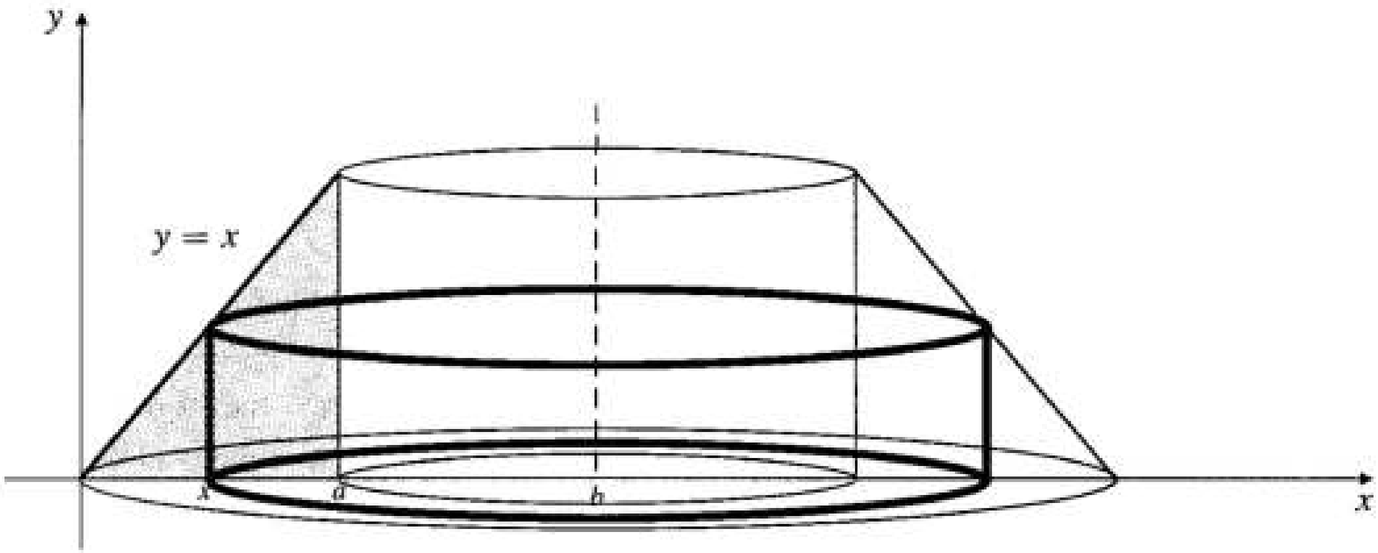
If region $R \longrightarrow$ is rotated about $\downarrow$		
the x-axis	use plane slices $V = \pi \int_a^b \left((g(x))^2 - (f(x))^2\right) dx$	use cylindrical shells $V = 2\pi \int_c^d y\left(k(y) - h(y)\right) dy$
the y-axis	use cylindrical shells $V = 2\pi \int_a^b x\left(g(x) - f(x)\right) dx$	use plane slices $V = \pi \int_c^d \left((k(y))^2 - (h(y))^2\right) dy$

Shell Formula for Revolution About a Vertical Line

$$V = \int_a^b 2\pi \begin{pmatrix} \text{shell} \\ \text{radius} \end{pmatrix} \begin{pmatrix} \text{shell} \\ \text{height} \end{pmatrix} dx.$$

Volumes by Slicing—Solids of Revolution

EXAMPLE The triangular region bounded by $y = x$, $y = 0$, and $x = a > 0$ is rotated about the line $x = b > a$. Find the volume of the solid so generated.



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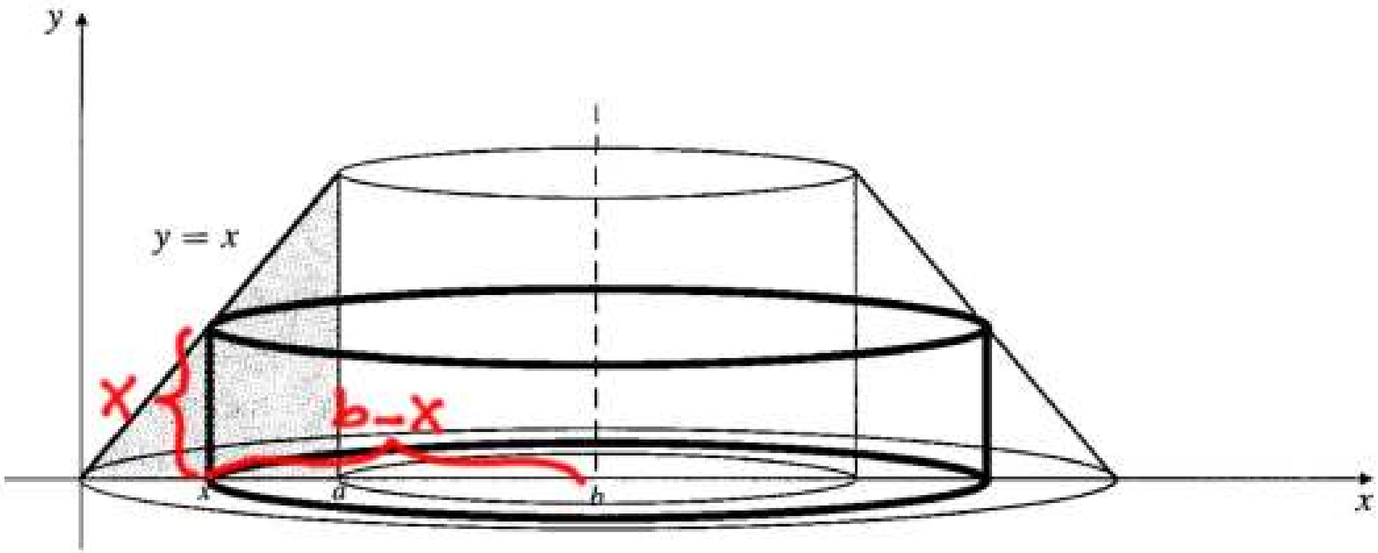

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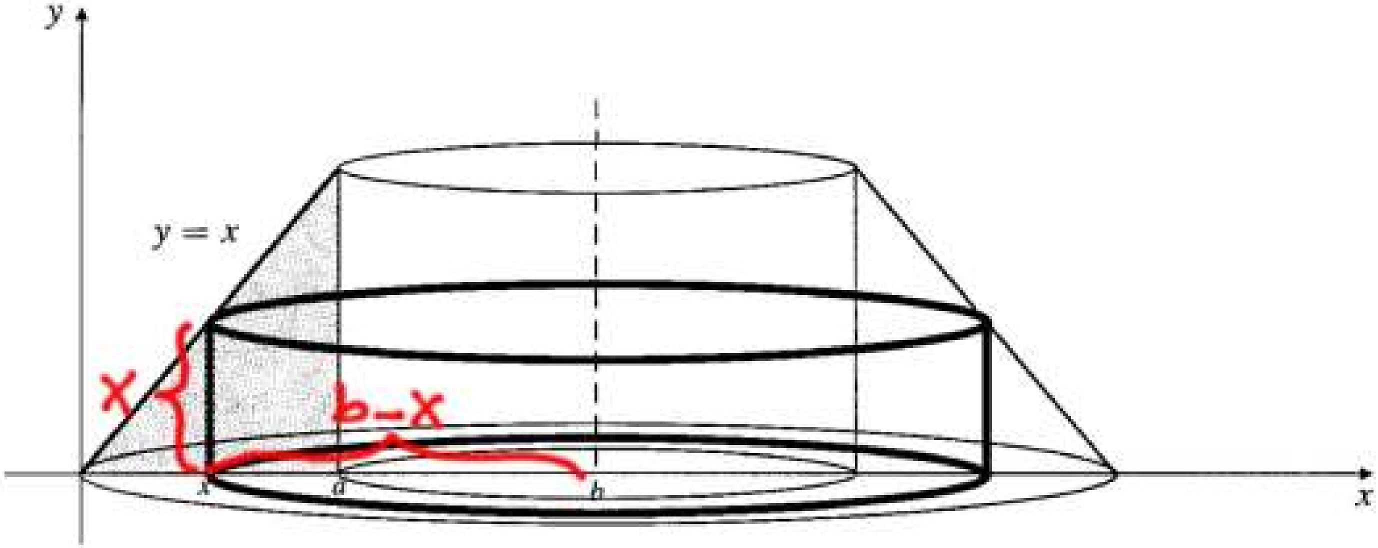
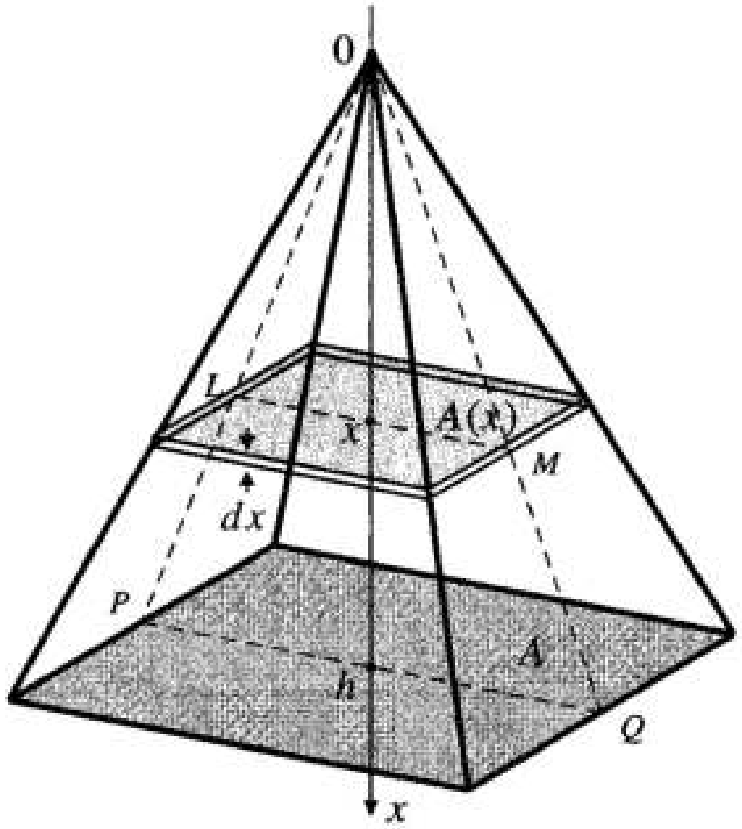


$$V = 2\pi \int_0^a (b - x)\, x\, dx = 2\pi \left(\frac{bx^2}{2} - \frac{x^3}{3} \right)\Bigg|_0^a = \pi \left(a^2 b - \frac{2a^3}{3} \right) \text{ cubic units.}$$

More Volumes by Slicing

EXAMPLE Verify the formula for the volume of a pyramid with rectangular base of area A and height h.



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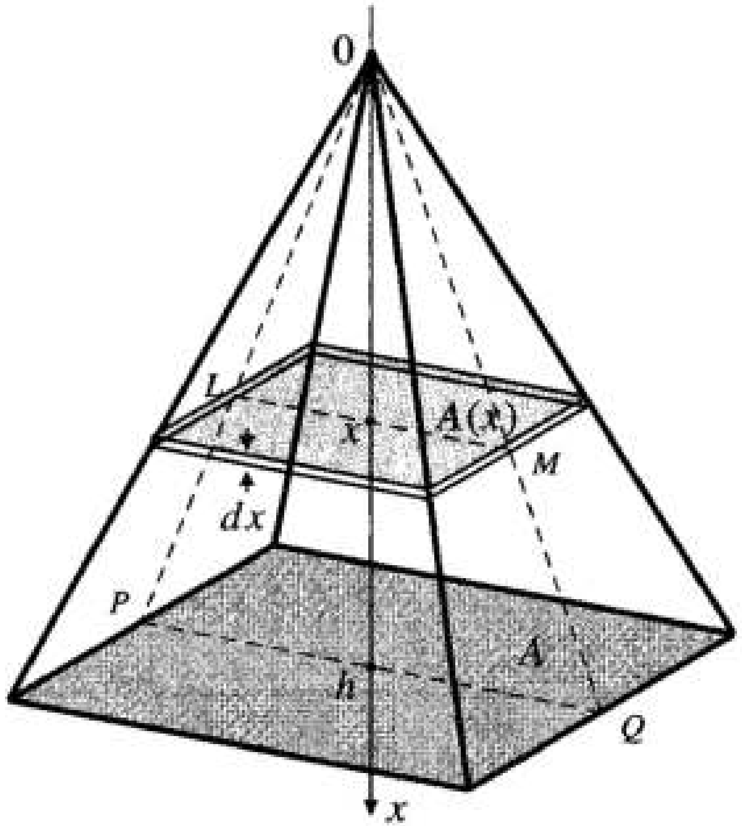

$$A(x) = \left(\frac{x}{h}\right)^2 A.$$

More Volumes by Slicing

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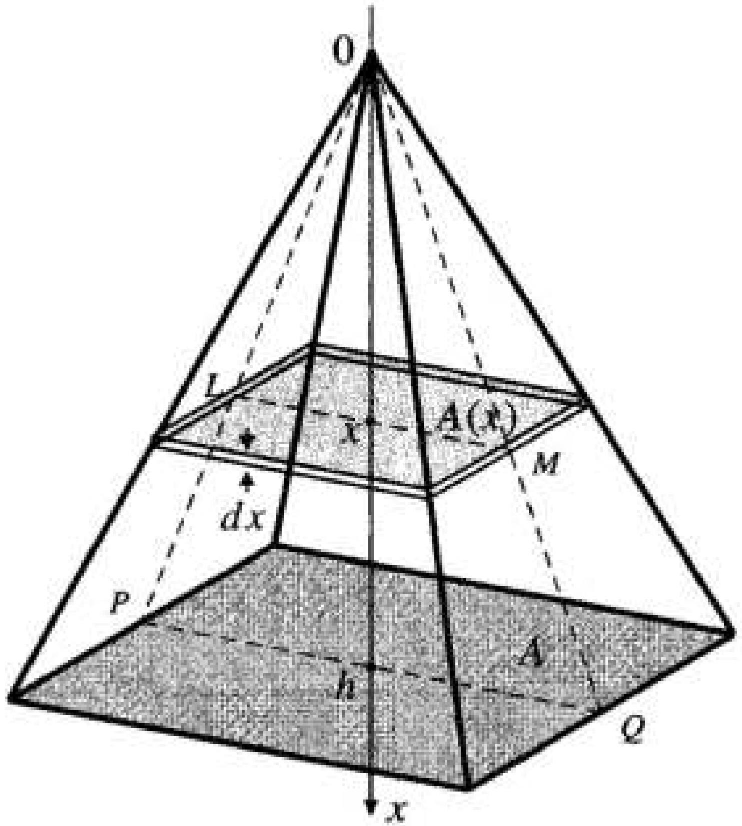
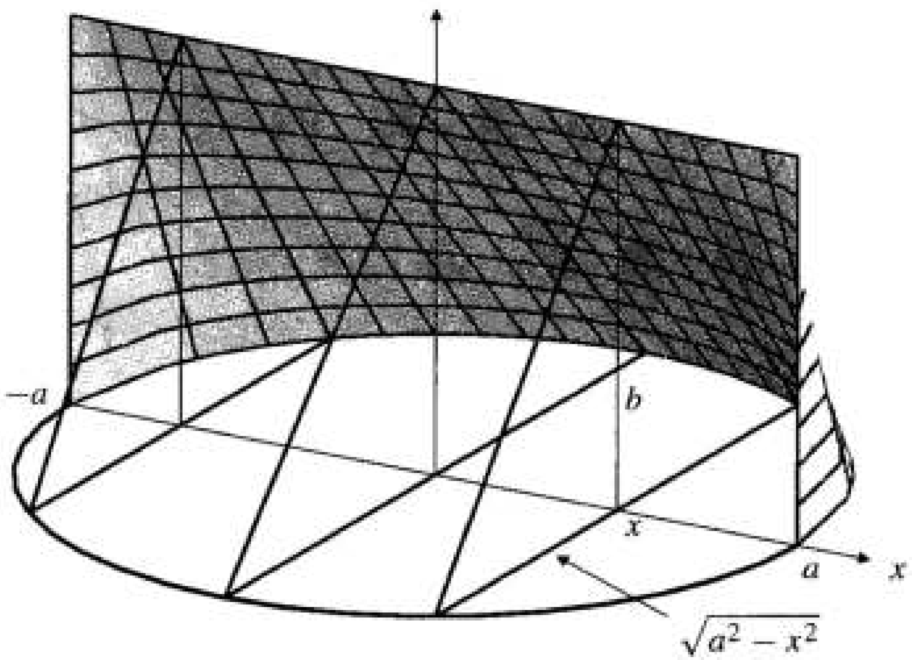


$$A(x) = \left(\frac{x}{h}\right)^2 A.$$

$$V = \int_0^h \left(\frac{x}{h}\right)^2 A\,dx = \frac{A}{h^2}\frac{x^3}{3}\bigg|_0^h = \frac{1}{3}Ah \text{ cubic units.}$$

More Volumes by Slicing

EXAMPLE A tent has a circular base of radius a metres and is supported by a horizontal ridge bar held at height b metres above a diameter of the base by vertical supports at each end of the diameter. The material of the tent is stretched tight so that each cross-section perpendicular to the ridge bar is an isosceles triangle. Find the volume of the tent.



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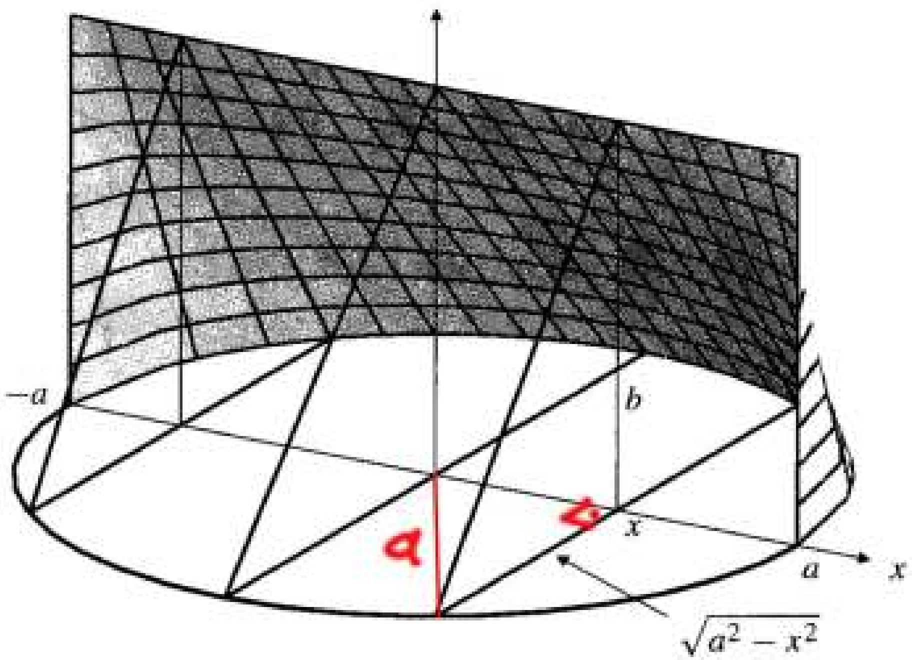

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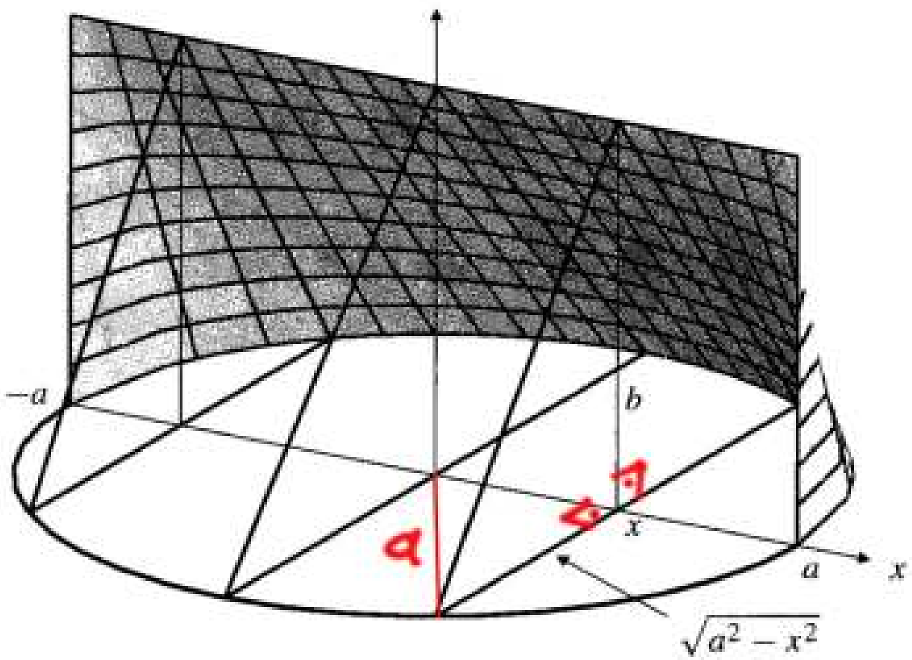

$$A(x) = \frac{1}{2}(2\sqrt{a^2 - x^2})b = b\sqrt{a^2 - x^2}$$

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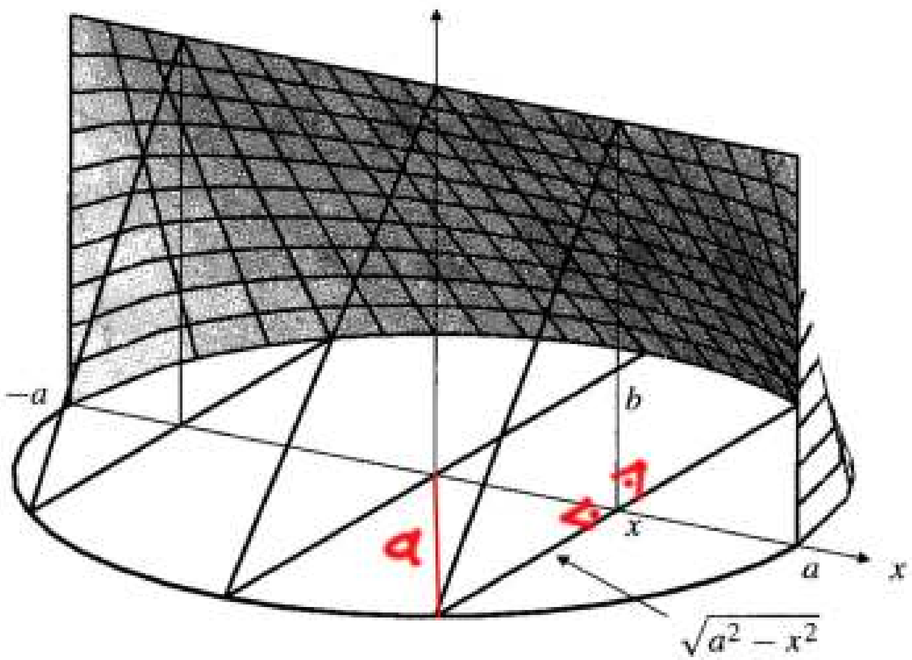

$$V = \int_{-a}^{a} b\sqrt{a^2 - x^2}\, dx = b\int_{-a}^{a} \sqrt{a^2 - x^2}\, dx$$

$$= b\frac{\pi a^2}{2} = \frac{\pi}{2}a^2 b \text{ m}^3.$$

$$A(x) = \frac{1}{2}(2\sqrt{a^2 - x^2})b = b\sqrt{a^2 - x^2}$$