Extreme Values

Critical Points, Singular Points, and Endpoints

Figure 4.17 suggests that a function $f(x)$ can have local extreme values only at points x of three special types:

(i) **critical points** of f (points x in $\mathcal{D}(f)$ where $f'(x) = 0$),

(ii) **singular points** of f (points x in $\mathcal{D}(f)$ where $f'(x)$ is not defined), and

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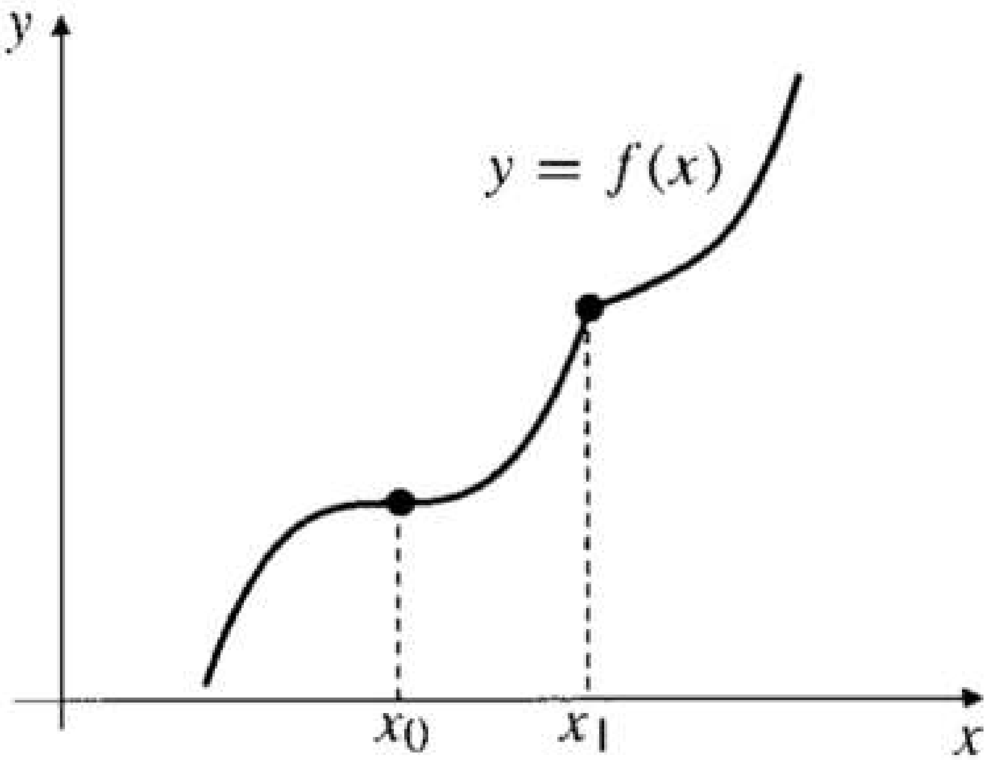
THEOREM

Locating extreme values

If the function f is defined on an interval I and has a local maximum (or local minimum) value at point $x = x_0$ in I, then x_0 must be either a critical point of f, a singular point of f, or an endpoint of I.

Extreme Values



A function need not have extreme values at a critical point or a singular point

Extreme Values

Finding Absolute Extreme Values

EXAMPLE

Find the maximum and minimum values of the function $g(x) = x^3 - 3x^2 - 9x + 2$ on the interval $-2 \le x \le 2$.

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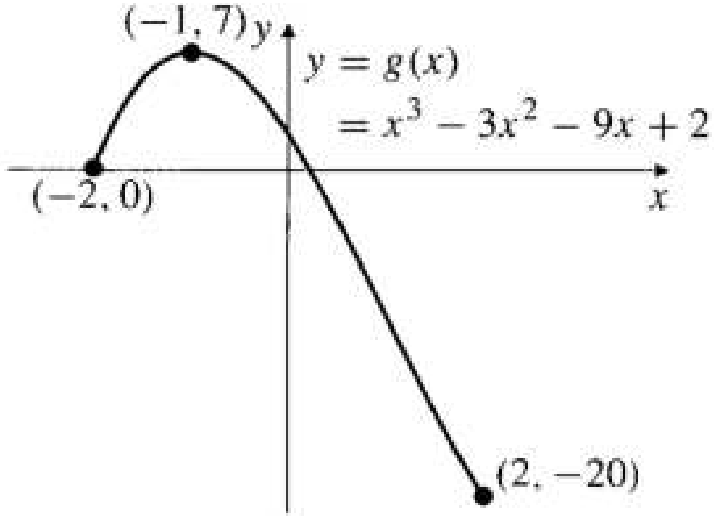

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max. min.

Extreme Values

THEOREM

The First Derivative Test

PART I. Testing interior critical points and singular points.

Suppose that f is continuous at x_0, and x_0 is not an endpoint of the domain of f.

(a) If there exists an open interval (a, b) containing x_0 such that $f'(x) > 0$ on (a, x_0) and $f'(x) < 0$ on (x_0, b), then f has a local maximum value at x_0.

(b) If there exists an open interval (a, b) containing x_0 such that $f'(x) < 0$ on (a, x_0) and $f'(x) > 0$ on (x_0, b), then f has a local minimum value at x_0.

PART II. Testing endpoints of the domain.

Suppose a is a left endpoint of the domain of f and f is right continuous at a.

(c) If $f'(x) > 0$ on some interval (a, b), then f has a local minimum value at a.

(d) If $f'(x) < 0$ on some interval (a, b), then f has a local maximum value at a.

Suppose b is a right endpoint of the domain of f and f is left continuous at b.

(e) If $f'(x) > 0$ on some interval (a, b), then f has a local maximum value at b.

(f) If $f'(x) < 0$ on some interval (a, b), then f has a local minimum value at b.

Extreme Values

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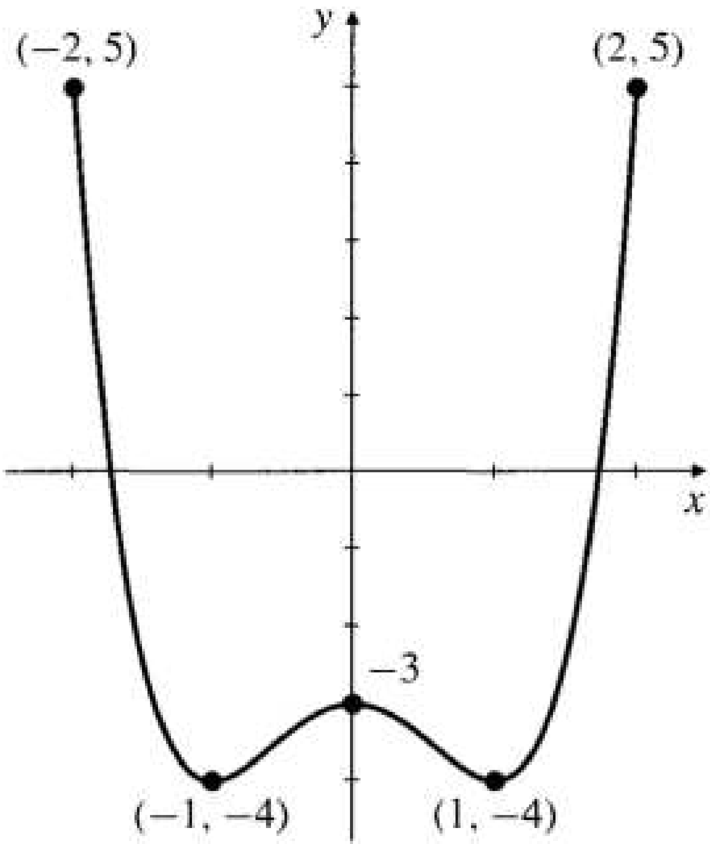
Solution We begin by calculating and factoring the derivative $f'(x)$:

$$f'(x) = 4x^3 - 4x = 4x(x^2 - 1) = 4x(x - 1)(x + 1).$$

The critical points are 0, -1, and 1. The corresponding values are $f(0) = -3$, $f(-1) = f(1) = -4$. There are no singular points. The values of f at the endpoints -2 and 2 are $f(-2) = f(2) = 5$.

Extreme Values



	EP		CP		CP		CP		EP
x	-2		-1		0		1		2
f'		$-$	0	$+$	0	$-$	0	$+$	
f	max	$\searrow$	min	$\nearrow$	max	$\searrow$	min	$\nearrow$	max

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Solution $f'(x) = 1 - \frac{2}{3}x^{-1/3} = \left(x^{1/3} - \frac{2}{3}\right)/x^{1/3}$. There is a singular point, $x = 0$, and a critical point, $x = 8/27$. The endpoints are $x = -1$ and $x = 2$. The values of f at these points are $f(-1) = -2$, $f(0) = 0$, $f(8/27) = -4/27$, and $f(2) = 2 - 2^{2/3} \approx 0.4126$.

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	EP		SP		CP		EP
x	-1		0		$8/27$		2
f'		$+$	undef	$-$	0	$+$	
f	min	$\nearrow$	max	$\searrow$	min	$\nearrow$	max

There are two local minima and two local maxima. The absolute maximum of f is $2 - 2^{2/3}$ at $x = 2$; the absolute minimum is -2 at $x = -1$.

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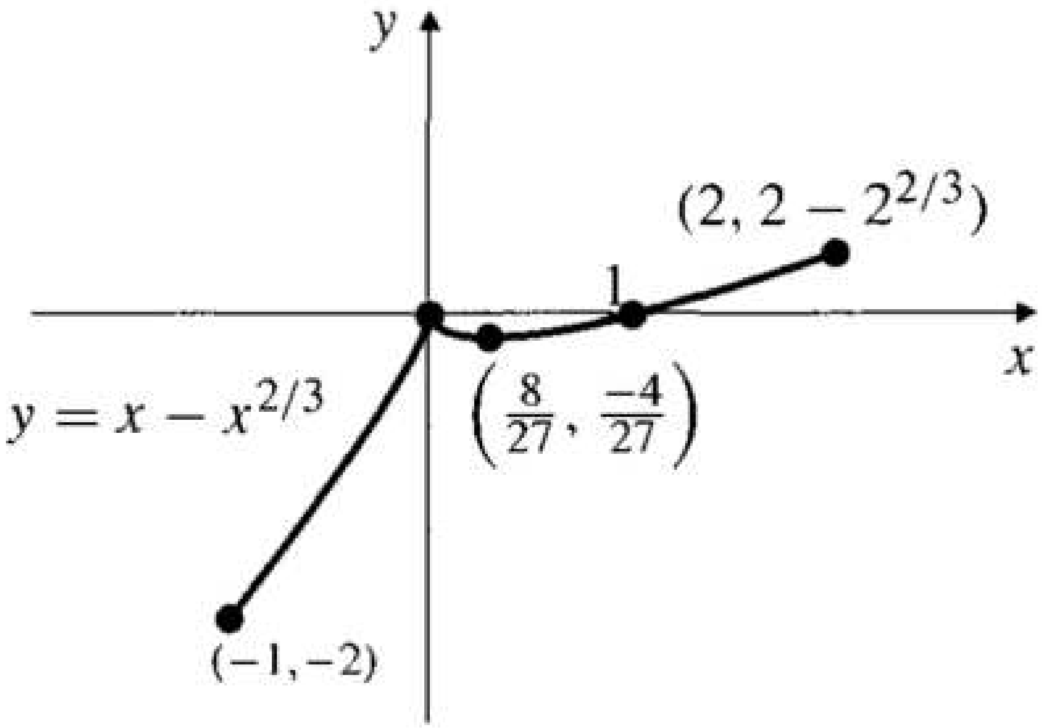

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Extreme Values

Functions Not Defined on Closed, Finite Intervals

THEOREM

Existence of extreme values on open intervals

If f is continuous on the open interval (a, b), and if

$$\lim_{x \to a+} f(x) = L \qquad \text{and} \qquad \lim_{x \to b-} f(x) = M,$$

then the following conclusions hold:

(i) If $f(u) > L$ and $f(u) > M$ for some u in (a, b), then f has an absolute maximum value on (a, b).

(ii) If $f(v) < L$ and $f(v) < M$ for some v in (a, b), then f has an absolute minimum value on (a, b).

In this theorem a may be $-\infty$, in which case $\lim_{x\to a+}$ should be replaced with $\lim_{x\to -\infty}$, and b may be ∞, in which case $\lim_{x\to b-}$ should be replaced with $\lim_{x\to\infty}$. Also, either or both of L and M may be either ∞ or $-\infty$.

Extreme Values

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Show that $f(x) = x + (4/x)$ has an absolute minimum value on the interval $(0, \infty)$, and find that minimum value.

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Solution We have

$$\lim_{x \to 0+} f(x) = \infty \qquad \text{and} \qquad \lim_{x \to \infty} f(x) = \infty.$$

Since $f(1) = 5 < \infty$, Theorem guarantees that f must have an absolute minimum value at some point in $(0, \infty)$.

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$f(2) = 4 \longrightarrow$ absolute min. value

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		CP		CP	
x		$-1/\sqrt{2}$		$1/\sqrt{2}$	
f'	$-$	0	$+$	0	$-$
f	$\searrow$	min	$\nearrow$	max	$\searrow$

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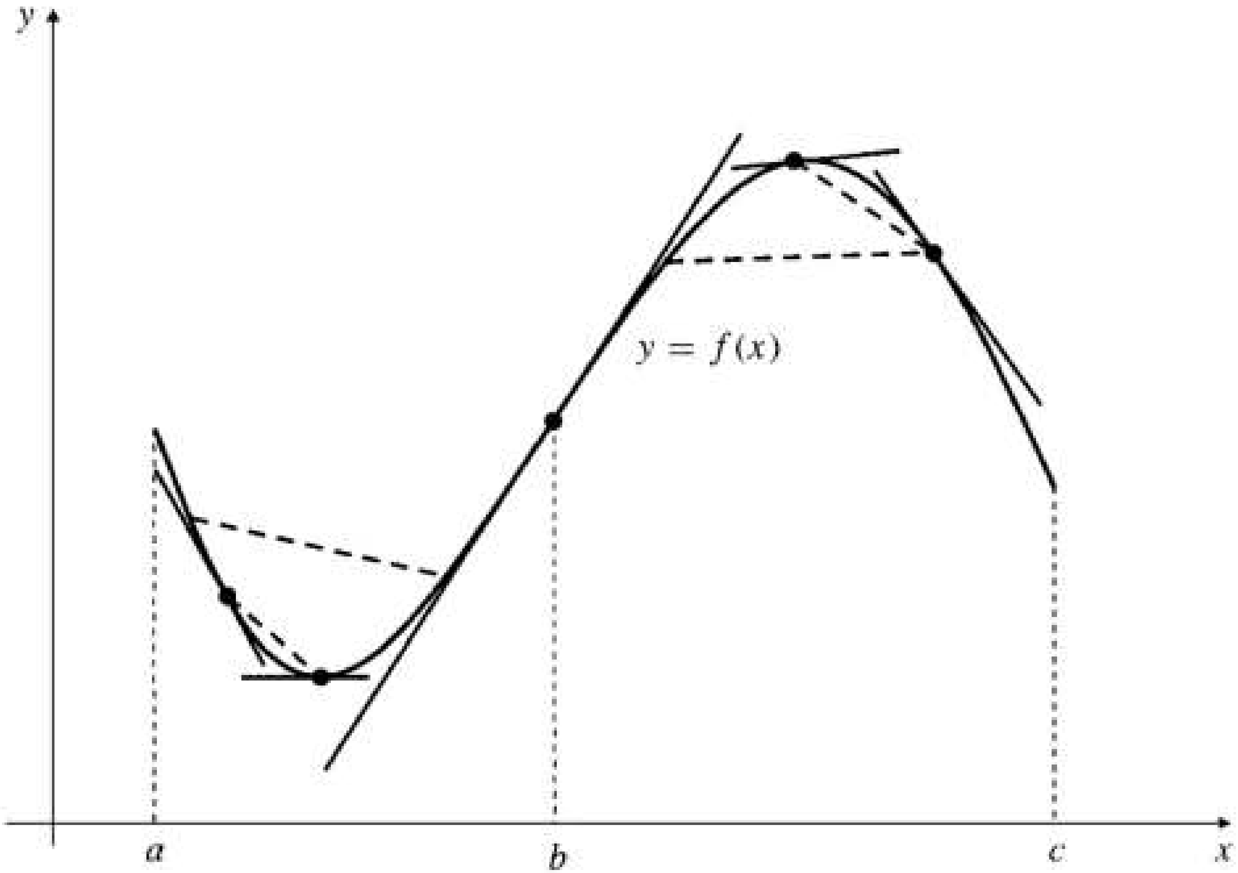
$\frac{1}{\sqrt{2e}}$ is the absolute max., $\frac{-1}{\sqrt{2e}}$ is the absolute min.

Concavity and Inflections

We say that the function f is **concave up** on an open interval I if it is differentiable there and the derivative f' is an increasing function on I. Similarly, f is **concave down** on I if f' exists and is decreasing on I.



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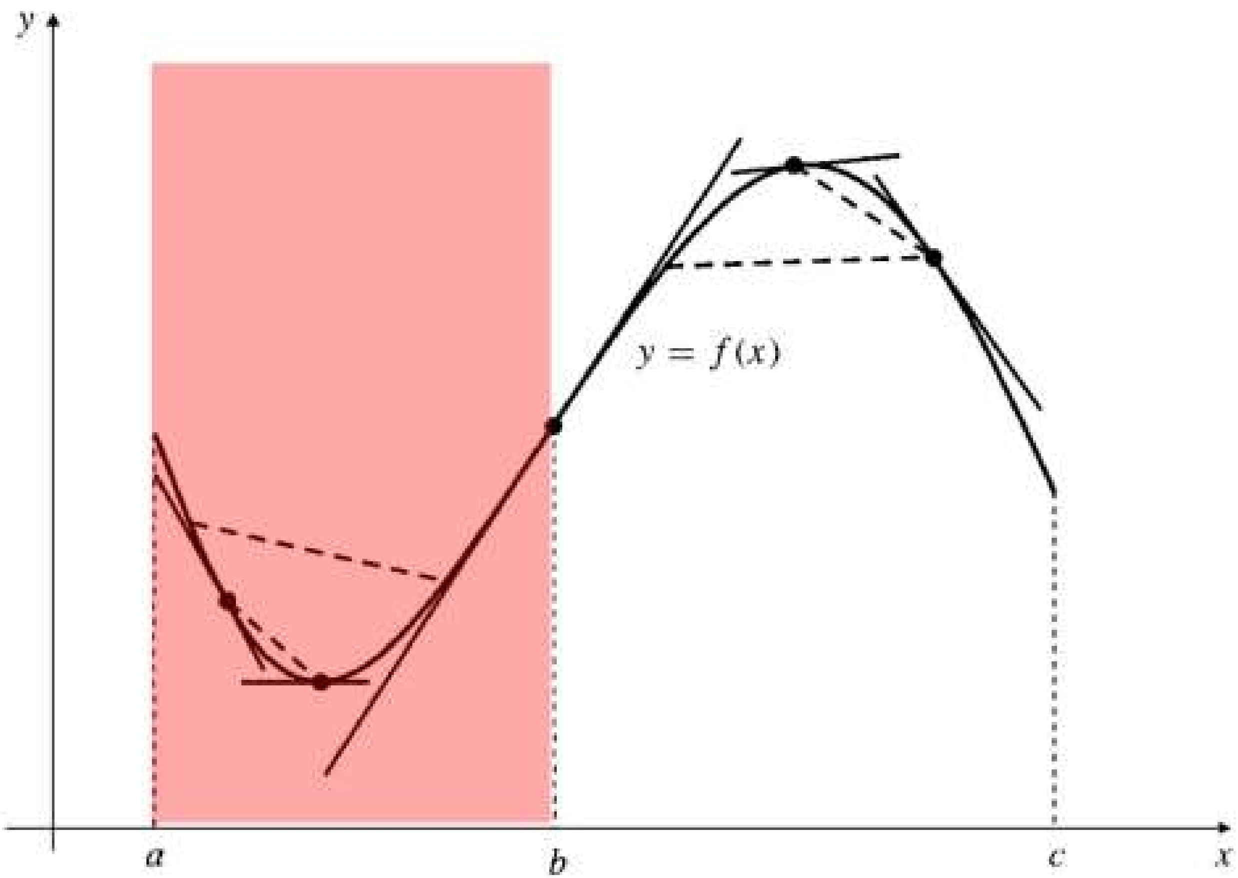

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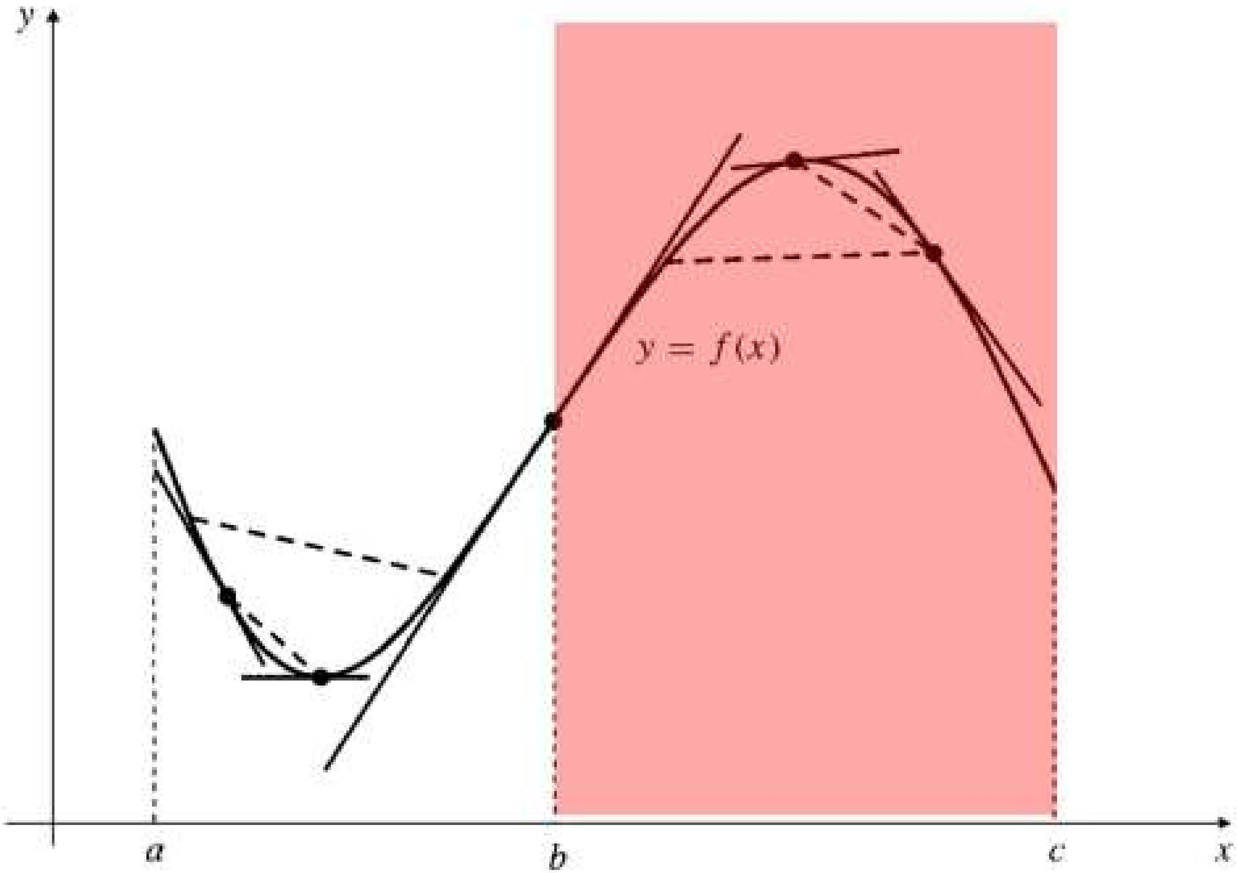


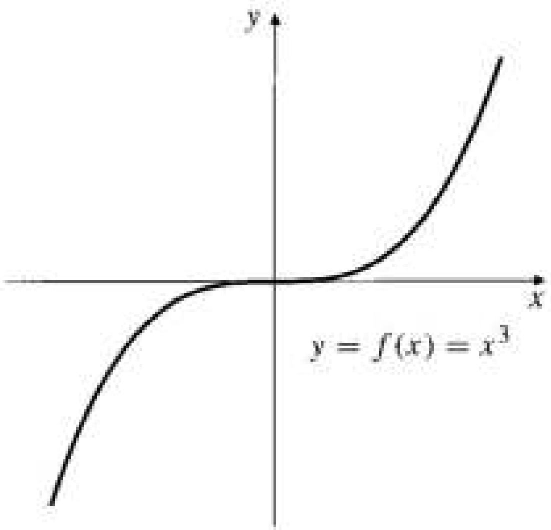
Concavity and Inflections

Inflection points

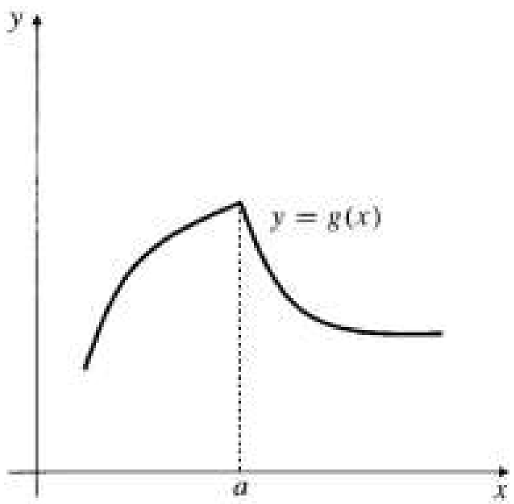
We say that the point $(x_0, f(x_0))$ *is* an **inflection point** of the curve $y = f(x)$ (or that the function f *has* an **inflection point** at x_0) if the following two conditions are satisfied:

(a) the graph of $y = f(x)$ has a tangent line at $x = x_0$, and

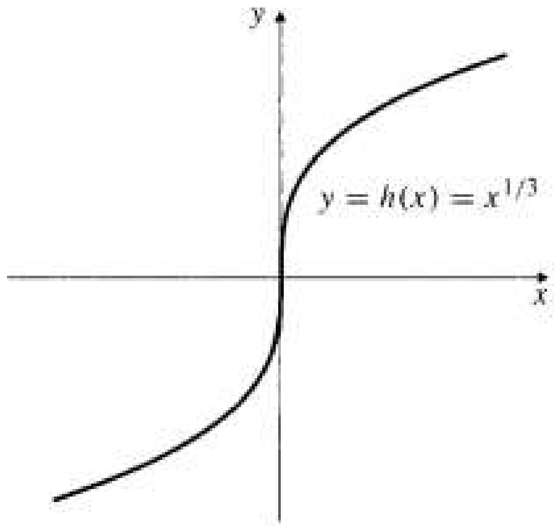
(b) the concavity of f is opposite on opposite sides of x_0.



$x = 0$ is a critical point of $f(x) = x^3$, and f has an inflection point there



The concavity of g is opposite on opposite sides of the singular point a, but its graph has no tangent and therefore no inflection point there



This graph of h has an inflection point at the origin even though $x = 0$ is a singular point of h

Concavity and Inflections

THEOREM

Concavity and the second derivative

(a) If $f''(x) > 0$ on interval I, then f is concave up on I.

(b) If $f''(x) < 0$ on interval I, then f is concave down on I.

(c) If f has an inflection point at x_0 and $f''(x_0)$ exists, then $f''(x_0) = 0$.

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		CP				CP	
x		0		1		3/2	
f'	$-$	0	$-$		$-$	0	$+$
f''	$+$	0	$-$	0	$+$		$+$
f	$\searrow$		$\searrow$		$\searrow$	min	$\nearrow$
	$\smile$	infl	$\frown$	infl	$\smile$		$\smile$

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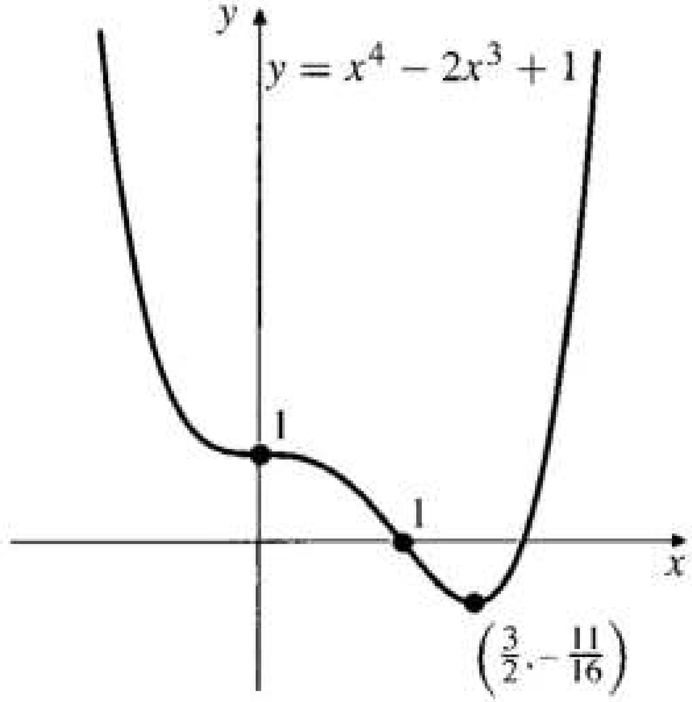
		CP				CP	
x		0		1		3/2	
f'	−	0	−		−	0	+
f''	+	0	−	0	+		+
f	↘		↘		↘	min	↗
	⌣	infl	⌢	infl	⌣		⌣



Concavity and Inflections

THEOREM

The Second Derivative Test

(a) If $f'(x_0) = 0$ and $f''(x_0) < 0$, then f has a local maximum value at x_0.

(b) If $f'(x_0) = 0$ and $f''(x_0) > 0$, then f has a local minimum value at x_0.

(c) If $f'(x_0) = 0$ and $f''(x_0) = 0$, no conclusion can be drawn; f may have a local maximum at x_0 or a local minimum, or it may have an inflection point instead.

Concavity and Inflections

EXAMPLE Find and classify the critical points of $f(x) = x^2 e^{-x}$.

Concavity and Inflections

EXAMPLE Find and classify the critical points of $f(x) = x^2e^{-x}$.

Solution We begin by calculating the first two derivatives of f:

$$f'(x) = (2x - x^2)e^{-x} = x(2-x)e^{-x} = 0 \quad \text{at } x = 0 \text{ and } x = 2,$$
$$f''(x) = (2 - 4x + x^2)e^{-x}$$
$$f''(0) = 2 > 0, \qquad f''(2) = -2e^{-2} < 0.$$

Thus, f has a local minimum value at $x = 0$ and a local maximum value at $x = 2$.