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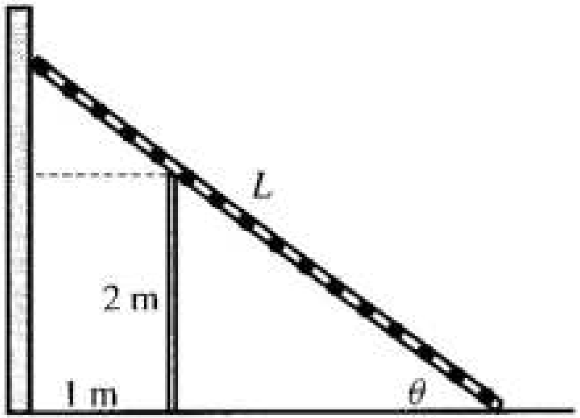

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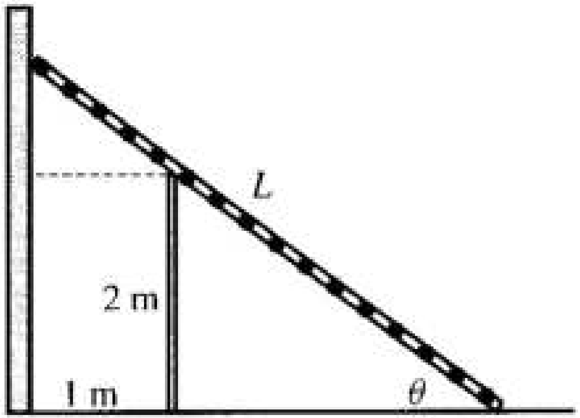

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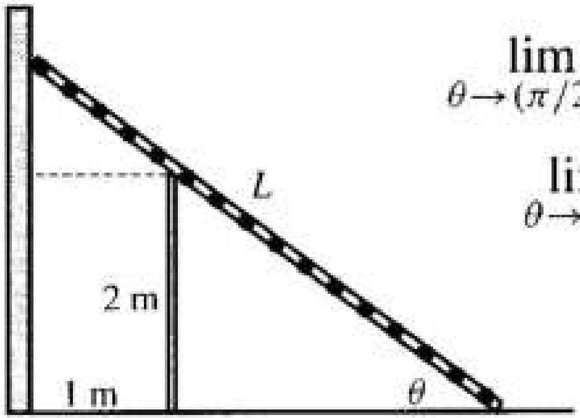

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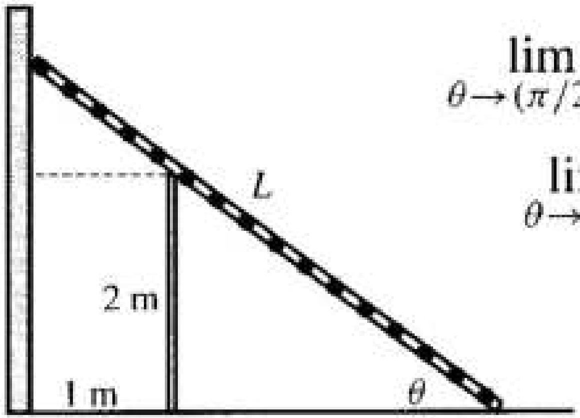

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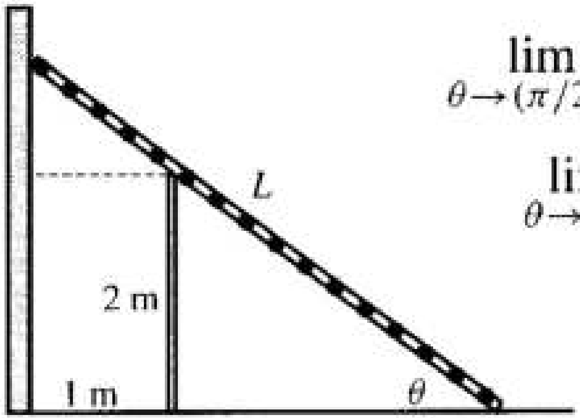

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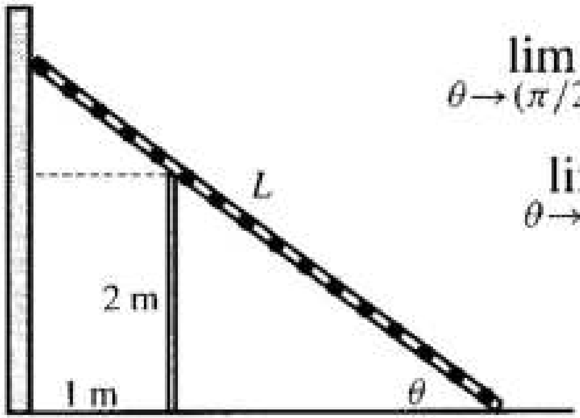

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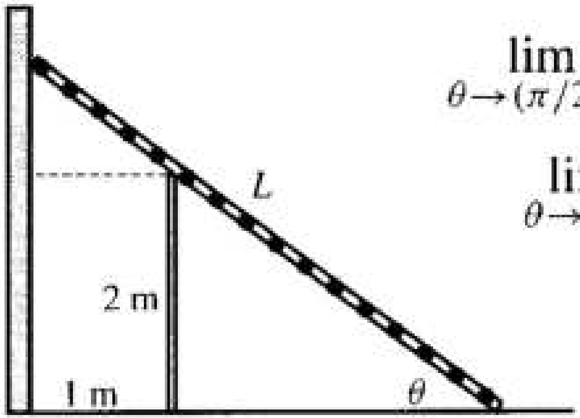

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Therefore the minimal value of $L(\theta)$ is

$$\frac{1}{\cos\theta} + \frac{2}{\sin\theta} = (1+2^{2/3})^{1/2} + 2\,\frac{(1+2^{2/3})^{1/2}}{2^{1/3}}$$

$$= \left(1+2^{2/3}\right)^{3/2} \approx 4.16.$$

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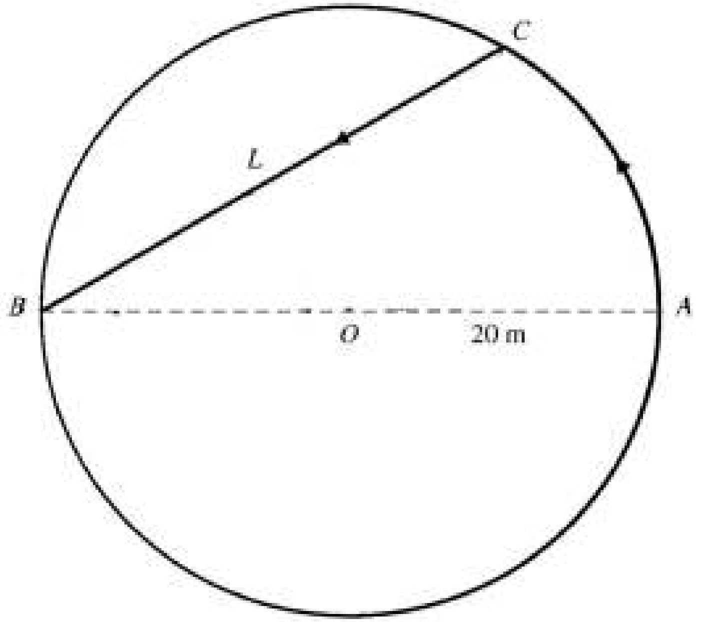

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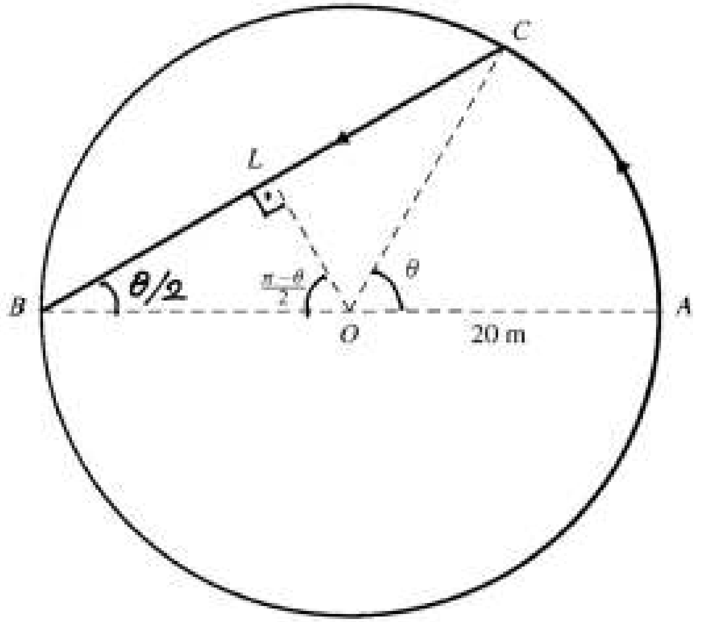

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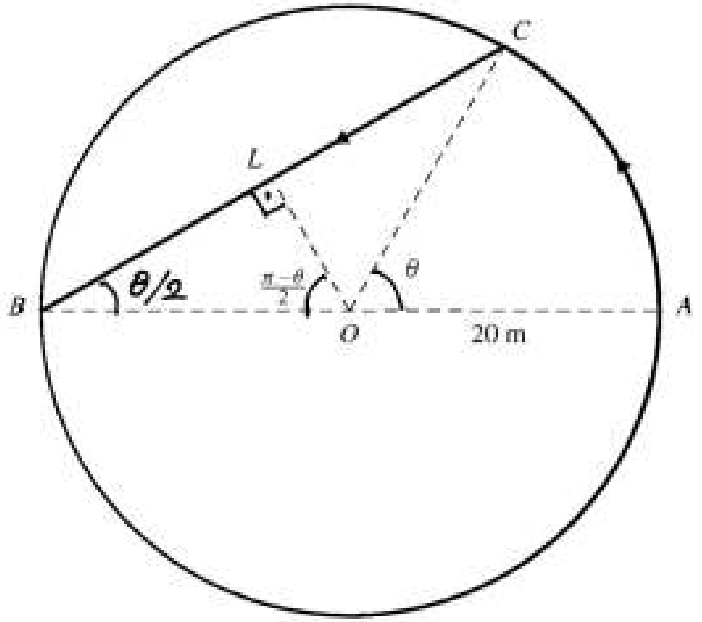

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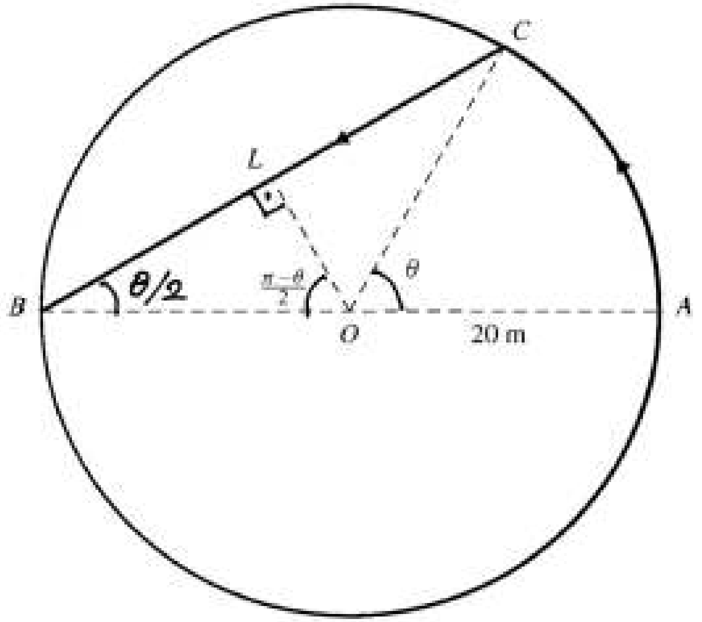

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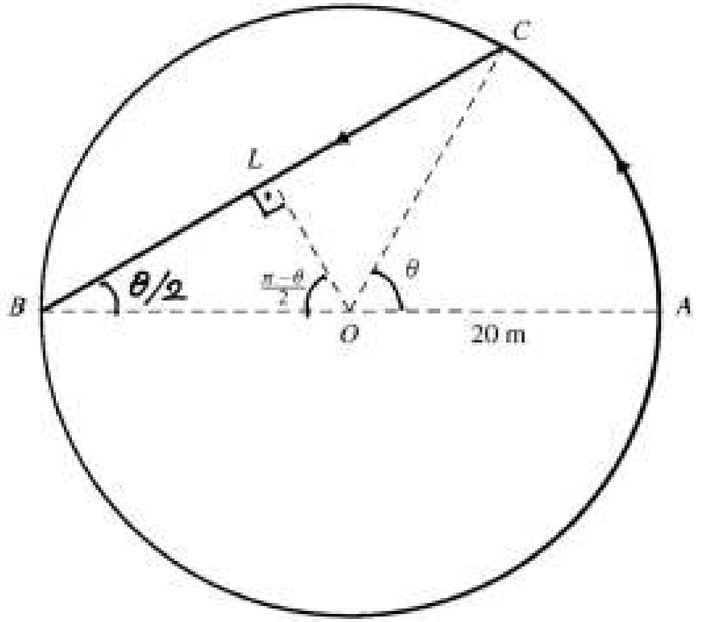

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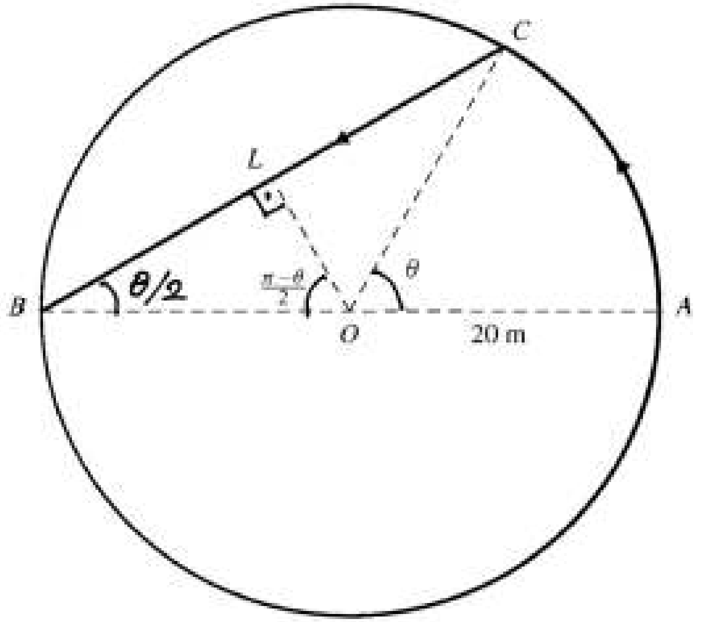

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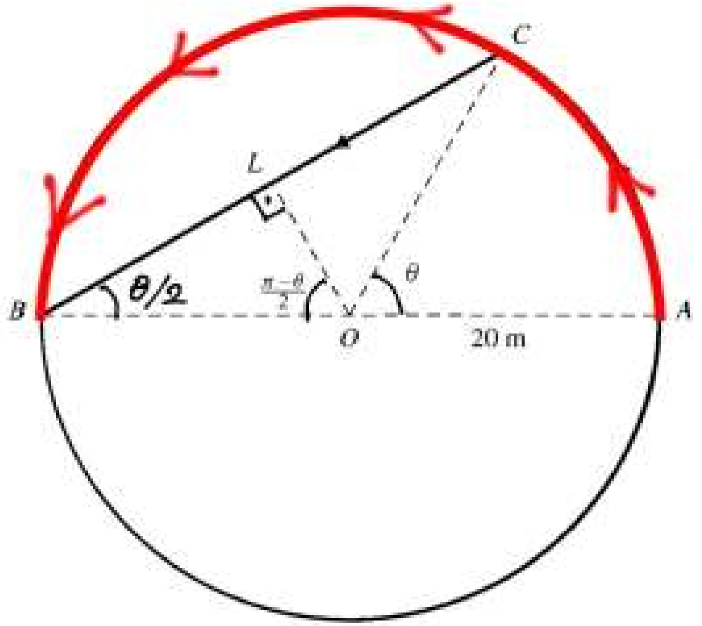

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Integration

Sums and Sigma Notation

Sigma notation

If m and n are integers with $m \le n$, and if f is a function defined at the integers $m, m+1, m+2, \ldots, n$, the symbol $\sum_{i=m}^{n} f(i)$ represents the sum of the values of f at those integers:

$$\sum_{i=m}^{n} f(i) = f(m) + f(m+1) + f(m+2) + \cdots + f(n).$$

The explicit sum appearing on the right side of this equation is the **expansion** of the sum represented in sigma notation on the left side.

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EXAMPLE $$\sum_{i=1}^{5} i^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 55.$$

Sums and Sigma Notation

Sometimes we use a subscripted variable a_i to denote the ith term of a general sum instead of using the functional notation $f(i)$:

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Sums and Sigma Notation

$$S = \sum_{i=1}^{n} i = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

$$\begin{aligned} S &= 1 + 2 + 3 + \cdots + (n-1) + n \\ S &= n + (n-1) + (n-2) + \cdots + 2 + 1 \end{aligned}$$

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S = & \boxed{n} & + & \boxed{(n-1)} & + & \boxed{(n-2)} & + \cdots + & \boxed{2} & + & \boxed{1} & \\
\hline
2S = & \boxed{(n+1)} & + & \boxed{(n+1)} & + & \boxed{(n+1)} & + \cdots + & \boxed{(n+1)} & + & \boxed{(n+1)} & = n(n+1)
\end{array}$$

Sums and Sigma Notation

THEOREM **Summation formulas**

(a) $$\sum_{i=1}^{n} 1 = \underbrace{1 + 1 + 1 + \cdots + 1}_{n \text{ terms}} = n.$$

(b) $$\sum_{i=1}^{n} i = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

(c) $$\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

(d) $$\sum_{i=1}^{n} r^{i-1} = 1 + r + r^2 + r^3 + \cdots + r^{n-1} = \frac{r^n - 1}{r - 1} \quad \text{if } r \neq 1.$$

Sums and Sigma Notation

(c) $$\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

PROOF

To prove (c) we write n copies of the identity

$$(k+1)^3 - k^3 = 3k^2 + 3k + 1,$$

one for each value of k from 1 to n, and add them up:

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3^3 & - & 2^3 & = & 3 \times 2^2 & + & 3 \times 2 & + & 1 \\
4^3 & - & 3^3 & = & 3 \times 3^2 & + & 3 \times 3 & + & 1 \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
n^3 & - & (n-1)^3 & = & 3(n-1)^2 & + & 3(n-1) & + & 1 \\
(n+1)^3 & - & n^3 & = & 3n^2 & + & 3n & + & 1 \\
\hline
(n+1)^3 & - & 1^3 & = & 3\left(\sum_{i=1}^{n} i^2\right) & + & 3\left(\sum_{i=1}^{n} i\right) & + & n \\
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Sums and Sigma Notation

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Evaluate $\displaystyle\sum_{k=m+1}^{n} (6k^2 - 4k + 3)$, where $1 \leq m < n$.

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Solution

$$\begin{aligned}\sum_{k=1}^{n}(6k^2 - 4k + 3) &= 6\sum_{k=1}^{n}k^2 - 4\sum_{k=1}^{n}k + 3\sum_{k=1}^{n}1 \\ &= 6\,\frac{n(n+1)(2n+1)}{6} - 4\,\frac{n(n+1)}{2} + 3n \\ &= 2n^3 + n^2 + 2n\end{aligned}$$

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&= 2n^3 + n^2 + 2n
\end{aligned}$$

Thus,

$$\begin{aligned}
\sum_{k=m+1}^{n} (6k^2 - 4k + 3) &= \sum_{k=1}^{n}(6k^2 - 4k + 3) - \sum_{k=1}^{m}(6k^2 - 4k + 3) \\
&= 2n^3 + n^2 + 2n - 2m^3 - m^2 - 2m.
\end{aligned}$$

Areas as Limits of Sums

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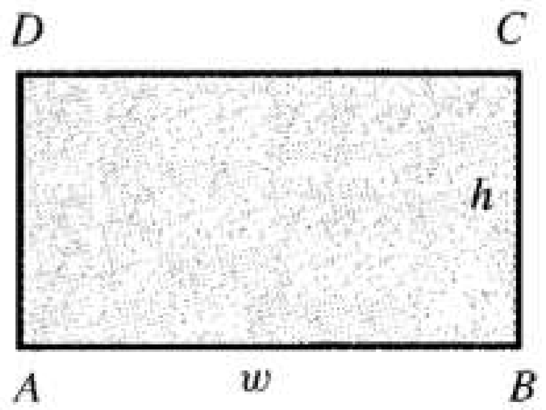

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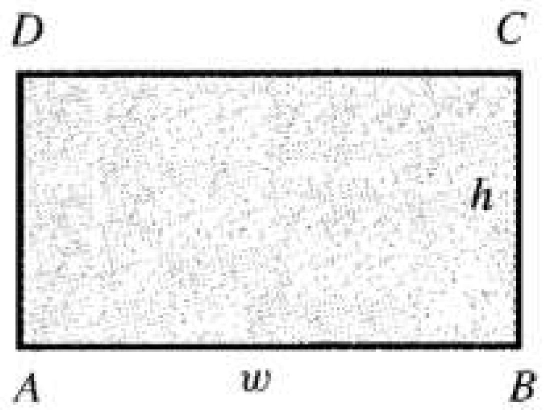

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(iii) The areas of congruent plane regions are equal.

Areas as Limits of Sums

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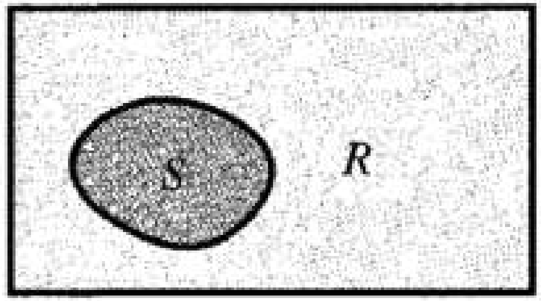

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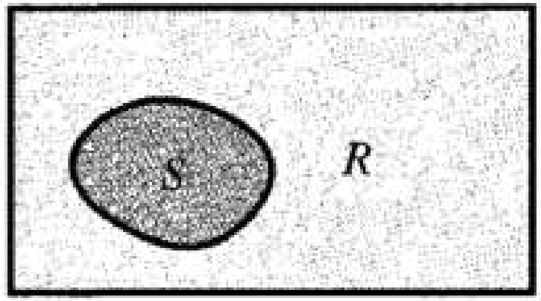

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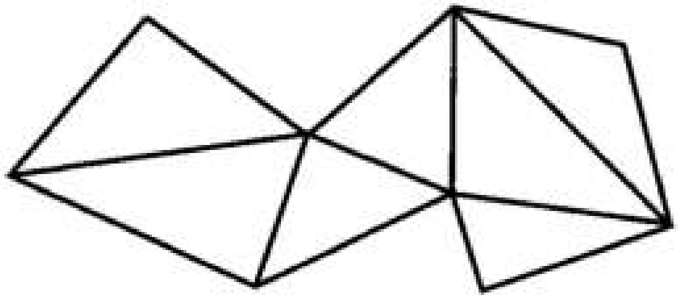
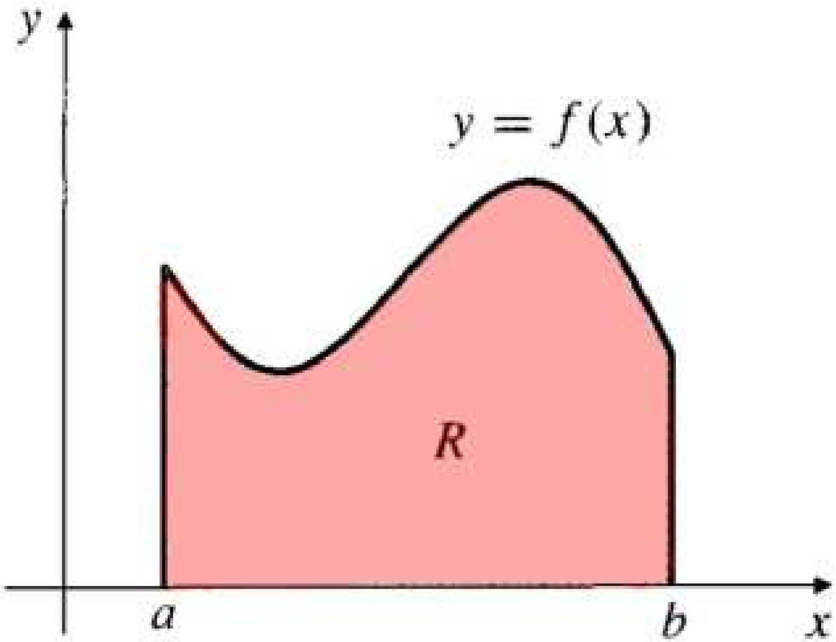


area of polygon =

sum of areas of triangles

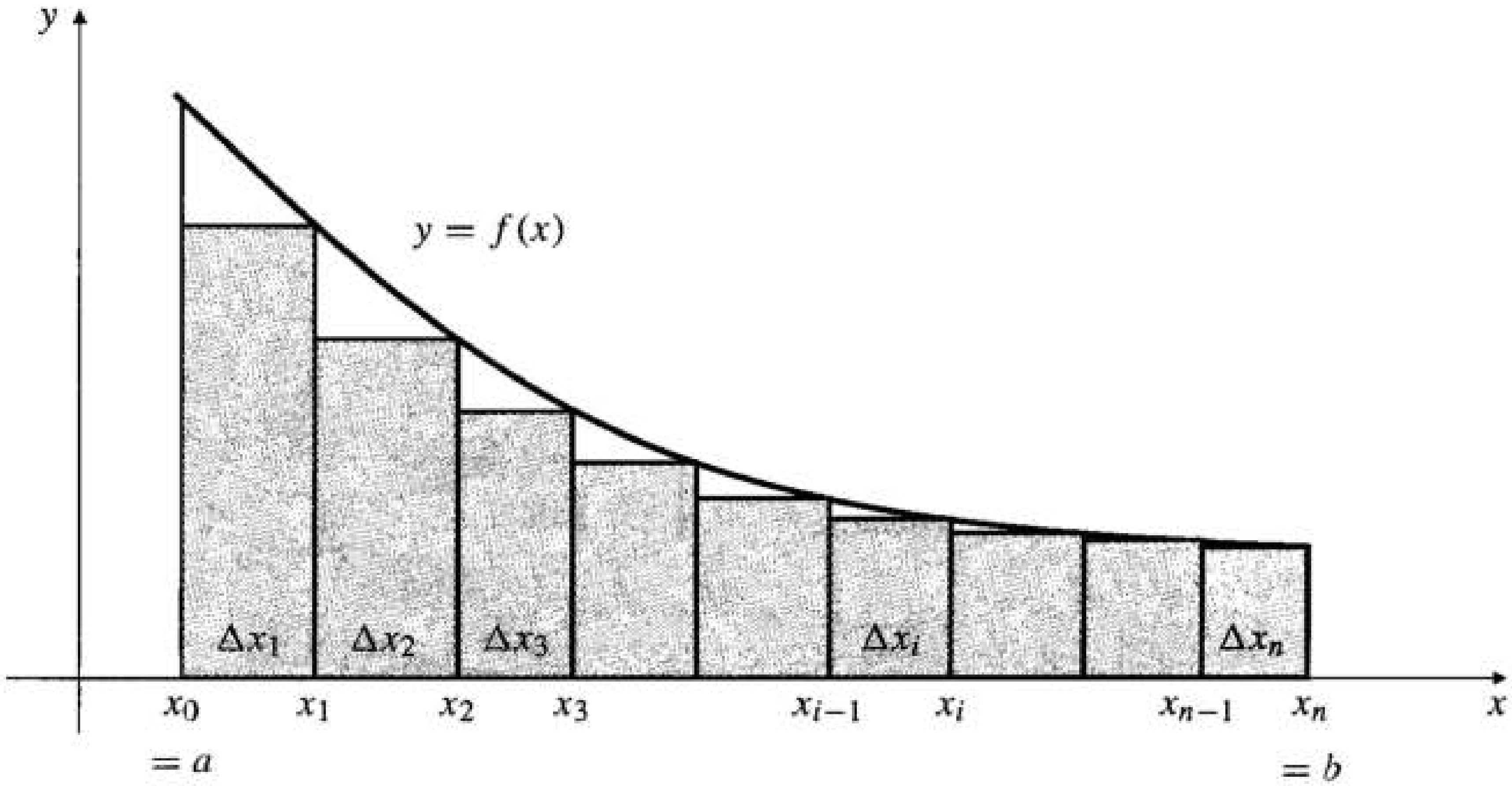
(v) If region R is a union of (finitely many) nonoverlapping regions, then the area of R is the sum of the areas of those regions.

Areas as Limits of Sums



The basic area problem: find the area of region R

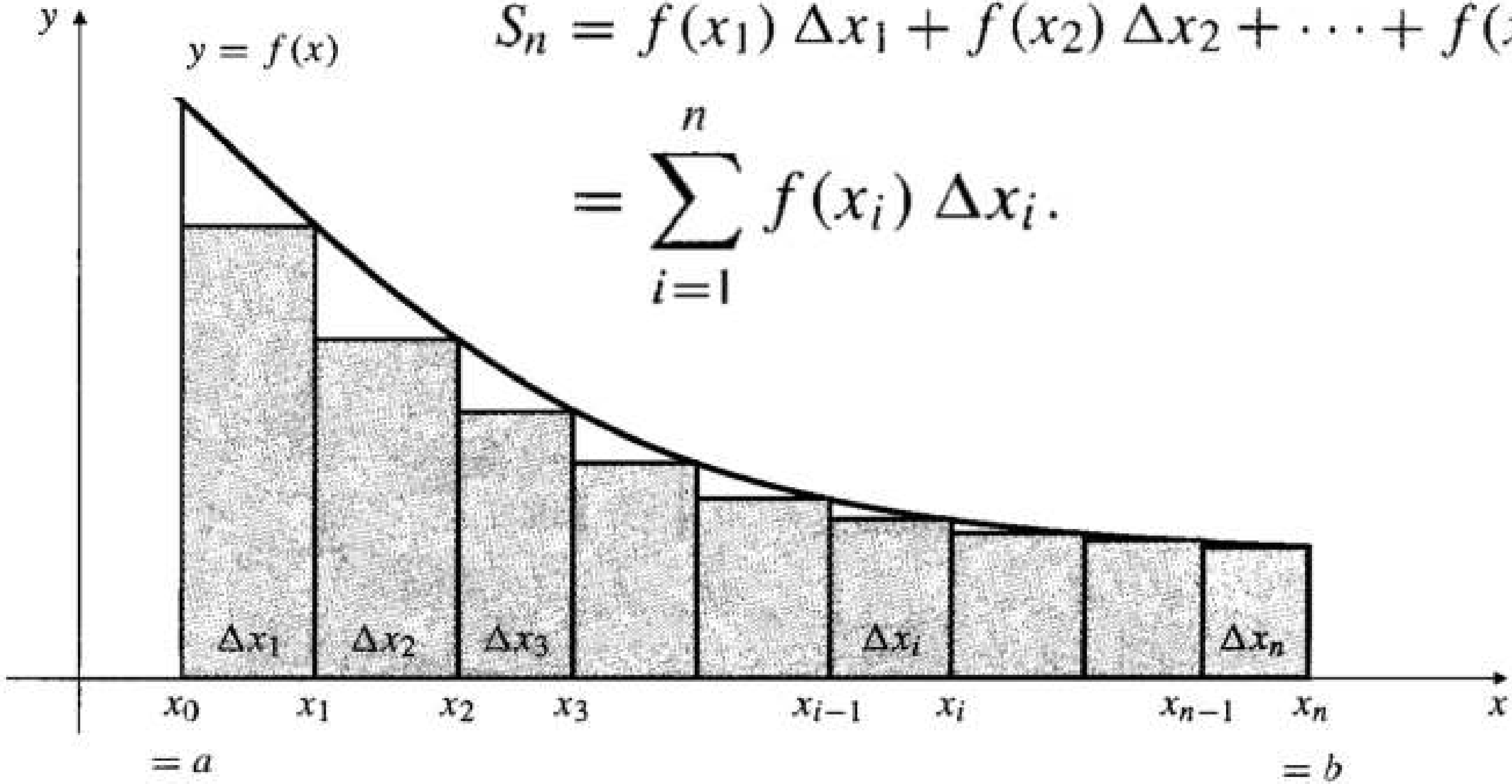
Areas as Limits of Sums



$$a = x_0 < x_1 < x_2 < x_3 < \cdots < x_{n-1} < x_n = b.$$

$$\Delta x_i = x_i - x_{i-1}, \qquad (i = 1, 2, 3, \ldots, n).$$

Areas as Limits of Sums



$$S_n = f(x_1)\,\Delta x_1 + f(x_2)\,\Delta x_2 + \cdots + f(x_n)\,\Delta x_n = \sum_{i=1}^{n} f(x_i)\,\Delta x_i.$$

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Areas as Limits of Sums

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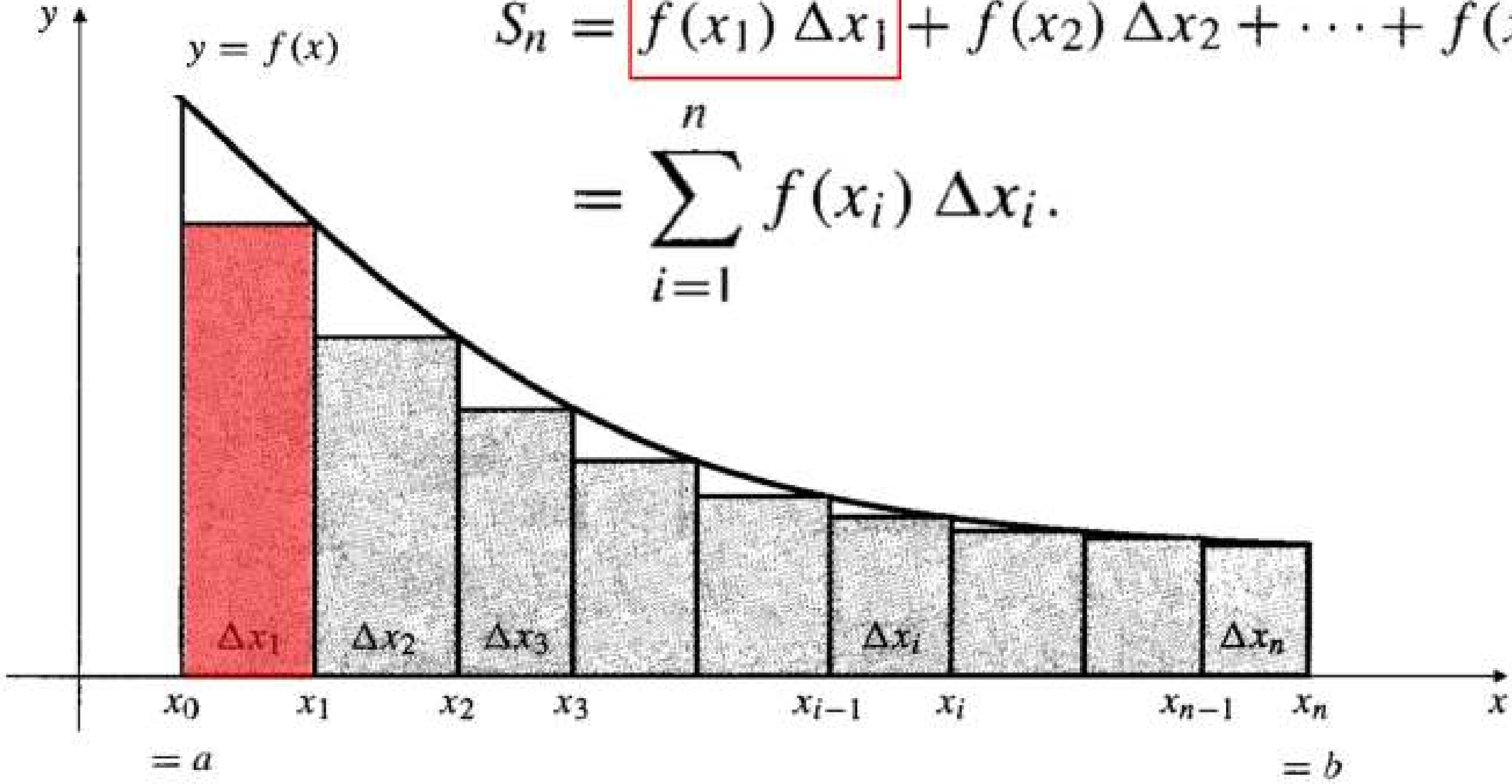

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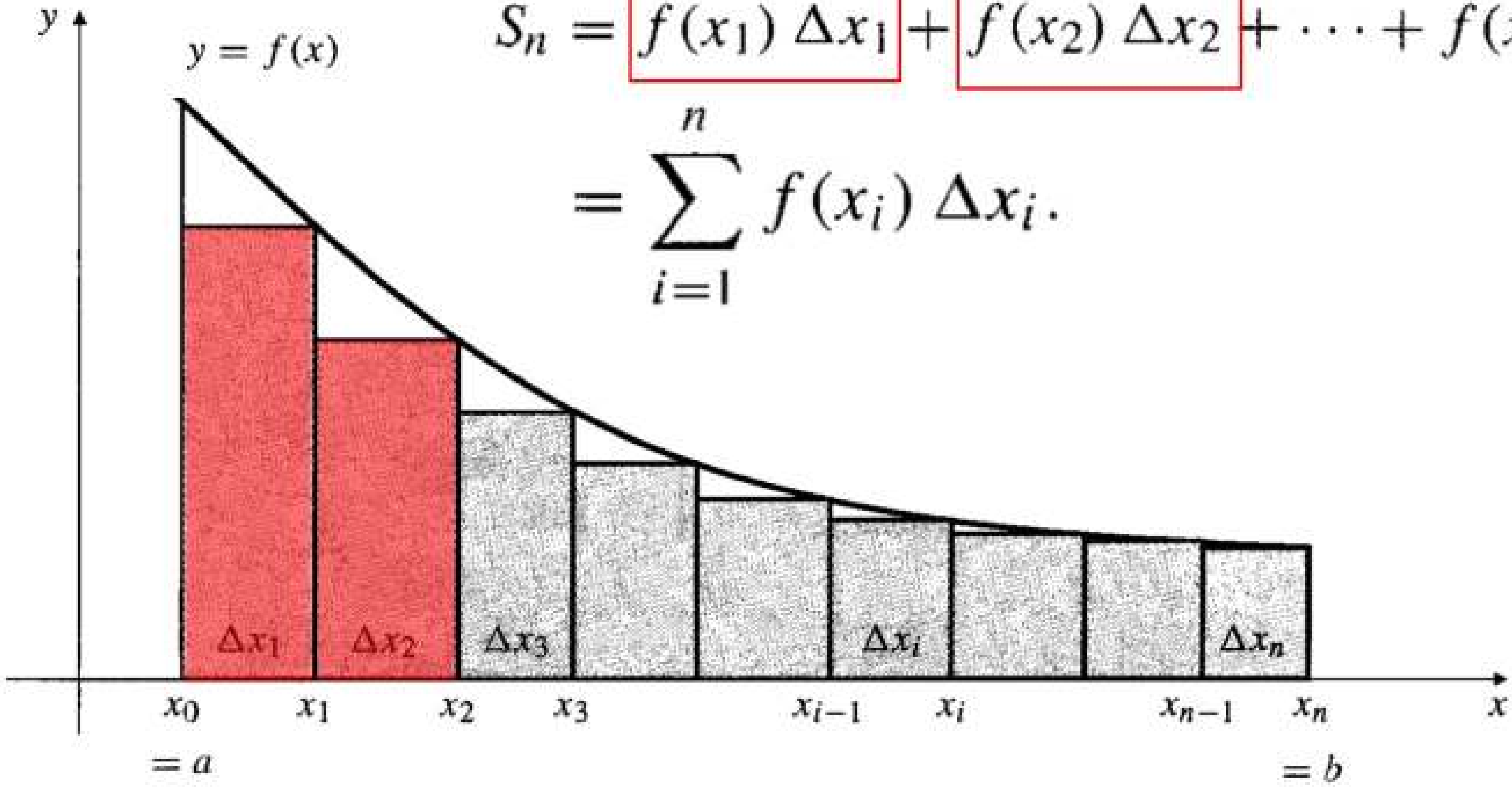

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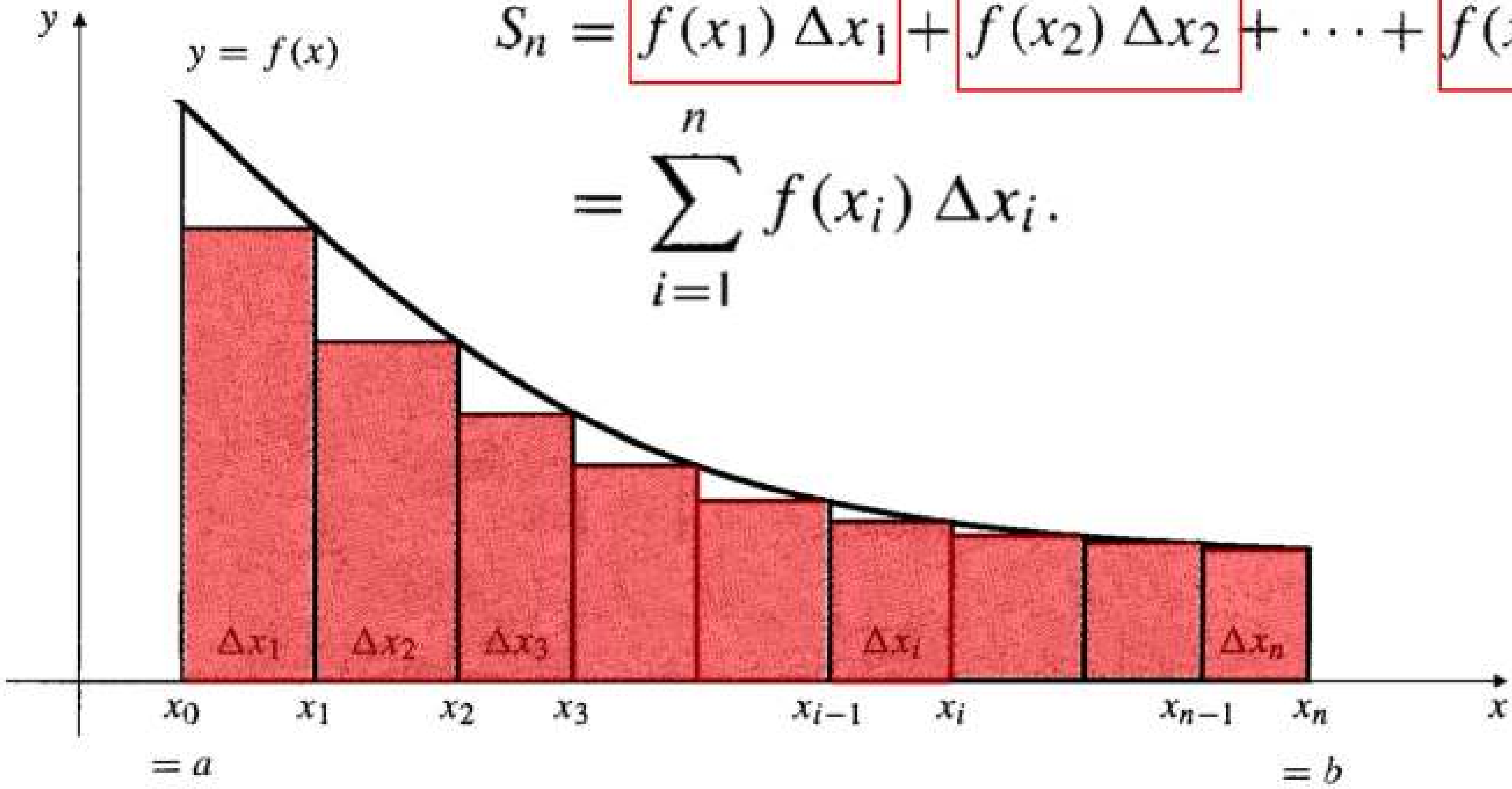

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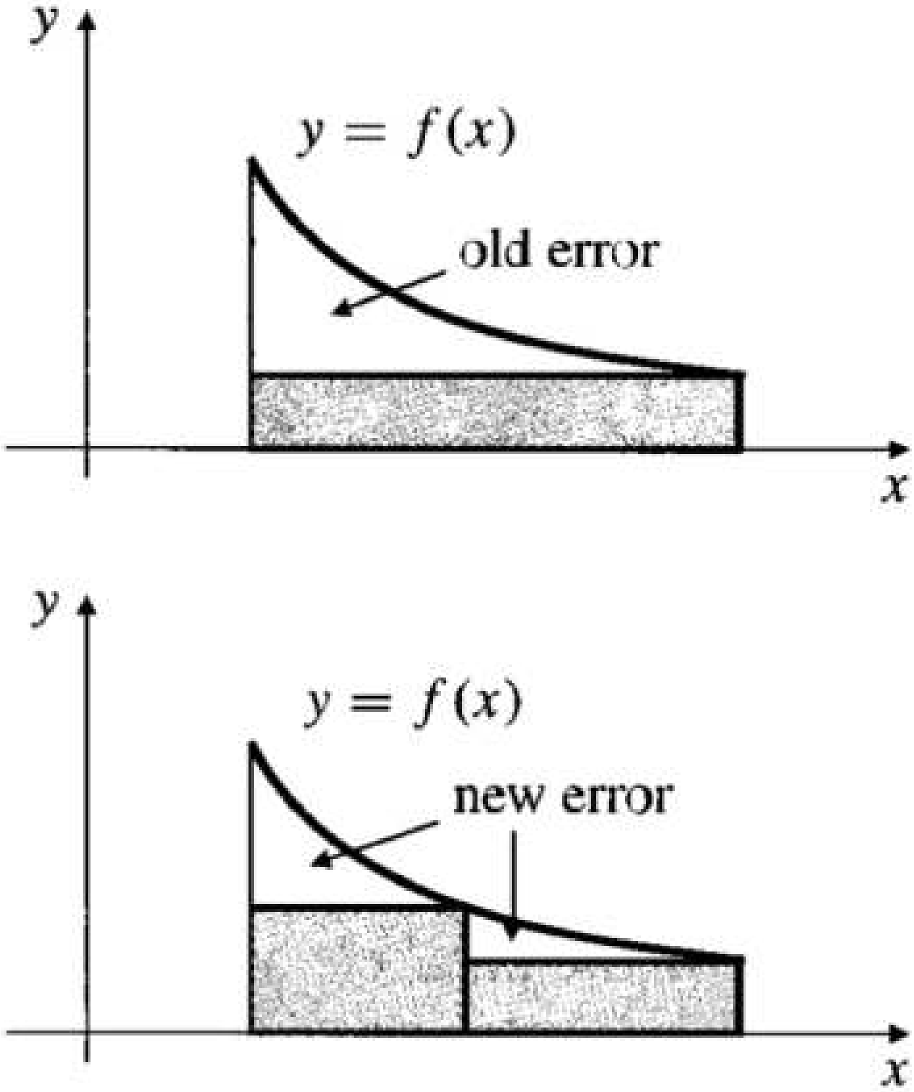
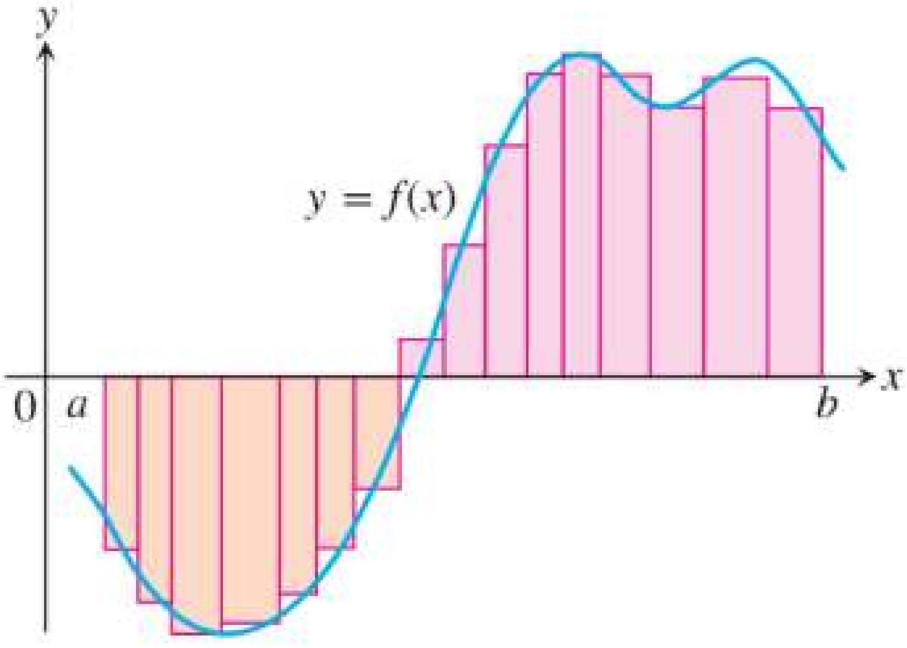
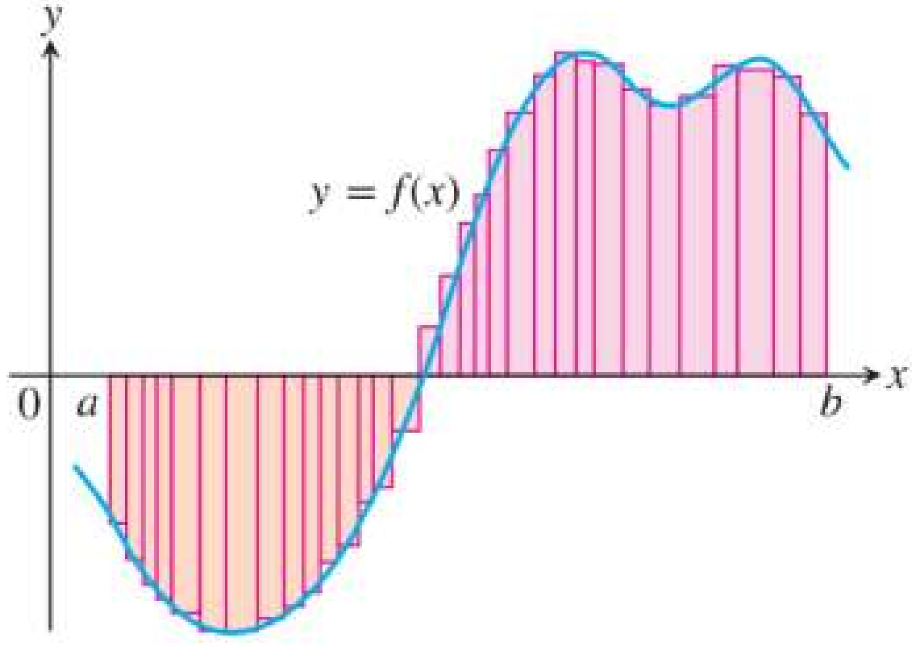


Using more rectangles makes the error smaller

Areas as Limits of Sums

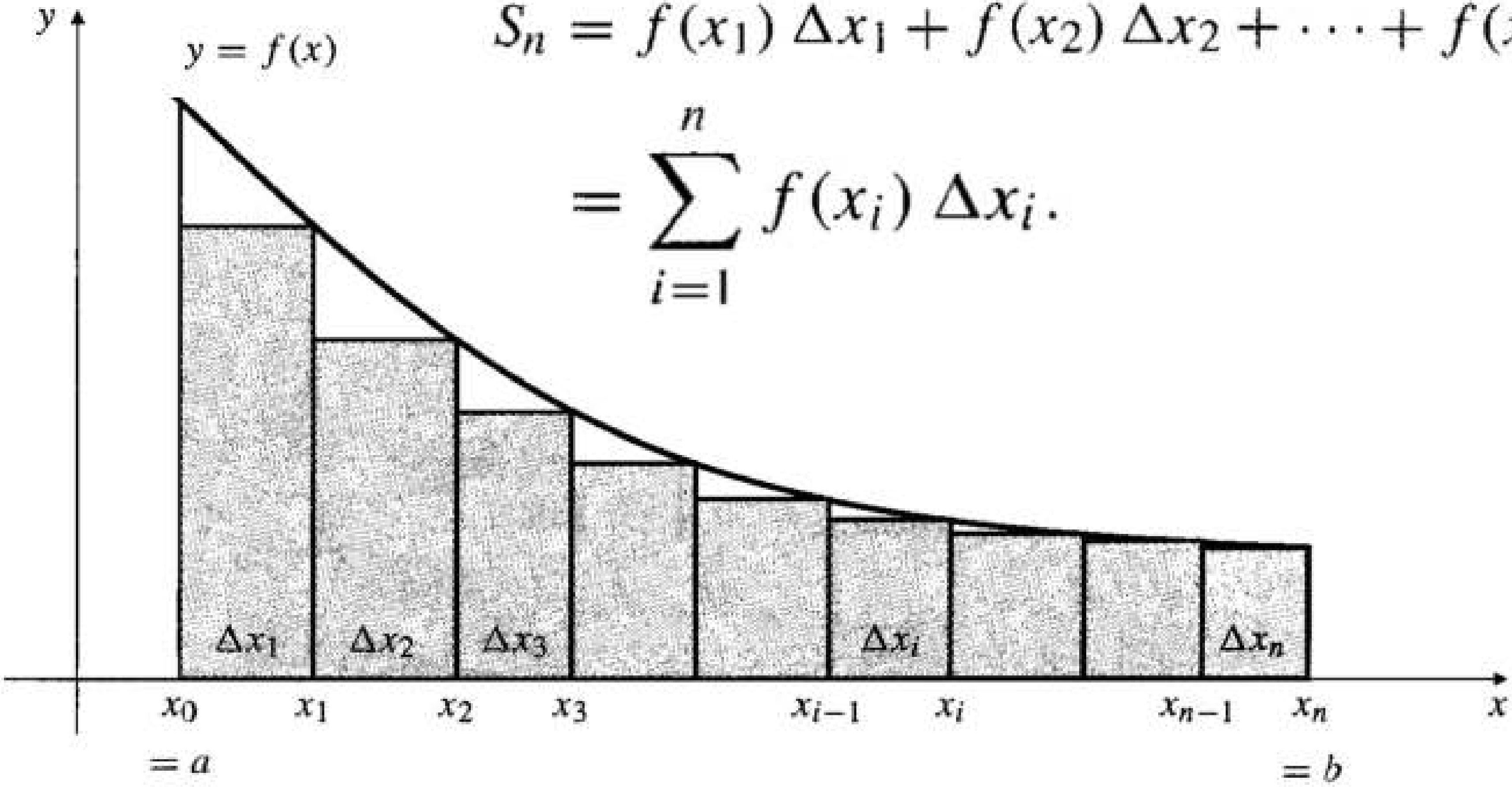


(a)



(b)

Areas as Limits of Sums



$$S_n = f(x_1)\,\Delta x_1 + f(x_2)\,\Delta x_2 + \cdots + f(x_n)\,\Delta x_n$$
$$= \sum_{i=1}^{n} f(x_i)\,\Delta x_i .$$

$$\text{Area of } R = \lim_{\substack{n \to \infty \\ \max \Delta x_i \to 0}} S_n .$$

Areas as Limits of Sums

EXAMPLE

Find the area A of the region lying under the straight line $y = x+1$, above the x-axis and between the lines $x = 0$ and $x = 2$.

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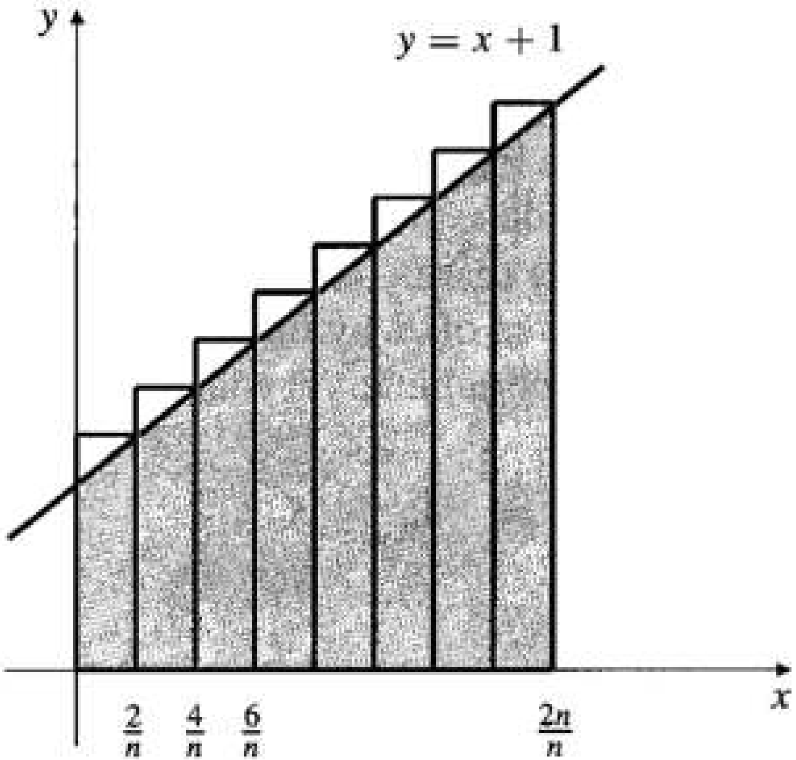

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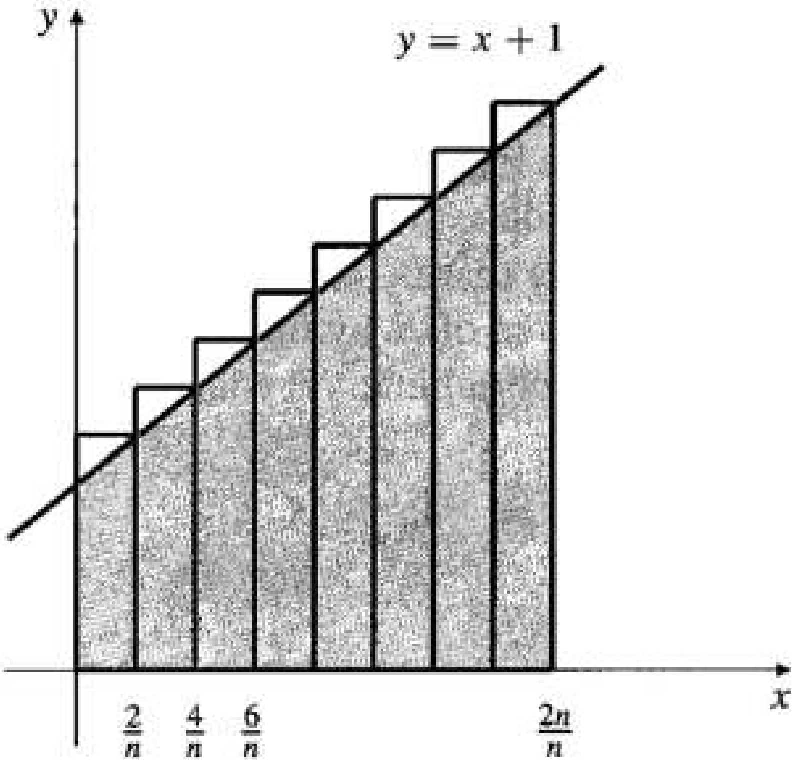

$$
\begin{aligned}
S_n &= \sum_{i=1}^{n} \left(\frac{2i}{n} + 1 \right) \frac{2}{n} \\
&= \left(\frac{2}{n} \right) \left[\frac{2}{n} \sum_{i=1}^{n} i + \sum_{i=1}^{n} 1 \right] \\
&= \left(\frac{2}{n} \right) \left[\frac{2}{n} \frac{n(n+1)}{2} + n \right] \\
&= 2 \frac{n+1}{n} + 2.
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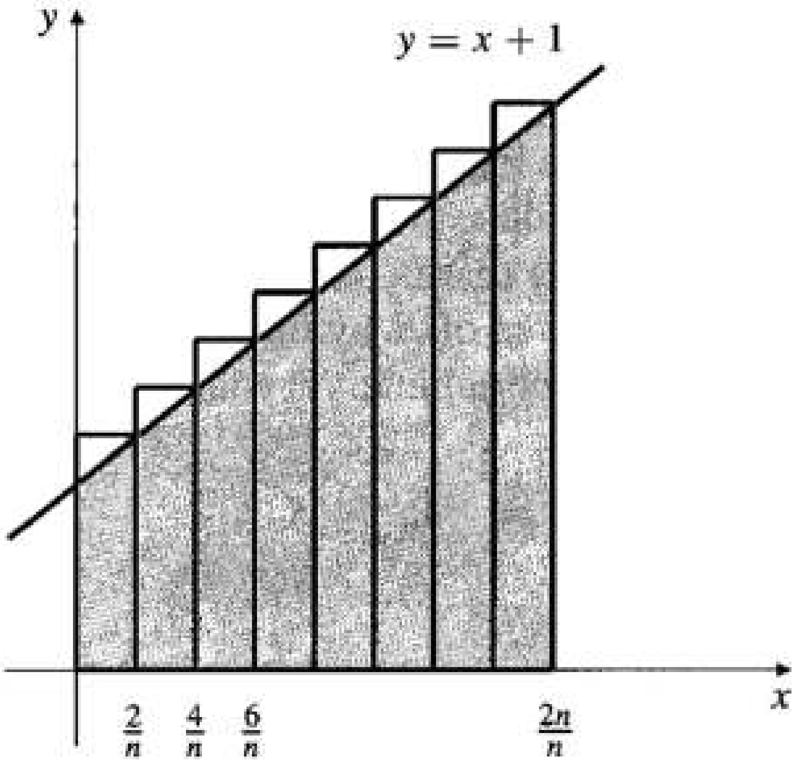

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\end{aligned}
$$

Therefore, the required area A is given by

$$A = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(2\,\frac{n+1}{n} + 2\right) = 2 + 2 = 4 \text{ square units.}$$

Areas as Limits of Sums

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Find the area of the region bounded by the parabola $y = x^2$ and the straight lines $y = 0$, $x = 0$, and $x = b$, where $b > 0$.

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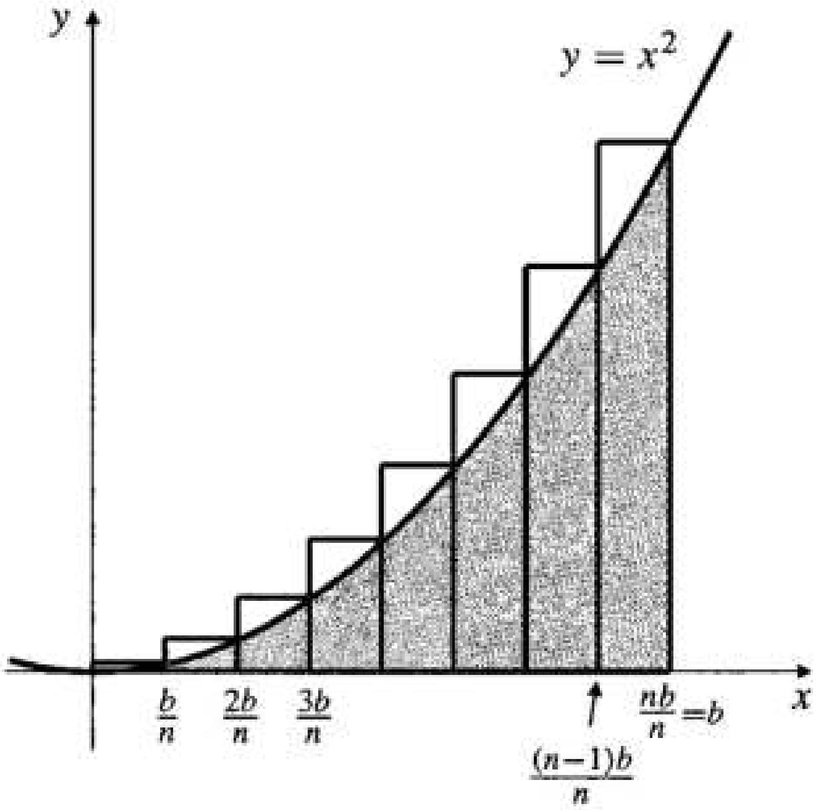

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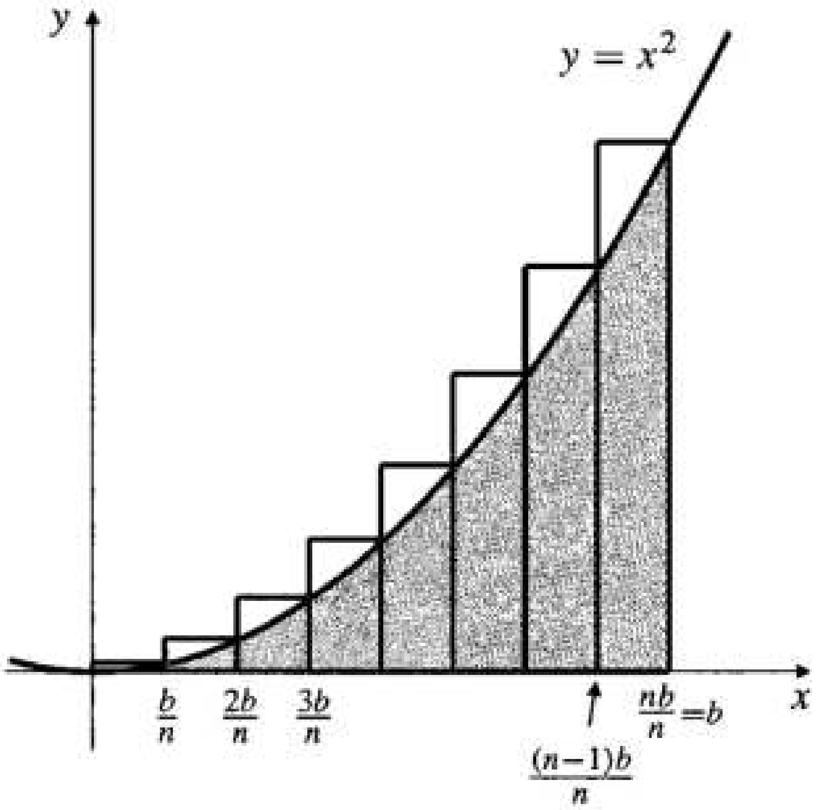

$$S_n = \sum_{i=1}^{n} \left(\frac{ib}{n}\right)^2 \frac{b}{n}$$

$$= \frac{b^3}{n^3} \sum_{i=1}^{n} i^2$$

$$= \frac{b^3}{n^3} \frac{n(n+1)(2n+1)}{6}$$

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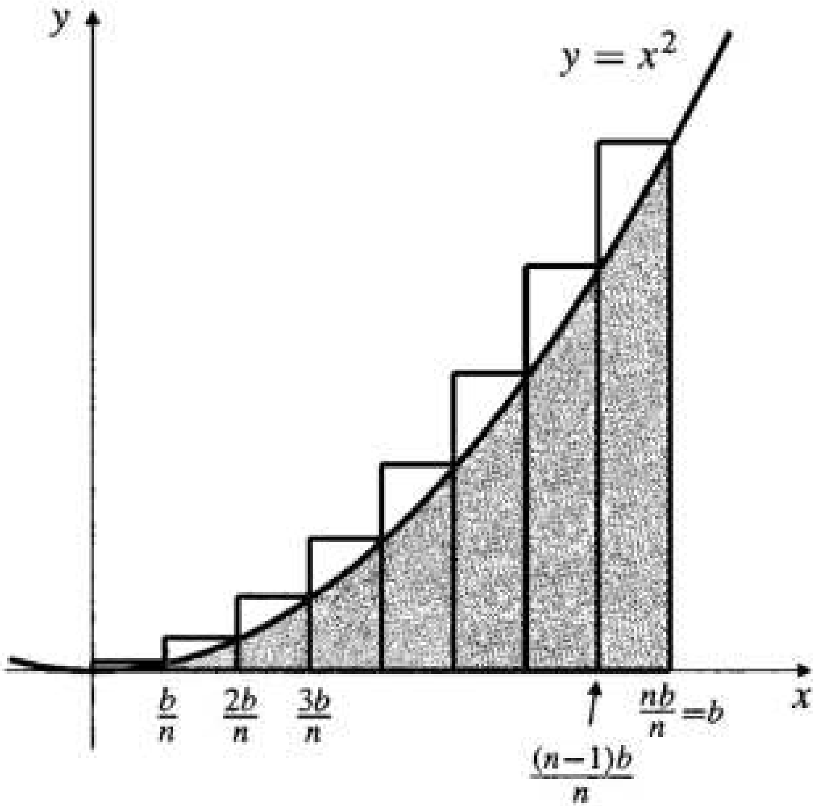

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Hence, the required area is

$$A = \lim_{n\to\infty} S_n = \lim_{n\to\infty} b^3 \frac{(n+1)(2n+1)}{6n^2} = \frac{b^3}{3} \text{ square units.}$$

The Definite Integral

Partitions

Let P be a finite set of points arranged in order between a and b on the real line, say

$$P = \{x_0, x_1, x_2, x_3, \ldots, x_{n-1}, x_n\},$$

where $a = x_0 < x_1 < x_2 < x_3 < \cdots < x_{n-1} < x_n = b$. Such a set P is called a **partition** of $[a, b]$.

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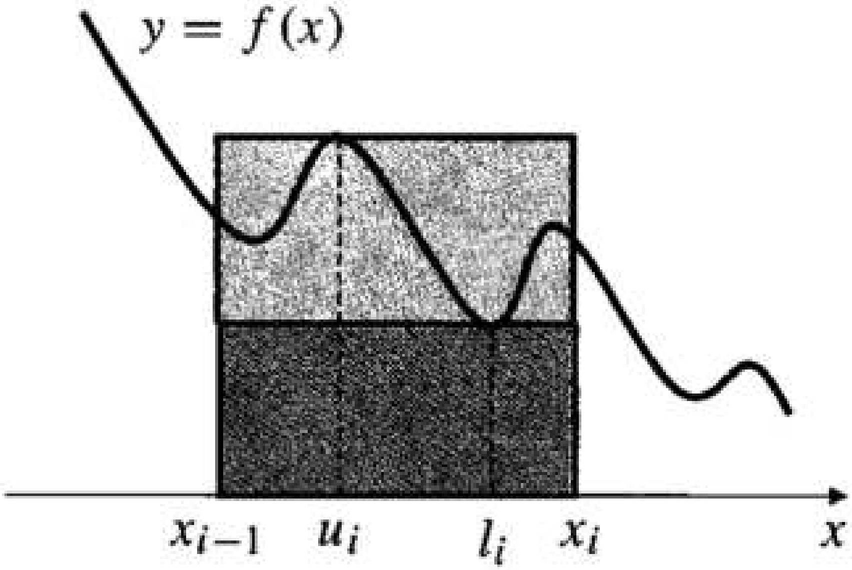
$$\Delta x_i = x_i - x_{i-1}, \qquad \text{(for } 1 \le i \le n)$$

and we call the greatest of these numbers Δx_i the **norm** of the partition P and denote it $\|P\|$:

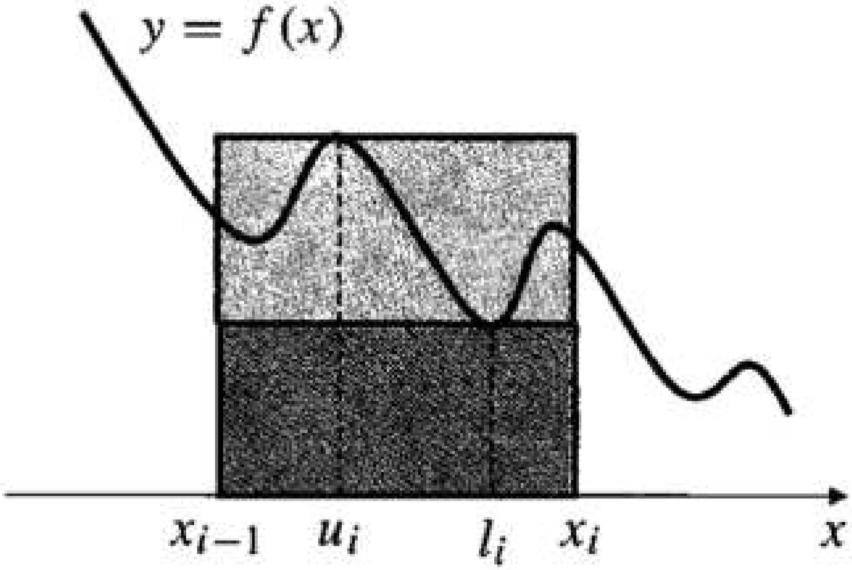
$$\|P\| = \max_{1 \le i \le n} \Delta x_i.$$

The Definite Integral

Riemann Sums



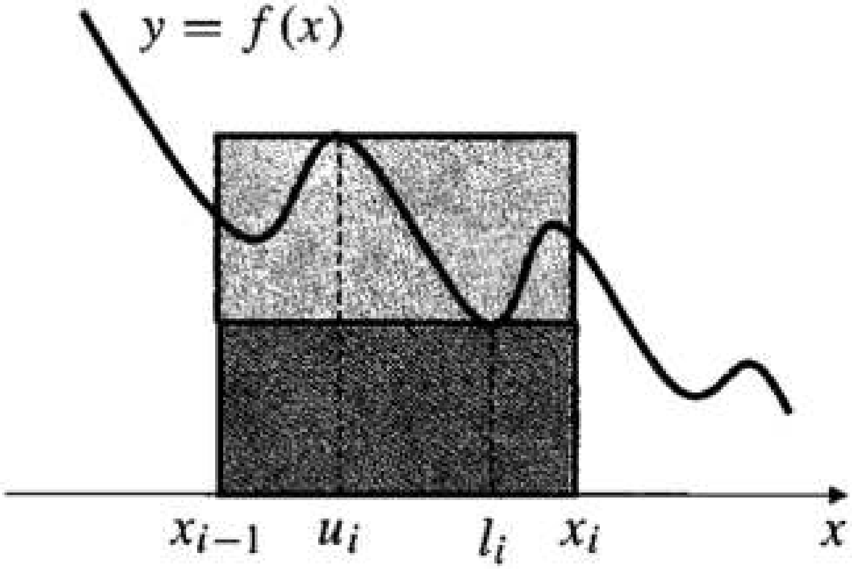
Riemann Sums



If A_i is that part of the area under $y = f(x)$ and above the x-axis that lies in the vertical strip between $x = x_{i-1}$ and $x = x_i$, then

$$f(l_i)\,\Delta x_i \le A_i \le f(u_i)\,\Delta x_i.$$

Riemann Sums



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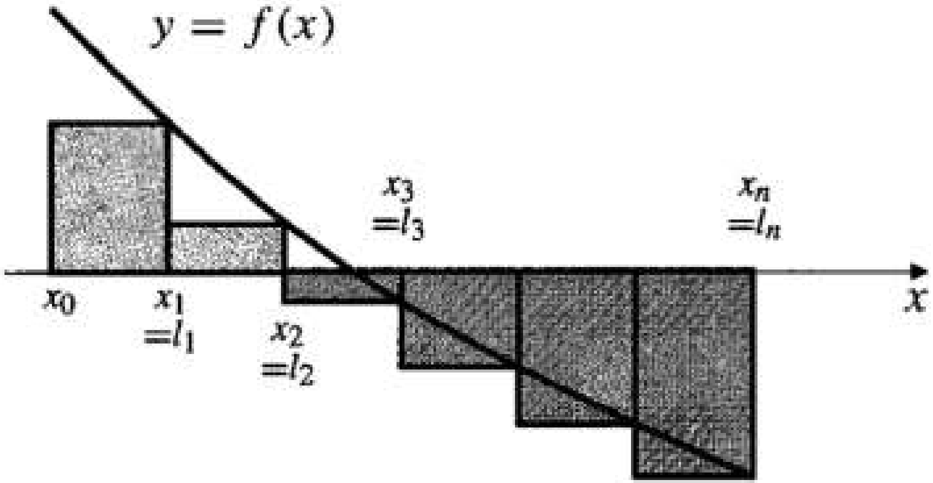
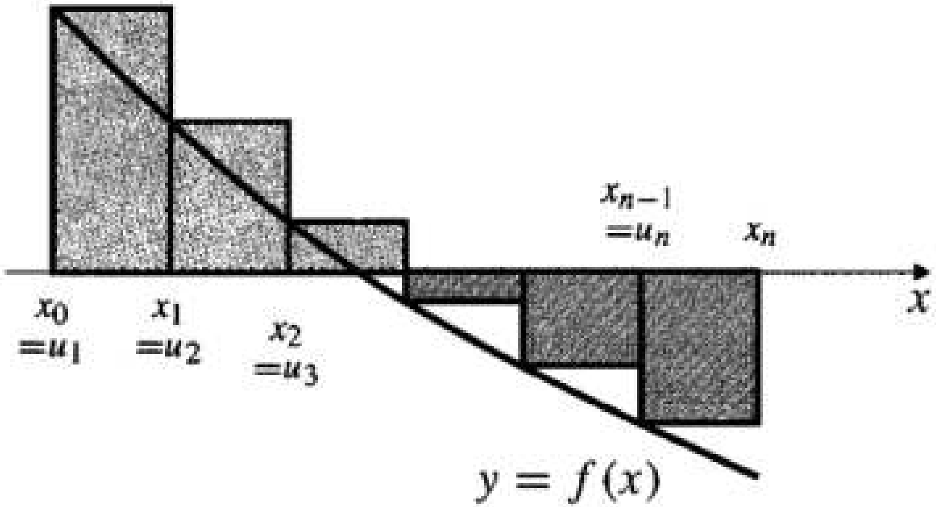
The **lower (Riemann) sum, $L(f, P)$**, and the **upper (Riemann) sum, $U(f, P)$**, for the function f and the partition P are defined by:

$$L(f, P) = f(l_1)\,\Delta x_1 + f(l_2)\,\Delta x_2 + \cdots + f(l_n)\,\Delta x_n = \sum_{i=1}^{n} f(l_i)\,\Delta x_i,$$

$$U(f, P) = f(u_1)\,\Delta x_1 + f(u_2)\,\Delta x_2 + \cdots + f(u_n)\,\Delta x_n = \sum_{i=1}^{n} f(u_i)\Delta x_i.$$

Riemann Sums





EXAMPLE A lower Riemann sum and an upper Riemann sum for a decreasing function f.

The Definite Integral

Riemann Sums

EXAMPLE Calculate the lower and upper Riemann sums for the function $f(x) = x^2$ on the interval $[0, a]$ (where $a > 0$), corresponding to the partition P_n of $[0, a]$ into n subintervals of equal length.

The Definite Integral

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Solution Each subinterval of P_n has length $\Delta x = a/n$, and the division points are given by $x_i = ia/n$ for $i = 0, 1, 2, \ldots, n$. Since x^2 is increasing on $[0, a]$, its minimum and maximum values over the ith subinterval $[x_{i-1}, x_i]$ occur at $l_i = x_{i-1}$ and $u_i = x_i$, respectively.

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$$
\begin{aligned}
L(f, P_n) &= \sum_{i=1}^{n} (x_{i-1})^2 \Delta x = \frac{a^3}{n^3} \sum_{i=1}^{n} (i-1)^2 \\
&= \frac{a^3}{n^3} \sum_{j=0}^{n-1} j^2 = \frac{a^3}{n^3} \frac{(n-1)n(2(n-1)+1)}{6} = \frac{(n-1)(2n-1)a^3}{6n^2}, \\
U(f, P_n) &= \sum_{i=1}^{n} (x_i)^2 \Delta x \\
&= \frac{a^3}{n^3} \sum_{i=1}^{n} i^2 = \frac{a^3}{n^3} \frac{n(n+1)(2n+1)}{6} = \frac{(n+1)(2n+1)a^3}{6n^2}.
\end{aligned}
$$

The Definite Integral