The Method of Substitution

Trigonometric Integrals

If the powers of $\sin x$ and $\cos x$ are both even, then we can make use of the *double-angle formulas*

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x) \qquad \text{and} \qquad \sin^2 x = \frac{1}{2}(1 - \cos 2x).$$

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EXAMPLE Evaluate $\int \sin^4 x\, dx$.

$$\begin{aligned}
\int \sin^4 x\, dx &= \frac{1}{4}\int (1 - \cos 2x)^2\, dx \\
&= \frac{1}{4}\int (1 - 2\cos 2x + \cos^2 2x)\, dx \\
&= \frac{x}{4} - \frac{1}{4}\sin 2x + \frac{1}{8}\int (1 + \cos 4x)\, dx \\
&= \frac{x}{4} - \frac{1}{4}\sin 2x + \frac{x}{8} + \frac{1}{32}\sin 4x + C \\
&= \frac{3}{8}x - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C
\end{aligned}$$

The Method of Substitution

Trigonometric Integrals

$$\int \sec^m x \tan^n x \, dx \qquad \text{or} \qquad \int \csc^m x \cot^n x \, dx, \quad \text{unless } m \text{ is odd and } n \text{ is even.}$$

The Method of Substitution

Trigonometric Integrals

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EXAMPLE **(Integrals involving secants and tangents)** Evaluate the following integrals:

(a) $\displaystyle\int \tan^2 x\, dx$, (b) $\displaystyle\int \sec^4 t\, dt$, and (c) $\displaystyle\int \sec^3 x \tan^3 x\, dx$.

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Solution

(a) $\int \tan^2 x \, dx = \int (\sec^2 x - 1) \, dx = \tan x - x + C.$

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Solution

(a) $$\int \tan^2 x \, dx = \int (\sec^2 x - 1) \, dx = \tan x - x + C.$$

(b) $$\int \sec^4 t \, dt = \int (1 + \tan^2 t) \sec^2 t \, dt \qquad \begin{array}{l} \text{Let } u = \tan t, \\ du = \sec^2 t \, dt. \end{array}$$

$$= \int (1 + u^2) \, du = u + \frac{1}{3}u^3 + C = \tan t + \frac{1}{3}\tan^3 t + C.$$

The Method of Substitution

Trigonometric Integrals

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(a) $\int \tan^2 x \, dx$, (b) $\int \sec^4 t \, dt$, and (c) $\int \sec^3 x \tan^3 x \, dx$.

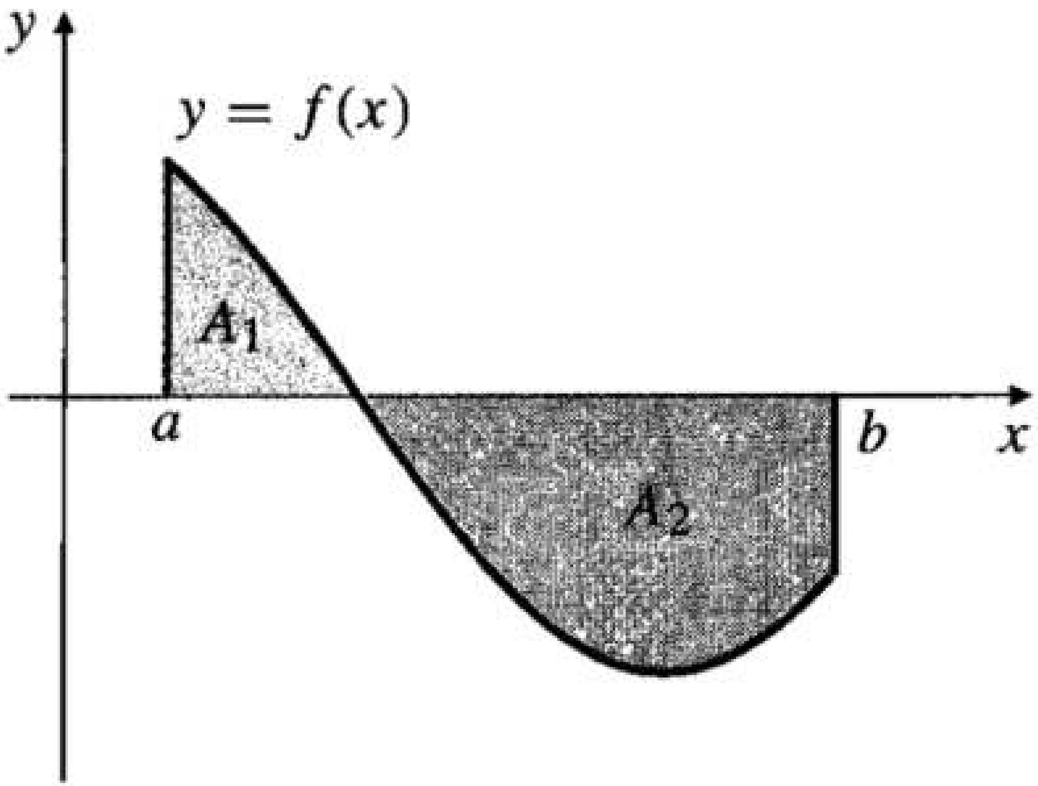
Solution

(c) $\int \sec^3 x \tan^3 x \, dx$

$$= \int \sec^2 x (\sec^2 x - 1) \sec x \tan x \, dx \qquad \text{Let } u = \sec x, \; du = \sec x \tan x \, dx.$$

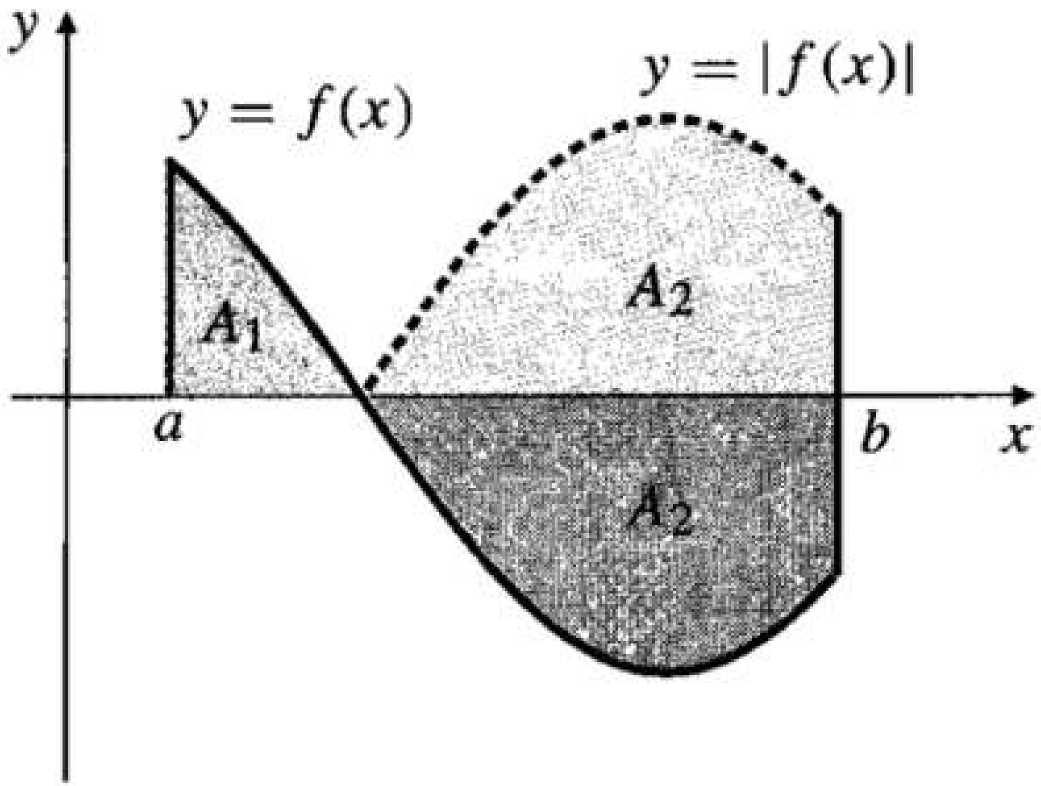
$$= \int (u^4 - u^2) \, du = \frac{u^5}{5} - \frac{u^3}{3} + C = \frac{1}{5}\sec^5 x - \frac{1}{3}\sec^3 x + C.$$

Areas of Plane Regions



$$\int_a^b f(x)\,dx = A_1 - A_2$$

Areas of Plane Regions



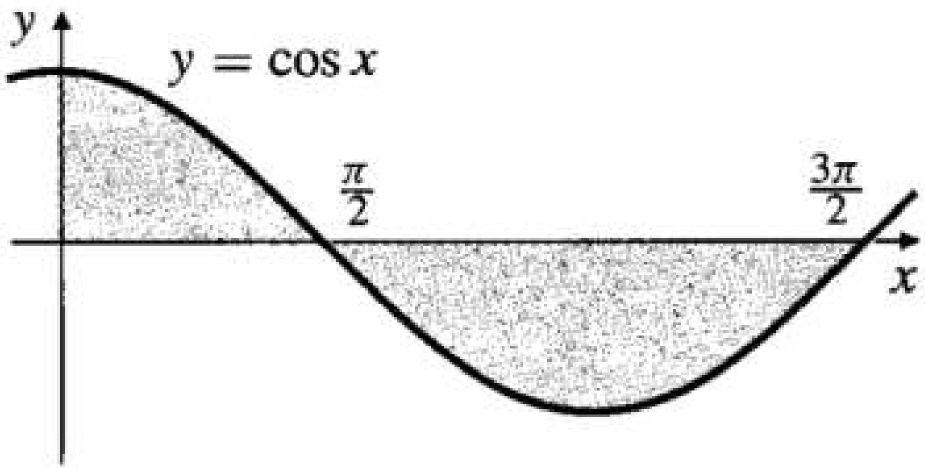
$$\int_a^b f(x)\,dx = A_1 - A_2$$

$$\int_a^b |f(x)|\,dx = A_1 + A_2$$

Areas of Plane Regions

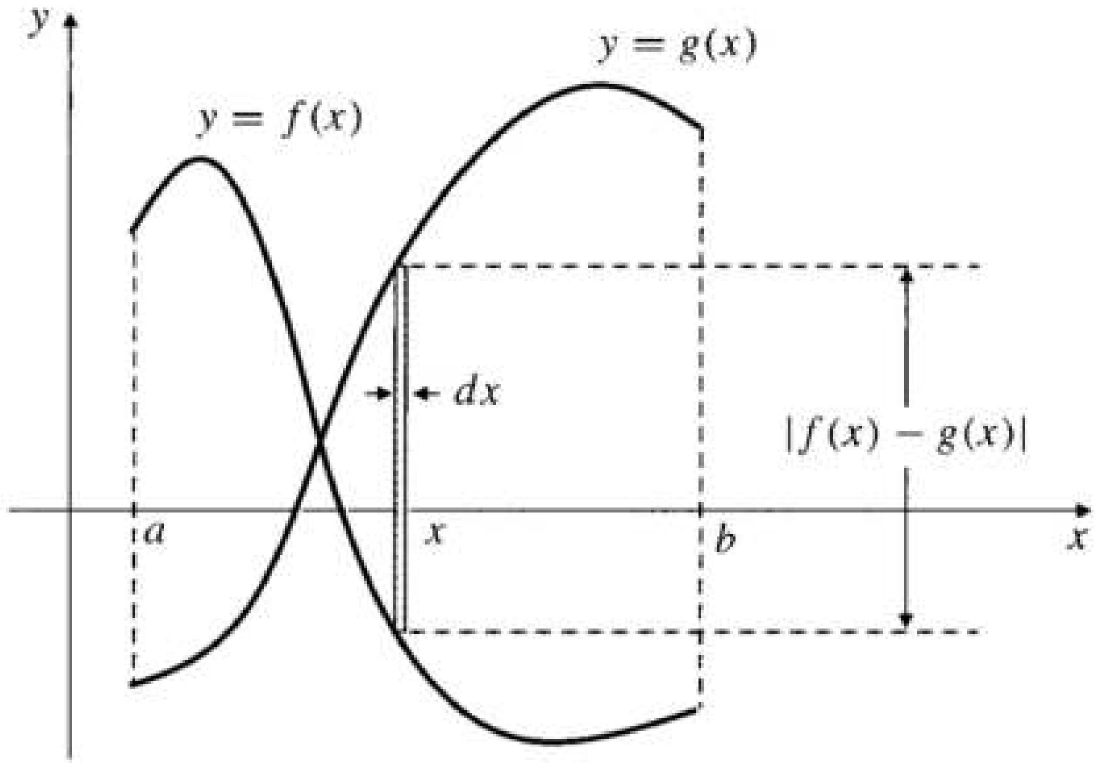
EXAMPLE The area bounded by $y = \cos x$, $y = 0$, $x = 0$, and $x = 3\pi/2$ is

$$\begin{aligned} A &= \int_0^{3\pi/2} |\cos x|\, dx \\ &= \int_0^{\pi/2} \cos x\, dx + \int_{\pi/2}^{3\pi/2} (-\cos x)\, dx \\ &= \sin x \Big|_0^{\pi/2} - \sin x \Big|_{\pi/2}^{3\pi/2} \\ &= (1-0) - (-1-1) = 3 \text{ square units.} \end{aligned}$$



Areas of Plane Regions

Areas Between Two Curves



$$A = \int_a^b |f(x) - g(x)|\,dx$$

Areas of Plane Regions

EXAMPLE

Find the area of the bounded, plane region R lying between the curves $y = x^2 - 2x$ and $y = 4 - x^2$.

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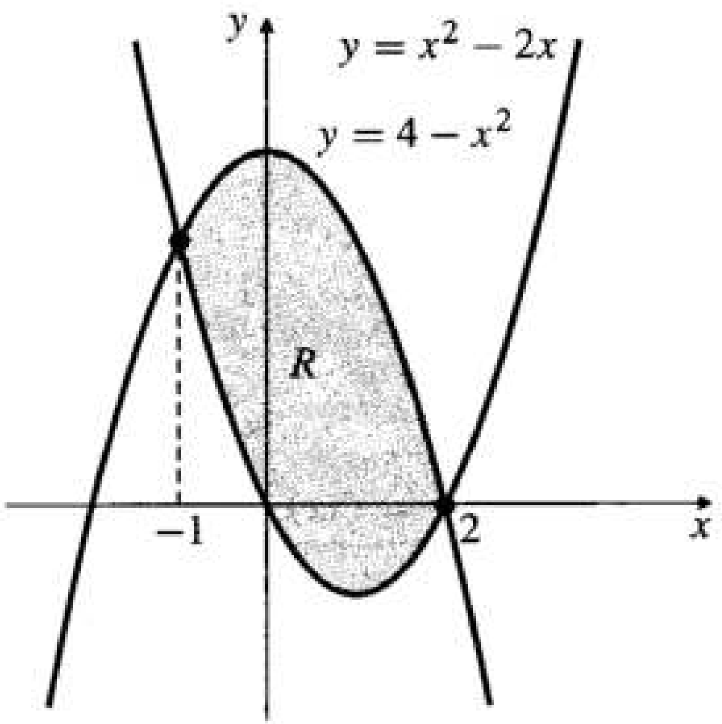

$$x^2 - 2x = y = 4 - x^2$$

$$2x^2 - 2x - 4 = 0$$

$$2(x - 2)(x + 1) = 0$$

$$x = 2 \text{ or } x = -1.$$

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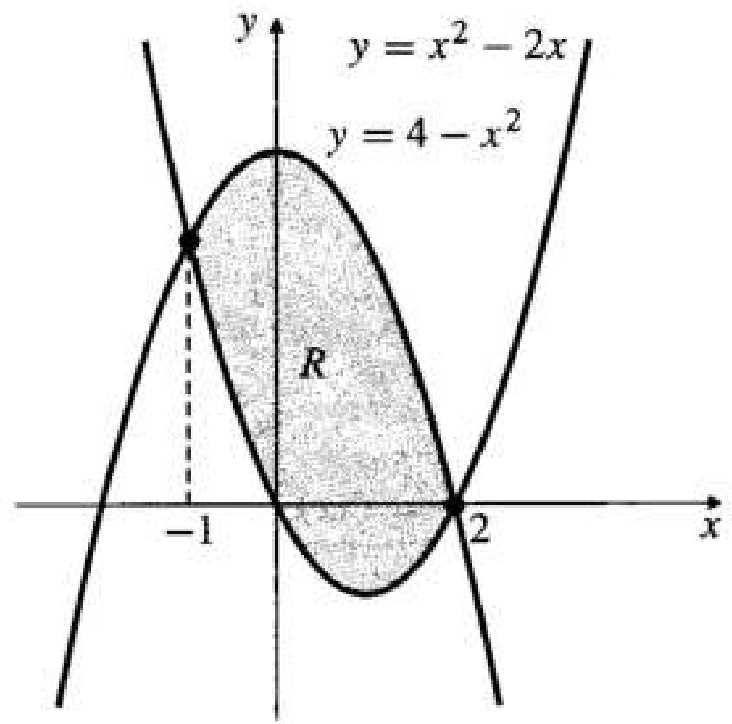

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$$2(x-2)(x+1) = 0$$

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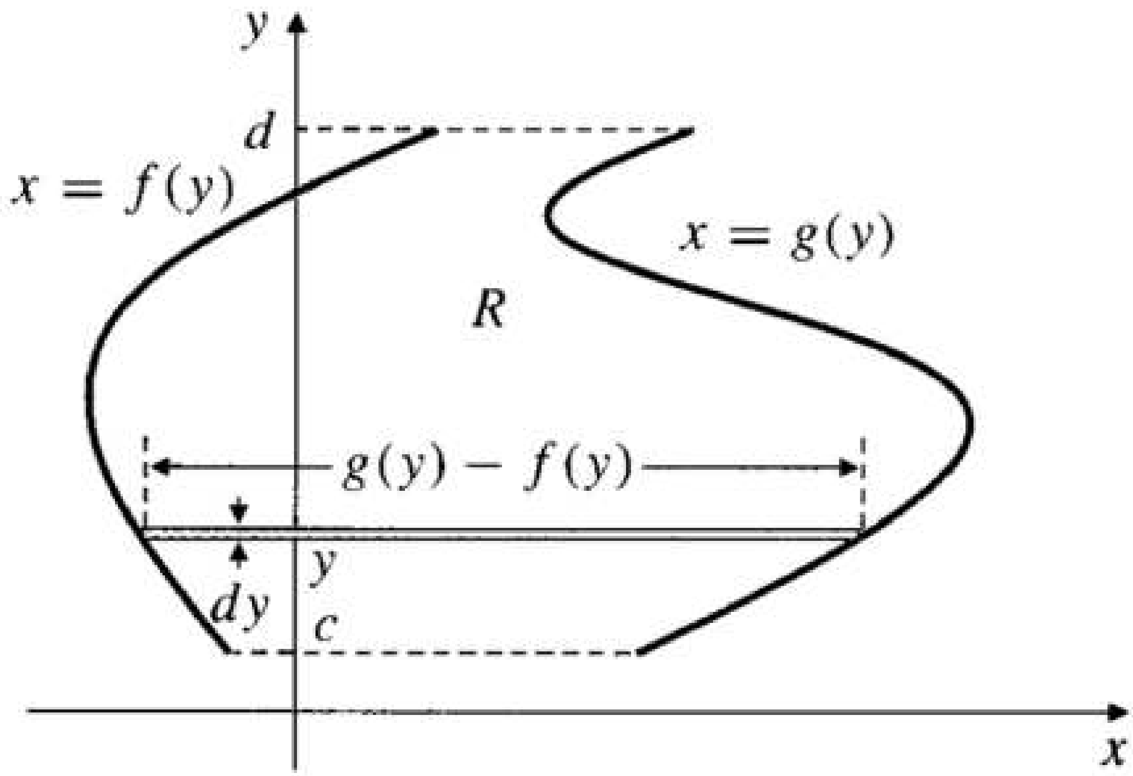
$$\begin{aligned}
A &= \int_{-1}^{2} \left((4 - x^2) - (x^2 - 2x)\right) dx \\
&= \int_{-1}^{2} (4 - 2x^2 + 2x)\, dx \\
&= \left(4x - \frac{2}{3}x^3 + x^2\right)\Big|_{-1}^{2} \\
&= 4(2) - \frac{2}{3}(8) + 4 - \left(-4 + \frac{2}{3} + 1\right) = 9 \text{ square units.}
\end{aligned}$$

Areas of Plane Regions

$$A = \int_c^d \big(g(y) - f(y)\big)\,dy.$$



EXAMPLE

Find the area of the plane region lying to the right of the parabola $x = y^2 - 12$ and to the left of the straight line $y = x$.

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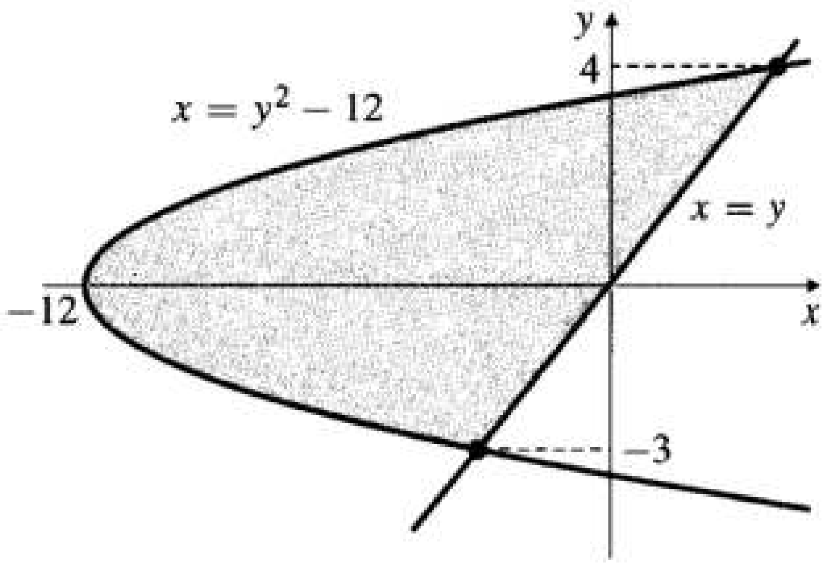

$$y^2 - 12 = x = y$$
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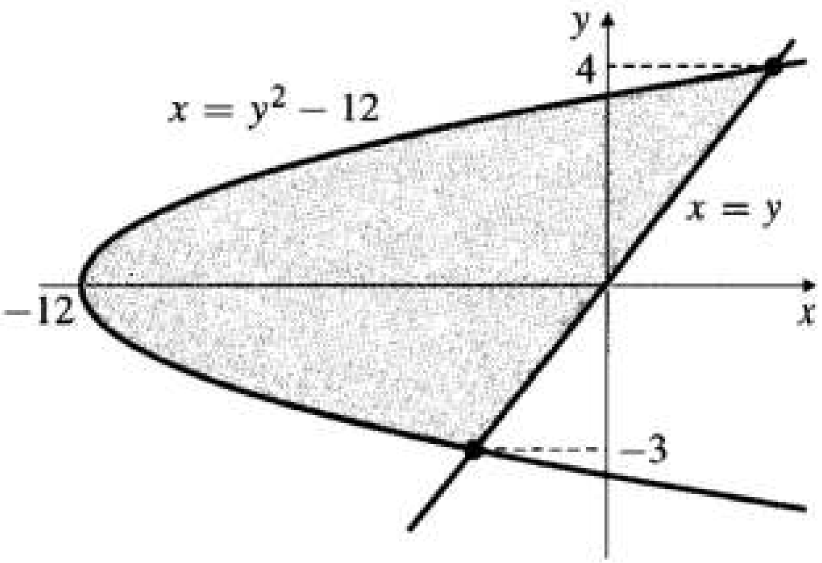

$$y^2 - 12 = x = y$$
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$$(y-4)(y+3) = 0 \quad \text{so } y = 4 \text{ or } y = -3.$$

$$A = \int_{-3}^{4} \left(y - (y^2 - 12)\right) dy = \left(\frac{y^2}{2} - \frac{y^3}{3} + 12y\right)\Bigg|_{-3}^{4}$$
$$= \frac{343}{6} \text{ square units.}$$

Areas of Plane Regions

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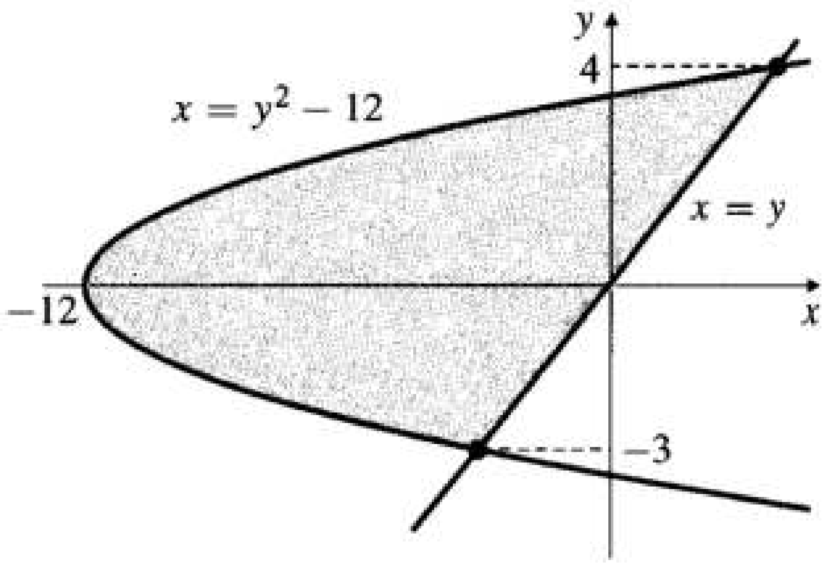

$$y^2 - 12 = x = y$$
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$$(y-4)(y+3) = 0 \quad \text{so } y = 4 \text{ or } y = -3.$$

$$A = \int_{-3}^{4} (y - (y^2 - 12))\, dy = \left(\frac{y^2}{2} - \frac{y^3}{3} + 12y \right) \Bigg|_{-3}^{4}$$
$$= \frac{343}{6} \text{ square units.}$$

integrating in the x direction,

$$A = \int_{-12}^{-3} (\sqrt{12+x} - (-\sqrt{12+x}))\, dx$$
$$+ \int_{-3}^{4} (\sqrt{12+x} - x)\, dx$$

TECHNIQUES OF INTEGRATION

Integration by Parts

Suppose that $U(x)$ and $V(x)$ are two differentiable functions. According to the Product Rule,

$$\frac{d}{dx}\left(U(x)V(x)\right) = U(x)\frac{dV}{dx} + V(x)\frac{dU}{dx}.$$

Integrating both sides of this equation and transposing terms, we obtain

$$\int U(x)\frac{dV}{dx}\,dx = U(x)V(x) - \int V(x)\frac{dU}{dx}\,dx$$

or, more simply,

$$\int U\,dV = UV - \int V\,dU.$$

Integration by Parts

EXAMPLE $\int xe^x \, dx$

Integration by Parts

EXAMPLE

$$\int \underbrace{x}_{u}\,\underbrace{e^x\,dx}_{dv}$$

Integration by Parts

EXAMPLE $\int xe^x \, dx$

Let $U = x, \quad dV = e^x \, dx.$

Then $dU = dx, \quad V = e^x.$

(i.e., $UV - \int V \, dU$)

Integration by Parts

EXAMPLE

$$\int xe^x\,dx$$

$$= xe^x - \int e^x\,dx$$

$$= xe^x - e^x + C.$$

$$\boxed{\begin{array}{l} \text{Let} \quad U = x, \quad dV = e^x\,dx. \\ \text{Then} \quad dU = dx, \quad V = e^x. \\ \text{(i.e., } UV - \int V\,dU) \end{array}}$$

Integration by Parts

EXAMPLE Use integration by parts to evaluate:

(a) $\int \ln x\, dx$, (b) $\int x^2 \sin x\, dx$, (c) $\int x \tan^{-1} x\, dx$, (d) $\int \sin^{-1} x\, dx$.

Solution

(a) $\int \ln x\, dx$

$$= x \ln x - \int x \frac{1}{x}\, dx$$

$$= x \ln x - x + C.$$

Let	$U = \ln x$,	$dV = dx$.
Then	$dU = dx/x$,	$V = x$.

Integration by Parts

EXAMPLE Use integration by parts to evaluate:

(a) $\int \ln x \, dx$, (b) $\int x^2 \sin x \, dx$, (c) $\int x \tan^{-1} x \, dx$, (d) $\int \sin^{-1} x \, dx$.

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Solution

(b) We have to integrate by parts twice this time:

$$\int x^2 \sin x\,dx \longleftarrow \boxed{\begin{array}{ll} \text{Let} \quad U = x^2, & dV = \sin x\,dx. \\ \text{Then } dU = 2x\,dx, & V = -\cos x. \end{array}}$$

Integration by Parts

EXAMPLE Use integration by parts to evaluate:

(a) $\int \ln x\,dx$, (b) $\int x^2 \sin x\,dx$, (c) $\int x \tan^{-1} x\,dx$, (d) $\int \sin^{-1} x\,dx$.

Solution

(b) We have to integrate by parts twice this time:

$$\int x^2 \sin x\,dx$$

Let	$U = x^2,$	$dV = \sin x\,dx.$
Then	$dU = 2x\,dx,$	$V = -\cos x.$

$$= -x^2 \cos x + 2\int x \cos x\,dx$$

Integration by Parts

EXAMPLE Use integration by parts to evaluate:

(a) $\int \ln x\, dx$, (b) $\int x^2 \sin x\, dx$, (c) $\int x \tan^{-1} x\, dx$, (d) $\int \sin^{-1} x\, dx$.

Solution

(b) We have to integrate by parts twice this time:

$$\int x^2 \sin x\, dx$$

Let	$U = x^2$,	$dV = \sin x\, dx$.
Then	$dU = 2x\, dx$,	$V = -\cos x$.

$$= -x^2 \cos x + 2\int x \cos x\, dx$$

Let	$U = x$,	$dV = \cos x\, dx$.
Then	$dU = dx$,	$V = \sin x$.

Integration by Parts

EXAMPLE Use integration by parts to evaluate:

(a) $\int \ln x\,dx$, (b) $\int x^2 \sin x\,dx$, (c) $\int x \tan^{-1} x\,dx$, (d) $\int \sin^{-1} x\,dx$.

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$$= -x^2 \cos x + 2\int x \cos x\,dx$$

$$\boxed{\begin{array}{lll} \text{Let} & U = x, & dV = \cos x\,dx. \\ \text{Then} & dU = dx, & V = \sin x. \end{array}}$$

$$= -x^2 \cos x + 2\left(x \sin x - \int \sin x\,dx\right)$$

$$= -x^2 \cos x + 2x \sin x + 2 \cos x + C.$$

Integration by Parts

EXAMPLE Use integration by parts to evaluate:

(a) $\int \ln x\, dx$, (b) $\int x^2 \sin x\, dx$, (c) $\int x \tan^{-1} x\, dx$, (d) $\int \sin^{-1} x\, dx$.

Solution

(c) $\int x \tan^{-1} x\, dx$

Let	$U = \tan^{-1} x,$	$dV = x\, dx.$
Then	$dU = dx/(1+x^2),$	$V = \frac{1}{2}x^2.$

$$= \frac{1}{2}x^2 \tan^{-1} x - \frac{1}{2}\int \frac{x^2}{1+x^2}\, dx$$

$$= \frac{1}{2}x^2 \tan^{-1} x - \frac{1}{2}\int \left(1 - \frac{1}{1+x^2}\right) dx$$

$$= \frac{1}{2}x^2 \tan^{-1} x - \frac{1}{2}x + \frac{1}{2}\tan^{-1} x + C.$$

Integration by Parts

EXAMPLE Use integration by parts to evaluate:

(a) $\int \ln x\,dx$, (b) $\int x^2 \sin x\,dx$, (c) $\int x \tan^{-1} x\,dx$, (d) $\int \sin^{-1} x\,dx$.

Solution

(d) $\int \sin^{-1} x\,dx$ ← $\boxed{\begin{array}{lll} \text{Let} & U = \sin^{-1} x, & dV = dx. \\ \text{Then} & dU = dx/\sqrt{1-x^2}, & V = x. \end{array}}$

$$= x \sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}}\,dx \qquad \begin{array}{l} \text{Let } u = 1 - x^2, \\ du = -2x\,dx \end{array}$$

$$= x \sin^{-1} x + \frac{1}{2}\int u^{-1/2}\,du$$

$$= x \sin^{-1} x + u^{1/2} + C = x \sin^{-1} x + \sqrt{1-x^2} + C.$$

Integration by Parts

EXAMPLE Evaluate $I = \int \sec^3 x\, dx$.

Solution Start by integrating by parts:

$$I = \int \sec^3 x\, dx \longleftarrow \boxed{\begin{array}{ll} \text{Let} \quad U = \sec x, & dV = \sec^2 x\, dx. \\ \text{Then } dU = \sec x \tan x\, dx, & V = \tan x. \end{array}}$$

$$= \sec x \tan x - \int \sec x \tan^2 x\, dx$$

$$= \sec x \tan x - \int \sec x (\sec^2 x - 1)\, dx$$

$$= \sec x \tan x - \int \sec^3 x\, dx + \int \sec x\, dx$$

$$= \sec x \tan x - I + \ln|\sec x + \tan x|.$$

This is an equation that can be solved for the desired integral I: Since $2I = \sec x \tan x + \ln|\sec x + \tan x|$, we have

$$\int \sec^3 x\, dx = I = \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln|\sec x + \tan x| + C.$$

Integration by Parts

EXAMPLE Find $I = \int e^{ax} \cos bx \, dx$.

Solution If either $a = 0$ or $b = 0$, the integral is easy to do, so let us assume $a \neq 0$ and $b \neq 0$. We have

$$I = \int e^{ax} \cos bx \, dx \longleftarrow \boxed{\begin{array}{l} \text{Let} \quad U = e^{ax}, \qquad dV = \cos bx \, dx. \\ \text{Then } dU = a\,e^{ax}\,dx, \quad V = (1/b)\sin bx. \end{array}}$$

$$= \frac{1}{b} e^{ax} \sin bx - \frac{a}{b} \int e^{ax} \sin bx \, dx \longleftarrow \boxed{\begin{array}{l} \text{Let} \quad U = e^{ax}, \qquad dV = \sin bx \, dx. \\ \text{Then } dU = ae^{ax}dx, \quad V = -(\cos bx)/b. \end{array}}$$

$$= \frac{1}{b} e^{ax} \sin bx - \frac{a}{b}\left(-\frac{1}{b} e^{ax} \cos bx + \frac{a}{b} \int e^{ax} \cos bx \, dx\right)$$

$$= \frac{1}{b} e^{ax} \sin bx + \frac{a}{b^2} e^{ax} \cos bx - \frac{a^2}{b^2} I.$$

Thus,

$$\left(1 + \frac{a^2}{b^2}\right) I = \frac{1}{b} e^{ax} \sin bx + \frac{a}{b^2} e^{ax} \cos bx + C_1$$

and

$$\int e^{ax} \cos bx \, dx = I = \frac{b\,e^{ax} \sin bx + a\,e^{ax} \cos bx}{b^2 + a^2} + C.$$

EXAMPLE **(A definite integral)**

$$\int_1^e x^3 (\ln x)^2\, dx \longleftarrow \boxed{\begin{array}{lll} \text{Let} & U = (\ln x)^2, & dV = x^3\, dx. \\ \text{Then} & dU = 2\ln x\,(1/x)\, dx, & V = x^4/4. \end{array}}$$

$$= \frac{x^4}{4}(\ln x)^2 \bigg|_1^e - \frac{1}{2}\int_1^e x^3 \ln x\, dx \longleftarrow \boxed{\begin{array}{lll} \text{Let} & U = \ln x, & dV = x^3\, dx. \\ \text{Then} & dU = dx/x, & V = x^4/4. \end{array}}$$

$$= \frac{e^4}{4}(1^2) - 0 - \frac{1}{2}\left(\frac{x^4}{4}\ln x \bigg|_1^e - \frac{1}{4}\int_1^e x^3\, dx\right)$$

$$= \frac{e^4}{4} - \frac{e^4}{8} + \frac{1}{8}\frac{x^4}{4}\bigg|_1^e = \frac{e^4}{8} + \frac{e^4}{32} - \frac{1}{32} = \frac{5}{32}e^4 - \frac{1}{32}.$$

Integration by Parts

EXAMPLE Obtain and use a reduction formula to evaluate

$$I_n = \int_0^{\pi/2} \cos^n x\, dx \qquad (n = 0,\ 1,\ 2,\ 3,\ \ldots).$$