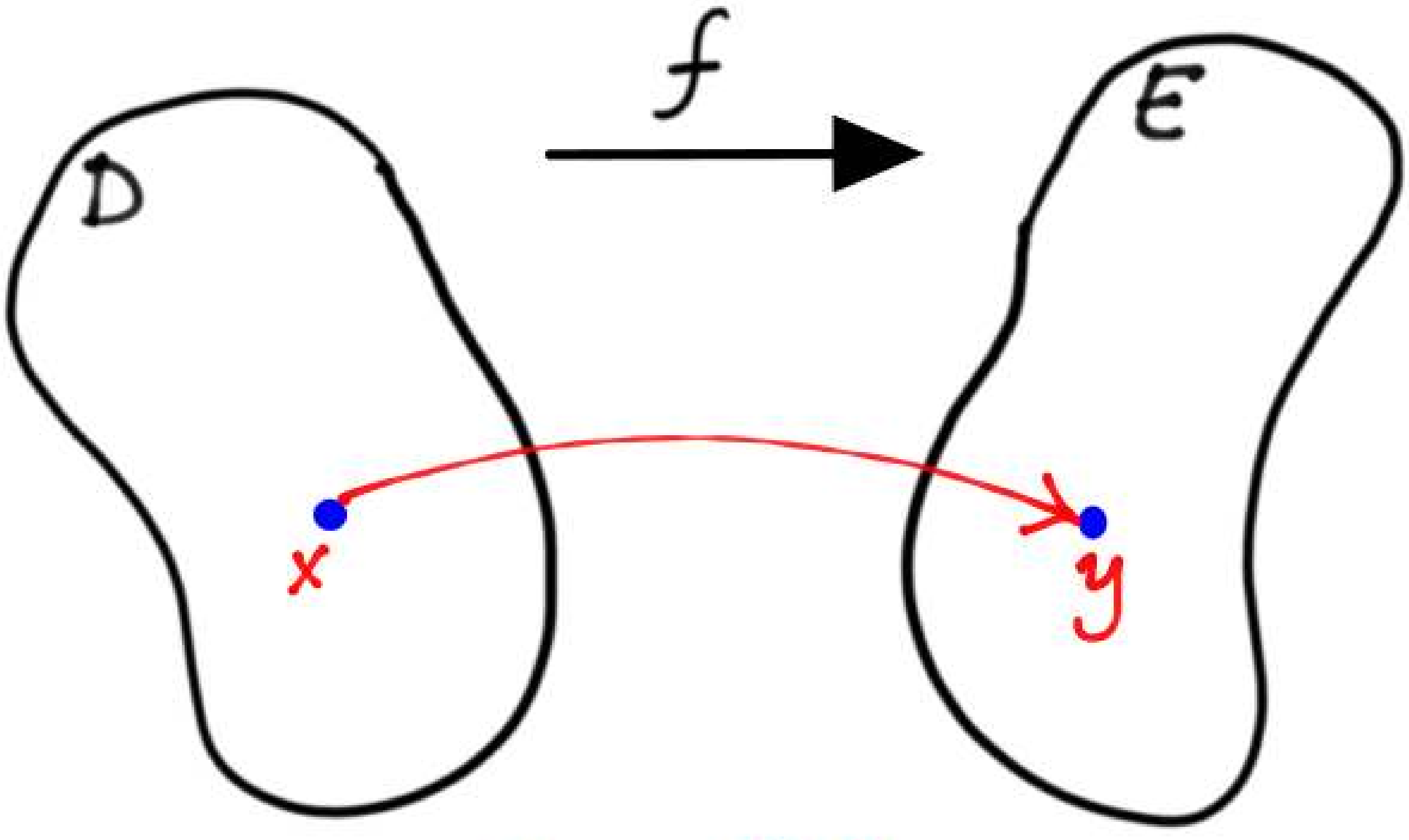
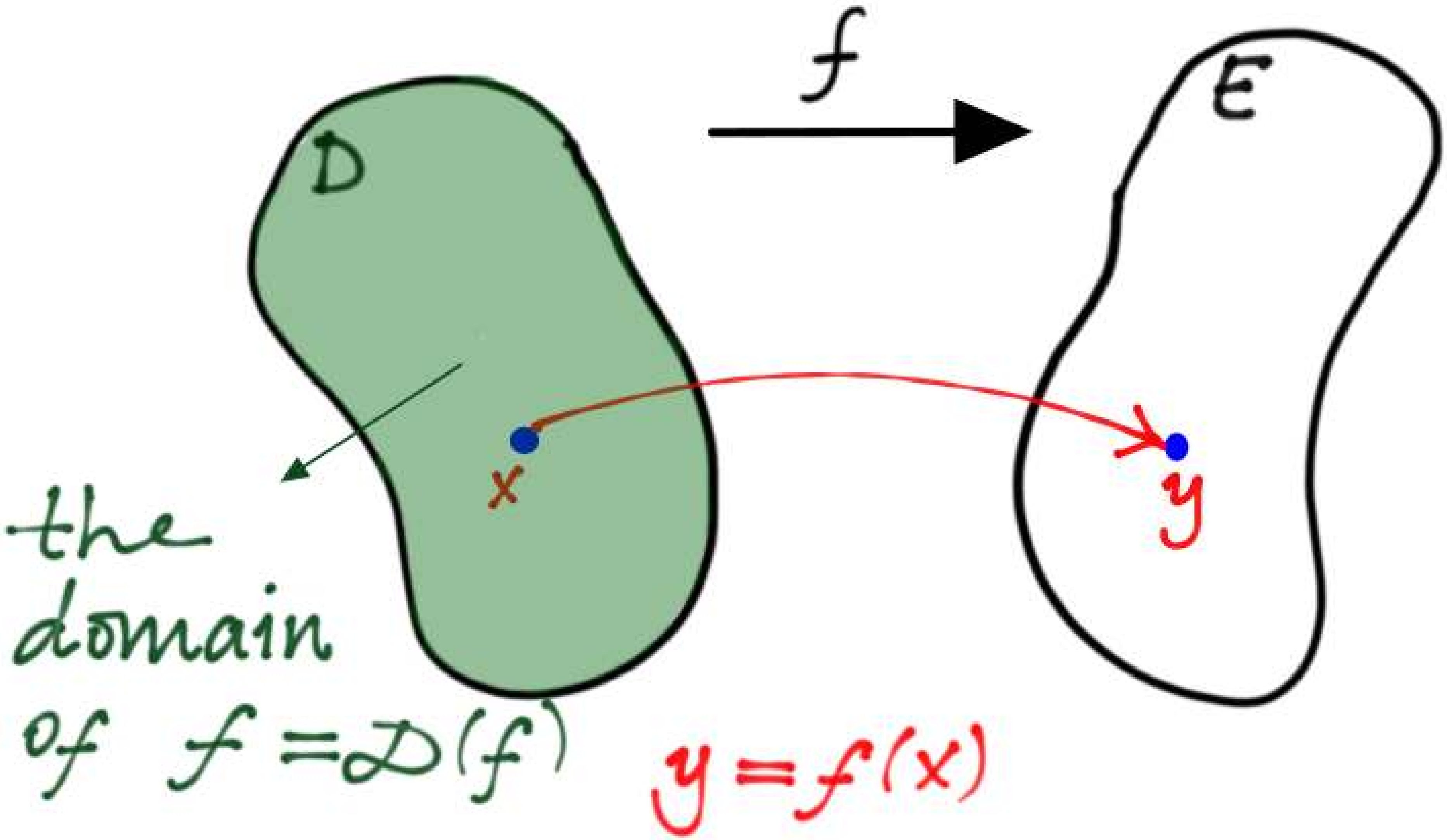
Functions and Their Graphs

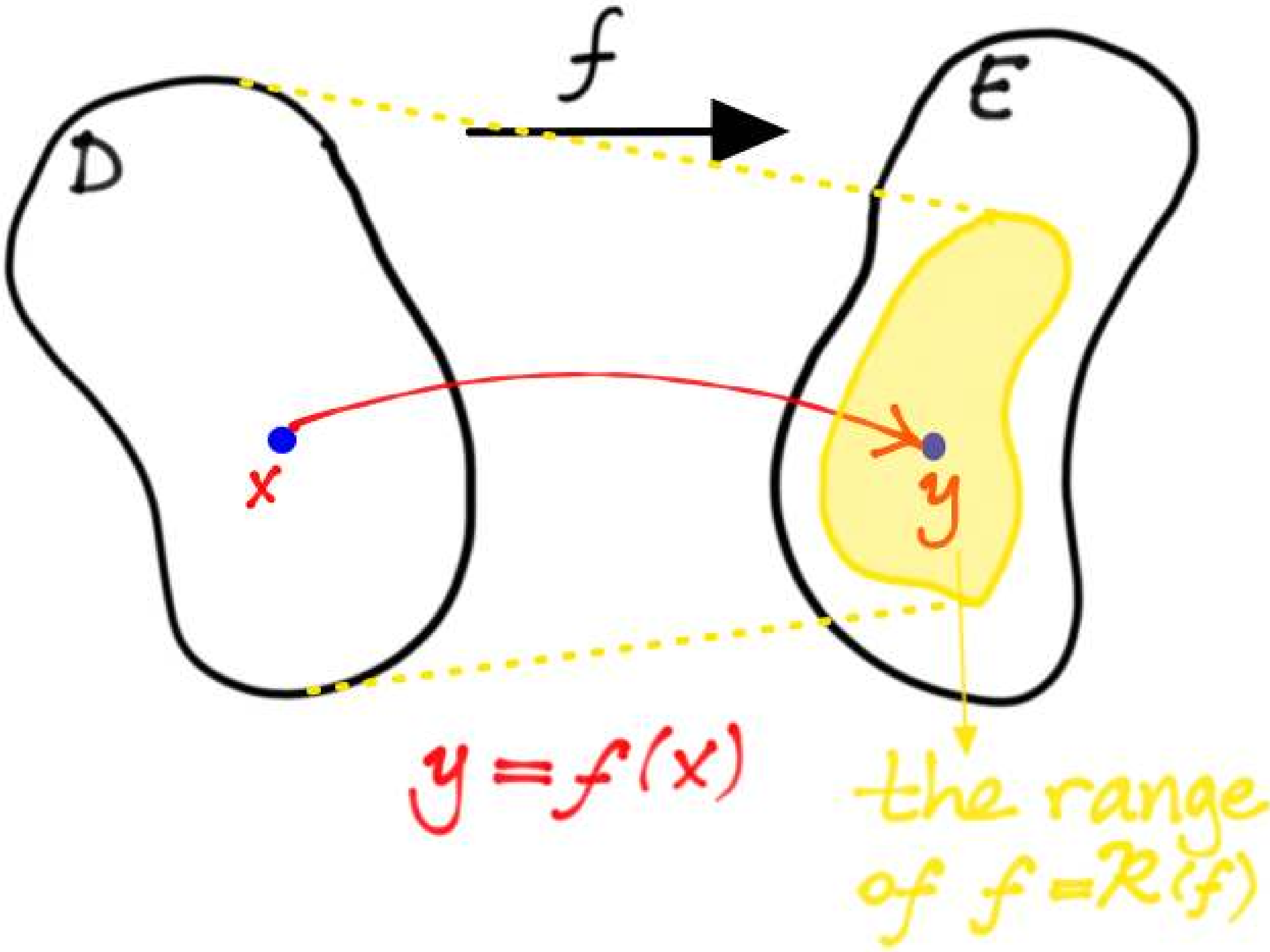


$y = f(x)$

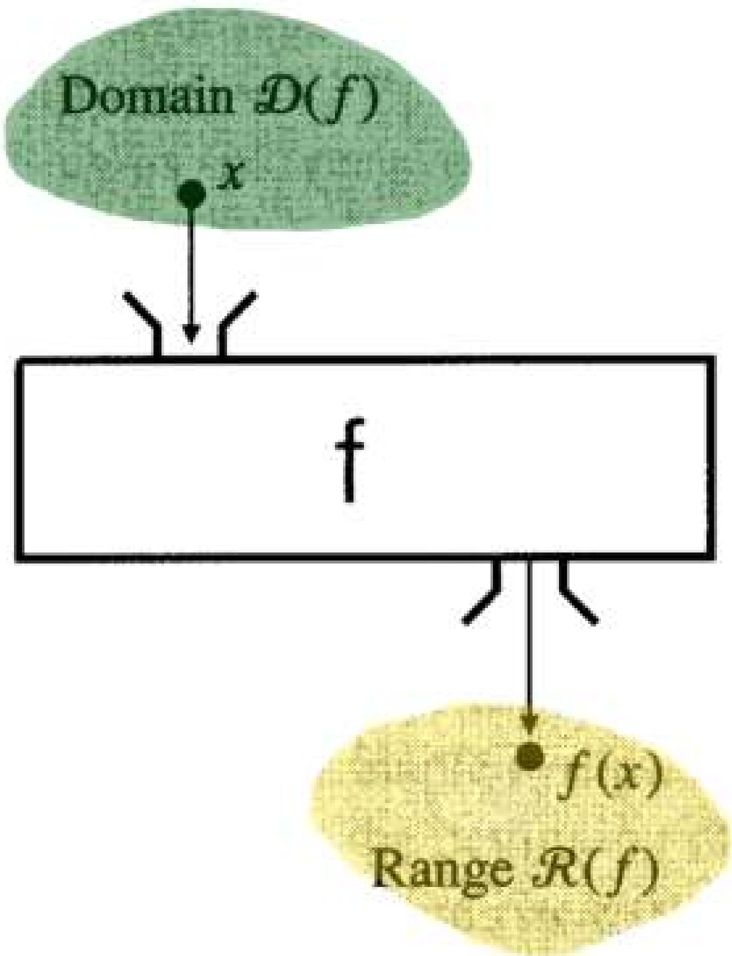
Functions and Their Graphs



Functions and Their Graphs



Functions and Their Graphs



A function machine

Functions and Their Graphs

$$y = f(x)$$

Functions and Their Graphs

$$y = f(\boxed{x})$$

independent variable

Functions and Their Graphs

$$\boxed{y} = f(x)$$

dependent
variable

Functions and Their Graphs

The function that converts any real number into its square can be denoted:

- by $y = x^2$

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Functions and Their Graphs

The function that converts any real number into its square can be denoted:

- by $y = x^2$
- by $f(x) = x^2$
- by $x \longrightarrow x^2$

The Domain Convention

Let f be a function.
Unless specified otherwise, the domain of f is meant to be the set of all real numbers for which $f(x)$ is a real number.

EXAMPLE. The domain of the function

$$f(x) = \sqrt{1 - x^2}$$

is the set of all real numbers x for which

$$1 - x^2 \geqslant 0.$$

Thus $\mathcal{D}(f) = [-1, 1]$.

Functions and Their Graphs

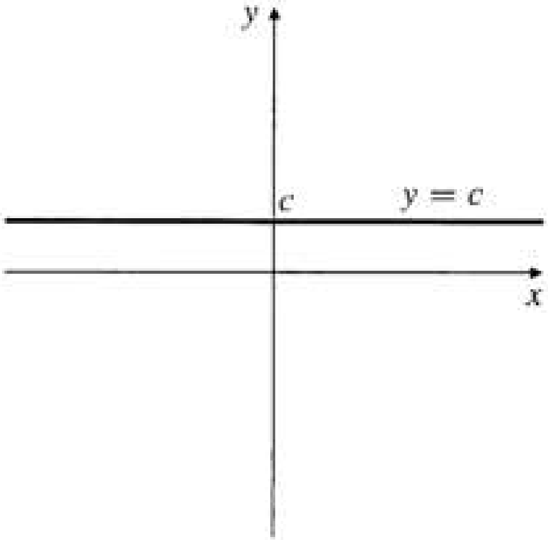
Graphs of Functions

"A picture is worth a thousand words."

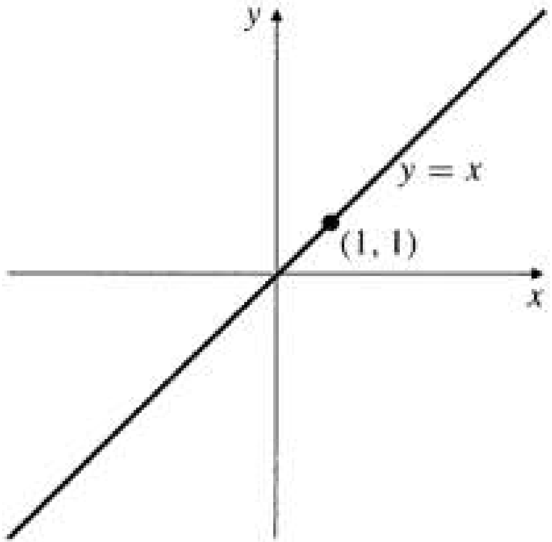
The <u>graph</u> of a function f is the set of points (x, y) in the Cartesian plane such that x lies in $\mathcal{D}(f)$ and $y = f(x)$.

Functions and Their Graphs

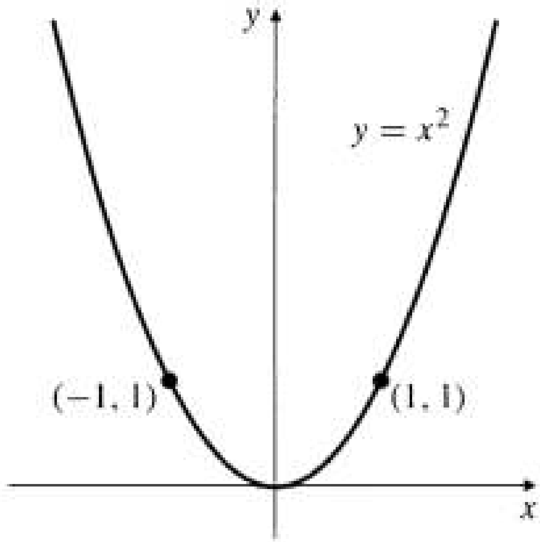
Graphs of Functions



The graph of a
constant function $f(x) = c$



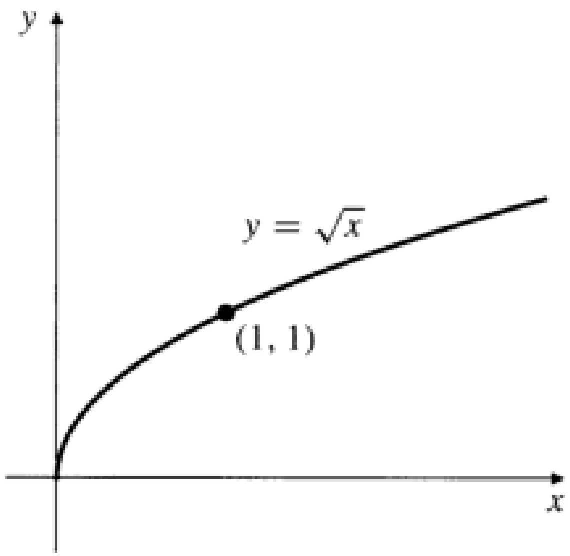
The graph of $f(x) = x$



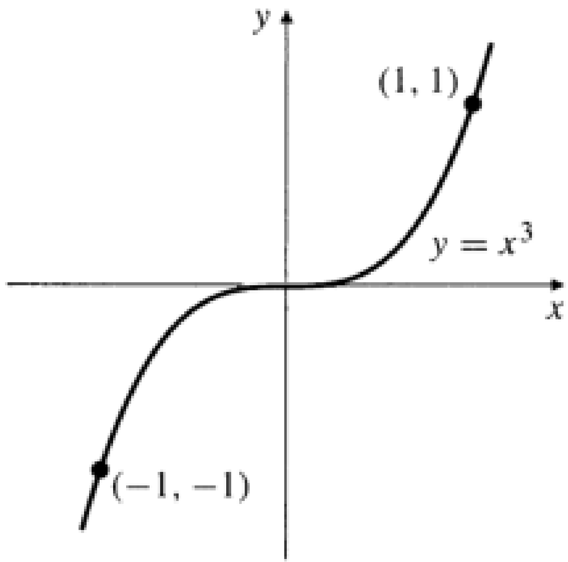
The graph of $f(x) = x^2$

Functions and Their Graphs

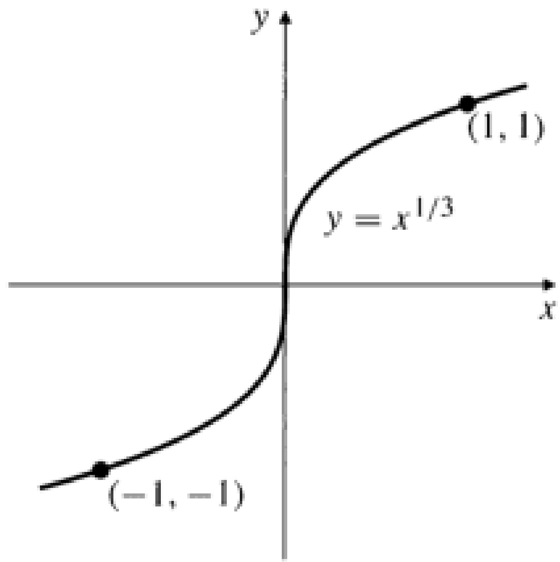
Graphs of Functions



The graph of $f(x) = \sqrt{x}$



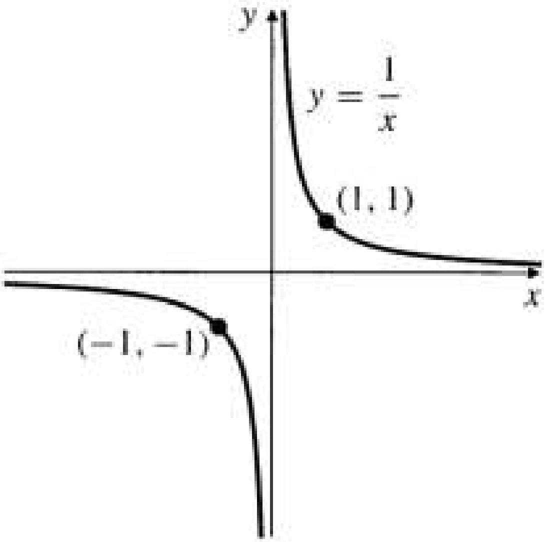
The graph of $f(x) = x^3$



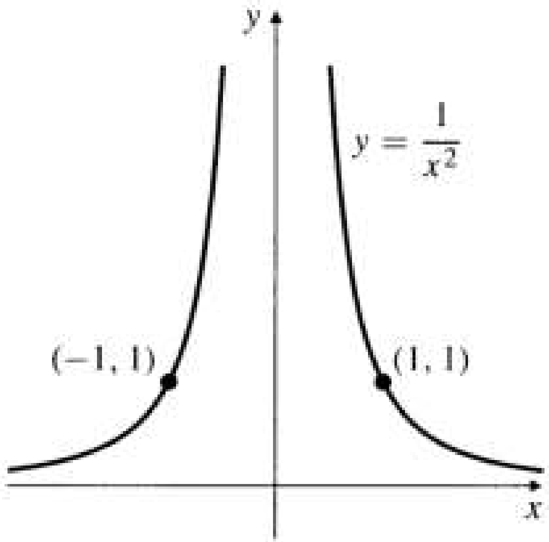
The graph of $f(x) = x^{1/3}$

Functions and Their Graphs

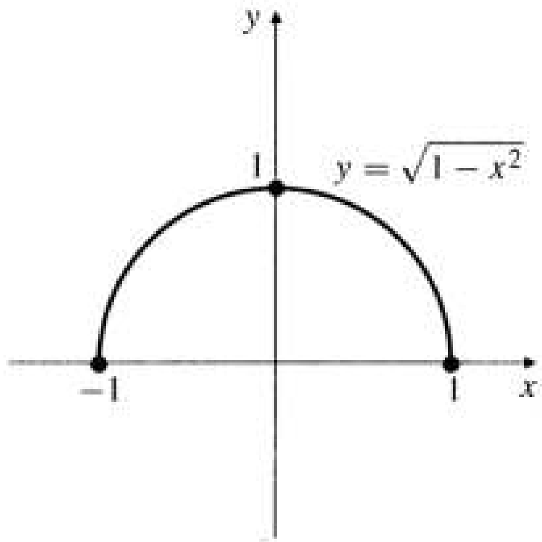
Graphs of Functions



The graph of $f(x) = 1/x$

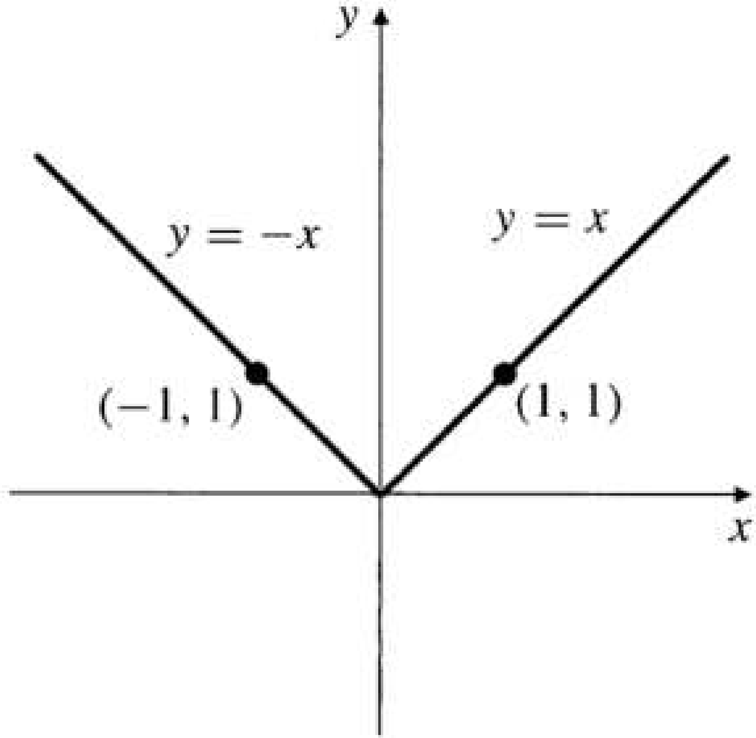


The graph of $f(x) = 1/x^2$



The graph of $f(x) = \sqrt{1 - x^2}$

Graphs of Functions



The graph of $f(x) = |x|$

Functions and Their Graphs

Graphs of Functions

EXAMPLE Sketch the graph of $y = 1 + \sqrt{x - 4}$.

Functions and Their Graphs

Graphs of Functions

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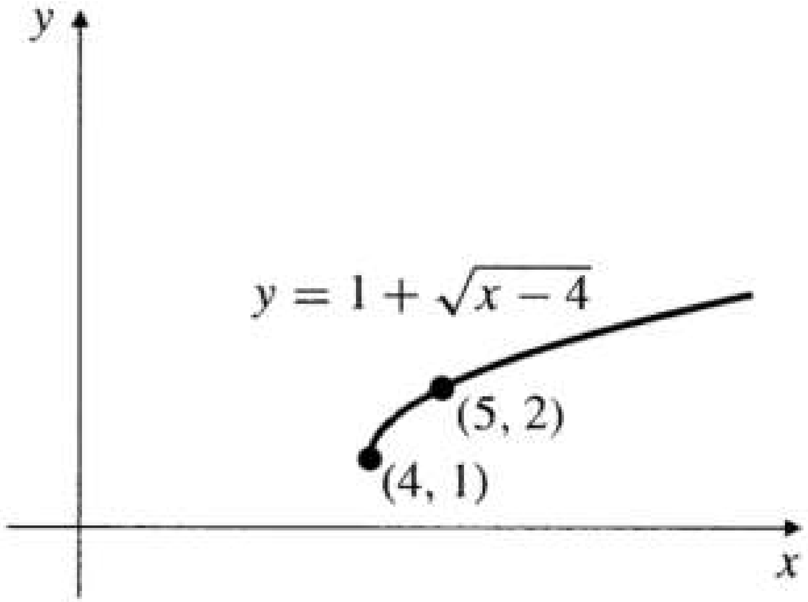
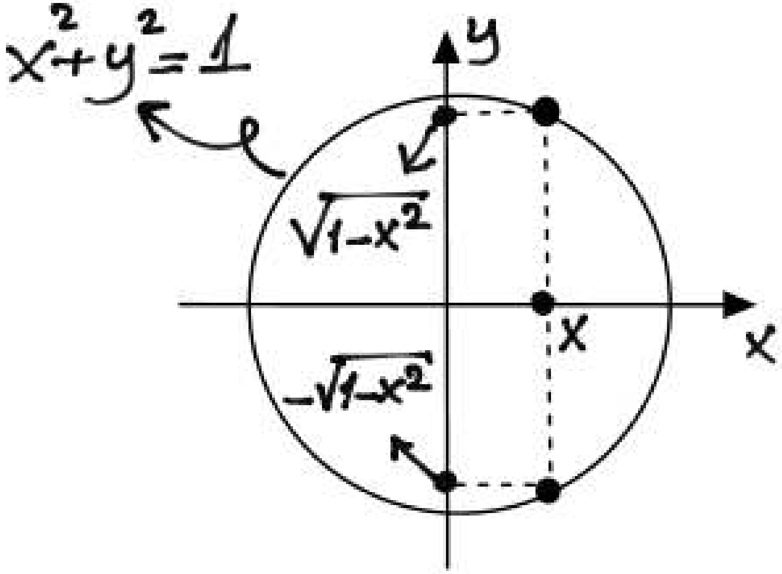


Graphs of Functions

EXAMPLE Not every curve is the graph of a function, e.g., $x^2+y^2=1$.



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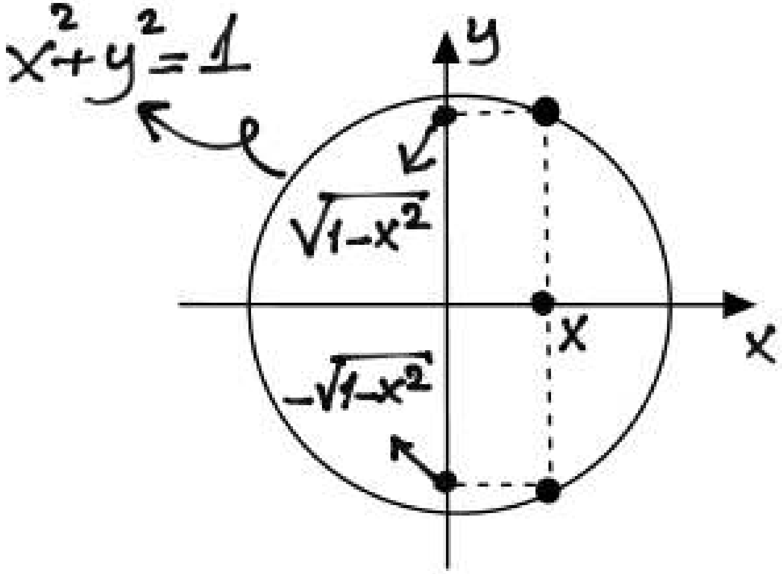
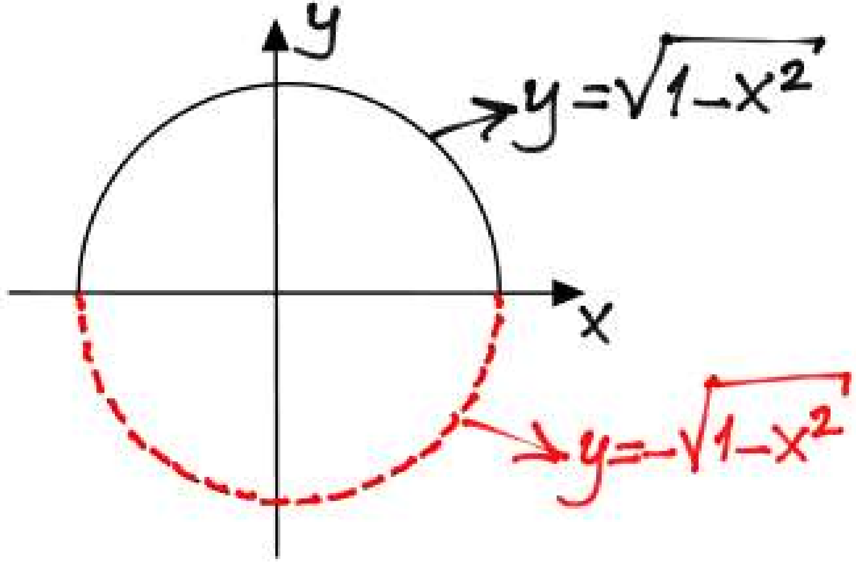




Functions and Their Graphs

Even and Odd Functions; Symmetry and Reflections

Even and odd functions

Suppose that $-x$ belongs to the domain of f whenever x does. We say that f is an **even function** if

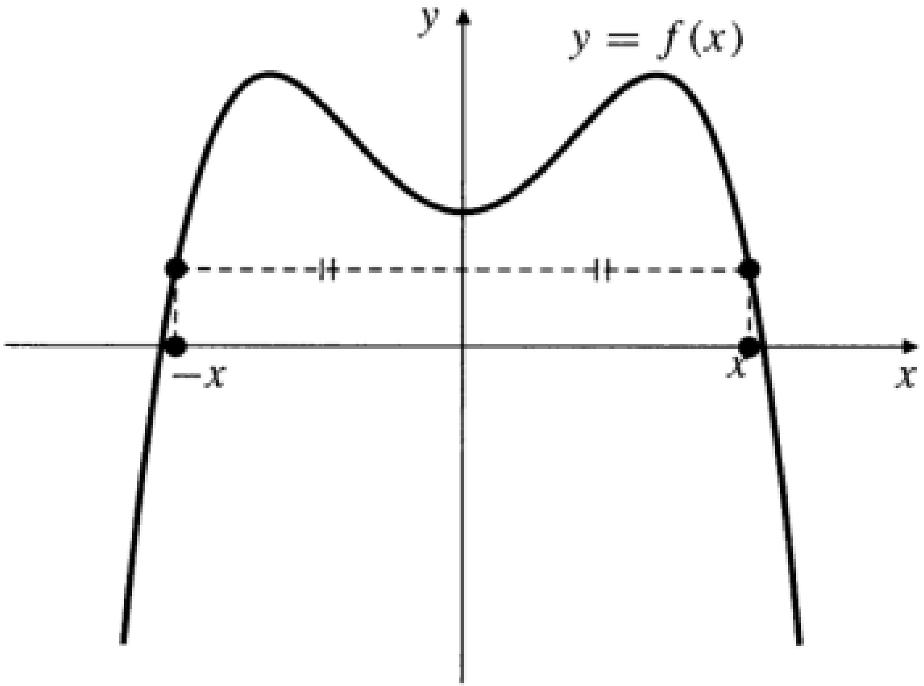
$$f(-x) = f(x) \qquad \text{for every } x \text{ in the domain of } f.$$

We say that f is an **odd function** if

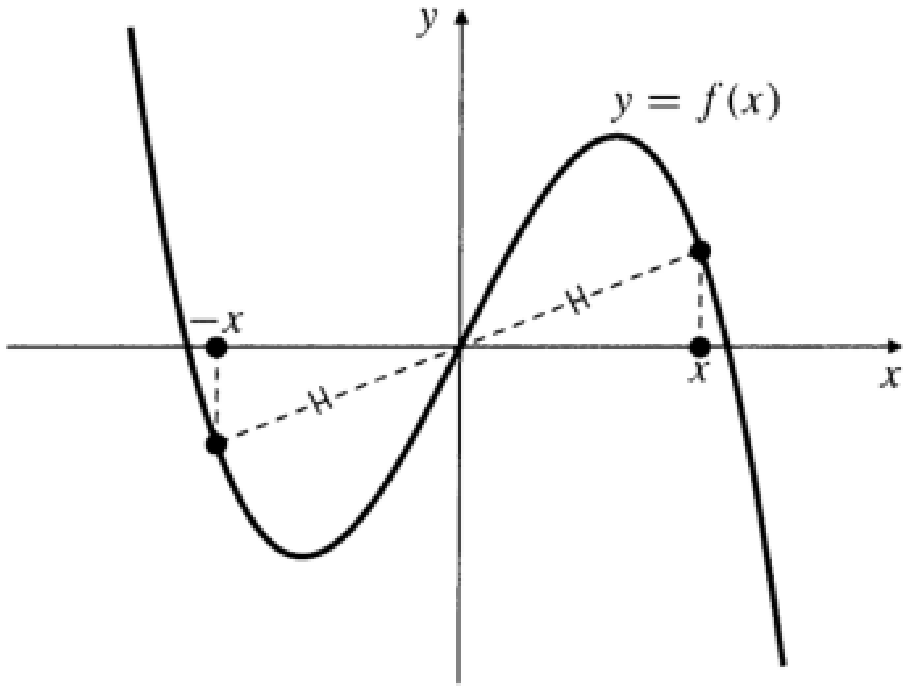
$$f(-x) = -f(x) \qquad \text{for every } x \text{ in the domain of } f.$$

Functions and Their Graphs

Even and Odd Functions; Symmetry and Reflections



The graph of an even function is symmetric about the y-axis



The graph of an odd function is symmetric about the origin

Reflections in Straight Lines

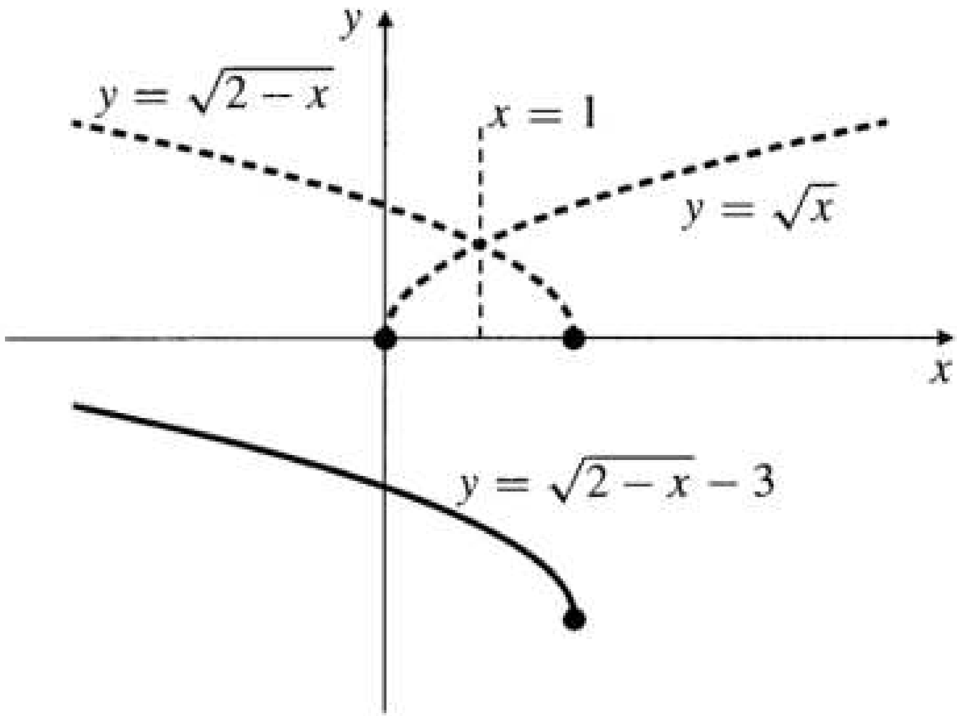
Reflections in special lines

1. Substituting $-x$ in place of x in an equation in x and y corresponds to reflecting the graph of the equation in the y-axis.
2. Substituting $-y$ in place of y in an equation in x and y corresponds to reflecting the graph of the equation in the x-axis.
3. Substituting $a - x$ in place of x in an equation in x and y corresponds to reflecting the graph of the equation in the line $x = a/2$.
4. Substituting $b - y$ in place of y in an equation in x and y corresponds to reflecting the graph of the equation in the line $y = b/2$.
5. Interchanging x and y in an equation in x and y corresponds to reflecting the graph of the equation in the line $y = x$.

Functions and Their Graphs

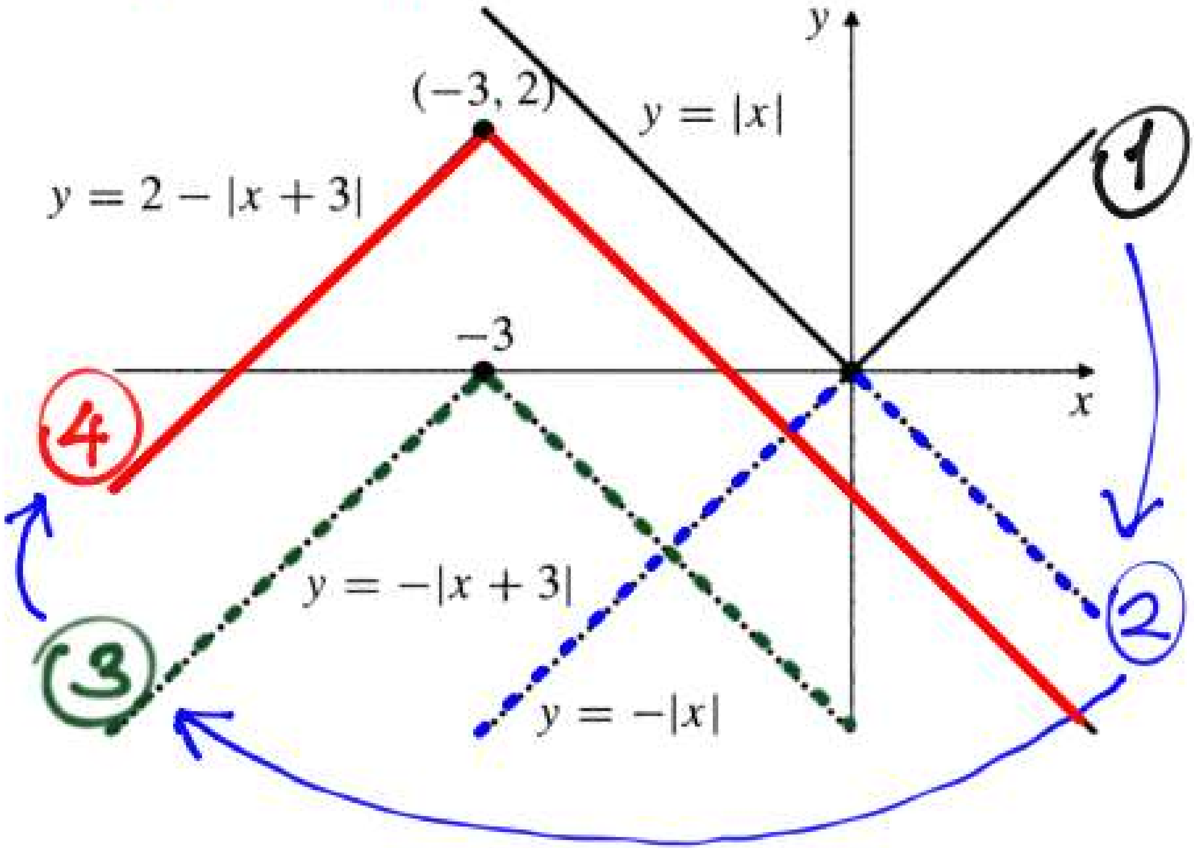
Reflections in Straight Lines

EXAMPLE Describe and sketch the graph of $y = \sqrt{2-x} - 3$.



Reflections in Straight Lines

Sketching the graph of the equation $y = 2 - |x+3|$.



Combining Functions to Make New Functions

Sums, Differences, Products, Quotients, and Multiples

If f and g are functions, then for every x that belongs to the domains of both f and g we define functions $f+g$, $f-g$, fg, and f/g by the formulas:

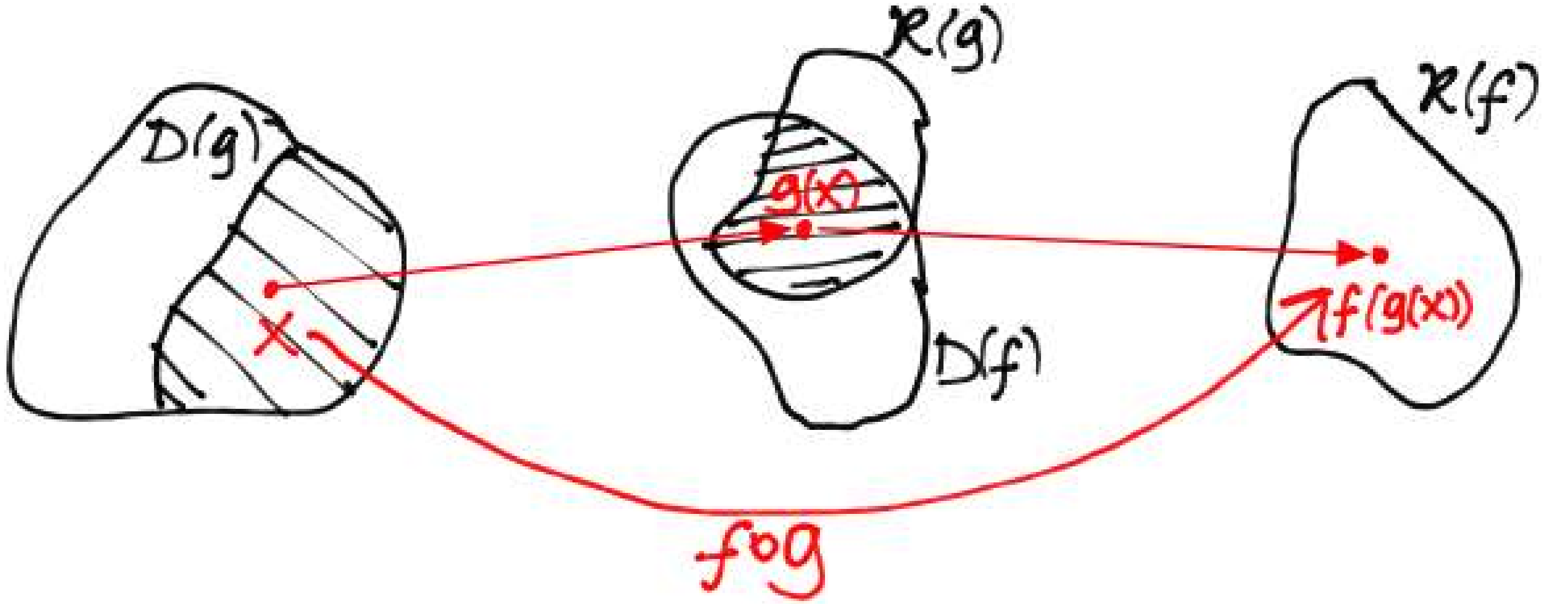
$$
\begin{aligned}
(f+g)(x) &= f(x)+g(x)\\
(f-g)(x) &= f(x)-g(x)\\
(fg)(x) &= f(x)g(x)\\
\left(\frac{f}{g}\right)(x) &= \frac{f(x)}{g(x)}, \qquad \text{where } g(x)\neq 0.
\end{aligned}
$$

Composite Functions

If f and g are two functions, the **composite** function $f \circ g$ is defined by

$$f \circ g(x) = f(g(x)).$$

The domain of $f \circ g$ consists of those numbers x in the domain of g for which $g(x)$ is in the domain of f. In particular, if the range of g is contained in the domain of f, then the domain of $f \circ g$ is just the domain of g.



Combining Functions to Make New Functions

Composite Functions

Composites of f and g and their domains

Function	Formula	Domain
f	$f(x) = \sqrt{x}$	$[0, \infty)$
g	$g(x) = x + 1$	$\mathbb{R}$
$f \circ g$	$f \circ g(x) = f(g(x)) = f(x+1) = \sqrt{x+1}$	$[-1, \infty)$
$g \circ f$	$g \circ f(x) = g(f(x)) = g(\sqrt{x}) = \sqrt{x} + 1$	$[0, \infty)$
$f \circ f$	$f \circ f(x) = f(f(x)) = f(\sqrt{x}) = \sqrt{\sqrt{x}} = x^{1/4}$	$[0, \infty)$
$g \circ g$	$g \circ g(x) = g(g(x)) = g(x+1) = (x+1)+1 = x+2$	$\mathbb{R}$

Composite Functions

For $h(x) = \frac{1-x}{1+x}$, $h \circ h(x) = x$. But the domain of the function $h \circ h$ is not $\mathbb{R}$! It is $\mathbb{R} \setminus \{-1\}$.

Piecewise Defined Functions

Functions defined by different formulas on different parts of the domain.

One simplest example is the absolute value function.

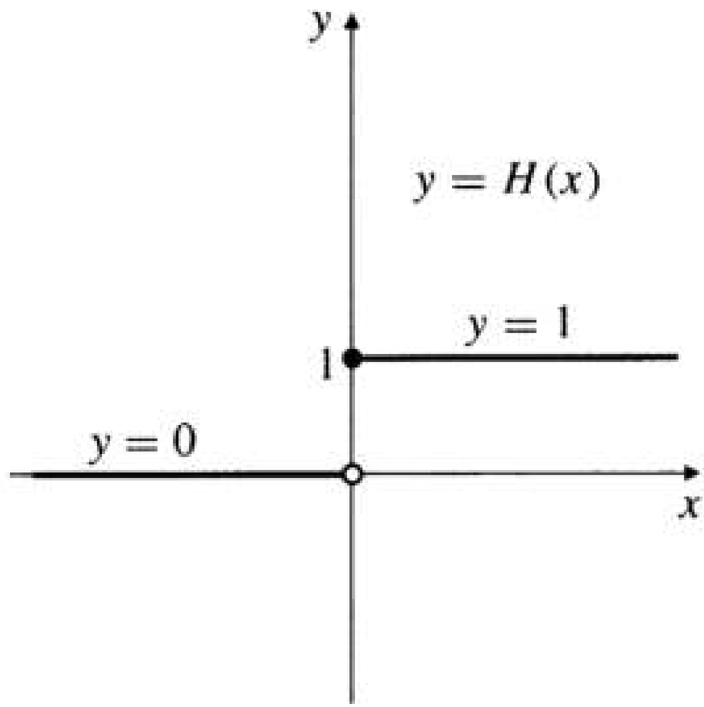
$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Piecewise Defined Functions

EXAMPLE

The Heaviside function. The Heaviside function (or unit step function) (Figure P.61) is defined by

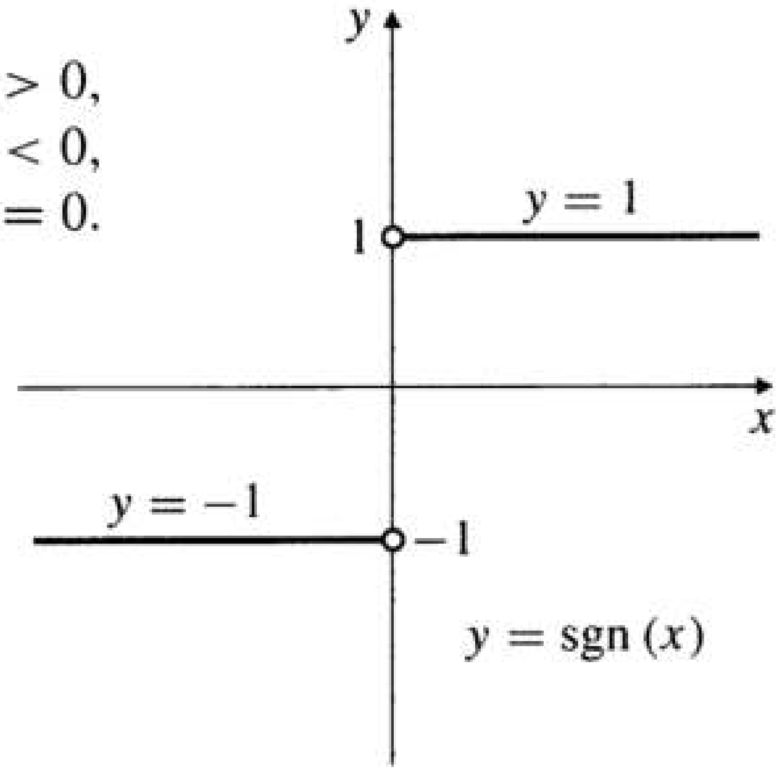
$$H(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0. \end{cases}$$



Piecewise Defined Functions

EXAMPLE **The signum function.**

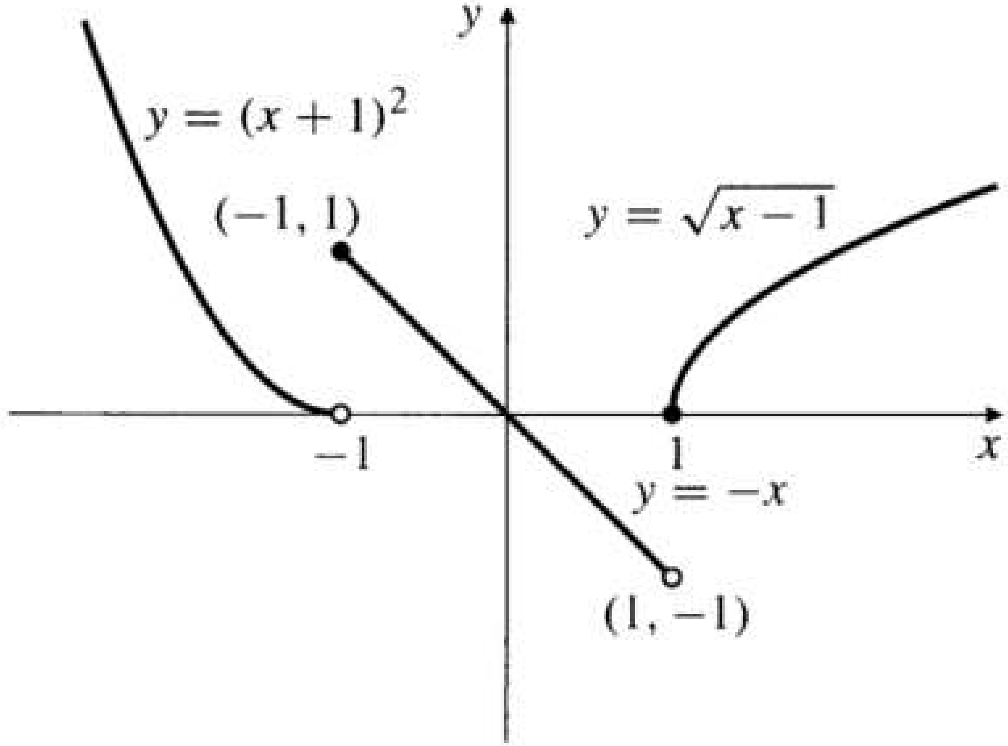
$$\operatorname{sgn}(x) = \frac{x}{|x|} = \begin{cases} 1 & \text{if } x > 0, \\ -1 & \text{if } x < 0, \\ \text{undefined} & \text{if } x = 0. \end{cases}$$



Piecewise Defined Functions

EXAMPLE

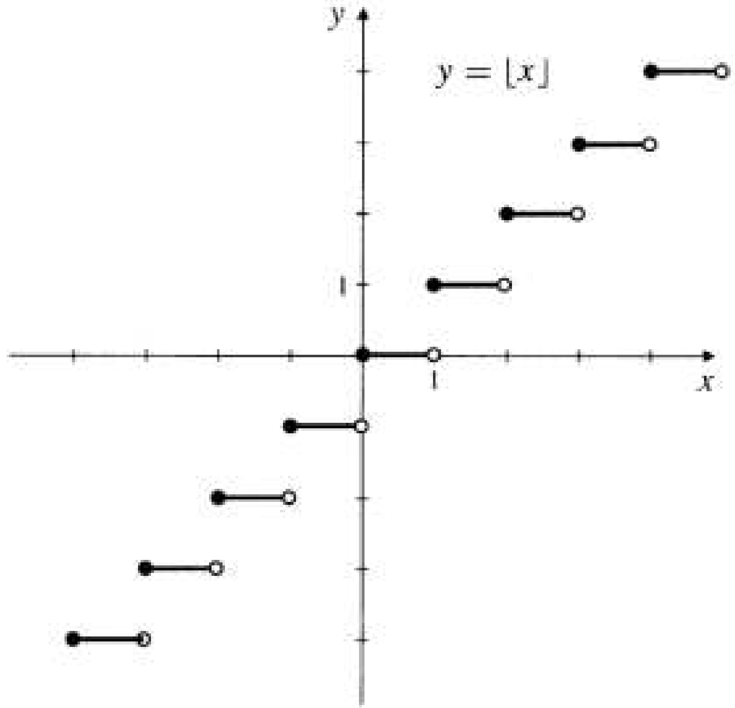
$$f(x) = \begin{cases} (x+1)^2 & \text{if } x < -1, \\ -x & \text{if } -1 \le x < 1, \\ \sqrt{x-1} & \text{if } x \ge 1, \end{cases}$$



Piecewise Defined Functions

EXAMPLE 10 **The greatest integer function.** The function whose value at any number x is the *greatest integer less than or equal to x* is called the **greatest integer function**. It is denoted $\lfloor x \rfloor$, or, in some books, $[x]$ or $[[x]]$.

$$\lfloor 2.4 \rfloor = 2, \qquad \lfloor 1.9 \rfloor = 1, \qquad \lfloor 0 \rfloor = 0, \qquad \lfloor -1.2 \rfloor = -2,$$
$$\lfloor 2 \rfloor = 2, \qquad \lfloor 0.2 \rfloor = 0, \qquad \lfloor -0.3 \rfloor = -1, \qquad \lfloor -2 \rfloor = -2.$$



Polynomials and Rational Functions

A **polynomial** is a function P whose value at x is

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0,$$

where a_n, a_{n-1}, ..., a_2, a_1, and a_0, called the **coefficients** of the polymonial, are constants and, if $n > 0$, then $a_n \neq 0$. The number n, the degree of the highest power of x in the polynomial, is called the **degree** of the polynomial. (The degree of the zero polynomial is not defined.)

Polynomials and Rational Functions

A rational function is a quotient

$$\frac{P(x)}{Q(x)}$$

where $P(x)$ and $Q(x)$ are polynomials and $Q(x) \neq 0$.

If $P(x)$ and $Q(x)$ are polynomials of degrees m and n, respectively, then we may write

$$\frac{P(x)}{Q(x)} = T(x) + \frac{R(x)}{Q(x)} \quad \text{(the division algorithm)}$$

where $T(x)$ and $R(x)$ are polynomials and the degree of $R(x)$ is less than the degree of $Q(x)$.

Polynomials and Rational Functions

EXAMPLE Write the division algorithm for $\dfrac{2x^3 - 3x^2 + 3x + 4}{x^2 + 1}$.

Roots, Zeros, and Factors

A number r is called a root or zero of a polynomial $P(x)$ if $P(r)=0$.

the polynomial x^3-x has three roots: $0, 1, -1$.

The fundamental theorem of algebra
Every polynomial of degree at least one has a root (which might be a complex number).

Polynomials and Rational Functions

Roots, Zeros, and Factors

The Factor Theorem

The number r is a root of the polynomial P of degree not less than 1 if and only if $x - r$ is a factor of $P(x)$.

Roots, Zeros, and Factors

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COROLLARY. Every polynomial of degree $n > 0$ has n roots.

Roots, Zeros, and Factors

The Factor Theorem

The number r is a root of the polynomial P of degree not less than 1 if and only if $x - r$ is a factor of $P(x)$.

COROLLARY. Every polynomial of degree $n>0$ has n roots.

EXAMPLE What is the degree of $P(x) = x^3(x^2 + 2x + 5)^2$? What are the roots of P and, what is the multiplicity of each root?

Miscellaneous Factorings

- Difference of squares: $\boxed{x^2 - a^2 = (x-a)(x+a)}$

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- More generally, a difference of nth powers for an integer $n>0$.

$$\boxed{x^n - a^n = (x-a)(x^{n-1}+ax^{n-2}+a^2x^{n-3}+\cdots+a^{n-2}x+a^{n-1})}$$

Miscellaneous Factorings

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$$\boxed{x^n - a^n = (x-a)(x^{n-1} + ax^{n-2} + a^2x^{n-3} + \cdots + a^{n-2}x + a^{n-1})}$$

- A sum of nth powers for an <u>odd</u> integer $n>0$:

$$\boxed{x^n + a^n = (x+a)(x^{n-1} - ax^{n-2} + a^2x^{n-2} - \cdots - a^{n-2}x + a^{n-1})}$$

Polynomials and Rational Functions

EXAMPLE Find the roots of the following polynomials:

(a) $x^3 - x^2 - 4x + 4$, (b) $x^4 + 3x^2 - 4$, (c) $x^5 - x^4 - x^2 + x$.