535.641 Mathematical Methods Assignment 1

Ben Minnick **Name**________________________________

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1. Find the solutions (x_1, x_2, x_3) for the linear system with various parameters (a, b).

$$\begin{aligned} 2x_1 + 3x_2 + ax_3 &= 1 \\ 2x_1 + 3x_2 + 2x_3 &= b \\ 4x_1 + 3x_2 + 2x_3 &= 4 \end{aligned}$$

   (a) $a = 4, \forall b \in \mathbb{R}$
   (b) $a = 2, b = 1$
   (c) $a = 2, \forall b \in \mathbb{R} \ni b \neq 1$

   Perform row reduction starting with the augmented matrix and ending with reduced row echelon form. Show each row reduction step.

   If the system has no solutions, explain why. If the system is consistent, find all solutions. In the case of infinitely many solutions, use parameters.

2. Let $A \in \mathbb{R}^{3\times 3}$ be an invertible matrix, and let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$ be column vectors. Define:

$$B = A + \mathbf{u}\mathbf{v}^T$$

Suppose A^{-1} is known and $1 + \mathbf{v}^T A^{-1}\mathbf{u} \neq 0$.

The **Sherman–Morrison identity** states:

$$\left(A + \mathbf{u}\mathbf{v}^T\right)^{-1} = A^{-1} - \frac{A^{-1}\mathbf{u}\mathbf{v}^T A^{-1}}{1 + \mathbf{v}^T A^{-1}\mathbf{u}}$$

This identity is useful to compute the inverse of a rank-1 update to a matrix whose inverse has previously been computed which for large matrices can result in considerable computational savings.

(a) Let

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

(i) Compute the rank of A

(ii) Compute the rank of $\mathbf{u}\mathbf{v}^T$

(b)

(i) Compute A^{-1}

(ii) Use the Sherman–Morrison identity to compute B^{-1}

(iii) Compute B^{-1} directly by first computing B, then inverting it. Verify that the result matches part (ii).

3. Consider the matrix A and vector $\mathbf{b}$.

$$A = \begin{bmatrix} 1 & -2 & -2 & -3 \\ 3 & -9 & 0 & -9 \\ -1 & 2 & 4 & 7 \\ -3 & -6 & 26 & 2 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 1 \\ 6 \\ 0 \\ 3 \end{bmatrix}$$

(a) Decompose the matrix $A = LU$ such that L and U are respectively lower and upper triangular matrices.

(b) Use your decomposition $A = LU$ to solve the system $A\mathbf{x} = \mathbf{b}$.

4. Consider a rectangular metal plate with known boundary temperatures. Assume the material is isotropic and the mesh (shown below) is uniform.

   Use the mean value property for heat conduction to determine the temperature at all interior nodes.

   Hint: See section 1.6 of Dr. Nakos' lectures notes.