Com S 311: Big O

Fall 20

1 Asymptotic Analysis

Let f and g are two functions from natural numbers to natural numbers.

We say that $f(n) \in O(g(n)$, if there exist a constant $c > 1$ and a natural number N such that for every $n > N$, $f(n) \leq cg(n)$. Even though this is the precise definition, in this notes we omit the phrase "and a natural number N such that for every $n > N$" and replace with for every n. This will make it bit easier to understand the concept of Big-O. The intuitive meaning of $f(n) \in O(g(n))$ is " f(n) is *smaller or equal* to g(n) if we *scale up* g(n) by a constant c". As such $f(n)$ may not be smaller than $g(n)$, however if we scale up $g(n)$ by multiplying by a constant $c \geq 1$, then it is smaller than $g(n)$.

Let us prove Big-O relationships among some functions.

Let us now prove that $3n^2 + 25n + 45 \in O(n^2)$. Note that $3n^2 + 25n + 45 \leq 3n^2 + 25n^2 + 45n^2 = 73n^2$. Now, let us take $c = 73$. Now, for every n

$$\begin{aligned}(3n^2 + 25n + 45) &\leq 73n^2 \\ &= c \times n^2\end{aligned}$$

Thus $3n^2 + 25n + 45 \in O(n^2)$.

We can prove Big-O relations using Calculus. We can use the following lemma to bring techniques from calculus to bear on this problem.

Lemma 1. *Let f and g be functions from the natural numbers to the natural numbers. If*

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$$

then $f \in O(g)$

You are not expected to prove this. This lemma will be helpful, especially in conjunction with L'Hopital's rule. In case you?ve forgotten, this is (a weaker version) of this rule.

Lemma 2. *(L' Hopital's rule). Let f and g be differentiable functions from the real numbers to the real numbers. If*

$$\lim_{n\to\infty} \frac{f'(n)}{g'(n)} = 0, \ \textit{and}$$

$$\lim_{n\to} f(n) = \infty, \ \ \textit{and}$$

$$\lim_{n\to} g(n) = \infty$$

then

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$$

Lets us use this prove the $\log x \in O(x^a)$ for every $a > 0$. Here log is to base 2. First note that the real valued functions $f(x) = \log x$ and $g(x) = n^a$ are differentiable, and

$$\lim_{x\to\infty} f(x) = \infty, \text{ and}$$

$$\lim_{x\to\infty} g(x) = \infty$$

We also know from calculus that

$$f'(x) = \frac{1}{\ln(2)x}$$

$$g'(x) = ax^{a-1}$$

So

$$\begin{aligned} \frac{f'(x)}{g'(x)} &= \frac{\frac{1}{\ln(2)x}}{ax^{a-1}} \\ &= \frac{1}{\ln(2)x^a} \end{aligned}$$

Now we know that $\lim_{n\to\infty} \frac{1}{\ln(2)x^a} = 0$. Thus $logx \in O(x^a)$.

We can show the following relations using the above method $n \in O(n \log n)$, $n \in O(n^2)$, $n^2 \in O(n^3) \cdots$. In general, we can show that $n^a \in O(n^b)$ for any real numbers $0 < a < b$. We can also that for every $k > 0$, $n^k \in O(2^n)$ using the above method. Try them on your own.

Let us prove couple of non-Big O relations. Lets prove that $n^3 \notin O(n^2)$. We prove by contradiction. Assume that $n^3 \in O(n^2)$. Thus there exist c and N such that

$$\forall n > N, n^2 \leq cn^3$$

this implies that

$$\forall n > N, n \leq c$$

Note that c is a constant thus it can not be that for every $n > N$, $n \leq c$. This is a contradiction. Thus $n^3 \notin O(n^2)$.

Now let us prove that $2^n \notin O(n^4)$. We prove by contradiction. Suppose not. Then there exists a constant c and N such that for every $n > N$, $2^n \leq cn^4$. Let us take log both sides. This implies that for every $n > N$, $n \leq \log c + 4 \log n$ for some c which implies that for every $n > N$, $n - 4 \log n \leq \log c$. Note that $\log c$ is a fixed constant (as c is a constant). However, $n - 4 \log n$ is an (monotonically) increasing function. Thus the difference between n and $4 \log n$ can not be bounded by any constant. This is a contradiction. Thus $2^n \notin O(n^4)$.