

COMS 474  
Homework 2

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Spring 2022

## Directions

- Due: Thursday February 10, 2022 at 9pm. Late submissions will be accepted for 24 hours after that time, with a 15% penalty. (the enforcement is strict, beginning at 9:01pm, except for extreme situations; having a poor wifi connection or minor computer problems is not sufficient for the penalty to be waived.)
- Upload the homework to Canvas as a single pdf file.
- If the graders cannot easily read your submission (writing is illegible, image is too dark, or if the contrast is too low) then you might receive a zero or only partial credit.
- Any non-administrative questions must be asked in office hours or (if a brief response is sufficient) Piazza.

## Problem 1

[13 points total (6,3,4)]

Book problem Chapter 3, problem 3 “Suppose we have a data set with five predictors,  $X_1 = GPA$ ,  $X_2 = IQ$ ,  $X_3 = Level$  1 for College and 0 for High School,  $X_4 =$

Interaction between GPA and IQ, and  $X_5 =$  Interaction between GPA and Level.

The response is starting salary after graduation (in thousands of dollars).

Suppose we use least squares to fit the model, and get  $\hat{\beta}_0 = 50$ ,  $\hat{\beta}_1 = 20$ ,  $\hat{\beta}_2 = 0.07$ ,  $\hat{\beta}_3 = 35$ ,  $\hat{\beta}_4 = 0.01$ ,  $\hat{\beta}_5 = -10$ ”

Note: for interactions, use products, e.g.  $X_4(i) = GPA(i) * IQ(i)$

1. Which answer is correct, and why?
  - (i) For a fixed value of IQ and GPA, high school graduates earn more, on average, than college graduates.
  - (ii) For a fixed value of IQ and GPA, college graduates earn more, on average, than high school graduates.
  - (iii) For a fixed value of IQ and GPA, high school graduates earn more, on average, than college graduates provided that the GPA is high enough.
  - (iv) For a fixed value of IQ and GPA, college graduates earn more, on average, than high school graduates provided that the GPA is high enough.

Answer *iii* is correct, by expanding the formula we get

$$\hat{Y} = \beta_0 + \beta_1 * (X_1) + \beta_2 * (X_2) + \beta_3 * (X_3) + \beta_4 * (X_4) + \beta_5 * (X_5)$$

And we know  $X_4 = X_1 * X_2$  and that  $X_5 = X_1 * X_3$  we can sub these in along with the  $\beta$  values to get

$$\hat{Y} = 50 + 20 * (X_1) + 0.07 * (X_2) + 35 * (X_3) + 0.01 * (X_1 * X_2) - 10 * (X_1 * X_3)$$

From there we can plug in values we know, such as  $X_3$  and see how it looks for highschoolers vs college students

Highschool:

$$\hat{Y} = 50 + 20 * (X_1) + 0.07 * (X_2) + 35 * (0) + 0.01 * (X_1 * X_2) - 10 * (X_1 * 0)$$

$$\hat{Y} = 50 + 20 * (X_1) + 0.07 * (X_2) + 0.01 * (X_1 * X_2)$$

College:

$$\hat{Y} = 50 + 20 * (X_1) + 0.07 * (X_2) + 35 * (1) + 0.01 * (X_1 * X_2) - 10 * (X_1 * 1)$$

$$\hat{Y} = 85 + 10 * (X_1) + 0.07 * (X_2) + 0.01 * (X_1 * X_2)$$

from this we can get down to if the GPA is less than 3.5, then college students would have more, however with a GPA of 3.5 or greater, a highschool student would have more.

2. Predict the salary of a college graduate with IQ of 110 and a GPA of 4.0.

Using the formula derived from above and assigning the following:

$X_2 = 110$  and  $X_1 = 4.0$  we get  $\hat{Y} = 85 + 10 * (4.0) + 0.07 * (110) + 0.01 * (4.0 * 110)$

$\hat{Y} = 137.1$  or about \$137,100

3. True or false: Since the coefficient for the GPA/IQ interaction term is very small, there is very little evidence of an interaction effect. Justify your answer.

False, the coefficient does not provide evidence for or against the interaction. We would need to test for a proper conclusion

## Problem 2

[12 points total (3 points each)]

Book problem Chapter 3, problem 4 “I collect a set of data ( $n = 100$  observations) containing a single predictor and a quantitative response. I then fit a linear regression model to the data, as well as a separate cubic regression, i.e.  $Y = \beta_0 + \beta_1 X + \beta_2 X^2 + \beta_3 X^3 + \epsilon$ .”

1. Suppose that the true relationship between  $X$  and  $Y$  is linear, i.e.  $Y = \beta_0 + \beta_1 X + \epsilon$ . Consider the training residual sum of squares (RSS) for the linear regression, and also the training RSS for the cubic regression. Would we expect one to be lower than the other, would we expect them to be the same, or is there not enough information to tell? Justify your answer.  
For the training data, it is harder to tell since a model with 3 features would be more flexible than a linear one, thus the RSS should be lesser in comparison. However, since it truly is linear, we may be fitting excessively and including noise/error.
2. Answer (a) using test rather than training RSS.
3. Suppose that the true relationship between  $X$  and  $Y$  is not linear, but we don't know how far it is from linear. Consider the training RSS for the linear regression, and also the training RSS for the cubic regression. Would we expect one to be lower than the other, would we expect them to be the same, or is there not enough information to tell? Justify your answer.
4. Answer (c) using test rather than training RSS.

### Problem 3

Suppose we have a data set with one feature  $X$  to predict another feature  $Y$ . Let  $n$  denote the number of samples. Let  $\bar{X}$  and  $\bar{Y}$  denote the average values of  $X$  and  $Y$  respectively in the dataset:

$$\bar{X} = \frac{1}{n} \sum_{i=1}^n X(i) \quad \bar{Y} = \frac{1}{n} \sum_{i=1}^n Y(i)$$

Let  $\beta_0^*$  and  $\beta_1^*$  denote the coefficients for the ordinary least squares (O.L.S) solution,

$$\{\beta_0^*, \beta_1^*\} = \hat{f}(x, y) = \arg \min_{\{\beta_0^*, \beta_1^*\}} \frac{1}{n} \sum_{i=1}^n \left( Y(i) - \beta_0 + \beta_1 * X(i) \right)^2$$

Their values are

$$\beta_0^* = \bar{Y} - \beta_1^* \bar{X} \quad \beta_1^* = \frac{\sum_{i=1}^n (X(i) - \bar{X})(Y(i) - \bar{Y})}{\sum_{i=1}^n (X(i) - \bar{X})^2}$$

Using those formulas, calculate the O.L.S. model's prediction for  $X = \bar{X}$  (i.e., the prediction  $\bar{Y}$  for a new sample for whose  $X$  feature has the value  $\bar{X}$ )

## Problem 4

Book problem Chapter 6, problem 1 “We perform best subset, orward step-wise, and backward stepwise selection on a single data set. For each approach, we obtain  $p + 1$  models, containing  $0, 1, 2, \dots, p$  predictors. Explain your answers”

Notes regarding the book’s pseudocode for Algorithms 6.1-6.3:

- “RSS” stands for “residual sum of squares” which is the (un-normalized) MSE,  
$$RSS = \sum_{i=1}^n (Y(i) - \bar{Y})^2$$

In the book’s pseudo-codes, “RSS” refers to training set RSS.

- “cross-validated prediction error” - you can read this as “Validation set MSE.”
- “ $R^2$ ” and “adjusted  $R^2$ ” - you can ignore these for this homework.

1. Which of the three models with  $k$  predictors has the smallest *training* RSS?
2. Which of the three models with  $k$  predictors has the smallest *test* RSS?
3. True or False:
  - (i) The predictors in the  $k$ -variable model identified by forward step-wise are a subset of the predictors in the  $(k + 1)$ -variable model identified by forward stepwise selection.
  - (ii) The predictors in the  $k$ -variable model identified by back-ward stepwise are a subset of the predictors in the  $(k + 1)$ -variable model identified by backward stepwise selection.
  - (iii) The predictors in the  $k$ -variable model identified by back-ward stepwise are a subset of the predictors in the  $(k + 1)$ -variable model identified by forward stepwise selection.
  - (iv) The predictors in the  $k$ -variable model identified by forward step-wise are a subset of the predictors in the  $(k + 1)$ -variable model identified by backward stepwise selection.

- (v) The predictors in the  $k$ -variable model identified by best subset are a subset of the predictors in the  $(k + 1)$ -variable model identified by best subset selection.