COMS 474

Homework 4

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Problem 1

[24 points total (5,5,3,3,4,4)]
Suppose you use lasso fit to a linear model for a data set. Let $\beta^*(\lambda)$ denote the lasso solution for a specific λ (i.e. the coeffcient vector you get for that λ).

Provide explanations for your answers to the following questions.

Part A

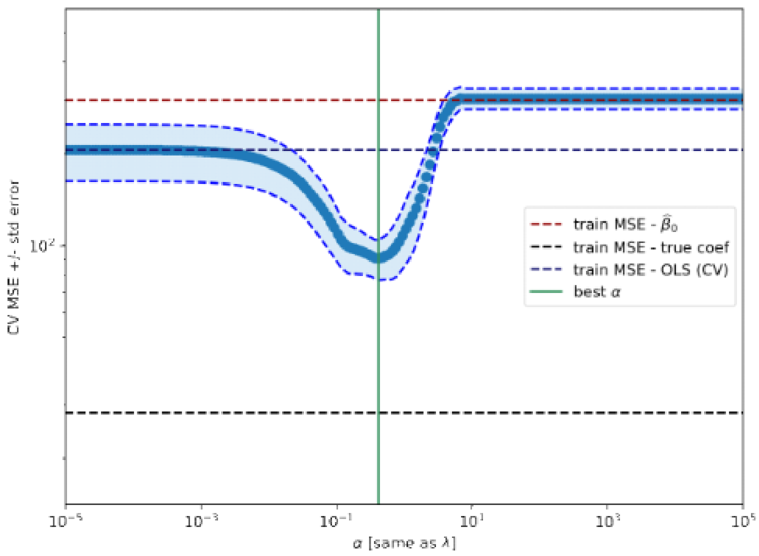
Describe how the training MSE changes as a function of λ, including $\lambda = 0$ and as $\lambda \to \infty$

The training MSE of the function $\beta^*(\lambda)$ would return the least squares fit when $\lambda = 0$. The value output does change though as λ grows in size, with the upper bound of ∞ being the most ideal constant that could be supplied to the equation, this is because at *some* point, λ will hit the absolute minimium of the function, which would be within the range of $0 \to \infty$ The training MSE would be higher overall.

Part B

Describe how the hold-out MSE changes as a function of λ, including $\lambda = 0$ and as $\lambda \to \infty$

The hold-out MSE *should* look similar to the training MSE and by applying a similar set of logic, the models should look relatively the same as it is from the same overall set of data. The resulting model should have an absolute minimium which would indicate what the most ideal λ would be. This value should be, if not exactly, then very close to where the training set's absolute minimium should be at. The image I have attached below is from the notes and displays what the models could look like as an example. A very large λ would cause the model to jump signifigantly. The hold-out MSE would be lower overall.



held out data MSE

λ

λ = 0 O.L.S.

Lambda --> infinity best constant

Part C

Describe $\beta^*(0)$.

When initially at $\beta^*(0)$, the function's value that it would return would simply be the L.S.F (least squares fit) value. The β changes as the λ changes as the solver tries to keep track of the penalties

Part D

Describe what happens to $\beta^*(0)$ as λ grows.

The model $\beta^*(0)$ would cover every possible value between $0 \rightarrow \infty$ and would eventually hit the functions aboslute minimium in a concave region. This is where it deviates off the null model and no longer just returns zero, and insteal will indicate what the absolute minimium, or the most ideal λ, would be.

Part E

If you used ridge regression instead of lasso, explain how your answers to (a).-(d). would differ.

If we were to use ridge regression, we should expect a bit of variance in the data since they approach model slightly differently. Another attribute of ridge regression is that it is not capable to have sharp points in the model, thus intersections at an axis is not common. This subsequently results in non-zero estimations. The models generated by this when comparing the training vs the hold-out data should look very similar to each other due to data distribution. Ridge regression looks at both extremes and steps through the function looking at each step from $0 \to \infty$. It also penalises $\lambda \to \infty$ signifigantly more than it would $\lambda = 0$. When$\lambda = 0$ it would return the Least Squares estimation, while when $\lambda \to \infty$ it estimates the model will approach zero and the shrinkage penalty grows.

Part F

We discussed the "constrained form" of lasso, with a constraint of the form

$$\sum_{i=1}^{p} |\beta_j| \leq t$$

Which value, or limiting value, of t coresponds to $\lambda = 0$ and which corresponds to $\lambda \to \infty$

The value of t would equate to $|\beta_1|^1 + |\beta_2|^1$ since t represents λ but in a constrained environment instead. The point at which the red oval from figure 6.7 from the book (attached below)

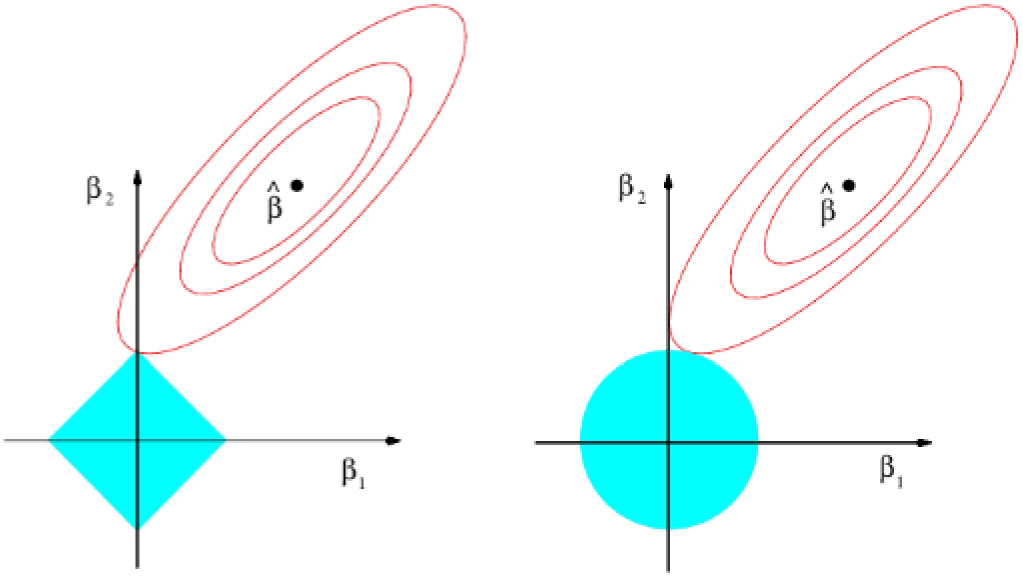


FIGURE 6.7. *Contours of the error and constraint functions for the lasso* (left) *and ridge regression* (right). *The solid blue areas are the constraint regions, $|\beta_1| + |\beta_2| \leq s$ and $\beta_1^2 + \beta_2^2 \leq s$, while the red ellipses are the contours of the RSS.*

Should $\lambda = 0$ then $t = |\beta_1|^1 + |\beta_2|^1$ since we are restricted by $\sum_{j=1}^{p} |\beta_j|$ having to be within the bound set by t and the expected output would quite small. For when $\lambda \to \infty$, then $t = 0$ since the solution of the Least Squares Fit since it would be encompassed by the range set by t.

Problem 2

[15 points]
You have already seen formulas for the best intercept in linear models when there are no features $p = 0$ and a single feature $p = 1$. You will now look at what happens with p features when we center the data.
Recall that "centering" a feature means subtracting its mean. For example, if the sample values for feature X_4 are $\begin{bmatrix}5\\0\\1\end{bmatrix}$, which has a mean of 2,
we could replace it with $\begin{bmatrix}5-2\\0-2\\1-2\end{bmatrix} = \begin{bmatrix}3\\-2\\-1\end{bmatrix}$ which has a mean of 0. Thus if feature X_4 is centred, then $\sum_{i=1}^{n} X_4(i) = 0$.

What is the value of the intercept β_0^* in the ordinary least squares solution, i.e.

$$(\beta_0^*, \beta_1^*, \ldots, \beta_p^*) = \underset{(\beta_0^*, \beta_1^*, \ldots, \beta_p^*) \epsilon \mathbb{R}^{p+1}}{\arg\min} \frac{1}{n} \sum_{i=1}^{n} \left(Y(i) - \beta_0 - \sum_{j=1}^{p} \beta_j X_j(i) \right)^2$$

when the features $X_1, \ldots, X_p$ are all centered? (e.g. $\sum_{i=1}^{n} X_j(i) = 0$ for $j = 1, \ldots, p$). (You do not need to use a second derivative test or solve for $\{\beta_1^*, \ldots, \beta_p^*\}$, just use the first derivative test $0 = \frac{\partial}{\partial \beta_0}$ MSE).

Starting with taking the partial

$$\frac{\partial}{\partial \beta_0} = \frac{2}{n} \sum_{i=1}^{n} \left(Y(i) - \beta_0 - \sum_{j=1}^{p} \beta_j X_j(i) \right) = 0$$

We then split up the summation of B_j and X_j

$$\frac{2}{n} \sum_{i=1}^{n} \left(Y(i) - \beta_0 - \sum_{j=1}^{p} \beta_j * \sum_{j=1}^{p} X_j(i) \right) = 0$$

We also have that $\sum_{i=1}^{n} X_j(i) = 0$ for $j = 1, \ldots, p$ which would lead to

$$\frac{2}{n} \sum_{i=1}^{n} (Y(i) - \beta_0 - \sum_{j=1}^{p} \beta_j * 0) = 0$$

$$\frac{2}{n} \sum_{i=1}^{n} (Y(i) - \beta_0 - 0) = 0$$

$$\frac{2}{n} \sum_{i=1}^{n} (Y(i) - \beta_0) = 0$$

Next we multiply each side by $\frac{2}{n}$ to get

$$\sum_{i=1}^{n} (Y(i) - \beta_0) = 0$$

From here we take β_0 out of the summation

$$\sum_{i=1}^{n} (Y(i)) - \beta_0 * n = 0$$

We can next move it over to the other side of the equation

$$\sum_{i=1}^{n} (Y(i)) = \beta_0 * n$$

Divide by n...

$$\frac{\sum_{i=1}^{n} (Y(i))}{n} = \beta_0$$

This end result of $\beta_0 = \frac{\sum_{i=1}^{n}(Y(i))}{n}$ or in other words, when β_0 is equal to the average value of all the samples is where the intercept would be at for the model when all of the features are centred.