Matrix determinant

Let $A = (a_{ij})_{n \times n}$ be a square matrix of order n then the $|A|$ called the determinant of the matrix A

(I) Determinant of 2x2 matrix

Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, then $|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$

$$|A| = a_{11} . a_{22} - a_{21} . a_{12}$$

(II) Determinant of 3x3 matrix

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then $|A| = \begin{array}{c} \begin{matrix} + & - & + \end{matrix} \\ \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \end{array}$

$$\begin{array}{c} \begin{matrix} + & - & + \end{matrix} \\ \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & \boxed{a_{22}} & \boxed{a_{23}} \\ a_{31} & \boxed{a_{32}} & \boxed{a_{33}} \end{vmatrix} \end{array} \Rightarrow a_{11} (a_{22} a_{3} - a_{32} a_{23})$$

$$\begin{array}{c} \begin{matrix} + & - & + \end{matrix} \\ \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ \boxed{a_{21}} & a_{22} & \boxed{a_{23}} \\ \boxed{a_{31}} & a_{32} & \boxed{a_{33}} \end{vmatrix} \end{array} \Rightarrow - a_{12} (a_{21} a_{3} - a_{31} a_{23})$$

$$\begin{array}{c} \begin{matrix} + & - & + \end{matrix} \\ \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ \boxed{a_{21}} & \boxed{a_{22}} & a_{23} \\ \boxed{a_{31}} & \boxed{a_{32}} & a_{33} \end{vmatrix} \end{array} \Rightarrow a_{13} (a_{21} a_{32} - a_{31} a_{32})$$

$$\therefore |A| = a_{11}(a_{22}a_{3} - a_{32}a_{23}) - a_{12}(a_{21}a_{3} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{31}a_{32})$$

Question : 01

Calculate the determinants of the following matrices

$$A = \begin{bmatrix} 2 & 3 & 7 \\ 5 & 6 & 9 \\ -3 & 2 & 8 \end{bmatrix} = 2(48-18) - 3(40+27) + 7(10+18)$$

$$= 60 - 201 + 196$$

$$= 256 - 201$$

$$= 55 \,//$$

$$B = \begin{bmatrix} 4 & 2 & 1 \\ 2 & 6 & 2 \\ 2 & 9 & 7 \end{bmatrix} = 4(42-18) - 2(14-4) + 1(18-12)$$

$$= 96 - 20 + 6$$

$$= 82 \,//$$

$$C = \begin{bmatrix} 4 & 2 \\ 5 & 6 \end{bmatrix} = 24 - 10 = 14 \,//$$

Properties of Determinant

A - $|A| = |A^t|$

B- Let A be a squire matrix

- If a row or Column is Completely zero, $|A| = 0$

$$\begin{vmatrix} 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{3} \end{vmatrix} = 0 \quad , \quad \begin{vmatrix} 0 & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{vmatrix} = 0$$

- If two rows or Columns is identical, $|A| = 0$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 0 \quad , \quad \begin{vmatrix} a_{11} & a_{11} & a_{13} \\ a_{21} & a_{21} & a_{23} \\ a_{31} & a_{31} & a_{33} \end{vmatrix} = 0$$

C- If A is a triangular matrix, then $|A|$ is product of diagonal elements.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{vmatrix} = a_{11} . a_{22} . a_{33}$$

D- If A is a squre matrix of order n and k is scalar, then $|KA| = k^n |A|$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3\times 3} , \quad |6A| = 6^3 \times |A|$$

Question :

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 4 & 0 \\ 3 & 5 & 7 \end{bmatrix}, \quad |C| = 1 \times 4 \times 7 = 28$$

Minor and Cofactors

$A = (a_{ij})_{n \times n}$ be a square matrix. Then M_{ij} denote a sub matrix of A with order $(n-1) \times (n-1)$ obtained by deleting its i^{th} row and j^{th} Column. the determinant $|M_{ij}|$ is called the minor of the element a_{ij} of A.

The cofactor of a_{ij} denoted by A_{ij} and is equal to $(-1)^{i+j} |M_{ij}|$

- we select the place
- if it is $(2,3)$ we delet 2nd row and 3rd Column.
- Then find the determinant for the remaining matrix. $\Rightarrow M_{2,3}$
- $|M_{2,3}|$ are the minors.
- Cofactor of $a_{2,3} \Rightarrow$ denoted as $A_{2,3}$

$$A_{2,3} = (-1)^{2+3} \cdot |M_{2,3}|$$

Example :

$$A = \begin{bmatrix} 3 & 1 & -1 \\ 2 & -2 & 0 \\ 1 & 2 & -1 \end{bmatrix}$$

$$M_{11} = \begin{bmatrix} -2 & 0 \\ 2 & -1 \end{bmatrix}, \quad |M_{11}| = 2, \quad A_{11} = 2$$

$$M_{12} = \begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix}, \quad |M_{12}| = -2, \quad A_{12} = 2$$

$$M_{13} = \begin{bmatrix} 2 & -2 \\ 1 & 2 \end{bmatrix}, \quad |M_{13}| = 4+2=6, \quad A_{13} = 6$$

$$M_{21} = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}, \quad |M_{21}| = -1+2=1, \quad A_{21} = -1$$

$$M_{22} = \begin{bmatrix} 3 & -1 \\ 1 & -1 \end{bmatrix}, \quad |M_{22}| = -3+1=-2, \quad A_{22} = -2$$

$$M_{23} = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}, \quad |M_{23}| = 6-1=5, \quad A_{23} = -5$$

$$M_{31} = \begin{bmatrix} 1 & -1 \\ -2 & 0 \end{bmatrix}, \quad |M_{31}| = -2, \quad A_{31} = -2$$

$$M_{32} = \begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix}, \quad |M_{32}| = 2, \quad A_{32} = -2$$

$$M_{33} = \begin{bmatrix} 3 & 1 \\ 2 & -2 \end{bmatrix}, \quad |M_{33}| = -8, \quad A_{33} = -8$$

$$A_{ij} = \begin{bmatrix} 2 & 2 & 6 \\ -1 & -2 & -5 \\ -2 & -2 & -8 \end{bmatrix}$$

Singular Matrix

- If A is a square matrix and $|A| = 0$
  Then it is a singular matrix.

Adjoin Matrix

$$\text{adj } A = (A_{ij})^t$$

$$A \cdot I = A \quad *$$

$$A(\text{adj } A) = (\text{adj } A)A = |A| \cdot I \quad *$$

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Inverse of Matrix

- If A and B are two matrices such that $AB = BA = I$

  then A, B is said to be the inverse of the each other.

  If $AB = BA = I$

  $A = B^{-1}$
  $B = A^{-1}$

- A^{-1} is the inverse of A

Existance of the inverse

if $|A| \neq 0$, then only matrix A can have an inverse.
Or in other words A shold be a non-singular matrix.

- If two columns or two rows are identical or if a column or row is completly zero, It's a singular matrix ($|A| = 0$) So, It doesn't have inverse.

Reverse law of the inverse of product

- If A and B are to non-singular matrices, then (AB) is also non singular

$$(AB)^{-1} = B^{-1} \cdot A^{-1}$$

A, B Inter Changed.
· B first

- If A is non-singular matrix, then

$$A(A^{-1}) = I \quad \text{——— } (2)$$

From (1) $\Rightarrow$ $(\text{adj } A) \cdot A = |A| \cdot I$

$$\frac{(\text{adj } A)}{|A|} = \frac{I}{A} \quad \text{——— } (3)$$

From (2) $\Rightarrow$ $A \cdot (A^{-1}) = I$

$$A^{-1} = \frac{I}{A} \quad \text{——— } (4)$$

By equating (3), (4) $\Rightarrow$

$$\frac{(\text{adj } A)}{|A|} = A^{-1}$$

Question: From $A = \begin{bmatrix} 3 & 1 & -1 \\ 2 & -2 & 0 \\ 1 & 2 & -1 \end{bmatrix}$

$$A_{ij} = \begin{bmatrix} 2 & 2 & 6 \\ -1 & -2 & -5 \\ -2 & -2 & -8 \end{bmatrix}$$

$$\text{adj } A = (A_{ij})^t$$

$$\text{adj } A = \begin{bmatrix} 2 & -1 & -2 \\ 2 & -2 & -2 \\ 6 & -5 & -8 \end{bmatrix}$$

$$|A| = -2(-1+2) - 2(-3+1)$$

$$= -2 + 4$$

$$|A| = 2 \,//$$

$$A^{-1} = \frac{\text{adj } A}{|A|} = \begin{bmatrix} 1 & -1/2 & -1 \\ 1 & -1 & -1 \\ 3 & -5/2 & -4 \end{bmatrix}$$

Equations

* $A_{ij} = (-1)^{i+j} \cdot |M_{ij}|$

$$A_{ij} = \begin{bmatrix} (-1)^{1+1} \cdot |M_{1,1}| & (-1)^{1+2} \cdot |M_{1,2}| & (-1)^{1+3} \cdot |M_{1,3}| \\ (-1)^{2+1} \cdot |M_{2,1}| & (-1)^{2+2} \cdot |M_{2,2}| & (-1)^{2+3} \cdot |M_{2,3}| \\ (-1)^{3+1} \cdot |M_{3,1}| & (-1)^{3+2} \cdot |M_{3,2}| & (-1)^{3+3} \cdot |M_{3,3}| \end{bmatrix}$$

* $\text{adj } A = (A_{ij})^t$

* $(\text{adj } A) A = |A| \cdot I$

* $A \cdot A^{-1} = I$

* $A^{-1} = \dfrac{\text{adj } A}{|A|}$

Question: $A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 2 & 2 \\ 1 & 1 & 2 \end{bmatrix}$

$|M_{11}| = 4-2 = 2 \quad (+) \quad , \quad A_{11} = 2$

$|M_{21}| = -1 \quad (-) \quad , \quad A_{21} = 1$

$|M_{31}| = -2 \quad (+) \quad , \quad A_{31} = -2$

$(-)\ |M_{12}| = -2-2 = -4, \quad A_{12} = 4$

$(+)\ |M_{22}| = 2-1 = 1, \quad A_{22} = 1$

$(-)\ |M_{32}| = 2+1 = 3, \quad A_{32} = -3$

$(+)\ |M_{13}| = -1-2 = -3, \quad A_{13} = -3$

$(-)\ |M_{23}| = 1, \quad A_{23} = -1$

$(+)\ |M_{33}| = 2, \quad A_{33} = 2$

$$A_{ij} = \begin{bmatrix} 2 & 4 & -3 \\ 1 & 1 & -1 \\ -2 & -3 & 2 \end{bmatrix}$$

$$\operatorname{adj} A = (A_{ij})^t = \begin{bmatrix} 2 & 1 & -2 \\ 4 & 1 & -3 \\ -3 & -1 & 2 \end{bmatrix}$$

$$\begin{aligned} |A| &= 1(4-2) + 1(-1-2) \\ &= 2 - 3 \\ &= -1 \end{aligned}$$

$$A^{-1} = \frac{\operatorname{adj} A}{|A|} = \begin{bmatrix} -2 & -1 & 2 \\ -4 & -1 & 3 \\ 3 & 1 & -2 \end{bmatrix}$$

Solving multiple variable equation using Matrix

$$3x_1 + 2x_2 + 3x_3 = 2$$

$$2x_1 + 1x_2 - x_3 = 6$$

$$4x_1 + 2x_2 - 3x_3 = 7$$