# Eigen Decomposition

## Eigen Values

I A is a matrix, then a non-zero vector $x$ in $R^n$ is called eigen vector of A if $Ax = \lambda x$ for som scalar. the scalar $\lambda$ is called an eigen value of A,

- If we want to have eigen values for matrix A, $|A - \lambda I| = 0$ should be satisfied.

Proof

If $\lambda$ is an eigen value of A, the there exist a non-zero vector $x$ such tact

$Ax = \lambda x$

$Ax = \lambda x$

$Ax = \lambda I x$ (where $I$ is an identity matrix)

$(A - \lambda I) x = 0$

$\therefore$ Here, $A - \lambda I$ should be zero.

* if $|A| = 0$ it is a non trivial solution.

Example :

$$A = \begin{bmatrix} 2 & 7 \\ 1 & -2 \end{bmatrix}_{2\times 2}$$

$$|A - \lambda I| = \left| \begin{bmatrix} 2 & 7 \\ 1 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right|$$

$$= \begin{vmatrix} 2-\lambda & 7 \\ 1 & -2-\lambda \end{vmatrix} = (2-\lambda)(-2-\lambda) - 7$$

$$= (2-\lambda)(-2-\lambda) - 7 = 0$$

$$-4 - 2\lambda + 2\lambda + \lambda^2 - 7 = 0$$

$$\lambda^2 = -11$$

$$\lambda = \pm\sqrt{11}$$

$$\therefore \lambda_1 = \sqrt{11}$$

$$\lambda_2 = -\sqrt{11}$$

Question : 1

$$B = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 4 \\ -1 & -1 & -2 \end{bmatrix}$$

B to have eigen values, $|B - \lambda I| = 0$

$$\left| \begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 4 \\ -1 & -1 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right| = 0$$

$$\begin{vmatrix} 2-\lambda & 1 & 1 \\ 2 & 3-\lambda & 4 \\ -1 & -1 & -2-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)\left[(3-\lambda)(-2-\lambda)+4\right) - (-4-2\lambda+4) + (-2+3-\lambda) = 0$$

$$= (2-\lambda)\left[(3-\lambda)(-2-\lambda)+4\right) + \lambda + 1 = 0$$

$$= (2-\lambda)\left[-6-3\lambda+2\lambda+\lambda^2+4)\right] + \lambda+1 = 0$$

$$= (2-\lambda)(\lambda^2-\lambda-2) + \lambda+1 = 0$$

$$= 2\lambda^2 - 2\lambda - 4 - \lambda^3 + \lambda^2 + 2\lambda + \lambda + 1 = 0$$

$$= -\lambda^3 + 3\lambda^2 + \lambda - 3 = 0$$

$$(-\lambda^3+\lambda) + (3\lambda^2-3) = 0$$

$$-\lambda(\lambda^2-1) + 3(\lambda^2-1) = 0$$

$$(\lambda^2-1)(3-\lambda) = 0$$

$$(\lambda+1)(\lambda-1)(3-\lambda) = 0$$

$\therefore \quad \lambda_1 = -1$

$\lambda_2 = 1$

$\lambda_3 = 3$

# Eigon Vector

Example:
$$A = \begin{bmatrix} -5 & 2 \\ -7 & 4 \end{bmatrix}$$

$$|A - \lambda I| = 0$$

$$\left| \begin{bmatrix} -5 & 2 \\ -7 & 4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = 0$$

$$\begin{vmatrix} -5-\lambda & 2 \\ -7 & 4-\lambda \end{vmatrix} = 0$$

$$(-5-\lambda)(4-\lambda) + 14 = 0$$

$$\lambda^2 + 5\lambda - 4\lambda - 20 + 14 = 0$$

$$\lambda^2 + \lambda - 6 = 0$$

$$(\lambda+3)(\lambda-2) = 0$$

$$\therefore \quad \lambda_1 = 2$$
$$\lambda_2 = -3$$

$$A\underline{x} = \lambda \underline{x}$$

$\lambda_2 = -3$

$$\begin{bmatrix} -5 & 2 \\ -7 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -3 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$-5x_1 + 2x_2 = -3x_1$$
$$-7x_1 + 4x_2 = -3x_2$$

$$2x_2 = 2x_1$$
$$x_2 = x_1 = t$$

$$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

eigen vector from $\lambda_2 = -3 \Rightarrow \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda_1 = 2$

$$\begin{bmatrix} -5 & 2 \\ -7 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 2\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$-5x_1 + 2x_2 = 2x_1 \quad \text{——— (1)}$$

$$-7x_1 + 4x_2 = 2x_2 \quad \text{——— (2)}$$

$(1) \Rightarrow \quad 2x_2 = 7x_1$

$(2) \Rightarrow \quad 2x_2 = 7x_1$

$\therefore$

$$R^n = \left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 7/2 \end{bmatrix}\right\}$$

$$x_1 = \frac{2}{7}x_2 = u$$

$$x_2 = \frac{7}{2}u$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} u \\ \frac{7}{2}u \end{bmatrix}$$

$$u\begin{bmatrix} 1 \\ 7/2 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y \\ \frac{7}{2}y \end{bmatrix} = y \begin{bmatrix} 1 \\ 7/2 \end{bmatrix}$$

Question :

$$B = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$$

$$[Bx = \lambda x] \quad \Big| \quad |B - \lambda I| = 0$$

$$\left| \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \right| = 0$$

$$\begin{vmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)\,[(2-\lambda)(3-\lambda) - 2] - 1\,[2 - 2(2-\lambda)]$$

$$= (1-\lambda)\,[(2-\lambda)(3-\lambda) - 2] - 2 + 4 - 2\lambda$$

$$= (1-\lambda)\,[6 - 5\lambda + \lambda^2 - 2] + 2 - 2\lambda$$

$$= (1-\lambda)\left[6-5\lambda+\lambda^2-2\right] + 2(1-\lambda)$$

$$= (1-\lambda)\left[6-5\lambda+\lambda^2-2+2\right]$$

$$= (1-\lambda)\left[\lambda^2-5\lambda+6\right]$$

$$= (1-\lambda)(\lambda-3)(\lambda-2)$$

$$\therefore \quad \lambda_1 = 1$$
$$\lambda_2 = 2$$
$$\lambda_3 = 3$$

$$B\underline{x} = \lambda \underline{x}$$

$\underline{\lambda_1 = 1}$

$$\begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 1\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\begin{bmatrix} x_1 - x_3 \\ x_1 + 2x_2 + x_3 \\ 2x_1 + 2x_2 + 3x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$x_1 - x_3 = x_1$$
$$x_3 = 0$$

$$x_1 + 2x_2 + x_3 = x_2$$
$$x_1 = -x_2$$

$\underline{\text{Assume } x_1 = a}$

$$x_2 = -a$$
$$x_3 = 0$$

$$\therefore \underline{x} = a\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$\lambda_2 = 2$

$$\begin{bmatrix} x_1 - x_3 \\ x_1 + 2x_2 + x_3 \\ 2x_1 + 2x_2 + 3x_3 \end{bmatrix} = 2\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$x_1 - x_3 = 2x_1$$
$$-x_3 = x_1$$

$$2x_1 + 2x_2 + 3x_3 = 2x_3$$

$$2x_1 + 2x_2 = -x_3$$
$$2x_1 + 2x_2 = x_1$$
$$2x_2 = -x_1$$
$$x_2 = -\tfrac{1}{2}x_1$$

$\underline{\text{Assume}}$ $x_1 = t$

$$x_2 = \frac{-t}{2} \qquad \therefore \underline{x} = t\begin{bmatrix} 1 \\ -1/2 \\ -1 \end{bmatrix}$$

$$x_3 = -t$$

$\lambda_3 = 3$

$$\begin{bmatrix} x_1 - x_3 \\ x_1 + 2x_2 + x_3 \\ 2x_1 + 2x_2 + 3x_3 \end{bmatrix} = 3\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$x_1 - x_3 = 3x_1$$
$$-x_3 = 2x_1$$

$$x_1 + 2x_2 + x_3 = 3x_2$$

$$x_1 + 2x_2 - 2x_1 = 3x_2$$

$$-x_1 = x_2$$

Assume $x_1 = m$

$$x_2 = -m$$

$$x_3 = -2m$$

$$\therefore \underline{x} = m \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$$

$$R_n = \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1/2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix} \right\}$$