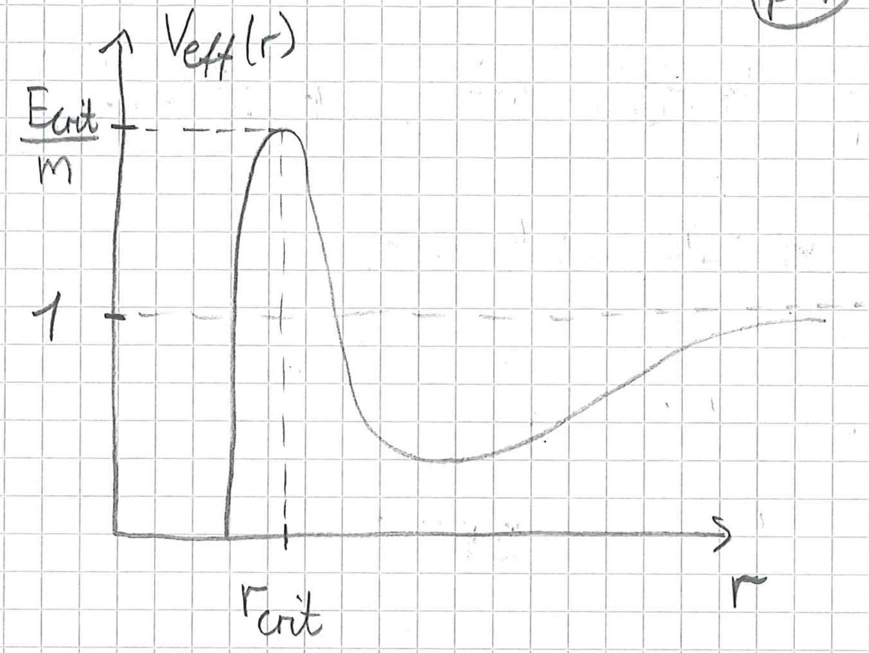
Part 9

p.1

Ex. 6

1.



2. We have $\frac{E}{m} = \left(1 - \frac{2M}{r}\right)\frac{dt}{d\tau}$

We can also write $\frac{dt}{d\tau} = \frac{dt}{dt_{sh}} \cdot \frac{dt_{sh}}{d\tau}$.

This gives

$\frac{E}{m} = \left(1 - \frac{2M}{r}\right)\frac{dt}{dt_{sh}}\frac{dt_{sh}}{d\tau}$. We know that the relationship between the time measured by a far away observer and the time measured by the shell observer is

$\frac{dt}{dt_{sh}} = \frac{1}{\sqrt{1-\frac{2M}{r}}}$. So

$\frac{E}{m} = \sqrt{1 - \frac{2M}{r}}\,\frac{dt_{sh}}{d\tau}$

For short time intervals dt_{sh} we can use special relativity. Then

$d\tau = \Delta t'$ and $dt_{sh} = \Delta t$, so

$$\frac{dt_{sh}}{d\tau} = \frac{\Delta t}{\Delta t'} = \frac{\gamma \Delta t'}{\Delta t'} = \gamma.$$ Where

γ uses the velocity v_{sh}, so we call it γ_{sh}. This gives

$$\boxed{\frac{E}{m} = \sqrt{1 - \frac{2M}{r}}\, \gamma_{sh}}$$

3. We know that in Schwarzschild geometry

$$V_{eff}(r) = \sqrt{\left(1 - \frac{2M}{r}\right)\left[1 + \frac{(L/m)^2}{r^2}\right]}.$$

We want to find the extremes, so we calculate the derivative

$$V_{eff}'(r) = \frac{1}{2\sqrt{\left(1 - \frac{2M}{r}\right)\left[1 + \frac{(L/m)^2}{r^2}\right]}} \cdot \left(\frac{2M}{r^2}\left[1 + \frac{(L/m)^2}{r^2}\right] + \left(1 - \frac{2M}{r}\right)\left(\frac{-2(L/m)^2}{r^3}\right)\right)$$

For the derivative to be 0, we must have

$$\frac{M}{r^2}\left[1+\frac{(L/m)^2}{r^2}\right]-\left(1-\frac{2M}{r}\right)\frac{(L/m)^2}{r^3}=0$$

$$M+\frac{M(L/m)^2}{r^2}+\frac{2M(L/m)^2}{r^2}-\frac{(L/m)^2}{r}=0$$

$$Mr^2-(L/m)^2r+3M(L/m)^2=0$$

$$r=\frac{(L/m)^2\pm\sqrt{(L/m)^4-12M^2(L/m)^2}}{2M}$$

$$r=\frac{(L/m)^2\left(1\pm\sqrt{1-\frac{12M^2}{(L/m)^2}}\right)}{2M}$$

$$\underline{\underline{r=\frac{(L/m)^2}{2M}\left(1\pm\sqrt{1-\frac{12M^2}{(L/m)^2}}\right)}}$$

We see from the graph of the potential that r_{crit} must be

$$\underline{r_{crit}=\frac{(L/m)^2}{2M}\left(1-\sqrt{1-\frac{12M^2}{(L/m)^2}}\right)}$$

4. We know that $\frac{L}{m} = r^2 \frac{d\phi}{d\tau}$.

$\frac{d\phi}{d\tau} = \frac{d\phi}{dt_{sh}} \cdot \frac{dt_{sh}}{d\tau}$. We already found $\frac{dt_{sh}}{d\tau} = \gamma_{sh}$.

$\frac{d\phi}{dt_{sh}}$ is the angular velocity of the object measured by the stationary shell observer. We know that $\frac{d\phi}{dt_{sh}} = \frac{V_{\phi sh}}{R}$, where $V_{\phi sh}$ is the tangential velocity measured by the shell observer. Since the angle between the radial vector and the velocity vector of the rocket is θ, we get $V_{\phi sh} = V_{sh} \sin\theta$. So

$\frac{d\phi}{dt_{sh}} = \frac{V_{sh}\sin\theta}{R}$. this gives

$$\frac{L}{m} = R^2 \frac{d\phi}{d\tau} = R^2 \frac{V_{sh} \sin\theta}{R} \gamma_{sh} =$$

$$\underline{\underline{R \gamma_{sh} V_{sh} \sin\theta}}$$

5. $\frac{L}{m} \doteq R\gamma_{sh} V_{sh} \sin\theta \approx \underline{37{,}824\,M}$

6. We have

$$\Delta r = \pm \sqrt{\left(\frac{E}{m}\right)^2 - \left[1 + \left(\frac{L/m}{r}\right)^2\right]\left(1 - \frac{2M}{r}\right)}\ \Delta\tau .$$

If we assume $L/m = 0$, and use $\Delta\tau = d\tau$ and $\Delta r = dr$, we get

$$d\tau = \frac{dr}{\sqrt{\left(\frac{E}{m}\right)^2 + \frac{2M}{r} - 1}} .$$

We can integrate this from $r = 2M$ to $r = 0$, to find the time T when the astronaut reaches the singularity if he measures $t = 0$ as he crosses the event horizon.

$$\int_{r=2M}^{0} d\tau = \int_{2M}^{0} \frac{dr}{\sqrt{\left(\frac{E}{m}\right)^2 + \frac{2M}{r} - 1}}$$

$$T = \left[\frac{r\sqrt{\left(\frac{E}{m}\right)^2 - 1 + \frac{2M}{r}}}{\left(\frac{E}{m}\right)^2 - 1} - \frac{2M\ln\left(2\sqrt{\left(\frac{E}{m}\right)^2-1}\, r\sqrt{\left(\frac{E}{m}\right)^2 - 1 + \frac{2M}{r}} + 2\left(\left(\frac{E}{m}\right)^2-1\right)r + 2M\right)}{2\left(\left(\frac{E}{m}\right)^2-1\right)^{\frac{3}{2}}} \right]_0^{2M} =$$

$$\lim_{r\to 0^+}\left(\frac{r\sqrt{\left(\frac{E}{m}\right)^2 - 1 + \frac{2M}{r}}}{\left(\frac{E}{m}\right)^2 - 1} - \frac{2M\ln\left(2\sqrt{\left(\frac{E}{m}\right)^2-1}\, r\sqrt{\left(\frac{E}{m}\right)^2 - 1 + \frac{2M}{r}} + 2\left(\left(\frac{E}{m}\right)^2-1\right)r + 2M\right)}{2\left(\left(\frac{E}{m}\right)^2-1\right)^{\frac{3}{2}}} \right) - \left(\frac{2M\frac{E}{m}}{\left(\frac{E}{m}\right)^2-1} - \frac{2M\ln\left(2\sqrt{\left(\frac{E}{m}\right)^2-1}\, 2M\frac{E}{m} + 4M\left(\left(\frac{E}{m}\right)^2-1\right) + 2M\right)}{2\left(\left(\frac{E}{m}\right)^2-1\right)^{\frac{3}{2}}} \right) =$$

$$-\frac{2M\ln(2M)}{2\left(\left(\frac{E}{m}\right)^2-1\right)^{\frac{3}{2}}} - \frac{2M\frac{E}{m}}{\left(\frac{E}{m}\right)^2-1} + \frac{2M\left(\ln 2M + \ln\left(2\sqrt{\left(\frac{E}{m}\right)^2-1}\,\frac{E}{m} + 2\left(\left(\frac{E}{m}\right)^2-1\right) + 1\right)\right)}{2\left(\left(\frac{E}{m}\right)^2-1\right)^{\frac{3}{2}}} =$$

$$\frac{M \ln\left(2\sqrt{\left(\frac{E}{m}\right)^2-1}\,\frac{E}{m} + 2\left(\left(\frac{E}{m}\right)^2-1\right)+1\right)}{\left(\left(\frac{E}{m}\right)^2-1\right)^{\frac{3}{2}}} - \frac{2M\frac{E}{m}}{\left(\frac{E}{m}\right)^2-1}$$

To get normal units we multiply $\frac{E}{m}$ by c^2. We get $\underline{\underline{T \approx 2{,}2 \cdot 10^{19}\,s}}$

7.