Part 9

Ex. 5 1. At one point, the x,y-coordinates of the satellites are

(−18044,213 , 17775,101) and

(−6739,196 , 24415,796). This

gives distances of

$$r_1 = \sqrt{x_1^2 + y_1^2} \approx 25328{,}795 \text{ km}$$

$$r_2 = \sqrt{x_2^2 + y_2^2} \approx 25328{,}795 \text{ km}$$

We get The height over the surface by subtracting the radius of the planet $R \approx 8733{,}662$

$$h = r - R \approx \underline{\underline{16595{,}133 \text{ km}}}$$

2. Since the orbits are circular, we can use

$$F_G = \frac{mv^2}{r}$$

$$\frac{GM}{r^2} = \frac{v^2}{r}$$

$$v = \sqrt{\frac{GM}{r}} \approx 5034{,}029 \text{ m/s}$$

3. $\vec{r}_1 = (21416{,}017 \; , \; -13524{,}129)$

$t_1 = 217{,}4716622$

$\vec{r}_2 = (25308{,}879 \; , \; -1004{,}230)$

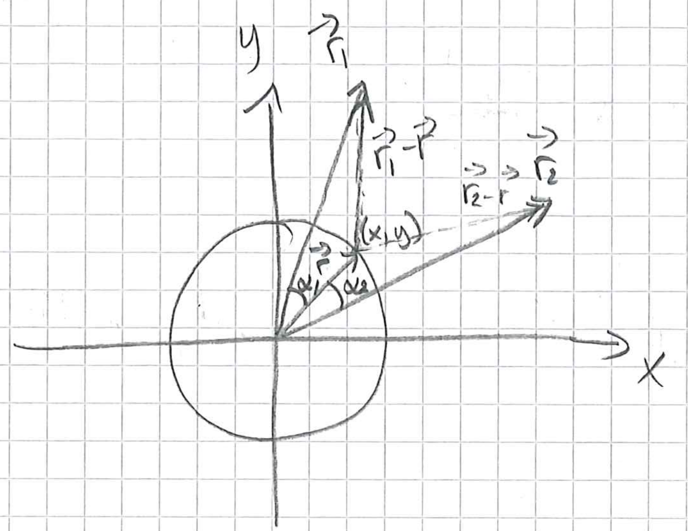
$t_2 = 217{,}4594303$

$\vec{r} = (x, y) \qquad t = 217{,}5634109$

I : $|\vec{r} - \vec{r}_1| = c(t - t_1)$

II : $|\vec{r} - \vec{r}_2| = c(t - t_2)$

We can draw the situation in a figure



We see that the distances $|\vec{r}_1 - \vec{r}|$ and $|\vec{r}_2 - \vec{r}|$ can be described using the law of cosines, and the angles α_1 and α_2 between $\vec{r}$ and $\vec{r}_1$ and $\vec{r}$ and $\vec{r}_2$.

$$|\vec{r}_1 - \vec{r}|^2 = R^2 + r_1^2 - 2Rr_1\cos\alpha_1$$

$$|\vec{r}_2 - \vec{r}|^2 = R^2 + r_2^2 - 2Rr_2\cos\alpha_2$$

which gives

$$c^2(t-t_1)^2 = R^2 + r_1^2 - 2Rr_1\cos\alpha_1$$

$$c^2(t-t_2)^2 = R^2 + r_2^2 - 2Rr_2\cos\alpha_2$$

$$\alpha_1 = \cos^{-1}\left(\frac{R^2 + r_1^2 - c^2(t-t_1)^2}{2Rr_1}\right)$$

$$\alpha_2 = \cos^{-1}\left(\frac{R^2 + r_2^2 - c^2(t-t_1)^2}{2Rr_2}\right)$$

we get $\alpha_1 \approx 1{,}16584521$ and $\alpha_2 \approx 2{,}18205506$

we find the angles between $\vec{r}_1$ and the x-axis and $\vec{r}_2$ and the x-axis by

$$\theta_1 = \tan^{-1} \frac{r_{1y}}{r_{1x}} \qquad \theta_2 = \tan^{-1} \frac{r_{2y}}{r_{2x}}$$

We get $\theta_1 \approx -0{,}563257$ and

$\theta_2 \approx -0{,}039658$.

Since α_1 and θ_1 are the angles between $\vec{r}_1$ and $\vec{r}$ and $\vec{r}_1$ and the x-axis, the angle between $\vec{r}$ and the x-axis must be

$\theta_1 \pm \alpha_1$. Similarly, the angle between $\vec{r}$ and the x-axis must be

$\theta_2 \pm \alpha_2$. We see that

$$\theta_1 - \alpha_1 = \theta_2 - \alpha_2 \approx -2{,}221709 \approx$$

$-127{,}2945552^\circ$. This must therefore be the angle between $\vec{r}$ and the x-axis. To find the coordinates of $\vec{r}$, we use $x = R \cdot \cos\theta_r$, $y = R \cdot \sin\theta_r$

$$\vec{r} = (-5291{,}8375 \ , \ -6947{,}9001)$$

4. We use the Schwarzschild line element to take both general and special relativity effects into account. We get for the planet

observer: $\Delta s^2 = \Delta \tau^2 = \Delta t_1^2 = \left(1 - \frac{2M}{r_1}\right)\Delta t^2$

$- \frac{\Delta r^2}{\left(1 - \frac{2M}{r}\right)} - r^2 \Delta\phi^2$. Since he is standing on the surface $\Delta r = 0$. Also, we assume the planet is not rotating, so $\Delta\phi^2 = 0$. This gives $\Delta t_1^2 = \left(1 - \frac{2M}{R}\right)\Delta t^2$.

Similarly, for an observer on the satellites we get

$$\Delta t_2^2 = \left(1 - \frac{2M}{r}\right)\Delta t^2 - r^2 \Delta \phi^2$$

Dividing both equations by Δt^2, we get

$$\left(\frac{\Delta t_1}{\Delta t}\right)^2 = 1 - \frac{2M}{R} \quad \text{and}$$

$$\left(\frac{\Delta t_2}{\Delta t}\right)^2 = 1 - \frac{2M}{r} - v^2,$$ where we used $\frac{\Delta \phi}{\Delta t} = \omega$, and $v = \omega r$, giving $r^2 \frac{\Delta \phi^2}{\Delta t^2} = v^2$. We divide the two equations by each other:

$$\frac{\Delta t_1}{\Delta t_2} = \sqrt{\frac{1 - \frac{2M}{R}}{1 - \frac{2M}{r} - v^2}}$$

$$\Delta t_1 \approx 0{,}999999999605 \, \Delta t_2 = \Delta t_2 - 3{,}95 \cdot 10^{-10} \Delta t_2$$

This means that when the satellites measured ~217s on their clocks, only $217 \cdot (1 - 3{,}95 \cdot 10^{-10})$ s had passed in the planet frame, so actually the signals have travelled farther than previously calulated. With the new $t - t_1$ and $t - t_2$, we get a position of

$r = (-5291{,}860 \;,\; -6947{,}883)$

which is ~ 28 m from the position we calculated without relativity.

6. We use the same methods again to find our position with and without relativity. In this case the received data is

$\vec{r}_1 = (-19190,887 \quad , \quad 16530,508)$

$t_1 = 19402,6095541$

$\vec{r}_2 = (-8354,542 \quad , \quad 23911,283)$

$t_2 = 19402,5971230$

$\vec{r} = (x, y) \qquad t = 19402,7005561$

Without accounting for relativity we get

$\vec{r} = (-5293,841 \quad , \quad -6946,374)$

This is approximately 2,5 km away from the position calculated earlier using relativity.

When accounting for relativity we get

$\vec{r} = (-5291,869 \quad , \quad -6947,876).$

This is app. 10m away from the previous position, probably because of numerical inaccuracy.