Problem Set 3

Math228B Numerical solutions to differential equations

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Problem 1

Problem 2

Part a

For this part, I simply plug in the equation for the linear transfinite interpolation in the lecture slides, and use $x(\xi) = \xi$ to transform the unit-square into Ω with a linear mesh. The resulting mesh is shown in Figure 1.

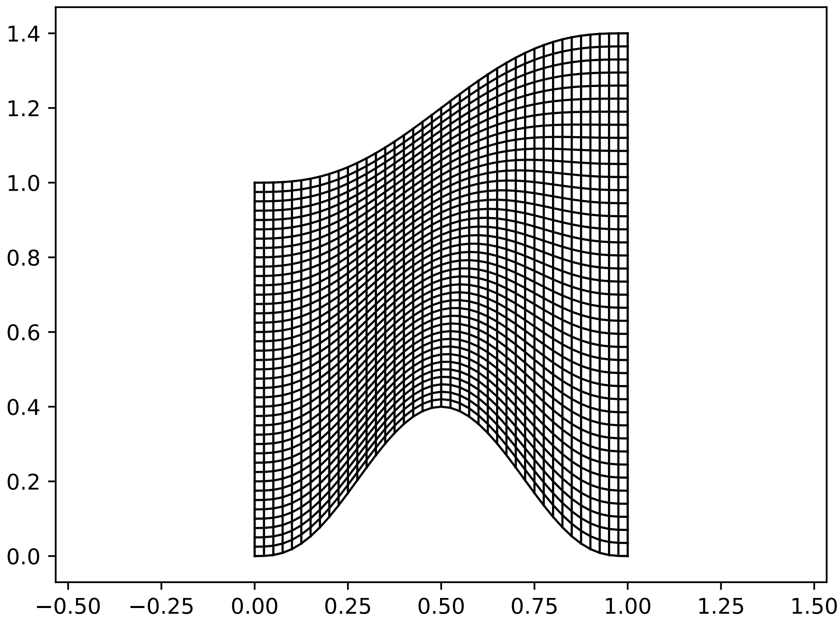

Figure 1: Linear transfinite interpolation of domain Ω.

Part b

Using cubic Hermite interpolation gives more degrees of freedom, which can be used to specify the normal vector of the grid at the boundaries. Here $\partial R = T\hat{n}$ will be used at all the boundaries, where $\hat{n}$ is the unit normal vector in the positive ξ and η directions, and T is a parameter. For $\xi = 0 \wedge \xi = 1$, the boundary is straight, so $\hat{n}_{lr} = [1, 0]$.

The normal vector at the bottom I find by rotating the tangential vector by $\pi/2$, like so:

$$n_{bottom}(x) = \begin{pmatrix} -dy_{bottom}(x) \\ dx \end{pmatrix} = \begin{pmatrix} -\partial_x(1 + Ax^3(6x^2 - 15x + 10)) \\ \partial_x x \end{pmatrix} = \begin{pmatrix} 64 \cdot 3Ax^2(1-x)^2(2x-1) \\ 1 \end{pmatrix}.$$

Equivalently, at the top, it is

$$n_{top}(x) = \begin{pmatrix} -\partial_x y_{top}(x) \\ \partial_x x \end{pmatrix} = \begin{pmatrix} -3Ax^2(6x^2 - 15x + 10) - A * x^3(12x - 15) \\ 1 \end{pmatrix}.$$

x is no longer linear in ξ, so I use the Hermite polynomial to transform it, as

$$x(\xi) = H_0(\xi) + T\tilde{H}_0(\xi) + T\tilde{H}_1(\xi).$$

Now every thing is ready to be put together, and the grid is given by

$$\hat{R}(\xi, \eta) = H_0(\eta)E(\xi, 0) + H_1(\eta)R(\xi, 1) + \tilde{H}_0(\eta)R_\eta(\xi, 0) + \tilde{H}_1(\eta)R_\eta(\xi, 1)$$

The resulting transform from the unit square is shown in Figure 2.

Problem 3

The complex-valued function

$$\omega(z) = \frac{2e^z - 3}{3e^z - 2} \tag{1}$$

is a conformal transformation from one grid to another. For $0 \leq \mathbf{Re}(z) = x \leq 1$, $0 \leq \mathbf{Im}(z) = y \leq 2\pi$, it maps the rectangle to a grid between two circles, centered at the real axis. This is shown in Figure 3. I will later show that the outer circle is the unit-circle ($r = 1$, $(x_0, y_0) = (0, 0)$), while the inner circle has radius $r = \frac{5e}{9e^2-4} \approx 0.22$, and is centered at $(c, 0)$, $c = \frac{6(e^2-1)}{9e^2-4} \approx 0.61$.

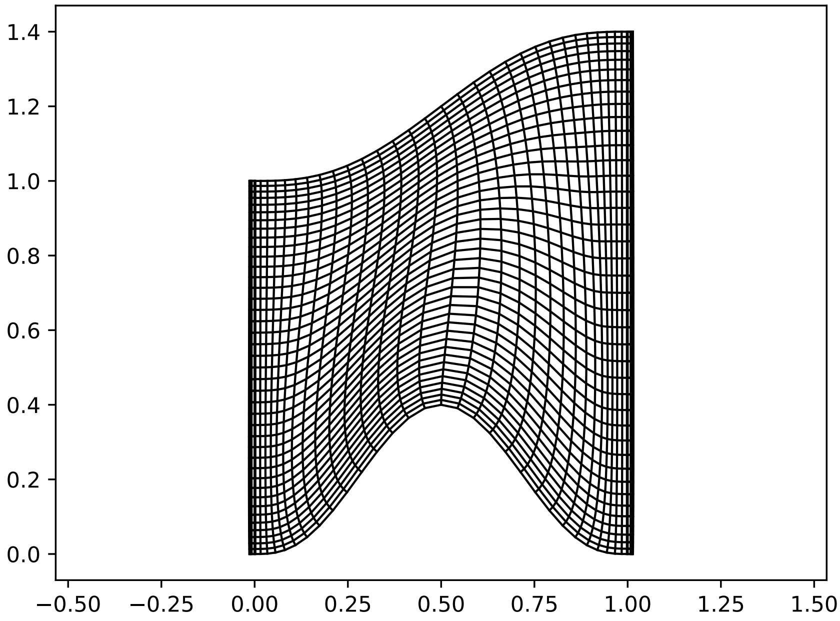

Figure 2: cubic Hermite transfinite interpolation of domain Ω with orthogonal boundaries.

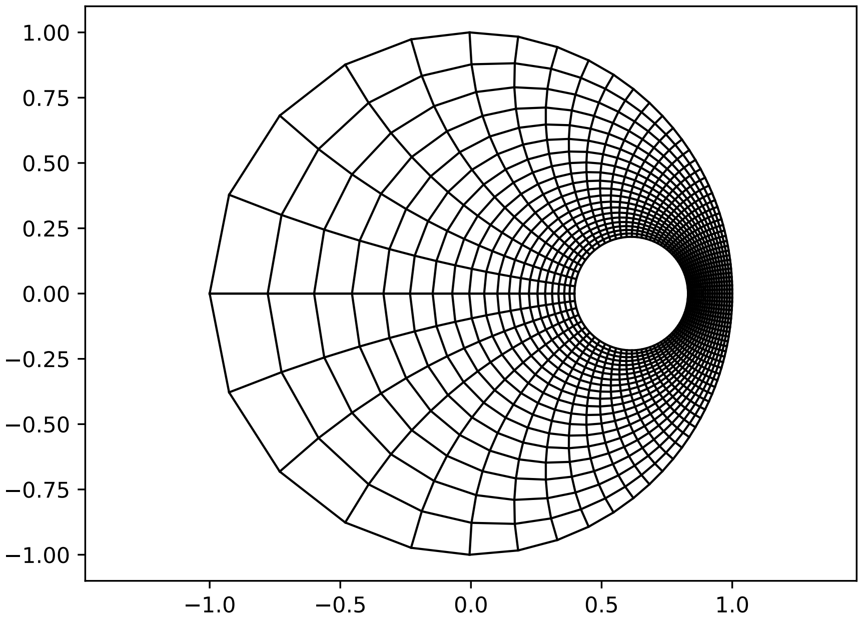

Figure 3: conformal mapping from $0 \leq x \leq 1, 0 \leq y \leq 2\pi$ by function $\omega(z)$ in (1)

It is known that a line constant in ξ will map to a circle. Further, $(\xi, \eta = 0)$ is the points on the real axis to the left of the centres, while $(\xi, \eta = \pi)$ is the real points to the right of the centres. The centre is then the average of these.

$$\begin{aligned}
c(\xi) =& \frac{\omega(\xi, 0) + \omega(\xi, \pi)}{2} \\
2c(\xi) =& \frac{2e^{\xi}e^{0} - 3}{3e^{\xi}e^{0} - 2} + \frac{2e^{\xi}e^{i\pi} - 3}{3e^{\xi}e^{i\pi} - 2} = \frac{2e^{\xi} - 3}{3e^{\xi} - 2} + \frac{2e^{\xi} + 3}{3e^{\xi} + 2} \\
=& \frac{(2e^{\xi} - 3)(3e^{\xi} + 2) + (2e^{\xi} + 3)(3e^{\xi} - 2)}{(3e^{\xi} - 2)(3e^{\xi} + 2)} \\
=& \frac{12(e^{2\xi} - 1)}{9e^{2\xi} - 4} \\
c(\xi) =& \frac{6(e^{2\xi} - 1)}{9e^{2\xi} - 4}
\end{aligned}$$

The radius of the circles can be found in a similar way.

$$\begin{aligned}
r(\xi) =& \omega(\xi, \pi) - c(\xi) \\
=& \frac{2e^{xi} + 3}{3e^{\xi} + 2} - \frac{6(e^{2\xi} - 1)}{9e^{2\xi} - 4} \\
=& \frac{5e^{\xi}}{9e^{2\xi} - 4}
\end{aligned}$$

With these functions, the top point as a function of the centre of the circle can be added to the plot in Figure 3. This is done in Figure 4, for $0 \leq \xi \leq 2\pi$, and $\pm r(\xi)$.

Problem 4

Followng the step-by-step instructions, I wrote a mesh-generator which used Delaunay to triangulate any polygon.

b

The boundary points are added by linear interpolation between each corner.

d

To determine if a triangle is outside the polygon, I make a test-point inside each triangle abc by $p_{test} = a + 0.5ab + 0.25bc$, and then the `inpolygon`-function.

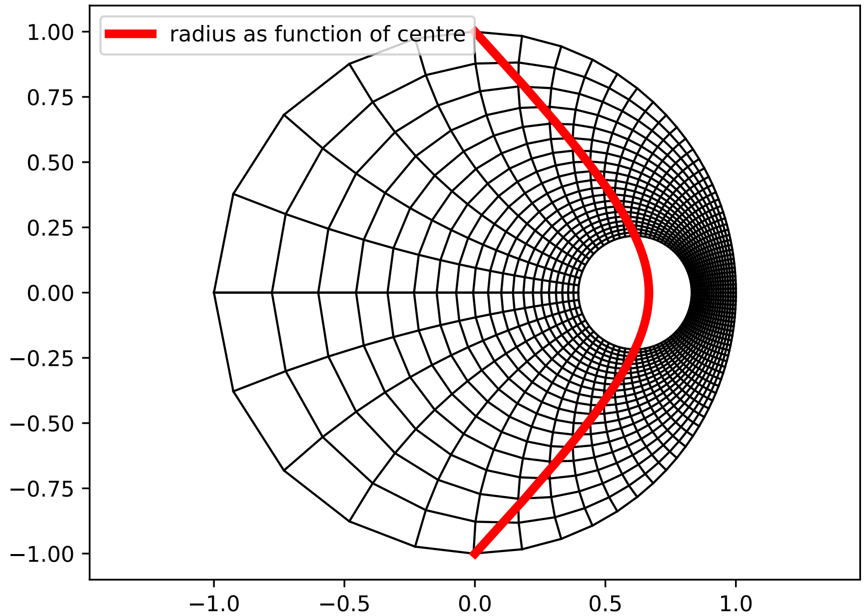

Figure 4: The conformal mapping of $\omega(z)$, along with the top-point of the circles, as function of the centre.

e

The area and circumcentre are calculated with formula from Wikipedia. This is repeated until no triangle is too large

h

The triangulation is refined by adding the centre point of each edge.

The final triangulated mesh for the parameters specified is shown in Figure 5. It shares a great resembelance to the one in Figure (h) in the problem description.

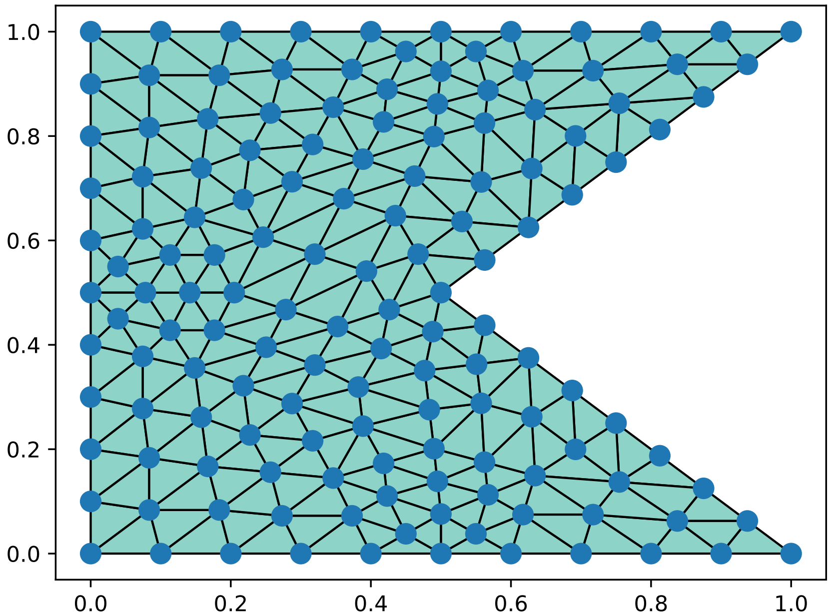

Figure 5: The triangulated mesh with specified parameters.