Problem Set 2

Math228B Numerical solutions to differential equations

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Problem 1

Part a, See previous problem set

Part b

We have the method

$$U_j^{n+1} = U_j^n + \frac{k\kappa}{2h^2}\left[U_{j-1}^n - 2U_j^n + U_{j+1}^n + U_{j-1}^{n+1} - 2U_j^{n+1} + U_{j+1}^{n+1}\right] - k\gamma\left[(1-\theta)U_j^n + \theta U_j^{n+1}\right], \tag{1}$$

which models diffusion with decay.

Let

$$U_j^n = e^{ijh\xi}, \quad U_j^{n+1} = g(\xi)U_j^n. \tag{2}$$

Von Neumann analysis says that the method is stable provided $|g| \leq 1$.

Inserting (2) into (1), we get the expression for $g(\xi)$ as

$$\begin{aligned}
ge^{ijh\xi} =& e^{ijh\xi} + \frac{k\kappa}{2h^2}\left[e^{i(j-1)h\xi} - 2e^{ijh\xi} + e^{i(j+1)h\xi} + ge^{i(j-1)h\xi} - 2ge^{ijh\xi} + ge^{i(j+1)h\xi}\right] \\
&- k\gamma\left[(1-\theta)e^{ijh\xi} + \theta g e^{ijh\xi}\right], \\
g =& 1 + \frac{k\kappa}{2h^2}\left[e^{-ih\xi} - 2 + e^{ih\xi} + ge^{-ih\xi} - 2g + ge^{ih\xi}\right] - k\gamma\left[(1-\theta) + \theta g\right], \\
g =& 1 + \frac{k\kappa}{h^2}\left[\cos(h\xi) - 1 + g\cos(h\xi) - g\right] + k\gamma(\theta - 1) - k\gamma\theta g, \\
g - \lambda g + k\gamma\theta g =& 1 + \lambda + k\gamma(\theta - 1), \qquad \text{where} \quad \lambda \equiv \frac{k\kappa}{h^2}\left(\cos(h\xi) - 1\right), \\
g =& \frac{1 + \lambda + k\gamma(\theta - 1)}{1 - \lambda + k\gamma\theta}.
\end{aligned}$$

Using the stability condition $|g| \leq 1$, we get

$$1 \geq \left| \frac{1 + \lambda + k\gamma(\theta - 1)}{1 - \lambda + k\gamma\theta} \right|, \tag{3}$$

$$|1 - \lambda + k\gamma\theta| \geq |1 + \lambda + k\gamma(\theta - 1)|\,.$$

In the second step I used the fact that λ is always negative or equal to 0, such that $1 - \lambda + k\gamma\theta > 1$. WolframAlpha gives that under the conditions $\lambda < 0$ and $k\gamma > 0$, both of which are obviously always true, then

$$\theta \geq \frac{k\gamma - 2}{2k\gamma} = \frac{1}{2} - \frac{1}{k\gamma}.$$

Again, as k and γ are both non-negative, the second term can be dropped, and the method is unconditionally stable for any

$$\theta \geq \frac{1}{2},$$

which is what was to be shown. (The conditions set are already true from the formulation of the problem).

c

Going back to (3), I now set $\theta = 0$, and do a similar analysis.

$$1 \geq \left| \frac{1 + \lambda - k\gamma}{1 - \lambda} \right|,$$

$$|1 - \lambda| \geq |1 + \lambda - k\gamma|.$$

Again, WolframAlpha, under the conditions that the solutions are real and $\lambda < 1$, both of which are again always true, gives the solution as $2\lambda \leq k\gamma \leq 2$. As $\lambda < 0$, the first inequality is always upheld. This gives unconditional stability for $k \leq 2/\gamma$, which is what was to be shown.

Problem 2

To solve the differential equation

$$u_t + au_x = 0,$$

one could use the following scheme

$$U_j^{n+1} = U_{j-2}^{n-1} - \left(\frac{ak}{h} - 1\right)(U_j^n - U_{j-2}^n).$$

Part a

To determine the order of accuracy of the method, I Taylor expand, and subtract the 2 sides from eachother. I Taylor expand to 3rd order.

$$\begin{aligned} U_j^n &= u(x,t) \equiv u \\ u(x,t+k) &= u(x-2h,t-k) - \left(\frac{ak}{h} - 1\right)(u(x,t) - u(x-2h,t)) \end{aligned}$$

$$\begin{aligned} u(x,t+k) =& u + ku_t + \frac{k^2}{2}u_{tt} + \frac{k^3}{6}u_{ttt} + \mathcal{O}(k^4) \\ u(x-2h,t) =& u - 2hu_x + \frac{4h^2}{2}u_{xx} - \frac{8h^3}{6}u_{xxx} + \mathcal{O}(h^4) \\ u(x-2h,t-k) =& u - 2hu_x - ku_t + \frac{1}{2}\left[4h^2u_{xx} + 2\cdot 2hku_{xt} + k^2u_{tt}\right] + \\ &+ \frac{1}{6}\left[-8h^3u_{xxx} - 3\cdot 4h^2ku_{xxt} - 3\cdot 2hk^2u_{xtt} - k^3u_{ttt}\right] + \mathcal{O}(h^4 + k^4) \end{aligned}$$

$$\begin{aligned} \tau =& u - 2hu_x - ku_t + \frac{1}{2}\left[4h^2u_{xx} + 2\cdot 2hku_{xt} + k^2u_{tt}\right] \\ &+ \frac{1}{6}\left[-8h^3u_{xxx} - 12h^2ku_{xxt} - 6hk^2u_{xtt} - k^3u_{ttt}\right] \\ &- \left(\frac{ak}{h} - 1\right)u + \left(\frac{ak}{h} - 1\right)\left[u - 2hu_x + \frac{4h^2}{2}u_{xx} - \frac{8h^3}{6}u_{xxx}\right] \\ &- \left(u + ku_t + \frac{k^2}{2}u_{tt} + \frac{k^3}{6}u_{ttt}\right) + \mathcal{O}(kh^3 + k^4) \\ =& \frac{k}{3}\left(\frac{2h^2}{a} + ak^2 - 3hk\right)u_{xxx} + \mathcal{O}(kh^3 + k^4) \\ =& \mathcal{O}(h^2k + k^3 + hk^2) \end{aligned}$$

where I have used the original equation $u_t = -au_x$ to cancel all frist-derivatives and second-derivatives of u, as well as many of the thrid-derivatives using the following relations.

$$u_t = -au_x$$

$$u_{tt} = -au_{xt} \quad u_{tx} = -au_{xx}$$

$$u_{ttt} = -au_{xtt} \quad u_{ttx} = -au_{xxt} \quad u_{txx} = -au_{xxx}$$

This seems to be a 3rd order method.

Part b

The true solution to the advection equation for a point is the value of the point upstream at the previous timestep. The skewed leapfrog method with $\nu \equiv ak/h = 1$ will depend on the point two steps back in time, and two upstream, so this is also okay. But for values near 1 it will depend on grid-point outside the true domain of dependence and thus not satisfy the CLF-condition.

Part c

Again, I use the definitions (2).

$$
\begin{aligned}
u_j^{n+1} =& u_{j-2}^{n-1} - (\nu - 1)(u_j^n - u_{j-2}^n), \quad \nu \equiv ak/h \\
& \Downarrow \\
ge^{i\xi hj} =& e^{i\xi h(j-2)}/g - (\nu - 1)(e^{i\xi hj} - e^{i\xi h(j-2)}) \\
g =& e^{-2i\xi h}/g - \nu + \nu e^{-2i\xi h} + 1 - e^{-2i\xi h} \\
\gamma^2 =& 1 - \nu\gamma e^{i\xi hj} + \nu g + \gamma e^{i\xi h} - g, \quad \gamma \equiv ge^{i\xi h} \\
g\gamma^2 =& g - \nu\gamma^2 + \nu g^2 + \gamma^2 - g^2 \\
\gamma^2(g + \nu - 1) =& g(1 + \nu g - g) \\
\gamma^2 =& g\frac{(1 + \nu g - g)}{(g + \nu - 1)}
\end{aligned}
$$

For $|g| = |\gamma| \geq 1$, I get that $\nu = 0 \wedge \nu = 1 \wedge |\nu| \geq 2$ will satisfy the sability condition. This was found by plotting $\gamma^2(\nu)$ for $|g| \geq 1$ and finding the regions where it was always less than 1.

This is similar to what I found in the previous part, but with the addition of courant numbers larger than 2.

Problem 3

Part a-e

Code, implemented in the Julia-code uploaded on bCourses. The functions I implemented for part b, c, d and e have different signatures than what was specified in the problem set.

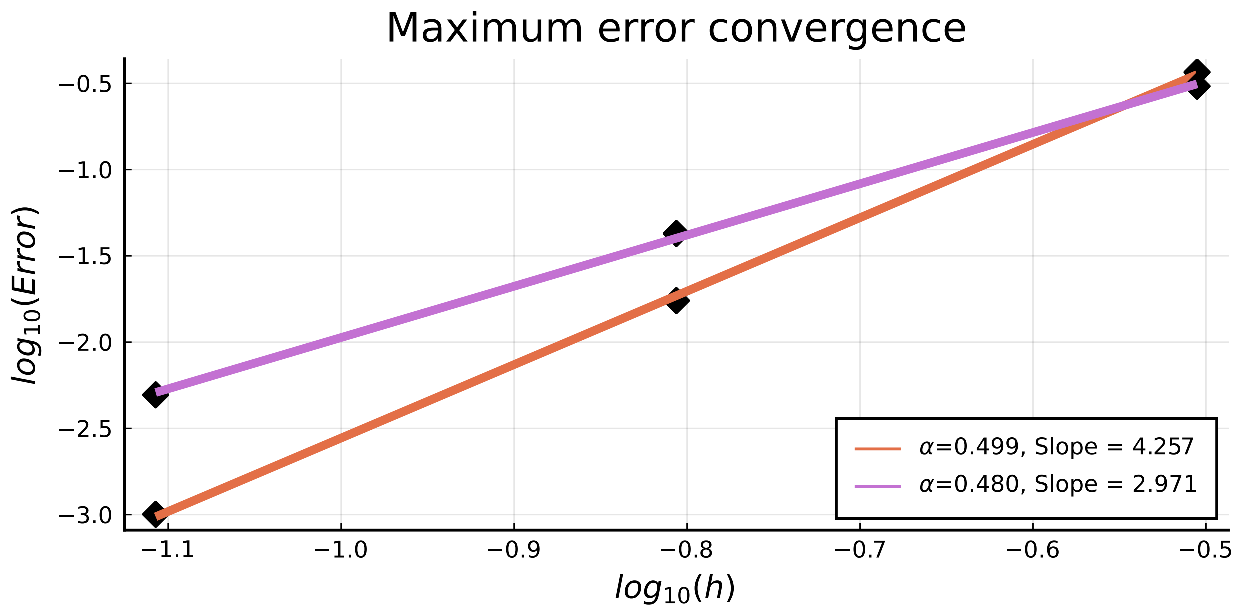

Figure 1: Convergence-plot of the error as the spatial step length gets smaller, for 2 different values of the filtering-parameter α.

I therefore also added some wrapper-functions with the specified signature so autograder would be happy. I have not tested these, but they are very small, so should work alright.

Part f

I implement a solver that sets up the system and iterates to the final solution. I use the max-norm to find the error from the exact solution among all solution components. Figure 1 shows a log-log plot of the error against the step length h for the 2 specified filter-parameters α. For $\alpha = 0.499$ I get that the slope is 4.257, while for $\alpha = 0.480$, it is 2.971.

$\alpha = 0.5$ corresponds to no filtering, so I get better results using less filtering for smaller step sizes. This is as expected, as filtering is not a correct operation when solving such differential equations, but is in this case needed to avoid the solution blowing up. Therefore, as I get a bigger grid, less filtering will be closer to the true solution, while for smaller grids, I need to filter more to maintain stability.

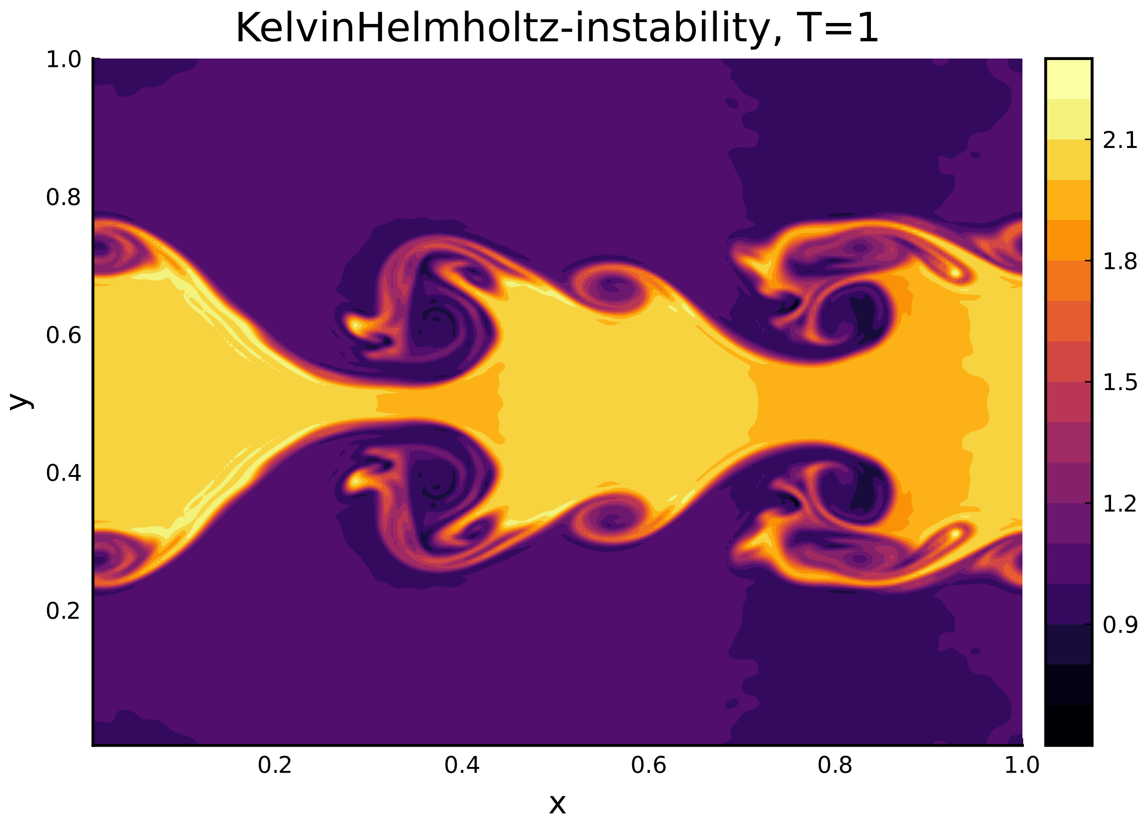

Figure 2: The simulated Kelvin-Helmholtz instability at $T = 1$.

Part g

I implement the specified intital conditions and use my RK4-solver. At $T = 1$, I plot the density of the system in the domain, which is shown in Figure 2. The results looks reasonable.