Final Project - Macroeconomics

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1 Simple Variant of Krussell Smith Method

1.1 Proof of Proposition 3

As stated in the appendix of Harenberg and Ludwig (2015), we find following proof of the proposition 3 in their paper:

Given: $\tilde{\beta} \in (0, 1/2] s.t. \beta = \frac{\tilde{\beta}}{1-\tilde{\beta}}$ and $\zeta_t > 0, \varrho_t > 0, \eta_{i,2,t} > 0 \forall i, t$
$E_\zeta = E_\varrho = E_{\eta_{i,2,t}} = 1$ and $\zeta_{t+1} \perp \zeta_t$, $\varrho_{t+1} \perp \varrho_t$,$\varrho_t \perp \zeta_t$,$\zeta_t \perp \eta_{i,2,t}$, $\varrho_t \perp \eta_{i,2,t}$
k_o given

1. Here we consider two periods of an HH, namely, working and pension period. At the beginning of a households life time it maximizes it's own utility. Therefore, saving and spending behavior is optimised in both periods. Hence, we know households saving behavior for first period agents equals assets position at the beginning of the next period:
$a_{2,t+1} = s(1-\tau)w_t = s(1-\tau)(1-\alpha)\Upsilon_t \zeta_t k_t^\alpha$

2. Given the assumption (1.) is true, the path of capital is:
$K_{t+1} = a_{2,t+1} = s(1-\tau)(1-\alpha)\Upsilon_t \zeta_t$, where $k_{t+1} = \frac{K_{t+1}}{\Upsilon_t(1+\lambda)}$
$k_{t+1} = \frac{1}{(1+\zeta)(1+\lambda)} s(1-\tau)(1-\alpha)\zeta_t$

3. Solving for the neoclassic HH problem leads to:
max $U_t = (1-\tilde{\beta})u(c_{1,t}) + b\tilde{eta}aE_t[u(c_{2,t+1})]$
FOC: $1 = \beta E_t[\frac{c_{1,t}(1+r_{t+1})}{c_{2,t+1}})] \Leftrightarrow 1 = \beta \frac{(1-s)}{s} \Phi \Leftrightarrow s(t) = \beta\Phi(t) \frac{}{1+\beta\Phi(t)}$

Where $\Phi(\tau) = E_t[\frac{1}{1+\frac{1-\alpha}{\alpha(1+\lambda)\rho_{t+1}}(\lambda\eta_{i,2,t+1}+\tau(1+\lambda)(1-\eta_{i,2,t+1})))} \leq 1$
The equation of the savings rate indicates dependence on tomorrows idiosyncratic shocks $\eta_{i,2,t+1}$ and aggregate shocks ρ_{t+1}.

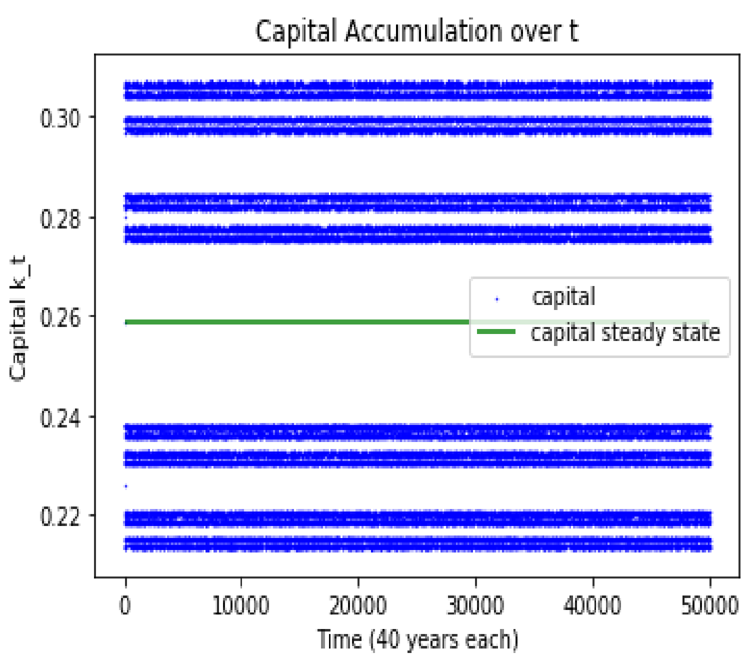
4. For $\lambda = 0$: $\Phi = 1$ for $\lambda > 0$: $\Phi = E_t[\frac{1}{1+x}]$;
$x = \frac{1-\alpha}{\alpha(1+\lambda)g_{t+1}}(\lambda\eta_{i,2,t+1} + \tau(1+\lambda(1-\eta_{i,2,t+1})))$
Hence, where $x \geq 0$, $E_t[\frac{1}{1+x}] \leq 1 \Leftrightarrow s \leq \frac{\beta}{1+\beta}$

1.2 Simulation Krussell Smith Algorithm

Following the above seen proposition 3 in python code.
In the very first exercise, we are asked to compute the capital accumulation path within the set up of the above mentioned paper by Harenberg and Ludwig using the closed form solution. The code starts by setting parameters and generating three different types of shocks, namely, aggregate (zeta and rho) and individual (eta) shocks. Aggregate shocks occur in a good and bad state, which is the expected value plus or minus respective standard deviation.
Eleven idiosyncratic shocks are generated along the quantecon tool named qwnorm, which resembles standard Guassian Quadrature method.
As shocks occur iid and households hold no knowledge about the future, they base their saving behavior on expectations over phi $\phi(t)$. Households know respective likelihoods for each shock. Along these likelihoods probable values of phi$\phi(t)$ are weighted for $\eta_{i,2,t+1}$ and ρ_{t+1}. Here, phi is constant and below 1 as stated in 1.1.3.
Along phi $\phi(t)$ a fixed saving rate is computed.

Starting in a steady state environment, where $k_t = k_t + 1$, allows to solve for k_{ss}: $ln(k_{ss}) = \frac{1}{1-\alpha} * (ln(s) + ln(1-\tau) + ln(1-\alpha) - ln(1+\lambda)$

For simulating $T = 50.000$ periods, we randomly draw equivalent number of zeta shocks and iterate forward based on k_t and ζ_t.



The simulation indicates an initial steady state of roughly $k_{ss} = 0.02$ (shown in green). All capital points above steady state are capital values for zeta simulating economic boom. Capital points below the steady state are driven by zeta taking low values simulating an economic recession. Further variation is driven by states of rho and eta. Hence, the highest capital values occur for simultaneous shocks where eta, rho and zeta are high.

1.3 Simple Krussell-Smith Algorithm

First, I solve for the theoretical values of $\psi_0(z)$ and $\psi_1(z)$. Based on these state-dependent values k_{t+1} can be computed given:
$ln(k_{t+1}) = \psi_0(z) + \psi_1(z) * ln(k_t)$.
Initial values for Ψ are computed along k_t which was derived in 1.2.
I find Ψ such that:

$$\Psi = \begin{pmatrix} \psi_0(z_r) & \psi_0(z_b) \\ \psi_1(z_r) & \psi_1(z_b) \end{pmatrix} = \begin{pmatrix} (-2.86986512) & (-2.61552057) \\ 0.3 & 0.3 \end{pmatrix}$$

In order to derive a $\Psi(z)$ that maximises households savings decisions over life time, an iteration process along optimised capital accumulation drives convergence for Ψ. The code employs following iteration steps:

1. Solving the household problem by applying budget constraints in the Euler equation yields:
$a_{2,t+1} = \frac{1}{1+\beta} * (\beta(1-\tau) * w_t - \lambda(1-\tau) * E_t(\frac{\eta_{t+1} w_{t+1}}{R_{t+1}}) - (1-\lambda) * E_t(\frac{b_{t+1}}{R_{t+1}}))$
where $w_t = (1-\alpha) *_t *k_t^\alpha * \zeta_t$
and $b_{t+1} = \tau * w_{t+1} * (\frac{1+\lambda}{1-\lambda})$ and $s(t, \zeta_t, k_t) = \frac{a_{2,t+1}}{(1-\tau)*(1-\alpha)**k_t^\alpha*\zeta}$

For each given k_t in k-grid and each state of zeta respective saving function is computed. Here, the saving matrix takes a dimension of (5x2). Given that the asset function depends on future shocks $\eta_{i,2,t+1}$, $\rho t + 1$ and $\zeta t + 1$ expectations are computed as weighted values based on the probability of the occurrence of each shock. k_{t+1} is for each k_t along Ψ.

2.In a second step, the saving policy function is applied in a simulation over $T = 50.000$ periods. Following equation determines capital accumulation in the economy:

$$ln(k_{t+1}) = ln(\frac{(1-\alpha)}{(1-\lambda)} + ln(s(t, \zeta_t, k_t)) + ln((1-\tau)) + ln(\zeta) + \alpha * ln(k_t)$$

Consequently, consumption can be derived s.t.:

$con_1 = 1 - s(t, \zeta_t, k_t)(1-\tau) * (1-\alpha) * \zeta_t * k_{t+1}^\alpha$
$con_2 = \beta con_{1,t} * (1 + \alpha * k_{t+1}^{\alpha-1} * \zeta_t * \rho_t)$

3. Thirdly, Ψ is updated along the new capital path derived in the simulation. By dividing the capital accumulation in a vector for all picks where zeta is high

and all states where zeta is low, give consecutive states k_{t+1} for each state. For both states of zeta a regression for a constant and k_t is executed for respective k_{t+1} to derive $\psi_0(z)$ and $\psi_1(z)$.

Ψ is updated along: $\Psi(z) = \omega\Psi_{(old)} + (1-\omega)\Psi_{(new)}$
where $\omega = 0.85$.

Results

For tau=0:

$$s(t, \zeta_t, k_t) = \begin{pmatrix} 0.1549 & 0.15555 & 0.156 & 0.1564 & 0.1567 \\ 0.1665 & 0.16685 & 0.1671 & 0.16735 & 0.1675 \end{pmatrix}$$

After X iterations Ψ converged:
$$\Psi_{result} = \begin{pmatrix} (-1.8163) & (-1.7663) \\ 0.2911 & 0.2939 \end{pmatrix}$$

The result shows similar tendencies as the initial guess. The sign and the magnitude seem fine to me. Slight differences for the constant value ψ_0 may occur due to the simulation process based on draws of s rather than exact shock combinations.

When computing the case of $\tau = 0.1$, I find controversial results. An increase in tau represents higher pension pay-out. One would assume that savings drop, but we find a slight increase in savings in comparison to the case for $tau = 0$. Consequently, households overall utility decreases. A potential explanation might be that an increase in taxes decreases income. The negative income effect is not balanced out by the substitution effect. Hence, tax distortion is greater than the utility gain of the household. This result become obvious when considering the average utility for t=0 being $u = 0.3044$ and for t=0.1 $u = 0.246$.

For tau=0.1:

$$s(t, \zeta_t, k_t) = \begin{pmatrix} 0.16155 & 0.16215 & 0.1626 & 0.16295 & 0.16325 \\ 0.17165 & 0.172 & 0.17225 & 0.17245 & 0.17265 \end{pmatrix}$$

$$\Psi_{result} = \begin{pmatrix} (-1.9293) & (-1.8161) \\ 0.2956 & 0.2966 \end{pmatrix}$$

2 Complex Krussell-Smith Algorithm

For the complex method, I heavily supported by Alex Wurdinger. He shared a solution of his code. Given my failed attempt to make a running program. I would like to explain my (theoretical) approach.

As seen in our last homework, a PAYG system is welfare improving for the entire economy as it insures agents against uncertainties in an incomplete market. In this exercise, we would like to examine the benefit of such a social system when not only idiosyncratic risk is present but also aggregate risk ζ.
The algorithm to solve such a model is similar to that stated above in 1.3. Main differences that complicate the computational process are: presence of heterogeneous agents (wealth), age $j \in (1, 80)$years and zeta as Markov chain process. Latter shock assumption implies that tomorrows expectations depend on today. Given the additional two states for the aggregate shock zeta, two dimensions need to be added to all policy functions, namely: cash on hand(endogenous grid), consumption, aggregate capital, assets and the value functions.
For the implementation a grid for $k_t mrw$ was computed along an initial guess of Ψ with dimension of (5x2). Hence, the state space is four-dimensional $(a, \eta, \zeta, K_{aggr})$. Additional dimensions are required to be add to all policy functions as stated above in the household problem. In comparison to only idiosyncratic shocks, we observe non-constant values for interest rates and wage rates. These depend on the history of shocks that determine the level of the aggregate capital stock (based on the distribution of asset holdings in the population). Income depends on interest rates and wage rates.

1. The household problem solves based on $a_{t+1}(a_t, \eta, \zeta, K_{aggr})$ asset holdings for a random draw of shocks ζ. In the household problem, a solution is derived by backward iteration given last periods spending behaviour of households. In the code, a solution for computed aggregate K needs to be adjusted to the grid of five capital values to apply computed policy functions.
The transition matrix weights the value function of tomorrow by respective likelihood for each possible combination of (k,zeta):
$K_{t+1} = \sum_j \int a_{t+1}(a_t, \eta, \zeta, K_{aggr}) * \Phi(a_{t+1}, z_{t+1}) * N_j$

An iteration process on the value function going from 80 year old to newborn provides recursively, the value function for tomorrow for each age j.

2. The simulation is heavily reduced to T=100 in comparison to the simple Krussell Smith method. This is due to the heavy computational steps required at each point in time. The simulation provides for each k_t k_{tmrw}. When repeating the simulation for iteration, a seed to control for random numbers support convergence.

3. In the regression we regress on respective k_{tmrmw} for a constant and k_t of either the sample of good state or bad state zeta. One could expect less precise regression coefficients as the sample size of $T = 100$ is small. Although the updating and iteration process is identical to the simple Krussell Smith method seen

above. This may increase the number of necessary iterations up to convergence.

Results:

The running code shows utility gains in comparison to no PAYS social insurance system. Although, increased taxes seem to distort the market outcome more than the provided insurance gains. This questions the extend of insurance a state should provide as there seem to exist a threshold where additional insurance distort the market more than the benefits coming along with it.