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The unique solvability of a complex 3D heat transfer problem

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ABSTRACT

Conductive–convective–radiative heat transfer in a scattering and absorbing medium with reflecting and radiating boundaries is considered. The P_1 approximation (diffusion model) is used for the simplification of the original problem. The existence of bounded states of the diffusion model is proved. The uniqueness of solutions is established under certain assumptions.

1. Introduction

This paper considers complex heat transfer processes that include conduction, convection and radiation heat exchanges. Heat conduction is a molecular process caused by non-uniform temperature distributions, whereas convection transfers heat due to macroscopic motions of media. Radiative heat transfer occurs through the emission and absorption of electromagnetic waves. All these processes are present in engineering objects such as gas turbines, combusting and cooling systems, industrial furnaces, boilers, etc.

The study of complex models involving all above mentioned kinds of heat transfer can be found in the monographs [10,9]. The work [16] considers the problem of glass cooling, where the radiative heat transfer plays a crucial role.

In papers [1–3,6], thermal properties of some semi-transparent and insulating materials are studied in context of coupled radiative and conductive heat transfer problems. The mathematical treatment of one-dimensional radiative and conductive thermal models is given in [13,12,5,4,15,7]. In particular, an existence and uniqueness theorem is proved in [5] in the case of isotropic scattering and non-reflecting boundaries.

A modified Monte Carlo method for the numerical treatment of nonlinear heat transfer in homogeneous layers with axisymmetric thermal radiative properties is proposed in [7]. The method is verified, and a comparison with the diffusion, P_1, approximation is performed in the case of isotropic scattering and reflecting boundaries. It occurs that the diffusion approximation provides a good description of the behavior of solutions. The method proposed is well appropriate for parallel computing on multiprocessor systems.

Papers [11,14] consider a three-dimensional model where the temperature state is governed by a transient heat transfer equation coupled with a steady-state radiative heat transfer equation (SP_1 approximation). The unique solvability of the corresponding initial boundary value problem is proved in the class of bounded functions.

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In the present work, a three-dimensional steady-state conductive, convective, and radiative heat transfer problem is considered. The P_1 (diffusion) approximation is used to reduce the original problem to a system of two local partial differential equations. The existence of bounded solutions of the diffusion model is proved. The uniqueness of states is established under certain assumptions.

2. Diffusion approximation for 3D problem of complex heat transfer

In this section, a model that includes an integro-differential equation describing the intensity of heat-radiation and a conductive–convective heat transfer equation is studied. The process is considered in a tree-dimensional convex bounded domain G with the boundary $\Gamma \in C^{0,1}$ consisting of three parts: Γ_1 being the flow impermeable solid part, Γ_2 the inflow part, and Γ_3 the outflow part. The ray directions are associated with points of the unit sphere $\Omega = \{\omega \in \mathbb{R}^3 : |\omega| = 1\}$.

The radiative heat transfer is described by the following equation:

$$\omega \cdot \nabla_r I(r,\omega) + \kappa I(r,\omega) = \frac{\kappa_s}{4\pi}\int_\Omega P(\omega,\omega')I(r,\omega')d\omega' + \kappa_a \frac{\sigma n^2 T^4(r)}{\pi}, \tag{1}$$

where $r \in G$ and $\omega \in \Omega$ denote spacial points and ray directions, respectively. Moreover, $\kappa := \kappa_s + \kappa_a$ is the extinction coefficient (the total attenuation factor), κ_s the scattering coefficient, κ_a the absorption coefficient, σ the Stefan–Boltzmann constant, n the index of refraction, $P(\omega,\omega')$ the phase function, $I(r,\omega)$ the intensity of radiation, and $T(r)$ the temperature. Assume for simplicity that κ, κ_s and κ_a are constants (do not depend on r).

Boundary conditions for Eq. (1) are written according to [9]. They express the effects of boundary emissivity and specular–diffuse reflection as follows:

$$\begin{aligned} &I(r,\omega) = \varepsilon(r)\frac{\sigma n^2}{\pi}T_0^4(r) + \rho^s(r)I(r,\omega_R) + \rho^d(r)\frac{1}{\pi}\int_{\omega'\cdot\mathbf{n}>0} I(r,\omega')\omega'\cdot\mathbf{n}\,d\omega', \quad r \in \Gamma_1,\\ &I(r,\omega) = 0, \quad r \in \Gamma_2 \cup \Gamma_3,\\ &\omega\cdot\mathbf{n} < 0, \quad \omega_R = \omega - 2(\omega\cdot\mathbf{n})\mathbf{n}. \end{aligned} \tag{2}$$

Here, ε is the emissivity coefficient, ρ^s and ρ^d are the coefficients of specular and diffuse reflection (notice that $\varepsilon + \rho^s + \rho^d = 1$), $\mathbf{n}$ denotes the outer normal to the corresponding boundary, ω_R is the direction of reflection, $T_0(r)$ a prescribed distribution of the boundary temperature.

The conductive–convective steady-state heat transfer equation is of the form

$$-k\Delta T(r) + \rho c_v \mathbf{v}(r)\cdot\nabla T(r) = -\nabla\cdot\int_\Omega \omega I(r,\omega)d\omega. \tag{3}$$

Here, $\mathbf{v}(r)$ is a prescribed velocity field, k the thermal conductivity, c_v the specific heat capacity, ρ the density. It is assumed that k, c_v, and ρ are constants.

Boundary conditions for Eq. (3) are specified as follows:

$$T(r) = T_0(r), \quad r \in \Gamma_1 \cup \Gamma_2, \qquad \frac{\partial T(r)}{\partial \mathbf{n}} = 0, \quad r \in \Gamma_3. \tag{4}$$

To normalize problem (1)–(4), denote

$$I(r,\omega) = \left(\frac{\sigma n^2}{\pi}T_{\max}^4\right)I^*(r,\omega), \qquad T(r) = T_{\max}\,\theta(r). \tag{5}$$

Substituting ansatz (5) into Eqs. (1)–(4) yields

$$\omega\cdot\nabla_r I^*(r,\omega) + \kappa I^*(r,\omega) = \frac{\kappa_s}{4\pi}\int_\Omega P(\omega,\omega')I^*(r,\omega')d\omega' + \kappa_a\theta^4(r), \tag{6}$$

$$\begin{aligned} &I^*(r,\omega) = \varepsilon(r)\Theta_0^4 + \rho^s(r)I^*(r,\omega_R) + \rho^d(r)\frac{1}{\pi}\int_{\omega'\cdot\mathbf{n}>0} I^*(r,\omega')\omega'\cdot\mathbf{n}d\omega', \quad r \in \Gamma_1,\\ &I^*(r,\omega) = 0, \quad r \in \Gamma_2\cup\Gamma_3,\\ &\omega\cdot\mathbf{n} < 0, \quad \omega_R = \omega - 2(\omega\cdot\mathbf{n})\mathbf{n}, \end{aligned} \tag{7}$$

$$-a\Delta\theta(r) + \mathbf{v}\cdot\nabla\theta(r) = -\nabla\cdot\frac{b}{4\pi}\int_\Omega \omega I^*(r,\omega)d\omega, \tag{8}$$

$$\theta(r) = \Theta_0(r), \quad r \in \Gamma_1\cup\Gamma_2, \qquad \frac{\partial\theta(r)}{\partial\mathbf{n}} = 0, \quad r \in \Gamma_3, \tag{9}$$

where

$$a = \frac{k}{\rho c_v}, \qquad b = \frac{4\sigma n^2 T_{\max}^3}{\rho c_v}, \qquad \Theta_0(r) = T_0(r)/T_{\max}.$$

Solving the system (6)–(9) requires significant computational efforts. To simplify this system, construct the diffusion (P_1) approximation of it. To this end, use the following ansatz:

$$I^*(r,\omega) \simeq \varphi(r) + \omega \cdot \mathbf{\Phi}(r), \tag{10}$$

which is the approximation of the intensity function by the sum of the first two terms of the Fourier series containing associated Legendre functions. It is reasonable to approximate the phase function also by two terms (see [9]) as follows:

$$P(\omega,\omega') = 1 + A\omega \cdot \omega'. \tag{11}$$

In (11), the coefficient $A \in [-1, 1]$ describes the anisotropy of scattering. The case $A = 0$ corresponds to the isotropic scattering.

Substitution of (10) and (11) into (6) yields the relation

$$\omega \cdot \nabla_r(\varphi(r) + \omega \cdot \mathbf{\Phi}(r)) + \kappa(\varphi(r) + \omega \cdot \mathbf{\Phi}(r)) = \kappa_s\left(\varphi(r) + \frac{1}{3}A\mathbf{\Phi}(r)\cdot\omega\right) + \kappa_a\theta^4(r). \tag{12}$$

The expansion of the residual in the Fourier–Legendre series and the requirement of vanishing the first two Fourier coefficients yield two first-order differential equations. For deriving the first equation, integrate (12) over Ω to obtain:

$$\frac{1}{3}\nabla \cdot \mathbf{\Phi}(r) + \kappa_a\varphi(r) = \kappa_a\theta^4(r). \tag{13}$$

The second equation is derived by multiplying (12) by ω and integrating the result over Ω, which yields:

$$\mathbf{\Phi}(r) = -\left(\kappa - \frac{A}{3}\kappa_s\right)^{-1}\nabla\varphi(r). \tag{14}$$

Then Eqs. (13) and (14) yield the second-order differential equation

$$-\alpha\Delta\varphi(r) + \kappa_a\varphi(r) = \kappa_a\theta^4(r), \tag{15}$$

where $\alpha = (3\kappa - A\kappa_s)^{-1}$.

To derive the boundary conditions for the diffusion approximation, substitute the ansatz (10) into the boundary condition (7), multiply the both sides by $\omega \cdot \mathbf{n}$, and integrate the result over the set of incoming, see (2), directions. This yields the conditions

$$\begin{aligned} &\alpha\frac{\partial\varphi(r)}{\partial\mathbf{n}} + \beta\varphi(r) = \beta\Theta_0^4(r), \quad r \in \Gamma_1, \\ &\alpha\frac{\partial\varphi(r)}{\partial\mathbf{n}} + \frac{1}{2}\varphi(r) = 0, \quad r \in \Gamma_2 \cup \Gamma_3, \end{aligned} \tag{16}$$

where

$$\beta = \frac{\varepsilon}{2(2-\varepsilon)}.$$

Now, substitute the ansatz (10) into the condition (8) to obtain:

$$-a\Delta\theta(r) + \mathbf{v}\cdot\nabla\theta(r) = -\frac{b}{3}\nabla\cdot\mathbf{\Phi}(r).$$

Taking (13) into account yields the following approximation of the convective–conductive heat transfer:

$$-a\Delta\theta(r) + \mathbf{v}\cdot\nabla\theta(r) + b\kappa_a\theta^4(r) = b\kappa_a\varphi(r), \tag{17}$$

$$\theta(r) = \Theta_0(r), \quad r \in \Gamma_1 \cup \Gamma_2, \qquad \frac{\partial\theta(r)}{\partial\mathbf{n}} = 0, \quad r \in \Gamma_3. \tag{18}$$

Thus, the system (15)–(18) is the diffusion approximation of the original model of complex heat transfer.

3. Weak solutions of the boundary-value problem

Determine the following functional space:

$$V = \{u \in H^1(G) : u(r) = 0 \text{ for } r \in \Gamma_1 \cup \Gamma_2\}.$$

Here and further, $H^s(G)$, $s \geq 0$ denotes the Sobolev space $W_2^s(G)$, and (f, g) denotes the scalar product in the space $L^2(G)$, i.e.

$$(f, g) = \int_G f(r)g(r)dr, \qquad \|f\|^2 = (f, f).$$

Suppose that $\mathbf{v} \in \left(L^\infty(G) \cap H^1(G)\right)^3$, $\Theta_0 \in L^\infty(\Gamma_1 \cup \Gamma_2)$, $\beta \in L^\infty(\Gamma_1)$, and the following conditions hold:

$$|\mathbf{v}| \leq v_0;\ \nabla \cdot \mathbf{v} = 0;\ v_n := \mathbf{v}(r) \cdot \mathbf{n}(r) \geq 0,\ r \in \Gamma_3;\ v_n(r) = 0,\ r \in \Gamma_1; \tag{19}$$

$$\begin{aligned} &0 \leq \Theta_0(r) \leq \Theta_1,\ r \in \Gamma_1 \cup \Gamma_2;\ 0 \leq \beta(r) \leq \beta_1,\ r \in \Gamma_1; \\ &\exists \widetilde{\theta} \in H^1(G),\ \widetilde{\theta}(r) = \Theta_0(r),\ r \in \Gamma_1 \cup \Gamma_2,\ 0 \leq \widetilde{\theta}(r) \leq \Theta_1,\ r \in G. \end{aligned} \tag{20}$$

Here, v_0, Θ_1, and β_1 are constants.

Definition 1. A pair $\{\theta, \varphi\} \in H^1(G) \times H^1(G)$ is called weak solution of the problem (15)–(18), if

$$a(\nabla\theta, \nabla u) + (\mathbf{v} \cdot \nabla\theta + b\kappa_a\theta^4 - b\kappa_a\varphi, u) = 0 \quad \text{for all } u \in V, \tag{21}$$

$$\alpha(\nabla\varphi, \nabla w) + \kappa_a(\varphi - \theta^4, w) + \int_{\Gamma_1} \beta(\varphi - \Theta_0^4)w d\Gamma + \frac{1}{2}\int_{\Gamma_2 \cup \Gamma_3} \varphi w d\Gamma = 0 \quad \text{for all } w \in H^1(G), \tag{22}$$

and $\theta(r) = \Theta_0(r)$ for $r \in \Gamma_1 \cup \Gamma_2$.

Notice that the term (θ^4, v) is defined for any function $v \in H^1(G)$, since $\theta \in H^1(G) \subset L^6(G)$, and therefore $\theta^4 \in L^{3/2}(G)$.

Theorem 1. *If conditions (19) and (20) hold, then there exists at least one weak solution $\{\theta, \varphi\}$ of the problem (15)–(18) such that*

$$0 \leq \theta \leq \Theta_1, \qquad 0 \leq \varphi \leq \Theta_1^4.$$

The proof of Theorem 1 is based on constructing a nonlinear operator whose fixed point is a weak solution of the problem (15)–(18).

4. The reduction of the boundary value problem to an operator equation

Consider the following two closed convex sets in $L^2(G)$:

$$K = \{\theta \in L^2(G) : 0 \leq \theta \leq \Theta_1\},$$
$$M = \{\varphi \in L^2(G) : 0 \leq \varphi \leq \Theta_1^4\}.$$

Consider first the boundary-value problem (15) and (16) assuming that $\theta \in K$ in (15) is a fixed function.

Lemma 1. *If $\theta \in K$ is a fixed function, the problem (15) and (16) is uniquely solvable in the sense of (22).*

Proof. The problem under consideration is a linear elliptic equation with a bounded right-hand side and the Robin boundary condition (16). The existence of a unique function $\varphi \in H^1(G)$ satisfying (22) follows from the Lax–Milgram lemma because the bilinear form

$$\alpha(\nabla\varphi, \nabla w) + \kappa_a(\varphi, w) + \int_{\Gamma_1} \beta\varphi w d\Gamma + \frac{1}{2}\int_{\Gamma_2 \cup \Gamma_3} \varphi w d\Gamma$$

is $H^1(G)$-elliptic, and the functional

$$w \to \kappa_a(\theta^4, w) + \int_{\Gamma_1} \beta\Theta_0^4 w d\Gamma$$

is continuous on $H^1(G)$. $\square$

Define an operator $F : K \to L^2(G)$ by the relation $F[\theta] := \varphi$, where φ is a unique weak solution of the problem (15) and (16) with θ being fixed.

Lemma 2. *The operator F is continuous, and $F[K] \subset M$.*

Proof. Let $\varphi_1 = F[\theta_1]$ and $\varphi_2 = F[\theta_2]$, where $\theta_1, \theta_2 \in K$. Denote $\theta = \theta_1 - \theta_2$ and $\varphi = \varphi_1 - \varphi_2$. Set $w = \varphi$ in (22) to obtain:

$$\alpha\|\nabla\varphi\|^2 + \int_{\Gamma_1} \beta\varphi^2 d\Gamma + \frac{1}{2}\int_{\Gamma_2\cup\Gamma_3} \varphi^2 d\Gamma + \kappa_a\|\varphi\|^2 = \kappa_a(\theta_1^4 - \theta_2^4, \varphi). \tag{23}$$

Hence

$$\|\varphi\| \le \|\theta_1^4 - \theta_2^4\| \le 4\Theta_1^3\|\theta\|,$$

which proves the continuity of the operator F.

Prove now that $F[K] \subset M$. Let $\varphi = F[\theta]$, where $\theta \in K$. Set $w = \psi := \max\{\varphi - \Theta_1^4, 0\}$ in (22) to obtain:

$$\alpha\|\nabla\psi\|^2 + \int_{\Gamma_1} \beta\left(\varphi - \Theta_0^4\right)\psi d\Gamma + \frac{1}{2}\int_{\Gamma_2\cup\Gamma_3} \varphi\psi d\Gamma + \kappa_a(\varphi - \theta^4, \psi) = 0. \tag{24}$$

It is easy to see that

$$\int_{\Gamma_1} \beta\left(\varphi - \Theta_0^4\right)\psi d\Gamma \ge \int_{\Gamma_1} \beta\psi^2 d\Gamma,$$

$$(\varphi - \theta^4, \psi) = \int_{\varphi \ge \Theta_1^4} \left(\varphi - \theta^4\right)\psi dr \ge \|\psi\|^2.$$

Combining the last two relations with (24) implies $\psi \equiv 0$, which means that $\varphi(r) \le \Theta_1^4, r \in G$. The estimation $\varphi(r) \ge 0, r \in G$, is proved analogously. Thus, $\varphi \in M$, and the lemma is proved. □

Now, consider the problem (17) and (18) with $\varphi \in M$ being a given function. For this problem, a weak solution θ is defined by Eq. (21) and the boundary condition $\theta(r) = \Theta_0(r), r \in \Gamma_1 \cup \Gamma_2$.

Lemma 3. *If $\varphi \in M$, and the conditions (19) and (20) hold, then there exists a unique weak solution $\theta \in K$ of the problem (17) and (18).*

Proof. Consider the following closed convex and bounded subset of the space $H^1(G)$:

$$\mathcal{K} = \{\zeta \in H^1(G) : \|\nabla\zeta\| \le R,\ \zeta(r) = \Theta_0(r) \text{ for } r \in \Gamma_1 \cup \Gamma_2\}.$$

Here, R is a positive "large" constant whose value will be appropriately chosen below.

Let $(H^1(G))'$ be the dual space of $H^1(G)$, and the angular brackets denote the duality. Define an operator $\mathcal{A} : \mathcal{K} \to (H^1(G))'$ by the relation

$$\langle\mathcal{A}\theta, u\rangle = a(\nabla\theta, \nabla u) + (\mathbf{v}\cdot\nabla\theta + b\kappa_a\theta^4 - b\kappa_a\varphi, u) \quad \text{for all } u \in H^1(G).$$

The restriction of this operator to the set $\mathcal{K}$ is bounded and possesses the following property: if a sequence $\{\theta_j\}_{j=1}^\infty \subset \mathcal{K}$ weakly converges to $\theta \in \mathcal{K}$ in $H^1(G)$, then

$$\liminf_{j\to\infty}\langle\mathcal{A}\theta_j, \theta_j - u\rangle \ge \langle\mathcal{A}\theta, \theta - u\rangle \quad \text{for all } u \in \mathcal{K}.$$

Therefore, the operator $\mathcal{A}$ is pseudomonotone (see [8]), and there exists a function $\theta \in \mathcal{K}$ such that

$$\langle\mathcal{A}\theta, \theta - \zeta\rangle \le 0 \quad \text{for all } \zeta \in \mathcal{K}. \tag{25}$$

Prove now that any function $\theta \in \mathcal{K}$ satisfying inequality (25) is a solution to the problem (17) and (18), if the constant R appearing in the definition of the set $\mathcal{K}$ is sufficiently large. Set $\zeta := \min\{\theta, \Theta_1\} = \theta - \eta$, where $\eta := \max\{\theta - \Theta_1, 0\}$. Notice that $\zeta \in \mathcal{K}$ and substitute it to inequality (25) to obtain the estimate

$$a\|\nabla\eta\|^2 + (\mathbf{v}\cdot\nabla\eta, \eta) + b\kappa_a(\theta^4 - \varphi, \eta) \le 0. \tag{26}$$

Since $\eta(r) = 0$ for $r \in \Gamma_1 \cup \Gamma_2$, then

$$(\mathbf{v}\cdot\nabla\eta, \eta) = \frac{1}{2}\int_{\Gamma_3} v_n\eta^2 d\Gamma \ge 0.$$

The condition $\varphi \le \Theta_1^4$ implies that

$$(\theta^4 - \varphi, \eta) \ge 0.$$

The last two inequalities and the estimate (26) show that $\eta \equiv 0$, and therefore $\theta \le \Theta_1$ in G.

Analogously, setting $\zeta := \max\{\theta, 0\}$ and using the inequality (25) yield the relation $\theta \geq 0$, $r \in G$. Thus, $\theta \in K$.
Further, if $R \geq \|\nabla\widetilde{\theta}\|$, where $\widetilde{\theta}$ is the function from the condition (20), then $\mathcal{K} \ni \widetilde{\theta}$. Setting $\zeta = \widetilde{\theta}$ in (25) yields

$$a(\nabla\theta, \nabla(\theta - \widetilde{\theta})) + (\mathbf{v} \cdot \nabla\theta, \theta - \widetilde{\theta}) \leq b\kappa_a(\varphi - \theta^4, \theta - \widetilde{\theta}).$$

Therefore,

$$\begin{aligned} a\|\nabla\theta\|^2 + \frac{1}{2}\int_{\Gamma_3} v_n(\theta - \widetilde{\theta})^2 d\Gamma &\leq a(\nabla\theta, \nabla\widetilde{\theta}) + (\mathbf{v} \cdot \nabla\widetilde{\theta}, \widetilde{\theta} - \theta) + b\kappa_a(\varphi - \theta^4, \theta - \widetilde{\theta}) \\ &\leq \frac{a}{2}\|\nabla\theta\|^2 + \frac{a}{2}\|\nabla\widetilde{\theta}\|^2 + v_0\|\nabla\widetilde{\theta}\|\Theta_1\sqrt{mes\,G} + b\kappa_a\Theta_1^5 mes\,G. \end{aligned}$$

Thus,

$$\|\nabla\theta\| \leq C_2, \tag{27}$$

where

$$C_2^2 = \|\nabla\widetilde{\theta}\|^2 + \frac{2}{a}\Theta_1\left(v_0\|\nabla\widetilde{\theta}\|\sqrt{mes\,G} + b\kappa_a\Theta_1^4 mes\,G\right).$$

Notice that, if $R > C_2$, then, for each function $u \in V$, there exists a sufficiently small $\varepsilon > 0$ such that $\zeta = \theta \pm \varepsilon u \in \mathcal{K}$. Hence, inequality (25) implies equation $\langle\mathcal{A}\theta, u\rangle = 0$, which means that (21) is valid for all $u \in V$. Thus, the existence of weak solutions of the problem (17) and (18) is proved. The above-described proof shows that all such solutions satisfy the inequality $0 \leq \theta \leq \Theta_1$ and the estimate (27). It is very important that the bound C_2 does not depend on functions $\varphi \in M$ yielding solutions θ.

To prove the uniqueness of solutions to the problem (17) and (18) in the class $H^1(G) \cap K$, it is sufficient to notice that the nonlinearity in Eq. (21) is monotone. Therefore, the difference ζ of two solutions satisfies the inequality

$$a\|\nabla\zeta\|^2 + \frac{1}{2}\int_{\Gamma_3} v_n\zeta^2 d\Gamma \leq 0,$$

which implies $\zeta \equiv 0$. $\square$

Lemma 3 allows us to define an operator $T : M \to K$ by setting $T[\varphi] := \theta$, where θ is a unique weak solution of the problem (17) and (18) corresponding to the function φ.

Lemma 4. *The operator T is continuous and has a relatively compact image.*

Proof. Let φ_1 and φ_2 be two arbitrary functions belonging to M. Set $\theta_1 = T[\varphi_1]$, $\theta_2 = T[\varphi_2]$, and $\zeta = \theta_1 - \theta_2$. Set $u = \zeta$ in Eq. (21) to obtain the equality

$$a\|\nabla\zeta\|^2 + \frac{1}{2}\int_{\Gamma_3} v_n\zeta^2 d\Gamma + b\kappa_a(\theta_1^4 - \theta_2^4, \theta_1 - \theta_2) = b\kappa_a(\varphi_1 - \varphi_2, \zeta).$$

Omit the second and third non-negative terms of the left-hand side to obtain the estimate

$$a\|\nabla\zeta\|^2 \leq b\kappa_a\|\varphi_1 - \varphi_2\|\ \|\zeta\|. \tag{28}$$

Since $\zeta(r) = 0$ for $r \in \Gamma_1 \cup \Gamma_2$, the Poincare inequality implies:

$$\|\zeta\|^2 \leq C(G)\|\nabla\zeta\|^2,$$

where the constant $C(G) > 0$ depends only on G. Therefore, the relation (28) yields the estimate

$$\|\theta_1 - \theta_2\| \leq \frac{b\kappa_a}{a}C(G)\|\varphi_1 - \varphi_2\|, \tag{29}$$

which proves the continuity of the operator T. The relative compactness of the image of T follows from the estimate (27) and the compactness of the embedding $H^1(G) \subset L^2(G)$. $\square$

Define an operator $A : K \to K$ as the superposition of the operators F and T, i.e. $A[\theta] := T[F[\theta]]$. By Lemmas 2 and 4, A is a continuous operator with relatively compact image. According to the Schauder fixed-point theorem, there is a function $\theta \in K$ such that $\theta = A[\theta]$. Obviously, the pair $\{\theta, F[\theta]\} \in K \times M$ is a weak solution of the problem (15)–(18). Theorem 1 is proved. $\square$

5. The uniqueness of solutions

Consider two weak solutions $\{\theta_1, \varphi_1\}$ and $\{\theta_2, \varphi_2\}$ of the problem (15)–(18) such that $\{\theta_i, \varphi_i\} \in K \times M$, $i = 1, 2$. Denote $\zeta = \theta_1 - \theta_2$, $\varphi = \varphi_1 - \varphi_2$. It follows from (23) that

$$\alpha\|\nabla\varphi\|^2 + \int_{\Gamma_1} \beta\varphi^2 d\Gamma + \frac{1}{2}\int_{\Gamma_2\cup\Gamma_3} \varphi^2 d\Gamma + \kappa_a\|\varphi\|^2 \le 4\kappa_a\Theta_1^3\|\zeta\|\,\|\varphi\|.$$

Accounting for the estimate (29) implies the inequality

$$\alpha\|\nabla\varphi\|^2 + \int_{\Gamma_1} \beta\varphi^2 d\Gamma + \frac{1}{2}\int_{\Gamma_2\cup\Gamma_3} \varphi^2 d\Gamma + \kappa_a\|\varphi\|^2 \le 4\kappa_a^2 b\Theta_1^3 C(G)\|\varphi\|^2/a. \tag{30}$$

Denote

$$\gamma(G) = \inf_{u\in H^1(G),\|u\|=1}\left\{\alpha\|\nabla u\|^2 + \int_{\Gamma_1}\beta u^2 d\Gamma + \frac{1}{2}\int_{\Gamma_2\cup\Gamma_3} u^2 d\Gamma\right\}.$$

Thus, if φ is not equal to zero, then the estimate (30) yields:

$$\gamma(G) + \kappa_a \le 4\kappa_a^2 b\Theta_1^3 C(G)/a.$$

The violation of the last inequality means that $\varphi = 0$, and we arrive at the following result.

Theorem 2. *If the conditions (19) and (20) hold, and the inequality*

$$4\kappa_a^2 b\Theta_1^3 C(G)/a - \kappa_a < \gamma(G) \tag{31}$$

is true, then there exists a unique weak solution $\{\theta, \varphi\}$ of the problem (15)–(18) in the class $K \times M$.

Notice that the positive quantity $\gamma(G)$ depends only on the coefficients α and β and the region G.

6. Example

Specify the estimate (31) in the case where the domain G is a parallelepiped, i.e. $G = (0, l_x) \times (0, l_y) \times (0, l_z)$. Let $\Gamma_1 = \Gamma \setminus \left(\overline{\Gamma_2} \cup \overline{\Gamma_3}\right)$, where

$$\Gamma_2 = \{(x, y, z) \in \mathbb{R}^3 : y = 0,\ x \in (0, l_x),\ z \in (0, l_z)\},$$
$$\Gamma_3 = \{(x, y, z) \in \mathbb{R}^3 : y = l_y,\ x \in (0, l_x),\ z \in (0, l_z)\}.$$

Since the function $\zeta \in V$ vanishes on the sets

$$\{(x, y, z) \in \mathbb{R}^3 :\ x = 0,\ l_x,\ y \in (0, l_y),\ z \in (0, l_z)\},$$
$$\{(x, y, z) \in \mathbb{R}^3 :\ z = 0,\ l_z,\ x \in (0, l_x),\ y \in (0, l_y)\},$$

then

$$\|\nabla\zeta\|^2 \ge \lambda_{\min}\|\zeta\|^2,$$

where $\lambda_{\min} = (1/l_x^2 + 1/l_z^2)\pi^2$ is the smallest eigenvalue of the 2D-Laplace operator considered on the rectangle $(0, l_x) \times (0, l_z)$ and subjected to the homogeneous Dirichlet boundary condition. Therefore, the constant $C(G)$ satisfies the inequality $C(G) \le (1/l_x^2 + 1/l_z^2)^{-1}/\pi^2$.

To estimate $\gamma(G)$, use the identity

$$\|u\|^2 = l_y\int_{\Gamma_3} u^2 d\Gamma - \int_G (u^2)_y\, y dr, \quad u \in H^1(G),$$

which implies:

$$\|u\|^2 \le l_y\int_{\Gamma_3} u^2 d\Gamma + 2l_y\|u\|\ \|u_y\| \le l_y\int_{\Gamma_3} u^2 d\Gamma + \frac{1}{2}\|u\|^2 + 2l_y^2\|\nabla u\|^2.$$

Therefore,

$$\frac{1}{2}\|u\|^2 \le 2l_y \max\left(1, \frac{l_y}{\alpha}\right)\left(\frac{1}{2}\int_{\Gamma_3} u^2 d\Gamma + \alpha\|\nabla u\|^2\right).$$

Further, assume that $\|u\| = 1$ to obtain the following estimate:

$$\gamma(G) \ge \inf_{u\in H^1(G),\|u\|=1}\left\{\alpha\|\nabla u\|^2 + \frac{1}{2}\int_{\Gamma_3} u^2 d\Gamma\right\} \ge \left(4l_y \max\left(1, \frac{l_y}{\alpha}\right)\right)^{-1}.$$

Therefore, the condition

$$\frac{4\kappa_a^2 b\Theta_1^3}{a\pi^2} \cdot \frac{l_x^2 l_z^2}{l_x^2 + l_z^2} - \kappa_a < \left(4l_y \max\left(1, \frac{l_y}{\alpha}\right)\right)^{-1} \tag{32}$$

ensures the fulfillment of inequality (31) of Theorem 2, and therefore provides the uniqueness of weak solutions of the problem (15)–(18).

Inequality (32) is valid either if the size of the region G is small, or in the case of small values of the absorption coefficient κ_a and of great heat conductivity of the medium.

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