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# Linear Regression

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Robert Haas

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**••• THIS IS STILL A DRAFT •••**

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**Project page:** <https://github.com/Haasrobertgmxnet/HeatConduction>  
**Contact:** [Haasrobert@gmx.net](mailto:Haasrobert@gmx.net)

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## Abstract

This documentation presents ...

## Contents

<b>1 Least Squares of correlated Data</b>	<b>2</b>
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## 1 Least Squares of correlated Data

Given a data set  $\{(x_i, y_i) \in \mathbb{R}^2, i \in \{1, \dots, n\}\}$  we ask for an approximate functional dependency of the form  $y_i \approx f_i = f(x_i)$  for all  $i \in \{1, \dots, n\}$ . This means,  $f$  is a mathematical function assigning each element  $x$  of an interval  $I \subset \mathbb{R}$  an unique  $y \in \mathbb{R}$ . For the rest of this documentation we assume that  $f$  is a linear function, i.e. there exist numbers  $\beta_0$  and  $\beta_1$  in  $\mathbb{R}$  such that  $f(x) = \beta_0 + \beta_1 x$ . To achieve an optimal  $f$  with optimal numbers  $\beta_0$  and  $\beta_1$  usually a least-square approach is used:

$$\min_{\beta_0, \beta_1} F(\beta_0, \beta_1) \text{ with } F(\beta_0, \beta_1) = \frac{1}{2} \sum_{i=1}^n |\beta_0 + \beta_1 x_i - y_i|^2. \quad (1)$$

For optimal  $\hat{\beta}_0$  and  $\hat{\beta}_1$  the necessary conditions read as

$$\frac{\partial F}{\partial \beta_1} = \sum_{i=1}^n (\hat{\beta}_0 + \hat{\beta}_1 x_i - y_i) x_i = 0 \text{ and } \frac{\partial F}{\partial \beta_0} = \sum_{i=1}^n (\hat{\beta}_0 + \hat{\beta}_1 x_i - y_i) = 0. \quad (2)$$

Now we need the following definitions:

$$\bar{x} = \frac{1}{n} \sum_{i=1}^n x_i, \text{ the arithmetic mean of } \{x_i\}_{i=1}^n, \quad (3)$$

$$\bar{y} = \frac{1}{n} \sum_{i=1}^n y_i, \text{ the arithmetic mean of } \{y_i\}_{i=1}^n, \quad (4)$$

$$\sigma_x = \sqrt{\nu_n \sum_{i=1}^n (x_i - \bar{x})^2}, \text{ the standard deviation of } \{x_i\}_{i=1}^n, \quad (5)$$

$$\sigma_y = \sqrt{\nu_n \sum_{i=1}^n (y_i - \bar{y})^2}, \text{ the standard deviation of } \{y_i\}_{i=1}^n, \quad (6)$$

$$\varrho_{x,y} = \frac{\nu_n \sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})}{\sigma_x \sigma_y}, \text{ the correlation coefficient of } \{x_i\}_{i=1}^n \text{ and } \{y_i\}_{i=1}^n. \quad (7)$$

Here  $\nu_n$  is either  $1/(n-1)$  if  $\{(x_i, y_i)\}_{i=1}^n$  is a sample, and  $\nu_n = 1/n$ , else. With this definitions the solutions  $\hat{\beta}_0$  and  $\hat{\beta}_1$  of (2) are  $\hat{\beta}_1 = \sigma_y \varrho_{x,y} / \sigma_x$  and  $\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}$ . We need some more definitions:

$$SS_{\text{tot}} = \sum_{i=1}^n (y_i - \bar{y})^2, \text{ the total sum of squares ,}$$

$$SS_{\text{reg}} = \sum_{i=1}^n (f_i - \bar{y})^2, \text{ the explained sum of squares ,}$$

$$SS_{\text{res}} = \sum_{i=1}^n (f_i - y_i)^2, \text{ the residual sum of squares .}$$

**Theorem 1.1.** *The equation  $SS_{\text{tot}} = SS_{\text{reg}} + SS_{\text{res}}$  is true.*

*Proof.* There is

$$\begin{aligned}
 SS_{\text{tot}} &= \sum_{i=1}^n (y_i - \bar{y})^2 \\
 &= \sum_{i=1}^n ((y_i - f_i) + (f_i - \bar{y}))^2 \\
 &= SS_{\text{reg}} + SS_{\text{res}} - 2 \sum_{i=1}^n (f_i - y_i)(f_i - \bar{y})
 \end{aligned}$$

It remains to show  $\sum_{i=1}^n (f_i - y_i)(f_i - \bar{y}) = 0$ . We have

$$\begin{aligned}
 \sum_{i=1}^n (f_i - y_i)(f_i - \bar{y}) &= \sum_{i=1}^n \left( (f_i - y_i)(\hat{\beta}_0 + \hat{\beta}_1 x_i) - (f_i - y_i)\bar{y} \right) \\
 &= (\hat{\beta}_0 - \bar{y}) \sum_{i=1}^n (f_i - y_i) + \hat{\beta}_1 \sum_{i=1}^n (f_i - y_i)x_i.
 \end{aligned}$$

Finally we have

$$\begin{aligned}
 \sum_{i=1}^n (f_i - y_i) &= \frac{\partial F}{\partial \beta_0} = \sum_{i=1}^n (\hat{\beta}_0 + \hat{\beta}_1 x_i - y_i) = 0, \\
 \sum_{i=1}^n (f_i - y_i)x_i &= \frac{\partial F}{\partial \beta_1} = \sum_{i=1}^n (\hat{\beta}_0 + \hat{\beta}_1 x_i - y_i)x_i = 0,
 \end{aligned}$$

as from (2). That is  $\sum_{i=1}^n (f_i - y_i)(f_i - \bar{y}) = 0$  and  $SS_{\text{tot}} = SS_{\text{reg}} + SS_{\text{res}}$ .  $\square$

The coefficient of determination  $R^2$  is given by

$$R^2 = 1 - \frac{SS_{\text{res}}}{SS_{\text{tot}}} = \frac{SS_{\text{reg}}}{SS_{\text{tot}}}.$$

**Theorem 1.2.** *The equation  $\varrho_{x,y}^2 = R^2$  is true.*

*Proof.* There is

$$\begin{aligned}
 \frac{SS_{\text{reg}}}{SS_{\text{tot}}} &= \frac{\sum_{i=1}^n (\hat{\beta}_0 + \hat{\beta}_1 x_i - \bar{y})^2}{SS_{\text{tot}}} \\
 &= \frac{\hat{\beta}_1^2 \sum_{i=1}^n (x_i - \bar{x})^2}{\sum_{i=1}^n (y_i - \bar{y})^2} \\
 &= \frac{\hat{\beta}_1^2 \sigma_x^2}{\sigma_y^2} \\
 &= \left( \frac{\sigma_y}{\sigma_x} \varrho_{x,y} \right)^2 \frac{\sigma_x^2}{\sigma_y^2} = \varrho_{x,y}^2
 \end{aligned}$$

$\square$