MAT-MEK 4270 Assignment 1

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1 Exact solution

We consider a plane-wave solution of the form

$$u(t, x, y) = e^{i(k_x x + k_y y - \omega t)}, \tag{1}$$

where $i = \sqrt{-1}$ is the imaginary unit.

The derivatives are

$$u_t = -i\omega u, \quad u_{tt} = -\omega^2 u, \quad u_x = ik_x u, \quad u_{xx} = -k_x^2 u, \quad u_y = ik_y u, \quad u_{yy} = -k_y^2 u.$$

Substituting these into the 2D wave equation

$$u_{tt} = c^2(u_{xx} + u_{yy}),$$

we obtain

$$-\omega^2 u = c^2(-k_x^2 - k_y^2)u,$$

which gives the continuous dispersion relation

$$\omega^2 = c^2(k_x^2 + k_y^2). \tag{2}$$

Hence, the plane wave satisfies the wave equation provided this relation holds.

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2 Dispersion coefficient

Assume $m_x = m_y$ so that $k_x = k_y = k$. A discrete plane wave can then be written as

$$u_{ij}^n = e^{i\left(kh(i+j) - \tilde{\omega} n \Delta t\right)}, \tag{3}$$

where $\tilde{\omega}$ is the numerical (discrete) dispersion coefficient.

We insert this into the standard second-order accurate finite-difference scheme

$$u_{ij}^{n+1} - 2u_{ij}^n + u_{ij}^{n-1} = (c\Delta t)^2 \Delta_h u_{ij}^n, \tag{4}$$

where the discrete Laplacian Δ_h is given by

$$\Delta_h u_{ij}^n = \frac{u_{i+1,j}^n - 2u_{ij}^n + u_{i-1,j}^n}{h^2} + \frac{u_{i,j+1}^n - 2u_{ij}^n + u_{i,j-1}^n}{h^2}. \tag{5}$$

Substituting the discrete plane wave yields

$$e^{-i\tilde{\omega}\Delta t} - 2 + e^{i\tilde{\omega}\Delta t} = (c\Delta t)^2 \frac{2\big(\cos(kh) + \cos(kh) - 2\big)}{h^2}.$$

Simplifying both sides gives

$$-4\sin^2\left(\frac{\tilde{\omega}\Delta t}{2}\right) = (c\Delta t)^2 \left[-\frac{8}{h^2}\sin^2\left(\frac{kh}{2}\right)\right].$$

Thus, the discrete dispersion relation becomes

$$\sin\left(\frac{\tilde{\omega}\Delta t}{2}\right) = \sqrt{2}\,C\,\sin\left(\frac{kh}{2}\right), \qquad C = \frac{c\Delta t}{h}. \tag{6}$$

For a CFL number $C = 1/\sqrt{2}$ we find

$$\sin\left(\frac{\tilde{\omega}\Delta t}{2}\right) = \sin\left(\frac{kh}{2}\right) \quad \Rightarrow \quad \tilde{\omega}\Delta t = kh.$$

Hence,

$$\tilde{\omega} = \frac{kh}{\Delta t} = \frac{kh}{h/(\sqrt{2}\,c)} = \sqrt{2}\,c\,k.$$

From the exact dispersion relation we have

$$\omega = c\sqrt{k_x^2 + k_y^2} = c\sqrt{2}\,k,$$

so that

$$\tilde{\omega} = \omega \quad when C = \frac{1}{\sqrt{2}}. \tag{7}$$