TP 6 Correction

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The model of the ordinary least squares is considered under the following hypotheses :

- H1 : The model is linear in $x_{i,t}$
- H2 : the values $x_{i,t}$ are observed without error
- H3 : $\mathbb{E}[u] = 0$ the mean of the error is equal to zero
- H4 : $\mathbb{E}[u^t u] = \sigma^2 I_n$, the variance of the error is constant
- H5 : $\mathbb{E}[u_t u_{t+1}]$, the errors are not correlated
- H6 : $Cov(x_{i,t}, u_t)$, the error is independent of the explanatory variable

1 Autocorrelation in the error

The violation of hypothesis H5 concerns time series where the off-diagonal elements of the covariance matrix of the errors are nonzero. In this case, the obtained OLS estimators are unbiased but no longer have a minimal variance. Thus, we must identify new estimators and techniques for detecting possible autocorrelation of the errors.

Exercise 1

We consider a model

$$i3t = \beta_0 + \beta_1 inft + \beta_2 deft + u_t$$

for $t = 1, \cdots, n$. From TP5, we know that the error term is an AR(1) process :

$$u_t = \rho u_{t-1} + e_t$$

with $|\rho| < 1$ (statbility conditon), and such that e_t are independent, zero mean random variables with variance σ_e^2. Thus, the form of u_t implies the autocorrelation in the error term.

1.1 Data transformation

We write the model in matrix form :

$$Y = X\beta + u$$
$$u_t = \rho u_{t-1} + e_t$$

Note that e are zero mean random variables with constant variance σ_e^2. Thus, $Var(u_t) = \frac{\sigma_e^2}{1-\rho^2} = \sigma_u^2$ and $\mathbb{E}[u_t u_{t-i}] = \rho^i \sigma_u^2$. The covariance matrix of residuals u (for AR(1) model) is of the form :

$$\Omega_u = \mathbb{E}[u^t u] = \frac{\sigma_e^2}{1-\rho^2} \cdot \begin{pmatrix} 1 & \rho & \rho^2 & \cdots & \rho^{T-1} \\ \rho & 1 & & & \\ \rho^2 & & & & \\ \cdots & & & & \\ \cdots & & & & \\ \rho^{T-1} & & & & 1 \end{pmatrix}$$

Since Ω_u is symmetric and positive definite, there exists matrix P such that $\Omega_u^{-1} = P^\intercal P$ (Choleski decomposition).The model $PY = PX + Pu$ has independent and homoskedactic errors. The matrix P for the covariance Ω_u of AR(1) process is of the form :

$$P = \begin{pmatrix} \sqrt{1-\rho^2} & 0 & 0 & \cdots & & 0 \\ -\rho & 1 & 0 & & & 0 \\ 0 & -\rho & 1 & & & 0 \\ \cdots & & & & & \\ \cdots & & & & & 0 \\ 0 & & & & -\rho & 1 \end{pmatrix}$$

We multiply Y by matrix P, ie. $Y \mapsto PY$ and compute $y_t - \rho y_{t-1}$ for $t = 2 \dots T$ and we take $\sqrt{1-\rho^2} y_1$ for the first observation. Code :

$$y = intdef(:, 2);$$
$$[n, k] = size(intdef)$$
$$X = [ones(n, 1), intdef(:, [3, 6])];$$

$$y\_ = [y(2 : n)]$$
$$y\_lag = [y(1 : n-1)]$$
$$new\_y = [sqrt(1 - rho^2) * y(1);$$
$$y\_ - rho * y\_lag]$$

$$X\_ = [X(2 : n, :)]$$
$$X\_lag = [X(1 : n-1, :)]$$
$$new\_X = [sqrt(1 - rho^2) * X(1, :);$$
$$X\_ - rho * X\_lag]$$
$$P = zeros(n, n);$$
$$v = repmat(1, 1, n)$$
$$P = P + diag(v, 0)$$
$$v = repmat(-rho, 1, n-1)$$
$$P = P + diag(v, -1)$$
$$P(1, 1) = sqrt(1 - rho^2)$$

$$new\_y\_bis = P * y$$
$$new\_X\_bis = P * X$$

We perform OLS estimation for transformed data :

$$[n, k] = size(new\_X)$$

$$beta = inv(new\_X' * new\_X) * new\_X' * new\_y$$

$$u = new\_y - new\_X * beta;$$

2 Granger causality

Consider two series y_t and z_t such that

$$y_t = \delta_0 + \alpha_1 y_{t-1} + \gamma_1 z_{t-1} + \alpha_2 y_{t-2} + \gamma_2 z_{t-2} + \cdots$$

and

$$z_t = \eta_0 + \beta_1 y_{t-1} + \rho_1 z_{t-1} + \beta_2 y_{t-2} + \rho_2 z_{t-2} + \cdots ,$$

where each equation contains an error that has zero expected value given past information on y and z. Such equations allow us to test whether, after controlling for past y, past z help to forecast y_t.

Generally, we say that z **Granger causes** y if

$$E(y_t|I_{t-1}) \neq E(y_t|J_{t-1}), \tag{1}$$

where I_{t-1} contains past information on y and z, and J_{t-1} contains only information on past y. When (1) holds, past z is useful, in addition to past y, for predicting y_t.

Exercise 2

We consider a model

$$i3_t = \beta_0 + \beta_1 inf_{t-1} + \beta_2 inf_{t-2} + \beta_3 def_{t-1} + \beta_4 def_{t-2} + u_t$$

for $t = 1 \ldots n$. Code :

$$y = intdef(:, 2);$$

$$[n, k] = size(intdef)$$

$$X = [intdef(:, [3, 6])];$$

$$y = intdef(3 : n, 2);$$

$$X\_lag = [X(2 : n-1, :)]$$

$$X\_lag2 = [X(1 : n-2, :)]$$

$$X = [X\_lag X\_lag2]$$

$$[n, k] = size(X)$$

$$beta = inv(X' * X) * X' * y$$

We want to test the significance of inflation and budget deficit (Granger causality test) : H_0 : $\beta_1 = \beta_2 = 0$ and $H_0 : \beta_3 = \beta_4 = 0$. Code :

$$SSR0 = u' * u;$$

$$X = X(:, [2, 4]);$$

$$beta1 = inv(X' * X) * X' * y;$$

$$u1 = y - X * beta1;$$
$$SSR1 = u1' * u1;$$

$$F = ((SSR1 - SSR0)/SSR0) * ((n - k)/2);$$
$$p = fdis\_prb(F, 2, n - k)$$

$$X = [X\_lagX\_lag2];$$
$$X = X(:, [1, 3]);$$
$$beta1 = inv(X' * X) * X' * y;$$
$$u1 = y - X * beta1;$$
$$SSR1 = u1' * u1;$$

$$F = ((SSR1 - SSR0)/SSR0) * ((n - k)/2);$$
$$p = fdis\_prb(F, 2, n - k)$$

We obtain respectively $p = 1.1491^{-14}$ and $p = 1.4639^{-4}$ thus, we strongly reject H_0 each time.