Notes on Raney's Lemmas

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In a 1960 paper, George Raney proved the first two lemmas below; the lemmas suppose we have a finite sequence of numbers meeting certain constraints, and provide the number of cycle shifts that contain all-positive partial sums (Raney 1960).

These notes expand on the ideas behind these lemmas. The final section of these notes discusses cyclic shifts for finite sequences of independent, uniformly random values; as far as I know, the work in that section is new.

I personally learned of these lemmas in chapter 7 of the book *Concrete Mathematics* (Knuth, Patashnik, and Graham 1998), which explores their applications to generating functions. The presentation of the lemmas here is based on the presentation in *Concrete Mathematics* rather than on Raney's original paper.

1 Integer sequences

Lemma 1 *Suppose* $\sum_{i=1}^{n} x_i = 1$*, where all* $x_i \in \mathbb{Z}$*. Extend the sequence by letting* $x_{n+p} = x_p$ *for* $1 \le p \le n$*. Then there is a unique* j*,* $1 \le j \le n$*, such that*

$$\sum_{i=j}^{j+k-1} x_i > 0; \quad 1 \le k \le n.$$

Intuitively, we can think of such an index j as a cyclic shift of the sequence that has partial sums that are all positive.

For example, the finite sequence $\langle x_1, \ldots, x_5 \rangle = \langle 3, -2, 4, -1, 1 \rangle$ offers $j = 5$ as the unique shift providing $\langle x_5, x_6 = x_1, \ldots, x_9 = x_4 \rangle = \langle 1, 3, -2, 4, -5 \rangle$ with partial sums $\langle 1, 4, 2, 6, 1 \rangle$ that are all positive.

Definitions Given a sequence $\langle x_1, \ldots, x_n \rangle$, it's useful to say that an index $i \in \{1, \ldots, n\}$ is a *positive-sum shift* if and only if the partial sums of

$\langle x_i, \ldots, x_n, x_1, \ldots, x_{i-1} \rangle$ are all positive. Since these notes focus on finite sequences, we'll implicitly use arbitrary indexes $x_j, j \in \mathbb{Z}$, to refer to x_k with $k \in \{1, \ldots, n\}, k \equiv j \pmod{n}$.

We'll use the subscript-free letter x to denote an entire finite sequence $\langle x_1, \ldots, x_n \rangle$. We'll write $\sigma(x)$ to indicate the number of indexes of x that are positive-sum shifts.

We can now concisely state a related result proved by Raney:

Lemma 2 *Suppose $\sum_{i=1}^{n} x_i = \ell$, where $x_i \in \mathbb{Z}$ and $x_i \leq 1$ for all i. Then $\sigma(x) = \ell$; that is, exactly ℓ indexes in $\{1, \ldots, n\}$ are positive-sum shifts.*

For example, let $\langle x_1, \ldots, x_8 \rangle = \langle -2, 1, 1, 0, -1, 1, 1, 1 \rangle$. Then $\sum x_i = 2$, and x_2, x_6 are the only positive-sum shifts:

shift	partial sums
$\langle x_2, \ldots \rangle = \langle 1, 1, 0, -1, 1, 1, 1, -2 \rangle$	$\langle 1, 2, 2, 1, 2, 3, 4, 2 \rangle$
$\langle x_6, \ldots \rangle = \langle 1, 1, 1, -2, 1, 1, 0, -1 \rangle$	$\langle 1, 2, 3, 1, 2, 3, 3, 2 \rangle$

Note that lemma 2 is not a strict generalization of lemma 1 as it adds the condition $x_i \leq 1$. This condition is necessary for lemma 2; without it we may have, for example, the one-element sequence $x = \langle 2 \rangle$ with sum $\ell = 2$ and $\sigma(x) = 1$.

Rather than proving the above two lemmas directly, we'll jump to the general case of real sequences x and prove strictly more general bounds on $\sigma(x)$ in that context.

2 Real sequences

In a moment we'll prove a general guarantee that $\sum x_i > 0 \Rightarrow \sigma(x) \geq 1$. In the context of a sequence x, it will be useful to write s_i to denote the i^{th} partial sum of x; that is, $s_0 = 0$, and

$$s_i = \sum_{j=1}^{i} x_j, \quad \text{for } i \geq 1.$$

We can define s_i for $i > n$ using the implicitly periodic sequence characterized by $x_{n+i} = x_i$.

Property 3 *Suppose $\sum_{i=1}^{n} x_i > 0$, where $x_i \in \mathbb{R}$. Let s_i denote the i^{th} partial sum of x, and let j be the largest index in $\{1, \ldots, n\}$ with $s_{j-1} = \min_{0 \leq i < n} s_i$. Then j is a positive-sum shift.*

Proof Let

$$s_i' = \sum_{k=j}^{j+i-1} x_k$$

denote the i^{th} partial sum of the shifted sequence $\langle x_j, \ldots, x_{j+n-1} \rangle$. Then, for $1 \le i \le n$,

$$s_i' = s_{j+i-1} - s_{j-1} \begin{cases} > 0 \text{ (by definition of } j) & \text{when } j+i-1 < n \\ = s_n + s_{j+i-1-n} - s_{j-1} \ge s_n > 0 & \text{when } j+i-1 \ge n. \end{cases}$$

□

Now we can assume without loss of generality that any sequence of real numbers $\langle x_1, \ldots, x_n \rangle$ with $\sum x_i > 0$ is already shifted so that all its partial sums $s_i > 0$ for $i > 0$. As we'll see in the next property, this assumption allows us to provide a nice general expression for $\sigma(x)$. This expression depends on the set $S(x)$, defined as $\{\min_{j \le i \le n} s_i \mid 1 \le j \le n\}$ for any finite sequence x with i^{th} partial sum s_i.

Property 4 *Suppose that x is a finite real sequence with i^{th} partial sum s_i, and that $s_i > 0$ for all $i > 0$. Then*

$$\sigma(x) = \#S(x) = \#\left\{ \min_{j \le i \le n} s_i \;\middle|\; 1 \le j \le n \right\}. \tag{1}$$

More specifically, an index j with $1 \le j \le n$ is a positive-sum shift iff

$$s_{j-1} < s_i \;\forall i : j \le i \le n. \tag{2}$$

Proof We'll start by supposing we have an index j with $1 \le j \le n$ and $s_{j-1} < s_i$ for all i with $j \le i \le n$; our goal is to show that such a j must be a positive-sum shift. Our approach will be similar to the proof of property 3.

Let s_i' denote the i^{th} partial sum of $\langle x_j, \ldots, x_{j+n-1} \rangle$:

$$s_i' = \sum_{k=j}^{j+i-1} x_k.$$

Then

$$s_i' = s_{j+i-1} - s_{j-1} \begin{cases} > 0 & \text{if } j+i-1 \le n, \\ = s_{j+i-1-n} + s_n - s_{j-1} > 0 & \text{if } j+i-1 > n; \end{cases}$$

the last inequality follows since $s_{j+i-1-n} > 0$ and $s_n > s_{j-1}$.

On the other hand, if $s_{j-1} \geq s_i$ for some i, j with $1 \leq j \leq i \leq n$, then $s'_{i-j+1} = s_i - s_{j-1} \leq 0$, so that j isn't a positive-sum shift. This completes the proof of the last part of the property.

Now let's verify that the set $S = S(x)$ from (1) has size $\sigma(x)$.

Let $j_1, \ldots, j_k$ be all the positive-sum shifts with $1 < j_i \leq n$; note that $k = \sigma(x) - 1$ since the trivial shift index 1 has been excluded. Let $T = \{s_{j_1-1}, \ldots, s_{j_k-1}, s_n\}$.

Notice that $j = n + 1$ trivially meets condition (2); combine this with the first part of the proof to see that all elements of T meet condition (2). This guarantees that all the elements are unique, so that $|T| = \sigma(x)$. This also means that $T \subset S$. Finally, observe that, for any $s_j \in S$, there's a largest index j' with $1 \leq j' \leq n$ and $s_{j'} = s_j$; this index j' meets condition (2), so that $S \subset T$, confirming that $|S| = |T| = \sigma(x)$. $\square$

Property 4 lends itself to a nice visual intuition. Consider the example sequence $\langle 2, -1, 2, 2, -3, 2, 1, 1, -1, -2 \rangle$ of length $n = 10$. Below is the line graph of its partial sums, starting with $s_0 = 0$.

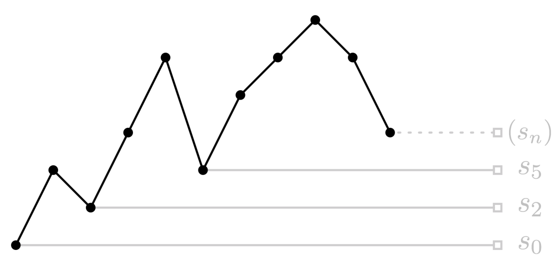

Figure 1: *Line graph of the partial sums s_i of the example sequence.*

Imagine an observer standing far to the right of the graph and looking directly to the left so they can only see along a perfectly horizontal line of sight. Below s_n, they can only see the three points s_0, s_2, and s_5. These are exactly the partial sums meeting condition (2), so that they correspond directly to all the positive-sum shifts of x, which have indexes 1, 3, and 6.

This visual intuition — that points visible-from-the-right and below s_n correspond exactly to the positive-sum shifts — extends to any sequence meeting the suppositions of property 4.

TEMP begin work on bounds

It's now possible to prove a simple general upper and lower bound for $\sigma(x)$ in the case that each x_i is an integer. We'll see below that these bounds provide both lemmas 1 and 2 as corollaries.

Property 5 *Suppose we have a finite integer sequence $x = \langle x_1, \ldots, x_n \rangle$ with*

$s_n > 0$. *Let* $m = \max_i x_i$. *Then*

$$\lceil s_n/m \rceil \leq \sigma(x) \leq s_n.$$

Proof Notice that we can work with any cyclic shift x' of x without changing s_n or m. Thus, using property 3, we can assume without loss of generality that $s_i > 0$ for $i > 0$.

Next, we can bound the elements of $S(x)$ via

$$0 < \min_{j \leq i \leq n} s_i \leq s_n$$

for all j with $1 \leq j \leq n$. So all elements of $S(x)$ are in the range $(0, s_n]$, and are integers. Hence $\#S(x) \leq s_n$, completing the proof of the upper bound.

Toward the lower bound, let's suppose that $S(x) = \{s_{j_1}, \ldots, s_{j_k}\}$ with each s_{j_i} meeting condition (2) and $0 < s_{j_i} < s_{j_{i+1}}$; refer to the proof of property 4 for more details on why we can suppose these conditions.

By our definition of s_{j_i}, we have

$$s_{j_i} = \min_{j_i \leq k \leq n} s_k \quad \text{and} \quad s_{j_{i+1}} = \min_{j_i+1 \leq k \leq n} s_k.$$

We also have $s_{j_i+1} - s_{j_i} \leq m$. Together, these last two facts tell us that $s_{j_{i+1}} - s_{j_i} \leq m$.

Note that $s_{j_k} = s_n$ so that $s_n - s_{j_{k-1}} \leq m \Rightarrow s_{j_{k-1}} \geq s_n - m$. This can be extended to see that $s_{j_{k-2}} \geq s_n - 2m$, and in general that

$$s_{j_{k-p}} \geq s_n - pm.$$

Our definition of m gives us that $s_{j_1} \leq m$, so $m \geq s_{j_1} \geq s_n - (k-1)m$, from which we can derive that

$$1 \geq s_n/m - (k-1) \quad \Rightarrow \quad k \geq s_n/m \quad \Rightarrow \quad k \geq \lceil s_n/m \rceil;$$

the last inequality uses the fact that $k = \sigma(x)$ is an integer. This completes the proof. □

TEMP end work on bounds

The Contraction Perspective

Next we'll consider a contraction operation that may shorten a sequence x while preserving $\sigma(x)$.

Call a sequence $x = \langle x_1, \ldots, x_n \rangle$ *sum-positive* iff $s_i > 0$ when $i > 0$. We'll say that a sequence $x' = \langle x'_1, \ldots, x'_{n-1} \rangle$ is a *contraction* of the length-n sum-positive sequence x iff there is some index j so that $x_{j+1} \leq 0$ and, for $1 \leq i \leq n-1$,

$$x'_i = \begin{cases} x_i & \text{if } i < j, \\ x_i + x_{i+1} & \text{if } i = j, \text{and} \\ x_{i+1} & \text{if } i > j. \end{cases}$$

For example. $x' = \langle 2, -1, 2 \rangle$ is a contraction of $x = \langle 3, -1, -1, 2 \rangle$ since the sequences are same except for the replacement of x_1, x_2 by their sum as x'_1, and $x_2 = -1 \leq 0$. The alternative sequence $x'' = \langle 3, -1, 1 \rangle$ is *not* a contraction as it replaces x_3, x_4 with their sum x''_3, but $x_4 = 2 > 0$.

Property 5 *If x' is a contraction of x, then x' is sum-positive and $\sigma(x') = \sigma(x)$.*

Proof Let j be the contracted index, so that $x'_j = x_j + x_{j+1}$ and $x_{j+1} \leq 0$.

Let s'_i denote the i^{th} partial sum of x'. Then

$$s'_i = \begin{cases} s_i & \text{if } 0 \leq i < j \\ s_{i+1} & \text{if } j \leq i \leq n-1. \end{cases}$$

So $s'_i > 0$ for $0 < i \leq n-1$, making x' sum-positive.

Since $x_{j+1} \leq 0$, $s_{j+1} \leq s_j$. This means that

$$\min_{k \leq i \leq n} s_i = \min_{k \leq i \leq n, i \neq j} s_i, \quad \text{and} \quad \min_{k \leq i \leq n} s'_i = \begin{cases} \min_{k \leq i \leq n} s_i & \text{if } k < j, \text{ and} \\ \min_{k+1 \leq i \leq n} s_i & \text{if } k \geq j, \end{cases}$$

for all k with $1 \leq k \leq n$. This last equality ensures that $S(x) = S(x')$, so that $\sigma(x) = \sigma(x')$ using property 4. This completes the proof. $\square$

3 Random sequences

Add more content here.

References

Knuth, Donald E., Oren Patashnik, and Ronald L. Graham. 1998. *Concrete Mathematics: A Foundation for Computer Science*. addison-wesley.

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