Notes on Raney's Lemmas

In a 1960 paper, George Raney proved the first two lemmas below; the lemmas suppose we have a finite sequence of numbers meeting certain constraints, and provide the number of cycle shifts that contain all-positive partial sums (Raney 1960).

These notes expand on the ideas behind these lemmas. The final section of these notes discusses cyclic shifts for finite sequences of independent, uniformly random values; as far as I know, the work in that section is new.

I personally learned of these lemmas in chapter 7 of the book Concrete Mathematics (Knuth, Patashnik, and Graham 1998), which explores their applications to generating functions.

Exact values for finite integer sequences

Lemma 1

Suppose $\sum_{i=1}^{n} x_i = 1$, where all $x_i \in \mathbb{Z}$. Extend the sequence by letting $x_{n+p} = x_p$ for $1 \le p \le n$. Then there is a unique j, $1 \le j \le n$, such that

$$\sum_{i=j}^{j+k-1} x_i > 0; \quad 1 \le k \le n.$$

Intuitively, we can think of such an index j as a cyclic shift of the sequence that has partial sums that are all positive.

For example, the finite sequence $\langle x_1, \ldots, x_5 \rangle = \langle 3, -2, 4, -1, 1 \rangle$ offers $j = 5$ as the unique shift providing $\langle x_5, x_6 = x_1, \ldots, x_9 = x_4 \rangle = \langle 1, 3, -2, 4, -5 \rangle$ with partial sums $\langle 1, 4, 2, 6, 1 \rangle$ that are all positive.

Given a sequence $\langle x_1, \ldots, x_n \rangle$, it's useful to say that an index $i \in \{1, \ldots, n\}$ is a *positive-sum shift* if and only if the partial sums of $\langle x_i, \ldots, x_n, x_1, \ldots, x_{i-1} \rangle$ are all positive. Since these notes focus on finite sequences, we'll also implicitly use arbitrary indexes $x_j, j \in \mathbb{Z}$, to refer to x_k with $k \in \{1, \ldots, n\}, k \equiv j \pmod{n}$.

Raney also proved the related result:

Lemma 2

Suppose $\sum_{i=1}^{n} x_i = \ell$, where $x_i \in \mathbb{Z}$ and $x_i \leq 1$ for all i. Then exactly ℓ indexes in $\{1, \ldots, n\}$ are positive-sum shifts.

For example, let $\langle x_1, \ldots, x_8 \rangle = \langle -2, 1, 1, 0, -1, 1, 1, 1 \rangle$. Then $\sum x_i = 2$, and x_2, x_6 are the only positive-sum shifts:

shift	partial sums
$\langle x_2, \ldots \rangle = \langle 1, 1, 0, -1, 1, 1, 1, -2 \rangle$	$\langle 1, 2, 2, 1, 2, 3, 4, 2 \rangle$
$\langle x_6, \ldots \rangle = \langle 1, 1, 1, -2, 1, 1, 0, -1 \rangle$	$\langle 1, 2, 3, 1, 2, 3, 3, 2 \rangle$

References

Knuth, Donald E., Oren Patashnik, and Ronald L. Graham. 1998. *Concrete Mathematics: A Foundation for Computer Science*. addison-wesley.

Raney, George. 1960. "Functional Composition Patterns and Power Series Reversion." *Transactions of the American Mathematical Society* 94: 441–51.