# Notes on Raney's Lemmas

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In a 1960 paper, George Raney proved the first two lemmas below; the lemmas suppose we have a finite sequence of numbers meeting certain constraints, and provide the number of cycle shifts that contain all-positive partial sums (Raney 1960).

These notes expand on the ideas behind these lemmas. The final section of these notes discusses cyclic shifts for finite sequences of independent, uniformly random values; as far as I know, the work in that section is new.

I personally learned of these lemmas in chapter 7 of the book *Concrete Mathematics* (Knuth, Patashnik, and Graham 1998), which explores their applications to generating functions. The presentation of the lemmas here is based on the presentation in *Concrete Mathematics* rather than on Raney's original paper.

## 1 Integer sequences

**Lemma 1** *Suppose $\sum_{i=1}^n x_i = 1$, where all $x_i \in \mathbb{Z}$. Extend the sequence by letting $x_{n+p} = x_p$ for $1 \le p \le n$. Then there is a unique $j$, $1 \le j \le n$, such that*

$$\sum_{i=j}^{j+k-1} x_i > 0; \quad 1 \le k \le n.$$

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Intuitively, we can think of such an index $j$ as a cyclic shift of the sequence that has partial sums that are all positive.

For example, the finite sequence $\langle x_1, \ldots, x_5 \rangle = \langle 3, -2, 4, -1, 1 \rangle$ offers $j = 5$ as the unique shift providing $\langle x_5, x_6 = x_1, \ldots, x_9 = x_4 \rangle = \langle 1, 3, -2, 4, -5 \rangle$ with partial sums $\langle 1, 4, 2, 6, 1 \rangle$ that are all positive.

**Definitions** Given a sequence $\langle x_1, \ldots, x_n \rangle$, it's useful to say that an index $i \in \{1, \ldots, n\}$ is a *positive-sum shift* if and only if the partial sums of

$\langle x_i, \ldots, x_n, x_1, \ldots, x_{i-1} \rangle$ are all positive. Since these notes focus on finite sequences, we'll implicitly use arbitrary indexes $x_j, j \in \mathbb{Z}$, to refer to $x_k$ with $k \in \{1, \ldots, n\}, k \equiv j \pmod{n}$.

We'll use the subscript-free letter $x$ to denote an entire finite sequence $\langle x_1, \ldots, x_n \rangle$. We'll write $\sigma(x)$ to indicate the number of indexes of $x$ that are positive-sum shifts.

We can now concisely state a related result proved by Raney:

**Lemma 2** *Suppose $\sum_{i=1}^{n} x_i = \ell$, where $x_i \in \mathbb{Z}$ and $x_i \leq 1$ for all $i$. Then $\sigma(x) = \ell$; that is, exactly $\ell$ indexes in $\{1, \ldots, n\}$ are positive-sum shifts.*

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For example, let $\langle x_1, \ldots, x_8 \rangle = \langle -2, 1, 1, 0, -1, 1, 1, 1 \rangle$. Then $\sum x_i = 2$, and $x_2, x_6$ are the only positive-sum shifts:

| shift | partial sums |
|---|---|
| $\langle x_2, \ldots \rangle = \langle 1, 1, 0, -1, 1, 1, 1, -2 \rangle$ | $\langle 1, 2, 2, 1, 2, 3, 4, 2 \rangle$ |
| $\langle x_6, \ldots \rangle = \langle 1, 1, 1, -2, 1, 1, 0, -1 \rangle$ | $\langle 1, 2, 3, 1, 2, 3, 3, 2 \rangle$ |

Note that lemma 2 is not a strict generalization of lemma 1 as it adds the condition $x_i \leq 1$. This condition is necessary for lemma 2; without it we may have, for example, the one-element sequence $x = \langle 2 \rangle$ with sum $\ell = 2$ and $\sigma(x) = 1$.

Rather than proving the above two lemmas directly, we'll jump to the general case of real sequences $x$ and prove strictly more general bounds on $\sigma(x)$ in that context.

## 2 Real sequences

In a moment we'll prove a general guarantee that $\sum x_i > 0 \Rightarrow \sigma(x) \geq 1$. In the context of a sequence $x$, it will be useful to write $s_i$ to denote the $i^{\text{th}}$ partial sum of $x$; that is, $s_0 = 0$, and

$$s_i = \sum_{j=1}^{i} x_j, \quad \text{for } i \geq 1.$$

We can define $s_i$ for $i > n$ using the implicitly periodic sequence characterized by $x_{n+i} = x_i$.

**Property 3** *Suppose $\sum_{i=1}^{n} x_i > 0$, where $x_i \in \mathbb{R}$. Let $s_i$ denote the $i^{\text{th}}$ partial sum of $x$, and let $j$ be the largest index in $\{1, \ldots, n\}$ with $s_{j-1} = \min_{0 \leq i < n} s_i$. Then $j$ is a positive-sum shift.*

**Proof** Let

$$s'_i = \sum_{k=j}^{j+i-1} x_k$$

denote the $i^{\text{th}}$ partial sum of the shifted sequence $\langle x_j, \ldots, x_{j+n-1} \rangle$. Then, for $1 \le i \le n$,

$$s'_i = s_{j+i-1} - s_{j-1} \begin{cases} > 0 \text{ (by definition of } j) & \text{when } j+i-1 < n \\ = s_n + s_{j+i-1-n} - s_{j-1} \ge s_n > 0 & \text{when } j+i-1 \ge n. \end{cases}$$

□

Now we can assume without loss of generality that any sequence of real numbers $\langle x_1, \ldots, x_n \rangle$ with $\sum x_i > 0$ is already shifted so that all its partial sums $s_i > 0$ for $i > 0$. As we'll see in the next property, this assumption allows us to provide a nice general expression for $\sigma(x)$. This expression depends on the set $S(x)$, defined as $\{\min_{j \le i \le n} s_i \mid 1 \le j \le n\}$ for any finite sequence $x$ with $i^{\text{th}}$ partial sum $s_i$.

**Property 4** *Suppose that $x$ is a finite real sequence with $i^{th}$ partial sum $s_i$, and that $s_i > 0$ for all $i > 0$. Then*

$$\sigma(x) = \#S(x) = \# \left\{ \min_{j \le i \le n} s_i \;\middle|\; 1 \le j \le n \right\}. \tag{1}$$

*More specifically, an index $j$ with $1 \le j \le n$ is a positive-sum shift iff*

$$s_{j-1} < s_i \; \forall i : j \le i \le n. \tag{2}$$

**Proof** We'll start by supposing we have an index $j$ with $1 \le j \le n$ and $s_{j-1} < s_i$ for all $i$ with $j \le i \le n$; our goal is to show that such a $j$ must be a positive-sum shift. Our approach will be similar to the proof of property 3.

Let $s'_i$ denote the $i^{\text{th}}$ partial sum of $\langle x_j, \ldots, x_{j+n-1} \rangle$:

$$s'_i = \sum_{k=j}^{j+i-1} x_k.$$

Then

$$s'_i = s_{j+i-1} - s_{j-1} \begin{cases} > 0 & \text{if } j+i-1 \le n, \\ = s_{j+i-1-n} + s_n - s_{j-1} > 0 & \text{if } j+i-1 > n; \end{cases}$$

the last inequality follows since $s_{j+i-1-n} > 0$ and $s_n > s_{j-1}$.

On the other hand, if $s_{j-1} \geq s_i$ for some $i, j$ with $1 \leq j \leq i \leq n$, then $s'_{i-j+1} = s_i - s_{j-1} \leq 0$, so that $j$ isn't a positive-sum shift. This completes the proof of the last part of the property.

Now let's verify that the set $S = S(x)$ from (1) has size $\sigma(x)$.

Let $j_1, \ldots, j_k$ be all the positive-sum shifts with $1 < j_i \leq n$; note that $k = \sigma(x) - 1$ since the trivial shift index 1 has been excluded. Let $T = \{s_{j_1-1}, \ldots, s_{j_k-1}, s_n\}$.

Notice that $j = n + 1$ trivially meets condition (2); combine this with the first part of the proof to see that all elements of $T$ meet condition (2). This guarantees that all the elements are unique, so that $|T| = \sigma(x)$. This also means that $T \subset S$. Finally, observe that, for any $s_j \in S$, there's a largest index $j'$ with $1 \leq j' \leq n$ and $s_{j'} = s_j$; this index $j'$ meets condition (2), so that $S \subset T$, confirming that $|S| = |T| = \sigma(x)$. □

Property 4 lends itself to a nice visual intuition. Consider the example sequence $\langle 2, -1, 2, 2, -3, 2, 1, 1, -1, -2 \rangle$ of length $n = 10$. Below is the line graph of its partial sums, starting with $s_0 = 0$.

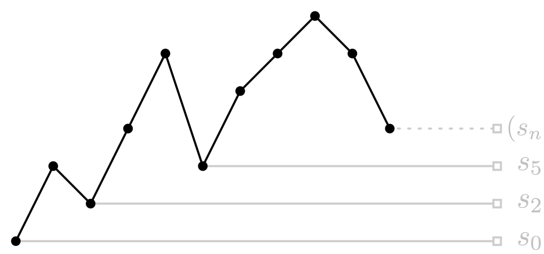


Figure 1: *Line graph of the partial sums $s_i$ of the example sequence.*

Imagine an observer standing far to the right of the graph and looking directly to the left so they can only see along a perfectly horizontal line of sight. Below $s_n$, they can only see the three points $s_0$, $s_2$, and $s_5$. These are exactly the partial sums meeting condition (2), so that they correspond directly to all the positive-sum shifts of $x$, which have indexes 1, 3, and 6.

This visual intuition — that points visible-from-the-right and below $s_n$ correspond exactly to the positive-sum shifts — extends to any sequence meeting the suppositions of property 4.

It's now possible to prove a simple general upper and lower bound for $\sigma(x)$ in the case that each $x_i$ is an integer. We'll see below that these bounds provide both lemmas 1 and 2 as corollaries.

**Property 5** *Suppose we have a finite integer sequence $x = \langle x_1, \ldots, x_n \rangle$ with $s_n > 0$. Let $m = \max_i x_i$. Then*

$$\lceil s_n/m \rceil \leq \sigma(x) \leq s_n.$$

**Proof idea** Here is the informal intuition behind the proof: We'll start by noticing that, for sum-positive $x$, $S(x) \subset (0, s_n]$; this is the basis used for the upper bound. The lower bound is based on the idea that each jump upwards from one $s_i \in S(x)$ to the next $s_j \in S(x)$ is limited by distance $m$. The smallest element in $S(x)$ can be at most $m$ above $s_0 = 0$, and the largest is necessarily $s_n$, so that there must be at least $s_n/m$ elements between the extremes.

**Proof** Notice that we can work with any cyclic shift $x'$ of $x$ without changing $s_n$ or $m$. Thus, using property 3, we can assume without loss of generality that $s_i > 0$ for $i > 0$.

Next, we can bound the elements of $S(x)$ via

$$0 < \min_{j \le i \le n} s_i \le s_n$$

for all $j$ with $1 \le j \le n$. So all elements of $S(x)$ are in the range $(0, s_n]$, and are integers. Hence $\#S(x) \le s_n$, completing the proof of the upper bound.

Toward the lower bound, let's suppose that $S(x) = \{s_{j_1}, \dots, s_{j_k}\}$ with each $s_{j_i}$ meeting condition (2) and $0 < s_{j_i} < s_{j_{i+1}}$. We know such $s_{j_i}$ exist as they are simply those partial sums in $s_{j_i} \in S(x)$ chosen so that $j_i = \max_{1 \le k \le n}\{k : s_k = s_{j_i}\}$.

By our definition of $s_{j_i}$, we have

$$s_{j_i} = \min_{j_i \le k \le n} s_k \;\text{ and }\; s_{j_{i+1}} = \min_{j_i+1 \le k \le n} s_k.$$

This means that

$$s_{j_{i+1}} - s_{j_i} = \min_{j_i+1 \le k \le n} s_k - s_{j_i} \le s_{j_i+1} - s_{j_i} \le m.$$

Note that $s_{j_k} = s_n$ so that $s_n - s_{j_{k-1}} \le m \Rightarrow s_{j_{k-1}} \ge s_n - m$. This can be extended to see that $s_{j_{k-2}} \ge s_n - 2m$, and in general that

$$s_{j_{k-p}} \ge s_n - pm.$$

Our definition of $m$ gives us that $s_{j_1} \le m$, so $m \ge s_{j_1} \ge s_n - (k-1)m$, from which we can derive that

$$1 \ge s_n/m - (k-1) \quad \Rightarrow \quad k \ge s_n/m \quad \Rightarrow \quad k \ge \lceil s_n/m \rceil;$$

the last inequality uses the fact that $k = \sigma(x)$ is an integer. This completes the proof. □

### The Contraction Perspective

Next we'll consider a contraction operation that may shorten a sequence $x$ while preserving $\sigma(x)$.

Call a sequence $x = \langle x_1, \dots, x_n \rangle$ *sum-positive* iff $s_i > 0$ when $i > 0$. We'll say that a sequence $x' = \langle x'_1, \dots, x'_{n-1} \rangle$ is a *contraction* of the length-$n$ sum-positive sequence $x$ iff there is some index $j$ so that $x_{j+1} \le 0$ and, for $1 \le i \le n-1$,

$$x'_i = \begin{cases} x_i & \text{if } i < j, \\ x_i + x_{i+1} & \text{if } i = j, \text{and} \\ x_{i+1} & \text{if } i > j. \end{cases}$$

For example. $x' = \langle 2, -1, 2 \rangle$ is a contraction of $x = \langle 3, -1, -1, 2 \rangle$ since the sequences are same except for the replacement of $x_1, x_2$ by their sum as $x'_1$, and $x_2 = -1 \le 0$. The alternative sequence $x'' = \langle 3, -1, 1 \rangle$ is *not* a contraction as it replaces $x_3, x_4$ with their sum $x''_3$, but $x_4 = 2 > 0$.

**Property 5** *If $x'$ is a contraction of $x$, then $x'$ is sum-positive and $\sigma(x') = \sigma(x)$.*

**Proof** Let $j$ be the contracted index, so that $x'_j = x_j + x_{j+1}$ and $x_{j+1} \le 0$.

Let $s'_i$ denote the $i^{\text{th}}$ partial sum of $x'$. Then

$$s'_i = \begin{cases} s_i & \text{if } 0 \le i < j \\ s_{i+1} & \text{if } j \le i \le n-1. \end{cases}$$

So $s'_i > 0$ for $0 < i \le n-1$, making $x'$ sum-positive.

Since $x_{j+1} \le 0$, $s_{j+1} \le s_j$. This means that

$$\min_{k \le i \le n} s_i = \min_{k \le i \le n, i \ne j} s_i, \text{ and } \min_{k \le i \le n} s'_i = \begin{cases} \min_{k \le i \le n} s_i & \text{if } k < j, \text{ and} \\ \min_{k+1 \le i \le n} s_i & \text{if } k \ge j, \end{cases}$$

for all $k$ with $1 \le k \le n$. This last equality ensures that $S(x) = S(x')$, so that $\sigma(x) = \sigma(x')$ using property 4. This completes the proof. $\square$

As long as a sum-positive sequence $x$ has any element $x_i \le 0$, we can apply a contraction to it to arrive at a shorter sequence. Thus, we can always apply a series of contractions to arrive at a sum-positive, length-$k$ sequence $x'$ with all elements $x'_i > 0$. All shifts of this $x'$ are positive-sum shifts, so that $\sigma(x') = k$, and thus $\sigma(x) = k$ as well. In other words, we can find $\sigma(x)$ by contracting $x$ until it can be contracted no more.

To add both intuition and more detail to this process, think of any sum-positive sequence $x$ as consisting of subsequences starting with $x_j$ for each positive-sum shift index $j$. For example, suppose sequence $x$ has positive-sum shift indexes 1, 3, and 4. Then

$$\langle x_1, x_2, x_3, x_4, x_5, x_6 \rangle = \langle x_1, x_2 \rangle \langle x_3 \rangle \langle x_4, x_5, x_6 \rangle.$$

Contractions effectively work within these subsequences. A contraction can never combine elements across a subsequence boundary because the first element of any subsequence must be positive. Our last proof showed that the set $S(x)$ is preserved by contraction. That proof also showed that the indexes $j_1, \ldots, j_k$ with $\{s_{j_1}, \ldots, s_{j_k}\} = S(x)$ are preserved as well, excepting a possible shift-by-one in the latter elements to account for the contraction. In other words, if an element $x_j$ begins a subsequence before a contraction, then it will be mapped to $x'_k$ with either $k = j$ or $k = j-1$, with $x'_k = x_j$, and with $x'_k$ also acting as a positive-sum shift for $x'$.

The final result of any maximal series of contractions is therefore deterministic: we must arrive at exactly the sequence of sums of the original subsequences. Using our last example sequence, we may make the following series of underlined contractions:

$$\begin{array}{cccc} & \langle x_1, x_2 \rangle & \langle x_3 \rangle & \langle x_4, x_5, x_6 \rangle \\ \to & \langle x_1, x_2 \rangle & \langle x_3 \rangle & \langle \underline{x_4 + x_5}, x_6 \rangle \\ \to & \langle \underline{x_1 + x_2} \rangle & \langle x_3 \rangle & \langle x_4 + x_5, x_6 \rangle \\ \to & \langle x_1 + x_2 \rangle & \langle x_3 \rangle & \langle x_4 + \underline{x_5 + x_6} \rangle. \end{array}$$

The final sequence has 3 positive elements, so we can't perform any more contractions. We have freedom in the order in which we execute those contractions, but the end result is independent of this order. Note that, although we've highlighted the subsequence structure here, we don't need to be aware of that structure to execute the contractions.

# 3 Random sequences

In this section, we'll consider a finite sequence $x$ of length $n$ whose elements $x_i$ are independent random variables, each uniformly distributed in the interval $[-1, 1]$.

The partial sums of such a sequence have two interesting properties:

1. If $i \neq j$, then $s_i \neq s_j$ with probability 1.
2. $\text{Prob}(s_i > s_j) = \text{Prob}(s_i < s_j)$.

To see the first property, suppose $i < j$ and that we're given the values of $x_1, \ldots, x_{j-1}$. If we are about to choose a random value $x_j \in [-1, 1]$, then there is at most one value which could give $s_i = s_j$, so that the event must have probability zero.

We can verify the second probaby by a symmetry argument. Notice that any fixed value for $x_i$ is as likely as its opposite value $-x_i$. This means that every measurable set of sequences $X$ has the same probability as its negation $\{-x = \langle -x_1, \ldots, -x_n \rangle \mid x \in X\}$. So the set $\{x \mid s_i < s_j\}$ has the same probability as the set $\{x \mid s_i > s_j\}$.

# References

Knuth, Donald E., Oren Patashnik, and Ronald L. Graham. 1998. *Concrete Mathematics: A Foundation for Computer Science*. addison-wesley.

Raney, George. 1960. "Functional Composition Patterns and Power Series Reversion." *Transactions of the American Mathematical Society* 94: 441–51.