Notes on Raney's Lemmas

In a 1960 paper, George Raney proved the following two lemmas on cyclic shifts of a finite sequence that contain positive-only partial sums (Raney 1960). This note expands on these lemmas.

I personally learned of these lemmas in chapter 7 of the book Concrete Mathematics (Knuth, Patashnik, and Graham 1998), which explores their applications to generating functions.

Exact values for finite integer sequences

Lemma 1

Suppose $\sum_{i=1}^{n} x_i = 1$, where all $x_i \in \mathbb{Z}$. Extend the sequence by letting $x_{n+p} = x_p$ for $1 \le p \le n$. Then there is a unique j, $1 \le j \le n$, such that

$$\sum_{i=j}^{j+k-1} x_i > 0; \quad 1 \le k \le n.$$

Intuitively, we can think of such an index j as a cyclic shift of the sequence that has partial sums that are all positive.

For example, the finite sequence $\langle x_1, \ldots, x_5 \rangle = \langle 3, -2, 4, -1, 1 \rangle$ offers $j = 5$ as the unique shift providing $\langle x_5, x_6 = x_1, \ldots, x_9 = x_4 \rangle = \langle 1, 3, -2, 4, -5 \rangle$ with partial sums $\langle 1, 4, 2, 6, 1 \rangle$ that are all positive.

Given a sequence $\langle x_1, \ldots, x_n \rangle$, it's useful to say that an index $i \in \{1, \ldots, n\}$ is a *positive-sum shift* if and only if the partial sums of $\langle x_i, \ldots, x_n, x_1, \ldots, x_{i-1} \rangle$ are all positive. Since this note focuses on finite sequences, we'll also implicitly use arbitrary indexes $x_j, j \in \mathbb{Z}$, to refer to x_k with $k \in \{1, \ldots, n\}, k \equiv j \pmod{n}$.

Raney also proved (TODO CHECK) the related result:

Lemma 2

Suppose $\sum_{i=1}^{n} x_i = \ell$, where $x_i \in \mathbb{Z}$ and $x_i \leq 1$ for all i. Then exactly ℓ indexes in $\{1, \ldots, n\}$ are positive-sum shifts.

For example, let $\langle x_1, \ldots, x_8 \rangle = \langle -2, 1, 1, 0, -1, 1, 1, 1 \rangle$. Then $\sum x_i = 2$, and x_2, x_6 are the only positive-sum shifts:

shift	partial sums
$\langle x_2, \ldots \rangle = \langle 1, 1, 0, -1, 1, 1, 1, -2 \rangle$	$\langle 1, 2, 2, 1, 2, 3, 4, 2 \rangle$
$\langle x_6, \ldots \rangle = \langle 1, 1, 1, -2, 1, 1, 0, -1 \rangle$	$\langle 1, 2, 3, 1, 2, 3, 3, 2 \rangle$

References

Knuth, Donald E., Oren Patashnik, and Ronald L. Graham. 1998. *Concrete Mathematics: A Foundation for Computer Science*. addison-wesley.

Raney, George. 1960. "Functional Composition Patterns and Power Series Reversion." *Transactions of the American Mathematical Society* 94: 441–51.