Notes on Raney's Lemmas

Tyler Neylon

251.2015

In a 1960 paper, George Raney proved the first two lemmas below; the lemmas suppose we have a finite sequence of numbers meeting certain constraints, and provide the number of cycle shifts that contain all-positive partial sums (Raney 1960).

These notes expand on the ideas behind these lemmas. The final section of these notes discusses cyclic shifts for finite sequences of independent, uniformly random values; as far as I know, the work in that section is new.

I personally learned of these lemmas in chapter 7 of the book *Concrete Mathematics* (Knuth, Patashnik, and Graham 1998), which explores their applications to generating functions. The presentation of the lemmas here is based on the presentation in *Concrete Mathematics* rather than on Raney's original paper.

1 Integer sequences

Lemma 1 *Suppose* $\sum_{i=1}^n x_i = 1$*, where all* $x_i \in \mathbb{Z}$*. Extend the sequence by letting* $x_{n+p} = x_p$ *for* $1 \le p \le n$*. Then there is a unique* j*,* $1 \le j \le n$*, such that*

$$\sum_{i=j}^{j+k-1} x_i > 0; \quad 1 \le k \le n.$$

Intuitively, we can think of such an index j as a cyclic shift of the sequence that has partial sums that are all positive.

For example, the finite sequence $\langle x_1, \ldots, x_5 \rangle = \langle 3, -2, 4, -1, 1 \rangle$ offers $j = 5$ as the unique shift providing $\langle x_5, x_6 = x_1, \ldots, x_9 = x_4 \rangle = \langle 1, 3, -2, 4, -5 \rangle$ with partial sums $\langle 1, 4, 2, 6, 1 \rangle$ that are all positive.

Definitions Given a sequence $\langle x_1, \ldots, x_n \rangle$, it's useful to say that an index $i \in \{1, \ldots, n\}$ is a *positive-sum shift* if and only if the partial sums of

$\langle x_i, \ldots, x_n, x_1, \ldots, x_{i-1} \rangle$ are all positive. Since these notes focus on finite sequences, we'll implicitly use arbitrary indexes $x_j, j \in \mathbb{Z}$, to refer to x_k with $k \in \{1, \ldots, n\}, k \equiv j \pmod{n}$.

We'll use the subscript-free letter x to denote an entire finite sequence $\langle x_1, \ldots, x_n \rangle$. We'll write $\sigma(x)$ to indicate the number of indexes of x that are positive-sum shifts.

We can now concisely state a related result proved by Raney:

Lemma 2 *Suppose $\sum_{i=1}^n x_i = \ell$, where $x_i \in \mathbb{Z}$ and $x_i \le 1$ for all i. Then $\sigma(x) = \ell$; that is, exactly ℓ indexes in $\{1, \ldots, n\}$ are positive-sum shifts.*

For example, let $\langle x_1, \ldots, x_8 \rangle = \langle -2, 1, 1, 0, -1, 1, 1, 1 \rangle$. Then $\sum x_i = 2$, and x_2, x_6 are the only positive-sum shifts:

shift	partial sums
$\langle x_2, \ldots \rangle = \langle 1, 1, 0, -1, 1, 1, 1, -2 \rangle$	$\langle 1, 2, 2, 1, 2, 3, 4, 2 \rangle$
$\langle x_6, \ldots \rangle = \langle 1, 1, 1, -2, 1, 1, 0, -1 \rangle$	$\langle 1, 2, 3, 1, 2, 3, 3, 2 \rangle$

Note that lemma 2 is not a strict generalization of lemma 1 as it adds the condition $x_i \le 1$. This condition is necessary for lemma 2; without it we may have, for example, the one-element sequence $x = \langle 2 \rangle$ with sum $\ell = 2$ and $\sigma(x) = 1$.

Rather than proving the above two lemmas directly, we'll jump to the general case of real sequences x and prove strictly more general bounds on $\sigma(x)$ in that context.

2 Real sequences

We start with a general guarantee that $\sum x_i > 0 \Rightarrow \sigma(x) \ge 1$. In the context of a sequence x, it will be useful to write s_i to denote the i^{th} partial sum of x; that is, $s_0 = 0$, and

$$s_i = \sum_{j=1}^{i} x_j, \quad \text{for } i \ge 1.$$

We can define s_i for $i > n$ using the implicitly periodic sequence characterized by $x_{n+i} = x_i$.

Property 3 *Suppose $\sum_{i=1}^n x_i > 0$, where $x_i \in \mathbb{R}$. Let s_i denote the i^{th} partial sum of x, and let j be the largest index in $\{1, \ldots, n\}$ with $s_{j-1} = \min_{0 \le i < n} s_i$. Then j is a positive-sum shift.*

Proof Let

$$s'_i = \sum_{k=j}^{j+i-1} x_k$$

denote the i^{th} partial sum of the shifted sequence $\langle x_j, \ldots, x_{j+n-1}\rangle$. Then, for $1 \le i \le n$,

$$s'_i = s_{j+i-1} - s_{j-1} \begin{cases} > 0 \text{ (by definition of } j) & \text{when } j+i-1 < n \\ = s_n + s_{j+i-1-n} - s_{j-1} \ge s_n > 0 & \text{when } j+i-1 \ge n. \end{cases}$$

□

Now we can assume without loss of generality that any sequence of real numbers $\langle x_1, \ldots, x_n\rangle$ with $\sum x_i > 0$ is already shifted so that all its partial sums $s_i > 0$ for $i > 0$. This allows us to provide a nice general expression for $\sigma(x)$.

Property 4 *Suppose that x is a finite real sequence with i^{th} partial sum s_i, and that $s_i > 0$ for all $i > 0$. Then*

$$\sigma(x) = \#\left\{ \min_{j \le i \le n} s_i \;\middle|\; 1 \le j \le n \right\}.$$

More specifically, an index j with $1 \le j \le n$ is a positive-sum shift iff $s_{j-1} < s_i$ for all i with $j \le i \le n$.

Proof We'll start by supposing we have an index j with $1 \le j \le n$ and $s_{j-1} < s_i$ for all i with $j \le i \le n$; our goal is to show that such a j must be a positive-sum shift. Our approach will be similar to the proof of property 3.

Let s'_i denote the i^{th} partial sum of $\langle x_j, \ldots, x_{j+n-1}\rangle$:

$$s'_i = \sum_{k=j}^{j+i-1} x_k.$$

Then

$$s'_i = s_{j+i-1} - s_{j-1} \begin{cases} > 0 & \text{if } j+i-1 \le n, \\ = s_{j+i-1-n} + s_n - s_{j-1} & \text{if } j+i-1 > n; \end{cases}$$

the last inequality follows since $s_{j+i-1-n} > 0$ and $s_n > s_{j-1}$.

On the other hand, if $s_{j-1} \ge s_i$ for some i, j with $1 \le j \le i \le n$, then $s'_{i-j+1} = s_i - s_{j-1} \le 0$, so that j isn't a positive-sum shift. This completes the proof of the last part of the property; next we'll justify the expression for $\sigma(x)$. □

(Next: note why the specifically case matches the $\sigma(x)$ value; proof; example illustration for intuition)

3 Random sequences

Add more content here.

References

Knuth, Donald E., Oren Patashnik, and Ronald L. Graham. 1998. *Concrete Mathematics: A Foundation for Computer Science*. addison-wesley.

Raney, George. 1960. "Functional Composition Patterns and Power Series Reversion." *Transactions of the American Mathematical Society* 94: 441–51.