SURREAL CANONICAL LINEAR ORDERS

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1. Summary

In this note, we'll define a linearly ordered set S_d for any ordinal d so that any linearly ordered set X of cardinality $\aleph_d$ can be embedded in S_d in an order-preserving manner. If one accepts the generalized continuum hypothesis, which states that any cardinals κ, λ cannot have $\lambda < \kappa < 2^\lambda$, then $||S_d|| = \aleph_d$, and clearly the S_d are of optimal size to act as a universal receiver of other same-sized linear orders. The universal embedding result (theorem **??**) uses the axiom of choice.

2. Definitions

We choose the von Neumann definition of *ordinal numbers*; informally, an ordinal d is the set of all previous ordinals. More formally, a set A is **transitive** iff $x \in A$ & $y \in x \implies y \in A$ (we'll use **bold** to indicate when a term is being defined). A set d is an **ordinal number** (or just an "ordinal"), written $d \in \mathcal{O}$ iff it is transitive and all its elements are ordinal numbers. It is a result that any set of ordinal numbers is well ordered by inclusion; this is what we mean when we write $x < y$ for $x, y \in \mathcal{O}$. It can be shown that every ordinal is of the form $\{d \in \mathcal{O} : d < e\}$ for some $e \in \mathcal{O}$, and that every well order is isomorphic to some $d \in \mathcal{O}$.

A **surreal** s is a function $s : d \to \{+, -\}$ for some $d \in \mathcal{O}$. For such a surreal, we call d the **domain** of it, and denote this as $D(s) = d$. It's clear that $s(e) = +$ or $-$ for $e \in d$; we will add the formal notation that $s(e) = \emptyset$ when $e \notin D(s)$.

Let $\boldsymbol{S_d}$ denote the set of surreals $\{s : ||D(s)|| < \aleph_d\}$.

We'll use the notation $\{< d\}$ to indicate the set of ordinals less than d. Technically, this is equal to d itself, but we will need to refer to $s(\{< d\}) = \{s(e) : e < d\}$ as a sequence of signs, as opposed to $s(d)$, indicating a single sign.

Consider surreals s and t. If they have $D(s) = D(t)$ and $s(d) = t(d)\ \forall\, d \in D(s)$, then we write $s = t$. Otherwise, let d be the first value in $D(s) \cup D(t)$ where $s(d) \neq t(d)$; we call d the **first difference coordinate**. We use the convention that $- < \emptyset < +$, for these definitions: $s < t$ iff $s(d) < t(d)$ and $s > t$ iff $s(d) > t(d)$. This can be considered a lexicographical order, and hence is a linear order on the surreals.

Given two orders X and Y, we say that X can be **order-embedded** in Y iff $\exists f : X \to Y$ with $x < y \implies f(x) < f(y)$; we call such f an **order-embedding**.

3. Main Results

Theorem 1. *Suppose $\aleph_d$ is either a successor cardinal, or is $\aleph_0$. Suppose e, f are ordinals of cardinality less than $\aleph_d$, and that $(s_i)_{i \in e} \subset S_d$ is increasing, $(t_j)_{j \in f} \subset S_d$ is decreasing, and $s_i < t_j\ \forall\, i, j$. Then there exists an element $r \in S_d$ such that $s_i < r < t_j\ \forall\, i, j$.*

Theorem 2 (Universal Embedding)**.** *If X is a linear order with $||X|| \leq \aleph_d$, then X can be order-embedded in S_d.*

4. Proofs

We will define a very useful limit idea on surreals that will eventually give us the proof of theorem **??**. From there, it is easy to get to theorem **??**.

4.1. **The Surreal Limit.** Suppose we have an increasing sequence of surreals $(s_i)_{i\in e}$ indexed by some ordinal e. Then we define the **surreal limit** s of (s_i), written as $s = \operatorname{slim}_{i\in e} s_i$, like this:

$$s(d) = \begin{cases} \lim_{i\in e} s_i(d) & \text{if } \exists\, k \in e : i, j \geq k \implies s_i(\{< d\}) = s_j(\{< d\}) \\ \emptyset & \text{otherwise.} \end{cases}$$

We must check that this definition makes sense. There are two things to check.

First, we need to see that $\lim s_i(d)$ exists, as used in the definition. This clause of the definition is only relevant when $s_i(\{< d\})$ "settles" on to a specific sign sequence. From that point onward, the sign at $s_i(d)$ can only change in the order $(-, \emptyset, +)$ as i increases, since the sequence is increasing. Thus there must be some tail t of the index set e such that $s_i(d)$ is constant for $i \in t$. (For any ordinal d, a **tail** of d is a subset $t \subset d$ which is upward-closed in d; in other words, $a \in t, b \in d, a < b \implies b \in t$.)

Second, we must check that if $s(d) = \emptyset$, then $s(f) = \emptyset$ for any $f > d$. Since $s(d) = \emptyset$, we know that $\forall\, k \in e\ \exists\, i, j \geq k : s_i(\{< d\}) \neq s_j(\{< d\})$. It follows immediately that this condition must also hold when $\{< d\}$ is replaced by any superset, including $\{< f\}$ for $f > d$. This completes the check that a surreal limit is well defined.

It is interesting to note that this limit exists for *any* increasing ordinal-indexed sequence, regardless of cardinality or boundedness.

Let's note some basic properties of the surreal limit. Let $s = \operatorname{slim} s_i$ and d be any ordinal in $D(s)$. Then each of the following is true:

$$\exists\, k \in e :\ \forall\, i \geq k,\ s_i(\{< d\}) = s(\{< d\}), \tag{1}$$

$$s_i(\{< d\}) = s(\{< d\}) \implies s(d) \geq s_i(d), \tag{2}$$

and

$$\lim_{i\in e} s_i(d) = s(d). \tag{3}$$

It is easy to verify each of these using the definition of s and the fact that s_i is increasing.

The surreal limit has some interesting properties which will lead us to the proof of theorem **??**. In each of these properties, $(s_i)_{i\in e}$ is an increasing sequence of surreals, indexed by an ordinal e.

For surreals s, t, say that s is a **prefix** of t iff $D(s) = \{< d\} \subset D(t)$ and $t(\{< d\}) = s$; intuitively, this means t "starts with" s, as a sign sequence.

Property 3. *Any surreal t with prefix $s = \operatorname{slim} s_i$ (including $s = t$) has $t \geq s_i\ \forall\, i \in e$.*

Proof. Suppose t has s as a prefix, and $t < s_i$ for some $i \in e$.

Let d be the first difference coordinate between t and s_i. If $d \in D(s)$, then $t(d) = s(d) \geq s_i(d)$ using property (**??**) above, contradicting that $t < s_i$.

So it must be the case that $d \notin D(s)$. This means that $s_i(D(s)) = s$, and in fact $s_j(D(s)) = s$ for any $j > i$; otherwise let $e \in D(s)$ be the first difference coordinate between $s_i < s_j$, and we would have a contradiction to property (**??**), setting $d = e$.

Let ordinal $f = D(s)$. By the definition of s, $s(f)$ must be defined (and $\neq \emptyset$) since $s_j(\{< f\}) = s_k(\{< f\})\ \forall\, j, k > i$. This is another contradiction since we chose f such that $s(f) = \emptyset$. The conclusion is that it is impossible for $t < s_i$. □

Note that if e is a limit ordinal and the s_i are strictly increasing, then this result also implies that $t > s_i$ for all $i \in e$, since if $t = s_i$ for any i, then we would get $t < s_{i+1}$, which is impossible.

Property 4. *Suppose $t < s = \operatorname{slim} s_i$, and s is not a prefix of t. Then $t < s_i$ for some $i \in e$.*

Proof. Let d be the first difference coordinate between s and t. We know $d \in D(s)$; otherwise s would be a prefix of t. By property (**??**) there is a k such that $i \geq k \implies s_i(\{< d\}) = s(\{< d\}) = t(\{< d\})$. By property (**??**), there is an $i \geq k$ such that $s_i(d) = s(d) > t(d)$, which means $s_i > t$. □

The next property is the essense of theorem **??**. Now is a good time to note that the definition of a surreal limit works equally well for decreasing sequences, and that the relevant properties apply in those cases with their inequalities reversed. We'll also say that surreals s, t are **relatives** if either s is a prefix of t, or vice versa.

Property 5. *Suppose $(s_i)_{i\in e}$ is an increasing sequence of surreals, $(t_i)_{i\in f}$ is decreasing, with $s_i < t_j \;\forall\, i \in e, j \in f$. Let $s = \operatorname{slim} s_i$ and $t = \operatorname{slim} t_j$. Then either $s < t$ or s, t are relatives.*

Proof. If $s < t$ or $s = t$, we are done, so suppose $t < s$, and that they are not relatives. Then the first difference coordinate d has $t(d) < s(d)$. We know that $s(d) \neq \emptyset$ and $t(d) \neq \emptyset$; otherwise, they would be relatives. Thus $t(d) = -$ and $s(d) = +$. Define the surreal r to have domain $\{< d\}$, with $r(\{< d\}) = s(\{< d\}) = t(\{< d\})$, so that $t < r < s$, and neither s nor t is a prefix of r.

Then property **??** provides some $s_i > r$ and some $t_j < r$. But this would mean $t_j < s_i$, a contradiction. Therefore, it's impossible to have $t < s$ while s, t are not relatives. □

The next property is not as elegant, but will be useful in proving theorem **??**.

Property 6. *If infinite cardinal C is a successor cardinal or $\aleph_0$, ordinal e has $||e|| < C$, and $||D(s_i)|| < C \;\forall\, i \in e$, then $||D(s)|| < C$ for $s = \operatorname{slim} s_i$.*

Proof. We have

$$D(s) \subset \max_{i\in e} D(s_i) \subset \sum D(s_i),$$

so $||D(s)|| \leq \sum_{i\in e} ||D(s_i)||$. If $C = \aleph_0$, then this is a finite sum of finite numbers, which is also clearly finite (i.e. $||D(s)|| < C$), and we would be done.

Otherwise, suppose C^- is the immediate successor of C. Then $||D(s_i)|| \leq C^-$ for all i, and $\sum ||D(s_i)|| \leq (C^-)^2 = C^-$, which completes the proof. □

4.2. **Proofs of the main results.** The theorems are restated here for convenience.

Theorem ??. *Suppose $\aleph_d$ is either a successor cardinal, or is $\aleph_0$. Suppose e, f are ordinals of cardinality less than $\aleph_d$, and that $(s_i)_{i\in e} \subset S_d$ is increasing, $(t_j)_{j\in f} \subset S_d$ is decreasing, and $s_i < t_j \;\forall\, i, j$. Then there exists an element $r \in S_d$ such that $s_i < r < t_j \;\forall\, i, j$.*

Proof. Let $s = \operatorname{slim} s_i$ and $t = \operatorname{slim} t_i$. By property **??**, both s and t are in S_d. By property **??**, either $s < t$ or one is the prefix of the other.

If they are not relatives, then $s < t$, and by property **??**, any surreal r in $[s, t]$ has $s_i < r < t_j \;\forall\, i, j$. We will construct a specific surreal $r \in S_d$ with $s < r < t$. In particular, let $d = D(s)$ and choose r with $r(\{< d\}) = s$ and $r(d) = +$, so that $s < r$. The first difference coordinate between s and t must be within $D(s)$ (otherwise they would be relatives), so we also have $r < t$. By property **??**, $||D(r)|| < \aleph_d$, so $r \in S_d$. Thus r satisfies the theorem when $s < t$.

Now suppose that s is a prefix of t, and let $r = t$. By property **??**, $s_i < r$ and $r < t_j$ for all $i \in e, j \in f$. By property **??**, $||D(r)|| < \aleph_d$, so again r satisfies the conditions of the theorem. If t is a prefix of s, we can similarly set $r = s$. This shows that the required surreal r exists in all cases. □

Theorem ??. *If X is a linear order with $||X|| \leq \aleph_d$, then X can be order-embedded in S_d.*

Proof. The axiom of choice (AoC) is used in this proof.

Suppose X, of size $\aleph_d$, is linearly ordered by $<_t$, and we give it a well order $<_w$ such that $||\{<_w x\}|| < \aleph_d \; \forall x \in X$. We will recursively define an order embedding $f : X \to S_d$, using transfinite induction on $<_w$.

Suppose f has been defined on $\{<_w x'\}$. Using x', we will define sequences $s_i, t_j \in S_d$ such that the condition $s_i < f(x') < t_j$ is necessary and sufficient for f to remain an order embedding.

Recursively define $s_i \in S_d$ as an element (using AoC) of $\{f(x) : x <_w x', x <_t x', s_j < f(x) \, \forall \, j < i\}$, unless this set is empty, in which case the sequence s_i terminates. Similarly define $t_j \in S_d$ as an element of $\{f(x) : x <_w x', x' <_t x, f(x) < t_k \, \forall \, k < j\}$, until that set is empty. It's possible that either or both of these sequences may be empty, which does not cause any difficulty. Note that s_i is increasing, and t_j is decreasing, and $f^{-1}(s_i) <_t x' <_t f^{-1}(t_j) \; \forall \, i, j$, so that $s_i < t_j$.

If $\aleph_d$ is a successor or $\aleph_0$, we can apply theorem **??** to find an element $r \in S_d$ with $s_i < r < t_j$, and we set $f(x') = r$. By transfinite induction, this results in an order embedding f which satisfies the theorem in this case.

If $\aleph_d = \cup_{e<d} \aleph_e$, then we note that the above construction is capable of extending *any* order embedding $f : \{< x'\} \to S_d$ by adding $f(x')$. For ordinal e, let x_e denote the $<_w$-first element of X with $||\{<_w x_e\}|| = \aleph_e$. Start by using the above method to find $f_0 : \{<_w x_0\} \to S_0$. Then at each step extend this from $\cup_{i<e} f_i$ to $f_e : \{<_w x_e\} \to S_e$ using the above technique. (This can be done since $i < j \implies S_i \subset S_j$.) Let f be the union of all these functions; this is an order embedding satisfying the theorem. $\square$

To conclude, we note that the size of S_d is

$$||S_d|| = \sum_{e<d} 2^{\aleph_e}.$$

(This notation is assuming the cardinals can be well-ordered, which is a consequence of the axiom of choice.) Under the generalized continuum hypothesis, $2^{\aleph_e} = \aleph_{e+1}$, and it follows that $||S_d|| = \aleph_d$.

5. Edit this paper

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