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Field and Wave Electromagnetics

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Preface

The many books on introductory electromagnetics can be roughly divided into two main groups. The first group takes the traditional development: starting with the experimental laws, generalizing them in steps, and finally synthesizing them in the form of Maxwell's equations. This is an inductive approach. The second group takes the axiomatic development: starting with Maxwell's equations, identifying each with the appropriate experimental law, and specializing the general equations to static and time-varying situations for analysis. This is a deductive approach. A few books begin with a treatment of the special theory of relativity and develop all of electromagnetic theory from Coulomb's law of force; but this approach requires the discussion and understanding of the special theory of relativity first and is perhaps best suited for a course at an advanced level.

Proponents of the traditional development argue that it is the way electromagnetic theory was unraveled historically (from special experimental laws to Maxwell's equations), and that it is easier for the students to follow than the other methods. I feel, however, that the way a body of knowledge was unraveled is not necessarily the best way to teach the subject to students. The topics tend to be fragmented and cannot take full advantage of the conciseness of vector calculus. Students are puzzled at, and often form a mental block to, the subsequent introduction of gradient, divergence, and curl operations. As a process for formulating an electromagnetic model, this approach lacks cohesiveness and elegance.

The axiomatic development usually begins with the set of four Maxwell's equations, either in differential or in integral form, as fundamental postulates. These are equations of considerable complexity and are difficult to master. They are likely to cause consternation and resistance in students who are hit with all of them at the beginning of a book. Alert students will wonder about the meaning of the field vectors and about the necessity and sufficiency of these general equations. At the initial stage students tend to be confused about the concepts of the electromagnetic model, and they are not yet comfortable with the associated mathematical manipulations. In any case, the general Maxwell's equations are soon simplified to apply to static fields, which allow the consideration of electrostatic fields and magnetostatic fields separately. Why then should the entire set of four Maxwell's equations be introduced at the outset?

It may be argued that Coulomb's law, though based on experimental evidence, is in fact also a postulate. Consider the two stipulations of Coulomb's law: that the charged bodies are very small compared with their distance of separation, and that the force between the charged bodies is inversely proportional to the square of their distance. The question arises regarding the first stipulation: How small must the charged bodies be in order to be considered "very small" compared with their distance? In practice the charged bodies cannot be of vanishing sizes (ideal point charges), and there is difficulty in determining the "true" distance between two bodies of finite dimensions. For given body sizes the relative accuracy in distance measurements is better when the separation is larger. However, practical considerations (weakness of force, existence of extraneous charged bodies, etc.) restrict the usable distance of separation in the laboratory, and experimental inaccuracies cannot be entirely avoided. This leads to a more important question concerning the inverse-square relation of the second stipulation. Even if the charged bodies were of vanishing sizes, experimental measurements could not be of an infinite accuracy no matter how skillful and careful an experimentor was. How then was it possible for Coulomb to know that the force was *exactly* inversely proportional to the *square* (not the 2.000001th or the 1.999999th power) of the distance of separation? This question cannot be answered from an experimental viewpoint because it is not likely that during Coulomb's time experiments could have been accurate to the seventh place. We must therefore conclude that Coulomb's law is itself a postulate and that it is a law of nature discovered and assumed on the basis of his experiments of a limited accuracy (see Section 3–2).

This book builds the electromagnetic model using an *axiomatic approach in steps*: first for static electric fields (Chapter 3), then for static magnetic fields (Chapter 6), and finally for time-varying fields leading to Maxwell's equations (Chapter 7). The mathematical basis for each step is Helmholtz's theorem, which states that a vector field is determined to within an additive constant if both its divergence and its curl are specified everywhere. Thus, for the development of the electrostatic model in free space, it is only necessary to define a single vector (namely, the electric field intensity **E**) by specifying its divergence and its curl as postulates. All other relations in electrostatics for free space, including Coulomb's law and Gauss's law, can be derived from the two rather simple postulates. Relations in material media can be developed through the concept of equivalent charge distributions of polarized dielectrics.

Similarly, for the magnetostatic model in free space it is necessary to define only a single magnetic flux density vector **B** by specifying its divergence and its curl as postulates; all other formulas can be derived from these two postulates. Relations in material media can be developed through the concept of equivalent current densities. Of course, the validity of the postulates lies in their ability to yield results that conform with experimental evidence.

For time-varying fields, the electric and magnetic field intensities are coupled. The curl **E** postulate for the electrostatic model must be modified to conform with

Faraday's law. In addition, the curl **B** postulate for the magnetostatic model must also be modified in order to be consistent with the equation of continuity. We have, then, the four Maxwell's equations that constitute the electromagnetic model. I believe that this gradual development of the electromagnetic model based on Helmholtz's theorem is novel, systematic, pedagogically sound, and more easily accepted by students.

In the presentation of the material, I strive for lucidity and unity, and for smooth and logical flow of ideas. Many worked-out examples are included to emphasize fundamental concepts and to illustrate methods for solving typical problems. Applications of derived relations to useful technologies (such as ink-jet printers, lightning arresters, electret microphones, cable design, multiconductor systems, electrostatic shielding, Doppler radar, radome design, Polaroid filters, satellite communication systems, optical fibers, and microstrip lines) are discussed. Review questions appear at the end of each chapter to test the students' retention and understanding of the essential material in the chapter. The problems in each chapter are designed to reinforce students' comprehension of the interrelationships between the different quantities in the formulas, and to extend their ability of applying the formulas to solve practical problems. In teaching, I have found the review questions a particularly useful device to stimulate students' interest and to keep them alert in class.

Besides the fundamentals of electromagnetic fields, this book also covers the theory and applications of transmission lines, waveguides and cavity resonators, and antennas and radiating systems. The fundamental concepts and the governing theory of electromagnetism do not change with the introduction of new electromagnetic devices. Ample reasons and incentives for learning the fundamental principles of electromagnetics are given in Section 1–1. I hope that the contents of this book, strengthened by the novel approach, will provide students with a secure and sufficient background for understanding and analyzing basic electromagnetic phenomena as well as prepare them for more advanced subjects in electromagnetic theory.

There is enough material in this book for a two-semester sequence of courses. Chapters 1 through 7 contain the material on fields, and Chapters 8 through 11 on waves and applications. In schools where there is only a one-semester course on electromagnetics, Chapters 1 through 7, plus the first four sections of Chapter 8 would provide a good foundation on fields and an introduction of waves in unbounded media. The remaining material could serve as a useful reference book on applications or as a textbook for a follow-up elective course. Schools on a quarter system could adjust the material to be covered in accordance with the total number of hours assigned to the subject of electromagnetics. Of course, individual instructors have the prerogative to emphasize and expand certain topics, and to deemphasize or delete certain others.

I have given considerable thought to the advisability of including computer programs for the solution of some problems, but have finally decided against it. Diverting students' attention and effort to numerical methods and computer software would distract them from concentrating on learning the fundamentals of electromagnetism. Where appropriate, the dependence of important results on the value of a parameter is stressed by curves; field distributions and antenna patterns are illustrated by graphs; and typical mode patterns in waveguides are plotted. The computer programs for obtaining these curves, graphs, and mode patterns are not always simple. Students in science and engineering are required to acquire a facility for using computers; but the inclusion of some cookbook-style computer programs in a book on the fundamental principles of electromagnetic fields and waves would appear to contribute little to the understanding of the subject matter.

This book was first published in 1983. Favorable reactions and friendly encouragements from professors and students have provided me with the impetus to come out with a new edition. In this second edition I have added many new topics. These include Hall effect, d-c motors, transformers, eddy current, energy-transport velocity for wide-band signals in waveguides, radar equation and scattering cross section, transients in transmission lines, Bessel functions, circular waveguides and circular cavity resonators, waveguide discontinuities, wave propagation in ionosphere and near earth's surface, helical antennas, log-periodic dipole arrays, and antenna effective length and effective area. The total number of problems has been expanded by about 35 percent.

The Addison-Wesley Publishing Company has decided to make this second edition a two-color book. I think the readers will agree that the book is handsomely produced. I would like to take this opportunity to express my appreciation to all the people on the editorial, production, and marketing staff who provided help in bringing out this new edition. In particular, I wish to thank Thomas Robbins, Barbara Rifkind, Karen Myer, Joseph K. Vetere, and Katherine Harutunian.

Chevy Chase, Maryland D. K. C.

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9

Theory and Applications of Transmission Lines

~~9~~–1 Introduction

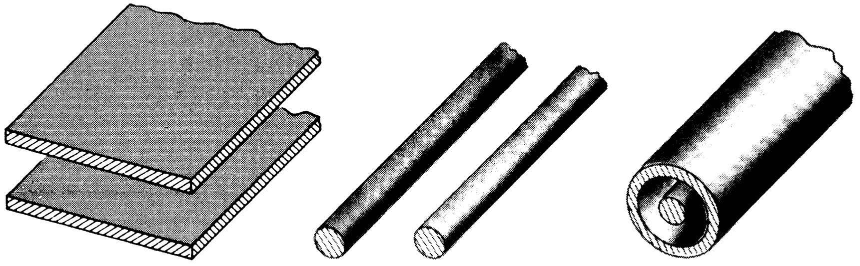
We have now developed an electromagnetic model with which we can analyze electromagnetic actions that occur at a distance and are caused by time-varying charges and currents. These actions are explained in terms of electromagnetic fields and waves. An isotropic or omnidirectional electromagnetic source radiates waves equally in all directions. Even when the source radiates through a highly directive antenna, its energy spreads over a wide area at large distances. This radiated energy is not guided, and the transmission of power and information from the source to a receiver is inefficient. This is especially true at lower frequencies for which directive antennas would have huge dimensions and therefore would be excessively expensive. For instance, at AM broadcast frequencies a single half-wavelength antenna (which is only mildly directive[†]) would be over a hundred meters long. At the 60 (Hz) power frequency a wavelength is 5 million meters or 5 (Mm)!

For efficient point-to-point transmission of power and information the source energy must be directed or guided. In this chapter we study transverse electromagnetic (TEM) waves guided by transmission lines. The TEM mode of guided waves is one in which **E** and **H** are perpendicular to each other and both are transverse to the direction of propagation along the guiding line. We discussed the propagation of unguided TEM plane waves in the last chapter. We will show in this chapter that many of the characteristics of TEM waves guided by transmission lines are the same as those for a uniform plane wave propagating in an unbounded dielectric medium.

The three most common types of guiding structures that support TEM waves are:

a) *Parallel-plate transmission line.* This type of transmission line consists of two parallel conducting plates separated by a dielectric slab of a uniform thickness.

[†] Principles of antennas and radiating systems will be discussed in Chapter 11.



(a) Parallel-plate transmission line. (b) Two-wire transmission line. (c) Coaxial transmission line.

FIGURE 9–1
Common types of transmission lines.

[See Fig. 9–1(a).] At microwave frequencies, parallel-plate transmission lines can be fabricated inexpensively on a dielectric substrate using printed-circuit technology. They are often called ***striplines.***

b) *Two-wire transmission line.* This transmission line consists of a pair of parallel conducting wires separated by a uniform distance. [See Fig. 9–1(b).] Examples are the ubiquitous overhead power and telephone lines seen in rural areas and the flat lead-in lines from a rooftop antenna to a television receiver.

c) *Coaxial transmission line.* This consists of an inner conductor and a coaxial outer conducting sheath separated by a dielectric medium. [See Fig. 9–1(c).] This structure has the important advantage of confining the electric and magnetic fields entirely within the dielectric region. No stray fields are generated by a coaxial transmission line, and little external interference is coupled into the line. Examples are telephone and TV cables and the input cables to high-frequency precision measuring instruments.

We should note that other wave modes more complicated than the TEM mode can propagate on all three of these types of transmission lines when the separation between the conductors is greater than certain fractions of the operating wavelength. These other transmission modes will be considered in the next chapter.

We will show that the TEM wave solution of Maxwell's equations for the parallel-plate guiding structure in Fig. 9–1(a) leads directly to a pair of transmission-line equations. The general transmission-line equations can also be derived from a circuit model in terms of the resistance, inductance, conductance, and capacitance per unit length of a line. The transition from the circuit model to the electromagnetic model is effected from a network with lumped-parameter elements (discrete resistors, inductors, and capacitors) to one with distributed parameters (continuous distributions of R, L, G, and C along the line). From the transmission-line equations, all the characteristics of wave propagation along a given line can be derived and studied.

The study of time-harmonic steady-state properties of transmission lines is greatly facilitated by the use of graphical charts, which avert the necessity of repeated calculations with complex numbers. The best known and most widely used graphical chart is the ***Smith chart***. The use of Smith chart for determining wave characteristics on a transmission line and for impedance matching will be discussed.

9–2 Transverse Electromagnetic Wave along a Parallel-Plate Transmission Line

Let us consider a y-polarized TEM wave propagating in the $+z$-direction along a uniform parallel-plate transmission line. Figure 9–2 shows the cross-sectional dimensions of such a line and the chosen coordinate system. For time-harmonic fields the wave equation to be satisfied in the sourceless dielectric region becomes the homogeneous Helmholtz's equation, Eq. (8–46). In the present case the appropriate phasor solution for the wave propagating in the $+z$-direction is

$$\mathbf{E} = \mathbf{a}_y E_y = \mathbf{a}_y E_0 e^{-\gamma z}. \tag{9–1a}$$

The associated $\mathbf{H}$ field is, from Eq. (8–31),

$$\mathbf{H} = \mathbf{a}_x H_x = -\mathbf{a}_x \frac{E_0}{\eta} e^{-\gamma z}, \tag{9–1b}$$

where γ and η are the propagation constant and the intrinsic impedance, respectively, of the dielectric medium. Fringe fields at the edges of the plates are neglected. Assuming perfectly conducting plates and a lossless dielectric, we have, from Chapter 8,

$$\gamma = j\beta = j\omega\sqrt{\mu\epsilon} \tag{9–2}$$

and

$$\eta = \sqrt{\frac{\mu}{\epsilon}}. \tag{9–3}$$

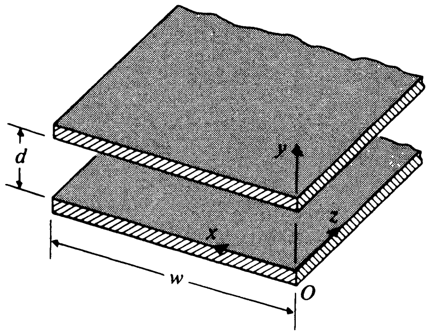


FIGURE 9–2
Parallel-plate transmission line.

The boundary conditions to be satisfied at the interfaces of the dielectric and the perfectly conducting planes are, from Eqs. (7–68a, b, c, and d), as follows:

At both $y = 0$ and $y = d$:

$$E_t = 0 \tag{9-4}$$

and

$$H_n = 0, \tag{9-5}$$

which are obviously satisfied because $E_x = E_z = 0$ and $H_y = 0$.

At $y = 0$ (lower plate), $\mathbf{a}_n = \mathbf{a}_y$:

$$\mathbf{a}_y \cdot \mathbf{D} = \rho_{s\ell} \quad \text{or} \quad \rho_{s\ell} = \epsilon E_y = \epsilon E_0 e^{-j\beta z}; \tag{9-6a}$$

$$\mathbf{a}_y \times \mathbf{H} = \mathbf{J}_{s\ell} \quad \text{or} \quad \mathbf{J}_{s\ell} = -\mathbf{a}_z H_x = \mathbf{a}_z \frac{E_0}{\eta} e^{-j\beta z}. \tag{9-7a}$$

At $y = d$ (upper plate), $\mathbf{a}_n = -\mathbf{a}_y$:

$$-\mathbf{a}_y \cdot \mathbf{D} = \rho_{su} \quad \text{or} \quad \rho_{su} = -\epsilon E_y = -\epsilon E_0 e^{-j\beta z}; \tag{9-6b}$$

$$-\mathbf{a}_y \times \mathbf{H} = \mathbf{J}_{su} \quad \text{or} \quad \mathbf{J}_{su} = \mathbf{a}_z H_x = -\mathbf{a}_z \frac{E_0}{\eta} e^{-j\beta z}. \tag{9-7b}$$

Equations (9–6) and (9–7) indicate that surface charges and surface currents on the conducting planes vary sinusoidally with z, as do E_y and H_x. This is illustrated schematically in Fig. 9–3.

Field phasors **E** and **H** in Eqs. (9–1a) and (9–1b) satisfy the two Maxwell's curl equations:

$$\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H} \tag{9-8}$$

and

$$\nabla \times \mathbf{H} = j\omega\epsilon\mathbf{E}. \tag{9-9}$$

Since $\mathbf{E} = \mathbf{a}_y E_y$ and $\mathbf{H} = \mathbf{a}_x H_x$, Eqs. (9–8) and (9–9) become

$$\frac{dE_y}{dz} = j\omega\mu H_x \tag{9-10}$$

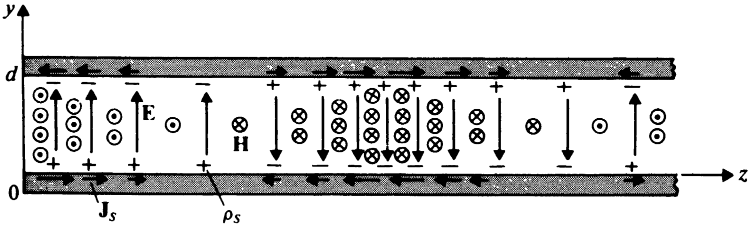


FIGURE 9–3
Field, charge, and current distributions along a parallel-plate transmission line.

and

$$\frac{dH_x}{dz} = j\omega\epsilon E_y. \tag{9–11}$$

Ordinary derivatives appear above because phasors E_y and H_x are functions of z only.

Integrating Eq. (9–10) over y from 0 to d, we have

$$\frac{d}{dz}\int_0^d E_y\,dy = j\omega\mu \int_0^d H_x\,dy$$

or

$$\begin{aligned} -\frac{dV(z)}{dz} &= j\omega\mu J_{su}(z)d = j\omega\left(\mu\frac{d}{w}\right)[J_{su}(z)w] \\ &= j\omega LI(z), \end{aligned} \tag{9–12}$$

where

$$V(z) = -\int_0^d E_y\,dy = -E_y(z)d$$

is the potential difference or voltage between the upper and lower plates,

$$I(z) = J_{su}(z)w$$

is the total current flowing in the $+z$ direction in the upper plate (w = plate width), and

$$\boxed{L = \mu\frac{d}{w} \qquad \text{(H/m)}} \tag{9–13}$$

is the inductance per unit length of the parallel-plate transmission line. The dependence of phasors $V(z)$ and $I(z)$ on z is noted explicitly in Eq. (9–12) for emphasis.

Similarly, we integrate Eq. (9–11) over x from 0 to w to obtain

$$\frac{d}{dz}\int_0^w H_x\,dx = j\omega\epsilon \int_0^w E_y\,dx$$

or

$$\begin{aligned} -\frac{dI(z)}{dz} &= -j\omega\epsilon E_y(z)w = j\omega\left(\epsilon\frac{w}{d}\right)[-E_y(z)d] \\ &= j\omega CV(z), \end{aligned} \tag{9–14}$$

where

$$\boxed{C = \epsilon\frac{w}{d} \qquad \text{(F/m)}} \tag{9–15}$$

is the capacitance per unit length of the parallel-plate transmission line.

Equations (9–12) and (9–14) constitute a pair of ***time-harmonic transmission-line equations*** for phasors $V(z)$ and $I(z)$. They may be combined to yield second-order differential equations for $V(z)$ and for $I(z)$:

$$\frac{d^2V(z)}{dz^2} = -\omega^2 LCV(z), \tag{9-16a}$$

$$\frac{d^2I(z)}{dz^2} = -\omega^2 LCI(z). \tag{9-16b}$$

The solutions of Eqs. (9–16a) and (9–16b) are, for waves propagating in the $+z$-direction,

$$V(z) = V_0 e^{-j\beta z} \tag{9-17a}$$

and

$$I(z) = I_0 e^{-j\beta z}, \tag{9-17b}$$

where the phase constant

$$\boxed{\beta = \omega\sqrt{LC} = \omega\sqrt{\mu\epsilon} \qquad \text{(rad/m)}} \tag{9-18}$$

is the same as that given in Eq. (9–2). The relation between V_0 and I_0 can be found by using either Eq. (9–12) or Eq. (9–14):

$$\boxed{Z_0 = \frac{V(z)}{I(z)} = \frac{V_0}{I_0} = \sqrt{\frac{L}{C}} \qquad (\Omega),} \tag{9-19}$$

which becomes, in view of the results of Eqs. (9–13) and (9–15),

$$\boxed{Z_0 = \frac{d}{w}\sqrt{\frac{\mu}{\epsilon}} = \frac{d}{w}\eta \qquad (\Omega).} \tag{9-20}$$

The quantity Z_0 is the impedance at any location that looks toward an infinitely long (no reflections) transmission line. It is called the ***characteristic impedance*** of the line. The ratio of $V(z)$ and $I(z)$ at any point on a finite line of any length terminated in Z_0 is Z_0.† For a parallel-plate transmission line with perfectly conducting plates of width w and separated by a lossless dielectric slab of thickness d, the characteristic impedance Z_0 is (d/w) times the intrinsic impedance η of the dielectric medium.

The velocity of propagation along the line is

$$\boxed{u_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{\mu\epsilon}} \qquad \text{(m/s)},} \tag{9-21}$$

which is the same as the phase velocity of a TEM plane wave in the dielectric medium.

† This statement will be proved in Section 9–4 (see Eq. 9–107).

9–2.1 LOSSY PARALLEL-PLATE TRANSMISSION LINES

We have so far assumed the parallel-plate transmission line to be lossless. In actual situations, loss may arise from two causes. First, the dielectric medium may have a nonvanishing loss tangent; second, the plates may not be perfectly conducting. To characterize these two effects, we define two new parameters: G, the conductance per unit length across the two plates; and R, the resistance per unit length of the *two* plate conductors.

The conductance between two conductors separated by a dielectric medium having a permittivity ϵ and an equivalent conductivity σ can be determined readily by using Eq. (5–81) when the capacitance between the two conductors is known. We have

$$G = \frac{\sigma}{\epsilon} C. \tag{9–22}$$

Use of Eq. (9–15) directly yields

$$\boxed{G = \sigma \frac{w}{d} \qquad \text{(S/m)}.} \tag{9–23}$$

If the parallel-plate conductors have a very large but finite conductivity σ_c (which must not be confused with the conductivity σ of the dielectric medium), ohmic power will be dissipated in the plates. This necessitates the presence of a nonvanishing axial electric field $\mathbf{a}_z E_z$ at the plate surfaces, such that the average Poynting vector

$$\mathscr{P}_{av} = \mathbf{a}_y p_\sigma = \tfrac{1}{2}\mathscr{R}e(\mathbf{a}_z E_z \times \mathbf{a}_x H_x^*) \tag{9–24}$$

has a y-component and equals the average power per unit area dissipated in each of the conducting plates. (Obviously the cross product of $\mathbf{a}_y E_y$ and $\mathbf{a}_x H_x$ does not result in a y-component.)

Consider the upper plate where the surface current density is $J_{su} = H_x$. It is convenient to define a ***surface impedance*** of an imperfect conductor, Z_s, as the ratio of the tangential component of the electric field to the surface current density at the conductor surface.

$$\boxed{Z_s = \frac{E_t}{J_s} \qquad (\Omega).} \tag{9–25}$$

For the upper plate we have

$$Z_s = \frac{E_z}{J_{su}} = \frac{E_z}{H_x} = \eta_c, \tag{9–26a}$$

where η_c is the intrinsic impedance of the plate conductor. Here we assume that both the conductivity σ_c of the plate conductor and the operating frequency are sufficiently high that the current flows in a very thin surface layer and can be represented by the surface current J_{su}. The intrinsic impedance of a good conductor has been given in Eq. (8–54). We have

$$Z_s = R_s + jX_s = (1 + j)\sqrt{\frac{\pi f \mu_c}{\sigma_c}} \qquad (\Omega), \tag{9–26b}$$

where the subscript c is used to indicate the properties of the conductor.

Substitution of Eq. (9–26a) in Eq. (9–24) gives

$$\begin{aligned} p_\sigma &= \tfrac{1}{2}\mathcal{R}e(|J_{su}|^2 Z_s) \\ &= \tfrac{1}{2}|J_{su}|^2 R_s \qquad (\text{W/m}^2). \end{aligned} \tag{9–27}$$

The ohmic power dissipated in a unit length of the plate having a width w is wp_σ, which can be expressed in terms of the total surface current, $I = wJ_{su}$, as

$$P_\sigma = wp_\sigma = \frac{1}{2} I^2 \left(\frac{R_s}{w}\right) \qquad (\text{W/m}). \tag{9–28}$$

Equation (9–28) is the power dissipated when a sinusoidal current of amplitude I flows through a resistance R_s/w. Thus, the effective series resistance per unit length for *both* plates of a parallel-plate transmission line of width w is

$$\boxed{R = 2\left(\frac{R_s}{w}\right) = \frac{2}{w}\sqrt{\frac{\pi f \mu_c}{\sigma_c}} \qquad (\Omega/\text{m}).} \tag{9–29}$$

Table 9–1 lists the expressions for the four *distributed parameters* (R, L, G, and C per unit length) of a parallel-plate transmission line of width w and separation d.

TABLE 9–1
Distributed Parameters of Parallel-Plate Transmission Line (Width = w, Separation = d)

Parameter	Formula	Unit
R	$\dfrac{2}{w}\sqrt{\dfrac{\pi f \mu_c}{\sigma_c}}$	Ω/m
L	$\mu \dfrac{d}{w}$	H/m
G	$\sigma \dfrac{w}{d}$	S/m
C	$\epsilon \dfrac{w}{d}$	F/m

We note from Eq. (9–26b) that surface impedance Z_s has a positive reactance term X_s that is numerically equal to R_s. If the total complex power (instead of its real part, the ohmic power P_σ, only) associated with a unit length of the plate is considered, X_s will lead to an ***internal series inductance*** per unit length $L_i = X_s/\omega = R_s/\omega$. At high frequencies, L_i is negligible in comparison with the external inductance L.

We note in the calculation of the power loss in the plate conductors of a finite conductivity σ_c that a nonvanishing electric field $\mathbf{a}_z E_z$ must exist. The very existence of this axial electric field makes the wave along a lossy transmission line strictly not TEM. However, this axial component is ordinarily very small in comparison to the transverse component E_y. An estimate of their relative magnitudes can be made as follows:

$$\frac{|E_z|}{|E_y|} = \frac{|\eta_c H_x|}{|\eta H_x|} = \sqrt{\frac{\epsilon}{\mu}}\,|\eta_c|$$
$$= \sqrt{\frac{\omega\epsilon\mu_c}{\mu\sigma_c}} = \sqrt{\frac{\omega\epsilon}{\sigma_c}},$$

where Eq. (8–54) has been used. For copper plates $[\sigma_c = 5.80 \times 10^7\ \text{(S/m)}]$ in air $[\epsilon = \epsilon_0 = 10^{-9}/36\pi\ \text{(F/m)}]$ at a frequency of 3 (GHz),

$$|E_z| \cong 5.3 \times 10^{-5}|E_y| \ll |E_y|.$$

Hence we retain the designation TEM as well as all its consequences. The introduction of a small E_z in the calculation of p_σ and R is considered a slight perturbation.

9–2.2 MICROSTRIP LINES

The development of solid-state microwave devices and systems has led to the widespread use of a form of parallel-plate transmission lines called microstrip lines or simply ***striplines***. A stripline usually consists of a dielectric substrate sitting on a grounded conducting plane with a thin narrow metal strip on top of the substrate, as shown in Fig. 9–4(a). Since the advent of printed-circuit techniques, striplines can be easily fabricated and integrated with other circuit components. However, because the results that we have derived in this section were based on the assumption of two wide

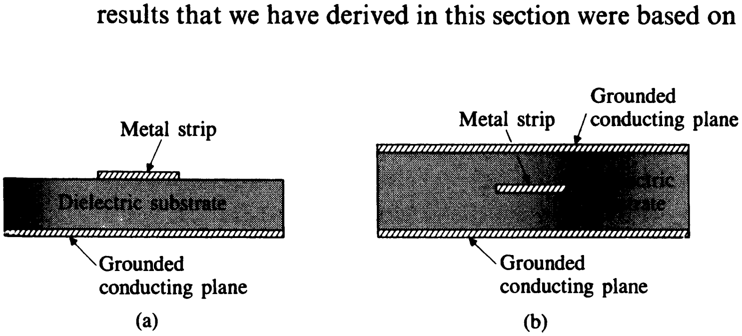


FIGURE 9–4
Two types of microstrip lines.

conducting plates (with negligible fringing effect) of equal width, they are not expected to apply here exactly. The approximation is closer if the width of the metal strip is much greater than the substrate thickness.

When the substrate has a high dielectric constant, a TEM approximation is found to be reasonably satisfactory. An exact analytical solution of the stripline in Fig. 9–4(a) satisfying all the boundary conditions is a difficult problem. Not all the fields will be confined in the dielectric substrate; some will stray from the top strip into the region outside of the strip, thus causing interference in the neighboring circuits. Semiempirical modifications to the formulas for the distributed parameters and the characteristic impedance are necessary for more accurate calculations.† All of these quantities tend to be frequency-dependent, and striplines are dispersive.

One method for reducing the stray fields of striplines is to have a grounded conducting plane on both sides of the dielectric substrate and to put the thin metal strip in the middle as in Fig. 9–4(b). This arrangement is known as a ***triplate line***. We can appreciate that triplate lines are more difficult and costly to fabricate and that the characteristic impedance of a triplate line is one-half of that of a corresponding stripline.

EXAMPLE 9–1 Neglecting losses and fringe effects and assuming the substrate of a stripline to have a thickness 0.4 (mm) and a dielectric constant 2.25, (a) determine the required width w of the metal strip in order for the stripline to have a characteristic resistance of 50 (Ω); (b) determine L and C of the line; and (c) determine u_p along the line. (d) Repeat parts (a), (b), and (c) for a characteristic resistance of 75 (Ω).

Solution

a) We use Eq. (9–20) directly to find w:

$$w = \frac{d}{Z_0}\sqrt{\frac{\mu}{\epsilon}} = \frac{0.4 \times 10^{-3}}{50} \frac{\eta_0}{\sqrt{\epsilon_r}}$$

$$= \frac{0.4 \times 10^{-3} \times 377}{50\sqrt{2.25}} = 2 \times 10^{-3} \quad \text{(m)}, \quad \text{or } 2 \quad \text{(mm)}.$$

b) $$L = \mu \frac{d}{w} = 4\pi 10^{-7} \times \frac{0.4}{2} = 2.51 \times 10^{-7} \quad \text{(H/m)}, \quad \text{or } 0.251 \quad (\mu\text{H/m}).$$

$$C = \epsilon_0 \epsilon_r \frac{w}{d} = \frac{10^{-9}}{36\pi} \times 2.25 \times \frac{2}{0.4} = 99.5 \times 10^{-12} \quad \text{(F/m)}, \quad \text{or } 99.5 \quad \text{(pF/m)}.$$

c) $$u_p = \frac{1}{\sqrt{\mu\epsilon}} = \frac{c}{\sqrt{\epsilon_r}} = \frac{c}{\sqrt{2.25}} = \frac{c}{1.5} = 2 \times 10^8 \quad \text{(m/s)}.$$

† See, for instance, K. F. Sander and G. A. L. Reed, *Transmission and Propagation of Electromagnetic Waves*, 2nd edition, Sec. 6.5.6, Cambridge University Press, New York, 1986.

d) Since w is inversely proportional to Z_0, we have, for $Z_0' = 75$ (Ω),

$$w' = \left(\frac{Z_0}{Z_0'}\right)w = \frac{50}{75} \times 2 = 1.33 \quad \text{(mm)}.$$

$$L' = \left(\frac{w}{w'}\right)L = \left(\frac{2}{1.33}\right) \times 0.251 = 0.377 \quad (\mu\text{H/m}).$$

$$C' = \left(\frac{w'}{w}\right)C = \left(\frac{1.33}{2}\right) \times 99.5 = 66.2 \quad \text{(pF/m)}.$$

$$u_p' = u_p = 2 \times 10^8 \quad \text{(m/s)}.$$

9–3 General Transmission-Line Equations

We will now derive the equations that govern general two-conductor uniform transmission lines that include parallel-plate, two-wire, and coaxial lines. Transmission lines differ from ordinary electric networks in one essential feature. Whereas the physical dimensions of electric networks are very much smaller than the operating wavelength, transmission lines are usually a considerable fraction of a wavelength and may even be many wavelengths long. The circuit elements in an ordinary electric network can be considered discrete and as such may be described by lumped parameters. It is assumed that currents flowing in lumped-circuit elements do not vary spatially over the elements, and that no standing waves exist. A transmission line, on the other hand, is a distributed-parameter network and must be described by circuit parameters that are distributed throughout its length. Except under matched conditions, standing waves exist in a transmission line.

Consider a differential length Δz of a transmission line that is described by the following four parameters:

R, resistance per unit length (both conductors), in Ω/m.

L, inductance per unit length (both conductors), in H/m.

G, conductance per unit length, in S/m.

C, capacitance per unit length, in F/m.

Note that R and L are series elements and G and C are shunt elements. Figure 9–5 shows the equivalent electric circuit of such a line segment. The quantities $v(z, t)$ and $v(z + \Delta z, t)$ denote the instantaneous voltages at z and $z + \Delta z$, respectively. Similarly, $i(z, t)$ and $i(z + \Delta z, t)$ denote the instantaneous currents at z and $z + \Delta z$, respectively. Applying Kirchhoff's voltage law, we obtain

$$v(z, t) - R\,\Delta z\, i(z, t) - L\,\Delta z\,\frac{\partial i(z, t)}{\partial t} - v(z + \Delta z, t) = 0, \tag{9–30}$$

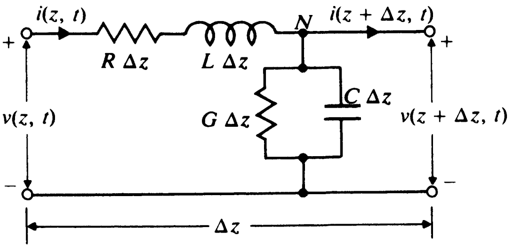


FIGURE 9–5
Equivalent circuit of a differential length Δz of a two-conductor transmission line.

which leads to

$$-\frac{v(z+\Delta z, t) - v(z, t)}{\Delta z} = Ri(z, t) + L\frac{\partial i(z, t)}{\partial t}. \tag{9–30a}$$

In the limit as $\Delta z \to 0$, Eq. (9–30a) becomes

$$\boxed{-\frac{\partial v(z, t)}{\partial z} = Ri(z, t) + L\frac{\partial i(z, t)}{\partial t}.} \tag{9–31}$$

Similarly, applying Kirchhoff's current law to the node N in Fig. 9–5, we have

$$i(z, t) - G\,\Delta z v(z+\Delta z, t) - C\,\Delta z\frac{\partial v(z+\Delta z, t)}{\partial t} - i(z+\Delta z, t) = 0. \tag{9–32}$$

On dividing by Δz and letting Δz approach zero, Eq. (9–32) becomes

$$\boxed{-\frac{\partial i(z, t)}{\partial z} = Gv(z, t) + C\frac{\partial v(z, t)}{\partial t}.} \tag{9–33}$$

Equations (9–31) and (9–33) are a pair of first-order partial differential equations in $v(z, t)$ and $i(z, t)$. They are the ***general transmission-line equations***.†

For harmonic time dependence the use of phasors simplifies the transmission-line equations to ordinary differential equations. For a cosine reference we write

$$v(z, t) = \mathcal{R}e[V(z)e^{j\omega t}], \tag{9–34a}$$

$$i(z, t) = \mathcal{R}e[I(z)e^{j\omega t}], \tag{9–34b}$$

where $V(z)$ and $I(z)$ are functions of the space coordinate z only and both may be complex. Substitution of Eqs. (9–34a) and (9–34b) in Eqs. (9–31) and (9–33) yields

† Sometimes referred to as the ***telegraphist's equations*** or ***telegrapher's equations***.

the following ordinary differential equations for phasors $V(z)$ and $I(z)$:

$$\boxed{\begin{aligned} -\frac{dV(z)}{dz} &= (R + j\omega L)I(z), \\ -\frac{dI(z)}{dz} &= (G + j\omega C)V(z). \end{aligned}} \qquad \begin{aligned} &(9\text{–}35a) \\ &(9\text{–}35b) \end{aligned}$$

Equations (9–35a) and (9–35b) are ***time-harmonic transmission-line equations***, which reduce to Eqs. (9–12) and (9–14) under lossless conditions ($R = 0$, $G = 0$).

9–3.1 WAVE CHARACTERISTICS ON AN INFINITE TRANSMISSION LINE

The coupled time-harmonic transmission-line equations, Eqs. (9–35a) and (9–35b), can be combined to solve for $V(z)$ and $I(z)$. We obtain

$$\boxed{\frac{d^2V(z)}{dz^2} = \gamma^2 V(z)} \qquad (9\text{–}36a)$$

and

$$\boxed{\frac{d^2I(z)}{dz^2} = \gamma^2 I(z),} \qquad (9\text{–}36b)$$

where

$$\boxed{\gamma = \alpha + j\beta = \sqrt{(R + j\omega L)(G + j\omega C)} \qquad (m^{-1})} \qquad (9\text{–}37)$$

is the ***propagation constant*** whose real and imaginary parts, α and β, are the ***attenuation constant*** (Np/m) and ***phase constant*** (rad/m) of the line, respectively. The nomenclature here is similar to that for plane-wave propagation in lossy media as defined in Section 8–3. These quantities are not really constants because, in general, they depend on ω in a complicated way.

The solutions of Eqs. (9–36a) and (9–36b) are

$$\begin{aligned} V(z) &= V^+(z) + V^-(z) \\ &= V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z}, \end{aligned} \qquad (9\text{–}38a)$$

$$\begin{aligned} I(z) &= I^+(z) + I^-(z) \\ &= I_0^+ e^{-\gamma z} + I_0^- e^{\gamma z}, \end{aligned} \qquad (9\text{–}38b)$$

where the plus and minus superscripts denote waves traveling in the $+z$- and $-z$-directions, respectively. Wave amplitudes (V_0^+, I_0^+) and (V_0^-, I_0^-) are related by Eqs. (9–35a) and (9–35b), and it is easy to verify (Problem P.9–5) that

$$\frac{V_0^+}{I_0^+} = -\frac{V_0^-}{I_0^-} = \frac{R + j\omega L}{\gamma}. \qquad (9\text{–}39)$$

For an infinite line (actually a semi-infinite line with the source at the left end) the terms containing the $e^{\gamma z}$ factor must vanish. There are no reflected waves; only the waves traveling in the $+z$-direction exist. We have

$$V(z) = V^+(z) = V_0^+ e^{-\gamma z}, \tag{9-40a}$$

$$I(z) = I^+(z) = I_0^+ e^{-\gamma z}. \tag{9-40b}$$

The ratio of the voltage and the current at any z for an infinitely long line is independent of z and is called the ***characteristic impedance*** of the line.

$$\boxed{Z_0 = \frac{R + j\omega L}{\gamma} = \frac{\gamma}{G + j\omega C} = \sqrt{\frac{R + j\omega L}{G + j\omega C}} \quad (\Omega).} \tag{9-41}$$

Note that γ and Z_0 are characteristic properties of a transmission line whether or not the line is infinitely long. They depend on R, L, G, C, and ω—not on the length of the line. An infinite line simply implies that there are no reflected waves.

There is a close analogy between the general governing equations and the wave characteristics of a transmission line and those of uniform plane waves in a lossy medium. This analogy will be discussed in the following example.

EXAMPLE 9–2 Demonstrate the analogy between the wave characteristics on a transmission line and uniform plane waves in a lossy medium.

Solution In a lossy medium with a complex permittivity $\epsilon_c = \epsilon' - j\epsilon''$ and a complex permeability $\mu = \mu' - j\mu''$ the Maxwell's curl equations (7–104a) and (7–104b) become

$$\nabla \times \mathbf{E} = -j\omega(\mu' - j\mu'')\mathbf{H}, \tag{9-42a}$$

$$\nabla \times \mathbf{H} = j\omega(\epsilon' - j\epsilon'')\mathbf{E}. \tag{9-42b}$$

If we assume a uniform plane wave characterized by an E_x that varies only with z, Eq. (9–42a) reduces to (see Eq. 8–12b)

$$\begin{aligned} -\frac{dE_x(z)}{dz} &= j\omega(\mu' - j\mu'')H_y \\ &= (\omega\mu'' + j\omega\mu')H_y. \end{aligned} \tag{9-43a}$$

Similarly, we obtain from Eq. (9–42b) the following relation:

$$-\frac{dH_y(z)}{dz} = (\omega\epsilon'' + j\omega\epsilon')E_x. \tag{9-43b}$$

Comparing Eqs. (9–43a) and (9–43b) with Eqs. (9–35a) and (9–35b), respectively, we recognize immediately the analogy of the governing equations for E_x and H_y of a uniform plane wave and those for V and I on a transmission line.

Equations (9–43a) and (9–43b) can be combined to give

$$\frac{d^2E_x(z)}{dz^2} = \gamma^2 E_x(z) \tag{9–44a}$$

and

$$\frac{d^2H_y(z)}{dz^2} = \gamma^2 H_y(z), \tag{9–44b}$$

which are entirely similar to Eqs. (9–36a) and (9–36b). The propagation constant of the uniform plane wave is

$$\gamma = \alpha + j\beta = \sqrt{(\omega\mu'' + j\omega\mu')(\omega\epsilon'' + j\omega\epsilon')}, \tag{9–45}$$

which should be compared with Eq. (9–37) for the transmission line. The intrinsic impedance of the lossy medium (the wave impedance of the plane wave traveling in the $+z$-direction) is (see Eq. 8–30)

$$\eta_c = \sqrt{\frac{\mu'' + j\mu'}{\epsilon'' + j\epsilon'}}, \tag{9–46}$$

which is analogous to the expression for the characteristic impedance of a transmission line in Eq. (9–41).

Because of the above analogies, many of the results obtained for normal incidence of uniform plane waves can be adapted to transmission-line problems, and vice versa.

The general expressions for the characteristic impedance in Eq. (9–41) and the propagation constant in Eq. (9–37) are relatively complicated. The following three limiting cases have special significance.

1. *Lossless Line* ($R = 0, G = 0$).

   a) Propagation constant:

$$\gamma = \alpha + j\beta = j\omega\sqrt{LC}; \tag{9–47}$$

$$\alpha = 0, \tag{9–48}$$

$$\beta = \omega\sqrt{LC} \qquad \text{(a linear function of } \omega\text{).} \tag{9–49}$$

   b) Phase velocity:

$$u_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{LC}} \qquad \text{(constant).} \tag{9–50}$$

   c) Characteristic impedance:

$$Z_0 = R_0 + jX_0 = \sqrt{\frac{L}{C}}; \tag{9–51}$$

$$R_0 = \sqrt{\frac{L}{C}} \qquad \text{(constant),} \tag{9–52}$$

$$X_0 = 0. \tag{9–53}$$

2. *Low-Loss Line* ($R \ll \omega L$, $G \ll \omega C$). The low-loss conditions are more easily satisfied at very high frequencies.

a) Propagation constant:

$$\gamma = \alpha + j\beta = j\omega\sqrt{LC}\left(1 + \frac{R}{j\omega L}\right)^{1/2}\left(1 + \frac{G}{j\omega C}\right)^{1/2}$$

$$\cong j\omega\sqrt{LC}\left(1 + \frac{R}{2j\omega L}\right)\left(1 + \frac{G}{2j\omega C}\right) \tag{9-54}$$

$$\cong j\omega\sqrt{LC}\left[1 + \frac{1}{2j\omega}\left(\frac{R}{L} + \frac{G}{C}\right)\right];$$

$$\alpha \cong \frac{1}{2}\left(R\sqrt{\frac{C}{L}} + G\sqrt{\frac{L}{C}}\right), \tag{9-55}$$

$$\beta \cong \omega\sqrt{LC} \qquad \text{(approximately a linear function of } \omega\text{).} \tag{9-56}$$

b) Phase velocity:

$$u_p = \frac{\omega}{\beta} \cong \frac{1}{\sqrt{LC}} \qquad \text{(approximately constant).} \tag{9-57}$$

c) Characteristic impedance:

$$Z_0 = R_0 + jX_0 = \sqrt{\frac{L}{C}}\left(1 + \frac{R}{j\omega L}\right)^{1/2}\left(1 + \frac{G}{j\omega C}\right)^{-1/2}$$

$$\cong \sqrt{\frac{L}{C}}\left[1 + \frac{1}{2j\omega}\left(\frac{R}{L} - \frac{G}{C}\right)\right]; \tag{9-58}$$

$$R_0 \cong \sqrt{\frac{L}{C}}, \tag{9-59}$$

$$X_0 \cong -\sqrt{\frac{L}{C}}\frac{1}{2\omega}\left(\frac{R}{L} - \frac{G}{C}\right) \cong 0. \tag{9-60}$$

3. *Distortionless Line* ($R/L = G/C$). If the condition

$$\frac{R}{L} = \frac{G}{C} \tag{9-61}$$

is satisfied, the expressions for both γ and Z_0 simplify.

a) Propagation constant:

$$\gamma = \alpha + j\beta = \sqrt{(R + j\omega L)\left(\frac{RC}{L} + j\omega C\right)}$$

$$= \sqrt{\frac{C}{L}}\,(R + j\omega L); \tag{9-62}$$

$$\alpha = R\sqrt{\frac{C}{L}}, \tag{9-63}$$

$$\beta = \omega\sqrt{LC} \qquad \text{(a linear function of } \omega\text{).} \tag{9-64}$$

b) Phase velocity:

$$u_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{LC}} \quad \text{(constant).} \tag{9–65}$$

c) Characteristic impedance:

$$Z_0 = R_0 + jX_0 = \sqrt{\frac{R + j\omega L}{(RC/L) + j\omega C}} = \sqrt{\frac{L}{C}}; \tag{9–66}$$

$$R_0 = \sqrt{\frac{L}{C}} \quad \text{(constant),} \tag{9–67}$$

$$X_0 = 0. \tag{9–68}$$

Thus, except for a nonvanishing attenuation constant, the characteristics of a distortionless line are the same as those of a lossless line—namely, a constant phase velocity ($u_p = 1/\sqrt{LC}$) and a constant real characteristic impedance ($Z_0 = R_0 = \sqrt{L/C}$).

A constant phase velocity is a direct consequence of the linear dependence of the phase constant β on ω. Since a signal usually consists of a band of frequencies, it is essential that the different frequency components travel along a transmission line at the same velocity in order to avoid distortion. This condition is satisfied by a lossless line and is approximated by a line with very low losses. For a lossy line, wave amplitudes will be attentuated, and distortion will result when different frequency components attenuate differently, even when they travel with the same velocity. The condition specified in Eq. (9–61) leads to both a constant α and a constant u_p—thus the name ***distortionless line***.

The phase constant of a lossy transmission line is determined by expanding the expression for γ in Eq. (9–37). In general, the phase constant is not a linear function of ω; thus it will lead to a u_p, which depends on frequency. As the different frequency components of a signal propagate along the line with different velocities, the signal suffers ***dispersion***. A general, lossy, transmission line is therefore *dispersive*, as is a lossy dielectric.

EXAMPLE 9–3 It is found that the attenuation on a 50 (Ω) distortionless transmission line is 0.01 (dB/m). The line has a capacitance of 0.1 (nF/m).

a) Find the resistance, inductance, and conductance per meter of the line.

b) Find the velocity of wave propagation.

c) Determine the percentage to which the amplitude of a voltage traveling wave decreases in 1 (km) and in 5 (km).

Solution

a) For a distortionless line,

$$\frac{R}{L} = \frac{G}{C}.$$

The given quantities are

$$R_0 = \sqrt{\frac{L}{C}} = 50 \quad (\Omega),$$

$$\alpha = R\sqrt{\frac{C}{L}} = 0.01 \quad (\text{dB/m})$$

$$= \frac{0.01}{8.69}\,(\text{Np/m}) = 1.15 \times 10^{-3} \quad (\text{Np/m}).$$

The three relations above are sufficient to solve for the three unknowns R, L, and G in terms of the given $C = 10^{-10}$ (F/m):

$$R = \alpha R_0 = (1.15 \times 10^{-3}) \times 50 = 0.057 \quad (\Omega/\text{m});$$

$$L = CR_0^2 = 10^{-10} \times 50^2 = 0.25 \quad (\mu\text{H/m});$$

$$G = \frac{RC}{L} = \frac{R}{R_0^2} = \frac{0.057}{50^2} = 22.8 \quad (\mu\text{S/m}).$$

b) The velocity of wave propagation on a distortionless line is the phase velocity given by Eq. (9–65).

$$u_p = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{(0.25 \times 10^{-6} \times 10^{-10}}} = 2 \times 10^8 \quad (\text{m/s}).$$

c) The ratio of two voltages a distance z apart along the line is

$$\frac{V_2}{V_1} = e^{-\alpha z}.$$

After 1 (km), $(V_2/V_1) = e^{-1000\alpha} = e^{-1.15} = 0.317$, or 31.7%.
After 5 (km), $(V_2/V_1) = e^{-5000\alpha} = e^{-5.75} = 0.0032$, or 0.32%. ■

9–3.2 TRANSMISSION-LINE PARAMETERS

The electrical properties of a transmission line at a given frequency are completely characterized by its four distributed parameters R, L, G, and C. These parameters for a parallel-plate transmission line are listed in Table 9–1. We will now obtain them for two-wire and coaxial transmission lines.

Our basic premise is that the conductivity of the conductors in a transmission line is usually so high that the effect of the series resistance on the computation of the propagation constant is negligible, the implication being that the waves on the line are approximately TEM. We may write, in dropping R from Eq. (9–37),

$$\gamma = j\omega\sqrt{LC}\left(1 + \frac{G}{j\omega C}\right)^{1/2}. \tag{9–69}$$

From Eq. (8–44) we know that the propagation constant for a TEM wave in a medium with constitutive parameters (μ, ϵ, σ) is

$$\gamma = j\omega\sqrt{\mu\epsilon}\left(1 + \frac{\sigma}{j\omega\epsilon}\right)^{1/2}. \tag{9–70}$$

But

$$\frac{G}{C} = \frac{\sigma}{\epsilon} \tag{9–71}$$

in accordance with Eq. (5–81); hence comparison of Eqs. (9–69) and (9–70) yields

$$\boxed{LC = \mu\epsilon.} \tag{9–72}$$

Equation (9–72) is a very useful relation, because if L is known for a line with a given medium, C can be determined, and vice versa. Knowing C, we can find G from Eq. (9–71). Series resistance R is determined by introducing a small axial E_z as a slight perturbation of the TEM wave and by finding the ohmic power dissipated in a unit length of the line, as was done in Subsection 9–2.1.

Equation (9–72), of course, also holds for a lossless line. ***The velocity of wave propagation on a lossless transmission line, $u_p = 1/\sqrt{LC}$, therefore, is equal to the velocity of propagation, $1/\sqrt{\mu\epsilon}$, of unguided plane wave in the dielectric of the line.*** This fact has been pointed out in connection with Eq. (9–21) for parallel-plate lines.

1. *Two-wire transmission line.* The capacitance per unit length of a two-wire transmission line, whose wires have a radius a and are separated by a distance D, has been found in Eq. (4–47). We have

$$\boxed{C = \frac{\pi\epsilon}{\cosh^{-1}(D/2a)} \quad \text{(F/m).}} \tag{9–73}^{\dagger}$$

From Eqs. (9–72) and (9–71) we obtain

$$\boxed{L = \frac{\mu}{\pi}\cosh^{-1}\left(\frac{D}{2a}\right) \quad \text{(H/m)}} \tag{9–74}^{\dagger}$$

and

$$\boxed{G = \frac{\pi\sigma}{\cosh^{-1}(D/2a)} \quad \text{(S/m).}} \tag{9–75}^{\dagger}$$

† $\cosh^{-1}(D/2a) \cong \ln(D/a)$ if $(D/2a)^2 \gg 1$.

To determine R, we go back to Eq. (9–28) and express the ohmic power dissipated per unit length of both wires in terms of p_σ. Assuming the current J_s (A/m) to flow in a very thin surface layer, the current in each wire is $I = 2\pi a J_s$, and

$$P_\sigma = 2\pi a p_\sigma = \frac{1}{2} I^2 \left(\frac{R_s}{2\pi a}\right) \quad \text{(W/m)}. \tag{9–76}$$

Hence the series resistance per unit length for both wires is

$$\boxed{R = 2\left(\frac{R_s}{2\pi a}\right) = \frac{1}{\pi a}\sqrt{\frac{\pi f \mu_c}{\sigma_c}} \quad (\Omega/\text{m}).} \tag{9–77}$$

In deriving Eqs. (9–76) and (9–77), we have assumed the surface current J_s to be uniform over the circumference of both wires. This is an approximation, inasmuch as the proximity of the two wires tends to make the surface current nonuniform.

2. *Coaxial transmission line.* The external inductance per unit length of a coaxial transmission line with a center conductor of radius a and an outer conductor of inner radius b has been found in Eq. (6–140):

$$\boxed{L = \frac{\mu}{2\pi} \ln \frac{b}{a} \quad \text{(H/m)}.} \tag{9–78}$$

From Eq. (9–72) we obtain

$$\boxed{C = \frac{2\pi\epsilon}{\ln (b/a)} \quad \text{(F/m)},} \tag{9–79}$$

and from Eq. (9–71),

$$\boxed{G = \frac{2\pi\sigma}{\ln (b/a)} \quad \text{(S/m)},} \tag{9–80}$$

where σ is the equivalent conductivity of the lossy dielectric. If one prefers, σ could be replaced by $\omega\epsilon''$ as in Eq. (7–112).

To determine R, we again return to Eq. (9–27), where J_{si} on the surface of the center conductor is different from J_{so} on the inner surface of the outer conductor. We must have

$$I = 2\pi a J_{si} = 2\pi b J_{so}. \tag{9–81}$$

The power dissipated in a unit length of the center and outer conductors are, respectively,

$$P_{\sigma i} = 2\pi a p_{\sigma i} = \frac{1}{2} I^2 \left(\frac{R_s}{2\pi a}\right), \tag{9–82}$$

TABLE 9–2
Distributed Parameters of Two-Wire and Coaxial Transmission Lines

Parameter	Two-Wire Line	Coaxial Line	Unit
R	$\dfrac{R_s}{\pi a}$	$\dfrac{R_s}{2\pi}\left(\dfrac{1}{a}+\dfrac{1}{b}\right)$	Ω/m
L	$\dfrac{\mu}{\pi}\cosh^{-1}\left(\dfrac{D}{2a}\right)$	$\dfrac{\mu}{2\pi}\ln\dfrac{b}{a}$	H/m
G	$\dfrac{\pi\sigma}{\cosh^{-1}(D/2a)}$	$\dfrac{2\pi\sigma}{\ln(b/a)}$	S/m
C	$\dfrac{\pi\epsilon}{\cosh^{-1}(D/2a)}$	$\dfrac{2\pi\epsilon}{\ln(b/a)}$	F/m

Note: $R_s = \sqrt{\pi f \mu_c/\sigma_c}$; $\cosh^{-1}(D/2a) \cong \ln(D/a)$ if $(D/2a)^2 \gg 1$. Internal inductance is not included.

$$P_{\sigma o} = 2\pi b p_{\sigma o} = \frac{1}{2} I^2 \left(\frac{R_s}{2\pi b}\right). \tag{9–83}$$

From Eqs. (9–82) and (9–83), we obtain the resistance per unit length:

$$\boxed{R = \frac{R_s}{2\pi}\left(\frac{1}{a}+\frac{1}{b}\right) = \frac{1}{2\pi}\sqrt{\frac{\pi f \mu_c}{\sigma_c}}\left(\frac{1}{a}+\frac{1}{b}\right) \qquad (\Omega/\text{m}).} \tag{9–84}$$

The R, L, G, C parameters for two-wire and coaxial transmission lines are listed in Table 9–2.

9–3.3 ATTENUATION CONSTANT FROM POWER RELATIONS

The attenuation constant of a traveling wave on a transmission line is the real part of the propagation constant; it can be determined from the basic definition in Eq. (9–37):

$$\alpha = \mathscr{Re}(\gamma) = \mathscr{Re}\left[\sqrt{(R + j\omega L)(G + j\omega C)}\right]. \tag{9–85}$$

The attenuation constant can also be found from a power relationship. The phasor voltage and phasor current distributions on an infinitely long transmission line (no reflections) may be written as (Eqs. (9–40a) and (9–40b) with the plus superscript dropped for simplicity):

$$V(z) = V_0 e^{-(\alpha + j\beta)z}, \tag{9–86a}$$

$$I(z) = \frac{V_0}{Z_0} e^{-(\alpha + j\beta)z}. \tag{9–86b}$$

The time-average power propagated along the line at any z is

$$
\begin{aligned}
P(z) &= \tfrac{1}{2}\mathcal{R}e[V(z)I^*(z)] \\
&= \frac{V_0^2}{2|Z_0|^2} R_0 e^{-2\alpha z}.
\end{aligned} \tag{9-87}
$$

The law of conservation of energy requires that the rate of decrease of $P(z)$ with distance along the line equals the time-average power loss P_L per unit length. Thus,

$$
\begin{aligned}
-\frac{\partial P(z)}{\partial z} &= P_L(z) \\
&= 2\alpha P(z),
\end{aligned}
$$

from which we obtain the following formula:

$$
\boxed{\alpha = \frac{P_L(z)}{2P(z)} \quad \text{(Np/m)}.} \tag{9-88}
$$

EXAMPLE 9–4

a) Use Eq. (9–88) to find the attenuation constant of a lossy transmission line with distributed parameters R, L, G, and C.

b) Specialize the result in part (a) to obtain the attenuation constants of a low-loss line and of a distortionless line.

Solution

a) For a lossy transmission line the time-average power loss per unit length is

$$
\begin{aligned}
P_L(z) &= \tfrac{1}{2}[|I(z)|^2 R + |V(z)|^2 G] \\
&= \frac{V_0^2}{2|Z_0|^2}(R + G|Z_0|^2)e^{-2\alpha z}.
\end{aligned} \tag{9-89}
$$

Substitution of Eqs. (9–87) and (9–89) in Eq. (9–88) gives

$$
\boxed{\alpha = \frac{1}{2R_0}(R + G|Z_0|^2) \quad \text{(Np/m)}.} \tag{9-90}
$$

b) For a low-loss line, $Z_0 \cong R_0 = \sqrt{L/C}$, Eq. (9–90) becomes

$$
\begin{aligned}
\alpha &\cong \frac{1}{2}\left(\frac{R}{R_0} + GR_0\right) \\
&= \frac{1}{2}\left(R\sqrt{\frac{C}{L}} + G\sqrt{\frac{L}{C}}\right),
\end{aligned} \tag{9-91}
$$

which checks with Eq. (9–55). For a distortionless line, $Z_0 = R_0 = \sqrt{L/C}$, Eq. (9–91) applies, and

$$\alpha = \frac{1}{2} R \sqrt{\frac{C}{L}} \left(1 + \frac{G}{R}\frac{L}{C}\right),$$

which, in view of the condition in Eq. (9–61), reduces to

$$\alpha = R\sqrt{\frac{C}{L}}. \tag{9–92}$$

Equation (9–92) is the same as Eq. (9–63). ■

9–4 Wave Characteristics on Finite Transmission Lines

In Subsection 9–3.1 we indicated that the general solutions for the time-harmonic one-dimensional Helmholtz equations, Eqs. (9–36a) and (9–36b), for transmission lines are

$$V(z) = V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z} \tag{9–93a}$$

and

$$I(z) = I_0^+ e^{-\gamma z} + I_0^- e^{\gamma z}, \tag{9–93b}$$

where

$$\frac{V_0^+}{I_0^+} = -\frac{V_0^-}{I_0^-} = Z_0. \tag{9–94}$$

For waves launched on an infinitely long line at $z = 0$ there can be only forward waves traveling in the $+z$-direction, and the second terms on the right side of Eqs. (9–93a) and (9–93b), representing reflected waves, vanish. This is also true for finite lines terminated in a characteristic impedance; that is, when the lines are ***matched.*** From circuit theory we know that *a maximum transfer of power from a given voltage source to a load occurs under "matched conditions" when the load impedance is the complex conjugate of the source impedance* (Problem P.9–11). In transmission line terminology, ***a line is matched when the load impedance is equal to the characteristic impedance (not the complex conjugate of the characteristic impedance) of the line.***

Let us now consider the general case of a finite transmission line having a characteristic impedance Z_0 terminated in an arbitrary load impedance Z_L, as depicted in Fig. 9–6. The length of the line is ℓ. A *sinusoidal* voltage source $V_g\underline{/0°}$ with an internal impedance Z_g is connected to the line at $z = 0$. In such a case,

$$\left(\frac{V}{I}\right)_{z=\ell} = \frac{V_L}{I_L} = Z_L, \tag{9–95}$$

which obviously cannot be satisfied without the second terms on the right side of Eqs. (9–93a) and (9–93b) unless $Z_L = Z_0$. Thus reflected waves exist on unmatched lines.

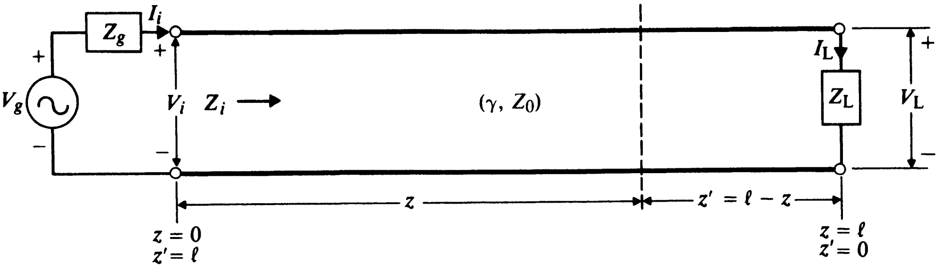


FIGURE 9–6
Finite transmission line terminated with load impedance Z_L.

Given the characteristic γ and Z_0 of the line and its length ℓ, there are four unknowns V_0^+, V_0^-, I_0^+, and I_0^- in Eqs. (9–93a) and (9–93b). These four unknowns are not all independent because they are constrained by the relations at $z = 0$ and at $z = \ell$. Both $V(z)$ and $I(z)$ can be expressed either in terms of V_i and I_i at the input end (Problem P.9–12), or in terms of the conditions at the load end. Consider the latter case.

Let $z = \ell$ in Eqs. (9–93a) and (9–93b). We have

$$V_L = V_0^+ e^{-\gamma\ell} + V_0^- e^{\gamma\ell}, \tag{9–96a}$$

$$I_L = \frac{V_0^+}{Z_0} e^{-\gamma\ell} - \frac{V_0^-}{Z_0} e^{\gamma\ell}. \tag{9–96b}$$

Solving Eqs. (9–96a) and (9–96b) for V_0^+ and V_0^-, we have

$$V_0^+ = \tfrac{1}{2}(V_L + I_L Z_0)e^{\gamma\ell}, \tag{9–97a}$$

$$V_0^- = \tfrac{1}{2}(V_L - I_L Z_0)e^{-\gamma\ell}. \tag{9–97b}$$

Substituting Eq. (9–95) in Eqs. (9–97a) and (9–97b), and using the results in Eqs. (9–93a) and (9–93b), we obtain

$$V(z) = \frac{I_L}{2}[(Z_L + Z_0)e^{\gamma(\ell - z)} + (Z_L - Z_0)e^{-\gamma(\ell - z)}], \tag{9–98a}$$

$$I(z) = \frac{I_L}{2Z_0}[(Z_L + Z_0)e^{\gamma(\ell - z)} - (Z_L - Z_0)e^{-\gamma(\ell - z)}]. \tag{9–98b}$$

Since ℓ and z appear together in the combination $(\ell - z)$, it is expedient to introduce a new variable $z' = \ell - z$, which is the distance measured backward from the load. Equations (9–98a) and (9–98b) then become

$$V(z') = \frac{I_L}{2}[(Z_L + Z_0)e^{\gamma z'} + (Z_L - Z_0)e^{-\gamma z'}], \tag{9–99a}$$

$$I(z') = \frac{I_L}{2Z_0}[(Z_L + Z_0)e^{\gamma z'} - (Z_L - Z_0)e^{-\gamma z'}]. \tag{9–99b}$$

We note here that although the same symbols V and I are used in Eqs. (9–99a) and (9–99b) as in Eqs. (9–98a) and (9–98b), the dependence of $V(z')$ and $I(z')$ on z' is different from the dependence of $V(z)$ and $I(z)$ on z.

The use of hyperbolic functions simplifies the equations above. Recalling the relations

$$e^{\gamma z'} + e^{-\gamma z'} = 2\cosh \gamma z' \qquad \text{and} \qquad e^{\gamma z'} - e^{-\gamma z'} = 2\sinh \gamma z',$$

we may write Eqs. (9–99a) and (9–99b) as

$$\boxed{V(z') = I_L(Z_L \cosh \gamma z' + Z_0 \sinh \gamma z'),} \tag{9–100a}$$

$$\boxed{I(z') = \frac{I_L}{Z_0}(Z_L \sinh \gamma z' + Z_0 \cosh \gamma z'),} \tag{9–100b}$$

which can be used to find the voltage and current at any point along a transmission line in terms of I_L, Z_L, γ, and Z_0.

The ratio $V(z')/I(z')$ is the impedance when we look toward the load end of the line at a distance z' from the load.

$$Z(z') = \frac{V(z')}{I(z')} = Z_0 \frac{Z_L \cosh \gamma z' + Z_0 \sinh \gamma z'}{Z_L \sinh \gamma z' + Z_0 \cosh \gamma z'} \tag{9–101}$$

or

$$\boxed{Z(z') = Z_0 \frac{Z_L + Z_0 \tanh \gamma z'}{Z_0 + Z_L \tanh \gamma z'} \quad (\Omega).} \tag{9–102}$$

At the source end of the line, $z' = \ell$, the generator looking into the line sees an ***input impedance*** Z_i.

$$\boxed{Z_i = (Z)_{\substack{z=0 \\ z'=\ell}} = Z_0 \frac{Z_L + Z_0 \tanh \gamma \ell}{Z_0 + Z_L \tanh \gamma \ell} \quad (\Omega).} \tag{9–103}$$

As far as the conditions at the generator are concerned, the terminated finite transmission line can be replaced by Z_i, as shown in Fig. 9–7. The input voltage V_i and input current I_i in Fig. 9–6 are found easily from the equivalent circuit in Fig. 9–7.

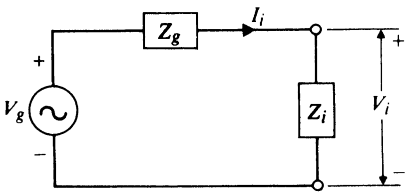


FIGURE 9–7
Equivalent circuit for finite transmission line in Figure 9–6 at generator end.

They are

$$V_i = \frac{Z_i}{Z_g + Z_i} V_g, \tag{9-104a}$$

$$I_i = \frac{V_g}{Z_g + Z_i}. \tag{9-104b}$$

Of course, the voltage and current at any other location on line cannot be determined by using the equivalent circuit in Fig. 9–7.

The average power delivered by the generator to the input terminals of the line is

$$(P_{av})_i = \tfrac{1}{2}\mathcal{R}e[V_i I_i^*]_{z=0,\, z'=\ell}. \tag{9-105}$$

The average power delivered to the load is

$$\begin{aligned}(P_{av})_L &= \tfrac{1}{2}\mathcal{R}e[V_L I_L^*]_{z=\ell,\, z'=0} \\ &= \frac{1}{2}\left|\frac{V_L}{Z_L}\right|^2 R_L = \frac{1}{2}|I_L|^2 R_L.\end{aligned} \tag{9-106}$$

For a lossless line, conservation of power requires that $(P_{av})_i = (P_{av})_L$.

A particularly important special case is when a line is terminated with its characteristic impedance—that is, when $Z_L = Z_0$. The input impedance, Z_i in Eq. (9–103), is seen to be equal to Z_0. As a matter of fact, the impedance of the line looking toward the load at any distance z' from the load is, from Eq. (9–102),

$$Z(z') = Z_0 \qquad (\text{for } Z_L = Z_0). \tag{9-107}$$

The voltage and current equations in Eqs. (9–98a) and (9–98b) reduce to

$$V(z) = (I_L Z_0 e^{\gamma\ell})e^{-\gamma z} = V_i e^{-\gamma z}, \tag{9-108a}$$

$$I(z) = (I_L e^{\gamma\ell})e^{-\gamma z} = I_i e^{-\gamma z}. \tag{9-108b}$$

Equations (9–108a) and (9–108b) correspond to the pair of voltage and current equations—Eqs. (9–40a) and (9–40b)—representing waves traveling in $+z$-direction, and there are no reflected waves. Hence, ***when a finite transmission line is terminated with its own characteristic impedance (when a finite transmission line is matched), the voltage and current distributions on the line are exactly the same as though the line has been extended to infinity.***

EXAMPLE 9–5 A signal generator having an internal resistance 1 (Ω) and an open-circuit voltage $v_g(t) = 0.3 \cos 2\pi 10^8 t$ (V) is connected to a 50 (Ω) lossless transmission line. The line is 4 (m) long, and the velocity of wave propagation on the line is 2.5×10^8 (m/s). For a matched load, find (a) the instantaneous expressions for the voltage and current at an arbitrary location on the line, (b) the instantaneous expressions for the voltage and current at the load, and (c) the average power transmitted to the load.

Solution

a) In order to find the voltage and current at an arbitrary location on the line, it is first necessary to obtain those at the input end ($z = 0$, $z' = \ell$). The given quantities are as follows:

$$\begin{aligned} V_g &= 0.3\underline{/0^\circ} \quad \text{(V)}, \qquad \text{a phasor with a cosine reference,} \\ Z_g &= R_g = 1 \quad (\Omega), \\ Z_0 &= R_0 = 50 \quad (\Omega), \\ \omega &= 2\pi \times 10^8 \quad \text{(rad/s)}, \\ u_p &= 2.5 \times 10^8 \quad \text{(m/s)}, \\ \ell &= 4 \quad \text{(m)}. \end{aligned}$$

Since the line is terminated with a matched load, $Z_i = Z_0 = 50$ (Ω). The voltage and current at the input terminals can be evaluated from the equivalent circuit in Fig. 9–7. From Eqs. (9–104a) and (9–104b) we have

$$V_i = \frac{50}{1 + 50} \times 0.3\underline{/0^\circ} = 0.294\underline{/0^\circ} \quad \text{(V)},$$

$$I_i = \frac{0.3\underline{/0^\circ}}{1 + 50} = 0.0059\underline{/0^\circ} \quad \text{(A)}.$$

Since only forward-traveling waves exist on a matched line, we use Eqs. (9–86a) and (9–86b) for the voltage and current, respectively, at an arbitrary location. For the given line, $\alpha = 0$ and

$$\beta = \frac{\omega}{u_p} = \frac{2\pi \times 10^8}{2.5 \times 10^8} = 0.8\pi \quad \text{(rad/m)}.$$

Thus,

$$V(z) = 0.294e^{-j0.8\pi z} \quad \text{(V)},$$

$$I(z) = 0.0059e^{-j0.8\pi z} \quad \text{(A)}.$$

These are phasors. The corresponding instantaneous expressions are, from Eqs. (9–34a) and (9–34b),

$$\begin{aligned} v(z, t) &= \mathcal{R}e[0.294e^{j(2\pi 10^8 t - 0.8\pi z)}] \\ &= 0.294 \cos(2\pi 10^8 t - 0.8\pi z) \quad \text{(V)}, \\ i(z, t) &= \mathcal{R}e[0.0059e^{j(2\pi 10^8 t - 0.8\pi z)}] \\ &= 0.0059 \cos(2\pi 10^8 t - 0.8\pi z) \quad \text{(A)}. \end{aligned}$$

b) At the load, $z = \ell = 4$ (m),

$$\begin{aligned} v(4, t) &= 0.294 \cos(2\pi 10^8 t - 3.2\pi) \quad \text{(V)}, \\ i(4, t) &= 0.0059 \cos(2\pi 10^8 t - 3.2\pi) \quad \text{(A)}. \end{aligned}$$

c) The average power transmitted to the load on a lossless line is equal to that at the input terminals.

$$\begin{aligned} (P_{av})_L &= (P_{av})_i = \tfrac{1}{2}\mathcal{R}e[V(z)I^*(z)] \\ &= \tfrac{1}{2}(0.294 \times 0.0059) = 8.7 \times 10^{-4}\ \text{(W)} = 0.87 \quad \text{(mW)}. \end{aligned}$$ ∎

9-4.1 TRANSMISSION LINES AS CIRCUIT ELEMENTS

Not only can transmission lines be used as wave-guiding structures for transferring power and information from one point to another, but at ultrahigh frequencies—UHF: frequency from 300 (MHz) to 3 (GHz), wavelength from 1 (m) to 0.1 (m)—they may serve as circuit elements. At these frequencies, ordinary lumped-circuit elements are difficult to make, and stray fields become important. Sections of transmission lines can be designed to give an inductive or capacitive impedance and are used to match an arbitrary load to the internal impedance of a generator for maximum power transfer. The required length of such lines as circuit elements becomes practical in the UHF range. At frequencies much lower than 300 (MHz) the required lines tend to be too long, whereas at frequencies higher than 3 (GHz) the physical dimensions become inconveniently small, and it would be advantageous to use waveguide components.

In most cases, transmission-line segments can be considered lossless: $\gamma = j\beta$, $Z_0 = R_0$, and $\tanh \gamma\ell = \tanh(j\beta\ell) = j\tan\beta\ell$. The formula in Eq. (9–103) for the input impedance Z_i of a lossless line of length ℓ terminated in Z_L becomes

$$\boxed{Z_i = R_0 \frac{Z_L + jR_0 \tan\beta\ell}{R_0 + jZ_L \tan\beta\ell} \qquad (\Omega).} \tag{9–109}$$

(Lossless line)

Comparison of Eq. (9–109) with Eq. (8–171) again confirms the similarity between normal incidence of a uniform plane wave on a plane interface and wave propagation along a terminated transmission line.

We now consider several important special cases.

1. *Open-circuit termination* ($Z_L \to \infty$). We have, from Eq. (9–109),

$$Z_{io} = jX_{io} = -\frac{jR_0}{\tan\beta\ell} = -jR_0 \cot\beta\ell. \tag{9–110}$$

Equation (9–110) shows that the input impedance of an open-circuited lossless line is purely reactive. The line can, however, be either capacitive or inductive because the function $\cot\beta\ell$ can be either positive or negative, depending on the value of $\beta\ell$ $(=2\pi\ell/\lambda)$. Figure 9–8 is a plot of $X_{io} = -R_0\cot\beta\ell$ versus ℓ. We see that X_{io} can assume all values from $-\infty$ to $+\infty$.

When the length of an open-circuited line is very short in comparison with a wavelength, $\beta\ell \ll 1$, we can obtain a very simple formula for its capacitive reactance by noting that $\tan\beta\ell \cong \beta\ell$. From Eq. (9–110) we have

$$Z_{io} = jX_{io} \cong -j\frac{R_0}{\beta\ell} = -j\frac{\sqrt{L/C}}{\omega\sqrt{LC}\,\ell} = -j\frac{1}{\omega C\ell}, \tag{9–111}$$

which is the impedance of a capacitance of $C\ell$ farads.

In practice, it is not possible to obtain an infinite load impedance at the end of a transmission line, especially at high frequencies, because of coupling to nearby objects and because of radiation from the open end.

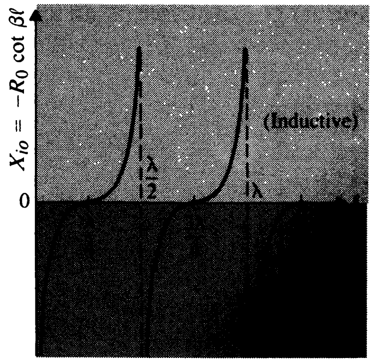


FIGURE 9–8
Input reactance of open-circuited transmission line.

2. *Short-circuit termination* ($Z_L = 0$). In this case, Eq. (9–109) reduces to

$$Z_{is} = jX_{is} = jR_0 \tan \beta\ell. \tag{9–112}$$

Since $\tan \beta\ell$ can range from $-\infty$ to $+\infty$, the input impedance of a short-circuited lossless line can also be either purely inductive or purely capacitive, depending on the value of $\beta\ell$. Figure 9–9 is a graph of X_{is} versus ℓ. We note that Eq. (9–112) has exactly the same form as that—Eq. (8–172)—of the wave impedance of the total field at a distance ℓ from a perfectly conducting plane boundary.

Comparing Figs. 9–8 and 9–9, we see that in the range where X_{io} is capacitive X_{is} is inductive, and vice versa. The input reactances of open-circuited and short-circuited lossless transmission lines are the same if their lengths differ by an odd multiple of $\lambda/4$.

When the length of a short-circuited line is very short in comparison with a wavelength, $\beta\ell \ll 1$, Eq. (9–112) becomes approximately

$$Z_{is} = jX_{is} \cong jR_0\beta\ell = j\sqrt{\frac{L}{C}}\,\omega\sqrt{LC}\ell = j\omega L\ell, \tag{9–113}$$

which is the impedance of an inductance of $L\ell$ henries.

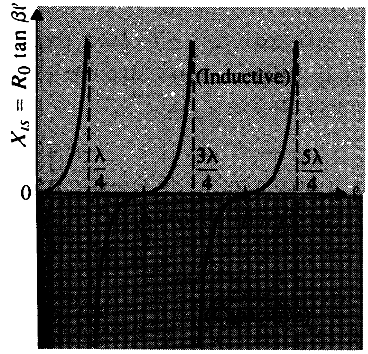


FIGURE 9–9
Input reactance of short-circuited transmission line.

3. *Quarter-wave section* ($\ell = \lambda/4$, $\beta\ell = \pi/2$). When the length of a line is an odd multiple of $\lambda/4$, $\ell = (2n-1)\lambda/4$, ($n = 1, 2, 3, \ldots$),

$$\beta\ell = \frac{2\pi}{\lambda}(2n-1)\frac{\lambda}{4} = (2n-1)\frac{\pi}{2},$$

$$\tan\beta\ell = \tan\left[(2n-1)\frac{\pi}{2}\right] \to \pm\infty,$$

and Eq. (9–109) becomes

$$\boxed{Z_i = \frac{R_0^2}{Z_L} \qquad \text{(Quarter-wave line).}} \tag{9–114}$$

Hence, ***a quarter-wave lossless line transforms the load impedance to the input terminals as its inverse multiplied by the square of the characteristic resistance.*** It acts as an impedance inverter and is often referred to as a ***quarter-wave transformer.*** An open-circuited, quarter-wave line appears as a short circuit at the input terminals, and a short-circuited quarter-wave line appears as an open circuit. Actually, if the series resistance of the line itself is not neglected, the input impedance of a short-circuited, quarter-wave line is an impedance of a very high value similar to that of a parallel resonant circuit. It is interesting to compare Eq. (9–114) with the formula for quarter-wave impedance transformation with multiple dielectrics, Eq. (8–182a).

4. *Half-wave section* ($\ell = \lambda/2$, $\beta\ell = \pi$). When the length of a line is an integral multiple of $\lambda/2$, $\ell = n\lambda/2$ ($n = 1, 2, 3, \ldots$),

$$\beta\ell = \frac{2\pi}{\lambda}\left(\frac{n\lambda}{2}\right) = n\pi,$$

$$\tan\beta\ell = 0,$$

and Eq. (9–109) reduces to

$$\boxed{Z_i = Z_L \qquad \text{(Half-wave line).}} \tag{9–115}$$

Equation (9–115) states that ***a half-wave lossless line transfers the load impedance to the input terminals without change.*** From Eq. (9–103) we observe that a half-wave line with loss does not have this property unless $Z_L = Z_0$.

By measuring the input impedance of a line section under open- and short-circuit conditions, we can determine the characteristic impedance and the propagation constant of the line. The following expressions follow directly from Eq. (9–103).

$$\textit{Open-circuited line, } Z_L \to \infty\text{:} \qquad Z_{io} = Z_0 \coth\gamma\ell. \tag{9–116}$$

$$\textit{Short-circuited line, } Z_L = 0\text{:} \qquad Z_{is} = Z_0 \tanh\gamma\ell. \tag{9–117}$$

From Eqs. (9–116) and (9–117) we have

$$\boxed{Z_0 = \sqrt{Z_{io}Z_{is}} \qquad (\Omega)} \tag{9–118}$$

and

$$\boxed{\gamma = \frac{1}{\ell}\tanh^{-1}\sqrt{\frac{Z_{is}}{Z_{io}}} \qquad (\mathrm{m}^{-1}).} \tag{9–119}$$

Equations (9–118) and (9–119) apply whether or not the line is lossy.

EXAMPLE 9–6 The open-circuit and short-circuit impedances measured at the input terminals of a lossless transmission line of length 1.5 (m), which is less than a quarter wavelength, are $-j54.6$ (Ω) and $j103$ (Ω), respectively. (a) Find Z_0 and γ of the line. (b) Without changing the operating frequency, find the input impedance of a short-circuited line that is twice the given length. (c) How long should the short-circuited line be in order for it to appear as an open circuit at the input terminals?

Solution The given quantities are

$$Z_{io} = -j54.6, \qquad Z_{is} = j103, \qquad \ell = 1.5.$$

a) Using Eqs. (9–118) and (9–119), we find

$$Z_0 = \sqrt{-j54.6(j103)} = 75 \quad (\Omega)$$

$$\gamma = \frac{1}{1.5}\tanh^{-1}\sqrt{\frac{j103}{-j54.6}} = \frac{j}{1.5}\tan^{-1} 1.373 = j0.628 \quad (\text{rad/m}).$$

b) For a short-circuited line twice as long, $\ell = 3.0$ (m),

$$\gamma\ell = j0.628 \times 3.0 = j1.884 \quad (\text{rad}).$$

The input impedance is, from Eq. (9–117),

$$\begin{aligned} Z_{is} &= 75 \tanh(j1.884) = j75 \tan 108^\circ \\ &= j75(-3.08) = -j231 \quad (\Omega). \end{aligned}$$

Note that Z_{is} for the 3 (m) line is now a capacitive reactance, whereas that for the 1.5 (m) line in part (a) is an inductive reactance. We may conclude from Fig. 9–9 that 1.5 (m) $< \lambda/4 <$ 3.0 (m).

c) In order for a short-circuited line to appear as an open circuit at the input terminals, it should be an odd multiple of a quarter-wavelength long:

$$\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{0.628} = 10 \quad (\text{m}).$$

Hence the required line length is

$$\begin{aligned} \ell &= \frac{\lambda}{4} + (n-1)\frac{\lambda}{2} \\ &= 2.5 + 5(n-1) \quad (\text{m}), \qquad n = 1, 2, 3, \ldots . \end{aligned}$$

So far in this subsection we have considered only open- and short-circuited lossless lines as circuit elements. We have seen in Figs. 9–8 and 9–9 that, depending on the length of the line, the input impedance of an open- or short-circuited lossless line can be either purely inductive or purely capacitive. Let us now examine the input impedance of a lossy line with a short-circuit termination. When the line length is a multiple of $\lambda/2$, the input impedance will not vanish as in Fig. 9–9. Instead, we have, from Eq. (9–117),

$$\begin{aligned} Z_{is} &= Z_0 \tanh \gamma\ell = Z_0 \frac{\sinh(\alpha + j\beta)\ell}{\cosh(\alpha + j\beta)\ell} \\ &= Z_0 \frac{\sinh \alpha\ell \cos \beta\ell + j \cosh \alpha\ell \sin \beta\ell}{\cosh \alpha\ell \cos \beta\ell + j \sinh \alpha\ell \sin \beta\ell}. \end{aligned} \tag{9–120}$$

For $\ell = n\lambda/2$, $\beta\ell = n\pi$, $\sin \beta\ell = 0$, Eq. (9–120) reduces to

$$Z_{is} = Z_0 \tanh \alpha\ell \cong Z_0(\alpha\ell), \tag{9–121}$$

where we have assumed a low-loss line: $\alpha\ell \ll 1$ and $\tanh \alpha\ell \cong \alpha\ell$. The quantity Z_{is} in Eq. (9–121) is small but not zero. At $\ell = n\lambda/2$ we have the condition of a series-resonant circuit.

When the length of a shorted lossy line is an odd multiple of $\lambda/4$, the input impedance will not go to infinity as indicated in Fig. 9–9. For $\ell = n\lambda/4$, $\beta\ell = n\pi/2$ (n = odd), $\cos \beta\ell = 0$, and Eq. (9–120) becomes

$$Z_{is} = \frac{Z_0}{\tanh \alpha\ell} \cong \frac{Z_0}{\alpha\ell}, \tag{9–122}$$

which is large but not infinite. We have the condition of a parallel-resonant circuit. It is a frequency-selective circuit, and we can determine the ***quality factor***, or Q, of such a circuit by first finding its ***half-power bandwidth***, or simply the ***bandwidth***. The bandwidth of a parallel-resonant circuit is the frequency range $\Delta f = f_2 - f_1$ around the resonant frequency f_0, where $f_2 = f_0 + \Delta f/2$ and $f_1 = f_0 - \Delta f/2$ are half-power frequencies at which the voltage across the parallel circuit is $1/\sqrt{2}$ or 70.7% of its maximum value at f_0 (assuming a constant-current source). Hence the associated power, which is proportional to $|Z_{is}|^2$ and is maximum at f_0, is one-half of its value at f_1 and f_2.

Let $f = f_0 + \delta f$, where δf is a small frequency shift from the resonant frequency. We have

$$\begin{aligned} \beta\ell &= \frac{2\pi f}{u_p}\ell = \frac{2\pi(f_0 + \delta f)}{u_p}\ell \\ &= \frac{n\pi}{2} + \frac{n\pi}{2}\left(\frac{\delta f}{f_0}\right), \qquad n = \text{odd}, \end{aligned} \tag{9–123}$$

$$\cos \beta\ell = -\sin\left[\frac{n\pi}{2}\left(\frac{\delta f}{f_0}\right)\right] \cong -\frac{n\pi}{2}\left(\frac{\delta f}{f_0}\right), \tag{9–124}$$

$$\sin \beta\ell = \cos\left[\frac{n\pi}{2}\left(\frac{\delta f}{f_0}\right)\right] \cong 1, \tag{9–125}$$

where we have assumed $(n\pi/2)(\delta f/f_0) \ll 1$. Substituting Eqs. (9–123), (9–124), and (9–125) in Eq. (9–120), noting that $\alpha\ell \ll 1$, and retaining only small terms of the first order, we obtain

$$Z_{is} = \frac{Z_0}{\alpha\ell + j\dfrac{n\pi}{2}\left(\dfrac{\delta f}{f_0}\right)} \tag{9–126}$$

and

$$|Z_{is}|^2 = \frac{|Z_0|^2}{(\alpha\ell)^2 + \left[\dfrac{n\pi}{2}\left(\dfrac{\delta f}{f_0}\right)\right]^2}. \tag{9–127}$$

At $f = f_0$, $\delta f = 0$, $|Z_{is}|^2$ is a maximum and equals $|Z_{is}|^2_{\max} = |Z_0|^2/(\alpha\ell)^2$. Thus,

$$\frac{|Z_{is}|^2}{|Z_{is}|^2_{\max}} = \frac{1}{1 + \left[\dfrac{n\pi}{2\alpha\ell}\left(\dfrac{\delta f}{f_0}\right)\right]^2}. \tag{9–128}$$

When $\delta f = \pm\Delta f/2$, we have the half-power frequencies f_2 and f_1, at which the ratio in Eq. (9–128) equals $\frac{1}{2}$, or

$$\frac{n\pi}{2\alpha\ell}\left(\frac{\Delta f}{2f_0}\right) = \frac{\beta}{2\alpha}\left(\frac{\Delta f}{f_0}\right) = 1, \qquad n = \text{odd}. \tag{9–129}$$

Therefore, the Q of the parallel-resonant circuit (a shorted lossy line having a length equal to an odd multiple of $\lambda/4$) is

$$\boxed{Q = \frac{f_0}{\Delta f} = \frac{\beta}{2\alpha}.} \tag{9–130}$$

Using the expressions of α and β for a low-loss line in Eqs. (9–55) and (9–56), we obtain

$$Q = \frac{\omega L}{R + GL/C} = \frac{1}{[(R/\omega L) + (G/\omega C)]}. \tag{9–131}$$

For a well-insulated line, $GL/C \ll R$, and Eq. (9–131) reduces to the familiar expression for the Q of a parallel-resonant circuit:

$$Q = \frac{\omega L}{R}. \tag{9–132}$$

In a similar manner an analysis can be made for the resonant behavior of an open-circuited low-loss transmission line whose length is an odd multiple of $\lambda/4$ (series resonance) or a multiple of $\lambda/2$ (parallel resonance). (See Problem P.9–21.)

EXAMPLE 9–7 The measured attenuation of an air-dielectric coaxial transmission line at 400 (MHz) is 0.01 (dB/m). Determine the Q and the half-power bandwidth of a quarter-wavelength section of the line with a short-circuit termination.

Solution At $f = 4 \times 10^8$ (Hz),

$$\lambda = \frac{c}{f} = \frac{3 \times 10^8}{4 \times 10^8} = 0.75 \quad \text{(m)},$$

$$\beta = \frac{2\pi}{\lambda} = \frac{2\pi}{0.75} = 8.38 \quad \text{(rad/m)},$$

$$\alpha = 0.01 \text{ (dB/m)} = \frac{0.01}{8.69} \quad \text{(Np/m)}.$$

Therefore,

$$Q = \frac{\beta}{2\alpha} = \frac{8.38 \times 8.69}{2 \times 0.01} = 3641,$$

which is much higher than the Q obtainable from any lumped-element parallel-resonant circuit at 400 (MHz). The half-power bandwidth is

$$\Delta f = \frac{f_0}{Q} = \frac{4 \times 10^8}{3641} = 0.11 \times 10^6 \quad \text{(Hz)}$$
$$= 0.11 \quad \text{(MHz)}, \quad \text{or } 110 \quad \text{(kHz)}.$$

9–4.2 LINES WITH RESISTIVE TERMINATION

When a transmission line is terminated in a load impedance Z_L different from the characteristic impedance Z_0, both an incident wave (from the generator) and a reflected wave (from the load) exist. Equation (9–99a) gives the phasor expression for the voltage at any distance $z' = \ell - z$ from the load end. Note that in Eq. (9–99a), the term with $e^{\gamma z'}$ represents the incident voltage wave and the term with $e^{-\gamma z'}$ represents the reflected voltage wave. We may write

$$\begin{aligned} V(z') &= \frac{I_L}{2}(Z_L + Z_0)e^{\gamma z'}\left[1 + \frac{Z_L - Z_0}{Z_L + Z_0}e^{-2\gamma z'}\right] \\ &= \frac{I_L}{2}(Z_L + Z_0)e^{\gamma z'}[1 + \Gamma e^{-2\gamma z'}], \end{aligned} \tag{9–133a}$$

where

$$\boxed{\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = |\Gamma|e^{j\theta_\Gamma} \qquad \text{(Dimensionless)}} \tag{9–134}$$

is the ratio of the complex amplitudes of the reflected and incident voltage waves at the load ($z' = 0$) and is called the ***voltage reflection coefficient*** of the load impedance Z_L. It is of the same form as the definition of the reflection coefficient in Eq. (8–140) for a plane wave incident normally on a plane interface between two dielectric media. It is, in general, a complex quantity with a magnitude $|\Gamma| \leq 1$. The current equation corresponding to $V(z')$ in Eq. (9–133a) is, from Eq. (9–99b),

$$I(z') = \frac{I_L}{2Z_0}(Z_L + Z_0)e^{\gamma z'}[1 - \Gamma e^{-2\gamma z'}]. \tag{9–133b}$$

The current reflection coefficient defined as the ratio of the complex amplitudes of the reflected and incident current waves, I_0^-/I_0^+, is different from the voltage reflection coefficient. As a matter of fact, the former is the negative of the latter, inasmuch as $I_0^-/I_0^+ = -V_0^-/V_0^+$, as is evident from Eq. (9–94). In what follows we shall refer only to the voltage reflection coefficient.

For a *lossless* transmission line, $\gamma = j\beta$, Eqs. (9–133a) and (9–133b) become

$$\begin{aligned} V(z') &= \frac{I_L}{2}(Z_L + R_0)e^{j\beta z'}[1 + \Gamma e^{-j2\beta z'}] \\ &= \frac{I_L}{2}(Z_L + R_0)e^{j\beta z'}[1 + |\Gamma| e^{j(\theta_\Gamma - 2\beta z')}] \end{aligned} \tag{9–135a}$$

and

$$I(z') = \frac{I_L}{2R_0}(Z_L + R_0)e^{j\beta z'}[1 - |\Gamma| e^{j(\theta_\Gamma - 2\beta z')}]. \tag{9–135b}$$

The voltage and current phasors on a lossless line are more easily visualized from Eqs. (9–100a) and (9–100b) by setting $\gamma = j\beta$ and $V_L = I_L Z_L$. Noting that $\cosh j\theta = \cos\theta$, and $\sinh j\theta = j\sin\theta$, we obtain

$$\boxed{\begin{aligned} V(z') &= V_L \cos\beta z' + jI_L R_0 \sin\beta z', \\ I(z') &= I_L \cos\beta z' + j\frac{V_L}{R_0}\sin\beta z'. \end{aligned}} \qquad \begin{aligned} &(9\text{–}136\text{a}) \\ &(9\text{–}136\text{b}) \end{aligned}$$

(Lossless line)

If the terminating impedance is purely resistive, $Z_L = R_L$, $V_L = I_L R_L$, the voltage and current magnitudes are given by

$$|V(z')| = V_L\sqrt{\cos^2\beta z' + (R_0/R_L)^2 \sin^2\beta z'}, \tag{9–137a}$$

$$|I(z')| = I_L\sqrt{\cos^2\beta z' + (R_L/R_0)^2 \sin^2\beta z'}, \tag{9–137b}$$

where $R_0 = \sqrt{L/C}$. Plots of $|V(z')|$ and $|I(z')|$ as functions of z' are standing waves with their maxima and minima occurring at fixed locations along the line.

Analogously to the plane-wave case in Eq. (8–147), we define the ratio of the maximum to the minimum voltages along a finite, terminated line as the ***standing-wave ratio*** **(SWR)**, S:

$$\boxed{S = \frac{|V_{max}|}{|V_{min}|} = \frac{1 + |\Gamma|}{1 - |\Gamma|} \quad \text{(Dimensionless).}} \tag{9–138}$$

The inverse relation of Eq. (9–138) is

$$\boxed{|\Gamma| = \frac{S-1}{S+1} \qquad \text{(Dimensionless).}} \tag{9–139}$$

It is clear from Eqs. (9–138) and (9–139) that on a lossless transmission line

$$\begin{aligned} &\Gamma = 0, && S = 1 && \text{when } Z_L = Z_0 \text{ (Matched load);} \\ &\Gamma = -1, && S \to \infty && \text{when } Z_L = 0 \text{ (Short circuit);} \\ &\Gamma = +1, && S \to \infty && \text{when } Z_L \to \infty \text{ (Open circuit).} \end{aligned}$$

Because of the wide range of S, it is customary to express it on a logarithmic scale: $20 \log_{10} S$ in (dB). Standing-wave ratio S defined in terms of $|I_{max}|/|I_{min}|$ results in the same expression as that defined in terms of $|V_{max}|/|V_{min}|$ in Eq. (9–138). A high standing-wave ratio on a line is undesirable because it results in a large power loss.

Examination of Eqs. (9–135a) and (9–135b) reveals that $|V_{max}|$ and $|I_{min}|$ occur together when

$$\theta_\Gamma - 2\beta z'_M = -2n\pi, \qquad n = 0, 1, 2, \ldots. \tag{9–140}$$

On the other hand, $|V_{min}|$ and $|I_{max}|$ occur together when

$$\theta_\Gamma - 2\beta z'_m = -(2n + 1)\pi, \qquad n = 0, 1, 2, \ldots. \tag{9–141}$$

For resistive terminations on a lossless line, $Z_L = R_L$, $Z_0 = R_0$, and Eq. (9–134) simplifies to

$$\Gamma = \frac{R_L - R_0}{R_L + R_0} \qquad \text{(Resistive load).} \tag{9–142}$$

The voltage reflection coefficient is therefore purely real. Two cases are possible.

1. $R_L > R_0$. In this case, Γ is positive real and $\theta_\Gamma = 0$. At the termination, $z' = 0$, and condition (9–140) is satisfied (for $n = 0$). This means that a voltage maximum (current minimum) will occur at the terminating resistance. Other maxima of the voltage standing wave (minima of the current standing wave) will be located at $2\beta z' = 2n\pi$, or $z' = n\lambda/2$ ($n = 1, 2, \ldots$) from the load.
2. $R_L < R_0$. Equation (9–142) shows that Γ will be negative real and $\theta_\Gamma = -\pi$. At the termination, $z' = 0$, and condition (9–141) is satisfied (for $n = 0$). A voltage minimum (current maximum) will occur at the terminating resistance. Other minima of the voltage standing wave (maxima of the current standing wave) will be located at $z' = n\lambda/2$ ($n = 1, 2, \ldots$) from the load. The roles of the voltage and current standing waves are interchanged from those for the case of $R_L > R_0$.

Figure 9–10 illustrates some typical standing waves for a lossless line with resistive termination.

The standing waves on an open-circuited line are similar to those on a resistance-terminated line with $R_L > R_0$, except that the $|V(z')|$ and $|I(z')|$ curves are now magnitudes of sinusoidal functions of the distance z' from the load. This is seen from

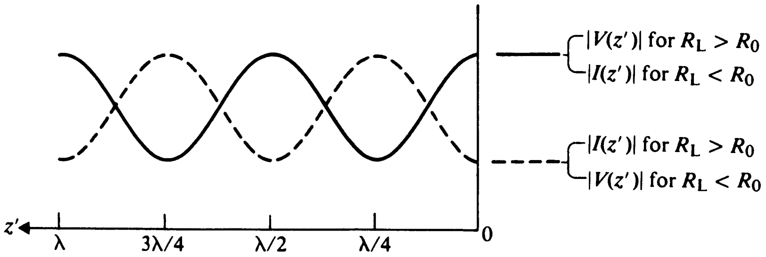


FIGURE 9–10
Voltage and current standing waves on resistance-terminated lossless lines.

Eqs. (9–137a) and (9–137b), by letting $R_L \to \infty$. Of course, $I_L = 0$, but V_L is finite. We have

$$|V(z')| = V_L|\cos \beta z'|; \tag{9–143a}$$

$$|I(z')| = \frac{V_L}{R_0}|\sin \beta z'|. \tag{9–143b}$$

All the minima go to zero. For an open-circuited line, $\Gamma = 1$ and $S \to \infty$.

On the other hand, the standing waves on a short-circuited line are similar to those on a resistance-terminated line with $R_L < R_0$. Here $R_L = 0$, $V_L = 0$, but I_L is finite. Equations (9–137a) and (9–137b) reduce to

$$|V(z')| = I_L R_0|\sin \beta z'|, \tag{9–144a}$$

$$|I(z')| = I_L|\cos \beta z'|. \tag{9–144b}$$

Typical standing waves for open- and short-circuited lossless lines are shown in Fig. 9–11.

EXAMPLE 9–8 The standing-wave ratio S on a transmission line is an easily measurable quantity. (a) Show how the value of a terminating resistance on a lossless line of known characteristic impedance R_0 can be determined by measuring S. (b) What

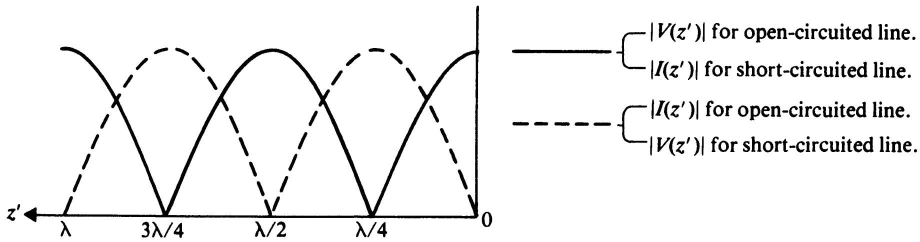


FIGURE 9–11
Voltage and current standing waves on open- and short-circuited lossless lines.

is the impedance of the line looking toward the load at a distance equal to one quarter of the operating wavelength?

Solution

a) Since the terminating impedance is purely resistive, $Z_L = R_L$, we can determine whether R_L is greater than R_0 (if there are voltage maxima at $z' = 0, \lambda/2, \lambda$, etc.) or whether R_L is less than R_0 (if there are voltage minima at $z' = 0, \lambda/2, \lambda$, etc.). This can be easily ascertained by measurements.

First, if $R_L > R_0$, $\theta_\Gamma = 0$. Both $|V_{max}|$ and $|I_{min}|$ occur at $\beta z' = 0$; and $|V_{min}|$ and $|I_{max}|$ occur at $\beta z' = \pi/2$. We have, from Eqs. (9–136a) and (9–136b),

$$|V_{max}| = V_L, \qquad |V_{min}| = V_L \frac{R_0}{R_L};$$

$$|I_{min}| = I_L, \qquad |I_{max}| = I_L \frac{R_L}{R_0}.$$

Thus,

$$\frac{|V_{max}|}{|V_{min}|} = \frac{|I_{max}|}{|I_{min}|} = S = \frac{R_L}{R_0}$$

or

$$R_L = SR_0. \tag{9–145}$$

Second, if $R_L < R_0$, $\theta_\Gamma = -\pi$. Both $|V_{min}|$ and $|I_{max}|$ occur at $\beta z' = 0$; and $|V_{max}|$ and $|I_{min}|$ occur at $\beta z' = \pi/2$. We have

$$|V_{min}| = V_L, \qquad |V_{max}| = V_L \frac{R_0}{R_L};$$

$$|I_{max}| = I_L, \qquad |I_{min}| = I_L \frac{R_L}{R_0}.$$

Therefore,

$$\frac{|V_{max}|}{|V_{min}|} = \frac{|I_{max}|}{|I_{min}|} = S = \frac{R_0}{R_L}$$

or

$$R_L = \frac{R_0}{S}. \tag{9–146}$$

b) The operating wavelength, λ, can be determined from twice the distance between two neighboring voltage (or current) maxima or minima. At $z' = \lambda/4$, $\beta z' = \pi/2$, $\cos \beta z' = 0$, and $\sin \beta z' = 1$. Equations (9–136a) and (9–136b) become

$$V(\lambda/4) = jI_L R_0,$$

$$I(\lambda/4) = j\frac{V_L}{R_0}.$$

(*Question:* What is the significance of the j in these equations?) The ratio of $V(\lambda/4)$ to $I(\lambda/4)$ is the input impedance of a quarter-wavelength, resistively terminated, lossless line.

$$Z_i(z' = \lambda/4) = R_i = \frac{V(\lambda/4)}{I(\lambda/4)}$$
$$= \frac{R_0^2}{R_L}.$$

This result is anticipated because of the impedance-transformation property of a quarter-wave line given in Eq. (9–114).

9–4.3 LINES WITH ARBITRARY TERMINATION

In the preceding subsection we noted that the standing wave on a *resistively terminated* lossless transmission line is such that a voltage maximum (a current minimum) occurs at the termination where $z' = 0$ if $R_L > R_0$, and a voltage minimum (a current maximum) occurs there if $R_L < R_0$. What will happen if the terminating impedance is not a pure resistance? It is intuitively correct to expect that a voltage maximum or minimum will not occur at the termination and that both will be shifted away from the termination. In this subsection we will show that information on the direction and amount of this shift can be used to determine the terminating impedance.

Let the terminating (or load) impedance be $Z_L = R_L + jX_L$, and assume the voltage standing wave on the line to look like that depicted in Fig. 9–12. We note that neither a voltage maximum nor a voltage minimum appears at the load at $z' = 0$. If we let the standing wave continue, say, by an extra distance ℓ_m, it will reach a minimum. The voltage minimum is where it should be if the original terminating impedance Z_L is replaced by a line section of length ℓ_m terminated by a pure resistance

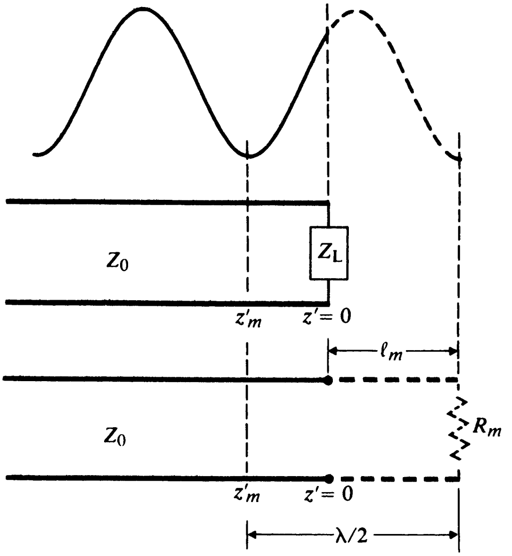


FIGURE 9–12
Voltage standing wave on a line terminated by an arbitrary impedance, and equivalent line section with pure resistive load.

$R_m < R_0$, as shown in the figure. The voltage distribution on the line to the left of the actual termination (where $z' > 0$) is not changed by this replacement.

The fact that any complex impedance can be obtained as the input impedance of a section of lossless line terminated in a resistive load can be seen from Eq. (9–109). Using R_m for Z_L and ℓ_m for ℓ, we have

$$R_i + jX_i = R_0 \frac{R_m + jR_0 \tan \beta\ell_m}{R_0 + jR_m \tan \beta\ell_m}. \tag{9–147}$$

The real and imaginary parts of Eq. (9–147) form two equations, from which the two unknowns, R_m and ℓ_m, can be solved (see Problem P.9–28).

The load impedance Z_L can be determined experimentally by measuring the standing-wave ratio S and the distance z'_m in Fig. 9–12. (Remember that $z'_m + \ell_m = \lambda/2$.) The procedure is as follows:

1. Find $|\Gamma|$ from S. Use $|\Gamma| = \dfrac{S-1}{S+1}$ from Eq. (9–139).
2. Find θ_Γ from z'_m. Use $\theta_\Gamma = 2\beta z'_m - \pi$ for $n = 0$ from Eq. (9–141).
3. Find Z_L, which is the ratio of Eqs. (9–135a) and (9–135b) at $z' = 0$:

$$Z_L = R_L + jX_L = R_0 \frac{1 + |\Gamma|e^{j\theta_\Gamma}}{1 - |\Gamma|e^{j\theta_\Gamma}}. \tag{9–148}$$

The value of R_m that, if terminated on a line of length ℓ_m, will yield an input impedance Z_L can be found easily from Eq. (9–147). Since $R_m < R_0$, $R_m = R_0/S$.

The procedure leading to Eq. (9–148) is used to determine Z_L from a measurement of S and of z'_m, the distance from the termination to the first voltage minimum. Of course, the distance from the termination to a voltage maximum, z'_M, could be used instead of z'_m. In that case, Eq. (9–140) should be used to find θ_Γ in Step 2 above.

EXAMPLE 9–9 The standing-wave ratio on a lossless 50 (Ω) transmission line terminated in an unknown load impedance is found to be 3.0. The distance between successive voltage minima is 20 (cm), and the first minimum is located at 5 (cm) from the load. Determine (a) the reflection coefficient Γ, and (b) the load impedance Z_L. In addition, find (c) the equivalent length and terminating resistance of a line such that the input impedance is equal to Z_L.

Solution

a) The distance between successive voltage minima is half a wavelength.

$$\lambda = 2 \times 0.2 = 0.4 \quad \text{(m)}, \qquad \beta = \frac{2\pi}{\lambda} = \frac{2\pi}{0.4} = 5\pi \quad \text{(rad/m)}.$$

Step 1: We find the magnitude of the reflection coefficient, $|\Gamma|$, from the standing-wave ratio $S = 3$.

$$|\Gamma| = \frac{S-1}{S+1} = \frac{3-1}{3+1} = 0.5.$$

Step 2: Find the angle of the reflection coefficient, θ_Γ, from

$$\theta_\Gamma = 2\beta z'_m - \pi = 2 \times 5\pi \times 0.05 - \pi = -0.5\pi \quad \text{(rad)},$$

$$\Gamma = |\Gamma| e^{j\theta_\Gamma} = 0.5e^{-j0.5\pi} = -j0.5.$$

b) The load impedance Z_L is determined from Eq. (9–148):

$$Z_L = 50\left(\frac{1 - j0.5}{1 + j0.5}\right) = 50(0.60 - j0.80) = 30 - j40 \quad (\Omega).$$

c) Now we find R_m and ℓ_m in Fig. 9–12. We may use Eq. (9–147),

$$30 - j40 = 50\left(\frac{R_m + j50 \tan \beta\ell_m}{50 + jR_m \tan \beta\ell_m}\right),$$

and solve the simultaneous equations obtained from the real and imaginary parts for R_m and $\beta\ell_m$. Actually, we know $z'_m + \ell_m = \lambda/2$ and $R_m = R_0/S$. Hence,†

$$\ell_m = \frac{\lambda}{2} - z'_m = 0.2 - 0.05 = 0.15 \quad \text{(m)}$$

and

$$R_m = \frac{50}{3} = 16.7 \quad (\Omega).$$

9–4.4 TRANSMISSION-LINE CIRCUITS

Our discussions on the properties of transmission lines so far have been restricted primarily to the effects of the load on the input impedance and on the characteristics of voltage and current waves. No attention has been paid to the generator at the "other end," which is the source of the waves. Just as the constraint (the boundary condition), $V_L = I_L Z_L$, which the voltage V_L and the current I_L must satisfy at the load end ($z = \ell$, $z' = 0$), a constraint exists at the generator end where $z = 0$ and $z' = \ell$. Let a voltage generator V_g with an internal impedance Z_g represent the source connected to a finite transmission line of length ℓ that is terminated in a load impedance Z_L, as shown in Fig. 9–6. The additional constraint at $z = 0$ will enable the voltage and current anywhere on the line to be expressed in terms of the source characteristics (V_g, Z_g), the line characteristics (γ, Z_0, ℓ), and the load impedance (Z_L).

The constraint at $z = 0$ is

$$V_i = V_g - I_i Z_g. \tag{9–149}$$

But, from Eqs. (9–133a) and (9–133b),

$$V_i = \frac{I_L}{2}(Z_L + Z_0)e^{\gamma\ell}[1 + \Gamma e^{-2\gamma\ell}] \tag{9–150a}$$

and

$$I_i = \frac{I_L}{2Z_0}(Z_L + Z_0)e^{\gamma\ell}[1 - \Gamma e^{-2\gamma\ell}]. \tag{9–150b}$$

---

† Another set of solutions to part (c) is $\ell'_m = \ell_m - \lambda/4 = 0.05$ (m) and $R'_m = SR_0 = 150$ (Ω). Do you see why?

Substitution of Eqs. (9–150a) and (9–150b) in Eq. (9–149) enables us to find

$$\frac{I_L}{2}(Z_L + Z_0)e^{\gamma\ell} = \frac{Z_0 V_g}{Z_0 + Z_g} \frac{1}{[1 - \Gamma_g \Gamma e^{-2\gamma\ell}]}, \tag{9–151}$$

where

$$\Gamma_g = \frac{Z_g - Z_0}{Z_g + Z_0} \tag{9–152}$$

is the ***voltage reflection coefficient*** at the generator end. Using Eq. (9–151) in Eqs. (9–133a) and (9–133b), we obtain

$$\boxed{V(z') = \frac{Z_0 V_g}{Z_0 + Z_g} e^{-\gamma z}\left(\frac{1 + \Gamma e^{-2\gamma z'}}{1 - \Gamma_g \Gamma e^{-2\gamma\ell}}\right).} \tag{9–153a}$$

Similarly,

$$\boxed{I(z') = \frac{V_g}{Z_0 + Z_g} e^{-\gamma z}\left(\frac{1 - \Gamma e^{-2\gamma z'}}{1 - \Gamma_g \Gamma e^{-2\gamma\ell}}\right).} \tag{9–153b}$$

Equations (9–153a) and (9–153b) are analytical phasor expressions for the voltage and current at any point on a finite line fed by a sinusoidal voltage source V_g. These are rather complicated expressions, but their significance can be interpreted in the following way. Let us concentrate our attention on the voltage equation (9–153a); obviously, the interpretation of the current equation (9–153b) is quite similar. We expand Eq. (9–153a) as follows:

$$\begin{aligned} V(z') &= \frac{Z_0 V_g}{Z_0 + Z_g} e^{-\gamma z}(1 + \Gamma e^{-2\gamma z'})(1 - \Gamma_g \Gamma e^{-2\gamma\ell})^{-1} \\ &= \frac{Z_0 V_g}{Z_0 + Z_g} e^{-\gamma z}(1 + \Gamma e^{-2\gamma z'})(1 + \Gamma_g \Gamma e^{-2\gamma\ell} + \Gamma_g^2 \Gamma^2 e^{-4\gamma\ell} + \cdots) \\ &= \frac{Z_0 V_g}{Z_0 + Z_g} [e^{-\gamma z} + (\Gamma e^{-\gamma\ell})e^{-\gamma z'} + \Gamma_g(\Gamma e^{-2\gamma\ell})e^{-\gamma z} + \cdots] \\ &= V_1^+ + V_1^- + V_2^+ + V_2^- + \cdots, \end{aligned} \tag{9–154}$$

where

$$V_1^+ = \frac{V_g Z_0}{Z_0 + Z_g} e^{-\gamma z} = V_M e^{-\gamma z}, \tag{9–154a}$$

$$V_1^- = \Gamma(V_M e^{-\gamma\ell})e^{-\gamma z'}, \tag{9–154b}$$

$$V_2^+ = \Gamma_g(\Gamma V_M e^{-2\gamma\ell})e^{-\gamma z}. \tag{9–154c}$$

$$\vdots$$

The quantity

$$V_M = \frac{Z_0 V_g}{Z_0 + Z_g} \tag{9–155}$$

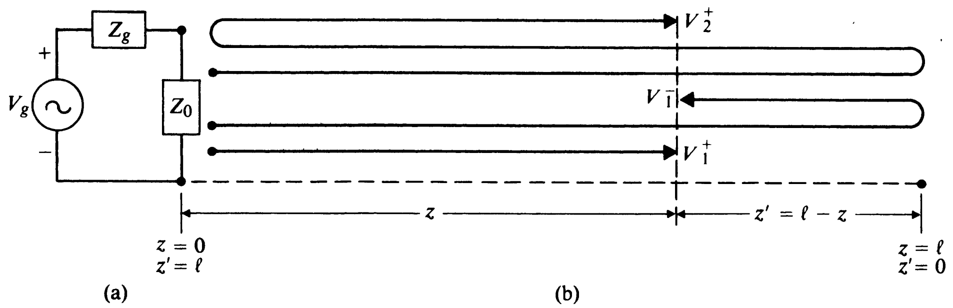


FIGURE 9–13
A transmission-line circuit and traveling waves.

is the complex amplitude of the voltage wave initially sent down the transmission line from the generator. It is obtained directly from the simple circuit shown in Fig. 9–13(a). The phasor V_1^+ in Eq. (9–154a) represents the initial wave traveling in the $+z$-direction. Before this wave reaches the load impedance Z_L, it sees Z_0 of the line as if the line were infinitely long.

When the first wave $V_1^+ = V_M e^{-\gamma z}$ reaches Z_L at $z = \ell$, it is reflected because of mismatch, resulting in a wave V_1^- with a complex amplitude $\Gamma(V_M e^{-\gamma\ell})$ traveling in the $-z$-direction. As the wave V_1^- returns to the generator at $z = 0$, it is again reflected for $Z_g \neq Z_0$, giving rise to a second wave V_2^+ with a complex amplitude $\Gamma_g(\Gamma V_M e^{-2\gamma\ell})$ traveling in $+z$-direction. This process continues indefinitely with reflections at both ends, and the resulting standing wave $V(z')$ is the sum of all the waves traveling in both directions. This is illustrated schematically in Fig. 9–13(b). In practice, $\gamma = \alpha + j\beta$ has a real part, and the attenuation effect of $e^{-\alpha\ell}$ diminishes the amplitude of a reflected wave each time the wave transverses the length of the line.

When the line is terminated with a matched load, $Z_L = Z_0$, $\Gamma = 0$, only V_1^+ exists, and it stops at the matched load with no reflections. If $Z_L \neq Z_0$ but $Z_g = Z_0$ (if the internal impedance of the generator is matched to the line), then $\Gamma \neq 0$ and $\Gamma_g = 0$. As a consequence, both V_1^+ and V_1^- exist, and V_2^+, V_2^- and all higher-order reflections vanish.

EXAMPLE 9–10 A 100 (MHz) generator with $V_g = 10\underline{/0°}$ (V) and internal resistance 50 (Ω) is connected to a lossless 50 (Ω) air line that is 3.6 (m) long and terminated in a $25 + j25$ (Ω) load. Find (a) $V(z)$ at a location z from the generator, (b) V_i at the input terminals and V_L at the load, (c) the voltage standing-wave ratio on the line, and (d) the average power delivered to the load.

Solution Referring to Fig. 9–6, the given quantities are

$$V_g = 10\underline{/0°} \quad \text{(V)}, \qquad Z_g = 50 \quad (\Omega), \qquad f = 10^8 \quad \text{(Hz)},$$
$$R_0 = 50 \quad (\Omega), \qquad Z_L = 25 + j25 = 35.36\underline{/45°} \quad (\Omega), \qquad \ell = 3.6 \quad \text{(m)}.$$

Thus,

$$\beta = \frac{\omega}{c} = \frac{2\pi 10^8}{3 \times 10^8} = \frac{2\pi}{3} \quad \text{(rad/m)}, \qquad \beta\ell = 2.4\pi \quad \text{(rad)},$$

$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{(25 + j25) - 50}{(25 + j25) + 50} = \frac{-25 + j25}{75 + j25} = \frac{35.36\underline{/135^\circ}}{79.1\underline{/18.4^\circ}}$$

$$= 0.447\underline{/116.6^\circ} = 0.447\underline{/0.648\pi},$$

$$\Gamma_g = 0.$$

a) From Eq. (9–153a) we have

$$V(z) = \frac{Z_0 V_g}{Z_0 + Z_g} e^{-j\beta z}[1 + \Gamma e^{-j2\beta(\ell - z)}]$$

$$= \frac{50(10)}{100} e^{-j2\pi z/3}[1 + 0.447 e^{j(0.648 - 4.8)\pi} e^{j4\pi z/3}]$$

$$= 5[e^{-j2\pi z/3} + 0.447 e^{j(2z/3 - 0.152)\pi}] \quad \text{(V)}.$$

We see that, because $\Gamma_g = 0$, $V(z)$ is the superposition of only two traveling waves, V_1^+ and V_1^-, as defined in Eq. (9–154).

b) At the input terminals,

$$V_i = V(0) = 5(1 + 0.447 e^{-j0.152\pi})$$

$$= 5(1.396 - j0.207)$$

$$= 7.06\underline{/-8.43^\circ} \quad \text{(V)}.$$

At the load,

$$V_L = V(3.6) = 5[e^{-j0.4\pi} + 0.447 e^{j0.248\pi}]$$

$$= 5(0.627 - j0.637) = 4.47\underline{/-45.5^\circ} \quad \text{(V)}.$$

c) The voltage standing-wave ratio (VSWR) is

$$S = \frac{1 + |\Gamma|}{1 - |\Gamma|} = \frac{1 + 0.447}{1 - 0.447} = 2.62.$$

d) The average power delivered to the load is

$$P_{av} = \frac{1}{2}\left|\frac{V_L}{Z_L}\right|^2 R_L = \frac{1}{2}\left(\frac{4.47}{35.36}\right)^2 \times 25 = 0.200 \quad \text{(W)}.$$

It is interesting to compare this result with the case of a matched load when $Z_L = Z_0 = 50 + j0$ (Ω). In that case, $\Gamma = 0$,

$$|V_L| = |V_i| = \frac{V_g}{2} = 5 \quad \text{(V)},$$

and a maximum average power is delivered to the load:

$$\text{Maximum } P_{av} = \frac{V_L^2}{2R_L} = \frac{5^2}{2 \times 50} = 0.25 \quad \text{(W)},$$

which is larger than the P_{av} calculated for the unmatched load in part (d) by an amount equal to the power reflected, $|\Gamma|^2 \times 0.25 = 0.05$ (W).

9–5 Transients on Transmission Lines

The discussion of the wave characteristics on transmission lines in the previous section was based on steady-state, single-frequency, time-harmonic sources and signals. We worked with voltage and current phasors. Quantities such as reactances (X), wavelength (λ), wavenumber (k), and phase constant (β) would lose their meaning under transient conditions. However, there are important practical situations in which the sources and signals are not time-harmonic and the conditions are not steady-state. Examples are digital (pulse) signals in computer networks and sudden surges in power and telephone lines. In this section we will consider the transient behavior of lossless transmission lines. For such lines ($R = 0$, $G = 0$), characteristic impedance becomes characteristic resistance $R_0 = 1/\sqrt{LC}$, and voltage and current waves propagate along the line with a velocity $u = 1/\sqrt{LC}$.

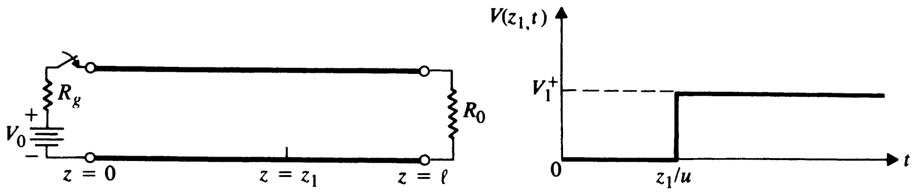
The simplest case is shown in Fig. 9–14(a), where a d-c voltage source V_0 is applied through a series (internal) resistance R_g at $t = 0$ to the input terminals of a lossless line terminated in a characteristic resistance R_0. Since the impedance looking into the terminated line is R_0, a voltage wave of magnitude

$$V_1^+ = \frac{R_0}{R_0 + R_g} V_0 \tag{9–156}$$

travels down the line in the $+z$-direction with a velocity $u = 1/\sqrt{LC}$. The corresponding magnitude, I_1^+, of the current wave is

$$I_1^+ = \frac{V_1^+}{R_0} = \frac{V_0}{R_0 + R_g}. \tag{9–157}$$

If we plot the voltage across the line at $z = z_1$ as a function of time, we obtain a delayed step function at $t = z_1/u$ as in Fig. 9–14(b). The current in the line at $z = z_1$



(a) Switch closed at $t = 0$. (b) Voltage at $z = z_1$.

FIGURE 9–14
A d-c source applied to a line terminated in characteristic resistance R_0 through a series resistance R_g.

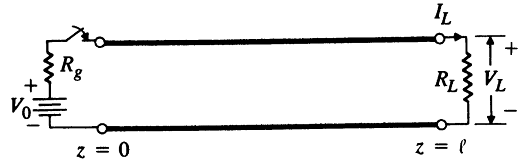


FIGURE 9–15
A d-c source applied to a terminated lossless line at $t = 0$ (general case).

has the same shape with a magnitude I_1^+ given in Eq. (9–157). When the voltage and current waves reach the termination at $z = \ell$, there are no reflected waves because $\Gamma = 0$. A steady state is established, and the entire line is charged to a voltage equal to V_1^+.

If both the series resistance R_g and the load resistance R_L are not equal to R_0, as in Fig. 9–15, the situation is more complicated. When the switch is closed at $t = 0$, the d-c source sends a voltage wave of magnitude

$$V_1^+ = \frac{R_0}{R_0 + R_g} V_0 \tag{9–158}$$

in the $+z$-direction with a velocity $u = 1/\sqrt{LC}$ as before because the V_1^+ wave has no knowledge of the length of the line or the nature of the load at the other end; it proceeds as if the line were infinitely long. At $t = T = \ell/u$ this wave reaches the load end $z = \ell$. Since $R_L \neq R_0$, a reflected wave will travel in the $-z$-direction with a magnitude

$$V_1^- = \Gamma_L V_1^+, \tag{9–159}$$

where

$$\Gamma_L = \frac{R_L - R_0}{R_L + R_0} \tag{9–160}$$

is the reflection coefficient of the load resistance R_L. This reflected wave arrives at the input end at $t = 2T$, where it is reflected by $R_g \neq R_0$. A new voltage wave having a magnitude V_2^+ then travels down the line, where

$$V_2^+ = \Gamma_g V_1^- = \Gamma_g \Gamma_L V_1^+. \tag{9–161}$$

In Eq. (9–161),

$$\Gamma_g = \frac{R_g - R_0}{R_g + R_0} \tag{9–162}$$

is the reflection coefficient of the series resistance R_g. This process will go on indefinitely with waves traveling back and forth, being reflected at each end at $t = nT$ $(n = 1, 2, 3, \ldots)$.

Two points are worth noting here. First, some of the reflected waves traveling in either direction may have a negative amplitude, since Γ_L or Γ_g (or both) may be negative. Second, except for an open circuit or a short circuit, Γ_L and Γ_g are less than unity. Thus the magnitude of the successive reflected waves becomes smaller and smaller, leading to a convergent process. The progression of the transient voltage waves on the lossless line in Fig. 9–15 for $R_L = 3R_0$ ($\Gamma_L = \frac{1}{2}$) and $R_g = 2R_0$ ($\Gamma_g = \frac{1}{3}$) is illustrated in Figs. 9–16(a), 9–16(b), and 9–16(c) for three different time intervals. The corresponding current waves are given in Figs. 9–16(d), 9–16(e), and 9–16(f), which are self-explanatory. The voltage and current at any particular location on the line in any particular time interval are just the *algebraic sums* ($V_1^+ + V_1^- + V_2^+ + V_2^- + \ldots$) and ($I_1^+ + I_1^- + I_2^+ + I_2^- + \ldots$), respectively.

It is interesting to check the ultimate value of the voltage across the load, $V_L = V(\ell)$, as t increases indefinitely. We have

$$\begin{aligned} V_L &= V_1^+ + V_1^- + V_2^+ + V_2^- + V_3^+ + V_3^- + \cdots \\ &= V_1^+(1 + \Gamma_L + \Gamma_g\Gamma_L + \Gamma_g\Gamma_L^2 + \Gamma_g^2\Gamma_L^2 + \Gamma_g^2\Gamma_L^3 + \cdots) \\ &= V_1^+[(1 + \Gamma_g\Gamma_L + \Gamma_g^2\Gamma_L^2 + \cdots) + \Gamma_L(1 + \Gamma_g\Gamma_L + \Gamma_g^2\Gamma_L^2 + \cdots)] \\ &= V_1^+\left[\left(\frac{1}{1-\Gamma_g\Gamma_L}\right) + \left(\frac{\Gamma_L}{1-\Gamma_g\Gamma_L}\right)\right] \\ &= V_1^+\left(\frac{1+\Gamma_L}{1-\Gamma_g\Gamma_L}\right). \end{aligned} \tag{9–163}$$

For the present case, $V_1^+ = V_0/3$, $\Gamma_L = \frac{1}{2}$, and $\Gamma_g = 1/3$, Eq. (9–163) gives

$$V_L = \tfrac{9}{5}V_1^+ = \tfrac{3}{5}V_0 \tag{9–163a}$$

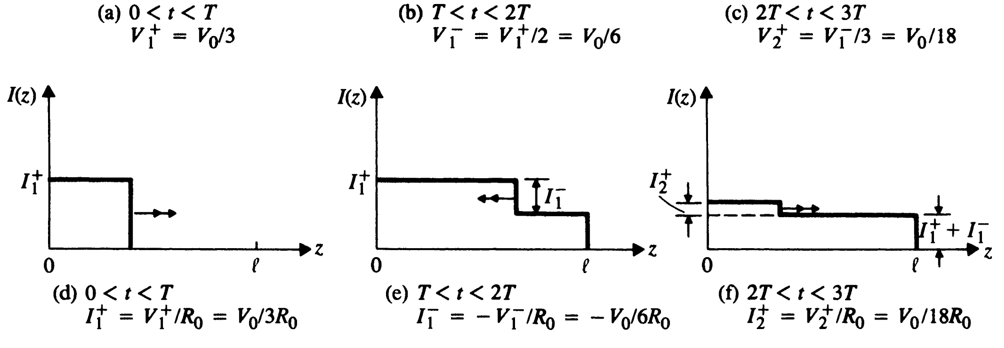


FIGURE 9–16
Transient voltage and current waves on transmission line in Fig. 9–15 for $R_L = 3R_0$ and $R_g = 2R_0$.

as $t \to \infty$. This result is obviously correct because, in the steady state, V_0 is divided between R_L and R_g in a ratio of 3 to 2. Similarly, we find

$$I_L = \left(\frac{1-\Gamma_L}{1-\Gamma_g\Gamma_L}\right)\frac{V_1^+}{R_0},$$

which yields

$$I_L = \frac{3}{5}\left(\frac{V_1^+}{R_0}\right) = \frac{V_0}{5R_0}, \tag{9-164}$$

as expected.

9-5.1 REFLECTION DIAGRAMS

The preceding step-by-step construction and calculation procedure of the voltage and current at a particular time and location on a transmission line with arbitrary resistive terminations tends to be tedious and difficult to visualize when it is necessary to consider many reflected waves. In such cases the graphical construction of a reflection diagram is very helpful. Let us first construct a ***voltage reflection diagram. A reflection diagram*** plots the time elapsed after the change of circuit conditions versus the distance z from the source end. The voltage reflection diagram for the transmission-line circuit in Fig. 9–15 is given in Fig. 9–17. It starts with a wave V_1^+ at $t = 0$ traveling from the source end ($z = 0$) in the $+z$-direction with a velocity $u = 1/\sqrt{LC}$. This wave is represented by the directed straight line marked V_1^+ from the origin. This line has a positive slope equal to $1/u$. When the V_1^+ wave reaches the load at $z = \ell$, a reflected wave $V_1^- = \Gamma_L V_1^+$ is created if $R_L \neq R_0$. The V_1^- wave travels in the $-z$-direction and is represented by the directed line marked $\Gamma_L V_1^+$ with a negative slope equal to $-1/u$.

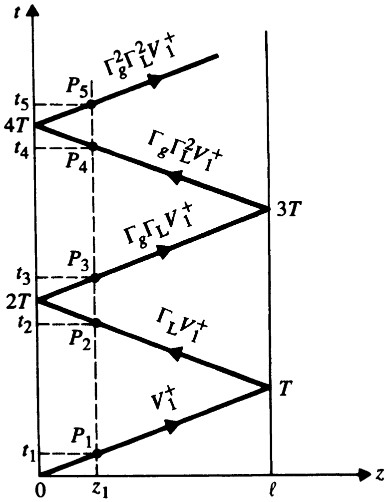


FIGURE 9–17
Voltage reflection diagram for transmission-line circuit in Fig. 9–15.

The V_1^- wave returns to the source end at $t = 2T$ and gives rise to a reflected wave $V_2^+ = \Gamma_g V_1^- = \Gamma_g \Gamma_L V_1^+$, which is represented by a second directed line with a positive slope. This process continues back and forth indefinitely. The voltage reflection diagram can be used conveniently to determine the voltage distribution along the transmission line at a given time as well as the variation of the voltage as a function of time at an arbitrary point on the line.

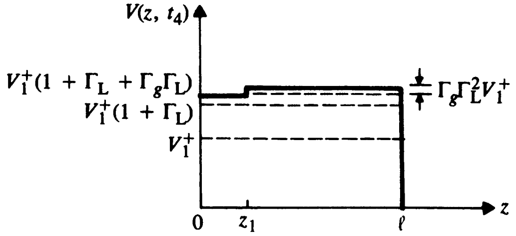
Suppose we wish to know the voltage distribution along the line at $t = t_4$ ($3T < t_4 < 4T$). We proceed as follows:

1. Mark t_4 on the vertical t-axis of the voltage reflection diagram.
2. Draw a horizontal line from t_4, intersecting the directed line marked $\Gamma_g \Gamma_L^2 V_1^+$ at P_4. (All directed lines above P_4 are irrelevant to our problem because they pertain to $t > t_4$.)
3. Draw a vertical line through P_4, intersecting the horizontal z-axis at z_1. The significance of z_1 is that in the range $0 < z < z_1$ (to the left of the vertical line) the voltage has a value equal to $V_1^+ + V_1^- + V_2^+ = V_1^+(1 + \Gamma_L + \Gamma_g \Gamma_L)$; and in the range $z_1 < z < \ell$ (to the right of the vertical line) the voltage is $V_1^+ + V_1^- + V_2^+ + V_2^- = V_1^+(1 + \Gamma_L + \Gamma_g \Gamma_L + \Gamma_g \Gamma_L^2)$. There is a voltage discontinuity equal to $\Gamma_g \Gamma_L^2 V_1^+$ at $z = z_1$.
4. The voltage distribution along the line at $t = t_4$, $V(z, t_4)$, is then as shown in Fig. 9–18(a), plotted for $R_L = 3R_0$ ($\Gamma_L = \frac{1}{2}$) and $R_g = 2R_0$ ($\Gamma_g = \frac{1}{3}$).

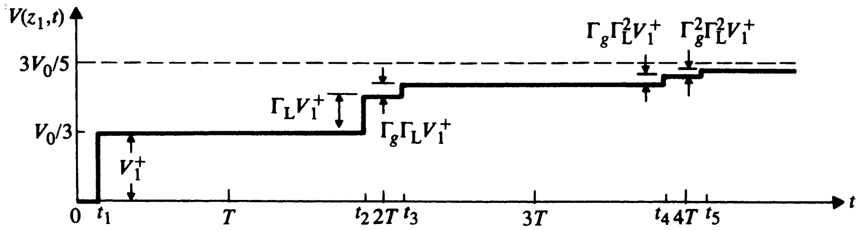
Next let us find the variation of the voltage as a function of time at the point $z = z_1$. We use the following procedure:

1. Draw a vertical line at z_1, intersecting the directed lines at points P_1, P_2, P_3, P_4, P_5, and so on. (There would be an infinite number of such intersection points if $R_L \neq R_0$ and $R_g \neq R_0$, as there would be an infinite number of directed lines if $\Gamma_L \neq 0$ and $\Gamma_g \neq 0$.)
2. From these intersection points, draw horizontal lines intersecting the vertical t-axis at t_1, t_2, t_3, t_4, t_5, and so on. These are the instants at which a new voltage wave arrives and abruptly changes the voltage at $z = z_1$.
3. The voltage at $z = z_1$ as a function of t can be read from the voltage reflection diagram as follows:

Time Range	Voltage	Voltage Discontinuity
$0 \le t < t_1$ ($t_1 = z_1/u$)	0	0
$t_1 \le t < t_2$ ($t_2 = 2T - t_1$)	V_1^+	V_1^+ at t_1
$t_2 \le t < t_3$ ($t_3 = 2T + t_1$)	$V_1^+(1 + \Gamma_L)$	$\Gamma_L V_1^+$ at t_2
$t_3 \le t < t_4$ ($t_4 = 4T - t_1$)	$V_1^+(1 + \Gamma_L + \Gamma_g \Gamma_L)$	$\Gamma_g \Gamma_L V_1^+$ at t_3
$t_4 \le t < t_5$ ($t_5 = 4T + t_1$)	$V_1^+(1 + \Gamma_L + \Gamma_g \Gamma_L + \Gamma_g \Gamma_L^2)$	$\Gamma_g \Gamma_L^2 V_1^+$ at t_4
$\vdots$	$\vdots$	$\vdots$



(a) $V(z, t_4)$ versus z;
$\Gamma_L = \frac{1}{2}$, $\Gamma_g = \frac{1}{3}$, $V_1^+ = V_0/3$.



(b) $V(z_1,t)$ versus t; $V(z_1, \infty) = 3V_0/5$.

FIGURE 9–18
Transient voltage on lossless transmission line for $R_L = 3R_0$ and $R_g = 2R_0$.

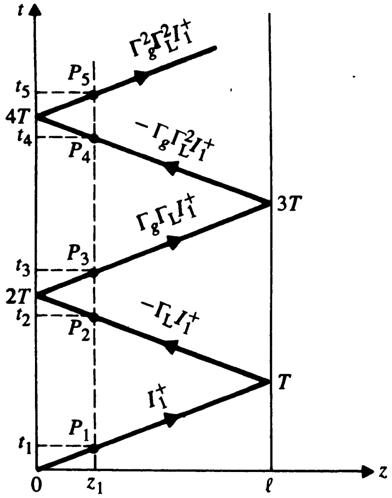


FIGURE 9–19
Current reflection diagram for transmission-line circuit in Fig. 9–15.

4. The graph of $V(z_1, t)$ is plotted in Fig. 9–18(b) for $\Gamma_L = \frac{1}{2}$ and $\Gamma_g = \frac{1}{3}$. When t increases indefinitely, the voltage at z_1 (and at all other points along the lossless line) will assume the value $3V_0/5$, as given in Eq. (9–163a).

Similar to the voltage reflection diagram in Fig. 9–17, a ***current reflection diagram*** for the transmission-line circuit of Fig. 9–15 can be constructed. This is shown in Fig. 9–19. Here we draw directed lines representing current waves. The essential difference between the voltage and current reflection diagrams is in the negative sign associated with the current waves traveling in the $-z$-direction on account of Eq. (9–94). The current reflection diagram can be used to determine the current distribution along the transmission line at a given time as well as the variation of the current as a function of time at a particular point on the line, following the same procedures outlined previously for voltage. For example, we can determine the current at $z = z_1$ by drawing a vertical line through z_1 in Fig. 9–19, intersecting the directed lines at points P_1, P_2, P_3, P_4, P_5, and so on, and by finding the corresponding times t_1, t_2, t_3, t_4, t_5, and so on, as before. Figure 9–20 is a plot of $I(z_1, t)$ versus t, which accompanies the $V(z_1, t)$ graph in Fig. 9–18(b). We see that they are quite dissimilar. The current along the line oscillates around the steady-state value of $V_0/5R_0$ (see Eq. 9–164) with successively smaller discontinuous jumps at t_1, t_2, t_3, t_4, t_5, etc.

We note two special cases here.

1. When $R_L = R_0$ (matched load, $\Gamma_L = 0$), the voltage and current reflection diagrams will each have only a single directed line, existing in the interval $0 < t < T$, irrespective of what R_g is.
2. When $R_g = R_0$ (matched source, $\Gamma_g = 0$) and $R_L \neq R_0$, the voltage and current reflection diagrams will each have only two directed lines, existing in the intervals $0 < t < T$ and $T < t < 2T$.

In both cases the determination of the transient behavior on the transmission line is much simplified.

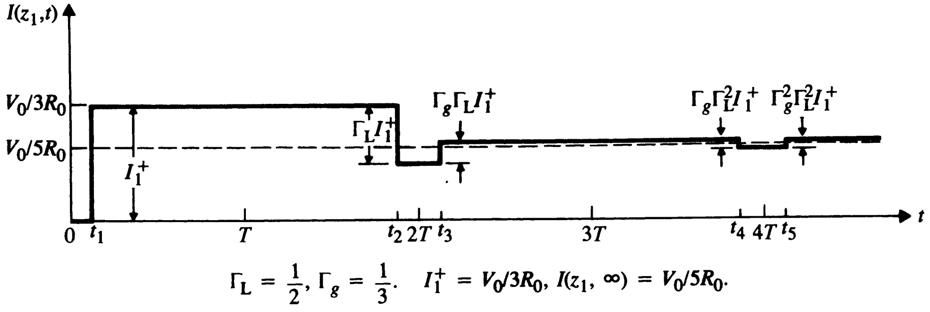


FIGURE 9–20
Transient current on lossless transmission line for $R_L = 3R_0$ and $R_g = 2R_0$.

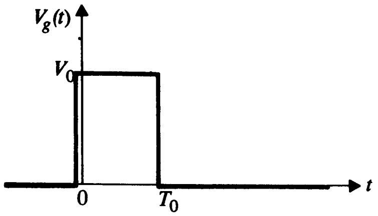


FIGURE 9–21
A rectangular pulse.

9–5.2 PULSE EXCITATION

So far, we have discussed the transient behavior of lossless transmission lines when the source is a sudden voltage surge in the form of a step function; that is,

$$v_g(t) = V_0 U(t), \tag{9–165}$$

where $U(t)$ denotes the unit step function

$$U(t) = \begin{cases} 0, & t < 0, \\ 1, & t > 0. \end{cases} \tag{9–166}$$

In many instances, such as in computer networks and pulse-modulation systems, the excitation may be in the form of pulses. The analysis of the transient behavior of a line with pulse excitation, however, does not present special difficulties because a rectangular pulse can be decomposed into two step functions. For example, the pulse of an amplitude V_0 lasting from $t = 0$ to $t = T_0$ shown in Fig. 9–21 can be written as

$$v_g(t) = V_0[U(t) - U(t - T_0)]. \tag{9–166a}$$

If $v_g(t)$ in Eq. (9–166a) is applied to a transmission line, the transient response is simply the superposition of the result obtained from a d-c voltage V_0 applied at $t = 0$ and that obtained from another d-c voltage $-V_0$ applied at $t = T_0$. We will illustrate this process by an example.

EXAMPLE 9–11 A rectangular pulse of an amplitude 15 (V) and a duration 1 (μs) is applied through a series resistance of 25 (Ω) to the input terminals of a 50 (Ω) lossless coaxial transmission line. The line is 400 (m) long and is short-circuited at the far end. Determine the voltage response at the midpoint of the line as a fuction of time up to 8 (μs). The dielectric constant of the insulating material in the cable is 2.25.

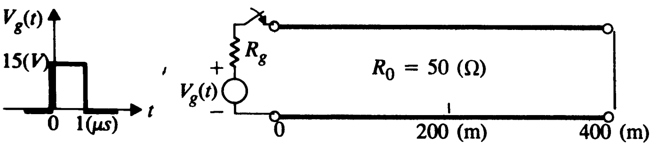


FIGURE 9–22
A pulse applied to a short-circuited line.

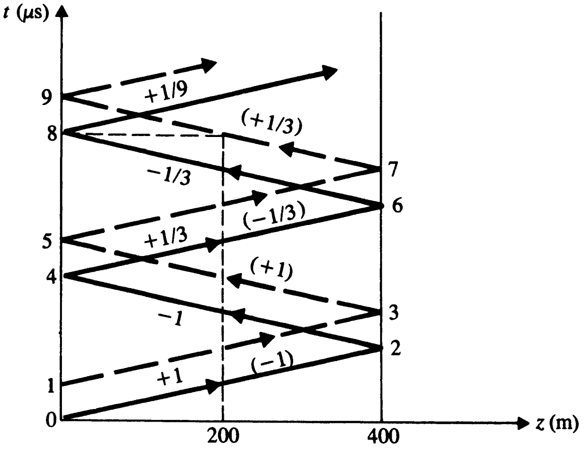


FIGURE 9–23
Voltage reflection diagram for Example 9–11.

Solution We have a situation as given in Fig. 9–22, where $R_g = 25$ (Ω) and $R_L = 0$. Also,

$$\Gamma_L = -1, \qquad \Gamma_g = \frac{25 - 50}{25 + 50} = -\frac{1}{3},$$

$$v_g(t) = 10[U(t) - U(t - 10^{-6})],$$

$$u = \frac{c}{\sqrt{\epsilon_r}} = \frac{3 \times 10^8}{\sqrt{2.25}} = 2 \times 10^8 \quad \text{(m/s)},$$

$$T = \frac{\ell}{u} = \frac{400}{2 \times 10^8} = 2 \times 10^{-6}\ \text{(s)} = 2 \quad (\mu\text{s}),$$

$$V_1^+ = \frac{15R_0}{R_0 + R_g} = \frac{15 \times 50}{50 + 25} = 10 \quad \text{(V)}.$$

A voltage reflection diagram is constructed in Fig. 9–23 for this problem. There are two sets of directed lines: The solid lines are for $+15$ (V) applied at $t = 0$, and the dashed lines are for -15 (V) applied at $t = 1$ (μs). Along each directed line is marked the amplitude of the wave (with the appropriate sign) normalized with respect to $V_1^+ = 10$ (V). The markings for the applied voltage $-15U(t - 10^{-6})$ are enclosed in brackets for easy reference. To obtain the voltage variation at the line's midpoint for the interval $0 < t \leq 8$ (μs), we draw a vertical line at $z = 200$ (m) and a horizontal line at $t = 8$ (μs). The voltage function due to $15U(t)$ can be read from the intersections of the vertical line with the solid directed lines. This is sketched as v_a in Fig. 9–24(a). Similarly, the voltage function due to $-15U(t - 10^{-6})$ is read from the intersections of the vertical line with the dashed directed lines; it is sketched as v_b in Fig. 9–24(b). The required response $v(200, t)$ for $0 < t \leq 8$ (μs) is then the sum of the responses, $v_a + v_b$, and is given in Fig. 9–24(c). ■

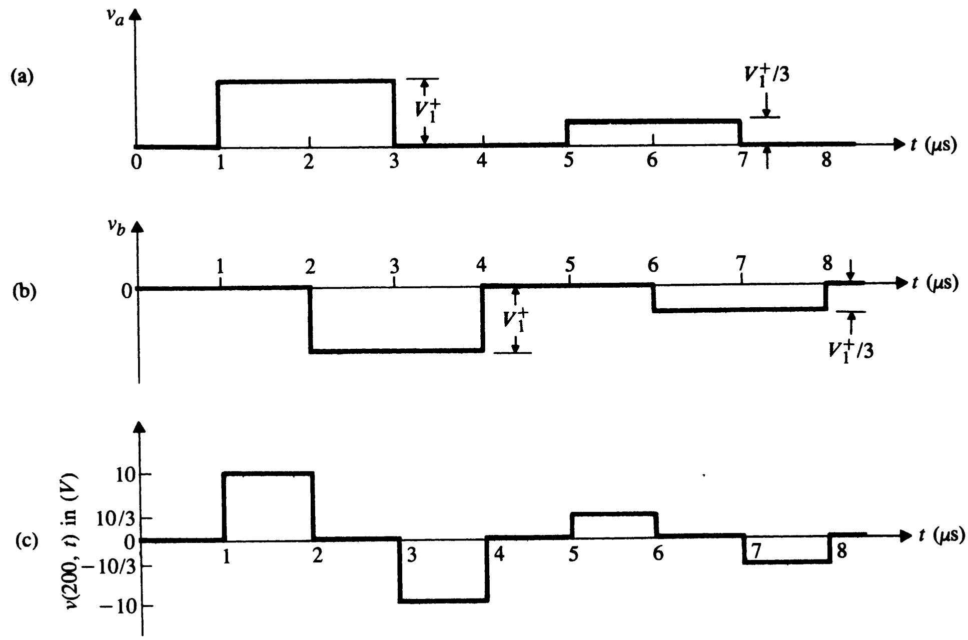


FIGURE 9–24
Voltage responses at the midpoint of the short-circuited line in Fig. 9–22 (Example 9–11).

9–5.3 INITIALLY CHARGED LINE

In our discussion of transients on transmission lines we have assumed that the lines themselves have no initial voltages or currents when an external source is applied. Actually, any disturbance or change in a transmission-line circuit will start transients along the line even without an external source if initial voltages and/or currents exist. We examine in this subsection a situation involving an initially charged line and develop a method of analysis.

Consider the following example.

EXAMPLE 9–12 A lossless, air-dielectric, open-circuited transmission line of characteristic resistance R_0 and length ℓ is initially charged to a voltage V_0. At $t = 0$ the line is connected to a resistance R. Determine the voltage across and the current in R as functions of time. Assume that $R = R_0$.

Solution This problem, as depicted in Fig. 9–25(a), can be analyzed by examining the circuits in Figs. 9–25(b), 9–25(c), and 9–25(d). The circuit in Fig. 9–25(b) is equivalent to that in Fig. 9–25(a). After the switch is closed, the conditions in the circuit in Fig. 9–25(b) are the same as the superposition of those shown in Figs. 9–25(c) and 9–25(d). But the circuit in Fig. 9–25(c) does not give rise to transients because of the opposing voltages; hence we use the circuit in Fig. 9–25(d) to study the transient behavior of the original circuit in Fig. 9–25(a). The line in the circuit in Fig. 9–25(d) is uncharged, and our problem has then been reduced to one with which we are already familiar.

When the switch is closed, a voltage wave of amplitude V_1^+ will be sent down the line in the $+z$-direction, where

$$V_1^+ = -\frac{R_0}{R + R_0} V_0 = -\frac{V_0}{2}.$$

At $t = \ell/c$, the V_1^+ wave reaches the open end, having reduced the voltage along the whole line from V_0 to $V_0/2$. At the open end, $\Gamma = 1$, and a reflected V_1^- wave is sent back in the $-z$-direction with $V_1^- = V_1^+ = -V_0/2$. This reflected wave returns to the sending end at $t = 2\ell/c$, reducing the voltage on the line to zero.

From Fig. 9–25(d),

where

$$I_R = -I_1,$$

$$I_1 = I_1^+ = \frac{V_1^+}{R_0} = -\frac{V_0}{2R_0} \qquad \text{for} \quad 0 \le t < 2\ell/c.$$

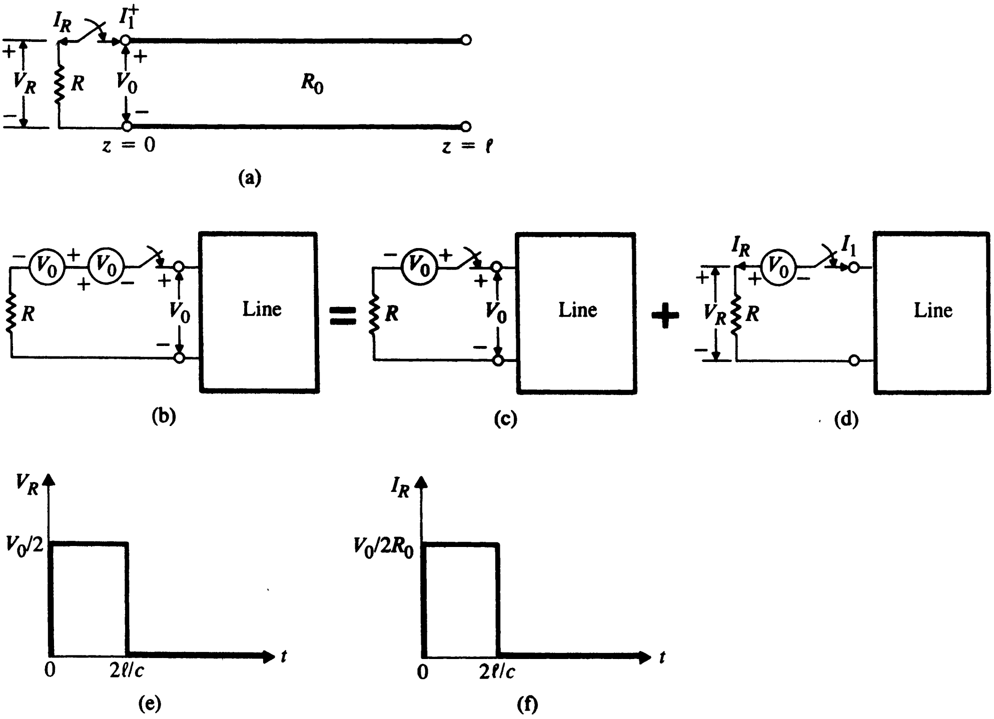


FIGURE 9–25
Transient problem of an open-circuited, initially charged line, $R = R_0$ (Example 9–12).

At $t = \ell/c$, I_1^+ reaches the open end, and the reflected I_1^- must make the total current there zero. Hence,

$$I_1^- = -I_1^+ = \frac{V_0}{2R_0},$$

which reaches the sending end at $t = 2\ell/c$ and reduces both I_1 and I_R to zero. Since $R = R_0$, there is no further reflection, and the transient state ends. As shown in Figs. 9–25(e) and 9–25(f), both V_R and I_R are a pulse of duration $2\ell/c$. We then have a way of generating a pulse by discharging a charged open-circuited transmission line, the width of the pulse being adjustable by changing ℓ. ∎

9–5.4 LINE WITH REACTIVE LOAD

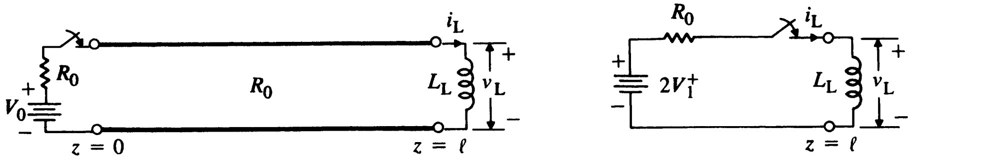
When the termination on a transmission line is a resistance different from the characteristic resistance, an incident voltage or current wave will produce a reflected wave of the same time dependence. The ratio of the amplitudes of the reflected and incident waves is a constant, which is defined as the reflection coefficient. If, however, the termination is a reactive element such as an inductance or a capacitance, the reflected wave will no longer have the same time dependence (no longer be of the same shape) as the incident wave. The use of a constant reflection coefficient is not feasible in such cases, and it is necessary to solve a differential equation at the termination in order to study the transient behavior. We shall consider the effect on the reflected wave of an inductive termination and a capacitive termination separately in this subsection.

Figure 9–26(a) shows a lossless line with a characteristic resistance R_0, terminated at $z = \ell$ with an inductance L_L. A d-c voltage V_0 is applied to the line at $z = 0$ through a series resistance R_0. When the switch is closed at $t = 0$, a voltage wave of an amplitude

$$V_1^+ = \frac{V_0}{2} \tag{9–167}$$

travels toward the load. Upon reaching the load at $t = \ell/u = T$, a reflected wave $V_1^-(t)$ is produced because of mismatch. It is the relation between $V_1^-(t)$ and V_1^+ that we wish to find. At $z = \ell$, the following relations hold for all $t \geq T$:

$$v_L(t) = V_1^+ + V_1^-(t), \tag{9–168}$$



(a) Transmission-line circuit with inductive termination (b) Equivalent circuit for the load end, $t \geqslant T$

FIGURE 9–26
Transient calculations for a lossless line with an inductive termination.

$$i_L(t) = \frac{1}{R_0}[V_1^+ - V_1^-(t)], \tag{9–169}$$

$$v_L(t) = L_L \frac{di_L(t)}{dt}. \tag{9–170}$$

Eliminating $V_1^-(t)$ from Eqs. (9–168) and (9–169), we obtain

$$v_L(t) = 2V_1^+ - R_0 i_L(t). \tag{9–171}$$

It is seen that Eq. (9–171) describes the application of Kirchhoff's voltage law to the circuit in Fig. 9–26(b), which is then the equivalent circuit at the load end for $t \geq T$. In view of Eq. (9–170), Eq, (9–171) leads to a first-order differential equation with constant coefficients:

$$L_L \frac{di_L(t)}{dt} + R_0 i_L(t) = 2V_1^+, \qquad t \geq T. \tag{9–172}$$

The solution of Eq. (9–172) is

$$i_L(t) = \frac{2V_1^+}{R_0}[1 - e^{-(t-T)R_0/L_L}], \qquad t \geq T, \tag{9–173}$$

which correctly gives $i_L(T) = 0$ and $i_L(\infty) = 2V_1^+/R_0$. The voltage across the inductive load is

$$v_L(t) = L_L \frac{di_L(t)}{dt} = 2V_1^+ e^{-(t-T)R_0/L_L}, \qquad t \geq T. \tag{9–174}$$

The amplitude of the reflected wave, $V_1^-(t)$, can be found from Eq. (9–168):

$$\begin{aligned} V_1^-(t) &= v_L(t) - V_1^+ \\ &= 2V_1^+[e^{-(t-T)R_0/L_L} - \tfrac{1}{2}], \qquad t > T. \end{aligned} \tag{9–175}$$

This reflected wave travels in the $-z$-direction. The voltage at any point $z = z_1$ along the line is V_1^+ before the reflected wave from the load end reaches that point, $(t - T) < (\ell - z_1)/u$, and equals $V_1^+ + V_1^-(t - T)$ after that.

In Figs. 9–27(a), 9–27(b), and 9–27(c) are plotted $i_L(t)$, $v_L(t)$, and $V_1^-(t)$ at $z = \ell$ using Eqs. (9–173), (9–174), and (9–175). The voltage distribution along the line for $T < t_1 < 2T$ is shown in Fig. 9–27(d). Obviously, the transient behavior on a transmission line with a reactive termination is more complicated than that with a resistive termination.

We follow a similar procedure in examining the transient behavior of a lossless line with a capacitive termination, shown in Fig. 9–28(a). The same Eqs. (9–167), (9–168), (9–169), and (9–171) apply at $z = \ell$, but Eq. (9–170) relating the load current $i_L(t)$ and load voltage $v_L(t)$ must now be changed to

$$i_L(t) = C_L \frac{dv_L(t)}{dt}. \tag{9–176}^\dagger$$

† The subscript roman L denotes load; it has nothing to do with inductance.

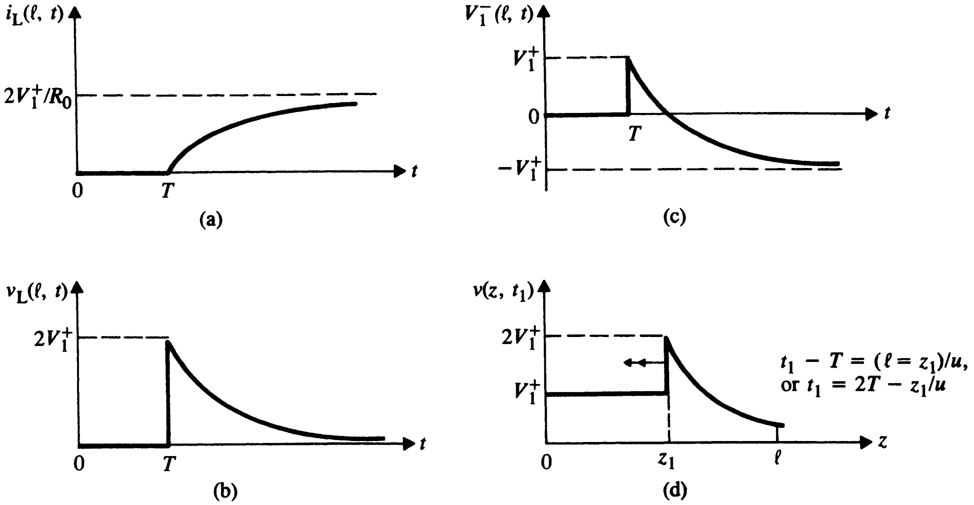


FIGURE 9–27
Transient responses of a lossless line with an inductive termination.

The differential equation to be solved at the load end is, by substituting Eq. (9–176) in Eq. (9–171),

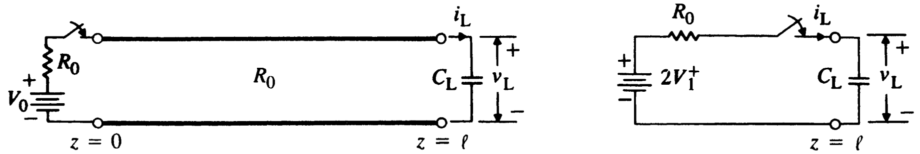
$$C_L \frac{dv_L(t)}{dt} + \frac{1}{R_0} v_L(t) = \frac{2}{R_0} V_1^+, \qquad t \geq T, \tag{9–177}$$

where $V_1^+ = V_0/2$, as given in Eq. (9–167). The solution of Eq. (9–177) is

$$v_L(t) = 2V_1^+ [1 - e^{-(t-T)/R_0C_L}], \qquad t \geq T. \tag{9–178}$$

The current in the load capacitance is obtained from Eq. (9–176):

$$i_L(t) = \frac{2V_1}{R_0} e^{-(t-T)/R_0C_L}, \qquad t \geq T. \tag{9–179}$$



(a) Transmission-line circuit with capacitive termination. (b) Equivalent circuit for the load end, $t \geqslant T$.

FIGURE 9–28
Transient calculations for a lossless line with a capacitive termination.

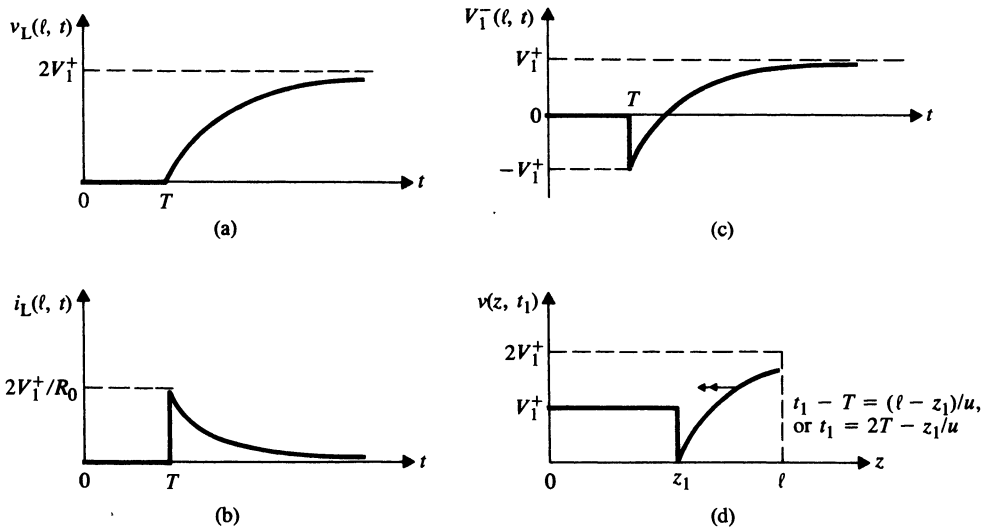


FIGURE 9–29
Transient responses of a lossless line with a capacitive termination.

Using Eq. (9–178) in Eq. (9–168), we find the amplitude of the reflected wave as a function of t:

$$V_1^-(t) = 2V_1^+\left[\tfrac{1}{2} - e^{-(t-T)/R_0C_L}\right], \qquad t \geq T. \tag{9–180}$$

The graphs of $v_L(t)$, $i_L(t)$, and $V_1^-(t)$ at $z = \ell$ are plotted in Figs. 9–29(a), 9–29(b), and 9–29(c) using Eqs. (9–178), (9–179), and (9–180), respectively. The voltage distribution along the line for $T < t_1 < 2T$ is shown in Fig. 9–29(d).

In this section we have discussed the transient behavior of only lossless transmission lines. For lossy lines, both the voltage and the current waves traveling in either direction will be attenuated as they proceed. This situation introduces additional complication in numerical computation, but the basic concept remains the same.

9–6 The Smith Chart

Transmission-line calculations—such as the determination of input impedance by Eq. (9–109), reflection coefficient by Eq. (9–134), and load impedance by Eq. (9–148)—often involve tedious manipulations of complex numbers. This tedium can be alleviated by using a graphical method of solution. The best known and most widely used graphical chart is the ***Smith chart*** devised by P. H. Smith.† Stated

† P H. Smith, "Transmission-line calculator," *Electronics*, vol. 12, p. 29, January 1939; and "An improved transmission-line calculator," *Electronics*, vol. 17, p. 130, January 1944.

succinctly, a Smith chart is a graphical plot of normalized resistance and reactance functions in the reflection-coefficient plane.

To understand how the Smith chart for a *lossless* transmission line is constructed, let us examine the voltage reflection coefficient of the load impedance defined in Eq. (9–134):

$$\Gamma = \frac{Z_L - R_0}{Z_L + R_0} = |\Gamma| e^{j\theta_\Gamma}. \tag{9–181}$$

Let the load impedance Z_L be normalized with respect to the characteristic impedance $R_0 = \sqrt{L/C}$ of the line.

$$z_L = \frac{Z_L}{R_0} = \frac{R_L}{R_0} + j\frac{X_L}{R_0} = r + jx \qquad \text{(Dimensionless)}, \tag{9–182}$$

where r and x are the normalized resistance and normalized reactance, respectively. Equation (9–181) can be rewritten as

$$\Gamma = \Gamma_r + j\Gamma_i = \frac{z_L - 1}{z_L + 1}, \tag{9–183}$$

where Γ_r and Γ_i are the real and imaginary parts, respectively, of the voltage reflection coefficient Γ. The inverse relation of Eq. (9–183) is

$$z_L = \frac{1 + \Gamma}{1 - \Gamma} = \frac{1 + |\Gamma| e^{j\theta_\Gamma}}{1 - |\Gamma| e^{j\theta_\Gamma}} \tag{9–184}$$

or

$$r + jx = \frac{(1 + \Gamma_r) + j\Gamma_i}{(1 - \Gamma_r) - j\Gamma_i}. \tag{9–185}$$

Multiplying both the numerator and the denominator of Eq. (9–185) by the complex conjugate of the denominator and separating the real and imaginary parts, we obtain

$$r = \frac{1 - \Gamma_r^2 - \Gamma_i^2}{(1 - \Gamma_r)^2 + \Gamma_i^2} \tag{9–186}$$

and

$$x = \frac{2\Gamma_i}{(1 - \Gamma_r)^2 + \Gamma_i^2}. \tag{9–187}$$

If Eq. (9–186) is plotted in the $\Gamma_r - \Gamma_i$ plane for a given value of r, the resulting graph is the locus for this r. The locus can be recognized when the equation is rearranged as

$$\left(\Gamma_r - \frac{r}{1 + r}\right)^2 + \Gamma_i^2 = \left(\frac{1}{1 + r}\right)^2. \tag{9–188}$$

It is the equation for a circle having a radius $1/(1 + r)$ and centered at $\Gamma_r = r/(1 + r)$ and $\Gamma_i = 0$. Different values of r yield circles of different radii with centers at different positions on the Γ_r-axis. A family of r-circles are shown in solid lines in Fig. 9–30. Since $|\Gamma| \leq 1$ for a lossless line, only that part of the graph lying within the unit circle on the $\Gamma_r - \Gamma_i$ plane is meaningful; everything outside can be disregarded.

Several salient properties of the r-circles are noted as follows:

1. The centers of all r-circles lie on the Γ_r-axis.
2. The $r = 0$ circle, having a unity radius and centered at the origin, is the largest.
3. The r-circles become progressively smaller as r increases from 0 toward ∞, ending at the ($\Gamma_r = 1$, $\Gamma_i = 0$) point for open-circuit.
4. All r-circles pass through the ($\Gamma_r = 1$, $\Gamma_i = 0$) point.

Similarly, Eq. (9–187) may be rearranged as

$$(\Gamma_r - 1)^2 + \left(\Gamma_i - \frac{1}{x}\right)^2 = \left(\frac{1}{x}\right)^2. \tag{9–189}$$

This is the equation for a circle having a radius $1/|x|$ and centered at $\Gamma_r = 1$ and $\Gamma_i = 1/x$. Different values of x yield circles of different radii with centers at different positions on the $\Gamma_r = 1$ line. A family of the portions of x-circles lying inside the $|\Gamma| = 1$ boundary are shown in dashed lines in Fig. 9–30. The following is a list of several salient properties of the x-circles.

1. The centers of all x-circles lie on the $\Gamma_r = 1$ line; those for $x > 0$ (inductive reactance) lie above the Γ_r-axis, and those for $x < 0$ (capacitive reactance) lie below the Γ_r-axis.
2. The $x = 0$ circle becomes the Γ_r-axis.

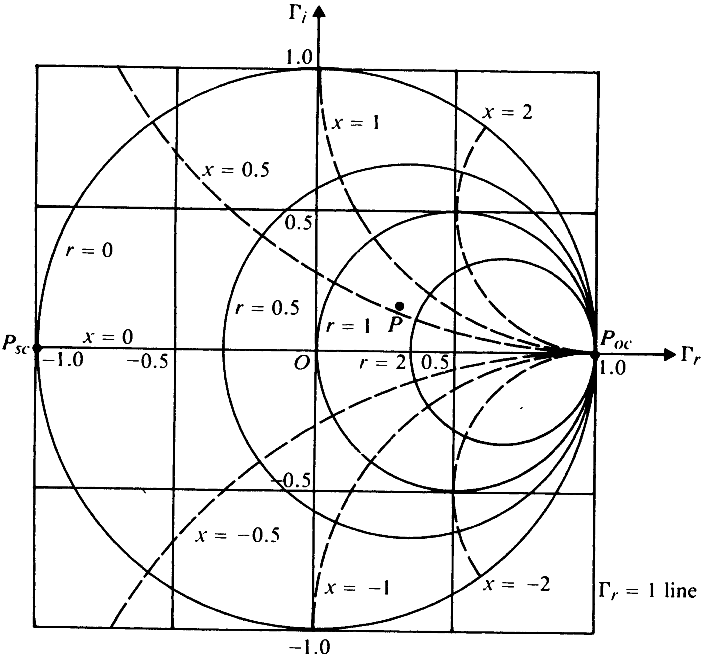


FIGURE 9–30
Smith chart with rectangular coordinates.

3. The x-circle becomes progressively smaller as $|x|$ increases from 0 toward ∞, ending at the ($\Gamma_r = 1, \Gamma_i = 0$) point for open-circuit.
4. All x-circles pass through the ($\Gamma_r = 1, \Gamma_i = 0$) point.

A Smith chart is a chart of r- and x-circles in the $\Gamma_r - \Gamma_i$ plane for $|\Gamma| \leq 1$. It can be proved that the r- and x-circles are everywhere orthogonal to one another. The intersection of an r-circle and an x-circle defines a point that represents a normalized load impedance $z_L = r + jx$. The actual load impedance is $Z_L = R_0(r + jx)$. Since a Simith chart plots the normalized impedance, it can be used for calculations concerning a lossless transmission line with an arbitrary characteristic impedance (resistance).

As an illustration, point P in Fig. 9–30 is the intersection of the $r = 1.7$ circle and the $x = 0.6$ circle. Hence it represents $z_L = 1.7 + j0.6$. The point P_{sc} at ($\Gamma_r = -1$, $\Gamma_i = 0$) corresponds to $r = 0$ and $x = 0$ and therefore represents a short-circuit. The point P_{oc} at ($\Gamma_r = 1, \Gamma_i = 0$) corresponds to an infinite impedance and represents an open-circuit.

The Smith chart in Fig. 9–30 is marked with Γ_r and Γ_i rectangular coordinates. The same chart can be marked with polar coordinates, such that every point in the Γ-plane is specified by a magnitude $|\Gamma|$ and a phase angle θ_Γ. This is illustrated in Fig. 9–31, where several $|\Gamma|$-circles are shown in dashed lines and some θ_Γ-angles are marked around the $|\Gamma| = 1$ circle. The $|\Gamma|$-circles are normally not shown on commercially available Smith charts; but once the point representing a certain $z_L = r + jx$ is located, it is a simple matter to draw a circle centered at the origin through the

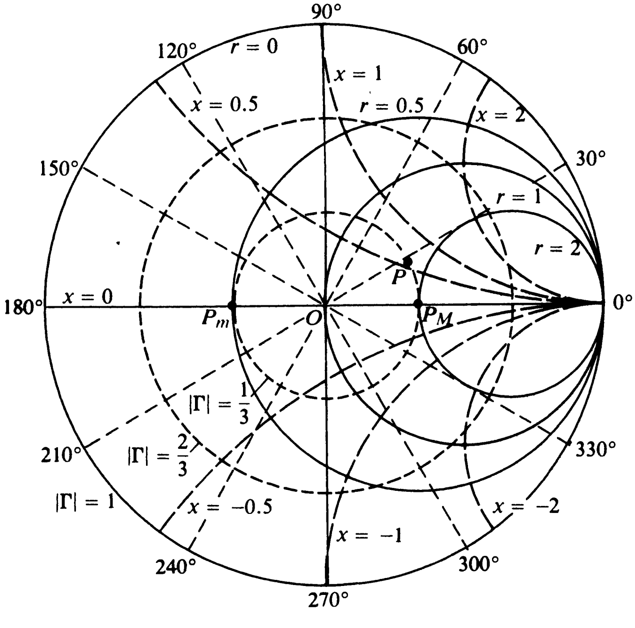


FIGURE 9–31
Smith chart with polar coordinates.

point. The fractional distance from the center to the point (compared with the unity radius to the edge of the chart) is equal to the magnitude $|\Gamma|$ of the load reflection coefficient; and the angle that the line to the point makes with the real axis is θ_Γ. This graphical determination circumvents the need for computing Γ by Eq. (9–183).

Each $|\Gamma|$-circle intersects the real axis at two points. In Fig. 9–31 we designate the point on the positive-real axis (OP_{oc}) as P_M and the point on the negative-real axis (OP_{sc}) as P_m. Since $x = 0$ along the real axis, P_M and P_m both represent situations with a purely resistive load, $Z_L = R_L$. Obviously, $R_L > R_0$ at P_M, where $r > 1$; and $R_L < R_0$ at P_m, where $r < 1$. In Eq. (9–145) we found that $S = R_L/R_0 = r$ for $R_L > R_0$. This relation enables us to say immediately, without using Eq. (9–138), that *the value of the r-circle passing through the point P_M is numerically equal to the standing-wave ratio.* Similarly, we conclude from Eq. (9–146) that *the value of the r-circle passing through the point P_m on the negative-real axis is numerically equal to* $1/S$. For the $z_L = 1.7 + j0.6$ point, marked P in Fig. 9–31, we find $|\Gamma| = \frac{1}{3}$ and $\theta_\Gamma = 28°$. At P_M, $r = S = 2.0$. These results can be verified analytically.

In summary, we note the following:

1. All $|\Gamma|$-circles are centered at the origin, and their radii vary uniformly from 0 to 1.
2. The angle, measured from the positive real axis, of the line drawn from the origin through the point representing z_L equals θ_Γ.
3. The value of the r-circle passing through the intersection of the $|\Gamma|$-circle and the positive-real axis equals the standing-wave ratio S.

So far we have based the construction of the Smith chart on the definition of the voltage reflection coefficient of the load impedance, as given in Eq. (9–134). The input impedance looking toward the load at a distance z' from the load is the ratio of $V(z')$ and $I(z')$. From Eqs. (9–133a) and (9–133b) we have, by writing $j\beta$ for γ for a lossless line,

$$Z_i(z') = \frac{V(z')}{I(z')} = Z_0\left[\frac{1 + \Gamma e^{-j2\beta z'}}{1 - \Gamma e^{-j2\beta z'}}\right]. \tag{9–190}$$

The normalized input impedance is

$$\begin{aligned} z_i = \frac{Z_i}{Z_0} &= \frac{1 + \Gamma e^{-j2\beta z'}}{1 - \Gamma e^{-j2\beta z'}} \\ &= \frac{1 + |\Gamma|e^{j\phi}}{1 - |\Gamma|e^{j\phi}}, \end{aligned} \tag{9–191}$$

where

$$\phi = \theta_\Gamma - 2\beta z'. \tag{9–192}$$

We note that Eq. (9–191) relating z_i and $\Gamma e^{-j2\beta z'} = |\Gamma|e^{j\phi}$ is of exactly the same form as Eq. (9–184) relating z_L and $\Gamma = |\Gamma|e^{j\theta_\Gamma}$. In fact, the latter is a special case of the former for $z' = 0$ ($\phi = \theta_\Gamma$). The magnitude, $|\Gamma|$, of the reflection coefficient and therefore the standing-wave ratio S, are not changed by the additional line length z'. Thus, just as we can use the Smith chart to find $|\Gamma|$ and θ_Γ for a given z_L at the load, we can keep $|\Gamma|$ constant and *subtract* (rotate in the *clockwise* direction) from θ_Γ an angle equal to $2\beta z' = 4\pi z'/\lambda$. This will locate the point for $|\Gamma|e^{j\phi}$, which determines z_i, the normalized input impedance looking into a lossless line of characteristic impedance R_0, length z', and a normalized load impedance z_L. Two additional scales in $\Delta z'/\lambda$ are usually provided along the perimeter of the $|\Gamma| = 1$ circle for easy reading of the

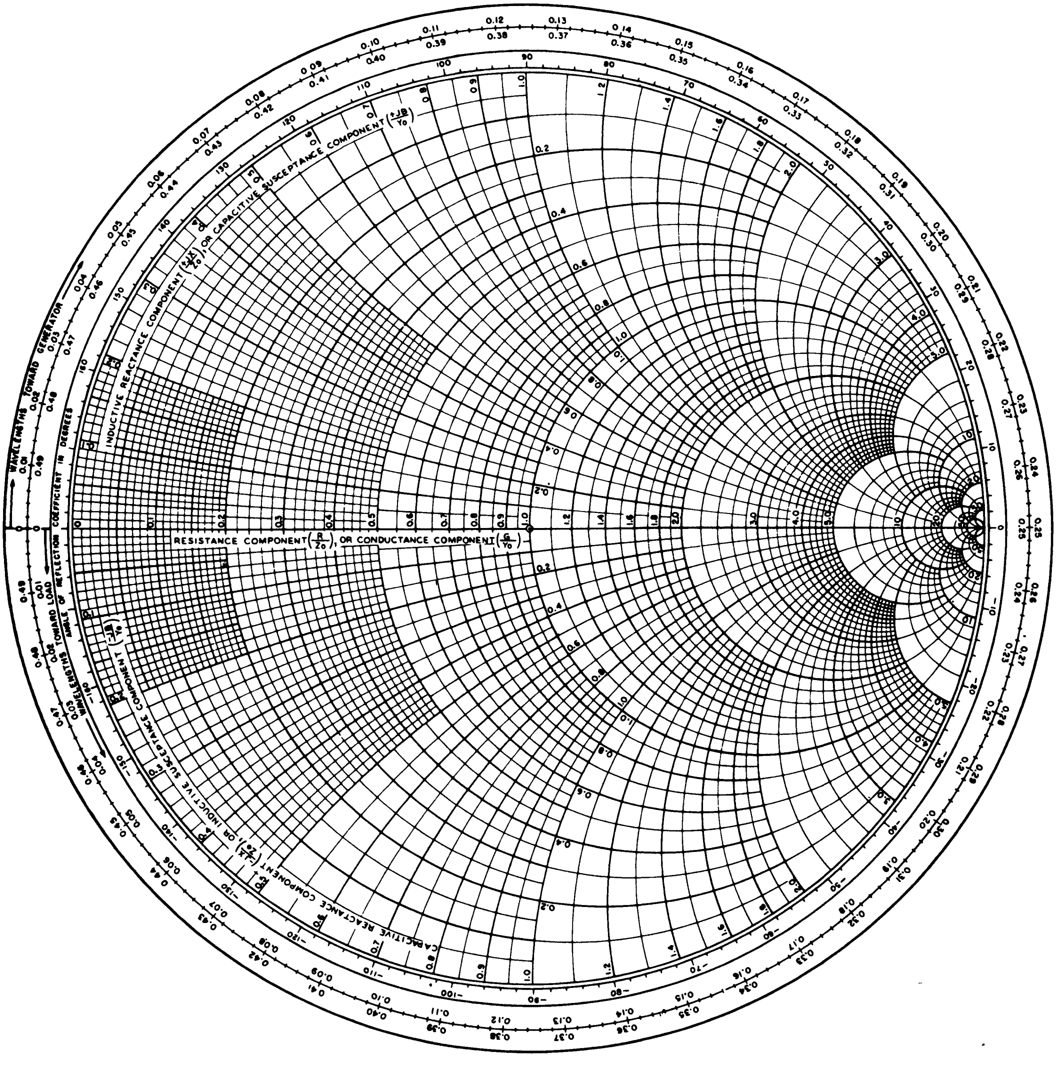


FIGURE 9–32
The Smith chart.

phase change $2\beta(\Delta z')$ due to a change in line length $\Delta z'$: The outer scale is marked "wavelengths toward generator" in the clockwise direction (increasing z'); and the inner scale is marked "wavelengths toward load" in the counterclockwise direction (decreasing z'). Figure 9–32 is a typical Smith chart, which is commercially available.† It has a complicated appearance, but it actually consists merely of constant-r and constant-x circles. We note that a change of half a wavelength in line length ($\Delta z' = \lambda/2$) corresponds to a $2\beta(\Delta z') = 2\pi$ change in ϕ. A complete revolution around a $|\Gamma|$-circle returns to the same point and results in no change in impedance, as was asserted in Eq. (9–115).

We shall illustrate the use of the Smith chart for solving some typical transmission-line problems by several examples.

EXAMPLE 9–13 Use the Smith chart to find the input impedance of a section of a 50 (Ω) lossless transmission line that is 0.1 wavelength long and is terminated in a short-circuit.

Solution Given

$$z_L = 0,$$
$$R_0 = 50 \quad (\Omega),$$
$$z' = 0.1\lambda.$$

1. Enter the Smith chart at the intersection of $r = 0$ and $x = 0$ (point P_{sc} on the extreme left of chart; see Fig. 9–33).
2. Move along the perimeter of the chart ($|\Gamma| = 1$) by 0.1 "wavelengths toward generator" in a clockwise direction to P_1.
3. At P_1, read $r = 0$ and $x \cong 0.725$, or $z_i = j0.725$. Thus, $Z_i = R_0 z_i = 50(j0.725) = j36.3$ (Ω). (The input impedance is purely inductive.)

This result can be checked readily by using Eq. (9–112):

$$Z_i = jR_0 \tan \beta\ell = j50 \tan\left(\frac{2\pi}{\lambda}\right)0.1\lambda$$
$$= j50 \tan 36° = j36.3 \quad (\Omega).$$

EXAMPLE 9–14 A lossless transmission line of length 0.434λ and characteristic impedance 100 (Ω) is terminated in an impedance $260 + j180$ (Ω). Find (a) the voltage reflection coefficient, (b) the standing-wave ratio, (c) the input impedance, and (d) the location of a voltage maximum on the line.

† All of the Smith charts used in this book are reprinted with permission of Emeloid Industries, Inc., New Jersey.

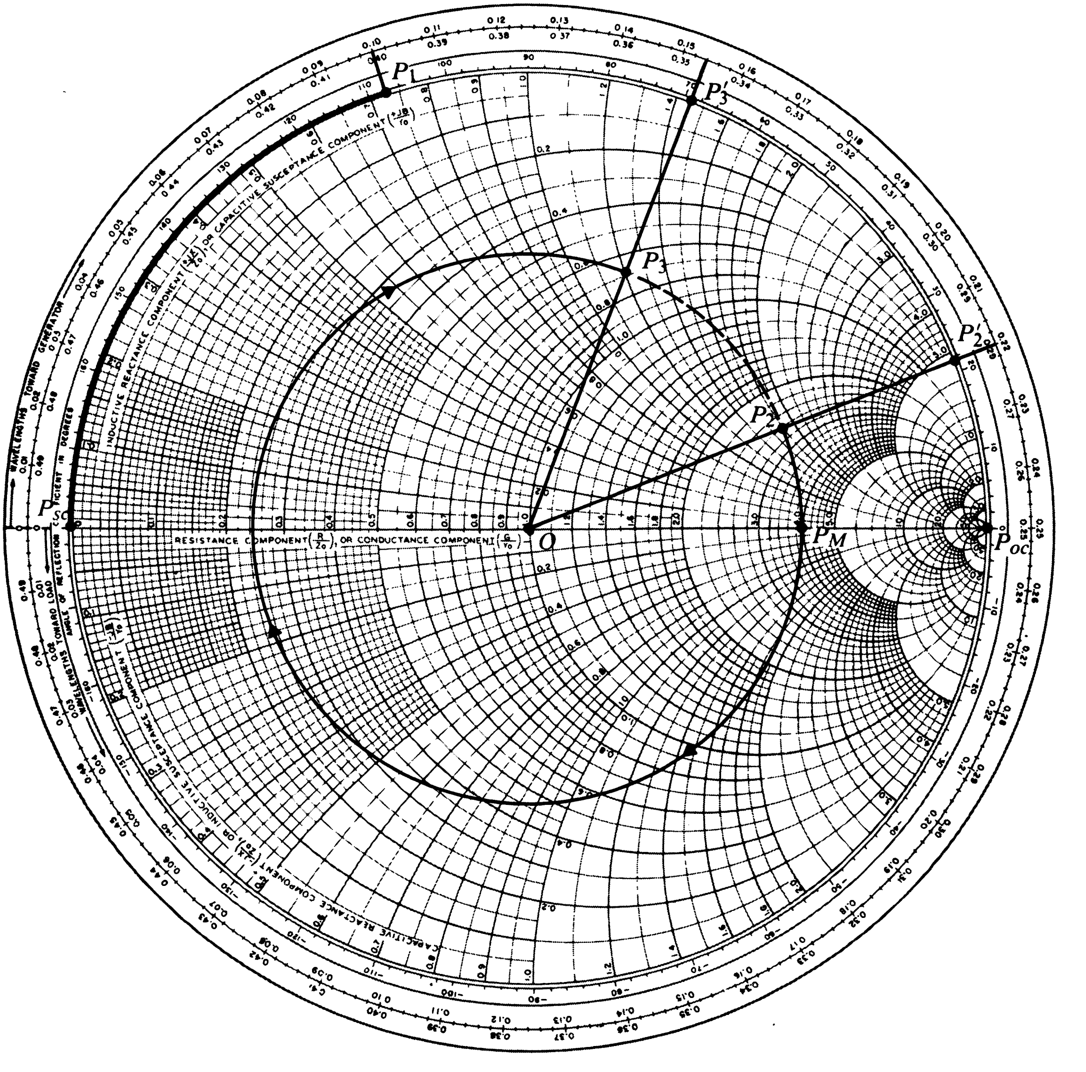


FIGURE 9–33
Smith-chart calculations for Examples 9–13 and 9–14.

Solution Given

$$z' = 0.434\lambda,$$
$$R_0 = 100 \quad (\Omega),$$
$$Z_L = 260 + j180 \quad (\Omega).$$

a) We find the voltage reflection coefficient in several steps:

1. Enter the Smith chart at $z_L = Z_L/R_0 = 2.6 + j1.8$ (point P_2 in Fig. 9–33).

2. With the center at the origin, draw a circle of radius $\overline{OP}_2 = |\Gamma| = 0.60$. (The radius of the chart $\overline{OP}_{sc}$ equals unity.)
3. Draw the straight line OP_2 and extend it to P_2' on the periphery. Read 0.220 on "wavelengths toward generator" scale. The phase angle θ_Γ of the reflection coefficient is $(0.250 - 0.220) \times 4\pi = 0.12\pi$ (rad) or 21°. (We multiply the change in wavelengths by 4π because angles on the Smith chart are measured in $2\beta z'$ or $4\pi z'/\lambda$. A half-wavelength change in line length corresponds to a complete revolution on the Smith chart.) The answer to part (a) is then

$$\Gamma = |\Gamma|e^{j\theta_\Gamma} = 0.60\underline{/21^\circ}.$$

b) The $|\Gamma| = 0.60$ circle intersects with the positive-real axis OP_{oc} at $r = S = 4$. Thus the voltage standing-wave ratio is 4.

c) To find the input impedance, we proceed as follows:

1. Move P_2' at 0.220 by a total of 0.434 "wavelengths toward generator," first to 0.500 (same as 0.000) and then further to 0.154[(0.500 − 0.220) + 0.154 = 0.434] to P_3'.
2. Join O and P_3' by a straight line which intersects the $|\Gamma| = 0.60$ circle at P_3.
3. Read $r = 0.69$ and $x = 1.2$ at P_3. Hence,

$$Z_i = R_0 z_i = 100(0.69 + j1.2) = 69 + j120 \quad (\Omega).$$

d) In going from P_2 to P_3, the $|\Gamma| = 0.60$ circle intersects the positive-real axis OP_{oc} at P_M where the voltage is a maximum. Thus a voltage maximum appears at $(0.250 - 0.220)\lambda$ or 0.030λ from the load.

EXAMPLE 9–15 Solve Example 9–9 by using the Smith chart. Given

$$\begin{aligned} R_0 &= 50 \quad (\Omega), \\ S &= 3.0, \\ \lambda &= 2 \times 0.2 = 0.4 \quad \text{(m)}, \\ \text{First voltage minimum at } z_m' &= 0.05 \text{ (m)}, \end{aligned}$$

find (a) Γ, (b) Z_L, (c) ℓ_m, and R_m (Fig. 9–12).

Solution

a) On the positive-real axis OP_{oc}, locate the point P_M at which $r = S = 3.0$ (see Fig. 9–34). Then $\overline{OP}_M = |\Gamma| = 0.5$ ($\overline{OP}_{oc} = 1.0$). We cannot find θ_Γ until we have located the point that represents the normalized load impedance.

b) We use the following procedure to find the load impedance on the Smith chart:

1. Draw a circle centered at the origin with radius $\overline{OP}_M$, which intersects with the negative-real axis OP_{sc} at P_m where there will be a voltage minimum.
2. Since $z_m'/\lambda = 0.05/0.4 = 0.125$, move from P_{sc} 0.125 "wavelengths toward load" in the counterclockwise direction to P_L'.
3. Join O and P_L' by a straight line, intersecting the $|\Gamma| = 0.5$ circle at P_L. This is the point representing the normalized load impedance.

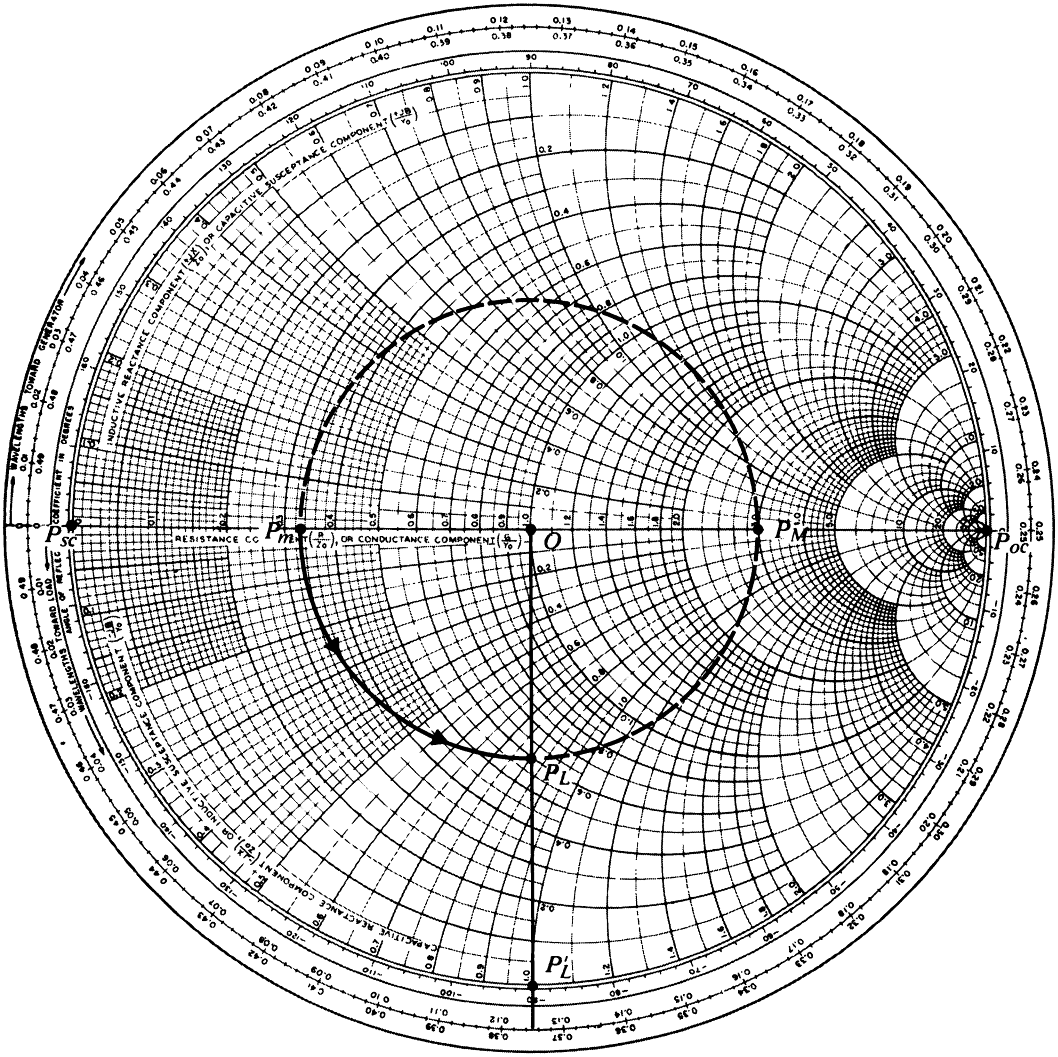


FIGURE 9–34
Smith-chart calculations for Example 9–15.

4. Read the angle $\angle P_{oc}OP'_L = 90° = \pi/2$ (rad). There is no need to use a protractor because $\angle P_{oc}OP'_L = 4\pi(0.250 - 0.125) = \pi/2$. Hence $\theta_\Gamma = -\pi/2$ (rad), or $\Gamma = 0.5\underline{/-90°} = -j0.5$.
5. Read at P_L, $z_L = 0.60 - j0.80$, which gives

$$Z_L = 50(0.60 - j0.80) = 30 - j40 \quad (\Omega).$$

c) The equivalent line length and the terminating resistance can be found easily:

$$\ell_m = \frac{\lambda}{2} - z'_m = 0.2 - 0.05 = 0.15 \quad \text{(m)},$$

$$R_m = \frac{R_0}{S} = \frac{50}{3} = 16.7 \quad (\Omega).$$

All the above results are the same as those obtained in Example 9–9, but no calculations with complex numbers are needed in using the Smith chart.

9–6.1 SMITH-CHART CALCULATIONS FOR LOSSY LINES

In discussing the use of the Smith chart for transmission-line calculations we have assumed the line to be lossless. This is normally a satisfactory approximation, since we generally deal with relatively short sections of low-loss lines. The lossless assumption enables us to say, following Eq. (9–191), that the magnitude of the $\Gamma e^{-j2\beta z'}$ term does not change with line length z' and that we can find z_i from z_L, and vice versa, by moving along the $|\Gamma|$-circle by an angle equal to $2\beta z'$.

For a lossy line of a sufficient length ℓ, such that $2\alpha\ell$ is not negligible in comparison to unity, Eq. (9–191) must be amended to read

$$\begin{aligned} z_i &= \frac{1 + \Gamma e^{-2\alpha z'} e^{-j2\beta z'}}{1 - \Gamma e^{-2\alpha z'} e^{-j2\beta z'}} \\ &= \frac{1 + |\Gamma| e^{-2\alpha z'} e^{j\phi}}{1 - |\Gamma| e^{-2\alpha z'} e^{j\phi}}, \qquad \phi = \theta_\Gamma - 2\beta z'. \end{aligned} \tag{9–193}$$

Hence, to find z_i from z_L, we cannot simply move along the $|\Gamma|$-circle; auxiliary calculations are necessary to account for the $e^{-2\alpha z'}$ factor. The following example illustrates what has to be done.

EXAMPLE 9–16 The input impedance of a short-circuited lossy transmission line of length 2 (m) and characteristic impedance 75 (Ω) (approximately real) is $45 + j225$ (Ω). (a) Find α and β of the line. (b) Determine the input impedance if the short-circuit is replaced by a load impedance $Z_L = 67.5 - j45$ (Ω).

Solution

a) The short-circuit load is represented by the point P_{sc} on the extreme left of the Smith impedance chart.

1. Enter $z_{i1} = (45 + j225)/75 = 0.60 + j3.0$ in the chart as P_1 (Fig. 9–35).
2. Draw a straight line from the origin O through P_1 to P'_1.
3. Measure $\overline{OP_1}/\overline{OP'_1} = 0.89 = e^{-2\alpha\ell}$. It follows that

$$\alpha = \frac{1}{2\ell} \ln\left(\frac{1}{0.89}\right) = \frac{1}{4} \ln 1.124 = 0.029 \quad \text{(Np/m)}.$$

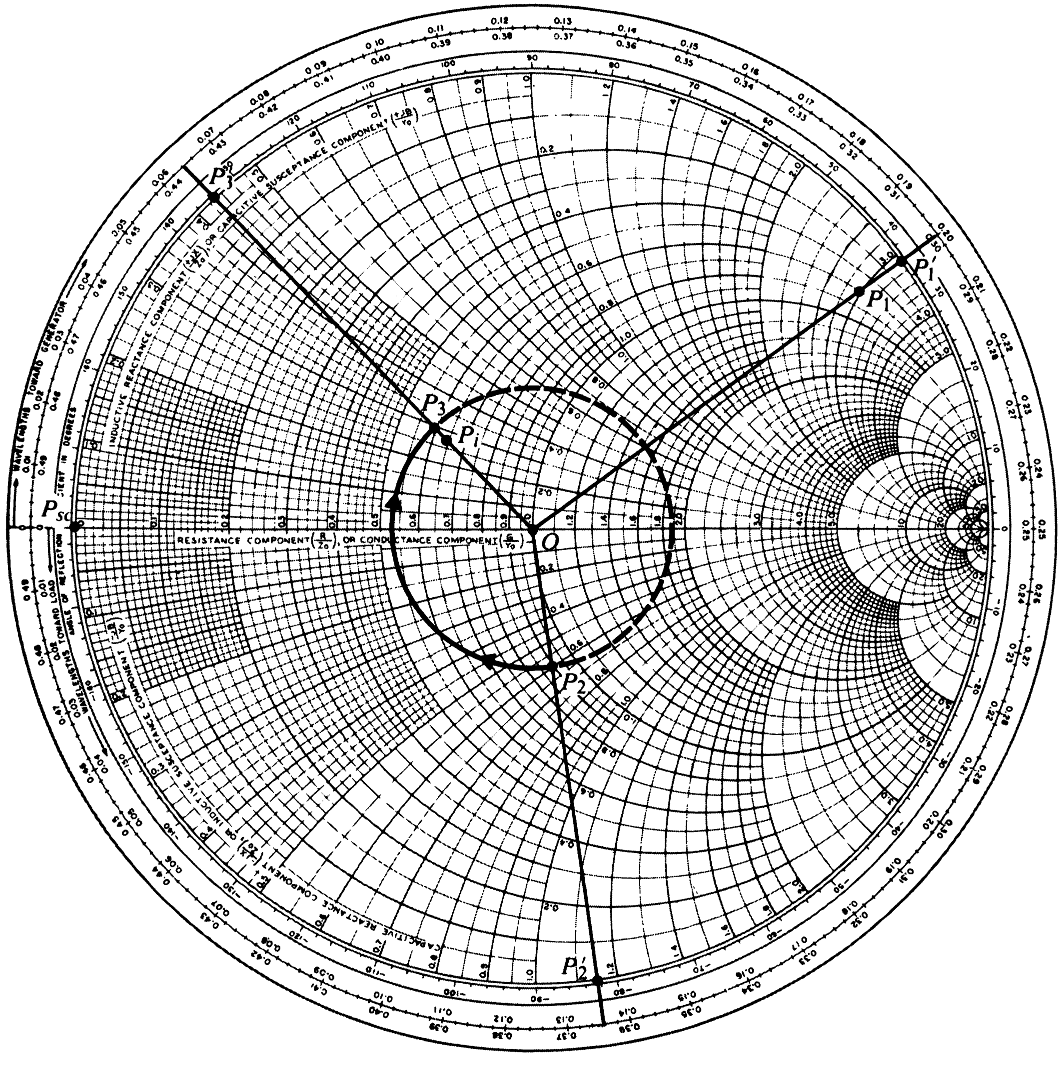


FIGURE 9–35
Smith-chart calculations for a lossy transmission line (Example 9–16).

4. Record that the arc $P_{sc}P_1'$ is 0.20 "wavelengths toward generator." We have $\ell/\lambda = 0.20$ and $2\beta\ell = 4\pi\ell/\lambda = 0.8\pi$. Thus,

$$\beta = \frac{0.8\pi}{2\ell} = \frac{0.8\pi}{4} = 0.2\pi \quad \text{(rad/m)}.$$

b) To find the input impedance for $Z_L = 67.5 - j45\ (\Omega)$:

1. Enter $z_L = Z_L/Z_0 = (67.5 - j45)/75 = 0.9 - j0.6$ on the Smith chart as P_2.

2. Draw a straight line from O through P_2 to P_2' where the "wavelengths toward generator" reading is 0.364.
3. Draw a $|\Gamma|$-circle centered at O with radius $\overline{OP_2}$.
4. Move P_2' along the perimeter by 0.20 "wavelengths toward generator" to P_3' at $0.364 + 0.20 = 0.564$ or 0.064.
5. Join P_3' and O by a straight line, intersecting the $|\Gamma|$-circle at P_3.
6. Mark on line OP_3 a point P_i such that $\overline{OP_i}/\overline{OP_3} = e^{-2\alpha\ell} = 0.89$.
7. At P_i, read $z_i = 0.64 + j0.27$. Hence,

$$Z_i = 75(0.64 + j0.27) = 48.0 + j20.3 \quad (\Omega).$$

9–7 Transmission-Line Impedance Matching

Transmission lines are used for the transmission of power and information. For radio-frequency power transmission it is highly desirable that as much power as possible is transmitted from the generator to the load and as little power as possible is lost on the line itself. This will require that the load be matched to the characteristic impedance of the line so that the standing-wave ratio on the line is as close to unity as possible. For information transmission it is essential that the lines be matched because reflections from mismatched loads and junctions will result in echoes and will distort the information-carrying signal. In this section we discuss several methods for impedance-matching on lossless transmission lines. We note parenthetically that the methods we develop will be of little consequence to power transmission by 60 (Hz) lines inasmuch as these lines are generally very short in comparison to the 5 (Mm) wavelength and the line losses are appreciable. Sixty-hertz power-line circuits are usually analyzed in terms of equivalent lumped electrical networks.

9–7.1 IMPEDANCE MATCHING BY QUARTER-WAVE TRANSFORMER

A simple method for matching a resistive load R_L to a lossless transmission line of a characteristic impedance R_0 is to insert a quarter-wave transformer with a characteristic impedance R_0' such that

$$R_0' = \sqrt{R_0 R_L}. \tag{9-194}$$

Since the length of the quarter-wave line depends on wavelength, this matching method is frequency-sensitive, as are all the other methods to be discussed.

EXAMPLE 9–17 A signal generator is to feed equal power through a lossless air transmission line with a characteristic impedance 50 (Ω) to two separate resistive loads, 64 (Ω) and 25 (Ω). Quarter-wave transformers are used to match the loads to the 50 (Ω) line, as shown in Fig. 9–36. (a) Determine the required characteristic impedances of the quarter-wave lines. (b) Find the standing-wave ratios on the matching line sections.

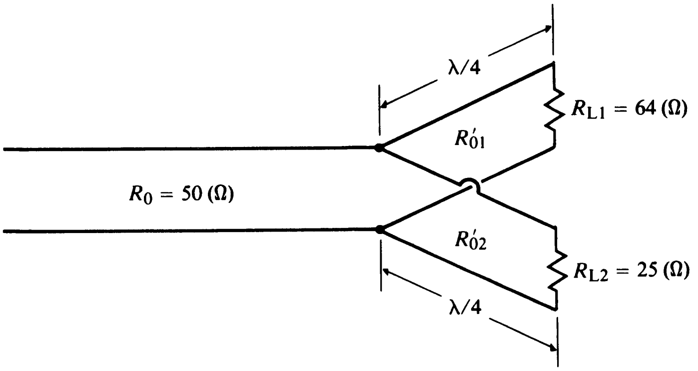


FIGURE 9–36
Impedance matching by quarter-wave lines (Example 9–17).

Solution

a) To feed equal power to the two loads, the input resistance at the junction with the main line looking toward each load must be equal to $2R_0$. $R_{i1} = R_{i2} = 2R_0 = 100\ (\Omega)$:

$$R'_{01} = \sqrt{R_{i1}R_{L1}} = \sqrt{100 \times 64} = 80 \quad (\Omega),$$
$$R'_{02} = \sqrt{R_{i2}R_{L2}} = \sqrt{100 \times 25} = 50 \quad (\Omega).$$

b) Under matched conditions there are no standing waves on the main transmission line ($S = 1$). The standing-wave ratios on the two matching line sections are as follows.

Matching section No. 1:

$$\Gamma_1 = \frac{R_{L1} - R'_{01}}{R_{L1} + R'_{01}} = \frac{64 - 80}{64 + 80} = -0.11,$$
$$S_1 = \frac{1 + |\Gamma_1|}{1 - |\Gamma_1|} = \frac{1 + 0.11}{1 - 0.11} = 1.25.$$

Matching section No. 2:

$$\Gamma_2 = \frac{R_{L2} - R'_{02}}{R_{L2} + R'_{02}} = \frac{25 - 50}{25 + 50} = -0.33,$$
$$S_2 = \frac{1 + |\Gamma_2|}{1 - |\Gamma_2|} = \frac{1 + 0.33}{1 - 0.33} = 1.99.$$

Ordinarily, the main transmission line and the matching line sections are essentially lossless. In that case, both R_0 and R'_0 are purely real, and Eq. (9–194) will have no solution if R_L is replaced by a complex Z_L. Hence quarter-wave transformers are not useful for matching a complex load impedance to a low-loss line.

In the following subsection we will discuss a method for matching an arbitrary load impedance to a line by using a single open- or short-circuited line section (a single stub) in parallel with the main line and at an appropriate distance from the load. Since it is more convenient to use admittances instead of impedances for parallel connections, we first examine how the Smith chart can be used to make admittance calculations.

Let $Y_L = 1/Z_L$ denote the load admittance. The normalized load impedance is

$$z_L = \frac{Z_L}{R_0} = \frac{1}{R_0 Y_L} = \frac{1}{y_L}, \tag{9–195}$$

where

$$\begin{aligned} y_L &= Y_L/Y_0 = Y_L/G_0 \\ &= R_0 Y_L = g + jb \qquad \text{(Dimensionless)} \end{aligned} \tag{9–196}$$

is the normalized load admittance having normalized conductance g and normalized susceptance b as its real and imaginary parts, respectively. Equation (9–195) suggests that a quarter-wave line with a unity normalized characteristic impedance will transform z_L to y_L, and vice versa. On the Smith chart we need only move the point representing z_L along the $|\Gamma|$-circle by a quarter-wavelength to locate the point representing y_L. Since a $\lambda/4$-change in line length ($\Delta z'/\lambda = \frac{1}{4}$) corresponds to a change of π radians ($2\beta\,\Delta z' = \pi$) on the Smith chart, ***the points representing z_L and y_L are*** then ***diametrically opposite to each other on the $|\Gamma|$-circle.*** This observation enables us to find y_L from z_L, and z_L from y_L, on the Smith chart in a very simple manner.

EXAMPLE 9–18 Given $Z_L = 95 + j20$ (Ω), find Y_L.

Solution This problem has nothing to do with any transmission line. In order to use the Smith chart we can choose an arbitrary normalizing constant; for instance, $R_0 = 50$ (Ω). Thus,

$$z_L = \tfrac{1}{50}(95 + j20) = 1.9 + j0.4.$$

Enter z_L as point P_1 on the Smith chart in Fig. 9–37. The point P_2 on the other side of the line joining P_1 and O represents y_L: $\overline{OP_2} = \overline{OP_1}$.

$$Y_L = \frac{1}{R_0} y_L = \frac{1}{50}(0.5 - j0.1) = 10 - j2 \quad \text{(mS)}.$$

EXAMPLE 9–19 Find the input admittance of an open-circuited line of characteristic impedance 300 (Ω) and length 0.04λ.

Solution

1. For an open-circuited line we start from the point P_{oc} on the extreme right of the impedance Smith chart, at 0.25 in Fig. 9–38.
2. Move along the perimeter of the chart by 0.04 "wavelengths toward generator" to P_3 (at 0.29).
3. Draw a straight line from P_3 through O, intersecting at P_3' on the opposite side.

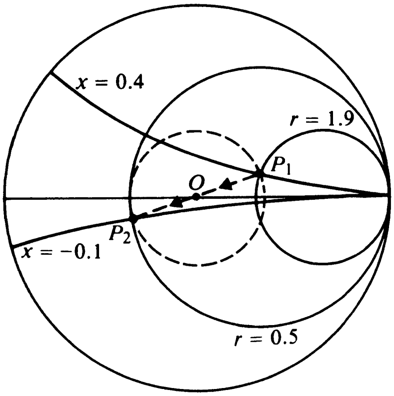


FIGURE 9–37
Finding admittance from impedance (Example 9–18).

4. Read at P'_3

$$y_i = 0 + j0.26.$$

Thus,

$$Y_i = \frac{1}{300}(0 + j0.26) = j0.87 \quad \text{(mS)}.$$

In the preceding two examples we have made admittance calculations by using the Smith chart as an impedance chart. The Smith chart can also be used as an admittance chart, in which case the r- and x-circles would be g- and b-circles. The points representing an open- and a short-circuit termination would be the points on the extreme left and the extreme right, respectively, on an admittance chart. For Example 9–19, we could then start from extreme left point on the chart, at 0.00 in Fig. 9–38, and move 0.04 "wavelengths toward generator" to P'_3 directly.

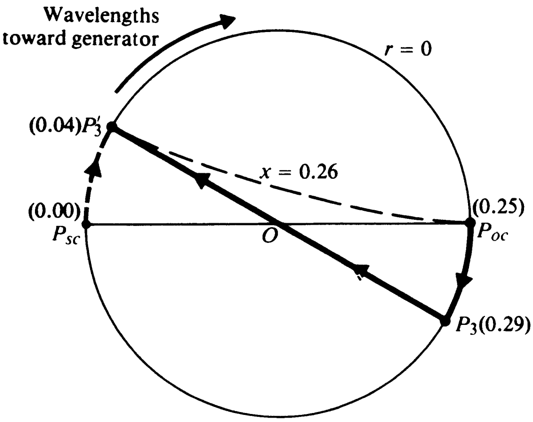


FIGURE 9–38
Finding input admittance of open-circuited line (Example 9–19).

9-7.2 SINGLE-STUB MATCHING

We now tackle the problem of matching a load impedance Z_L to a lossless line that has a characteristic impedance R_0 by placing a single short-circuited stub in parallel with the line, as shown in Fig. 9-39. This is the ***single-stub method*** for impedance matching. We need to determine the length of the stub, ℓ, and the distance from the load, d, such that the impedance of the parallel combination to the right of points B–B' equals R_0. Short-circuited stubs are usually used in preference to open-circuited stubs because an infinite terminating impedance is more difficult to realize than a zero terminating impedance for reasons of radiation from an open end and coupling effects with neighboring objects. Moreover, a short-circuited stub of an adjustable length and a constant characteristic resistance is much easier to construct than an open-circuited one. Of course, the difference in the required length for an open-circuited stub and that for a short-circuited stub is an odd multiple of a quarter-wavelength.

The parallel combination of a line terminated in Z_L and a stub at points B–B' in Fig. 9-39 suggest that it is advantageous to analyze the matching requirements in terms of admittances. The basic requirement is

$$\begin{aligned} Y_i &= Y_B + Y_s \\ &= Y_0 = \frac{1}{R_0}. \end{aligned} \tag{9-197}$$

In terms of normalized admittances, Eq. (9-197) becomes

$$1 = y_B + y_s, \tag{9-198}$$

where $y_B = R_0 Y_B$ is for the load section and $y_s = R_0 Y_s$ is for the short-circuited stub. However, since the input admittance of a short-circuited stub is purely susceptive, y_s is purely imaginary. As a consequence, Eq. (9-198) can be satisfied only if

$$y_B = 1 + jb_B \tag{9-199}$$

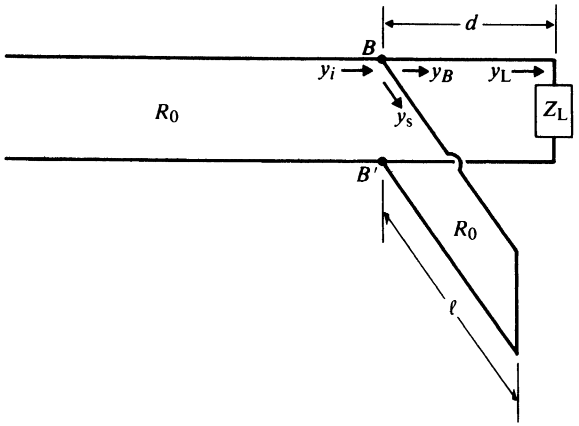


FIGURE 9-39
Impedance matching by single-stub method.

and

$$y_s = -jb_B, \tag{9-200}$$

where b_B can be either positive or negative. Our objectives, then, are to find the length d such that the admittance, y_B, of the load section looking to the right of terminals B–B' has a *unity real part* and to find the length ℓ_B of the stub required to *cancel the imaginary part.*

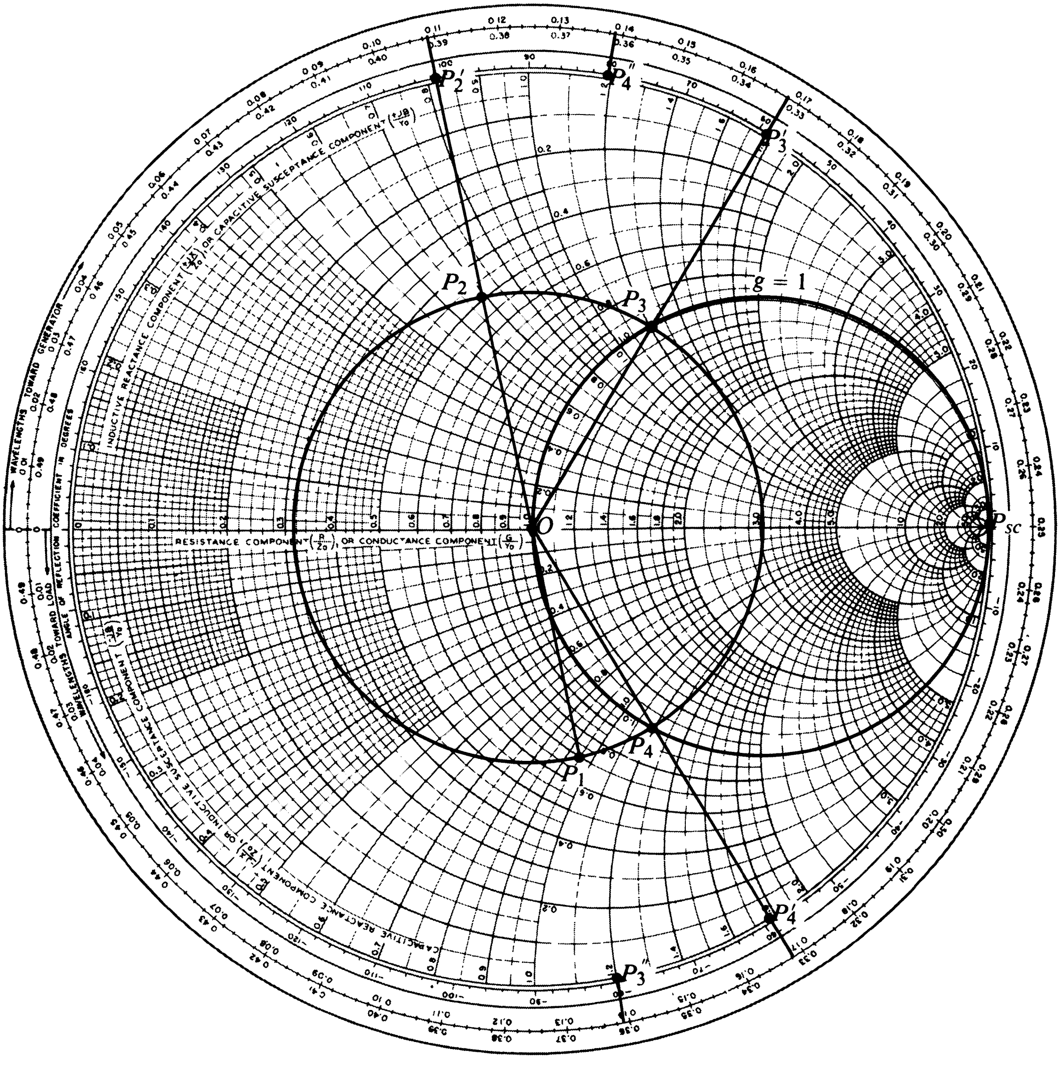


FIGURE 9–40
Construction for single-stub matching on Smith admittance chart (Example 9–20).

Using the Smith chart as an admittance chart, we proceed as follows for single-stub matching:

1. Enter the point representing the normalized load admittance y_L.
2. Draw the $|\Gamma|$-circle for y_L, which will intersect the $g = 1$ circle at two points. At these points, $y_{B1} = 1 + jb_{B1}$ and $y_{B2} = 1 + jb_{B2}$. Both are possible solutions.
3. Determine load-section lengths d_1 and d_2 from the angles between the point representing y_L and the points representing y_{B1} and y_{B2}.
4. Determine stub lengths ℓ_{B1} and ℓ_{B2} from the angles between the short-circuit point on the extreme right of the chart to the points representing $-jb_{B1}$ and $-jb_{B2}$, respectively.

The following example will illustrate the necessary steps.

EXAMPLE 9–20 A 50 (Ω) transmission line is connected to a load impedance $Z_L = 35 - j47.5$ (Ω). Find the position and length of a short-circuited stub required to match the line.

Solution Given

$$R_0 = 50\ (\Omega)$$
$$Z_L = 35 - j47.5\ (\Omega)$$
$$z_L = Z_L/R_0 = 0.70 - j0.95.$$

1. Enter z_L on the Smith chart as P_1 (Fig. 9–40).
2. Draw a $|\Gamma|$-circle centered at O with radius $\overline{OP}_1$.
3. Draw a straight line from P_1 through O to point P_2' on the perimeter, intersecting the $|\Gamma|$-circle at P_2, which represents y_L. Note 0.109 at P_2' on the "wavelengths toward generator" scale.
4. Note the two points of intersection of the $|\Gamma|$-circle with the $g = 1$ circle.

$$\text{At } P_3\text{:}\quad y_{B1} = 1 + j1.2 = 1 + jb_{B1};$$
$$\text{At } P_4\text{:}\quad y_{B2} = 1 - j1.2 = 1 + jb_{B2}.$$

5. Solutions for the position of the stub:

$$\text{For } P_3 \text{ (from } P_2' \text{ to } P_3'\text{):}\quad d_1 = (0.168 - 0.109)\lambda = 0.059\lambda;$$
$$\text{For } P_4 \text{ (from } P_2' \text{ to } P_4'\text{):}\quad d_2 = (0.332 - 0.109)\lambda = 0.223\lambda.$$

6. Solutions for the length of short-circuited stub to provide $y_s = -jb_B$:

For P_3 (from P_{sc} on the extreme right of chart to P_3'', which represents $-jb_{B1} = -j1.2$):

$$\ell_{B1} = (0.361 - 0.250)\lambda = 0.111\lambda;$$

For P_4 (from P_{sc} to P_4'', which represents $-jb_{B2} = j1.2$):

$$\ell_{B2} = (0.139 + 0.250)\lambda = 0.389\lambda.$$

In general, the solution with the shorter lengths is preferred unless there are other practical constraints. The exact length, ℓ_B, of the short-circuited stub may require fine adjustments in the actual matching procedure; hence the shorted matching sections are sometimes called ***stub tuners.***

The use of Smith chart in solving impedance-matching problems avoids the manipulation of complex numbers and the computation of tangent and arc-tangent functions; but graphical constructions are needed, and graphical methods have limited accuracy. Actually, the analytical solutions of impedance-matching problems are relatively simple, and easy access to a computer may diminish the reliance on the Smith chart and, at the same time, yield more accurate results.

For the single-stub matching problem illustrated in Fig. 9–39 we have, from Eq. (9–109).

$$z_B = \frac{(r_L + jx_L) + jt}{1 + j(r_L + jx_L)t}, \tag{9–201}$$

where

$$t = \tan \beta d. \tag{9–202}$$

The normalized input admittance to the right of points B–B' is

$$y_B = \frac{1}{z_B} = g_B + jb_B, \tag{9–203}$$

where

$$g_B = \frac{r_L(1 - x_L t) + r_L t(x_L + t)}{r_L^2 + (x_L + t)^2} \tag{9–204}$$

and

$$b_B = \frac{r_L^2 t - (1 - x_L t)(x_L + t)}{r_L + (x_L + t)^2}. \tag{9–205}$$

A perfect match requires the simultaneous satisfaction of Eqs. (9–199) and (9–200). Equating g_B in Eq. (9–204) to unity, we have

$$(r_L - 1)t^2 - 2x_L t + (r_L - r_L^2 - x_L^2) = 0. \tag{9–206}$$

Solving Eq. (9–206), we obtain

$$t = \begin{cases} \dfrac{1}{r_L - 1}\{x_L \pm \sqrt{r_L[(1 - r_L)^2 + x_L^2]}\}, & r_L \neq 1, \quad \text{(9–207a)} \\[2ex] -\dfrac{x_L}{2}, & r_L = 1. \quad \text{(9–207b)} \end{cases}$$

The required length d can be found from Eqs. (9–202), (9–207a), and (9–207b):

$$\frac{d}{\lambda} = \begin{cases} \dfrac{1}{2\pi} \tan^{-1} t, & t \geq 0, \quad \text{(9–208a)} \\[2ex] \dfrac{1}{2\pi}(\pi + \tan^{-1} t), & t < 0. \quad \text{(9–208b)} \end{cases}$$

Similarly, from Eqs. (9–200) and (9–205), we obtain

$$\frac{\ell}{\lambda} = \begin{cases} \dfrac{1}{2\pi}\tan^{-1}\left(\dfrac{1}{b_B}\right), & b_B \geq 0, \quad (9\text{–}209\text{a}) \\[2ex] \dfrac{1}{2\pi}\left[\pi + \tan^{-1}\left(\dfrac{1}{b_B}\right)\right], & b_B < 0. \quad (9\text{–}209\text{b}) \end{cases}$$

For a given load impedance, both d/λ and ℓ/λ can be determined easily on a scientific calculator. It is also a simple matter to write a general computer program for the single-stub matching problem. More accurate answers to the problem in Example 9–20 ($r_L = 0.70$ and $x_L = -0.95$) are

$$d_1 = 0.05894469\lambda, \qquad \ell_{B1} = 0.11117792\lambda,$$
$$d_2 = 0.22347730\lambda, \qquad \ell_{B2} = 0.38882208\lambda.$$

Of course, such accuracies are seldom needed in an actual problem; but these answers have been obtained easily without a Smith chart.

9–7.3 DOUBLE-STUB MATCHING

The method of impedance matching by means of a single stub described in the preceding subsection can be used to match any arbitrary, nonzero, finite load impedance to the characteristic resistance of a line. However, the single-stub method requires that the stub be attached to the main line at a specific point, which varies as the load impedance or the operating frequency is changed. This requirement often presents practical difficulties because the specified junction point may occur at an undesirable location from a mechanical viewpoint. Furthermore, it is very difficult to build a variable-length coaxial line with a constant characteristic impedance. In such cases an alternative method for impedance-matching is to use two short-circuited stubs attached to the main line at fixed positions, as shown in Fig. 9–41. Here, the distance d_o is fixed and arbitrarily chosen (such as $\lambda/16$, $\lambda/8$, $3\lambda/16$, $3\lambda/8$, etc.), and the lengths of the two stub tuners are adjusted to match a given load impedance Z_L to the main line. This scheme is the ***double-stub method*** for impedance matching.

In the arrangement in Fig. 9–41 a stub of length ℓ_A is connected directly in parallel with the load impedance Z_L at terminals A–A', and a second stub of length ℓ_B is attached at terminals B–B' at a fixed distance d_o away. For impedance matching with a main line that has a characteristic resistance R_0, we demand the total input admittance at terminals B–B', looking toward the load, to equal the characteristic conductance of the line; that is,

$$Y_i = Y_B + Y_{sB} = Y_0 = \frac{1}{R_0}. \tag{9–210}$$

In terms of normalized admittances, Eq. (9–210) becomes

$$1 = y_B + y_{sB}. \tag{9–211}$$

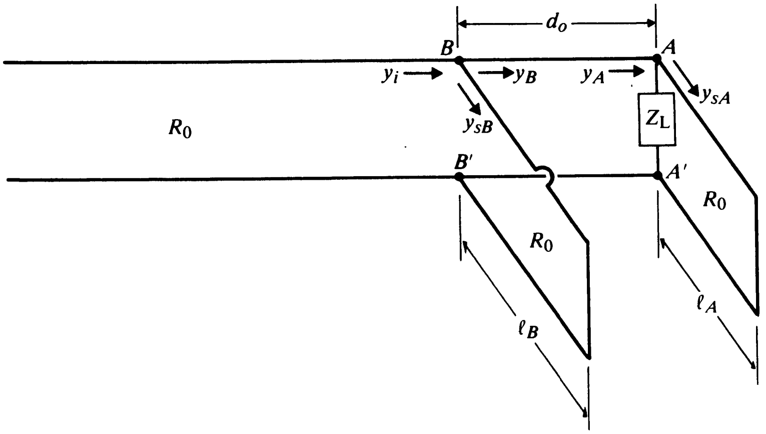


FIGURE 9–41
Impedance matching by double-stub method.

Now, since the input admittance y_{sB} of a short-circuited stub is purely imaginary, Eq. (9–211) can be satisfied only if

$$y_B = 1 + jb_B \tag{9–212}$$

and

$$y_{sB} = -jb_B. \tag{9–213}$$

Note that these requirements are exactly the same as those for single-stub matching.

On the Smith admittance chart the point representing y_B must lie on the $g = 1$ circle. This requirement must be translated by a distance d_o/λ "wavelengths toward load"; that is, y_A at terminals A–A' must lie on the $g = 1$ circle rotated by an angle $4\pi d_o/\lambda$ in the counterclockwise direction. Again, since the input admittance y_{sA} of the short-circuited stub is purely imaginary, the real part of y_A must be solely contributed by the real part of the normalized load admittance, g_L. The solution (or solutions) of the double-stub matching problem is then determined by the intersection (or intersections) of the g_L-circle with the rotated $g = 1$ circle. The procedure for solving a double-stub matching problem on the Smith admittance chart is as follows.

1. Draw the $g = 1$ circle. This is where the point representing y_B should be located.
2. Draw this circle rotated in the counterclockwise direction by d_o/λ "wavelengths toward load." This is where the point representing y_A should be located.
3. Enter the $y_L = g_L + jb_L$ point.
4. Draw the $g = g_L$ circle, intersecting the rotated $g = 1$ circle at one or two points where $y_A = g_L + jb_A$.
5. Mark the corresponding y_B-points on the $g = 1$ circle: $y_B = 1 + jb_B$.
6. Determine stub length ℓ_A from the angle between the point representing y_A and the point representing y_L.

7. Determine stub length ℓ_B from the angle between the point representing $-jb_B$ and P_{sc} on the extreme right.

EXAMPLE 9–21 A 50 (Ω) transmission line is connected to a load impedance $Z_L = 60 + j80$ (Ω). A double-stub tuner spaced an eighth of a wavelength apart is used to match the load to the line, as shown in Fig. 9–41. Find the required lengths of the short-circuited stubs.

Solution Given $R_0 = 50$ (Ω) and $Z_L = 60 + j80$ (Ω), it is easy to calculate

$$y_L = \frac{1}{z_L} = \frac{R_0}{Z_L} = \frac{50}{60 + j80} = 0.30 - j0.40.$$

(We could find y_L on the Smith chart by locating the point diametrically opposite to $z_L = (60 + j80)/50 = 1.20 + j1.60$, but this would clutter up the chart too much.) We follow the procedure outlined above, using a Smith admittance chart.

1. Draw the $g = 1$ circle (Fig. 9–42).
2. Rotate this $g = 1$ circle by $\frac{1}{8}$ "wavelengths toward load" in the counterclockwise direction. The angle of rotation is $4\pi/8$ (rad) or 90°.
3. Enter $y_L = 0.30 - j0.40$ as P_L.
4. Mark the two points of intersection, P_{A1} and P_{A2}, of the $g_L = 0.30$ circle with the rotated $g = 1$ circle.

$$\text{At } P_{A1}, \text{ read } \quad y_{A1} = 0.30 + j0.29;$$
$$\text{At } P_{A2}, \text{ read } \quad y_{A2} = 0.30 + j1.75.$$

5. Use a compass centered at the origin O to mark the points P_{B1} and P_{B2} on the $g = 1$ circle corresponding to the points P_{A1} and P_{A2}, respectively.

$$\text{At } P_{B1}, \text{ read } \quad y_{B1} = 1 + j1.38;$$
$$\text{At } P_{B2}, \text{ read } \quad y_{B2} = 1 - j3.5.$$

6. Determine the required stub lengths ℓ_{A1} and ℓ_{A2} from

$$(y_{sA})_1 = y_{A1} - y_L = j0.69, \qquad \ell_{A1} = (0.096 + 0.250)\lambda = 0.346\lambda \text{ (Point } A_1\text{)},$$
$$(y_{sA})_2 = y_{A2} - y_L = j2.15, \qquad \ell_{A2} = (0.181 + 0.250)\lambda = 0.431\lambda \text{ (Point } A_2\text{)}.$$

7. Determine the required stub lengths ℓ_{B1} and ℓ_{B2} from

$$(y_{sB})_1 = -j1.38, \qquad \ell_{B1} = (0.350 - 0.250)\lambda = 0.100\lambda \text{ (Point } B_1\text{)},$$
$$(y_{sB})_2 = j3.5, \qquad \ell_{B2} = (0.206 + 0.250)\lambda = 0.456\lambda \text{ (Point } B_2\text{)}.$$

Examination of the construction in Fig. 9–42 reveals that if the point P_L, representing the normalized load admittance $y_L = g_L + jb_L$, lies within the $g = 2$ circle (if $g_L > 2$), then the $g = g_L$ circle does not intersect with the rotated $g = 1$ circle, and no solution exists for double-stub matching with $d_o = \lambda/8$. This region for no solution varies with the chosen distance d_o between the stubs (Problem P.9–52). In such cases,

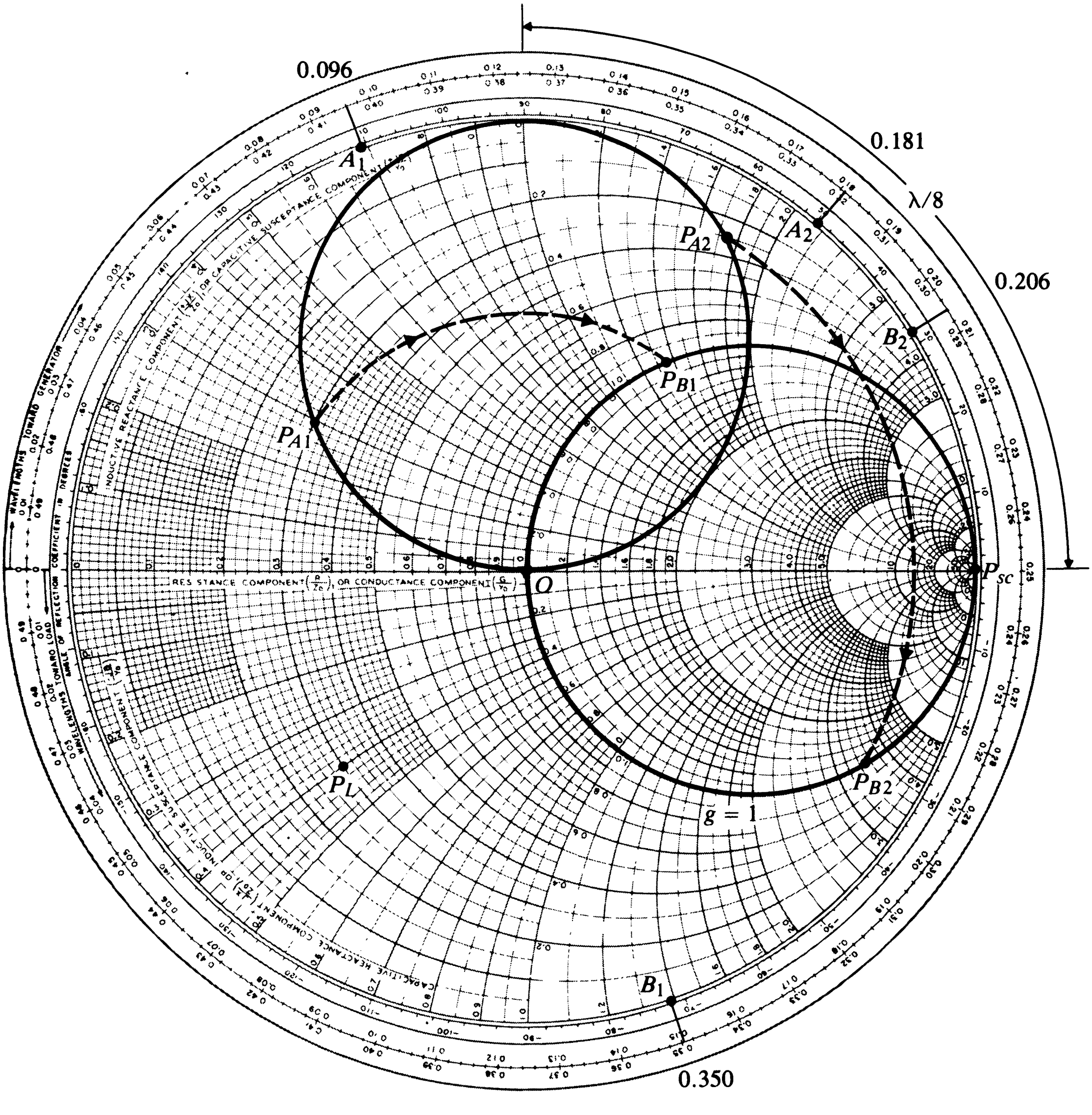


FIGURE 9–42
Construction for double-stub matching on Smith admittance chart.

impedance matching by the double-stub method can be achieved by adding an appropriate line section between Z_L and terminals A–A', as illustrated in Fig. 9–43 (Problem P.9–51).

An analytical solution of the double-stub impedance matching problem is, of course, also possible, albeit more involved than that of the single-stub problem devel-

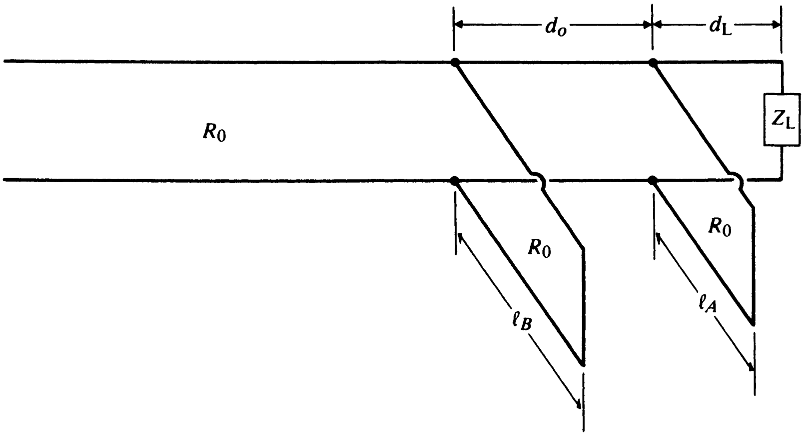


FIGURE 9–43
Double-stub impedance matching with added load-line section.

oped in the preceding subsection. The more ambitious reader may wish to obtain such an analytical solution and write a computer program for determining d_L/λ, ℓ_A/λ, and ℓ_B/λ in terms of z_L and d_o/λ.†

Review Questions

R.9–1 Discuss the similarities and dissimilarities of uniform plane waves in an unbounded media and TEM waves along transmission lines.

R.9–2 What are the three most common types of guiding structures that support TEM waves?

R.9–3 Compare the advantages and disadvantages of coaxial cables and two-wire transmission lines.

R.9–4 Write the transmission-line equations for a lossless parallel-plate line supporting TEM waves.

R.9–5 What are *striplines*?

R.9–6 Describe how the characteristic impedance of a parallel-plate transmission line depends on plate width and dielectric thickness.

R.9–7 Compare the velocity of TEM-wave propagation along a parallel-plate transmission line with that in an unbounded medium.

R.9–8 Define *surface impedance*. How is surface impedance related to the power dissipated in a plate conductor?

† D. K. Cheng and C. H. Liang, "Computer solution of double-stub impedance-matching problems," *IEEE Transactions on Education*, vol. E-25, pp. 120–123, November 1982.

R.9–9 State the difference between the surface resistance and the resistance per unit length of a parallel-plate transmission line.

R.9–10 What is the essential difference between a transmission line and an ordinary electric network?

R.9–11 Explain why waves along a lossy transmission line cannot be purely TEM.

R.9–12 What is a *triplate line*? How does the characteristic impedance of a triplate line compare with that of a corresponding stripline? Explain.

R.9–13 Write the general transmission-line equations for arbitrary time dependence and for time-harmonic time dependence.

R.9–14 Define *propagation constant* and *characteristic impedance* of a transmission line. Write their general expressions in terms of R, L, G, and C for sinusoidal excitation.

R.9–15 What is the phase relationship between the voltage and current waves on an infinitely long transmission line?

R.9–16 What is meant by a "distortionless line"? What relation must the distributed parameters of a line satisfy in order for the line to be distortionless?

R.9–17 Is a distortionless line lossless? Is a lossy transmission line dispersive? Explain.

R.9–18 Outline a procedure for determining the distributed parameters of a transmission line.

R.9–19 Show how the attenuation constant of a transmission line is determined from the propagated power and the power lost in the line per unit length.

R.9–20 What does "matched transmission line" mean?

R.9–21 On what factors does the input impedance of a transmission line depend?

R.9–22 What is the input impedance of an open-circuited lossless transmission line if the length of the line is (a) $\lambda/4$, (b) $\lambda/2$, and (c) $3\lambda/4$?

R.9–23 What is the input impedance of a short-circuited lossless transmission line if the length of the line is (a) $\lambda/4$, (b) $\lambda/2$, and (c) $3\lambda/4$?

R.9–24 Is the input reactance of a transmission line $\lambda/8$ long inductive or capacitive if it is (a) open-circuited, and (b) short-circuited?

R.9–25 On a line of length ℓ, what is the relation between the line's characteristic impedance and propagation constant and its open- and short-circuit input impedances?

R.9–26 What is a "quarter-wave transformer"? Why is it not useful for matching a complex load impedance to a low-loss line?

R.9–27 What is the input impedance of a lossless transmission line of length ℓ that is terminated in a load impedance Z_L if (a) $\ell = \lambda/2$, and (b) $\ell = \lambda$?

R.9–28 Discuss how a section of an open-circuited or short-circuited low-loss transmission line can be used to provide a parallel-resonant circuit.

R.9–29 Define the *bandwidth* and the *quality factor*, Q, of a parallel resonant circuit.

R.9–30 Define *voltage reflection coefficient*. Is it the same as "current reflection coefficient"? Explain.

R.9–31 Define *standing-wave ratio*. How is it related to voltage and current reflection coefficients?

R.9–32 What are Γ and S for a line with an open-circuit termination? A short-circuit termination?

R.9–33 Where do the minima of the voltage standing wave on a lossless line with a resistive termination occur (a) if $R_L > R_0$ and (b) if $R_L < R_0$?

R.9–34 Explain how the value of a terminating resistance can be determined by measuring the standing-wave ratio on a lossless transmission line.

R.9–35 Explain how the value of an arbitrary terminating impedance on a lossless transmission line can be determined by standing-wave measurements on the line.

R.9–36 A voltage generator having an internal impedance Z_g is connected at $t = 0$ to the input terminals of a lossless transmission line of length ℓ. The line has a characteristic impedance Z_0 and is terminated with a load impedance Z_L. At what time will a steady state on the line be reached if (a) $Z_g = Z_0$ and $Z_L = Z_0$, (b) $Z_L = Z_0$ but $Z_g \neq Z_0$, (c) $Z_g = Z_0$ but $Z_L \neq Z_0$, and (d) $Z_g \neq Z_0$ and $Z_L \neq Z_0$?

R.9–37 A battery of voltage V_0 is applied through a series resistance R_g to the input terminals of a lossless transmission line having a characteristic resistance R_0 and a load resistance R_L at the far end. What is the amplitude of the first transient voltage wave traveling from the battery to the load? What is the amplitude of the first reflected voltage wave from the load to the battery?

R.9–38 In Question R.9–37, what are the amplitudes of the first current wave traveling from the battery to the load and the first reflected current wave from the load to the battery?

R.9–39 What are *reflection diagrams* of transmission lines? For what purposes are they useful?

R.9–40 How do the voltage and current reflection diagrams of a terminated line differ?

R.9–41 A d-c voltage is applied to a lossless transmission line. Under what conditions will the transient voltage and current distributions along the line have different shapes? Under what conditions will they have the same shape?

R.9–42 Why is the concept of reflection coefficients not useful in analyzing the transient behavior of a transmission line terminated in a reactive load?

R.9–43 What is a Smith chart and why is it useful in making transmission-line calculations?

R.9–44 Where is the point representing a matched load on a Smith chart?

R.9–45 For a given load impedance Z_L on a lossless line of characteristic impedance Z_0, how do we use a Smith chart to determine (a) the reflection coefficient, and (b) the standing-wave ratio?

R.9–46 Why does a change of half a wavelength in line length correspond to a complete revolution on a Smith chart?

R.9–47 Given an impedance $Z = R + jX$, what procedure do we follow to find the admittance $Y = 1/Z$ on a Smith chart?

R.9–48 Given an admittance $Y = G + jB$, how do we use a Smith chart to find the impedance $Z = 1/Y$?

R.9–49 Where is the point representing a short-circuit on a Smith admittance chart?

R.9–50 Is the standing-wave ratio constant on a transmission line even when the line is lossy? Explain.

R.9–51 Can a Smith chart be used for impedance calculations on a lossy transmission line? Explain.

R.9–52 Why is it more convenient to use a Smith chart as an admittance chart for solving impedance-matching problems than to use it as an impedance chart?

R.9–53 Why is it desirable to achieve an impedance match in a transmission line?

R.9–54 Explain the single-stub method for impedance matching on a transmission line.

R.9–55 Explain the double-stub method for impedance matching on a transmission line.

R.9–56 Compare the relative advantages and disadvantages of the single-stub and the double-stub methods of impedance matching.

R.9–57 Why are the stubs used in impedance matching usually of the short-circuited type instead of the open-circuited type?

Problems

P.9–1 Neglecting fringe fields, prove analytically that a y-polarized TEM wave that propagates along a parallel-plate transmission line in $+z$-direction has the following properties: $\partial E_y/\partial x = 0$ and $\partial H_x/\partial y = 0$.

P.9–2 The electric and magnetic fields of a general TEM wave traveling in the $+z$-direction along a transmission line may have both x- and y-components, and both components may be functions of the transverse dimensions.

a) Find the relations among $E_x(x, y)$, $E_y(x, y)$, $H_x(x, y)$, and $H_y(x, y)$.

b) Verify that all the four field components in part (a) satisfy the two-dimensional Laplace's equation for static fields.

P.9–3 Consider lossless stripline designs for a given characteristic impedance.

a) How should the dielectric thickness, d, be changed for a given plate width, w, if the dielectric constant, ϵ_r, is doubled?

b) How should w be changed for a given d if ϵ_r is doubled?

c) How should w be changed for a given ϵ_r if d is doubled?

d) Will the velocity of propagation remain the same as that for the original line after the changes specified in parts (a), (b), and (c)? Explain.

P.9–4 Consider a transmission line made of two parallel brass strips—$\sigma_c = 1.6 \times 10^7$ (S/m)—of width 20 (mm) and separated by a lossy dielectric slab—$\mu = \mu_0$, $\epsilon_r = 3$, $\sigma = 10^{-3}$ (S/m)—of thickness 2.5 (mm). The operating frequency is 500 MHz.

a) Calculate the R, L, G, and C per unit length.

b) Compare the magnitudes of the axial and transverse components of the electric field.

c) Find γ and Z_0.

P.9–5 Verify Eq. (9–39).

P.9–6 Show that the attenuation and phase constants for a transmission line with perfect conductors separated by a lossy dielectric that has a complex permittivity $\epsilon = \epsilon' - j\epsilon''$ are, respectively,

$$\alpha = \omega\sqrt{\frac{\mu\epsilon'}{2}}\left[\sqrt{1 + \left(\frac{\epsilon''}{\epsilon'}\right)^2} - 1\right]^{1/2} \quad \text{(Np/m)}, \tag{9–214}$$

$$\beta = \omega\sqrt{\frac{\mu\epsilon'}{2}}\left[\sqrt{1 + \left(\frac{\epsilon''}{\epsilon'}\right)^2} + 1\right]^{1/2} \quad \text{(rad/m)}. \tag{9–215}$$

P.9–7 In the derivation of the approximate formulas of γ and Z_0 for low-loss lines in Subsection 9–3.1, all terms containing the second and higher powers of $(R/\omega L)$ and $(G/\omega C)$

were neglected in comparison with unity. At lower frequencies, better approximations than those given in Eqs. (9–54) and (9–58) may be required. Find new formulas for γ and Z_0 for low-loss lines that retain terms containing $(R/\omega L)^2$ and $(G/\omega C)^2$. Obtain the corresponding expression for phase velocity.

P.9–8 Obtain approximate expressions for γ and Z_0 for a lossy transmission line at very low frequencies such that $\omega L \ll R$ and $\omega C \ll G$.

P.9–9 The following characteristics have been measured on a lossy transmission line at 100 MHz:

$$Z_0 = 50 + j0 \quad (\Omega),$$
$$\alpha = 0.01 \quad (\text{dB/m}),$$
$$\beta = 0.8\pi \quad (\text{rad/m}).$$

Determine R, L, G, and C for the line.

P.9–10 It is desired to construct uniform transmission lines using polyethylene ($\epsilon_r = 2.25$) as the dielectric medium. Assuming negligible losses, (a) find the distance of separation for a 300 (Ω) two-wire line, where the radius of the conducting wires is 0.6 (mm); and (b) find the inner radius of the outer conductor for a 75 (Ω) coaxial line, where the radius of the center conductor is 0.6 (mm).

P.9–11 Prove that a maximum power is transferred from a voltage source with an internal impedance Z_g to a load impedance Z_L over a lossless transmission line when $Z_i = Z_g^*$, where Z_i is the impedance looking into the loaded line. What is the maximum power-transfer efficiency?

P.9–12 Express $V(z)$ and $I(z)$ in terms of the voltage V_i and current I_i at the input end and γ and Z_0 of a transmission line (a) in exponential form, and (b) in hyperbolic form.

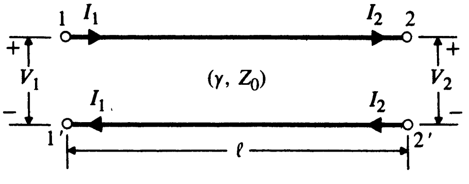
P.9–13 Consider a section of a uniform transmission line of length ℓ, characteristic impedance Z_0, and propagation constant γ between terminal pairs 1–1′ and 2–2′ shown in Fig. 9–44(a). Let (V_1, I_1) and (V_2, I_2) be the phasor voltages and phasor currents at terminals 1–1′ and 2–2′, respectively.

a) Use Eqs. (9–100a) and (9–100b) to write the equations relating (V_1, I_1) and (V_2, I_2) in the form

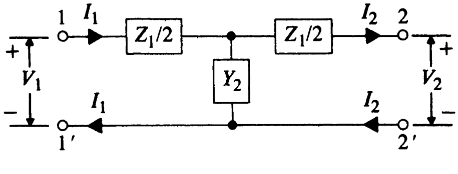
$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_2 \\ I_2 \end{bmatrix}. \tag{9–216}$$

Determine A, B, C, and D, and note the following relations:

$$A = D \tag{9–217}$$



(a) A line section of length ℓ.



(b) An equivalent two-port symmetrical T-network.

FIGURE 9–44
Equivalence of a line section and a symmetrical two-port network.

and

$$AD - BC = 1. \tag{9-218}$$

b) Because of Eqs. (9–216), (9–217), and (9–218), the line section in Fig. 9–44(a) can be replaced by an equivalent two-port symmetrical T-network shown in Fig. 9–44(b). Prove that

$$Z_1 = \frac{2}{C}(A-1) = 2Z_0 \tanh\frac{\gamma\ell}{2} \tag{9-219}$$

and

$$Y_2 = C = \frac{1}{Z_0}\sinh \gamma\ell. \tag{9-220}$$

P.9–14 A d-c generator of voltage V_g and internal resistance R_g is connected to a lossy transmission line characterized by a resistance per unit length R and a conductance per unit length G.

a) Write the governing voltage and current transmission-line equations.
b) Find the general solutions for $V(z)$ and $I(z)$.
c) Specialize the solutions in part (b) to those for an infinite line.
d) Specialize the solutions in part (b) to those for a finite line of length ℓ that is terminated in a load resistance R_L.

P.9–15 A generator with an open-circuit voltage $v_g(t) = 10 \sin 8000\pi t$ (V) and internal impedance $Z_g = 40 + j30$ (Ω) is connected to a 50 (Ω) distortionless line. The line has a resistance of 0.5 (Ω/m), and its lossy dielectric medium has a loss tangent of 0.18%. The line is 50 (m) long and is terminated in a matched load. Find (a) the instantaneous expressions for the voltage and current at an arbitrary location on the line, (b) the instantaneous expressions for the voltage and current at the load, and (c) the average power transmitted to the load.

P.9–16 The input impedance of an open- or short-circuited *lossy* transmission line has both a resistive and a reactive component. Prove that the input impedance of a very short section ℓ of a slightly lossy line ($\alpha\ell \ll 1$ and $\beta\ell \ll 1$) is approximately

a) $Z_{in} = (R + j\omega L)\ell$ with a short-circuit termination.
b) $Z_{in} = (G - j\omega C)/[G^2 + (\omega C)^2]\ell$ with an open-circuit termination.

P.9–17 Find the input impedance of a low-loss quarter-wavelength line ($\alpha\lambda \ll 1$)

a) terminated in a short circuit.
b) terminated in an open circuit.

P.9–18 A 2 (m) lossless air-spaced transmission line having a characteristic impedance 50 (Ω) is terminated with an impedance $40 + j30$ (Ω) at an operating frequency of 200 (MHz). Find the input impedance.

P.9–19 The open-circuit and short-circuit impedances measured at the input terminals of an air-spaced transmission line 4 (m) long are $250\angle{-50^\circ}$ (Ω) and $360\angle{20^\circ}$ (Ω), respectively.

a) Determine Z_0, α, and β of the line.
b) Determine R, L, G, and C.

P.9–20 Measurements on a 0.6 (m) lossless coaxial cable at 100 (kHz) show a capacitance of 54 (pF) when the cable is open-circuited and an inductance of 0.30 (μH) when it is short-circuited.

a) Determine Z_0 and the dielectric constant of its insulating medium.
b) Calculate the X_{io} and X_{is} at 10 (MHz).

P.9–21 Starting from the input impedance of an open-circuited lossy transmission line in Eq. (9–116), find the expressions for the half-power bandwidth and the Q of a low-loss line with $\ell = n\lambda/2$.

P.9–22 A lossless quarter-wave line section of characteristic impedance R_0 is terminated with an inductive load impedance $Z_L = R_L + jX_L$.

a) Prove that the input impedance is effectively a resistance R_i in parallel with a capacitive reactance X_i. Determine R_i and X_i in terms of R_0, R_L, and X_L.
b) Find the ratio of the magnitude of the voltage at the input to that at the load (*voltage transformation ratio*, $|V_{in}|/|V_L|$) in terms of R_0 and Z_L.

P.9–23 A 75 (Ω) lossless line is terminated in a load impedance $Z_L = R_L + jX_L$.

a) What must be the relation between R_L and X_L in order that the standing-wave ratio on the line be 3?
b) Find X_L, if $R_L = 150$ (Ω).
c) Where does the voltage minimum nearest to the load occur on the line for part (b)?

P.9–24 Consider a lossless transmission line.

a) Determine the line's characteristic resistance so that it will have a minimum possible standing-wave ratio for a load impedance $40 + j30$ (Ω).
b) Find this minimum standing-wave ratio and the corresponding voltage reflection coefficient.
c) Find the location of the voltage minimum nearest to the load.

P.9–25 A lossy transmission line with characteristic impedance Z_0 is terminated in an arbitrary load impedance Z_L.

a) Express the standing-wave ratio S on the line in terms of Z_0 and Z_L.
b) Find in terms of S and Z_0 the impedance looking toward the load at the location of a voltage maximum.
c) Find the impedance looking toward the load at a location of a voltage minimum.

P.9–26 A transmission line of characteristic impedance $R_0 = 50$ (Ω) is to be matched to a load impedance $Z_L = 40 + j10$ (Ω) through a length ℓ' of another transmission line of characteristic impedance R_0'. Find the required ℓ' and R_0' for matching.

P.9–27 The standing-wave ratio on a lossless 300 (Ω) transmission line terminated in an unknown load impedance is 2.0, and the nearest voltage minimum is at a distance 0.3λ from the load. Determine (a) the reflection coefficient Γ of the load, (b) the unknown load impedance Z_L, and (c) the equivalent length and terminating resistance of a line, such that the input impedance is equal to Z_L.

P.9–28 Obtain from Eq. (9–147) the formulas for finding the length ℓ_m and the terminating resistance R_m of a lossless line having a characteristic impedance R_0 such that the input impedance equals $Z_i = R_i + jX_i$.

P.9–29 Obtain an analytical expression for the load impedance Z_L connected to a line of characteristic impedance Z_0 in terms of standing-wave ratio S and the distance, z_m'/λ, of the voltage minimum closest to the load.

P.9–30 A sinusoidal voltage generator with $V_g = 0.1\underline{/0°}$ (V) and internal impedance $Z_g = R_0$ is connected to a lossless transmission line having a characteristic impedance $R_0 = 50$ (Ω). The line is ℓ meters long and is terminated in a load resistance $R_L = 25$ (Ω). Find (a) V_i, I_i, V_L, and I_L; (b) the standing-wave ratio on the line; and (c) the average power delivered to the load. Compare the result in part (c) with the case where $R_L = 50$ (Ω).

P.9–31 Consider a lossless transmission line of a characteristic impedance R_0. A time-harmonic voltage source of an amplitude V_g and an internal impedance $R_g = R_0$ is connected to the input terminals of the line, which is terminated with a load impedance $Z_L = R_L + jX_L$. Let P_{inc} be the average incident power associated with the wave traveling in the $+z$-direction.

a) Find the expression for P_{inc} in terms of V_g and R_0.
b) Find the expression for the average power P_L delivered to the load in terms of V_g and the reflection coefficient Γ.
c) Express the ratio P_L/P_{inc} in terms of the standing-wave ratio S.
d) For $V_g = 100$ (V), $R_g = R_0 = 50$ (Ω), $Z_L = 50 - j25$ (Ω) determine P_{inc}, Γ, S, P_L, $|V_L|$, and $|I_L|$.

P.9–32 A sinusoidal voltage generator $v_g = 110 \sin \omega t$ (V) and internal impedance $Z_g = 50$ (Ω) is connected to a quarter-wave lossless line having a characteristic impedance $R_0 = 50$ (Ω) that is terminated in a purely reactive load $Z_L = j50$ (Ω).

a) Obtain the voltage and current phasor expressions $V(z')$ and $I(z')$.
b) Write the instantaneous voltage and current expressions $v(z', t)$ and $i(z', t)$.
c) Obtain the instantaneous power and the average power delivered to the load.

P.9–33 A d-c voltage V_0 is applied at $t = 0$ directly to the input terminals of an open-circuited lossless transmission line of length ℓ as in Fig. 9–45. Sketch the voltage and current waves on the line (in the manner of Fig. 9–16) for the following time intervals:

a) $0 < t < T\,(=\ell/u)$
b) $T < t < 2T$
c) $2T < t < 3T$
d) $3T < t < 4T$

What happens after $t = 4T$?

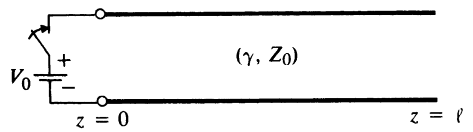


FIGURE 9–45
A d-c voltage applied to an open-circuited line (Problem P.9–33).

P.9–34 A 100 (V) d-c voltage is applied at $t = 0$ to the input terminals of a lossless coaxial cable ($R_{01} = 50$ (Ω), dielectric constant of insulation $\epsilon_{r1} = 2.25$) through an internal resistance $R_g = R_{01}$. The cable is 200 (m) long and is connected to a lossless two-wire line ($R_{02} = 200$ (Ω), $\epsilon_{r2} = 1$), which is 400 (m) long and is terminated in its characteristic resistance.

a) Describe the transient behavior of the system and find the amplitudes of all reflected and transmitted voltage and current waves.
b) Sketch the voltage and current as functions of t at the midpoint of the coaxial cable.
c) Repeat part (b) at the midpoint of the two-wire line.

P.9–35 A d-c voltage V_0 is applied at $t = 0$ to the input terminals of an open-circuited air-dielectric line of a length ℓ through a series resistance equal to $R_0/2$, where R_0 is the characteristic resistance of the line.

a) Draw the voltage and current reflection diagrams.
b) Sketch $V(0, t)$ and and $I(0, t)$.
c) Sketch $V(\ell/2, t)$ and $I(\ell/2, t)$.

P.9–36 A d-c voltage V_0 is applied at $t = 0$ directly to the input terminals of a lossless air-dielectric transmission line of a length ℓ. The line has a characteristic resistance R_0 and is terminated in a load resistance $R_L = 2R_0$.
a) Draw the voltage and currrent reflection diagrams.
b) Sketch $V(\ell, t)$ and $I(\ell, t)$.
c) Sketch $V(z, 2.5T)$ and $I(z, 2.5T)$, where $T = \ell/u$.

P.9–37 For the problem in Example 9–11, determine and sketch $i(200, t)$.

P.9–38 A lossless, air-dielectric, open-circuited transmission line of characteristic resistance R_0 and length ℓ is initially charged to a voltage V_0. At $t = 0$ the line is connected to a resistance R, as shown in Fig. 9–46. Determine $V_R(t)$ and $I_R(t)$ for $0 < t < 5\ell/c$:
a) if $R = 2R_0$,
b) if $R = R_0/2$.

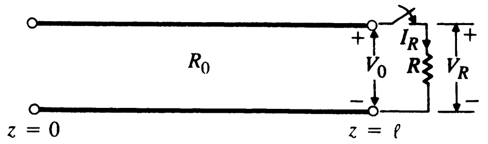


FIGURE 9–46
An initially charged line connected to a resistance (Problem 9–38).

P.9–39 Refer to Fig. 9–26(a) but change the load from a pure inductance to a series combination of $R_L = 10\ (\Omega)$ and $L_L = 48\ (\mu H)$. Assume that $V_0 = 100$ (V), $R_0 = 50\ (\Omega)$, $\ell = 900$ (m), and $u = c$.
a) Find the expressions for the current in and the voltage across the load as functions of t.
b) Sketch the current and voltage distributions along the transmission line at $t_1 = 4\ (\mu s)$.

P.9–40 Refer to Fig. 9–28(a) but change the load from a pure capacitance to a parallel combination of $C_L = 14$ (nF) and $R_L = 1000\ (\Omega)$. Assume that $V_0 = 100$ (V), $R_0 = 50\ (\Omega)$, $\ell = 900$ (m), and $u = c$.
a) Find the expressions for the current in and the voltage across the load as functions of t.
b) Sketch the current and voltage distributions along the transmission line at $t_1 = 4\ (\mu s)$.

P.9–41 The Smith chart, constructed on the basis of Eqs. (9–188) and (9–189) for a lossless transmission line, is restricted to a unit circle because $|\Gamma| \le 1$. In the case of a lossy line, Z_0 is a complex quantity, and so, in general, is the normalized load impedance $z_L = Z_L/Z_0$.
a) Show that the phase angle of z_L, θ_L, lies between $\pm 3\pi/4$.
b) Show that $|\Gamma|$ may be greater than unity.
c) Prove that max. $|\Gamma| = 2.414$.

P.9–42 The characteristic impedance of a given lossless transmission line is 75 (Ω). Use a Smith chart to find the input impedance at 200 (MHz) of such a line that is (a) 1 (m) long and open-circuited, and (b) 0.8 (m) long and short-circuited. Then (c) determine the corresponding input admittances for the lines in parts (a) and (b).

P.9–43 A load impedance $30 + j10$ (Ω) is connected to a lossless transmission line of length 0.101λ and characteristic impedance 50 (Ω). Use a Smith chart to find (a) the standing-wave ratio, (b) the voltage reflection coefficient, (c) the input impedance, (d) the input admittance, and (e) the location of the voltage minimum on the line.

P.9–44 Repeat Problem P.9–43 for a load impedance $30 - j10$ (Ω).

P.9–45 In a laboratory experiment conducted on a 50 (Ω) lossless transmission line terminated in an unknown load impedance, it is found that the standing-wave ratio is 2.0. The successive voltage minima are 25 (cm) apart, and the first minimum occurs at 5 (cm) from the load. Find (a) the load impedance, and (b) the reflection coefficient of the load. (c) Where would the first voltage minimum be located if the load were replaced by a short-circuit?

P.9–46 The input impedance of a short-circuited lossy transmission line of length 1.5 (m) ($<\lambda/2$) and characteristic impedance 100 (Ω) (approximately real) is $40 - j280$ (Ω).

a) Find α and β of the line.

b) Determine the input impedance if the short-circuit is replaced by a load impedance $Z_L = 50 + j50$ (Ω).

c) Find the input impedance of the short-circuited line for a line length 0.15λ.

P.9–47 A dipole antenna having an input impedance of 73 (Ω) is fed by a 200 (MHz) source through a 300 (Ω) two-wire transmission line. Design a quarter-wave two-wire air line with a 2 (cm) spacing to match the antenna to the 300 (Ω) line.

P.9–48 The single-stub method is used to match a load impedance $25 + j25$ (Ω) to a 50 (Ω) transmission line.

a) Find the required length and position of a short-circuited stub made of a section of the same 50 (Ω) line.

b) Repeat part (a) assuming that the short-circuited stub is made of a section of a line that has a characteristic impedance of 75 (Ω).

P.9–49 A load impedance can be matched to a transmission line also by using a single stub placed in series with the load at an appropriate location, as shown in Fig. 9–47. Assuming that $Z_L = 25 + j25$ (Ω), $R_0 = 50$ (Ω), and $R_0' = 35$ (Ω), find d and ℓ required for matching.

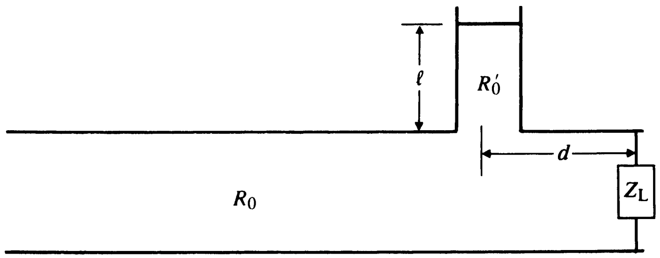


FIGURE 9–47
Impedance matching by a series stub (Problem P.9–49).

P.9–50 The double-stub method is used to match a load impedance $100 + j100$ (Ω) to a lossless transmission line of characteristic impedance 300 (Ω). The spacing between the stubs is $3\lambda/8$, with one stub connected directly in parallel with the load. Determine the lengths of the stub tuners (a) if they are both short-circuited, and (b) if they are both open-circuited.

P.9–51 If the load impedance in Problem P.9–50 is changed to $100 + j50$ (Ω), one discovers that a perfect match using the double-stub method with $d_0 = 3\lambda/8$ and one stub connected directly across the load is not possible. However, the modified arrangement shown in Fig. 9–43 can be used to match this load with the line.

a) Find the minimum required additional line length d_L.

b) Find the required lengths of the short-circuited stub tuners, using the minimum d_L found in part (a).

P.9–52 The double-stub method shown in Fig. 9–41 cannot be used to match certain loads to a line with a given characteristic impedance. Determine the regions of load admittances on a Smith admittance chart for which the double-stub arrangement in Fig. 9–41 cannot lead to a match for $d_o = \lambda/16$, $\lambda/4$, $3\lambda/8$, and $7\lambda/16$.