probability basic def

problem

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CHAPTER 7

PROBABILITY: A MEASURE OF UNCERTAINTY

7.1 UNDERSTANDING PROBABILITY

Throughout our lives, we are confronted with decision-making situations that involve an uncertain future. The concept of this uncertainty is as old as civilization itself. The subject most useful in effectively dealing with such uncertainties is contained under the heading of probability. The term 'probability' is an estimate of the proportion of one or more uncertain experimental outcomes when the experiment is performed at random. Here are some examples that involve uncertainty:

- What is the chance that the country will experience severe flood this year if dredging of the major rivers is not undertaken?
- What is the likelihood that the new vaccine will be more effective than the old one in controlling tuberculosis?
- How likely is that the proposed Padma Bridge will be completed by the end of the year 2015?
- How likely it is that tomorrow will be a sunny day?
- What is the probability that the stock market will show an abrupt rise soon after the forthcoming budget announcement?

These are some of the queries we usually encounter in our every day life. In most cases, we respond to these questions as saying "good chance", "very likely", "quite possible", or "almost sure". These are all qualitative answers, which often are not enough to realize their relative importance. What we need is to quantify these answers in order to assess their relative weights. Probability theory enables us to attach appropriate numerical values or measurements to these queries with certain degree of confidence. For example, when we say that tomorrow will certainly be a 'sunny day', we mean that there is a 100% chance to have a sunny day. When the weather forecast indicates possibility of 'no sunny day', we mean that there is 'near zero' chance to have a sunny day. A 50% probability indicates that sunny day is just as likely to occur as not. In the foregoing example, 'sunny day' is an **event**, and we are interested to measure numerically the occurrence of this event with certain degree of confidence. In this particular case, the numerical measures are seen to vary from 0, implying no possibility of a sunny day, to 100%, implying a sunny day sure to occur.

Probability and Possibility

Sometimes a distinction is made between **probability** and **possibility**. Possibility precedes probability. We cannot make educated guess or prediction unless we know at least 'what is possible'. We can hardly predict who is going to win the next world cup football, for example, unless we know who the possible participants are. Possibility thus refers to exhaustiveness of outcomes, while probability refers to their predictive behavior.

7.2 PROBABILITY: HISTORICAL PERSPECTIVES

The concept of probability originates from gambling. The idea of gambling has a long history. By about the year 3500 BC, games of chance played with bone-objects, that could be considered precursors of dice, were apparently highly developed in Egypt and elsewhere. Cubical dice with markings virtually identical to those on modern dice have been found in Egyptian tombs dating from 2000 BC (Degroot 1984). Gambling with dice has been popular ever since that time and played an important role in the early development of the theory of probability.

It is believed that the French mathematicians Blaise Pascal (1623-1662) and Pieree Fermat (1601-1665) were the pioneers in developing mathematical theory of probability. They succeeded in deriving exact probabilities for certain gambling problems involving dice. Before their

works, Girolamo Cardano (1501-1576) and Galieo Galiei (1564-1642) contributed significantly to the development of mathematical theory of probability.

The theory of probability has been developed steadily since the seventeenth century and has been widely applied in diverse fields of study. Today, probability theory is an important tool in most areas of engineering, science and management. Probability theory is being extensively applied in fields such as medicine, metrology, marketing, earthquake prediction, human behavior, computer science and law.

Before we discuss this important topic in details, we shall first look into the idea of a few concepts and preliminaries that are basic, fundamental and occupy a central position in understanding probability and evolving theory of probability. These include, among others, set, permutation, combination, and a few other basic elements, which are closely related to the theory of probability.

7.3 A REVIEW OF SET AND SET NOTATIONS

The algebra of set plays an important role in understanding many concepts in modern mathematics and statistics. Set theory provides a basis for the discussion of probability theory and its application enhances our knowledge on the concepts related to statistical inference, such as interval estimation and hypothesis testing. It is also used to develop quite precise definitions of the important notions of relation and function in mathematics. However, our discussion of this important concept will be confined only to a few simple theorems and operations in an integrated manner so as to make them relevant to the study of probability theory.

Description and Notations

A **set** is simply a well-defined list or collection of distinct objects. The chairs in your classroom form a set so do the files and the various directories in the hard disk of your personal computer. The English alphabet is a set of 26 letters; the natural number between 1 and 10 inclusive is a set of 10 integers.

The individual objects of a set are **elements**, or **members**. The set is the collection of its elements. Thus the letter *m*, say, is an element of the set of English alphabet.

The elements of a set must be **distinct**; that is, each element must appear once and only once. Once an object is recorded in a set, it is not repeated.

For instance, the set of letters in the word **statistics** is a set with the five distinct letters *a, c, i, s,* and *t* as its members. The set of scores of 6 students in an examination, 650, 645, 651, 650, 647, and 651, does not contain all the six scores, but the four distinct scores 645, 647, 650 and 651.

A set must be **well-defined**. This means that it must be specified so that you can clearly determine whether a certain object is a member of the set or not. For any given object, there must be an unequivocal "yes" or "no" answer to the question: Is this object an element of the set? Of these two answers, one and only one is correct.

To describe a set, we use a capital letter, such as *I, A, R, S, T* or *U* and a small letter, such as *a, b, x, y,* or *z* to symbolize a particular element of the set. The symbol $\in$ is used to denote "is an element of," or "belongs to." Thus $x \in S$ is read "x is an element of the set S or "x belongs to S"

There are two different ways in which sets may be specified. One of these is the tabular, or enumeration method, that simply lists all of the elements of the set. The names of individual elements are separated by commas and are enclosed in braces. For example,

$$A = \{a, b, c, d, e\}$$

is a complete itemization of the set A, which indicates that A consists of the letters a, b, c, d, and e.

An alternative way of specifying a set is an algebraic rule or a description by which you can decide what objects are or are not members. Thus the set S may be specified by

$$S = \{x|x \text{ is a real number}\}$$

The symbol "|" is read as "such that," and the above expression indicates that S is the set of all elements x such that x is a real number. Similarly, the expression

$$Q = \{i|i \text{ is an integer and } 0 \le i \le 10\}$$

is read "Q is the set of all elements i such that i is an integer between 0 and 10 inclusive."

There may be more than one condition about x, for example.

$$S = \{x|x^2 = 2, x>0\}$$

the set of a single number, namely, $\{\sqrt{2}\}$.

Specifying a set by a rule or description is most frequently encountered, and it is obviously far more convenient than listing all the elements entailed by the tabular method.

Universal Set

A **universal set** is the set of all elements that may possibly be considered in a particular discussion: It is the frame of reference for the discussion. Unless otherwise stated, all sets under investigation are assumed to be subsets of the universal set. Sometimes it is a small set consisting of only a few elements; for example, in the toss of two dice, the universal set U is

$$U = \{x|x \text{ is sum of points uppermost on the two dice}\}$$

or more explicitly

$$U = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$$

In human population studies, the universal set may be defined to be a set of all the people of the world.

Venn Diagram

A helpful scheme to illustrate the relationship between sets and set operations is the **Venn diagram** (named after the English logistician John Venn, 1834 –1923). The universal set is usually represented by an enclosed plane geometrical figure such as a square or rectangle within which small regions, such as circles or parts or circles, represent subsets. Sometimes a given universal set has disjoint subsets that together make up the universal set.

Empty Set

Any set that has no element in it is called an empty set, or null set. It is designated by ϕ, a zero with a slash through it. (It is a Scandinavian letter and not the Greek letter phi.).

The theory of sets requires the concept of an empty set just as ordinary mathematics needs the number zero. An empty set is a subset of every set. We define this as follows:

When we write $A=\phi$, we mean, "A is an empty set." When we write $A=\{0\}$, we mean "A is the set whose only member is zero". The latter is not an empty set. The set of people who are taller than 10 feet is an empty set. So is the set of woman presidents of Bangladesh, while the set of woman prime ministers of Bangladesh is not an empty set.

Subset

If every element of the set A is also an element of the set B, then A is said to be a subset of B. We write this statement as "$A \subseteq B$."

The set $A=\{1, 3, 6\}$ is a subset of the set $B=\{1, 2, 3, 4, 5, 6\}$. Similarly, the set $C=\{2\}$ is a subset of the set $S=\{x | x^2-4=0\}$.

The definition of a subset implies that a set can be a subset of itself, that is, $A \subseteq A$. The null set ϕ is considered to be a subset of every set.

How many subsets can then be constructed from a set of n elements? The answer is 2^n. In particular, a set with 1 element has 2 subsets; a set with 2 elements has 4 subsets and so on. In each case, the set of subsets will contain the null set and the set itself.

Example 7.1: Enumerate the subsets of the set $A=\{x\}$. The subsets are $B=\{x\}$ and $C=\phi$. Similarly, the set $S=\{a, b\}$ has 4 subsets: $\{a\}$, $(b\}$, $\{a, b\}$ and ϕ. Likewise, $\{a\}$, $\{b\}$, $\{c\}$, $\{a, b\}$, $\{a, c\}$, $\{b, c\}$, $\{a, b, c\}$ and ϕ are the subsets of the set $U=\{a, b, c\}$.

Proper Subset

If there is at least one element of B that is not in A, then A is said to be a proper subset of B, and we write it as $A \subset B$.

We call A a proper subset of B if, first, A is a subset of B and, secondly if A is not equal to B. Symbolically, A is a proper subset of B if $A \subset B$ and $A \neq B$.

The above definition implies that any subset of a set is a proper subset of the set if the subset is not the set itself. Thus in Example 7.1, all the subsets of $S=\{a, b\}$ except $\{a, b\}$ are proper subsets. Similarly, the set U has 7 proper subsets, viz. $\{a\}$, $\{b\}$, $(c\}$, $\{a, b\}$, $\{a, c\}$, $\{b, c\}$, and ϕ.

Equal Set

Two sets A and B are said to be equal, or $A = B$, if and only if $A \subseteq B$ and $B \subseteq A$.

Thus if $A=\{1, 2, 5\}$, $B=\{1, 5, 2\}$, $C=\{2, 5, 1\}$, $D=\{2, 1, 5\}$, $E=\{5, 1, 2\}$ and $F=\{5, 2, 1\}$, the sets A, B, C, D, E, and F are all equal.

Unit Set

If a set contains only one element, it is called a unit set or singleton set. If $A= \{x | 3x=6\}$, then $A = \{2\}$ is a unit set.

Finite and Infinite Sets

A set may contain a finite or an infinite number of objects. A set is called finite if its elements are equal in number to some specifiable nonnegative integers. It is called infinite if the number of its elements is greater than any positive integer.

The set of the days of the week or the set of all students enrolled at Dhaka City College in a given academic session is finite. So is the set of all names listed in the Dhaka telephone directory in a specific year, and the set of all major rivers on the earth. The number of members of each of these sets contains is quite large, yet they are all finite. On the other hand, the set of all natural numbers or of real numbers, the set of all points on a straight line, and the set of all stars in the universe are examples of infinite sets.

7.4 OPERATIONS WITH SETS

Just as operations on numbers result in new numbers, operations on sets lead to new sets. Suppose that the sets A, B, C, and so on are subsets of the universal set U, then these can be operated on to form new sets, which are also subsets of U.

Intersection of Sets

One of the operations is the intersection of two or more sets. Given the sets A and B, which are subsets of U, the intersection of A and B, written as "$A \cap B$," is the set of all elements that are members of both sets. "$A \cap B$," is read as "A cap B". "A intersect B," or "A intersection B.". Formally, $A \cap B = \{x | x \in A \text{ and } x \in B\}$

For example, if.

$$U = \{1, 2, 3, 4, 5, 6\}, \ A = \{1, 2, 5, 6\}, \ B = \{2, 4, 5, 6\}$$

then

$$A \cap B = \{2, 5, 6\}$$

since these are the elements that are members of both A and B. Figure 7.1 is a conventional Venn diagram. The shaded area depicted in the diagram represents the intersection of A and B. If A and B have no elements in common, their intersection is an empty set, or $A \cap B=\phi$, which is shown in Figure 7.2. Then A and B are said to be **disjoint** or **mutually exclusive**.

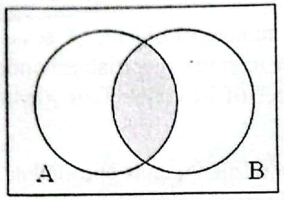


Figure 7.1: $A \cap B$ is shaded

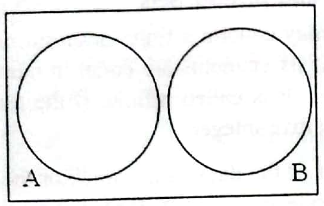


Figure 7.2: $A \cap B = \phi$

The intersection of sets defined above can be extended to any finite number of sets. For example, the intersection of A, B, and C is

$$A \cap B \cap C = \{x | x \in A \text{ and } x \in B \text{ and } x \in C\}$$

Thus if

$U = \{1, 2, 3, 4, 5, 6\}$, $A = \{1, 2, 5, 6\}$, $B = \{2, 4, 5, 6\}$, $C = \{1, 5, 6\}$,

then

$$A \cap B \cap C = \{5, 6\}.$$

Also if A were all female students, B were all those 19 years of age, and C were those enrolled in statistical inference, then $A \cap B \cap C$ would be the set of all girls 19 years of age enrolled in statistical inference.

We illustrate the case of three events by a Venn diagram in Figure 7.3 below.

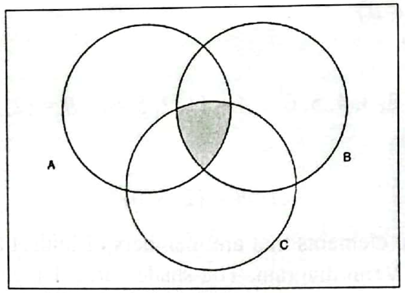


Figure 7.3: $A \cap B \cap C$ is shaded

Union of Set

Another operation is the union of two or more sets. The union of two sets A and B is the set of all elements belonging to A or B or both. The union of

A and B is denoted by $A \cup B$, read as "A union B." or "A cup B". Symbolically, $A \cup B = \{x | x \in A \text{ or } x \in B\}$

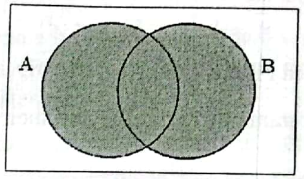


Figure 7.4: $A \cup B$ is shaded

The definition of union of two sets can be extended to any number of sets. For three sets A, B and C the Venn diagram will be as in Figure 7.5.

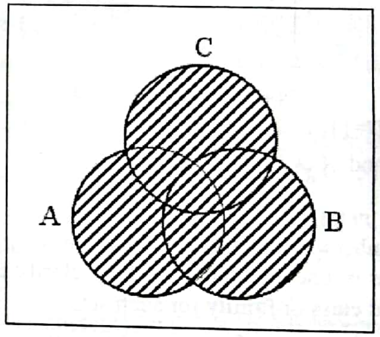


Figure 7.5: $A \cup B \cup C$ is shaded

Example 7.2: Let U be a set defined as $U=\{1, 2, 3, 4, 5, 6\}$. Let two of its subsets be A and B defined as $A=\{1, 4, 6\}$ and $B=\{2, 4, 6\}$. Then by definition

$$A \cup B = \{1, 2, 4, 6\}.$$

Example 7.3: If $U = \{x | x \text{ is a positive integer}\}$

$$A = \{x | x \text{ is a positive even integer}\}$$

$$B = \{x | x \text{ is a positive even integer smaller than } 10\}$$

then

$$A \cup B = \{x | x \text{ is a positive even integer}\}$$

Complementation

The complement $\overline{A}$ of a set A is defined to be a set that contains all outcomes in the universal set U which do not belong to A. Symbolically,

$$\overline{A} = \{x | x \in U, x \notin A\}$$

Such a set is displayed in Figure 7.6.

Note that A and $\overline{A}$ are mutually exclusive and their union is the universal set.

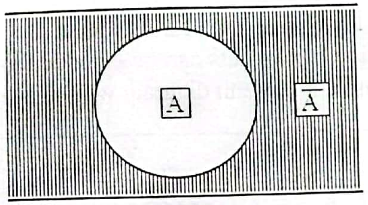


Figure 7.6: $\overline{A}$ is shaded

Example 7.4: Let $U=\{1, 2, 3, 4, 5, 6\}$ and $A=\{1, 4, 6\}$. Then $\overline{A}=\{2, 3, 5\}$ so that $A \cap \overline{A} = \phi$ and $A \cup \overline{A} = U$.

Class of Sets

Frequently the members of a set are sets themselves. For example, each line in a set of lines is a set of points. To help clarify these situations, we usually use the word **class** or **family** for such sets.

A set whose members are sets themselves, is called a class or family of sets.

The set

$$\mathcal{A}=\{\{a\},\{b\},\{c\},\{a,b\},\{b,c\}\}$$

is a family or class of sets.

Power Set

The class of all subsets of a set A is called the power set of A and is denoted by $\varphi(A)$.

If $A = \{1, 2, 3\}$, then the power set of A is

$$\varphi(A). = \{\phi, \{1\} \{2\} \{3\} \{1, 2\}\{1, 3\} \{2, 3\} \{1, 2, 3\}\}$$

The general rule is that if A is finite and has n elements; the power set will consist of 2^n elements including the null set.

Cartesian Product Set

Just as a collection of objects constitutes a set, a collection of pairs of objects constitutes a special kind of set. Let us use (x, y) to symbolize a pair of objects. If the first object is a member of some set A and the second object is a member of some set B, that is $x \in A$ and $x \in B$, then (x, y) is called an **ordered pair**. This ordered pair is different from the set $\{x, y\}$ for which the order is immaterial.

Let A and B represent two distinct finite sets. It is then possible to identify all the possible pairs (x, y); each pair relates an element of A to an element of B. These possible pairs constitute a set called the **Cartesian Product** set of A and B (named after Rene Descartes, 1596–1650).

Thus if A and B are two sets, then the product set of A and B, denoted by $A \times B$, contains all ordered pairs (x, y), where $x \in A$, $y \in B$.

The product of a set A by itself is denoted by A^2 and this concept may be extended to any finite number of sets.

Example 7.5: Let $A = \{1, 2, 3\}$ and $B= \{4, 5\}$. Then the product set of A and B is $A \times B =\{ (1, 4), (1, 5), (2, 4), (2, 5), (3, 4), (3, 5)\}$.

It is not necessary that the sets A and B in a Cartesian product set contain the same kind of elements. For instance, we may have two sets A and B defined as follows:

$$A = \{a | a \text{ is a student}\}, \text{ and } B = \{b | b \text{ is an age in years}\}$$

Then

$$A \times B = \{(a, b) | a \in A, b \in B\}$$

is the set of all possible ordered pairs of a student and an age.

Differences of Sets

The difference of the sets A and B denoted by $A \backslash B$, is the set of elements which belongs to A but not to B. Symbolically,

$$A \backslash B = \{x | x \in A, x \notin B\}$$

The set $A \backslash B$ in fact is the same as the set $A \cap \overline{B}$. Figure 7.7 below shows the difference of the two sets A and B

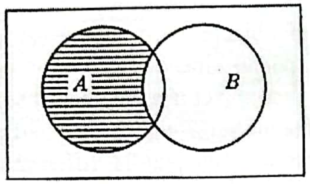


Figure 7.7: Venn diagram showing $A\backslash B$

7.5 LAWS OF SETS

The algebra of numbers is closely related to the algebra of sets and hence most of the fundamental laws for addition and multiplication in the ordinary algebra of numbers have their analogs for union and intersection in the algebra of sets. The basic laws of set theory, together with their analogs in the algebra of real numbers, are stated in this section.

Commutative Laws

Exactly analog to the real numbers, unions and intersection of sets are commutative. Thus if A and B are two sets, then we have

$$A \cup B = B \cup A \text{ and } A \cap B = B \cap A$$

for any sets A and B.

Associative Laws

We know that addition and multiplication of real numbers are associative. Analogously, the **associative law** holds for the union and intersection of sets too. If A, B, and C represent three sets, we have

$$(A \cup B) \cup C = A \cup (B \cup C)$$

and

$$(A \cap B) \cap C = A \cap (B \cap C)$$

for any sets A, B, and C.

Distributive Laws

For set algebra, we have two distributive laws

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

and

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

The second law is in contrast with the algebra of real numbers, where addition does not distribute across multiplication (i.e. $a+(b\times c)\neq(a+b)\times(a+c)$). As you see, in the algebra of sets, union does distribute across intersection.

Identity Laws

In ordinary algebra "0" is called an identity number with respect to addition because zero added to any number k gives the same number as sum; that is, $k+0 = k$. In set algebra the union of any set A and the null set ϕ is the same set A; that is,

$$A \cup \phi = A$$

In ordinary algebra "1" is an identity number with respect to multiplication, since multiplying any number "a" by "1" yields the same number as product, that is, $k \times 1 = k$. In set algebra the intersection of any set A and the universal set U is the same set A; that is,

$$A \cap U = A$$

Because of these similarities, we sometimes refer to $A \cap B$ as the **logical sum**, and $A \cap B$ as the **logical product**.

Other two relations are

$$A \cup U = U \text{ and } A \cap \phi = \phi$$

Idempotent Laws

As already indicated in the discussion of distributive laws, the analogy with the algebra of real numbers is not perfect. This is further shown in the **idempotent** laws: If k is a real number, then $k+k = 2k$, and $k \times k = k^2$. If A is a set, then we have two idempotent laws:

$$A \cup A = A \text{ and } A \cap A = A$$

for any set A.

Complement Laws

For every subset A of a universal set U, there is one and only one complement of A, namely, $\overline{A}$ have the following properties:

$$A \cup \overline{A} = U \text{ and } A \cap \overline{A} = \phi$$

Furthermore, the complement of the complement of $\overline{A}$ is equal to A itself; that is, $\overline{\overline{A}} = A$ for any set A.

De Morgan's Laws

The laws state that if $A \cup B$ is equal to the intersection of the complement of A with the complement of B and that the complement of $A \cap B$ is equal to the union of the complement of A and the complement of B. Symbolically,

$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$

and

$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$

7. 6 RANDOM PHENOMENON AND RELATED CONCEPTS

Probability theory is important and has diverse applications because of certain common properties possessed by many social and physical phenomena. Among such phenomena, are the numbers of births or deaths of a population during a given period, the number of defective products in a production process, frequency of telephone calls in a certain hour of the day, the number of vehicle accidents on some highway during a certain period and countless others. Each of these phenomena possesses a common property: that its observed outcome may not always be the same under the same set of circumstances and uniform conditions. In other words, there is no **functional** or **deterministic regularity** in its outcome. For example, the number of fatal accidents on the Asian highway in July may be quite different from those in January though circumstances were similar. The number of defective bulbs produced by the same set of workers, same machines, and same materials by an electric company may vary from run to run, though conditions remain similar.

However, the observed outcomes may differ in such a way that there is a **probabilistic regularity**. This means that in a sufficiently large number of observations, there exists a number between 0 and 1 representing the relative frequencies, with which the different outcomes may be observed. A phenomenon that possesses this probabilistic, though not deterministic regularity is often referred to as a **random phenomenon** or **random process**.

In deterministic phenomenon, on the other hand, we deal with relationships of the type, say, exhibited by Newton's law of gravity, which states that

> Every particle in the universe attracts every other particle with a force directly proportional to the product of their masses and inversely proportional to the square of the distance between them.

Other examples are Ohm's law, Boyle's law and Newton's law of motion.

Random Experiment

Using the terminology of probability, we define an **experiment** to be any process, which generates well-defined outcomes or observations. The process must be defined so that on any single repetition of the experiment, **one and only one** of the possible outcomes will occur. The possible outcomes for an experiment are called **experimental outcomes.** But this outcome is not known in advance with certainty. The outcomes thus possess the property or principle of randomness. **Randomness** is the result of a mechanical process intended to ensure that individual biases, either known or unknown in nature, do not influence or interfere with the outcome of the experiment. Owing to this property, the outcomes of the experiment are **chance measurements** or **random observations** and the associated experiment is the **random (or statistical) experiment.**

***Definition* 7.1:** *A random experiment is an experiment in which*

(a) *All possible outcomes of the experiment are known in advance;*
(b) *Any performance of the experiment results in a outcome that is not exactly known in advance;*
(c) *The experiment can be repeated under identical conditions.*

In probability theory, we study this uncertainty of a random experiment.

Example 7.6: A fair die with six faces marked 1, 2, ...,6 is tossed once. This is an experiment with six possible outcomes {1, 2, ...,6}. But we are uncertain about whether a 2 or a 6 will land when tossed. This makes the experiment a random experiment.

Example 7.7: We record the time in hours that an electric bulb takes before it burns out. Any non- negative number is a conceivable outcome of this experiment. Since the exact hours that the bulb will take to burn out is unknown, the experiment is a random experiment.

Example7.8: An experiment consists of counting the number of bacteria in a portion of a food, we might observe a countable but infinite number of bacteria, such as 0 bacteria, 1 bacterium, 2 bacteria, and so on. This experiment represents a random experiment.

Sample Space

The first step in analyzing a particular experiment is to carefully define the experimental outcomes. When we have defined all possible experimental outcomes, we have identified what we call the sample space for the experiment. That is, the sample space can be thought of as the set of all possible experimental outcomes.

***Definition* 7.2**: *A sample space of an experiment is a set or collection of all possible outcomes of the same experiment such that any outcome of the experiment corresponds to exactly one element in the set. A sample space is usually denoted by the symbol S.*

Discrete and Continuous Sample Space

The examples examined above have the property that they consist of either a finite or a countable number of sample points. In the die-tossing example above, there are six (a finite number) sample points. The number of sample points associated with the bacteria counting experiment is infinite, but the number of distinct sample points can be put into one-to-one correspondence with the integers (i.e. the number of sample points is countable). Such sample spaces are said to be **discrete**.

***Definition* 7.3**: *If a sample space contains a finite number of possibilities or an unending sequence with as many elements as there are whole numbers, it is called a discrete sample space.*

The outcomes of some experiments may be neither finite nor countable. Measurements such as weight, height, distance, time all occur in intervals, and thus we have an infinite and uncountable number of such measurements in the sample space that cannot be equated to the number of whole numbers. Example 7.8 (longevity of electric bulbs) is an example of continuous sample space.

***Definition* 7.4**: *If a sample space contains an infinite number of possibilities equal to the number of points on a line segment, it is called a continuous sample space.*

Example 7.9: If an experiment consists of tossing two coins and noting whether they land heads (H) or tails (T), then the set S is

$$S = \{HH, HT, TH, TT\} \qquad \text{... (a)}$$

which provides a list that represents the possible outcomes of one toss, where the first letter in any pair designates the outcome of the first coin and the second letter that for the second coin. It is evident that every outcome of the experiment corresponds to exactly one element of the set (a) above. Thus the outcome HH, for example, says that both coins land heads, while the outcome TH says that the first coin lands head and the

second coin a tail. When three coins are tossed, the corresponding sample space is of the form

$$S = \{HHH, HHT, HTH, HTT, THH, THH, THT, TTH, TTT\}$$

A tree diagram is drawn for the sample space as shown in Figure 7.8.

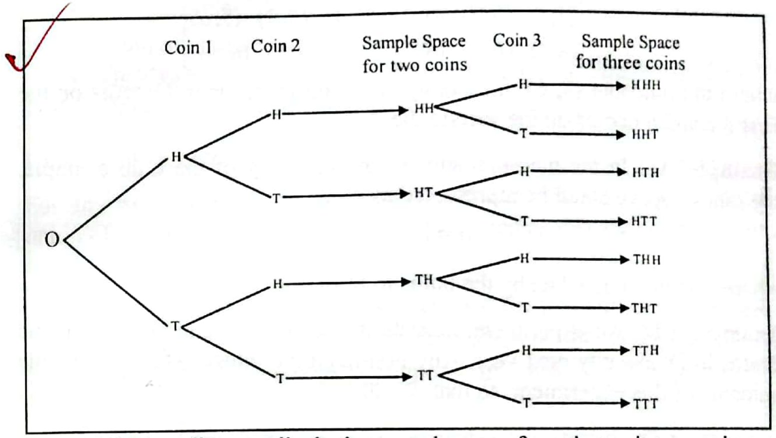


Figure 7.8: Tree diagram displaying sample space for coin tossing experiments

Example 7.10: If the experiment consists of rolling a die, then the sample space can be presented as follows:

$$S = \{x \mid x = 1, 2, 3, ..., 6\}$$

where x represents the number appearing on the uppermost face of the die. A more fundamental sample space for the above experiment is as follows:

$$S = \{1, 2, 3, 4, 5, 6\}$$

Example 7.11: A businessman may make a profit, encounter a loss or may have on a break-even point while running his business. With these possible outcomes, the possible sample space is

$$S = \{\text{Profit, Loss, Break-even}\}$$

Example 7.12: If the experiment involves rolling a pair of dice, then the resulting sample space is of the following form:

$$S=\begin{Bmatrix} (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6)\\ (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6)\\ (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6)\\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6)\\ (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6)\\ (6,1) & (6,2) & (6,3) & (6,4 & (6,5) & (6,6) \end{Bmatrix}$$

where the outcome (3, 6), for example, is said to occur if 3 occurs on the first die and 6 occurs on the second die.

Example7.13: In the measurement of the longevity of the bulb example, the sample space could be represented as

$$S=\{x \mid x\geq 0\}$$

where x is the time taken by the bulbs to burn out.

Example 7.14: An experiment consists in recording the number of traffic deaths in Dhaka city next year. Any non-negative integer is a conceivable outcome of this experiment; so that $S=\{0,1,2,....\}$

Example 7.15: For the microbiology example of counting bacteria in a food specimen, let E_1 correspond to observing 0 bacteria, E_1 for two bacterium, and so on, then

$$S=\{E_0, E_1, E_2, ...\}$$

because no integer number of bacteria can be rules out as a possible outcome.

Event

When an experiment is performed, it can result in one or more experimental outcomes, which are called **events**. In our discussions, events will be denoted by capital letters. As we will see, certain concepts from set theory are useful for expressing the relationships between various events associated with an experiment. Because sets are collections of points, we associate a distinct point, called a **sample point,** with each and every simple event associated with an experiment.

***Definition* 7.5**: *An event in a discrete sample space S is a collection of sample points. That is, any subset of S is known as an event.*

- In Example 7.9, if $A=\{HH, HT\}$, then A is an event that the first coin lands on heads.
- In Example 7.11, $A=\{\text{Loss}\}$, then A is an event that the businessman will incur a loss while running his business.
- In Example 7.12, if $A=\{(3,1\,(2,2)\,(1,3)\}$, then A is the event that sum of points on the dice equals 4.
- In Example 7.14, we can define an event A with fewer than 200 traffic deaths so that $A=\{0,1,2,...199\}$.

Union and Intersection of Events

For any two events A and B, we define the new event $A\cup B$, called the union of events A and B to consist of all outcomes that are either in A or in B or in both A and B.

- In Example 7.9, if A is the event that the first coin falls head and B is the event that second coin falls tail, then $A=\{HH, HT\}$ and $B=\{HT, TT\}$, then the union of the events A and B is $A\cup B=\{HH, HT, TT\}$.
- In Example 7.10, if A is the events of all odd numbers and B is the event of all numbers less 3, then $A=\{1,3,5\}$ and $B=\{1, 2\}$. Hence the union of A and B is $A\cup B=\{1,2,3,5\}$.

For any two events A and B, the **intersection** of A and B is the new event that consists of all events that are common to both A and B. The symbolic representation of the intersection of A and B is $A\cap B$. Thus if in two-die experiment (Example 7.9) $A=\{HH, HT\}$ and $B=\{HT, TT\}$, then the intersection of the events A and B is $A\cap B=\{HT\}$.

We may similarly define union and intersection of more than two events. Thus if S is a sample space of first 10 natural numbers, A is an event of all odd numbers, B is the event of all numbers less than 4 and C is the event of numbers divisible by 3, then the sample space is $S=\{1, 2, 3, .4, 5, 6, 7, 8, 9, 10\}$, and the events A, B, and C are as follows:

$$A=\{1,3,5,7,9\}, B=\{1,2,3\}\ \ C=\{3,,6,9\}.$$

And then the union of A, B, and C is $A\cup B\cup C=\{1,2,3,5,6,7,9\}$ while the intersection of A, B, C is a new event $A\cap B\cap C=\{3\}$.

Note that certain concepts from set theory are useful for expressing the relationships between various events associated with an experiment. Because sets are collections of points, we associate a distinct point, called **sample point**, with each and every simple event with an experiment. The Venn diagram illustrating the union and intersection of events vis-à-vis sets have been displayed in Figure 7.1–Figure 7.5.

Simple Event and Compound Event

A **simple event** is an event that cannot be decomposed. Each simple event corresponds to one and only one sample point.

Thus we can think of a simple event as a set consisting of a single event – namely the single sample point associated with the event. The sample space S associated with the die-tossing experiment consists of six sample points to the six simple events e_1, e_2, e_3, e_4, e_5, and e_6. That is

$$S=\{e_1, e_2, e_3, e_4, e_5, e_6\}$$

Very frequently, we are interested not in any particular simple event but rather the sets or groups of such events. Any such set is simply called **compound event.** For example, in a single die-tossing experiment, the event A (observing an odd number) will occur if and only if one of the simple events with the numbers 1, 3, and 5 occurs. Thus $A = \{e_1, e_3, e_5\}$ is a compound event. Note that A can be decomposed into three simple events with the numbers 1, 3, and 5.

Clearly, for experiments with discrete sample space, compound events can be viewed as collections (sets) of sample points or, equivalently, as union of the sets of single sample points corresponding to the appropriate simple events. This ensures that we can also express A as follows:

$$A = e_1 \cup e_3 \cup e_5$$

Complementary Event

For any event A, we define the new event $\overline{A}$, referred to as the **complement** of A, to consist of all outcomes in the sample space S that are not in A. That is, $\overline{A}$ will occur if and only if A does not occur. With reference to the die tossing experiment, if A is an event of all odd numbers, then its complement, denoted by $\overline{A}$, will consist of all even numbers, That is if $A = \{1,3,5\}$, then $\overline{A} = \{2,4,6\}$. Note that $A \cup \overline{A} = S$ and $A \cap \overline{A} = \phi$

Sure Event and Impossible Event

An event is called a **sure event** if it is certain to happen, while an event is an impossible event, which is impossible to happen. Occurrence of any of the events from 1 to 6 inclusive when a six-sided die is tossed, is a sure event, while occurrence of 7 is an impossible event. In set notation, ϕ is sometimes is used to represent an **impossible** event and S the **certain** or **sure** event.

Equally Likely Events

Two or more vents are said to be equally likely if they have the same chance of occurrence. Examples of equally likely events will be evident once we define probability of an event.

Mutually Exclusive Events

If $A \cap B=\phi$, it is impossible to observe an elementary event that is in both A and B. This means that events A and B cannot both happen simultaneously. Two such events are said to be **mutually exclusive** or **disjoint**. It follows that the events A and B are disjoint or mutually exclusive if and only if $A \cap B = \phi$.

Exhaustive Events

On the other hand, if the union of A and B form the sample space i.e. $A \cup B=S$, we say that the events are **exhaustive.** If both $A \cap B = \phi$ and $A \cup B = S$ are true, we say that S is partitioned into the events A and B.

Event Space

From our foregoing discussion, it appears that the total number of events associated with a given experiment depends on the number of elements in the sample space. If a sample space consists of n elements, then how many events (or subsets) can be formed? Since an event is a subset of a sample space, there will be a total of 2^n subsets or events. The space consisting of these 2^n events is known as the **event space**. The sample space S and the empty set ϕ are also the subsets of this event space. When a coin is tossed, the sample space is $S = \{H, T\}$. For this experiment, the event space will consist of $2^2 = 4$ events. If Ω stands for an event space, then

$$\Omega = \{H, T, S, \phi\}.$$

Thus an event space may be defined as the class of all events associated with a given experiment.

Function

A function is a special kind of notation. If each element of a set S is paired with **exactly one** element of another set T, then the relation is called a **function.** In other words, if each element of the domain is associated with one and only one element of the range, this association is said to be a function or a functional relation. Let S be set of 4 cities of Bangladesh namely Dhaka, Chittagong, Khulna and Rajshahi and T be the set of their mayors. Since every city has one and only one mayor, the relation that each city is paired with a person as its mayor is a functional relation or a function. If we let $S=\{1, 3, 5\}$ and $T=\{2, 4, 6\}$, we may have a relation $R=\{(1, 2), (3, 4), (5, 6\}$. Then R is a function.

7. 7 COUNTING RULES: PERMUTATION AND COMBINATION

The students are already familiar with the concept of experiment and its outcomes. In many experiments, the number of outcomes in the sample space S is so large that a complete listing of these outcomes is almost impossible. In such cases, it is convenient to have a method of determining the number of outcomes in the sample space S and in various events in S without compiling a list of all these outcomes. In this section, some of these rules will be presented in brief.

Multiplication Rule

The multiplication rule states that:

> If an experiment can be performed in n_1 ways and if for each of these, a second operation can be performed in n_2 ways and each of the first two operations, a third operation can be performed in n_3 ways, and so forth, then the number of ways in which the sequence of k operations can be performed is given by

$$n_1 \times n_2 \times n_3. \times \ldots \times n_k$$

Furthermore, if the number of ways in which each of the k operations can be performed remains the same, say n, the total number of ways would be

$$n \times n \times n. \times \ldots \times n = n^k.$$

Example 7.16: Determine the number of sample points in the sample space when a pair of die is rolled once.

Solution: The first die can land in six ways. For each of these six ways, the second die also can land in six ways. Therefore the number of ways in which the pair of dice can land is $6\times 6=36$ (see example 7.12).

Example 7.17: Suppose that there are three different routes from city A to city B and the five different routes from city B to city C. Then the number of different routes from A to C that pass through B is $3\times 5=15$.

Example 7.18: If six coins are tossed, determine the number of sample points.

Solution: Each outcome in S will consist of a sequence of 6 heads and tails, such as *HHTHHT*. Since there are two possible outcomes for each of the two coins, the total number of outcomes in S will be $2^6=64$.

Permutation

A permutation is an arrangement of all or part of a set of objects. If there are n objects in the set, the number of permutations will depend on r, the number of objects you want to select and arrange.

Consider the three letters a, b, and c. The possible permutations of all these three letters are *abc*, *acb*, *bac*, *bca*, *cab*, and *cba*. Thus we see that there are six distinct arrangements. There are three positions to be filled from the letters a, b, and c. Therefore, we have three choices for the first position, then two for the second, and leaving only one choice for the last position, giving a total of $3\times2\times1=6$ permutations. In general n distinct objects can be arranged in $n(n-1)(n-2)\ldots\ldots\ 3\times2\times1$ ways. We represent this product term by the symbol $n!$, which is read "*n factorial*" and is called the permutation of n things taken all together. This is denoted by ${}_nP_n$. Thus

$${}_nP_n = n(n-1)(n-2)\ldots\ldots\ 3\times2\times1 = n!$$

Now suppose that r is less than n. It can be shown that the number of permutations of n distinct objects taken r at a time is given by

$${}_nP_r = n(n-1)\ldots.(n-r+1)\frac{(n-r)(n-r-1)\ldots\ldots\ldots(2)(1)}{(n-r)(n-r-1)\ldots\ldots\ldots(2)(1)}$$

$$= \frac{n!}{(n-r)!}$$

Example 7.19: How many permutations of four letters a, b, c and d can be made taking two at a time?

Solution: By the multiplication rule, we have two positions to fill with 4 choices for the first and 3 choices for the second for a total of 4×3=12 permutations. When enumerated, the possible arrangements are: *ab, ac, ad, ba, ca, da, bc, cb, bd, db, cd,* and *dc*. Here n=4, and r=2. Hence applying the general rule, the number of permutations of n distinct objects taken 2 at a time is

$$_4P_2=\frac{4!}{(4-2)!}=\frac{4\times3\times2\times1}{2\times1}=12.$$

Permutations of Objects Not All Different

So far we have considered the permutations of objects, which are all different. But when one or more objets are repeated, the number of permutations will need adjustment. For instance, consider the word *zoo*, in which '*o*' is repeated 2 times. Therefore the permutations of z, o, o will be 3!/ 2!=3, which are all different. These are **zoo, ozo** and **ooz.** If it had been a word **zip** (say), we would have 6 distinct permutations. Similarly, for the 10-letter word **statistics**, where '*s*' is repeated 3 times, '*t*' 3 times and '*i*' two times, the number of distinct permutations would be 10!/3! 3! 2!. The following rule can now be stated for repeated objects:

Theorem 7.1: *The number of distinct permutations of n things of which n_1 are of one kind, n_2 of a second kind...... and n_k of a kth kind is*

$$\frac{n!}{n_1!n_2!.........n_k!}$$

Example 7.20: There are 9 birthday candles, of which four are yellow, three are red and two are blue. How many ways these candles can be arranged in 9 positions?

Solution: Using Theorem 7.1, the number of distinct arrangements is

$$\frac{9!}{4!3!2!}=1260.$$

Combination

Very often we are interested in the number of ways of selecting r objects from n without regard to order of arrangements. These selections are called **combinations**. Here the arrangements *ab* and *ba* are regarded as the same. Thus, with the 3 letters *a, b, c*, the number of permutations taking two at a time, the arrangements are *ab, ba, ac, ca, bc* and *cb*. But if the order of the arrangements are disregarded (*i.e. ba=ab, ac=ca, bc=cb*), the number of combinations will be 3. A combination is actually a partition with two cells. the one cell containing r objects selected and the other cell containing (n–r) objects that are left.

Let ^nC_r denote the number of combinations of n objects taken r at a time irrespective of order. The symbol ^nC_r is sometimes called binomial coefficient. We have noted earlier that for each set of r things, there are r! permutations. Since combination of r things is a set with r elements, $^nC_r\,r!$ must be equal to the number of permutations of n things taken r at a time. Thus

$$^nC_r\,r!=\frac{n!}{(n-r)!}$$

or

$$^nC_r=\frac{n!}{r!(n-r)!}$$

It is helpful to note that

$$^nC_r={}^nC_{n-r}$$

In particular

$$^{10}C_7={}^{10}C_3,\ {}^{50}C_{48}={}^{50}C_2,\text{ etc}$$

Example 7.21: There are 20 people in a room of whom 12 are men and 8 are women. A committee of 3 is to be formed from them. How many ways can this be done? If it is desired that the committee will consist of 2 men and 1 woman, in how many ways can this be done?

Solution: It is evident that the order of the selected people is of no importance. Therefore this is a problem of combination with n=20 and r=3. Thus

$$^{20}C_3=\frac{20!}{3!(20-3)!}=\frac{20!}{3!(17)!}$$
$$=\frac{20\times19\times18\times17!}{3!\,17!}=1140.$$

The number of ways in which 2 men and 1 woman will be in the committee is

$$^{12}C_2\times{}^8C_1=66\times8=528$$

7.8 ASSIGNING PROBABILITIES TO EXPERIMENTAL OUTCOMES

In assigning probabilities to experimental outcomes, there are various acceptable approaches, however, regardless of the approach taken, the following two basic requirements must be satisfied:

- The probability values assigned to each experimental outcome (sample point) must be between 0 and 1. That is, if A stands for the experimental outcome and P(A) for the probability of the outcome, we must have

$$0 \le P(A_i) \le 1 \qquad \text{for all } i$$

- The probabilities of all the experimental outcomes must sum to 1. For example, if a sample space has k sample points, then we must have

$$\sum_{i=1}^{k} P(A_i) = 1$$

There are usually three different approaches to compute probability of any specified event. These are

- Classical approach
- Relative frequency approach and
- Subjective approach

We provide below a brief description of these approaches.

~~7.8.1~~ Classical Approach

In most probability problems, the computation of probability is merely a problem in counting. This is true especially, if the elementary events have a uniform probability distribution, that is, when all the experimental outcomes are equally likely. This is the so-called **classical method.** As a simple example, consider the experiment of tossing a fair coin, where there are two equally likely outcomes: head (H) and tail (T). Therefore, logic suggests that the probability of observing a head, denoted by P(H) is 0.5 and that the probability of observing a tail denoted by P(T), is also 0.5. Further note that each of the probabilities lies between 0 and 1 and that $P(H)+P(T)=1$, because H and T are all of the experimental outcomes.

***Definition* 7.6:** *If a random experiment can result in n(S) mutually exclusive, exhaustive and equally likely outcomes and if n(A) of these outcomes are favorable to an event A, then the probability of A is the ratio of n(A) to n(S). In symbol,*

$$\boxed{P(A) = \frac{n(A)}{n(S)}}$$

The definition under classical approach referred to above is also known as **a priori** or **mathematical** definition of probability.

The following examples illustrate the applications of classical definition:

Example 7.22: A bag contains 4 white and 6 red balls. A ball is drawn at random from the bag. What is the probability that it is red? That it is white? Are the events obtaining a red ball and obtaining a white ball equally likely?

Solution: A possible sample space for this experiment is $S=\{w_1, w_2, w_3, w_4, r_1, r_2, r_3, r_4, r_5, r_6\}$, where w stands for the white ball and r for the red ball. Let R be the event that the ball is red and W be the event that the ball is white. Here $n(S)=10$, $n(W)=4$, and $n(R)=6$. Hence

$$\text{(i)}\ P(R) = \frac{n(R)}{n(S)} = \frac{6}{10} = 0.6,\ \text{(ii)}\ P(W) = \frac{n(W)}{n(S)} = \frac{4}{10} = 0.4$$

Since $P(R) \neq P(W)$, the occurrences of the events R and W are not equally likely.

Example 7.23: An ordinary die is rolled once. Find the probability that (i) an even number occurs and (ii) a number greater than 4 occurs

Solution: Let $S=\{1, 2, 3, 4, 5, 6\}$. If A denotes an even number and B a number greater than 4, then $A=\{2, 4, 6\}$ and $B=\{5, 6\}$, then

$$\text{(i)}\ P(A) = \frac{n(A)}{n(S)} = \frac{3}{6} = \frac{1}{2} \text{ and (ii) } P(B) = \frac{n(B)}{n(S)} = \frac{2}{6} = \frac{1}{3}.$$

Example 7.24: A newly married couple plans to have two children, and suppose that each child is equally likely to be a boy or a girl. In order to find a sample space for this experiment, let B denote that a child is a boy and G denote that a child is a girl. Then one possible sample space that can be formed is

$$S = \{BB,\ BG,\ GB,\ GG\}$$

The double BG, for instance represents the outcome 'the older child is a boy', while 'the younger one is a girl'.

(a) What is the probability that the couple will have two boys?
(b) What is the probability that the couple will have one boy and one girl?

(c) What is the probability that the couple will have at most one boy?

Solution: Let A_1, A_2, and A_3 be the events that the couple will have two boys, one boy one girl, and at most one boy respectively so that

$$A_1 = \{BB\},\ A_2 = \{BG, GB\},\ A_3 = \{BG, GB, GG\}.$$

Since by assumption, all the points in S are equally likely, that is $P(BB) = P(BG) = P(GB) = P(GG) = 1/4$, we can use classical approach to compute the required probabilities. Hence

$$\text{(a) } P(A_1) = \frac{n(A_1)}{n(S)} = P(BB) = \frac{1}{4}$$

$$\text{(b) } P(A_2) = \frac{n(A_2)}{n(S)} = P(BG) + P(GB) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

$$\text{(c) } P(A_3) = \frac{n(A_3)}{n(S)} = P(BG) + P(GB) + P(GG) = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{3}{4}$$

Example 7.25: A fair coin is tossed three times. Compute the probability that (a) exactly two tosses result in heads, and (b) at most one toss results in head.

Solution: The experiment consists of observing the outcomes (H or T) for each of the three tosses of the coin. One of the ways of presenting the sample space is as follows:

$$S = \{HHH,\ HHT,\ HTH,\ HTT,\ THH,\ THT,\ TTH,\ TTT\}$$

Because the coin is fair, we would expect the outcomes to be equally likely. That is, if A_i represents the i-th outcome, then

$$P(A_i) = \frac{1}{8}, \quad i = 1, 2, \ldots, 8$$

For (a), let the event of interest be A, so that

$$A = \{HHT, HTH, THH\} \text{ and hence}$$

$$P(A) = P(HHT) + P(HTH) + P(IHH) = \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{3}{8}$$

For (b), let the event of interest be B, so that

$$B = \{TTT, THT, TTH, HTT\}, \text{ and hence}$$

$$P(B) = P(TTT) + P(THT) + P(TTH) + P(HTT) = \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2}$$

Example 7.26: For coin tossing experiment in Example 7.24, find the probability of obtaining (a) exactly two runs and (b) Less than two runs.

Solution: Any unbroken sequence of like letters is called a run, even though the sequence has only one letter. Thus the outcome HHH has one run while the outcome HTT has two runs. We now enumerate the number of runs in the above coin tossing experiment in a tabular form as below:

Outcome	Event	Number of runs
HHH	A_1	1
HHT	A_2	2
HTH	A_3	3
HTT	A_4	2
THH	A_5	2
THT	A_6	3
TTH	A_7	2
TTT	A_8	1

Let X denote the number of runs. Hence for (a), there are four cases which are favorable to (a) and hence the required probability is

$$P(X = 2) = P(A_2) + P(A_4) + P(A_5) + P(A_7) = \frac{4}{8} = \frac{1}{2}$$

Similarly, for (b) the event of interest is $X < 2$ and number of cases favorable to this event is 2, so that the required probability is.

$$P(X < 2) = P(A_1) + P(A_8) = \frac{2}{8} = \frac{1}{4}$$

Example 7.27: A businessman has a stock of 8400 baby wears imported from 5 different countries. The distribution of the wears was as follows:

Country	Number of wears
USA	1500
India	1200
China	2700
Korea	1000
Thailand	2000
Total	8400

A piece of baby wear was selected at random. What is the probability that it was imported from (i) USA, from (ii) China, and (iii) either from India or from Thailand?

Solution: Using classical definition of probability, we find that

$$P(\text{USA}) = \frac{1500}{8400} = 0.18, \quad P(\text{China}) = \frac{2700}{8400} = 0.32$$

$$P(\text{India or Thailand}) = \frac{1200}{8400} + \frac{2000}{8400} = 0.13 + 0.24 = 0.37$$

Example 7.28: A leap year consists of 366 days with 29 days in February. If a leap year is selected at random, what is the probability that the selected leap year will consist of 53 Saturdays?

Solution: A leap year consists of 52 complete weeks ($52 \times 7 = 364$ days) plus two additional days with the following possible combinations of the days of the week:

(a) Sunday and Monday
(b) Monday and Tuesday
(c) Tuesday and Wednesday
(d) Wednesday and Thursday
(e) Thursday and Friday
(f) Friday and Saturday
(g) Saturday and Sunday

Clearly, two of the 7 combinations above, contain Saturday. Thus if A is the event that one of these combinations will consist of Saturday, then

$$P(A) = \frac{2}{7}.$$

We observe that by classical definition, the probability of any event is a number between 0 and 1 inclusive. If the event is certain not to occur, its probability is zero, implying that the event in question is an impossible event. Such an event is usually denoted by ϕ, so that $P(\phi)=0$. If it is certain to happen, its probability is 1. Thus the probability of obtaining a 7 in tossing a die is 0. The probability that the die will show up a number less than 7 is a sure event (which we denote by S), and hence its probability is 1.

The classical method was developed originally in the analysis of gambling problems, where the assumption of equally likely outcomes is often reasonable. In many problems, however, this assumption is not valid. Hence alternative methods of assigning probabilities are required. One such approach is relative frequency approach, which we discuss below.

7.8.2 The Relative Frequency Approach

Very often probability is interpreted to be a **long-run relative frequency**. As an example, consider repeatedly tossing a coin. If we get 6 heads in the first 10 tosses, then the relative frequency of heads is 6/10=0.6. We now increase the number of tosses and record the number of heads so obtained as a result of tossing the coin. Suppose the results of these tosses were as shown below:

Number of tosses	Observed frequency of head	Observed relative frequency of head	Long-run relative frequency
10	6	0.6000	0.5
100	58	0.5800	0.5
1000	556	0.5560	0.5
10000	5067	0.5067	0.5

Since the relative frequency of heads is getting closer and closer to 0.5, we might estimate that the probability of obtaining a head when tossing the coin is 0.5. When we say this, we mean that, if we tossed the coin an infinitely large number of times (that is the number tosses approaching infinity), the relative frequency of heads obtained would approach 0.5. Of course, in actuality, it is impossible to toss a coin (or performing an experiment) an indefinitely large number of times. Therefore, a relative frequency interpretation of probability is a mathematical idealization. To summarize, suppose that A is an experimental outcome that might occur when a particular experiment is performed. We assume that the relative frequency of A stabilizes to a certain idealized value p as the number of repetitions of the experiment becomes large and this stabilized value is referred to as the **probability of the event** A under reference. This leads us to define probability of an event under relative frequency approach as follows:

***Definition* 7.7**: *If an experiment is repeated n times under similar conditions as a result of which an event A occurs m times, then the limit of the ratio m/n tends to an idealized value as n becomes infinitely large. This idealized value is called the probability of the event A. Symbolically,*

$$P(A) = \operatorname*{Limit}_{n \to \infty}\left(\frac{m}{n}\right)$$

More specific interpretation of this definition is that, if an experiment is repeated a large number of times, there is a high probability that the proportion of repeats producing a specific event will be very close to the probability of that event. The third column of the above table shows this feature. As a further illustration of the idea of the relationship between the relative frequency and probability of an event, we show below a simulated result of flipping a perfect die 10,000 times by Ross (2005: 154).

	Face of the die showing					
	1	2	3	4	5	6
Frequency	1724	1664	1628	1648	1672	1664
Relative frequency	.1724	.1664	.1628	.1648	.1672	.1662

Note: 1/6=0.166667

The definition provided under relative frequency approach is also known as **aposteriori** or **statistical** or **empirical** definition of probability

Conceptually, the frequency definition of probability is a more appropriate definition of probability. It is based on empirical observations and thus one feels confident to rely on this as a definition of probability. Consider the following examples to see how this definition works in understanding real life problems.

Example 7.29: Suppose you want to predict that a student being admitted in the first year honors class in Economics will belong to tribal area in any particular year. If our admission records of several years in the past reveals that 12 percent of the admitted students come from tribal areas, then it might be reasonable to assume that the probability of a tribal student being admitted in the class is approximately 0.12.

Example 7.30: The Dean of Science has noticed that, according to past records, only 55% of the students who begin a program successfully graduate from the programs 4 years later. We choose a name at random from the list of beginning students to evaluate the chance that he will successfully graduate from the program in 4 years. Basing probabilities on the statistical record, the student has a 55% chance and, hence, a probability of 55/100, or more simply 11/20 of graduating successfully. This is a problem that falls under the frequency interpretation of probability.

Limitations of Frequency Definition

Despite its superiority and efficiency over the classical definition, there are several shortcomings of the frequency interpretation of probability. Some of these are as follows:

(i) An experiment cannot always be repeated under similar conditions. These conditions must be specified precisely.

(ii) It may be sometimes dangerous to repeat an experiment under similar conditions. Consider the tossing of a fair coin. If it can be made that the coin is repeated under similar conditions, then the coin may almost always result in heads or tails. Hence the coin must not be completely controlled, but must have some random feature.

(iii) The term 'large number of times' is too vague to interpret. There is no definite indication of an actual number that would be considered large enough. Moreover, it is not practically feasible to repeat the experiment an infinite number of times under identical conditions.

(iv) It is stated that the relative frequency should tend to a stable value p, but no limit is specified for the permissible variation from p.

~~7.8.3~~ Subjective Approach

The two general types of probability discussed above have one important point in common: they both require a conceptual experiment in which the various outcomes can occur under somewhat uniform conditions. Such a situation may not exist in reality. For example, we might like to know: what is the probability that Awami League will win the next three consecutive national elections and form the government? What is the probability that a severe cyclone will hit coastal areas by the year 2015? Such queries are legitimate and may be included in what is known as **subjective probability.**

Under subjective approach, the probability that a person assigns to a possible outcome of some experiment represents his own judgment of the likelihood that the outcome will be obtained. This judgment will be based on that person's degree of beliefs, his experience, guesses, intuition and prior information about the process. Another person might hold a different view and thus might assign a different probability to the same process. It is in this sense; the subjective probability has no objective basis and thus is neither true nor unique. You may decide to purchase a car before the financial budget is announced in next July, because you are anticipating that new tax will be imposed on the import of car. Your friend may rule out such possibility and thus may defer his decision until next December. We may now define subjective probability as follows:

***Definition 7.8**: Subjective probability is the probability that an individual assigns to an event on the basis of his/her own experience, judgment, guesses, intuition, prior information or beliefs.*

7.8.4 Axioms of Probability

Although relative frequency does not provide a rigorous definition of probability, any definition applicable to the real world should agree with our intuitive notion of the relative frequencies of events. On analyzing the relative frequency concept of probability, we observe that three conditions must hold:

1. The relative frequency of occurrence of any event must be greater than or equal to zero.
2. The relative frequency of the entire sample space S must add to unity.
3. If two events are mutually exclusive, the relative frequency of their union is the sum of their respective relative frequencies.

The above three conditions form the basis of axiomatic definition of probability, which we present below.

Suppose S is a sample space associated with an experiment. To every event A, in S, (A is a sub-set of S), we assign a number, $P(A)$, called the probability of A, so that the following axioms hold:

Axiom 1: $P(A) \geq 0$

Axiom 2: $P(S) = 1$

Axiom 3: If $A_1, A_2, ...$ *form a sequence of pair-wise mutually exclusive events in S (that is* $A_i \cap A_j = \phi$ if $i \neq j$*), then*

$$P(A_1 \cup A_2 \cup A_3 \cup ...) = \sum_{i=1}^{\infty} P(A_i)$$

We can show that Axiom 3, which is stated in terms of an infinite sequence of events, implies a similar property for a finite sequence of events. Specifically if $A_1, A_2, , A_n$ are pair-wise mutually exclusive, then

$$P(A_1 \cup A_2 \cup A_3 \cup ... \cup A_n) = \sum_{i=1}^{n} P(A_i)$$

Axiom 1 is referred as the axiom of **positiveness**, which states that the probability of every event must be non-negative. If $\overline{A}$ stands for the complement of A, then this axiom also assumes that $P(\overline{A}) \geq 0$. Axiom 2 is referred to as the axiom of **certainty** and states that if an event is certain to occur, then the probability of that event is 1. Axiom 3 is known as the axiom of **additivity** or **unions.**

Example 7.31: A manufacturer has five seemingly identical cars available for shipment. Unknown to the manufacturer, 2 of the 5 cars have defective steering. A particular order calls for supplying randomly two of the cars. Find the probability that (i) both of the cars had defective steering, (ii) none had defective steering, and (iii) one had defective steering.

Solution: Let the two cars with defective steering be labeled D_1 and D_2 and the three cars with good steering be labeled G_1, G_2 and G_3 respectively. The sample space is then set up as follows:

$$S = \{D_1, D_2, G_1, G_2, G_3\}$$

The random selection of the two cars will have the following combinations of simple events:

$$A_1 = \{D_1, D_2\},\ A_2 = \{D_1, G_1\},\ A_3 = \{D_1, G_2\},\ A_4 = \{D_1, G_3\},\ A_5 = \{D_2, G_1\},$$
$$A_6 = \{D_2, G_2\},\ A_7 = \{D_2, G_3\},\ A_8 = \{G_1, G_2\},\ A_9 = \{G_1, G_3\},\ A_{10} = \{G_2, G_3\}$$

(i) Let A be the event that both the cars had defective steering, so that $A == \{A_1\}$

Since the cars were selected at random, any pair of cars is as likely to be selected any other pair. Thus, $P(A_i) = 1/10$, $i = 1, 2, ..., 10$. Hence

$$P(A) = \frac{n(A)}{n(S)} = \frac{1}{10}.$$

(ii) Let B be the event that both had good steering, then the number of elements in B will be 3: $B = \{A_8, A_9, A_{10}\}$. Hence

$$P(B) = \frac{n(B)}{n(S)} = \frac{3}{10}.$$

(iii) Let C stand for the event that the shipment will have one defective and one non-defective steering. Then $C = \{A_2, A_3, A_4, A_5, A_6, A_7\}$, so that

$$P(C) = \frac{n(C)}{n(S)} = \frac{6}{10}.$$

7.9 THE ODDS

Sometimes the relative chances of an event A to its complement $\overline{A}$ are expressed in terms of the **odds** in favor of A. We denote these odds by O_A defined ad

$$O_A = \frac{P(A)}{P(\overline{A})}$$

Sometimes these are also called **odds against** $\overline{A}$.

On the other hand, if we know that the odds in favor of A are a/b, we can compute $P(A)$ and $P(\bar{A})$ as follows:

$$\frac{P(A)}{P(\bar{A})}=\frac{a}{b}$$

This gives

$$P(A)=\frac{a}{b}P(\bar{A}).$$

Since A and $\bar{A}$ are complementary to each other and exhaustive,

$$P(A)+P(\bar{A})=1$$

or

$$\frac{a}{b}P(\bar{A})+P(\bar{A})=1$$

This leads to the following solution:

$$P(A)=\frac{a}{a+b} \text{ and } P(\bar{A})=\frac{b}{a+b} \qquad (*)$$

In terms of a and b, we define the odds as follows:

***Definition* 7.9**: *If an experiment results in n(S) equally likely outcomes, such that 'a' is favorable to an event A and 'b' is favorable to its complement $\bar{A}$, then the odds in favor of A are a/b. Symbolically*

$$O_A=\frac{a}{b} \qquad \text{... (**)}$$

This is also referred to as the odds against $\bar{A}$. Logically, the odds against A (vis a vis odds in favor of $\bar{A}$) are b/a which we write as

$$O_{\bar{A}}=\frac{b}{a} \qquad \text{... (***)}$$

Example 7.32: From a class of 40 students with 25 girls, one student is chosen at random. What is the probability that a boy is chosen? Find the odds in favor of choosing a boy and interpret the result.

Solution: Since the student is chosen by lot, there are 40 equally likely outcomes. Of these, 15 correspond to the event 'a boy is chosen'. Consequently, $n(S)=40$ and $n(A)=15$, where A is the event of interest. Hence by definition

$$P(A)=\frac{n(A)}{n(S)}=\frac{15}{40}=\frac{3}{8}.$$

The odds in favor of the boy to be chosen is

$$O_A=\frac{P(A)}{P(\bar{A})}=\frac{3}{5}=60\%.$$

This implies that a boy has 100-60=40% less chance than a girl to be chosen.

The odds in favor of $\bar{A}$ is expressed as follows:

$$O_{\bar{A}}=\frac{P(\bar{A})}{P(A)}=\frac{5}{3}=1.67.$$

This means that a girl has 67% more chance than a boy to be chosen.

Example 7.33: If a political candidate has only 35% chance of winning an election, what are the odds against winning the election for the candidate?

Solution: If A stands for the event 'winning the election'

$$P(A)=0.35 \text{ and } P(\bar{A})=0.65$$

Hence the odds against winning the election for the candidate is

$$O_{\bar{A}}=\frac{P(\bar{A})}{P(A)}=\frac{0.65}{0.35}=\frac{13}{7}$$

This means that the candidate is $(13/7-1)\times 100=85.7\%$ more likely to loose the election than winning.

Example 7.34: The odds are two to one that A wins, when A and B play tennis. Find their respective chances of winning.

Solution: It is given that

$$\frac{P(A)}{P(B)}=\frac{a}{b}=\frac{2}{1}$$

By virtue of (*)

$$P(A)=\frac{a}{a+b}=\frac{2}{2+1}=\frac{2}{3} \text{ and } P(B)=\frac{b}{a+b}=\frac{1}{2+1}=\frac{1}{3}$$

7.10 JOINT PROBABILITY

Two or more events form a **joint event** if all of them occur simultaneously and probability of these joint events are called the **joint probabilities**. Thus all events of the form $A\cap B$, $A\cap B\cap C$, $A\cap B\cap C\cap D$ or

$A_1 \cap A_2 \cap A_3 \cap ... \cap A_n$ are joint events. Thus if A is the event "smoker" and B "heart disease patient", then $A \cap B$ is a joint event describing that a randomly chosen person is a smoker who suffers from heart disease. Similarly, if a randomly chosen person reads "The Times", "The New Nations" and "The Daily Star", we can construct the joint event of the form $A \cap B \cap C$ for those who read all these three dailies.

Example7.35: Suppose a sample space consists of 500 persons and are distributed according to their sex and employment status as shown in the accompanying table.

Sex	Employment status		
	Employed (E)	Unemployed (U)	Total
Male (M)	255	20	275
Female (F)	80	145	225
Total	335	165	500

One of these 500 persons was selected at random. We define the following simple events:

M: The selected person is a male
F: The selected person is a female
E: The selected person is employed
U: The selected person is unemployed

The joint events that can be formed are

$M \cap E$: The selected person is a male and employed
$M \cap U$: The selected person is a male and unemployed
$F \cap E$: The selected person is a female and employed
$F \cap U$: The selected person is a female and unemployed

Using the tabulated values, we can find several probabilities, such as $P(M)$, probability that the selected person is a male, $P(M \cap E)$, probability that the selected person is male and employed and so on. Since the totals $n(M)$, $n(F)$, $n(E)$ and $n(U)$ all appear in the margins of the table, they are called **marginal totals** and the corresponding probabilities $P(M)$, $P(F)$, $P(E)$ and $P(U)$ are all called **marginal probabilities.**

From the given table, we have four marginal totals:

$$n(M) = 275,\ n(F) = 225,\ n(E) = 335,\ n(U) = 165$$

Hence the marginal probability that a randomly chosen person will be a male is

$$P(M) = \frac{n(M)}{n(S)} = \frac{275}{500} = 0.55$$

Similarly the marginal probability that a randomly chosen person will be employed is

$$P(E) = \frac{n(E)}{n(S)} = \frac{335}{500} = 0.67$$

What is the probability that a randomly chosen person is a male and at the same time employed? This is a joint probability and the associated event of interest is $M \cap E$. Thus

$$P(M \cap E) = \frac{n(M \cap E)}{n(S)} = \frac{255}{500} = 0.51$$

Similarly, we can find the probability that the selected male is unemployed:

$$P(M \cap U) = \frac{n(M \cap U)}{n(S)} = \frac{20}{500} = 0.04$$

Note that the marginal probability can also be computed as a sum of the two joint probabilities:

$$P(M) = P(M \cap E) + P(M \cap U) = .51 + .04 = 0.55$$

as ought to be.

Example 7.36: In an office of 100 employees, 75 read English, 50 read Bangla dailies and 40 read both. An employee is selected at random. What is the probability that the selected employee

(a) Reads English newspaper?
(b) Reads at least one of the papers?
(c) Reads none?
(d) Reads English but not Bangla?

Solution: Let us define the above events:

E = Reads English
B = Reads Bangla only
$\bar{E} \cap \bar{B}$ = Reads none
$B \cap \bar{E}$ = Reads Bangla but not English

The number of cases favorable to the above events can be placed in a tabular form as follows:

	E	$\overline{E}$	Total
B	$n(B\cap E)=40$	$n(B\cap \overline{E})=?$	$n(B)=50$
$\overline{B}$	$n(\overline{B}\cap E)=?$	$n(\overline{B}\cap \overline{E})=?$	$n(\overline{B})=50$
Total	$n(E)=75$	$n(\overline{E})=25$	$n(S)=100$

[illegible]

$$P(E)=\frac{n(E)}{n(S)}=\frac{75}{100}=0.75$$

(b) The probability that the selected employee reads at least one (i.e. either E or B or both) is

$$P(E\cup B)=P(E)+P(B)-P(E\cap B)$$
$$=\frac{75}{100}+\frac{50}{100}-\frac{40}{100}=\frac{85}{100}=0.85$$

(c) The probability that the selected employee reads none (i.e. neither E nor B)

$$P(\overline{E}\cap \overline{B})=P(\overline{E\cup B})=1-P(E\cup B)=1-0.85=0.15$$

(d) The probability that the selected employee reads Bangla but not English is

$$P(B\cap\overline{E})=\frac{n(B)-n(B\cap E)}{n(S)}=\frac{50-40}{100}=\frac{10}{100}=0.10$$

7.11 CONDITIONAL PROBABILITY

The probability of an event A when it is known that some other event B has occurred is called a **conditional probability** and is denoted by $P(A|B)$. The symbol $P(A|B)$ is usually read as 'the probability that A occurs given that B occurs' or simply probability of A given B, where the slash '|' stands for 'given that'. In general $P(A|B)$ is not equal to $P(A)$.

With two events A and B, the most fundamental formula to compute conditional probability for A given B is

$$P(A\mid B)=\frac{P(A\cap B)}{P(B)},\quad P(B)\neq 0 \qquad \dots (a)$$

and that for B given A is

$$P(B\mid A)=\frac{P(A\cap B)}{P(A)},\quad P(A)\neq 0$$

It thus follows from the above equations that for two dependent events A and B,

$$P(A\cap B)=P(A)P(B|A)$$
$$=P(B)P(A|B)$$

[illegible]

This rule is frequently referred to as the **multiplication law**, **multiplication theorem** or **law of compound probability**. [illegible] be stated more [illegible]:

Definition 7.10: For two events A and B, the probability of their simultaneous occurrence is equal to the product of the unconditional probability of A and the conditional probability of B, given that A has actually occurred [illegible]

$$P(A\cap B)=P(A)\cdot P(B|A)$$

We can extend the above multiplication [illegible] events 3 or more. For 3 events A_1, A_2, and A_3,

$$P(A_1\cap A_2\cap A_3)=P(A_1)P(A_2|A_1)P(A_3|A_1\cap A_2)$$

where $A_3|A_1\cap A_2$ is read as "A_3 occurs given that A_1 and A_2 have already occurred".

For k events, the rule is as follows:

$$P(A_1\cap A_2\cap\dots\cap A_k)=P(A_1)P(A_2|A_1)P(A_3|A_1\cap A_2)\dots P(A_k|A_1\cap\dots\cap A_{k-1})$$

Refer to the table [illegible] in the preceding section. Suppose that the selected person was known to be a male. We now ask: what is the probability under the changed situation that he is employed? This is a problem of conditional probability and we symbolically write this as $P(E|M)$.

The probability $P(E|M)$ can be computed once $P(E\cap M)$ and $P(M)$ are known from the original sample space:

$$P(E\mid M)=\frac{P(E\cap M)}{P(M)}=\frac{.51}{.55}=0.93$$

An alternative way of computing $P(E|M)$ is to use the **reduced sample space** M, which is a part of S. To accomplish this, note that

$$[illegible] = \frac{n(E \cap M)}{n([illegible])} \text{ and } P(M) = \frac{n(M)}{n([illegible])}$$

Substituting [illegible]se quantities in ([illegible])

$$P(E|M) = \frac{n(E \cap M)}{[illegible](M)}$$

Referring once again to the tabular values

$$P(E|M) = \frac{n(E \cap M)}{[illegible]} = \frac{255}{[illegible]} = 0.93$$

as before.

Example 7.37: A pair of dice is [illegible]. Find [illegible] probab[illegible] sum of the points on the two dice is 10 or greater if a 5 appears on the first die.

Solution: Let A be the event that sum of the points on the two dice is 10 or greater and B be the event that a 5 appears on the first toss. Symbolically we want to eval[illegible] conditional probability $P(A|B)$.

Now

$$A = \{(4, 6), (5, 5), (5, 6), (6, 4), (6, 5), (6, 6)\},$$

$$[illegible] (5, 4), [illegible]$$

$$A \cap B = \{(5, 5), (5, 6)\}$$

Hence

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{2/[illegible]}{6/36} = \frac{[illegible]}{3}$$

Alternatively, if B is considered as a reduced sample space, then only two sample points, viz. (5, [illegible]) and (5, 6) are favorable to the event that the sum [illegible] 10 or [illegible] sample points in B. [illegible] probability is

$$P(A|B) = \frac{[illegible]}{[illegible]} = \frac{1}{3}$$

as [illegible] to [illegible]

Example 7.38: The probability that a married man watches a certain TV show is 0.4 and that his wife [illegible] 0.5. The probability that a man watches the [illegible] given [illegible] wife does [illegible] 7. Find

(a) The probability that a married couple watches the show.

(b) The probability that a wife watches the show given that her husband does.

(c) The probability that at least one of the partners will watch the show.

[illegible] Let [illegible] events H and W as [illegible]

H: Husband watches the show

W: Wife watches the show

We are given that

$$P(H) = 0.4,\ P(W) = 0.5 \text{ and } P(H|W) = 0.7.$$

(a) The probability that the couple watches the show is

$$P(W \cap H) = P(W)\,P(H|W) = 0.5 \times 0.7 = 0.35$$

(b) The conditional probability that a wife watches the show given that her husband also watches

$$P(W|H) = \frac{P(W \cap H)}{P(H)} = \frac{0.35}{0.40} = 0.875$$

(c) The probability that at least one (either H or W or both) watches [illegible]

$$[illegible]$$

$$= 0.40 + 0.50 - 0.35 = 0.55$$

Example 7.39: A box contains 7 red balls and 3 black balls. Three balls are drawn from the box one after the other. Find the probability that the fi[illegible] and the th[illegible] black [illegible] the next draw if (ii) the balls are not replaced.

Solution: Let [illegible] drawing a black ball. If a ball is replaced before the next draw, then the subsequent drawings are not affected [illegible] the number of balls in [illegible] remain the same. [illegible] probability. The symbols R_1, R_2, B_3 are the events representing first ball, second ball [illegible] third [illegible] that

$$P(R_1 \cap R_2 \cap B_3) = P(R_1) \times P(R_2) \times P(B_3)$$

$$= \left(\frac{7}{[illegible]}\right)[illegible]\left(\frac{3}{[illegible]}\right) = \frac{14}{[illegible]}$$

When the balls are not returned to the bag before the next dra[illegible] case of conditional probability, when the number of balls will decrease [illegible] the required probability [illegible]

$$P(R_1 [illegible]) [illegible] P(B [illegible]$$

$$= \left(\frac{7}{10}\right)\left(\frac{6}{9}\right)\left(\frac{3}{8}\right) = \frac{[illegible]}{40}$$

Example 7.40: A coin is tossed until a head appears or it has been tossed three times. Given that the head does not appear on the first toss, what is the probability that the coin is tossed three times?

Solution: A sample space for the experiment is $S=\{H, TH, TTH, TTT\}$
The associated probabilities are

$$P(H)=\frac{1}{2},\ P(TH)=\frac{1}{4},\ P(TTH)=\frac{1}{8},\ P(TTT)=\frac{1}{8}$$

Let A be the event that the coin is tossed 3 times and B be the events that no heads appear on the first toss so that

$A=\{TTH, TTT\}$, $B=\{TH, TTH, TTT\}$, and hence $A\cap B=\{TTH, TTT\}$

The associated probabilities are

$$P(A)=\frac{1}{8}+\frac{1}{8}=\frac{1}{4},\ P(B)=\frac{1}{4}+\frac{1}{8}+\frac{1}{8}=\frac{1}{2} \text{ and } P(A\cap B)=\frac{1}{8}+\frac{1}{8}=\frac{1}{4}$$

Hence the required conditional probability is

$$P(A\mid B)=\frac{P(A\cap B)}{P(B)}=\frac{1}{2}. \rightarrow \text{खतम}$$

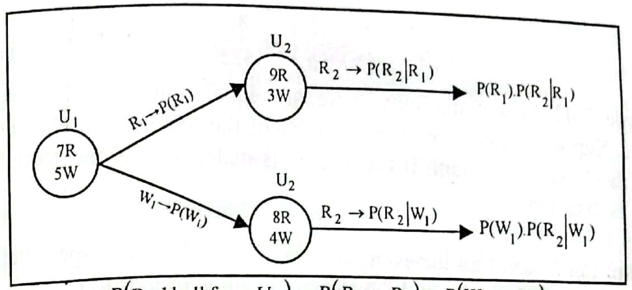
W

Example 7.41: Suppose we have two urns U_1 and U_2. The first urn contains 5 white balls and 7 red balls, while the second urn contains 3 white balls and 8 red balls. One ball is transferred from the first urn to the second urn unseen and then a ball is drawn from the second urn. What is the probability that this ball is red?

Solution: The ball to be drawn from the second urn with a specified color is conditional upon the color of the ball to be transferred from U_1. If a white ball is transferred from U_1, then U_2 will consist of 4 white balls and 8 red balls. On the other hand, if the ball transferred from U_1 happens to be red, then U_2 will consist of 3 white balls and 9 red balls. Then the probability that the ball drawn from U_2 will be red is given by summing the probabilities of following two events:

(a) When the ball transferred is red
(b) When the ball transferred is white

Let R_1, R_2 and W_1 represent, respectively the drawing of a red ball from U_1 and a white ball from U_2. We are then interested in the union of the two mutually exclusive events $R_1\cap R_2$ and $W_1\cap R_2$. The computation of the probability of this problem is facilitated by referring to the following diagram:



$$P(\text{Red ball from } U_2)=P(R_1\cap R_2)+P(W_1\cap R_2)$$

where

$$P(R_1\cap R_2)=P(R_1)P(R_2\mid R_1)=\frac{7}{12}\left(\frac{9}{12}\right)=\frac{63}{144}.$$

$$P(W_1\cap R_2)=P(W_1)P(R_2\mid W_1)=\frac{5}{12}\left(\frac{8}{12}\right)=\frac{40}{144}.$$

so that

$$P(\text{Red ball from } U_2)=\frac{63}{144}+\frac{40}{144}=\frac{103}{144}.$$

What would be the probability of drawing a white ball from the second urn if other conditions remain the same? Students are advised to draw a tree diagram and compute the probability of this event.

Example 7.42: Three cards are drawn from in succession, without replacement, from an ordinary deck of playing cards. Find the probability that the first card is a red ace; the second card is a 10 or a jack and the third card is greater than 3 but less than 7.

Solution: Let us define the events

A : the first card is a red ace
B : the second card is a 10 or a jack
C : the third card is greater than 3 but less than 7.

Now

$$P(A)=\frac{2}{52},\ P(B\mid A)=\frac{8}{51},\ P(C\mid A\cap B)=\frac{12}{50}.$$

Hence employing multiplicative rule of probability for three event

$$P(A\cap B\cap C)=P(A)P(B\mid A)P(C\mid A\cap B)$$

$$=\left(\frac{2}{52}\right)\left(\frac{8}{51}\right)\left(\frac{12}{50}\right)=\frac{8}{5525}$$

Example 7.43 In a community there are equal number of males and females. Suppose 5% of the males and 2% of the females are disabled. A person is chosen at random. If this person is male, what is the probability that he is disabled?

Solution: Let D stand for the event 'disabled' and M and F respectively for male and female. As males and females are in equal proportion,

$$P(M)=P(F)=0.5.$$

Also

$$P(M\cap D)=.05,\ P(F\cap D)=.02.$$

We want the conditional probability that the selected person is disabled:

$$P(D\mid M)=\frac{P(D\cap M)}{P(M)}=\frac{P(M\cap D)}{P(M)}=\frac{0.05}{0.5}=0.1$$

7.12 INDEPENDENCE OF EVENTS

Suppose two events A and B occur in a manner that occurrence or non-occurrence of either of them has no relation to, and no influence on the occurrence or non-occurrence of the other. Under this condition, we say that the events A and B occur independently of one another. Given this situation, it is natural to assume that the probability that both A and B will occur is equal to the product of their individual probabilities. Symbolically,

$$P(A\cap B) = P(A)\times P(B).$$

This leads to arrive at the following definition of independence of two events:

***Definition* 7.11:** *If A and B are two events and if the occurrence of A does not affect, and is not affected by the occurrence of B, then A and B are said to be independent. In other words, two events are said to be independent if and only if* $P(A\cap B)=P(A)\times P(B)$.

Thus as a rule, probability of the joint occurrence of the events A and B must be equal to the product of their individual probabilities, if the events A and B are independent.

Example 7.44: Two ideal coins are tossed. Let A be the event 'head on the first coin' and B the event that 'head on the second coin. A sample space for this experiment is

$$S=\{HH, HT, TH, TT\}.$$

We define two events A and B as follows:

$$A=\{HH, HT\} \text{ and } B=\{HH, TH\}$$

The intersection of these events is

$$A\cap B=\{HH\}$$

It follows that $P(A\cap B)=\frac{1}{4}$ and that $P(A)=\frac{1}{2}$ and $P(B)=\frac{1}{2}$. Clearly

$$P(A\cap B)=P(A)\times P(B).$$

By definition, the events A and B are independent, implying that occurrence of head on the first coin does not influence the occurrence of head on the second coin.

Example 7.45: Three coins are tossed. Show that the events "heads on the first coin" and the event "tails on the last two" are independent.

Solution: We construct a sample S space for the above experiment.

$$S=\{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$

Let A denote the event "head on the first coin" and B denote event "tails on the last two coins". Then

$A=\{HHT, HHT, HTH, HTT\}$ and $B=\{HTT, TTT\}$, so that their intersection is $A\cap B=\{HTT\}$

Hence

$$P(A)=\frac{4}{8}=\frac{1}{2},\ P(B)=\frac{2}{8}=\frac{1}{4} \text{ and } P(A\cap B)=\frac{1}{8}$$

Since

$$P(A)\times P(B)=\frac{1}{2}\times\frac{1}{4}=\frac{1}{8}=P(A\cap B),$$

the events "heads on the first coin" and "tails on the last two" are independent.

Example7.46: A fire brigade has two fire engines operating independently. The probability that a specific fire engine is available when needed is 0.99.

(a) What is the probability that an engine is available when needed?
(b) What is the probability that neither is available when needed?

Solution: Let A be the event that the first engine is available when needed and B be the event that the second engine is available when needed. Then $P(A)=P(B)=0.99$. Given this, the probability that both of them will be available is $P(A\cap B)=P(A)\times P(B)=.99\times.99=.9801$, since they operate independently.

(a) Here the event of interest is $A\cup B$, so that

$$P(A\cup B)=P(A)+P(B)-P(A\cap B)$$
$$=0.99+0.99-0.9801$$
$$=0.9999$$

(b) In set notation, this event is $\bar{A}\cap\bar{B}$, which equals $\overline{A\cup B}$, so that the required probability is

$$P(\overline{A\cup B})=1-P(A\cup B)=1-.9999=0.0001$$

Example 7.47: If A is any event in a sample space S, show that A and S are independent.

Solution: From algebra of set, $A\cap S=A$ so that $P(A\cap S)=P(A)$. Since $P(S)=1$, $P(A)\times P(S)=P(A)$. But $P(A)=P(A\cap S)$. Hence A and S are independent. Alternatively,

$$P(A\mid S)=\frac{P(A\cap S)}{P(S)}=\frac{P(A)}{1}=P(A).$$

Hence the proof.

Example 7.48: In a community, 36% of the families own a dog and 22% of the families own both a dog and a cat. If a randomly selected family owns a dog, what is the probability that it owns a cat too?

Solution: Let us define the events of interest as follows:

D : Family owns a dog
C : Family owns a cat

Then

$$P(D)=.36 \text{ and } P(D\cap C)=.22$$

Since being owner of a dog and owner of cat are independent,

$$P(D\cap C)=P(D)P(C)=.22$$

Hence

$$P(C)=\frac{.22}{P(D)}=\frac{.22}{.36}=0.61$$

Independence of More than Two Events

The multiplication rule for independent events extends very simply to three or more independent events. For three events we have the following rule:

If A, B, C are all independent of each other (i.e. the occurrence of any one is not affected by the occurrence of any combination of the others), then $P(A\cap B\cap C)=P(A)\times P(B)\times P(C)$

This result can naturally be extended to n events $A_1, A_2, \ldots, A_n$. We state this result as saying that n events are independent provided the probability of their intersections is equal to the product of their individual probabilities. That is

$$P(A_1\cap A_2\cap\ldots\cap A_n)=P(A_1)\times P(A_2)\times\ldots\times P(A_n) \quad (7.1)$$

But what about the independence of the pairs of events, such as (A_1 and A_2) or (A_2 and A_3) or (A_4 and A_5)? It is possible that the events A_1, A_2, A_3.. are not independent (i.e. equation (7.1) is not satisfied), even though all possible combinations of the events are independent. For example, we might have $P(A_1\cap A_2)=P(A_1)\times P(A_2)$ or $P(A_1\cap A_2\cap A_3)=P(A_1)\times P(A_2)\times P(A_3)$ or any other combination of the events $A_1, A_2, A_3, \ldots, A_n$, satisfying this relation but not 7.1. In such instances, we raise the question of complete **independence**, which we define below:

Definition 7.12: *The n events are said to be completely independent if and only if every combination of these events, taken any number at a time, is independent.*

If every combination other than the one in (7.1) is independent, then we say that the events are **pairwise** independent but not completely **independent**.

For three events A_1, A_2, A_3, the complete independence ensures that the following equations are satisfied:

$$P(A_1\cap A_2)=P(A_1)\times P(A_2)$$
$$P(A_1\cap A_3)=P(A_1)\times P(A_3)$$
$$P(A_2\cap A_3)=P(A_2)\times P(A_3)$$
$$P(A_1\cap A_2\cap A_3)=P(A_1)\times P(A_2)\times P(A_3)$$

If first three of the above relations are satisfied but not the last one, we say that the events A_1, A_2, and A_3 are pairwise independent but not completely independent.

It is important to note that any combination of the events formed by replacing one or two or three..... by their complements, will also make the above results valid.

Example 7.49: Two coins are tossed. If A is the event "head on the first coin", B is the event "head on the second coin" and C is the event "coins fall alike", then show that the events A, B, and C are pairwise independent but not completely independent.

Solution: The sample space is $S=\{HH, HT, TH, TT\}$ and the events A, B, C are as follows

$$A=\{HH, HT\},\ B=\{HH, TH\} \text{ and } C=\{HH, TT\}.$$

and

$$A\cap B=\{HH\},\ A\cap C=\{HH\},\ B\cap C=\{HH\},\ A\cap B\cap C=\{HH\}.$$

And the associated probabilities are

$$P(A)=P(B)=P(C)=\frac{1}{2}, \text{ and } P(A\cap B)=P(A\cap C)=P(B\cap C)=\frac{1}{4}$$

which shows that

$$P(A)\times P(B)=P(A\cap B),\ P(A)\times P(C)=P(A\cap C) \text{ and } P(B)\times P(C)=P(B\cap C)$$

Hence the events are pairwise independent. But

$$P(A\cap B\cap C)=\frac{1}{4} \text{ and } P(A)\times P(B)\times P(C)=\frac{1}{8}.$$

showing that

$$P(A\cap B\cap C)\neq P(A)\times P(B)\times P(C)$$

Hence they are not independent taken altogether, in other words, not completely independent.

7.13 SELECTED THEOREMS ON PROBABILITY

On the basis of the axioms stated earlier, we state and prove the following theorems:

***Theorem* 7.2:** *For any event A,* $P(\bar{A})=1-P(A)$

Proof: Since A and $\bar{A}$ are exhaustive, their union constitute the sample space S. That is $A\cup\bar{A}=S$. Consequently

$$P(A\cup\bar{A})=P(S)=1 \qquad \text{[by Axiom 2]}$$

Since A and $\bar{A}$ are disjoint

$$P(A\cup\bar{A})=P(A)+P(\bar{A})=1$$

Hence

$$P(\bar{A})=1-P(A)$$

***Theorem* 7.3:** *If* ϕ *is an empty set,* $P(\phi)=0$.

Proof: Since S and $\bar{S}$ are exhaustive, $S\cup\bar{S}=S$, we have

$$P(S\cup\bar{S})=P(S)$$

Also S and $\bar{S}$ are mutually exclusive, so that

$$P(S)+P(\bar{S})=P(S)$$

Hence it follows that

$$P(\phi)=0$$

***Theorem* 7.4:** $P(A)\leq 1$.

Proof: Since by Axiom 1, $P(\bar{A})\geq 0$, and by Axiom, $P(A)+P(\bar{A})=1$, it follows that

$$P(A)\leq 1.$$

***Theorem* 7.5:** If $A\cap B\neq\phi$, *then* $P(A\cup B)=P(A)+P(B)-P(A\cap B)$.

This theorem is variously known as the **addition theorem** (law), **additive law** or **law of total probability**.

Proof: Look at the following Venn diagram.

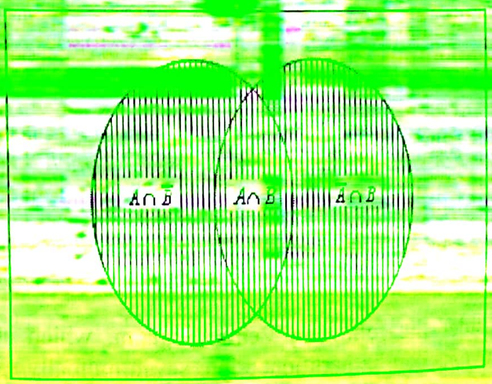


Figure 7.9: Union of A & B: $A\cup B$

The union of A and B can be partitioned as follows:

$$A \cup B = (A \cap \overline{B}) \cup B.$$

Thus by Axiom 3

$$\begin{aligned} P(A \cup B) &= P(A \cap \overline{B}) + P(B) \\ &= P(A) - P(A \cap B) + P(B) \\ &= P(A) + P(B) - P(A \cap B) \end{aligned}$$

Hence the proof

We can also write $(A \cup B)$ as follows:

$$(A \cup B) = (\overline{A} \cap B) \cup A$$

and proceed to prove the theorem as above

Example 7.50: Two coins are tossed. A is the event 'getting two heads' and B is the event 'second coin shows head'. Evaluate $P(A \cup B)$.

Solution: The sample space for this experiment is

$$S = \{HH, HT, TH, TT\}$$

and the events A, B and $A \cap B$ are

$$A = \{HH\},\ B = \{HH, TH\},\ A \cap B = \{HH\}$$

and the associated probabilities are

$$P(A) = \frac{1}{4},\ P(B) = \frac{2}{4},\ P(A \cap B) = \frac{1}{4}$$

and hence

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ &= \frac{1}{4} + \frac{2}{4} - \frac{1}{4} = \frac{1}{2} \end{aligned}$$

Theorem7.6. *For any three events, A, B, C, which are not mutually exclusive,*

$$\begin{aligned} P(A \cup B \cup C) &= P(A) + P(B) + P(C) - P(A \cap B) \\ &\quad - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C) \quad \ldots (a) \end{aligned}$$

Proof: Consider the following **Venn diagram**, which shows how the union of 3 events can be partitioned into seven mutually exclusive events $a_1, a_2, a_3, \ldots a_7$.

We shall denote the probabilities of these events by the values $P(a_1)$, $P(a_2), P(a_3), \ldots P(a_7)$ respectively as indicated in the diagram.

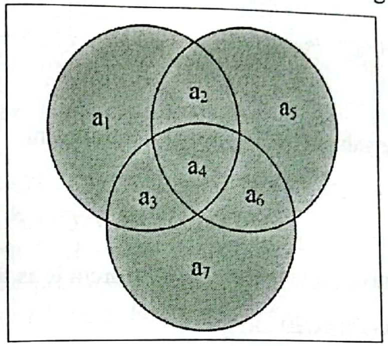


Figure 7.10: Union of A, B and C: $A \cup B \cup C$

Then

$$P(A \cup B \cup C) = P(a_1) + P(a_2) \ldots + P(a_7) = \sum_{i=1}^{7} P(a_i)$$

And we must show that the right-hand side of (1) is also equal to $\Sigma P(a_i)$

Since

$$\begin{aligned} A &= a_1 \cup a_2 \cup a_3 \cup a_4,\ B = a_2 \cup a_4 \cup a_5 \cup a_6 \text{ and} \\ C &= a_3 \cup a_4 \cup a_6 \cup a_7. \\ P(A) &= P(a_1 \cup a_2 \cup a_3 \cup a_4) = P(a_1) + P(a_2) + P(a_3) + P(a_4) \\ P(B) &= P(a_2 \cup a_4 \cup a_5 \cup a_6) = P(a_2) + P(a_4) + P(a_5) + P(a_6) \\ P(C) &= P(a_3 \cup a_4 \cup a_6 \cup a_7) = P(a_3) + P(a_4) + P(a_6) + P(a_7) \end{aligned}$$

Adding

$$\begin{aligned} P(A) + P(B) + P(C) &= P(a_1) + 2P(a_2) + 2P(a_3) + 3P(a_4) + P(a_5) \\ &\quad + 2P(a_6) + P(a_7) \\ &= [P(a_1) + P(a_2) + \ldots\ldots\ldots + P(a_7)] \\ &\quad + [P(a_2) + P(a_4)] + [P(a_3) + P(a_4)] \\ &\quad + [P(a_4) + P(a_6)] - P(a_4). \\ &= \sum_{i=1}^{7} p(a_i) + [P(a_2) + P(a_4)] + [P(a_3) + P(a_4)] \\ &\quad + [P(a_4) + P(a_6)] - P(a_4) \quad \ldots\ldots\ldots\ldots (2) \end{aligned}$$

But an examination of the Venn diagram shows that

$$\sum_{i=1}^{7} P(a_i) = P(A \cup B \cup C)$$

$$P(a_4)=P(A\cap B\cap C),\ P(a_2)+P(a_4)=P(A\cap B),\ P(a_3)+P(a_4)=P(A\cap C)$$
$$P(a_4)+P(a_6)=P(B\cap C)$$

Substituting these values in (2) above and transposing

$$P(A\cup B\cup C)=P(A)+P(B)+P(C)-P(A\cap B)$$
$$-P(A\cap C)-P(B\cap C)+P(A\cap B\cap C)$$

An alternative way of proving the above theorem is as follows:

$$P(A\cup B\cup C)=P[(A\cup B)\cup C)]$$
$$=P(A\cup B)+P(C)-P[(A\cup B)\cap C]$$
$$=P(A)+P(B)-P(A\cap B)+P(C)-P[(A\cap C)\cup(B\cap C)]$$

The last term of the above expression can be written as

$$P[(A\cap C)\cup(B\cap C)]=P(A\cap C)+P(B\cap C)-P[(A\cap C)\cap(B\cap C)]$$
$$=P(A\cap C)+P(B\cap C)-P(A\cap B\cap C)$$

On substitution, the result follows.

***Note**: If A, B, C are mutually exclusive, the above relation reduces to*

$$P(A\cup B\cup C)=P(A)+P(B)+P(C$$

For four events A, B, C and D, which are not mutually exclusive, the formula can be extended as follows:

$$P(A\cup B\cup C\cup D)=P(A)+P(B)+P(C)+P(D)-P(A\cap B)-P(A\cap C)$$
$$-P(A\cap D-P(B\cap C)-P(B\cap D)-P(C\cap D)$$
$$+P(A\cap B\cap C)+P(A\cap B\cap D)+P(A\cap C\cap D)$$
$$+P(B\cap C\cap D)-P(A\cap B\cap C\cap D)$$

Note: For n events, $A_1, A_2, \ldots A_n$, which are not mutually exclusive

$$P(A_1\cup A_2\cup\ldots\cup A_n)=\sum_{i=1}^{n}P(A_i)-\sum_{i<j}^{n}P(A_i\cap A_j)+\sum_{i<j<k}^{n}P(A_i\cap A_j\cap A_k)$$
$$-\sum_{i<j<k<l}^{n}P(A_i\cap A_j\cap A_k\cap A_l)+\ldots$$
$$+(-1)^{n+1}P((A_i\cap A_j\cap A_k\cap\ldots\cap A_n)$$

From the above expression, it follows that

$$P(A_1\cup A_2\cup\ldots\cup A_k)\le P(A_1)+P(A_2)+\ldots+P(A_k)$$

Occasionally, this is referred to as **Boole's inequality**.

When $A_1, A_2, \ldots A_n$ are all mutually exclusive,

$$P(A_1\cup A_2\cup\ldots A_n)=P(A_1)+P(A_2)+\ldots+P(A_n)$$

***Theorem 7.7**: For any event A, prove that*

$$0\le P(A)\le 1.$$

Proof: It is known from Axiom 1 that $P(A)\ge 0$. If $P(A)>1$, then it follows from Theorem 1 that $P(\bar{A})<0$. Since these results contradict Axiom 1 (which states that the probability of every event must be non-negative), it must also be true that $P(A)\le 1$. Hence the proof.

***Theorem 7.8**: If $A\subset B$, then $P(A)\le P(B)$*

The theorem states that if A is contained in B, then probability of A happening can not exceed the probability of happening of B.

Proof: As illustrated in figure below, the event B may be treated as the union of two disjoint events A and $B\cap\bar{A}$ $\bar{A}$

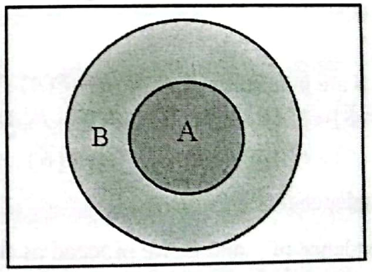


Figure 7.11: Venn diagram displaying $A\subset B$

Therefore

$$P(B)=P(A)+P(B\cap\bar{A})\bar{A})$$

Since $P(B\cap\bar{A})\ge 0$, it follows that

$$P(A)\le P(B)$$

***Theorem 7.9** For two events A and B, prove that*

$$P[(\bar{A}\cap B)\cup(A\cap\bar{B})]=P(A\cup B)-P(A\cap B)$$

Proof: Look at the Venn diagram. Since the events $\overline{A} \cap B$ and $A \cap \overline{B}$ are mutually exclusive.

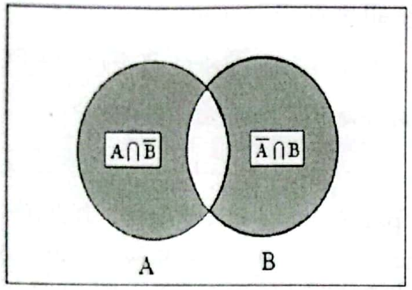


Figure 7.12: Venn diagrams displaying $\overline{A} \cap B$ and $A \cap \overline{B}$

$$\begin{aligned} P[(\overline{A} \cap B) \cup (A \cap \overline{B})] &= P(\overline{A} \cap B) + P(A \cap \overline{B}) \\ &= P(A) - P(A \cap B) + P(B) - P(A \cap B) \\ &= P(A) + P(B) - 2P(A \cap B) \\ &= P(A \cup B) - P(A \cap B) \end{aligned}$$

Hence the proof.

***Theorem* 7.10:** *If the events A and B are independent, so are $\overline{A}$ and $\overline{B}$, A and $\overline{B}$, $\overline{A}$ and B*

Proof: Since A and B are independent, $P(A \cap B) = P(A)\,P(B)$ and

$$\begin{aligned} P(A \cap \overline{B}) = P(A) - P(A \cap B) &= P(A) - P(A)P(B) \\ &= P(A)[1 - P(B)] = P(A).P(\overline{B}) \end{aligned}$$

Hence A and $\overline{B}$ are independent.

To prove the independence of $\overline{\ }$ and B, we proceed as follows:

$$\begin{aligned} P(\overline{A} \cap B) = P(B) - P(A \cap B) &= P(B) - P(A)P(B) \\ &= P(B)[1 - P(A)] = P(\overline{A}).P(B) \end{aligned}$$

To prove that $\overline{A}$ and $\overline{B}$ are independent, we find that

$$\begin{aligned} P(\overline{A} \cap \overline{B}) = P(\overline{A}) - P(\overline{A} \cap B) &= P(\overline{A}) - P(\overline{A})P(B) \\ &= P(\overline{A})[1 - P(B)] = P(\overline{A}).P(B) \end{aligned}$$

Hence $\overline{A}$ and $\overline{B}$ are independent.

We wish to clearly emphasize that independence of events should never be confused with disjoint or mutually exclusive events. If two events, each

with non-zero probability, are mutually exclusive, they are obviously dependent, since the occurrence of one will automatically preclude the occurrence of the other. Similarly, if A and B are independent and $P(A)>0$, $P(B)>0$, the A and B cannot be mutually exclusive.

Corollary: If A and B are independent events, then

$$P(A \mid B) = P(A) \qquad \text{if } P(B) > 0$$

and

$$P(B \mid A) = P(A) \qquad \text{if } P(A) > 0$$

Example 7.51: A certain retail shop accepts either the American Express or the VISA credit card. A total of 25 percent of its customers carry an American Express card, 60 percent carry VISA credit card and 15 percent carry both. What is the probability that a randomly chosen customer will have at least one of these cards? What is the probability that the customer has neither an American Express nor a VISA card? Are the events 'accepting an American card' and accepting a VISA card' independent?

Solution: Let A be the event that the customer has an American card and B be the event that he has a VISA card. Then

$$P(A) = 0.25 \quad P(B) = 0.60 \quad P(A \cap B) = 0.15$$

Using the addition law of probability, the desired probability $P(A \cup B)$ is

$$P(A \cup B) = 0.25 + 0.60 - 0.15 = 0.70$$

We conclude that 70 percent of the customers carry at least one of the cards that it will accept.

The probability of having neither is $P(\overline{A} \cap \overline{B})$, which equals $P(\overline{A \cup B})$:

$$P(\overline{A \cup B}) = 1 - P(A \cup B) = 1 - 0.70 = 0.30$$

Example 7.52: Of the total students of a women's college, 60% wear neither a ring nor a necklace, 20% wear a ring, and 30% wear a necklace. If one of the women is randomly chosen, find the probability that she is wearing (a) A ring or a necklace (b) Both.

Solution: Let R and N respectively denote the events that a woman wears a ring and a necklace. We are given that

$$P(R) = 0.20, \quad P(N) = 0.30 \text{ and } P(\overline{A} \cap \overline{N}) = 0.60$$

The probability that she is wearing a ring or a necklace is

$$P(R\cup N)=1-P(\overline{R\cup N})=1-P(\overline{R}\cap\overline{N})=0.40$$

The probability that she wears both is

$$P(R\cap N)=P(R)+P(N)-P(R\cup N)=0.20+0.30-0.40=0.10$$

Example 7.53: A newly married couple is planning to have two children and suppose that each child is equally likely to be a boy or a girl. Construct a sample space and find the probability that the couple will have (a) two boys, (b) one boy and one girl and (c) at least one girl.

Solution: We let B denote that the child is a boy and G denote that the child is a girl. The 'double' BG, for instance, represents the outcome 'first child is boy and the second is a girl'. Then one possible sample space is

$$S=\{BB,BG,GB,GG\}$$

where each of the outcomes is equally likely so that

$$P(BB)=P(BG)=P(GB)=P(GG)=\frac{1}{4}$$

(a) The probability that the couple will have two boys is

$$P(BB)=\frac{1}{4}$$

(b) The probability that the couple will have one boy and one girl

$$P(BG)+P(GB)=\frac{1}{4}+\frac{1}{4}=\frac{1}{2}$$

(c) The probability that the couple will have at least one boy

$$P(BB)+P(GB)+P(BG)=\frac{3}{4}$$

Example 7.54: A school teacher has received five complementary tickets to watch a final football match to be played between Abahoni Krira Chakra and Mohamedan Sporting Club. He decides to give these tickets to five of his best students: Ariq (A), Bashir (B), Chandon (C), Delwar (D) and Erfan (E). What is the probability that (i) both Ariq and Bashir are chosen, (ii) both Chandon and Erfan are chosen, (iii) Bashir, Chandon, and Delwar are chosen?

Solution: There are ${}^5C_3$ or 10 possible selections of the 5 persons taken 3 at a time, so that a sample space is

$$S=\{ABC,ABD,ABE,ACD,ACE,ADE,BCD,BCE,BDE,CDE\}$$

Let A_1, A_2, and A_3 be the events of interest, so that

$$A_1=\{ABC,ABD,ABE\},\ A_2=\{ACE.BCE,CDE\}\text{and } A_3=\{BCD\}$$

Since the three students are chosen at random, we assign to each sample point of S a probability 1/10. Since the events are mutually exclusive, we have

$$P(A_1\cup A_2\cup A_3)=P(A_1)+P(A_2)+P(A_3)=\frac{3}{10}+\frac{3}{10}+\frac{1}{10}=\frac{7}{10}.$$

Example 7.55: In a poker hand consisting of 5 cards, find the probability of holding 2 aces and 3 jacks.

Solution: The number of ways of being dealt 2 aces from 4 is ${}^4C_2=6$ and the number of ways of being dealt 3 jacks from 4 is ${}^4C_3=4$. The total number of 5-card poker hands is ${}^{52}C_5=2598960$. Now if A is the event of getting 2 aces and 3 jacks in a 5-card poker hand, then

$$P(A)=\frac{{}^4C_2\times{}^4C_3}{{}^{52}C_5}=\frac{24}{2598960}=0.9\times10^{-5}$$

Example 7.56: The probability that A will die within the next 20 years is 0.025, and that B will die within the next 20 years is 0.030. What is the probability that both A and B will die within the next 20 years? That A will die but B will not die? That neither will die? That at least one will die?

Solution: We assume that dying of A and that of B are independent. Then

(a) the probability that both will die is

$$P(A\cap B)=P(A)\times P(B)=0.025\times0.030=0.00075$$

(b) the probability that A will die but B will not is

$$P(A\cap\overline{B})=P(A)\times P(\overline{B})=0.025\times0.97=0.02425$$

(c) the probability that neither A nor B will die is

$$P(\overline{A}\cap\overline{B})=P(\overline{A})\times P(\overline{B})=0.975\times0.970=0.94575$$

(d) At least one of them will die

$$P(A\cup B)=P(A)+P(B)-P(A\cap B)$$
$$=0.025+0.030-0.00075=0.05425$$

B2|W1
B2|B1

A

Example 7.57: A bag contains 4 white balls and 3 black balls and a second bag contains 3 white balls and 5 black balls. One ball is drawn from the

[W1∩B2]+(B1∩B2)]

first bag and placed unseen in the second bag. A ball is now drawn from the second bag. What is the probability that this ball is black? White?

Solution: Let B_1 and W_1 respectively represent the drawing of a black ball and a white ball from bag 1 and B_2 represents the drawing a black ball from bag 2. Then our interest is the union of the two mutually exclusive events $B_1 \cap B_2$ and $W_1 \cap B_2$. Thus, the required probability is

$$P[(B_1 \cap B_2) \cup (W_1 \cap B_2)] = P[(B_1 \cap B_2) + (W_1 \cap B_2)]$$
$$= P(B_1)P(B_2 \mid B_1) + P(W_1)P(B_2 \mid W_1)$$
$$= \left(\frac{3}{7}\right)\left(\frac{6}{9}\right) + \left(\frac{4}{7}\right)\left(\frac{5}{9}\right) = \frac{38}{63}$$

where $B_1 \cap B_2$, for example, denotes the event of drawing a black ball from bag 1and also a second back from bag 2. Similarly, $W_1 \cap B_2$ is the event of drawing a white ball from bag 1 and a black ball from bag 2.

Example 7.58: The event C is twice as likely as A, and B is as likely as A and C together. The events are mutually exclusive and exhaustive. Find the probabilities of the events A, B and C.

Solution: It is evident from the statement of the problem that

$$P(C) = 2P(A) \text{ and } P(B) = P(A) + P(C) = 3P(A)$$

Since the events are mutually exclusive and exhaustive (together the constitute the sample space)

$$P(A) + P(B) + P(C) = 1$$

Substituting the values of $P(B)$ and $P(C)$ in the above equation, we have $P(A) = 1/6$. Consequently $P(C) = 1/3$ and $P(B) = 1/2$

Example 7.59: From a lot of 20 radios, a sample of 3 is randomly selected for inspection. If there are 6 defective radios in the lot, what is the probability that the sample (a) is composed entirely of defectives, (b) is composed entirely of non-defectives, (c) is composed of one defective and two non-defectives?

Solution: (a) By combinatorial rule, the number of possible equally likely samples is

$$^{20}C_3 = 1140 \text{ ways.}$$

A sample composed entirely of defective radios can be selected in

$$^6C_3 \times {}^{14}C_0 = 20 \text{ ways.}$$

Hence

$$P(3 \text{ defective radios}) = \frac{20}{1140}$$

(b) A sample consisting of entirely non-defectives can be selected in

$$^{14}C_3 \times {}^6C_0 = 364 \text{ ways.}$$

Hence

$$P(3 \text{ non-defective radios}) = \frac{364}{1140}$$

(c) A sample composed of one defective and two non-defectives can be selected in

$$^6C_1 \times {}^{14}C_2 = 546 \text{ ways}$$

Hence

$$P(1 \text{ defective and 2 non-defectives}) = \frac{546}{1140}$$

Example 7.60: The probability that a person picked at random from a population will exhibit the symptom of certain disease is 0.2, and that the probability that a person picked at random has the disease is 0.23. The probability that a person who has the symptom also has the disease is 0.18. A person selected at random from the population does not have the symptom. What is the probability that the person has the disease?

Solution: Let S be the event that the person has the symptom and D be the event that a person has actually the disease. Then the conditional probability that that the person has the disease given that he did not show any symptom is

$$P(D \mid \bar{S}) = \frac{P(D \cap \bar{S})}{P(\bar{S})}$$

Now

$$P(D) = P(D \cap S) + P(D \cap \bar{S}) = 0.23$$

But

$$P(D \cap S) = 0.18 \text{ and } P(\bar{S}) = 1 - 0.02 = 0.80$$

so that

$$P(D \cap \bar{S}) = 0.23 - 0.18 = 0.05$$

Hence

$$P(D \mid \bar{S}) = \frac{0.05}{0.80} = 0.0625$$

Example 7.61: It is reported that for the Bangladeshi adult population as a whole, 50 percent are above normal weight, 25 percent have high blood pressure, and 65 percent either are above ideal weight or have high blood pressure. If A is the event that a randomly chosen person is above his/her normal weight and B is the event that this person has high blood pressure, are A and B independent?

Solution: Here $P(A)=0.50, P(B)=0.25, P(A\cup B)=0.65$. The events A and B will be independent if and only if $P(A\cap B)=P(A)\times P(B)$. For the problem in hand, the following relationship can be used:

$$P(A\cup B)=P(A)+P(B)-P(A\cap B)$$

from which

$$P(A\cap B)=0.10$$

But $P(A)\times P(B)=0.50\times 0.25=0.125\neq P(A\cap B)$. Hence A and B are not independent.

Example 7.62: A total of 500 married couples were polled about whether their monthly salaries exceeded Tk.25000. The information obtained on this is shown in the accompanying table.

	Husband	
Wife	Less than 25000	More than 25000
Less than 25000	212	198
More than 25000	36	54

(a) What is the probability that the husband earns less than 25000?
(b) What is the probability that a wife who earns less than 25000, her husband also earns this amount?
(c) What is the conditional probability that the wife earns more than 25000 given that her husband earns more than this amount?

Solution: For (a)

$$P(a)=\frac{212+36}{500}=\frac{248}{500}$$

To calculate the probability for (b), we note that there are 212 couples satisfying this event so that

$$P(b)=\frac{212}{500}$$

To compute the probability in (c), the reduced sample space consists of 198+54=252 cases, among which 198 are favorable to the event, so that

$$P(c)=\frac{198}{198+54}=\frac{198}{252}$$

7.14 BAYES' THEOREM

Very often we begin our probability analysis with initial or **prior** probability estimates for specific event of interest. Then, from sources, such as a sample, a special report or document, we obtain some additional information about the events. Given this new information, we want to revise and up-date the prior probability values. The new and revised probabilities for the events are referred to as posterior probabilities. **Bayes' theorem**, which we will deal with here, provides a means of computing these revised probabilities. We cite an example below to illustrate the use of Bayes' theorem.

7.14.1 Understanding Bayes' Rule

Suppose an Electric Company has two machines A and B for manufacturing electric bulbs. The authority is seriously concerned with the quality of bulbs manufactured by these two machines. The company receives complaints from the dealers that they frequently receive defective bulbs. Once a dealer reports to the authority with a defective product. The manager now wants to make a probabilistic guess as to which machine could have produced this bulb. For this purpose he needs some prior information. Office record shows that, of the total output, 60 percent are produced by machine A and the remaining 40 percent by machine B. Thus we can reasonably assume that the probability that a bulb taken at random from the stock is produced by machine A is 0.60 and that it is produced by machine B is 0.40. These, in our probabilistic terminology, are referred to as the **prior probabilities.** These are prior probabilities, because we know these probabilities before it is known that the selected item is defective. We designate these probabilities by $P(A)$ and $P(B)$. But these prior information are not enough to come to a valid conclusion about the problem in hand. Which machine is to be blamed for manufacturing defective bulbs? The manager wants to make some educated guess on this. Further scrutinizing of the office records reveals that 3 percent of the bulbs produced by machine A in the recent past were defectives, while this proportion is 5 percent for machine B. If D stands for the event that the bulb is defective, then $P(D|A)$ will be the conditional probability of getting a defective bulb, if it is known in advance that the bulb was produced by machine A. This in the present instance is .03. Similarly $P(D|B)$ =.05. By definition, $P(D)=P(D|A)+P(D|B)$, because the defective bulb could be produced by either of the machines. Given the defective bulb (D) in hand, the manager now wants to know: what is the chance that this bulb was produced by machine A? By machine B? Symbolically, we want to evaluate $P(A|D)$ and $P(B|D)$. These probabilities are the modified probabilities of $P(A)$ and $P(B)$ respectively and are referred to as the **posterior probabilities**, which the Bayes' theorem deals with. $P(A|D)$ and $P(B|D)$ are posterior probabilities, because these are the probabilities of the events after it is known that the selected item is defective. Note that the events of interest are mutually exclusive and collectively exhaustive. Thus $P(A \cap B)=0$ and the sum of their probabilities $P(A)+P(B)=1$. Also since the defective items are either produced by A or B, so that $P(A|D)+P(B|D)=1$. In all applications of Bayes' theorem, the events of interest will be mutually exclusive and collectively exhaustive.

7.14.2 Toward Developing Bayes' Theorem

Let us now see how the formula for Bayes' theorem is developed. For this purpose, we make use of the foregoing example.

Since the probability $P(A|D)$ that we are seeking for, is a conditional probability, we can start with the definition of conditional probability expressed as follows:

$$P(A \mid D)=\frac{P(A \cap D)}{P(D)}$$

Using the multiplication rule, we can replace $P(A \cap D)$ by $P(A) \times P(D|A)$ so that

$$P(A \mid D)=\frac{P(A) P(D \mid A)}{P(D)} \quad \text{... (7.6)}$$

A similar formula can be presented for $P(B|D)$, the conditional probability of B given D

$$P(B \mid D)=\frac{P(B) P(D \mid B)}{P(D)} \quad \text{... (7.7)}$$

The above equations are the simplest forms of Bayes' theorem involving two events A and B.

The numerators of (7.6) and (7.7) are easy to compute since $P(A)$, $P(B)$, $P(D|A)$ and $P(D|B)$ are all known. But how can we compute $P(D)$, the probability that the bulb is defective? To compute $P(D)$, consider the following partitioning of the sample space.

Machine	D	$\bar{D}$
A	$A \cap D$	$A \cap \bar{D}$
B	$B \cap D$	$B \cap \bar{D}$

Clearly

$$D=(A \cap D) \cup (B \cap D)$$

and hence

$$\begin{aligned} P(D) &= P(A \cap D)+P(B \cap D) \\ &= P(A) P(D \mid A)+P(B) P(D \mid B) \end{aligned} \quad \text{... (7.8)}$$

Substituting (7.8) in (7.6) and (7.7)

$$P(A \mid D)=\frac{P(A)\, P(D \mid A)}{P(A) P(D \mid A)+P(B) P(D \mid B)} \quad \text{... (7.9)}$$

and

$$P(B \mid D)=\frac{P(B) P(D \mid B)}{P(A) P(D \mid A)+P(B) P(D \mid B)} \quad \text{... (7.10)}$$

which are the alternative forms of Bayes' rule for two events.

For three events A, B and C, the Bayes' formulae will assume the following form:

$$P(A \mid D)=\frac{P(A)\, P(D \mid A)}{P(A) P(D \mid A)+P(B) P(D \mid B)+P(C) P(D \mid C)}$$

and so on.

For k events $A_1, A_2, \ldots, A_k$

$$P(A \mid D)=\frac{P(A_i)\, P(D \mid A_i)}{\sum_{i=1}^{k} P(A_i) P(D \mid A_i)} \quad \text{... (7.11)}$$

where

$P(A_i)$ = prior probability of event A_i, $i=1, 2, \ldots, k$.
$P(D|A_i)$ = conditional probability of D given A_i
$P(A_i|D)$ = posterior probability of A_i given D

We now provide a general proof of the Bayes' theorem.

Theorem 7.11: *Let $A_1, A_2, \ldots, A_k$ be k mutually exclusive events forming partitions of the sample space S of an experiment.. Let B be any event of S such that $P(B)>0$, for $i=1,2,\ldots,k$. Then*

$$P(A_i \mid B) = \frac{P(A_i)P(B \mid A_i)}{\sum_{i=1}^{k} P(A_i)P(B \mid A_i)}$$

Proof: The left-hand side of the above equation can be written as

$$P(A_i \mid B) = \frac{P(A_i \cap B)}{P(B)}$$

Since the events $A_1, A_2, \ldots A_k$ are mutually exclusive, their union is the sample space S. This implies that $A_1 \cup A_2 \cup \ldots \cup A_k = S$. Further, if B is any other event in S, then the events $A_1 \cap B, A_2 \cap B, \ldots, A_k \cap B$ will form partitions of B, as illustrated in Figure 7.9 below. Hence we can write

$$B = (A_1 \cap B) \cup (A_2 \cap B) \cup \ldots \cup (A_k \cap B). \quad \ldots (*)$$

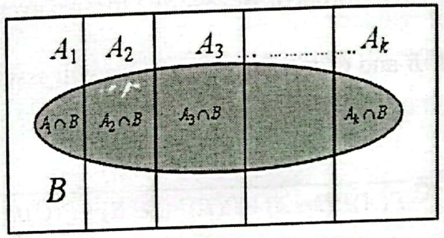


Figure 7.13: Intersection of B with $A_1, A_2 \ldots A_k$

Furthermore, since the events on the right-hand side of the equation (*) are disjoint,

$$P(B) = P(A_1 \cap B) + P(A_2 \cap B) + \ldots + P(A_k \cap B)$$
$$= \sum_{i=1}^{k} P(A_i \cap B)$$

But for $P(A_i) > 0$

$$P(A_i \cap B) = P(A_i)P(B \mid A_i),$$

so that

$$P(B) = \sum P(A_i)P(B \mid A_i)$$

Hence

$$P(A_i \mid B) = \frac{P(A_i \cap B)}{P(B)} = \frac{P(A_i)P(B \mid A_i)}{\sum_{i=1}^{k} P(A_i)P(B \mid A_i)}$$

This proves the theorem.

Example 7.63: An opinion poll in Dhaka city shows that 45 percent of the city dwellers support Awami League (A), 40 percent support the Bangladesh Nationalist Party (B) and the remaining 15 percent support Communist and other parties (C). Previous records reveal that in city elections, 65 percent of the Awami League, 80 percent of the Bangladesh Nationalist Party and 50 percent of the Communist or other party supporters turned up to cast their votes. A person in the City is chosen at random and it is learned that he did not cast his vote in the last election. What is the probability that he is a supporter of A? B? C?

Solution: The prior probabilities are $P(A) = 0.45$, $P(B) = 0.40$ and $P(C) = 0.15$. If V stands for the event: Casting vote, then the conditional probabilities are

$$P(V|A) = 0.65, \; P(V|B) = 0.80 \text{ and } P(V|C) = 0.50$$

and the conditional probabilities for not-voting will be

$$P(\bar{V} \mid A)) = 0.35, \; P(\bar{V} \mid B) = 0.20 \text{ and } P(\bar{V} \mid C) = 0.50$$

Hence the probability that a person chosen at random did not cast his vote, denoted by $P(\bar{V})$ is

$$P(\bar{V}) = P(A \cap \bar{V}) + P(B \cap \bar{V}) + P(C \cap \bar{V})$$
$$= P(A)P(\bar{V} \mid A) + P(B)P(\bar{V} \mid B) + P(C)P(\bar{V} \mid C)$$
$$= 0.45 \times 0.35 + 0.40 \times 0.20 + 0.15 \times 0.50 = 0.3125$$

We need now to obtain $P(A|\bar{V})$, $P(B|\bar{V})$ and $P(C|\bar{V})$.

Using Bayes' rule

$$P(A \mid \bar{V}) = \frac{P(A \cap \bar{V})}{P(\bar{V})} = \frac{P(A)P(\bar{V} \mid A)}{P(\bar{V})} = \frac{(0.45)(0.35)}{0.3125} = 0.504$$

Similarly, the probabilities of the other two conditional events can be computed, which appear below:

$$P(B \mid \bar{V}) = \frac{P(B \cap \bar{V})}{P(\bar{V})} = \frac{P(B)P(\bar{V} \mid A)}{P(\bar{V})} = \frac{(0.40)(0.20)}{0.3125} = 0.256$$

$$P(C \mid \bar{V}) = \frac{P(C \cap \bar{V})}{P(\bar{V})} = \frac{P(C)P(\bar{V} \mid C)}{P(\bar{V})} = \frac{(0.15)(0.50)}{0.3125} = 0.240$$

Example 7.64: The percentages of students favoring a 4-year honors course in three different universities were as follows: Dhaka University: 21 percent, Rajshahi University: 45 percent and Chittagong University: 75 percent. If a university is chosen at random and a student is selected from this university also at random, what is the probability that the student so selected will be in favor of introducing a 4-year honors course in the university? Given that the student is in favor of a 4-year course, what is the probability that he comes from Dhaka University? From Chittagong University? From Rajshahi University?

Solution: If D, C and R stand respectively for the events that a student is selected from Dhaka, Chittagong and Rajshahi Universities, then

$$P(D)=P(C)=P(R)=\frac{1}{3}$$

Denoting the event 'favoring' by F, we have the following conditional probabilities:

$$P(F|D)=0.21,\ P(F|C)=0.45 \text{ and } P(F|R)=0.75$$

where for example, $P(F|D)$ is read as: 'probability that the student so selected, will favor the issue given that he/she belongs to Dhaka University'. Thus, if $P(F)$ denotes the probability that a student chosen at random is in favor of a 4-year course,

$$\begin{aligned}P(F)&=P(D\cap F)+P(C\cap F)+P(R\cap F)\\&=P(D)P(F\mid D)+P(C)P(F\mid C)+P(R)P(F\mid R)\\&=\frac{1}{3}(0.21+0.45+0.75)=0.47\end{aligned}$$

We can now use Bayes' theorem to compute the probability that a student chosen at random will belong to the Dhaka University, given that he/she is in favor of a 4-year degree course. If it is denoted by $P(D|F)$,

$$P(D\mid F)=\frac{P(D\cap F)}{P(F)}=\frac{P(D)P(F\mid D)}{P(F)}=\frac{\frac{1}{3}(0.21)}{0.47}=0.15$$

Similarly, $P(C|F)$ can be computed as

$$P(C\mid F)=\frac{P(C\cap F)}{P(F)}=\frac{P(C)P(F\mid C)}{P(F)}=\frac{\frac{1}{3}(0.45)}{0.47}=0.32$$

and finally

$$P(R\mid F)=\frac{P(R\cap F)}{P(F)}=\frac{P(R)P(F\mid R)}{P(F)}=\frac{\frac{1}{3}(0.75)}{0.47}=0.53$$

Since the events are mutually exclusive and exhaustive, you can also evaluate $P(R|F)$ from the relation $P(R|F)+P(C|F)+P(D|F)=1$ from which $P(R|F)=1-[P(C|F)+P(D|F)]=0.53$

Example 7.65: An insurance company believes that people can be classified into two classes–those who are prone to have accident and those who are not. The data indicate that an accident prone person will meet an accident in a one-year period is 0.1; and the probability to all others is 0.05. Suppose that the probability is 0.2 that a new policyholder is accident-prone.

a) What is the probability that a new policyholder will have an accident in the first year?

b) If a new policyholder has an accident in the first year, what is the probability that he/she is accident-prone?

Solution: Let E be the event that the new policyholder is accident-prone, and A denote the event that he/she has an accident in the first year. Then we can compute $P(A)$ by conditioning on whether the person is accident-prone:

$$\begin{aligned}P(A)&=P(E\cap A)+P(\bar{E}\cap A)\\&=P(E)P(A\mid E)+P(\bar{E})P(A\mid \bar{E})\\&=(0.2)(0.1)+(0.80)(0.05)=0.06\end{aligned}$$

This value implies that there is a 6 percent chance that a new policyholder will meet an accident in one- year time.

In the second case, the conditional probability is

$$P(E\mid A)=\frac{P(E\cap A)}{P(A)}=\frac{P(E)P(A\mid E)}{P(A)}=\frac{(0.2)(0.1)}{0.06}=0.33$$

Example 7.66: A blood test is 90 percent effective in detecting a certain disease when the disease is present. However, the test also yields a **false-positive** result for 5 percent of the healthy patients tested. (That is, if a healthy person is tested, then with probability 0.05 the test will say that this person has the disease.) Suppose 1 percent of the population has the disease. Find the conditional probability that a randomly chosen person actually has the disease given that his test result is positive.

Solution: Let D denote the event that the person has the disease, and let E be the event that the test is positive. We want to determine $P(D|E)$, which can be evaluated by applying Bayes' theorem:

$$P(D\mid E)=\frac{P(D\cap E)}{P(E)}=\frac{P(D)P(E\mid D)}{P(D)P(E\mid D)+P(\bar{D})P(E\mid \bar{D})}$$

$$=\frac{(0.01)(0.90)}{(0.01)(0.90)+(0.99)(0.05)}=0.15$$

Thus, there is a 15 percent chance that a randomly chosen person from the population who tests positive actually has the disease.

Example 7.67: In attempting a question on multiple-choice test, a student either knows the answer or guesses. Let p be probability that he knows the answer and $1-p$ the probability that he guesses. Assume that a student who guesses at the answer will be correct with probability $1/m$, where m is the number of multiple-choice alternatives. What is the probability that a student knew the answer to a question given that he answered it correctly? If the test consisted of 10 multiple –choice alternatives and if the probability that student knew the answer, then find the probability that a student knew the answer given that he correctly answered the question.

Solution: Let C denote the event that the student answers the question correctly and the K the event that he actually knows the answer. Then

$$P(K\mid C)=\frac{P(K\cap C)}{P(C)}=\frac{P(K)P(C\mid K)}{P(K)P(C\mid K)+P(\bar{K})P(C\mid \bar{K})}$$

$$=\frac{p}{p+(1-p)\left(\frac{1}{m}\right)}=\frac{mp}{1+(m-1)p}$$

Specifically, if m=10, and p=0.5, then $P(K\mid C)=\dfrac{10(0.5)}{1+(10-1)(0.5)}=0.91$

Thus there is a 91% chance that a student, who correctly answered the question, had correct knowledge of the answer.

Example 7.68: In the submission of an income tax return, a tax- payer has applied for the allowance of certain expenses, for which the legitimacy is subject to verification. A three-member review board operating on a rotation schedule receives applications for consideration. The first board, consisting mainly of members have vast experience, approves only 25% of such claims without verification. The second board, consisting mainly of midterm individuals, approves 50% of such claims. The third board with relatively newer members dubious about refuting claims approves 70% of such cases. There is a 20% chance that the application will be considered by the first board, a 50% chance that it will be considered by the second board, and a 30% chance that it will be considered by the third board.

a) What is the chance that the application will be approved?

b) If the application is approved, what is the probability that it was considered by (i) the first board (ii) the second board (iii) the third board?

Solution: The approval may be granted by any one of three boards. Hence if A denotes the event that the application will be approved, and B_1, B_2 and B_3 denote the considerations by the first, second and third boards respectively, the required probabilities are:

(a) $$P(A)=P(B_1\cap A)+P(B_2\cap A)+P(B_3\cap A)$$
$$=P(B_1)P(A\mid B_1)+P(B_2)P(A\mid B_2)+P(B_3)P(A\mid B_3)$$
$$=.20\times.25+.50\times.50+.30\times.70=0.51$$

(b) $$P(B_1\mid A)=\frac{P(B_1)P(A\mid B_1)}{P(A)}=\frac{0.20\times0.25}{0.51}=0.098$$

$$P(B_2\mid A)=\frac{P(B_2)P(A\mid B_2)}{P(A)}=\frac{0.50\times0.50}{0.51}=0.490$$

$$P(B_3\mid A)=\frac{P(B_3)P(A\mid B_3)}{P(A)}=\frac{0.30\times0.70}{0.51}=0.412$$

7.15 MISCELLANEOUS EXAMPLES

This section is designed to present a few more worked out examples covering various approaches of probability computations. Before we do so, we summarize below the basic steps involved in obtaining probabilities of outcomes associated with an experiment:

1. Set up a sample space of all possible outcomes
2. Define event or events of interest consistent with the sample points in the experiment.
3. Assign probabilities to the sample points enumerated as the elements of the sample space. Note that in a sample space of n

equally likely outcomes, we assign probability $1/n$ to each sample point. Further, these probabilities must add to 1.

4. Assign appropriate weights to the sample points such that the sum of the weights add to 1 if the sample points in the sample space are not equally likely,
5. Finally, add the probabilities assigned to the elements of the sub-set of the sample space that are favorable to the event A to obtain the probability of the event A.

Example 7.69: Three coins are tossed simultaneously. Set up a sample space for this experiment and obtain the probability that

(*a*) All are heads
(*b*) No heads occur
(*c*) Two or more heads occur

Solution: The sample space is

$$S = \{HHH, HHT, HTH, HTT\ THH, THT, TTH, TTT\}$$

There are 8 elementary events in the sample space, which are all equally likely.

The assignment of the probabilities to the events defined in (*a*), (*b*) and (*c*) will thus follow the classical approach.

(a) Let A be the event that all are heads. Referring to S, we find that

$$A = \{HHH\} \quad \therefore P(A) = \frac{n(A)}{n(S)} = \frac{1}{8}$$

(b) Let B be the event that no heads occur. Thus

$$B = \{TTT\} \quad \therefore \ P(B) = \frac{n(B)}{n(S)} = \frac{1}{8}$$

(c) Let C be the event that two or more heads occur, so that

$$C = \{HHT, HTH, THH, HHH\} \quad \therefore \ P(C) = \frac{n(C)}{n(S)} = \frac{4}{8} = \frac{1}{2}$$

Example 7.70: Examination results of 150 students showed that 95 students passed mathematics, 75 students passed economics and 135 students passed at least one of the above subjects. A student is selected at random. What is the probability that the student

(*a*) Passed both mathematics and economics?
(*b*) Failed both the subjects?
(*c*) Passed mathematics but failed economics?

Solution: Let us define two events M and S as follows:

M= the student passed mathematics and
E= the student passed economics

Evidently, their complements $\overline{M}$ and $\overline{E}$ will denote events that the student failed in mathematics and economics respectively. We may now use the following symbols to represent the events defined in (*a*), (*b*) and (*c*) above:

$M \cap E$ = Passing both mathematics and economics;
$\overline{M} \cap \overline{E}$ = Failing both mathematics and economics;
$M \cap \overline{E}$ = Passing mathematics but failing economics.

Given the information and notations, we can construct the following table:

Event	M	$\overline{M}$	Total
E	$n(M \cap E)$	$n(\overline{M} \cap E)$	75
$\overline{E}$	$n(M \cap \overline{E})$	$n(\overline{M} \cap \overline{E})$	75
Total	95	55	150

It is evident that $n(M)=95$, $n(E) = 75$. Also $n(M \cup E) = 135$, $n(S)=150$
Thus

$$P(M) = \frac{n(M)}{n(S)} = \frac{95}{150} = \frac{19}{30}$$

$$P(E) = \frac{n(E)}{n(S)} = \frac{75}{150} = \frac{1}{2}$$

The probability of passing at least one of the subjects is $P(M \cup E)$, where

$$P(M \cup E) = \frac{n(M \cup E)}{n(S)} = \frac{135}{150} = \frac{9}{10}.$$

Hence

(a) $P(M \cap E) = P(M) + P(E) - P(M \cup E) = \frac{95}{150} + \frac{75}{150} - \frac{135}{150} = \frac{7}{30}$

(b) $P(\overline{M} \cap \overline{E}) = 1 - P(M \cup E) = 1 - \frac{9}{10} = \frac{1}{10}$

(c) $P(M \cap \overline{E}) = P(M) - P(M \cap E) = \frac{19}{30} - \frac{7}{30} = \frac{2}{5}$

Example 7.71: Three horses A, B, and C are on a race. A is twice as likely to win as B, and B is twice as likely to win as C. What are their respective chances of winning?

Solution: Let $P(A)$, $P(B)$ and $P(C)$ stand for the probabilities of A, B, and C winning respectively. As stated in the problem, the events are not equally likely. If $P(C)=k$, then $P(B)=2P(C)=2k$. Since A is twice as likely to win as $B, P(A)=4k$.

The winning of A, B and C are mutually exclusive and exhaustive. Hence

$$P(A \cup B \cup C) = 1.$$

Thus

$$P(A)+P(B)+P(C)=1.$$

Or

$$4k+2k+k=1, \text{ giving } k=\frac{1}{7}$$

Hence

$$P(A)=4k=\frac{4}{7},\ P(B)=2k=\frac{2}{7}, \text{ and } P(C)=\frac{1}{7}$$

Example 7.72: A coin is so loaded that heads are twice as likely to appear as tails. If the coin is tossed once, find the probability that the coin falls tail. Also find the odds in favor of head.

Solution: Let $P(T)=k$, then $P(H)=2k$. Since the falling of a head and falling of a tail are mutually exclusive and exhaustive,

$$P(H)+P(T)=1 \text{ Or } k+2k=1, \text{ giving } k=\frac{1}{3}.$$

Hence

$$P(T)=\frac{1}{3}, \text{ and } P(H)=\frac{2}{3}$$

The odds in favor of head is defined as follows:

$$O_H=\frac{P(H)}{P(T)}=\frac{2/3}{1/3}=2:1.$$

Example 7.73: A cubical die is so loaded that an odd number is thrice as likely to appear as an even number. Find the probability that in a single toss, (i) a 4 occurs and (ii) a number greater than 4 occurs.

Solution: A sample space for this experiment is

$$S=\{1. 2. 3. 4. 5. 6\}$$

For a fair coin, each of the above elements has a probability of 1/6 of its occurrence. But in the present case, the die has been deliberately made biased and as a result, the numbers on the die are not equally likely. There are three even numbers, 2, 4, 6 and three odd numbers, 1, 3, 5. Thus if x is used to denote the number on the faces of the die,

$$P(x)=k, \quad \text{if } x \text{ is even}$$
$$=3k, \quad \text{if } x \text{ is odd}$$

But

$$\sum P(x=r)=1,\ r=1, 2. ...,6$$

Or

$$P(1)+P(2)+P(3)+P(4)+P(5)+P(6)=1$$

Or

$$3k+k+3k+k+3k+k=1$$

Solving for k

$$k=\frac{1}{12}$$

This shows that $P(x)=\frac{1}{12}$, when x is even, and $P(x)=\frac{3}{12}$, when x is odd.

Thus,

$$P(x=4)=\frac{1}{12}$$

and

$$P(x>4)=P(5)+P(6)=\frac{3}{12}+\frac{1}{12}=\frac{1}{3}.$$

Example 7.74: The probability that a student passes history is 2/3 and that he passes economics is 4/9. If the probability of passing at least one course is 4/5, what is the probability that he will pass both courses?

Solution: Let H be the event that the student passes history and E be the event that he passes economics. With these notations, we compute $P(H \cap E)$, the probability of passing both the subjects.

By addition law

$$P(H \cup E)=P(H)+P(E)-P(H \cap E)$$

from which the probability of passing both the subjects is

$$P(H \cap E)=P(H)+P(E)-P(H \cup E)$$
$$=\frac{2}{3}+\frac{4}{9}-\frac{4}{5}=\frac{14}{45}$$

Example 7.75: Two ordinary six-sided dice were tossed. Set up a sample space for this experiment and hence find the probability that

(a) Sum of the numbers on the two dice is 7.
(b) Numbers on die 1 are greater than the numbers on die 2.
(c) First die shows an even number.
(d) Numbers on both the dice are the same, (i.e. throwing a double).
(e) Difference in the outcomes of two dice is 2.

Solution: The experiment is equivalent to throwing a single dice twice. Table below shows a sample space that lists the possible outcomes of the experiment.

D_1 ⇓ D_2⇒	1	2	3	4	5	6
1	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
2	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
3	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
4	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
5	(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
6	(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

Each row of the table corresponds to a fixed value of the first die labeled D_1 and each column corresponds to a fixed value of the second die labeled D_2. These values are the outcomes of the two dice. Thus, the entry (3, 4) in the third row and the fourth column represents the event "first die shows a 3, while the second die shows a 4". If x_1 and x_2 stand for the outcomes of the first die and second die respectively, then the entire sample space S is the set of ordered pairs (x_1, x_2) with x_1 and x_2 each taking the values 1, 2, 3, 4, 5 or 6:

$$S=\{x_1, x_2 | 1 \le x_1 \le 6,\ 1 \le x_2 \le 6\}$$

Since the 6 admissible values of x_1 may be associated with each of the 6 admissible values of x_2, there will be 6×6 or 36 possible outcomes of the experiment. So long as the dice are well balanced and thrown under uniform conditions, these outcomes are all equally likely. This leads us to assign probability 1/36 to each outcome in the sample space. We now compute the probabilities of the events defined in (*a*), (*b*), (*c*) and (*d*). In expressing the verbal statements associated with the events defined in (*a*), (*b*), (*c*) and (*d*), we shall translate them into algebraic conditions in terms of x_1 and x_2.

(a) $P(\text{sum is } 7) = P(x_1 + x_2 = 7) = ?$

In evaluating this probability, we count the cases, which make sum of the points on the two dice equal to 7. There are 6 such cases. These are

(1, 6), (2, 5), (3, 4), (4, 3), ((5, 2), and (6, 1)

Thus

$$P(x_1 + x_2 = 7) = \frac{6}{36} = \frac{1}{6}$$

(c) $P(\text{points on the first die are greater than the second}) = P(x_1 > x_2)$

By simple counting, we find that there are 15 such cases, which are favorable to this event. These are

(2, 1), (3, 1), (4, 1), (5, 1), (6, 1),
(3, 2), (4, 2), (5, 2), (6, 2), (4, 3),
(5, 3), (6, 3), (5, 4), (6, 4), (6, 5).

Thus the required probability is

$$P(x_1 > x_2) = \frac{15}{36} = \frac{5}{12}$$

(*c*) Let E_1 stand for the event that the first die shows an even number. We then compute $P(E_1)$. Counting of the sample points of S shows that there are 18 such points. These are:

(2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6),
(4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6),
(6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6).

Hence

$$P(E_1) = \frac{18}{36} = \frac{1}{2}$$

(*d*) $P(\text{throwing a double}) = P(x_1 = x_2)$?

The elements that satisfy this condition lie on the diagonal line. There are 6 such cases. These are

(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6).

Thus

$$P(x_1 = x_2) = \frac{6}{36} = \frac{1}{6}$$

(*e*) The event is $|x_1 - x_2 = 2|$. The following 8 outcomes are favorable to this event:

(1, 3), (2, 4), (3, 1), (3, 5), (4, 2), (4, 6), (5, 3), (6, 4).

Hence

$$P\{|x_1 - x_2| = 2\} = \frac{8}{36} = \frac{2}{9}$$

Example 7.76: Three bulbs are chosen at random from a lot of 15 bulbs, of which 5 are defective. Find the probability that

(*a*) None are defective,

(*b*) Exactly one is defective

(*c*) At least one is defective.

Solution: If D stands for a defective bulb and $\overline{D}$ for a non-defective bulb, then a sample space for this experiment is of the form

$$S = \{D, \overline{D}\}$$

The sample space will consist of $5D$'s and $10\,\overline{D}$'s.

(a) Total number of ways in which 3 bulbs can be chosen is ${}^{15}C_3$

The choice of 3 non-defective bulbs then must come from the sub-set of 10 non-defective bulbs. The number of ways in which this selection can be made is ${}^{10}C_3$. Thus

$$P(3\,\overline{D}) = \frac{{}^{10}C_3}{{}^{15}C_3} = \frac{120}{455} = \frac{24}{91}.$$

(b) If exactly one bulb is defective, then the other two must be non-defective. The number of ways in which one defective and 2 non-defective bulbs can be chosen is ${}^{5}C_1 \times {}^{10}C_2$, so that

$$P(1D, 2\overline{D}) = \frac{{}^{5}C_1 \times {}^{10}C_2}{{}^{15}C_3} = \frac{45}{91}$$

(c) The event "at least one defective" can be computed as follows:

$$P(\text{at least one defective}) = 1 - P(\text{none defective})$$
$$= 1 - \frac{24}{91} = \frac{67}{91}.$$

The generalization of the foregoing example may be made as follows:

Suppose that we have a group of n objects, of which 'a' are of kind A, and 'b' are of another kind B, so that $a + b = n$. From these n objects, we choose a sample of 'r' objects. What is the probability that the sample will consist of exactly x of kind A (*i.e.* x A's)? Here we have ${}^{n}C_r$ possible samples. Of these, ${}^{a}C_x \times {}^{b}C_{r-x}$ are exactly of kind A. Therefore

$$P(x\,A's) = \frac{{}^{a}C_x \times {}^{b}C_{r-x}}{{}^{n}C_r}$$

Example 7.77: A picnic party is conducting a raffle draw where 50 tickets are to be sold –one per participant. There are three prizes to be awarded. If the 4 organizers of the raffle each buys one ticket, what is the probability that the 4 organizers

a) Win all the prizes?

b) Win exactly 2 of the prizes

c) Win exactly one of the prizes?

d) Win none of the prizes?

Solution: There are ${}^{50}C_3 = 19600$ ways to choose the winners. Since the choice is random, each of the 19600sample points is equally likely.

a) There are ${}^{4}C_3 = 4$ ways for the organizers to win all of the prizes. Hence the desired probability is

$$P(a) = \frac{{}^{4}C_3}{{}^{50}C_3} = \frac{4}{19600}.$$

b) The organizers can win exactly 2 of the prizes if 1 of the other 46 participants wins 1 prize. There are ${}^{4}C_2 \times {}^{46}C_1 = 276$ ways for this to occur. Hence, the desired probability is

$$P(b) = \frac{{}^{4}C_2 \times {}^{46}C_1}{{}^{50}C_3} = \frac{276}{19600}$$

c) Likewise, the organizers can win exactly 1 of the prizes if 2 of the other 46 participants win 2 prizes. There are ${}^{4}C_1 \times {}^{46}C_2 = 4140$ ways for this to occur. Hence, the desired probability is

$$P(c) = \frac{{}^{4}C_1 \times {}^{46}C_2}{{}^{50}C_3} = \frac{4140}{19600}$$

d) Similarly, the organizers win none of the prizes if 3 of the other 46 participants win all the 3 prizes. There are ${}^{4}C_0 \times {}^{46}C_3 = 15180$ ways for this to occur. Hence, the desired probability is

$$P(d) = \frac{{}^{4}C_0 \times {}^{46}C_3}{{}^{50}C_3} = \frac{15180}{19600}$$

Example 7.78: A class contains 10 men and 20 women of whom half of the men and half of the women have brown eyes. A person is chosen at random. What is the probability that the person is either a man or has brown eyes? Are the events "being a male" and "having brown eyes" independent?

Solution: To solve this problem, you can construct a table as follows:

	Man (M)	Woman (W)	Total
Brown (B)	$n(M \cap B)=5$	$n(W \cap B)=10$	$n(B)=15$
Not brown ($\bar{B}$)	$n=(M \cap \bar{B})=5$	$n(W \cap \bar{B})=10$	$n(\bar{B})=15$
Total	$n(M)=10$	$n(W)=20$	$n(S)=30$

Our problem is to compute $P(M \cup B)$ where.

$$P(M \cup B)=P(M)+P(B)-P(M \cap B)$$

From the above table, we have,

$$P(M)=\frac{n(M)}{n(S)}=\frac{10}{30}=\frac{1}{3}$$

$$P(B)=\frac{n(B)}{n(S)}=\frac{15}{30}=\frac{1}{2}$$

$$P(M \cap B)=\frac{n(M \cap B)}{n(S)}=\frac{5}{30}=\frac{1}{6}$$

Thus

$$P(M \cup B)=\frac{1}{3}+\frac{1}{2}-\frac{1}{6}=\frac{2}{3}.$$

To check the independence of M and B, we must show that $P(M) \times P(B)=P(M \cap B)$, which for the present case is true. Hence M and B are independent implying that sex has nothing to do with the eye color.

Example 7.79: A class contains 6 girls and 10 boys. If three students are chosen at random from the class to form a picnic committee, find the probability that

(a) all the 3 are boys,
(b) exactly 2 are boys and
(c) at least one is a boy.

Solution: A committee of 3 students can be formed in $^{16}C_3 = 560$ ways. Thus

$$\text{(a) } P(3B,0G)=\frac{^{10}C_3 \times {^6C_0}}{^{16}C_3}=\frac{3}{14}$$

$$\text{(b) } P(2B,1G)=\frac{^{10}C_2 \times {^6C_1}}{^{16}C_3}=\frac{27}{56}$$

$$\text{(c) } P(\text{Atleast one boy})=P(1B,2G)+P(2B,1G)+P(3B,0G)$$

$$=\frac{15}{56}+\frac{27}{56}+\frac{3}{14}=\frac{27}{28}$$

Example 7.80: Afreen can solve a problem, whose probability of doing so is 1/4, while the probability that Nawar can solve the same problem is 2/5. A problem is chosen at random from a book. What is the probability that the problem will be solved?

Solution: We need here that at least one of the two persons will be able to solve the problem. Symbolically, we want to evaluate $P(A \cup B)$.

$$P(A \cup B)=P(A)+P(B)-P(A \cap B)$$

Here $P(A \cap B)$ is the probability that both Afreen (A) and Nawar (B) can solve the problem. Since A and B can independently solve the problem,

$$P(A \cap B)=P(A) \times P(B)=\left(\frac{1}{4}\right)\left(\frac{2}{5}\right)=\frac{1}{10}.$$

Thus

$$P(A \cup B)=\frac{1}{4}+\frac{2}{5}-\frac{1}{10}=\frac{11}{20}$$

Example 7.81: A lot consists of 10 radios of which 4 are defectives. A buyer will accept a lot of 10 radios if a sample of 2 picked at random from the lot shows no defective. What is the probability that the buyer will accept the lot?

Solution: The buyer can choose 2 radios from the entire lot in $^{10}C_2=45$ ways. Since the lot contains 4 defectives and 6 non-defectives, the buyer can have a sample of two non-defective radios if the sample comes only from the 6 non-defectives This choice can be made in $^{6}C_2=15$ ways. The probability of accepting the lot $P(A)$ is thus

$$P(A)=\frac{^6C_2}{^{10}C_2}=\frac{15}{45}=\frac{1}{3}$$

Example 7.82: In a class of 180 students, all of whom took English and History, 15 failed History, 10 failed English and 5 failed both.

(a) Find the probability that a student chosen at random failed History but passed English.

(b) What is the conditional probability that a student, who passed English, also passed History?

Solution: If E stands for the event that a student chosen at random passed English and H for the event that the student passed History, then we can construct the following table for computation of the required probabilities

Events	H	$\overline{H}$	Total
E	160	10	170
$\overline{E}$	5	5	10
Total	165	15	180

where $\overline{H}$ and $\overline{E}$ stand respectively for the complements of the events H and E, representing the failure in the respective subjects..

We are given that

$$n(S)=180,\ n(\overline{H})=15,\ n(\overline{E})=10 \text{ and } n(\overline{E}\cap\overline{H})=5$$

(i) The probability that the student failed History but passed English is

$$P(\overline{H}\cap E)=\frac{n(\overline{H}\cap E)}{n(S)}$$

Since

$$n(\overline{H}\cap E)=n(\overline{H})-n(\overline{H}\cap\overline{E})=15-5=10$$

we have

$$P(\overline{H}\cap E)=\frac{n(\overline{H}\cap E)}{n(S)}=\frac{10}{180}=\frac{1}{18}$$

(ii) This is the case of conditional probability of the event $H|E$.

$$P(H\mid E)=\frac{n(H\cap E)/n(S)}{n(E)/n(S)}=\frac{n(H\cap E)}{n(E)}$$

From the table, we find that $n(H\cap E)=160$. Hence

$$P(H\mid E)=\frac{n(H\cap E)}{n(E)}=\frac{160}{170}=\frac{16}{17}$$

Example 7.83: A group of 20 students went on a picnic. Five got sun burnt, 8 got bitten by mosquitoes and 10 returned home safely without any mishap. What is the probability that (a) a boy, who got sun burnt was ignored by the mosquitoes? (b) a mosquito bitten boy also got sun burnt?

Solution: Let us define two events S and M as follows:

$$S=\text{got sun burnt. } M=\text{bitten by mosquitoes.}$$

If $\overline{S}$ and $\overline{M}$ stand respectively for 'not sun burnt' and 'not bitten by mosquitoes', the following tabular presentation may be used to evaluate the required probabilities.

	S	$\overline{S}$	Total
M	3	5	8
$\overline{M}$	2	10	12
Total	5	15	20

(a) With the notations defined above, $\overline{M}|S$ is the event that a fire burnt boy was ignored by the mosquitoes. Hence using the reduced sample space,

$$P(\overline{M}\mid S)=\frac{n(\overline{M}\cap S)}{n(S)}=\frac{2}{5}$$

(b) Here $S|M$ is the event that a boy who is known to be mosquito bitten, was also fire burnt. Hence using the reduced sample space M,

$$P(S\mid M)=\frac{n(S\cap M)}{n(M)}=\frac{3}{8}$$

Example 7.84: Suppose two machines I and II in a factory operate independently of each other. Past experience showed that during a given 8-hour time, machine I remains inoperative one third of the time and machine II does so about one-fourth of the time. What is the probability that at least one of the machines will become inoperative during the given period?

Solution: Let A be the event that machine I will become inoperative during the period and B be the event that machine II will remain inoperative during the same period. Clearly, $A\cap B$ is the event that both the machines will remain inoperative.

(i) Assuming independence of A and B

$$P(A\cap B)=P(A)P(B)=\left(\frac{1}{3}\right)\left(\frac{1}{4}\right)=\frac{1}{12}$$

(ii) $A\cup B$ is the event that at least one of the two machines will remain inoperative and hence

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{1}{3} + \frac{1}{4} - \frac{1}{12} = \frac{1}{2}$$

Example 7.85: Three groups of children I, II and III, contain respectively 3 girls and 1 boy, 2 girls and 2 boys, 1girl and 3 boys. One child is selected from each group at random. What is the chance that the three selected consist of 1 girl and 2 boys?

Solution: Let B stand for a boy and G for a girl. The selection of 1 boy and 2 girls can thus be made from the following sequences:

$$(G\ B\ B)\ (B\ G\ B)\ (B\ B\ G)$$

where the sequence (G B B), for example, stands for the event that a girl will be selected from group I, a boy from group II and a boy from group III. Since choice of the children can be made independently between groups, the probability of the sequence (G B B) is

$$P(GBB)=P(G)\ P(B)\ P(B)=\left(\frac{3}{4}\right)\left(\frac{2}{4}\right)\left(\frac{3}{4}\right)=\frac{18}{64}$$

Similarly

$$P(BGB)=P(B)\ P(G)\ P(B)=\left(\frac{1}{4}\right)\left(\frac{2}{4}\right)\left(\frac{3}{4}\right)=\frac{6}{64}$$

and

$$P(BBG)=P(B)\ P(B)\ P(G)=\left(\frac{1}{4}\right)\left(\frac{2}{4}\right)\left(\frac{1}{4}\right)=\frac{2}{64}$$

Since the above combinations are mutually exclusive, the required probability is

$$\text{P(one girl and two boys)}= P(GBB)+ P(BGB) + P(BBG)$$
$$=\frac{1}{64}(18+6+2)=\frac{13}{32}$$

Example 7.86: A and B are two events associated with an experiment. Suppose that $P(A)$=0.4, $P(A \cup B)$=0.7. If $P(B)=p$, find the value of p for which (i) A and B are mutually exclusive and (ii) A and B are independent.

Solution: (i) For A and B to be mutually exclusive, $A \cap B = \phi$

Hence

$$P(A\cup B)=P(A)+P(B), \text{ or } 0.7=0.4+p, \text{ from which, } p=0.3$$

(ii) For A and B to be independent $P(A\cap B)=P(A)\times P(B)$. Therefore

$$P(A\cup B)=P(A)+P(B) - P(A)\times P(B)$$

Since $P(A\cup B)$=0.7, $P(A)$=0.4,

$$0.7 = 0.4 + P(B) - 0.4 \times P(B)$$

whence $P(B) = 0.5$

Example 7.87: A football player is known to miss a penalty shot in three out of four shots whereas another player is known to miss a shot in two out of three shots. Find the probability of missing a shot when they both try.

Solution: Let A denote the event that the first player misses the shot, and B the event that the second player misses the shot. Then

$$P(A)=\frac{3}{4} \text{and } P(B)=\frac{2}{3}$$

Since the events are not mutually exclusive, the probability that at least one of the two players will miss the shot, is, by addition law of probability:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
$$=\frac{3}{4}+\frac{2}{3}-\frac{3}{4}\times\frac{2}{3}=\frac{11}{12}$$

Example 7.88: Three men A, B, and C toss a coin in succession for a prize to be given to the one who first obtains heads. Show that their respective chances of winning are 4/7, 2/7 and 1/7.

Solution: If A starts first, then the following sequence of outcomes will be favorable to his winning:

$$\{H\}, (TTTH\}, \{TTTTTTH\}, \{TTTTTTTTTH\}\ldots\ldots$$

As an interpretation of the above sequence, note that the first outcome H denotes that, if A obtains a head, he wins. Thus

$$P(H)=\frac{1}{2}$$

If A fails to obtain head (in other words, he obtains a tail), then B and C will have to obtain tails when their turns come and then the turn of A comes, when he must obtain a head for his winning. This sequence is represented by the outcome $\{TTTH\}$. Thus,

$$P(TTTH)=\left(\frac{1}{2}\right)^4$$

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