Fundamental Concepts: Vectors

Movement of a system is commonly described into two basic quantities, i.e. scalars and vectors.

Scalar - is a physical quantity with magnitude only (Volume, Temperature, Pressure)
Vector - is a physical quantity with both magnitude and direction

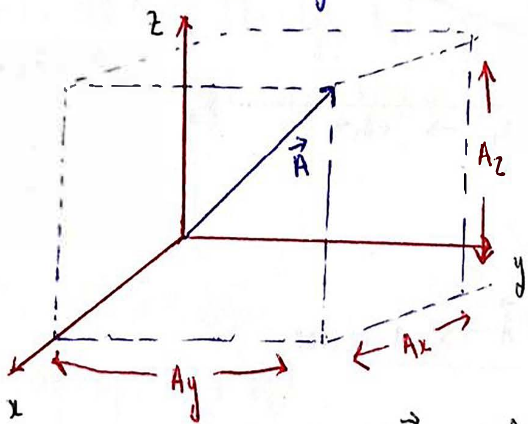
* Mathematically vectors combine each other according to the parallelogram rule of addition.

* Vectors plays a major role in simplifying most complicated physical systems into simple systems which can be understood by everyone.

Vectors mathematically are written as:

$$\vec{A}\ ,\ \underline{A}\ ,\ \mathbf{A};\ (\text{i.e. } \vec{F} = m\vec{a}\ ;\ \underline{F} = m\underline{a})$$

- It is represented diagrammatically as following:



$\vec{A}$ can be represented as a set of three scalars
$A = A_x, A_y, A_z$

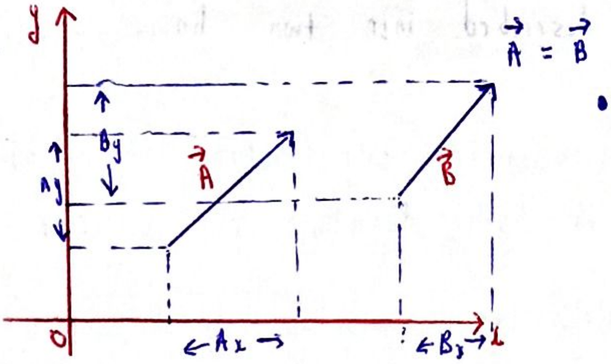
$\vec{A} = x\hat{x} + y\hat{y} + z\hat{z}$ (in terms of a unit vector)

$\vec{A} = (A_x, A_y, A_z)$, components of vectors

* The concept of space and time is regarded to be the fundamental in understanding the concept / behaviour of the universe.

Vector Algebra / Vector Rules

I. Equality of Vectors.



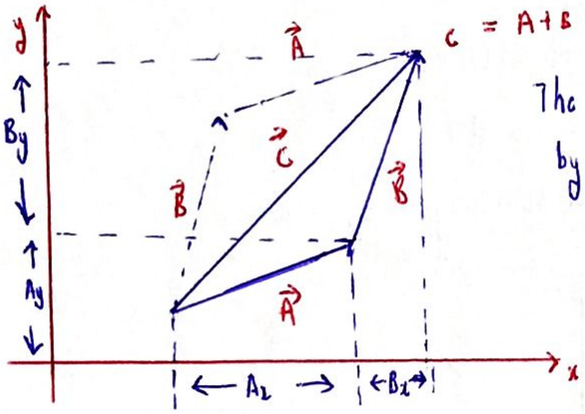
- They are equal because:
  1. A ∥ B and have the same length.

$(A_x, A_y, A_z) = (B_x, B_y, B_z)$

thus:

$A_x = B_x$; $A_y = B_y$; $A_z = B_z$

II. Addition of vectors



$C = A + B = B + A$

The addition of two vectors is defined by the equation:

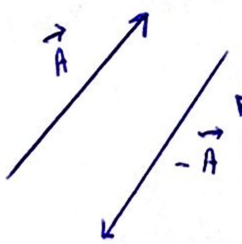
$$\vec{A} + \vec{B} = (A_x, A_y, A_z) + (B_x, B_y, B_z)$$
$$= (A_x + B_x, A_y + B_y, A_z + B_z)$$

III. Multiplication by a scalar

If c is a scalar and $\vec{A}$ is a vector:

$$c\vec{A} = c(A_x, A_y, A_z) = (cA_x, cA_y, cA_z) = \vec{A}c$$

The negative of a Vector



Vector Substraction

$A - B = A + (-1)B = (A_x - B_x,\ A_y - B_y,\ A_z - B_z)$

Substraction of a given vector $\vec{B}$ from the vector $\vec{A}$ is equivalent to adding $-\vec{B}$ to $\vec{A}$.

The Null Vector

The vector $0 = (0,0,0)$ is called null vector. The direction of the null vector is undefined.

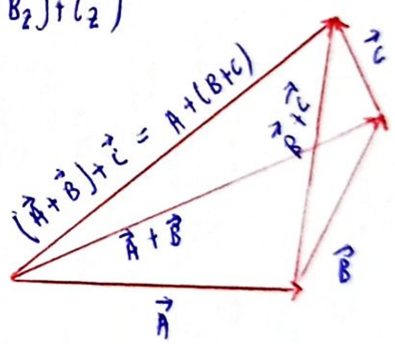
The Commutative Low of addition

$$\vec{A} + \vec{B} = \vec{B} + \vec{A}$$

because $A_x + B_x = B_x + A_x$ and similarly for the y and z components

The Associative Low

$$\vec{A} + (\vec{B} + \vec{C}) = (A_x + (B_x + C_x),\ A_y + (B_y + C_y),\ A_z + (B_z + C_z))$$
$$= ((A_x + B_x) + C_x,\ (A_y + B_y) + C_y,\ (A_z + B_z) + C_z)$$
$$= (\vec{A} + \vec{B}) + \vec{C}$$



The distribution Low

$$c(\vec{A} + \vec{B}) = c(A_x + B_x,\ A_y + B_y,\ A_z + B_z)$$
$$= (c(A_x + B_x),\ c(A_y + B_y),\ c(A_z + B_z))$$
$$= cA_x + cB_x,\ cA_y + cB_y,\ cA_z + cB_z$$
$$= c\vec{A} + c\vec{B}$$

Magnitude of a Vector

- The magnitude of a vector $\vec{A}$ denoted by $|A|$ is defined as the square root of the sum of the squares of the components, namely,

$$A = |\vec{A}| = (A_x^2 + A_y^2 + A_z^2)^{1/2}$$

Unit Coordinate Vectors

A unit vector whose magnitude is unity.

$\hat{e}_x = (1,0,0)$, $\hat{e}_y = (0,1,0)$, $\vec{e}_z = (0,0,1)$ ← coordinate vector of basis vectors.

or $\hat{i} = e_x$, $\hat{j} = e_y$, $\hat{k} = e_z$

Find the sum and magnitude of two vectors

$\vec{A} = (1,0,2)$ and $\vec{B} = (0,1,1)$

Sum: $\vec{A} + \vec{B} = (1,1,3)$

magnitude: $|\vec{A} + \vec{B}| = (1^2 + 1^2 + 3^2)^{1/2} = \sqrt{11}$

Summary: (LAWS)

Equality : $\vec{A} = \vec{B}$

Addition : $\vec{A} + \vec{B} = (A_x, A_y, A_z) + (B_x, B_y, B_z)$

Substraction : $\vec{A} - \vec{B} = \vec{A} + (-\vec{B})$

Multiplication : $c\vec{A} = c(A_x, A_y, A_z)$

Distribution : $c(\vec{A} + \vec{B}) = c\vec{A} + c\vec{B}$

Magnitude : $\vec{A} = (A_x^2 + A_y^2 + A_z^2)^{1/2}$

The Scalar Product

- Given two vectors $\vec{A}$ and $\vec{B}$, the scalar product or "dot" product $\vec{A}\cdot\vec{B}$, is the scalar defined by the equation:

$$\vec{A}\cdot\vec{B} = A_x B_x + A_y B_y + A_z B_z \quad -①$$

- The scalar multiplication is commutative,

$$\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A} \quad -②$$

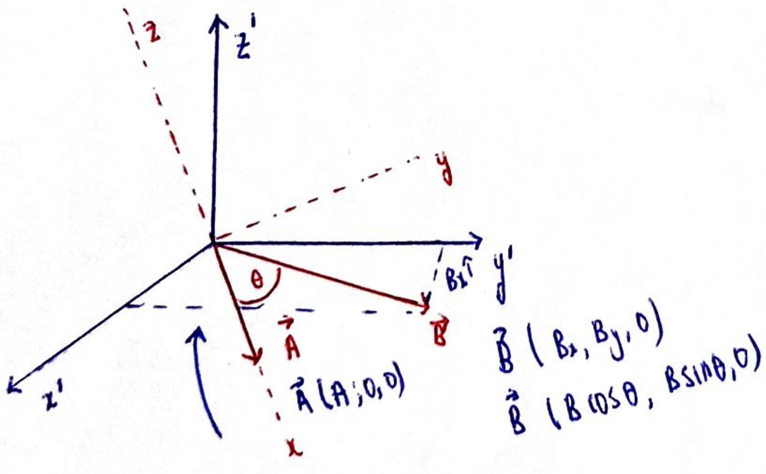
Because $A_x B_x = B_x A_x$ and so on. It is also distributive

$$\vec{A}\cdot(\vec{B}+\vec{C}) = \vec{A}\cdot\vec{B} + \vec{A}\cdot\vec{C} \quad -③$$

If we apply equation 1

$$\begin{aligned}\vec{A}\cdot(\vec{B}+\vec{C}) &= A_x(B_x+C_x) + A_y(B_y+C_y) + A_z(B_z+C_z)\\ &= A_xB_x + A_xC_x + A_yB_y + A_yC_y + A_zB_z + A_zC_z\\ &= A_xB_x + A_yB_y + A_zB_z + A_xC_x + A_yC_y + A_zC_z\\ &= \vec{A}\cdot\vec{B} + \vec{A}\cdot\vec{C}\end{aligned}$$

The dot product $\vec{A}\cdot\vec{B}$ has a simple geometrical interpretation and can be used to calculate the angle θ between those two vectors.



Using the diagram:

$$\vec{A}\cdot\vec{B} = A_xB_x = A(B\cos\theta)$$

$$\vec{A}\cdot\vec{B} = |A||B|\cos\theta$$

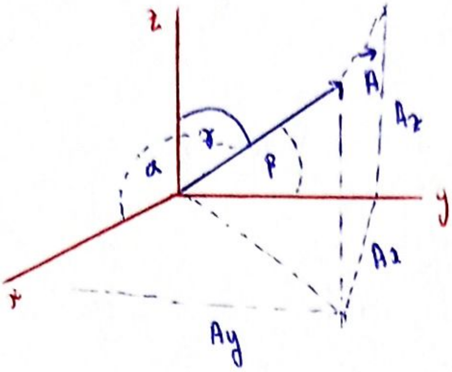
$$\boxed{\begin{aligned}\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1\\ \hat{i}\cdot\hat{j} = \hat{i}\cdot\hat{k} = \hat{j}\cdot\hat{k} = 0\end{aligned}}$$

The significance of this equation is to prove that the cosine of the angle between two line segments is given by:

$$\cos\theta = \frac{\vec{A}\,\vec{B}}{|A||B|}$$

Alternative definition of dot product.

Expressing Ay Vector as the Product of its magnitude by a unit vector.



$$\vec{A} = A\left(\frac{A_x}{A}\hat{i} + \frac{A_y}{A}\hat{j} + \frac{A_z}{A}\hat{k}\right) \quad *$$

$$\frac{A_x}{A} = \cos\alpha \;;\; \frac{A_y}{A} = \cos\beta \;;\; \frac{A_z}{A} = \cos\gamma$$

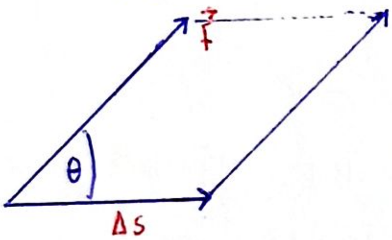
Equation * can be written:

$$\vec{A} = A\left(\cos\alpha\,\hat{i} + \cos\beta\,\hat{j} + \cos\gamma\,\hat{k}\right)$$

Example 1.4.1.

Componet of a vector: Work

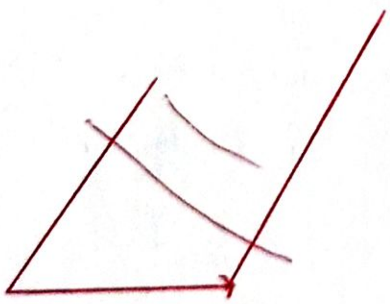
Object under the action of constant force undergoes linear displacement Δs. We just need to determine work done by the force is given by the product of the component of the force $\vec{F}$.



$$\Delta W = (F\cos\theta)\,\Delta s$$

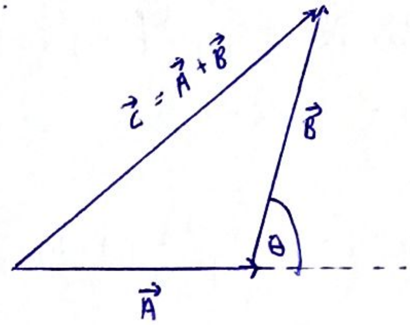
$$\Delta W = \vec{F} \cdot \Delta s$$

Ex 1.4.2.



Example 1-4.2

Law of Cosines
Take the dot product of $\vec{C}$ with itself.



$$\vec{C}\cdot\vec{C} = (\vec{A}+\vec{B})\cdot(\vec{A}+\vec{B})$$
$$= \vec{A}\cdot\vec{A} + 2\vec{A}\cdot\vec{B} + \vec{B}\cdot\vec{B}$$
$$C^2 = A^2 + 2AB\cos\theta + B^2$$

The Vector Product

Given vector $\vec{A}$ and $\vec{B}$ the vector product $\vec{A}\times\vec{B}$

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \begin{vmatrix} A_y & A_z \\ B_y & B_z \end{vmatrix}\hat{i} + \begin{vmatrix} A_z & A_x \\ B_z & B_x \end{vmatrix}\hat{j} + \begin{vmatrix} A_x & A_y \\ B_x & B_y \end{vmatrix}\hat{k}$$

$$= (A_yB_z - A_zB_y)\hat{i} + (A_zB_x - A_xB_z)\hat{j} + (A_xB_y - A_yB_x)\hat{k}$$

Defined as the vector whose components are given by the equation

$$\boxed{\vec{A}\times\vec{B} = (A_yB_z - A_zB_y,\ A_zB_x - A_xB_z,\ A_xB_y - A_yB_x)}$$

Rules for cross multiplication

$$\boxed{\begin{aligned} \vec{A}\times\vec{B} &= -\vec{B}\times\vec{A} \\ \vec{A}\times(\vec{B}+\vec{C}) &= \vec{A}\times\vec{B}+\vec{A}\times\vec{C} \\ n(\vec{A}\times\vec{B}) &= (n\vec{A})\times\vec{B} = \vec{A}\times(n\vec{B}) \end{aligned}}$$

$$\hat{i}\times\hat{i} = \hat{j}\times\hat{j} = \hat{k}\times\hat{k} = 0$$
$$\hat{j}\times\hat{k} = \hat{i} = -\hat{k}\times\hat{j}$$
$$\hat{i}\times\hat{j} = \hat{k} = -\hat{j}\times\hat{i}$$
$$\hat{k}\times\hat{i} = \hat{j} = -\hat{i}\times\hat{k}$$

* The cross product of two parallel vectors is null

The magnitude of the cross product.

$$|\vec{A} \times \vec{B}|^2 = (A_y B_z - A_z B_y)^2 + (A_z B_x - A_x B_z)^2 + (A_x B_y - A_y B_x)^2$$

$$|\vec{A} \times \vec{B}|^2 = (A_y B_z - A_z B_y)(A_y B_z - A_z B_y) + (A_z B_x - A_x B_z)(A_z B_x - A_x B_z) + (A_x B_y - A_y B_x)$$

$$\cancel{(A_y B_z)^2 - 2 A_z B_y + (A_z B_y)^2 + (A_z B_x)^2} + (A_x B_y - A_y B_x)$$

$$= [(A_y B_z)^2 - A_y B_z A_z B_y + (A_z B_y)^2 + (A_z B_x)^2 - A_z B_x A_x B_z + (A_x B_z)^2 + (A_x B_y)^2 - A_x B_y A_y B_x + (A_y B_x)^2]$$

$$\Rightarrow |\vec{A} \times \vec{B}|^2 = (A_x^2 + A_y^2 + A_z^2)(B_x^2 + B_y^2 + B_z^2) - (A_x B_x + A_y B_y + A_z B_z)^2$$

$$|\vec{A} \times \vec{B}|^2 = A^2 B^2 - (\vec{A} \cdot \vec{B})^2$$

$$|\vec{A} \times \vec{B}|^2 = A^2 B^2 - [AB \cos\theta]^2$$

θ is the angle between vector $\vec{A}$ and $\vec{B}$

$$= A^2 B^2 - A^2 B^2 \cos^2\theta$$

$$|\vec{A} \times \vec{B}|^2 = A^2 B^2 (1 - \cos^2\theta)$$

$$|\vec{A} \times \vec{B}| = AB (1 - \cos^2\theta)^{1/2} = \underline{AB \sin\theta}$$

Triple product

The expression

$\vec{A}\cdot(\vec{B}\times\vec{C})$ [scalar triple product of $\vec{A}$, $\vec{B}$ and $\vec{C}$]

↑ because is the dot product of two vectors

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = \begin{array}{c} \begin{array}{ccc} \hat{i} & \hat{j} & \hat{k} \end{array} \\ \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} \end{array}$$

Determinant

$$= A_x(B_yC_z - B_zC_y)\hat{i} + A_y(B_xC_z - B_zC_x)\hat{j} + A_z(B_xC_y - B_yC_x)\hat{k}$$

Because of exchange of terms of two rows or of two columns of a determinant changes it sign but not its absolute value, we can derive the following useful equation:

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = (\vec{A}\times\vec{B})\cdot\vec{C}$$

Thus, the dot product and the cross may be interchanged in the scalar triple product.

$\vec{A}\times(\vec{B}\times\vec{C})$ is called a vector triple product.

$$\boxed{\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})}$$

The change of coordinate system, The transformation Matrix

There are two ways of describing a vector namely:

① Geometric approach - vector as an arrow $\vec{A}$

② Algebric approach - Vector as components of cartesian coordinates

However, both approaches are not very satisfactory

In that case we need to follow the approach of mathematicians and define a vector as a set of 3 components that transforms in the same manner as a displacement when we change the coordinate.

The displacement vector is the model for the behavior of all vectors. A more precise definition is that a vector in space has no prefered direction and no prefered location. As a result a physical quantity such as displacement or force should be independent of the coordinate system choosen.

To understand this better let's consider, the diagrams below:

A simple vector in space is shown below without any rotation (Fig A). In fig B we are rotateing along the z - axis.

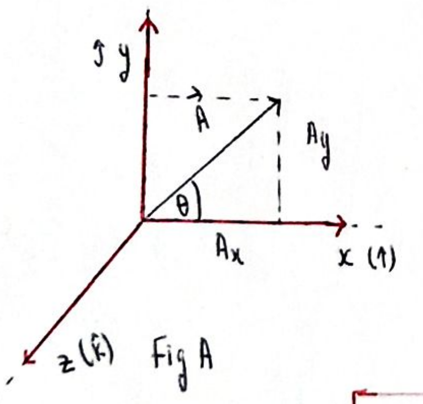


Fig A

$\Longrightarrow$

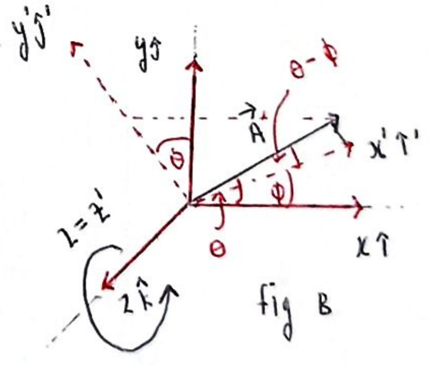


fig B

$$\begin{cases} A_y = A\sin\theta \\ A_x = A\cos\theta \\ A_z = 1 \end{cases}$$

The big question is how can we get the components of the new coordinate system?

Using fig B, we can find the components:

$$\begin{cases} A_y' = A\sin(\theta - \phi) \\ A_x' = A\cos(\theta - \phi) \\ A_z' = A_z = 1 \end{cases}$$

$$\left.\begin{aligned}\cos(A-B) &= \cos A\cos B + \sin A\sin B\\ \sin(A-B) &= \sin A\cos B - \cos A\sin B\end{aligned}\right\}$$

$$A_y' = A\sin(\theta-\phi)$$
$$A_y' = A[\sin\theta\cos\phi - \cos\theta\sin\phi]$$
$$A_y' = A\sin\theta\cos\phi - A\cos\theta\sin\phi$$
$$A_y' = A_y\cos\phi - A_x\sin\phi$$
$$\therefore A_y' = -A_x\sin\phi + A_y\cos\phi$$

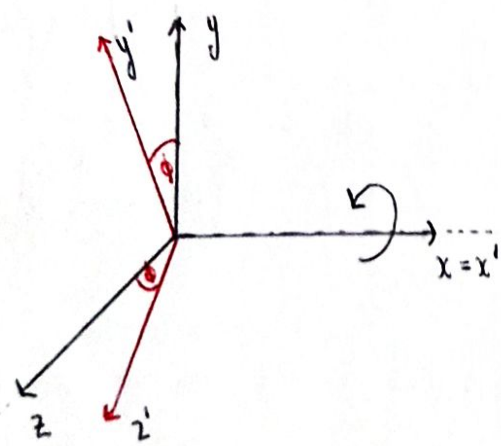
$$A_x' = A\cos(\theta-\phi)$$
$$A_x' = A[\cos\theta\cos\phi + \sin\theta\sin\phi]$$
$$A_x' = A\cos\theta\cos\phi + A\sin\theta\sin\phi$$
$$A_x' = A_y\cos\phi + A_x\sin\phi$$
$$\therefore A_x' = A_x\sin\phi + A_y\cos\phi$$

Thus:

$$\begin{bmatrix} A_x' \\ A_y' \\ A_z \end{bmatrix} = \overbrace{\begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}}^{\text{Transformation Matrix (T)}} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

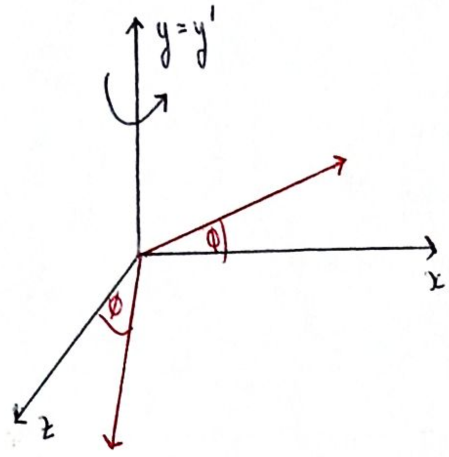
Summary of coordinate transformation in 3D

Rotation about the x-axis



$$\begin{bmatrix} A_x' \\ A_y' \\ A_z' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

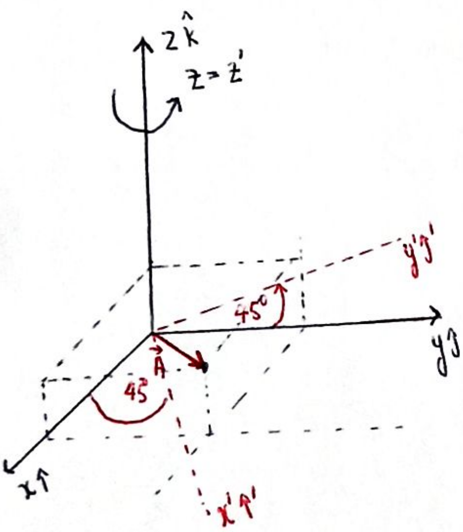
Rotate about the y-axis



$$\begin{bmatrix} A_x' \\ A_y' \\ A_z' \end{bmatrix} = \begin{bmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

Example 1.8.1

Express the vector $\vec{A} = 3\hat{\imath} + 2\hat{\jmath} + \hat{k}$ in terms of the triad $\hat{\imath}'\hat{\jmath}'\hat{k}$, where the $x'y'$- axes are rotated 45° around the z-axis.



$$\begin{bmatrix} A_x' \\ A_y' \\ A_z' \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

$$\begin{bmatrix} A_x' \\ A_y' \\ A_z \end{bmatrix} = \begin{bmatrix} \cos 45^\circ & \sin 45^\circ & 0 \\ -\sin 45^\circ & \cos 45^\circ & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} A_x' \\ A_y' \\ A_z' \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

$$\therefore\ A_x' = \frac{1}{\sqrt{2}}(3) + \frac{1}{\sqrt{2}}(2) = \frac{5}{\sqrt{2}}$$

$$A_y' = -\frac{3}{\sqrt{2}} + \frac{2}{\sqrt{2}} = -\frac{1}{\sqrt{2}}$$

$$A_z' = 1$$

Thus:

$$\vec{A} = \frac{5}{\sqrt{2}}\hat{\imath}' - \frac{1}{\sqrt{2}}\hat{\jmath}' + \hat{k}$$

[Problem 1.13 and 1.14] Tut 2

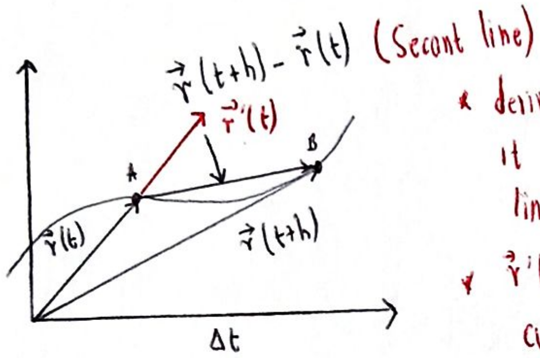
Derivative of a Vector

Defination: If $\vec{r}(t)$ is a value vector function, the derivative of $\vec{r}$ with respect to t is:

$$\dot{\vec{r}} = \vec{r}\,'(t) = \frac{d\vec{r}}{dt} = \frac{d}{dt}(\vec{r}(t)) \leftarrow \text{Notation}$$

$$r'(t) = \lim_{h\to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h} \quad [\text{First principle}]$$

Geometrically



* derivative gives a tangent and it has to point in the direction of a secant line
* $\vec{r}\,'(t)$ is a vector tangent to the parameter curve at tip $\vec{r}(t)$.

Consider a vector $\vec{A}$, whose components are function of a single variable U. The vector may represent position, velocity and so on. The parameter U is usually the time (t), but it can be any quantity that determines the components of $\vec{A}$.

$$\vec{A}(u) = A_x(u)\hat{i} + A_y(u)\hat{j} + A_z(u)\hat{k}$$

$$\frac{d\vec{A}}{du} = \lim_{\Delta u\to 0} \frac{\vec{A}(u+\Delta u) - \vec{A}(u)}{\Delta u}$$

$$\begin{cases} \Delta A_x = A_x(u+\Delta u) - A_x(u) \\ \Delta A_y = A_y(u+\Delta u) - A_y(u) \\ \Delta A_z = A_z(u+\Delta u) - A_z(u) \end{cases}$$

$$\therefore \frac{d\vec{A}}{du} = \lim_{\Delta u\to 0}\frac{\Delta A_x}{\Delta u}\hat{i} + \lim_{\Delta u\to 0}\frac{\Delta A_y}{\Delta u}\hat{j} + \lim_{\Delta u\to 0}\frac{\Delta A_z}{\Delta u}$$

$$\therefore = \frac{dA_x}{du}\hat{i} + \frac{dA_y}{du}\hat{j} + \frac{dA_z}{du}\hat{k}$$

The derivative of a vector is a vector whose cartesian components are ordinary derivatives.

$$\frac{d}{du}(\vec{A}+\vec{B}) = \frac{d\vec{A}}{du} + \frac{d\vec{B}}{du}$$

$$\frac{d(n\vec{A})}{du} = \frac{dn}{du}\vec{A} + n\frac{d\vec{A}}{du}$$

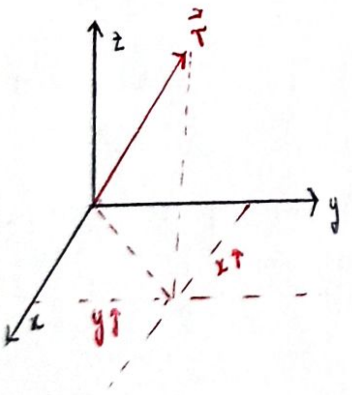
$$\frac{d(\vec{A}\cdot\vec{B})}{du} = \frac{d\vec{A}}{du}\cdot\vec{B} + \vec{A}\cdot\frac{d\vec{B}}{du}$$

$$\frac{d(\vec{A}\times\vec{B})}{du} = \frac{d\vec{A}}{du}\times\vec{B} + \vec{A}\frac{d\vec{B}}{du}$$

Position vector of a particle: Velocity and Acceleration in Rectangular coordinate

Position Vector: It is a vector that represents the position of any object / particle at any instant of time in a coordinate system or in any reference frame.

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$



The components of a position vector of a moving particle as a function of time

$$x = x(t), \quad y = y(t), \quad z = z(t)$$

If $\vec{r}$ is the position vector of a moving particle:

$$\vec{V} = \frac{d\vec{r}}{dt} = \frac{d}{dt}(x\hat{i} + y\hat{j} + \hat{k}z)$$

$$= \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}$$

$$= \dot{x}\hat{i} + \dot{y}\hat{j} + \dot{z}\hat{k}$$

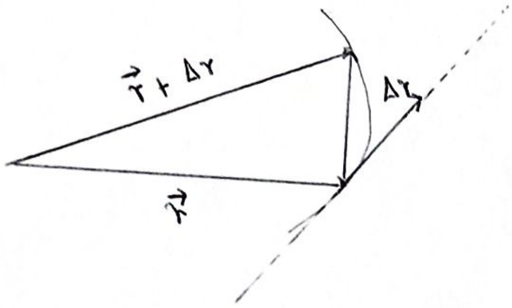
[The dot indicate the differentiation with respect to t].

The geometrical representation of the velocity vector can be considered as following.

* Let's suppose we have a moving particle.

At time t, the position vector $\vec{r}(t)$.

At time $t+\Delta t$, position vector is $\vec{r}(t+\Delta t)$



$\therefore \Delta\vec{r} = \vec{r}(t+\Delta t) - \vec{r}(t)$

from first principle we can find the tangent $(\vec{v})$

$$\frac{\Delta\vec{r}}{\Delta t} = \frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t}$$

$$\lim_{\Delta t \to 0} \frac{\Delta\vec{r}}{\Delta t} = \frac{d\vec{r}}{dt} = \vec{v}$$

The magnitude of the velocity is called the speed. The speed in rectangular component is just

$$v = |\vec{v}| = \sqrt{(\dot{x})^2 + (\dot{y})^2 + (\dot{z})^2}$$

Acceleration:

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}(\dot{x}\hat{i} + \dot{y}\hat{j} + \dot{z}\hat{k})$$

$$= \underline{\ddot{x}\hat{i} + \ddot{y}\hat{j} + \ddot{z}\hat{k}}$$

[The acceleration is a vector quantity whose components in rectangular coordinates are the second derivatives of the positional coordinates of a moving particle.]

Problem 1.18 As example.

A buzzing fly moves in a helical path given by the equation

$$\vec{r}(t) = \hat{i}\, b\sin wt + \hat{j}\, b\cos wt + \hat{k}\, ct^2 \quad ; \quad w = \frac{\Delta\theta}{\Delta t}$$

$\uparrow$ angular velocity ; $\swarrow$ change in angular rotation

Show that the magnitude of the acceleration of the fly is constant (b, w and c) are constants.

$$\vec{r}(t) = b\sin wt\,\hat{i} + b\cos wt\,\hat{j} + ct^2\,\hat{k}$$

$$\dot{\vec{r}}(t) = \vec{v} = \frac{d\vec{r}}{dt} = bw\cos wt\,\hat{i} - bw\sin wt\,\hat{j} + 2ct\,\hat{k}$$

$$\ddot{\vec{r}}(t) = \vec{a} = \frac{d\vec{v}}{dt} = -\hat{i}\, bw^2\sin wt - \hat{j}\, bw^2\cos wt + 2c\hat{k}$$

$$a = |\vec{a}| = \left[(-bw^2\sin wt)^2 + (-bw^2\cos wt)^2 + (2c)^2\right]^{1/2}$$

$$= \left[b^2 w^4 \sin^2 wt + b^2 w^4 \cos^2 wt + 4c^2\right]^{1/2}$$

$$= \left[b^2 w^4 (\sin^2 wt + \cos^2 wt) + 4c^2\right]^{1/2}$$

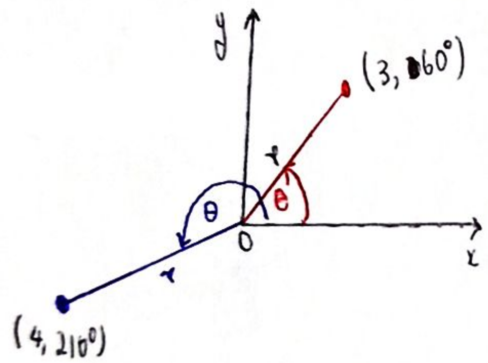
$$= \sqrt{b^2 w^4 + 4c^2} \Rightarrow a = \text{constant}$$

General Definition

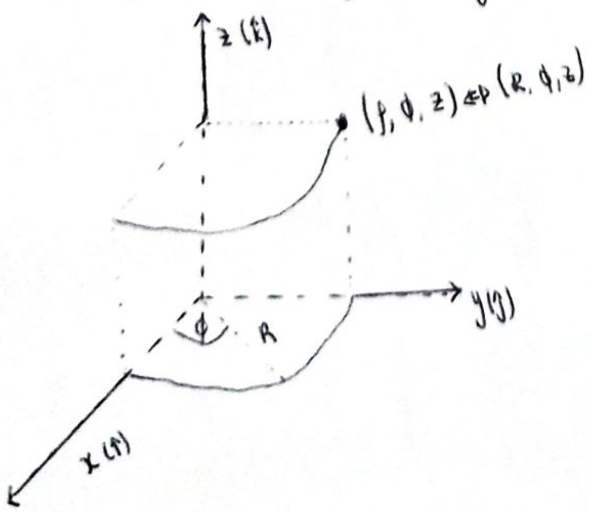
A coordinate system: Is a system designed to establish positions with respect to given reference points. The coordinate system consists of one or more reference points, the style of measurement (linear or angular) from those reference points, and the directions (or axes) in which those measurements will be taken.

For example in satellite navigation various coordinate (reference) systems are used to precisely define the satellite and user locations. In this section, different coordinate systems will be classified and later see how to derive their properties for real world applications.

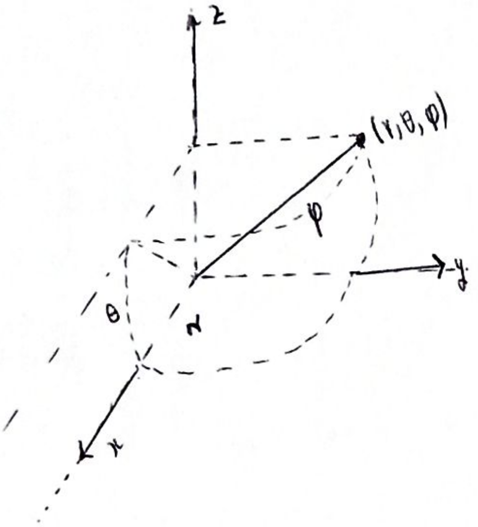
Polar Coordinate system: This is a 2D coordinate system in which each point on a plane is determined by a distance from a fixed point and an angle from a fixed direction.



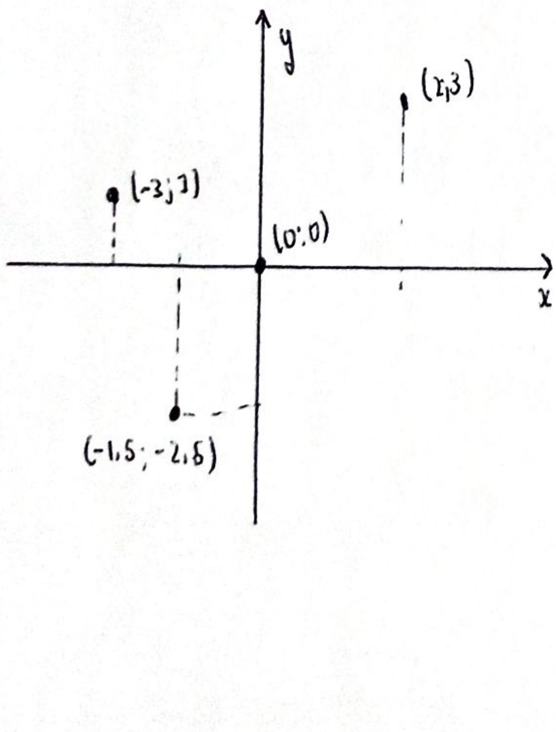
Cylindrical coordinate system: Is a 3D coordinate system, where each point is specified by the two polar coordinates of its perpendicular projection onto fixed plane, and by its (signed) distance from that plane.

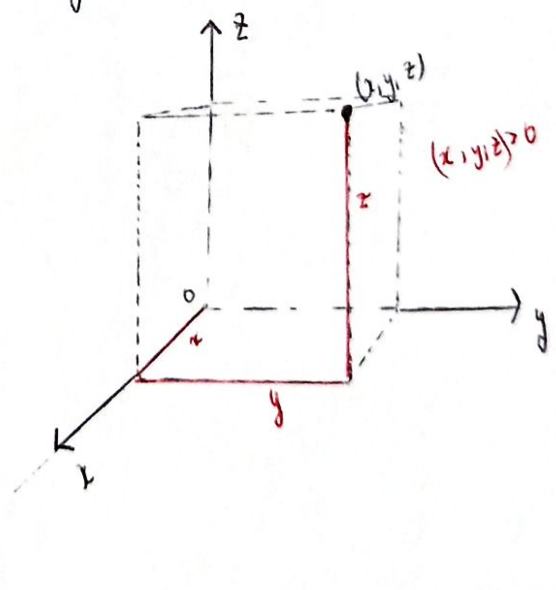


Spherical coordinate system: Is a coordinate system for 3D space where the position of a point is specified by 3 numbers, namely the radial distance of that point from a fixed origin, its elevation angle measured from a fixed plane, and the azimuth angle of its orthogonal projection on that plane.



Cartesian coordinate system: specifies each point uniquely in a plane by a pair of numerical coordinates, which are the signed distance from the point of two fixed perpendicular directed lines, measured in the same unit of length.





Velocity and Acceleration in Plane Polar Coordinates

* Vectors depends on vectors and vectors depends on coordinate system.

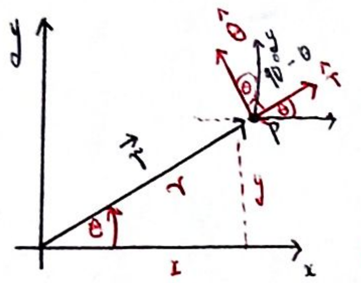


Fig 1.

$\vec{v} = \dot{\vec{r}} = \dfrac{d\vec{r}}{dt} \Rightarrow$ $\vec{r} = x\hat{\imath} + y\hat{\jmath}$ ⇐ cartesian coordinate

$\dfrac{d\vec{r}}{dt} = \dfrac{dx}{dt}\hat{\imath} + \dfrac{dy}{dt}\hat{\jmath} = V_x\hat{x} + V_y\hat{y}$

$\vec{r} = |r|\hat{r}$

OR

$x = r\cos\theta$
$y = r\sin\theta$

$d\vec{r} = \hat{x}\dfrac{dx}{dt} + x\dfrac{d\hat{x}}{dt}^{\,=0} + \hat{y}\dfrac{dy}{dt} + y\dfrac{d\hat{y}}{dt}^{\,=0}$

$\hat{r} = \cos\theta\hat{x} + \sin\theta\hat{y}$

$\hat{\theta} \perp \hat{r}$, $\hat{\theta} = -\sin\theta\hat{x} + \cos\theta\hat{y}$

$\hat{x} = \langle 1,0,0 \rangle$, $\hat{y} = \langle 0,1,0 \rangle$

In Polar coordinate

* Use the angle θ and scalar r in figure 1
* $\hat{\theta}$ and $\hat{r}$ both changes as the position of P changes

$\vec{r} = r\hat{r}$ ↙ plane polar

* As θ increases even $\hat{r}$ changes direction

$$\frac{d\vec{r}}{dt} = \frac{dr}{dt}\hat{r} + r\frac{d\hat{r}}{dt} \quad \text{— ①}$$

$$\frac{d\hat{r}}{dt} = \frac{d}{dt}(\cos\theta)\hat{x} + \cancel{\cos\theta\frac{d\hat{x}}{dt}} + \frac{d}{dt}(\sin\theta)\hat{y} + \sin\theta\frac{d\hat{y}}{dt}$$

$$= -\sin\theta\frac{d\theta}{dt}\hat{x} + \cos\theta\frac{d\theta}{dt}\hat{y}$$

$$= \dot{\theta}(-\sin\theta\hat{x} + \cos\theta\hat{y})$$

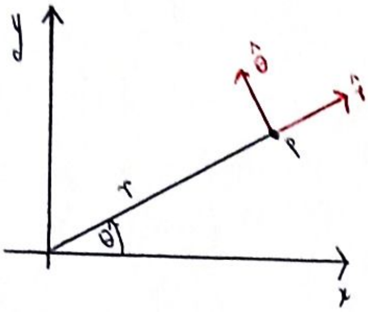
$$= \dot{\theta}\hat{\theta}$$

Using equation 1, we can simply write the velocity in polar coordinate:

$$\boxed{\frac{d\vec{r}}{dt} = \frac{dr}{dt}\hat{r} + r\frac{d\hat{r}}{dt} = \underset{\uparrow\, m}{\dot{r}\hat{r}} + \underset{\uparrow\, \text{rad/s}}{r\dot{\theta}\hat{\theta}}}$$

Book written as.
$V = \dot{r}e_r + r\dot{\theta}e_\theta$

Acceleration in Polar Coordinates



$$\hat{r} = \cos\theta\,\hat{x} + \sin\theta\,\hat{y}$$
$$\hat{\theta} = -\sin\theta\,\hat{x} + \cos\theta\,\hat{y}$$
$$\dot{\vec{r}} = \dot{r}\,\hat{r} + r\dot{\theta}\,\hat{\theta}$$

$$\frac{d\hat{\theta}}{dt} = -\cos\theta\,\dot{\theta}\,\hat{x} - \sin\theta\,\dot{\theta}\,\hat{y}$$
$$= -\dot{\theta}\,(\cos\theta\,\hat{x} + \sin\theta\,\hat{y})$$
$$= -\dot{\theta}\,\hat{r}$$

From defination: $\frac{d\hat{r}}{dt} = \dot{\theta}\,\hat{\theta}$

$$\vec{a} = \frac{d\dot{\vec{r}}}{dt} \;;\; \frac{d(\dot{\vec{r}})}{dt} = (\dot{r}\,\hat{r} + r\dot{\theta}\,\hat{\theta})$$

$$= \ddot{r}\,\hat{r} + \frac{d\hat{r}}{dt}\dot{r} + \dot{r}\dot{\theta}\,\hat{\theta} + r\ddot{\theta}\,\hat{\theta} + r\dot{\theta}\,\frac{d\hat{\theta}}{dt}$$
$$= \ddot{r}\,\hat{r} + \dot{r}\dot{\theta}\,\hat{\theta} + \dot{r}\dot{\theta}\,\hat{\theta} + r\ddot{\theta}\,\hat{\theta} - r\dot{\theta}\dot{\theta}\,\hat{r}$$
$$= \boxed{\hat{r}\,(\ddot{r} - r\dot{\theta}^2) + \hat{\theta}\,(2\dot{r}\dot{\theta} + r\ddot{\theta})}$$

acceleration in polar co-ordinate

radial component

transverse component

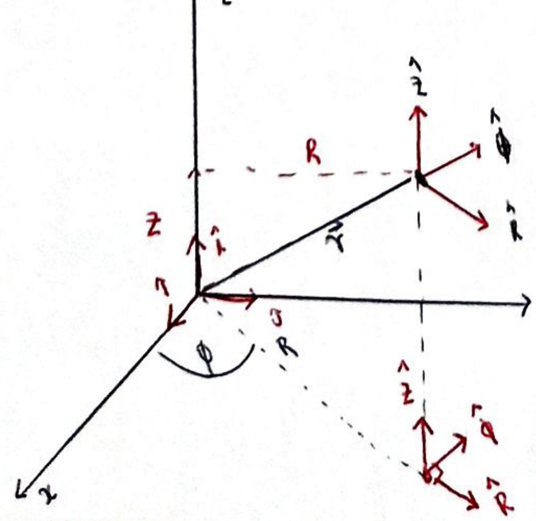
Velocity and Acceleration in Cylindrical and spherical Coordinates

Cylindrical coordinates:

In the case of 3D motion, the position of a particle can be described in cylindrical coordinates R, ϕ, z.

$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$ ← cartesian co-ordinate system

$$\vec{v} = \frac{d\vec{r}}{dt}$$



$x = R\cos\phi$

$y = R\sin\phi$

$z = z$

$\hat{R} = \cos\phi\,\hat{x} + \sin\phi\,\hat{y}$

$\hat{\phi} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y}$

$\hat{z} = \hat{z}$

$$\hat{R} = \frac{\partial\vec{r}}{\partial R} \Big/ \left|\frac{\partial \vec{r}}{\partial R}\right| \;;\; \hat{\phi} = \frac{\partial\vec{r}}{\partial\phi} \Big/ \left|\frac{\partial\vec{r}}{\partial\phi}\right|$$

Thus:

$$\vec{r} = R\cos\phi\,\hat{x} + R\sin\phi\,\hat{y} + z\,\hat{z}$$

$$= R(\cos\phi\,\hat{x} + \sin\phi\,\hat{y}) + z\hat{z}$$

$$= R\hat{R} + z\hat{z}$$ (cylindrical co-ordinate system)

Determining the velocity:

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt}(R\hat{R} + z\hat{z})$$

$$= \frac{d}{dt}(R\hat{R}) + \frac{d}{dt}(z\hat{z})$$

$$= \dot{R}\hat{R} + R\frac{d\hat{R}}{dt} + \dot{z}\hat{z} + z\overbrace{\frac{d\hat{z}}{dt}}^{=0} \quad \text{——— } *$$

$$\hat{R} = \cos\phi\,\hat{x} + \sin\phi\,\hat{y}$$

$$\frac{d\hat{R}}{dt} = -\sin\phi\,\dot{\phi}\,\hat{x} + \dot{\phi}\cos\phi\,\hat{y}$$

$$= \dot{\phi}(-\sin\phi\,\hat{x} + \cos\phi\,\hat{y})$$

$$= \dot{\phi}\hat{\phi} \quad \text{——— } **$$

Sub (**) into (*)

$$\therefore \boxed{\vec{v} = \dot{R}\hat{R} + R\dot{\phi}\hat{\phi} + \dot{z}\hat{z}}$$

Acceleration in cylindrical co-ordinate system:

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d\dot{\vec{r}}}{dt}$$

$$= \frac{d}{dt}(\dot{R}\hat{R} + R\dot{\phi}\hat{\phi} + \dot{z}\hat{z})$$

$$= \frac{d}{dt}(\dot{R}\hat{R}) + \frac{d}{dt}(R\dot{\phi}\hat{\phi}) + \frac{d}{dt}(\dot{z}\hat{z})$$

$$= \ddot{R}\hat{R} + \dot{R}\frac{d\hat{R}}{dt} + \dot{R}\dot{\phi}\hat{\phi} + R\dot{\phi}\frac{d\hat{\phi}}{dt} + R\ddot{\phi}\hat{\phi} + \ddot{z}\hat{z} \quad \text{——— } ***$$

$$\hat{\phi} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y}$$

$$\frac{d\hat{\phi}}{dt} = -\cos\phi\,\dot{\phi}\,\hat{x} - \sin\phi\,\dot{\phi}\,\hat{y}$$

$$= -(\cos\phi\,\hat{x} + \sin\phi\,\hat{y})\,\dot{\phi} = -\hat{R}\,\dot{\phi} \quad \text{——} \quad ****$$

sub (****) into (***)

$$\vec{a} = \ddot{R}\hat{R} + \dot{R}\frac{d\hat{R}}{dt} + \dot{R}\dot{\phi}\hat{\phi} + R\ddot{\phi}\hat{\phi} + R\dot{\phi}\frac{d\hat{\phi}}{dt} + \ddot{z}\hat{z}$$

$$= \ddot{R}\hat{R} + \dot{R}\,\dot{\phi}\hat{\phi} + \dot{R}\dot{\phi}\hat{\phi} + R\ddot{\phi}\hat{\phi} + R\dot{\phi}(-\hat{R}\dot{\phi}) + \ddot{z}\hat{z}$$

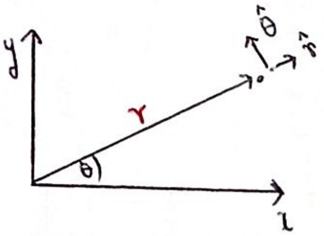
$$= \ddot{R}\hat{R} + 2\dot{R}\dot{\phi}\hat{\phi} + R\ddot{\phi}\hat{\phi} - R\hat{R}(\dot{\phi})^2 + \ddot{z}\hat{z}$$

$$= \boxed{\hat{R}(\ddot{R} - R\dot{\phi}^2) + (2\dot{R}\dot{\phi} + R\ddot{\phi})\hat{\phi} + \ddot{z}\hat{z}}$$

Acceleration in cylindrical coordinate.

Unit vector in cylindrical Coordinates [TASK-FOR-STUDENT]

$$\hat{R} = \frac{\frac{\partial\vec{r}}{\partial R}}{\left|\frac{\partial\vec{r}}{\partial R}\right|} \qquad \hat{\phi} = \frac{\frac{\partial\vec{r}}{\partial\phi}}{\left|\frac{\partial\vec{r}}{\partial\phi}\right|}$$



$x = r\cos\theta$
$y = r\sin\theta$
$z = z$

$$\vec{r} = r\cos\theta\,\hat{x} + r\sin\theta\,\hat{y} + z\hat{z}$$

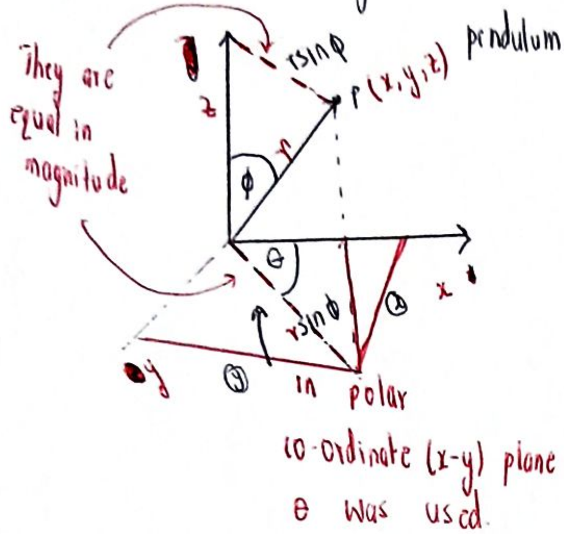
$$\hat{r} = \frac{\partial\vec{r}}{\partial r} = \cos\theta\,\hat{x} + \sin\theta\,\hat{y} + 0z$$

$$\left|\frac{\partial\vec{r}}{\partial r}\right| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$$

$$\therefore\ \hat{r} = \cos\theta\,\hat{x} + \sin\theta\,\hat{y}$$

Spherical Coordinates

* This helps when something is confined to a spherical system, one of the typical example is the pendulum moving up and down while rotating also.



same thing

* We need to find (x,y,z) interms of (r,θ,ϕ) or (ρ,θ,ϕ)

$$\left.\begin{aligned} x &= r\sin\phi\cos\theta \\ y &= r\sin\phi\sin\theta \\ z &= r\cos\phi \end{aligned}\right\}$$

$$\left.\begin{aligned} \vec{r} &= x\hat{x} + y\hat{y} + z\hat{z} \\ \vec{V} &= \dot{x}\hat{x} + \dot{y}\hat{y} + \dot{z}\hat{z} \end{aligned}\right\} \text{C.C.}$$

for spherical coordinates

$$\vec{r} = r\hat{r}$$

$$\vec{V} = \dot{r}\hat{r} + r\frac{d\hat{r}}{dt} \ ?$$

$$\hat{r} = \frac{\partial\vec{r}}{\partial r} \bigg/ \left|\frac{\partial\vec{r}}{\partial r}\right| \quad ; \quad \vec{r} = r\sin\phi\cos\theta\,\hat{x} + r\sin\phi\sin\theta\,\hat{y} + r\cos\phi\,\hat{z}$$

$$\frac{\partial\vec{r}}{\partial r} = \sin\phi\cos\theta\,\hat{x} + \sin\phi\sin\theta\,\hat{y} + \cos\phi\,\hat{z}$$

$$\left|\frac{\partial\vec{r}}{\partial r}\right| = \left[\sin^2\phi\cos^2\theta + \sin^2\phi\sin^2\theta + \cos^2\phi\right]^{1/2}$$

$$= \left[\sin^2\phi\left(\cos^2\theta + \sin^2\theta\right) + \cos^2\phi\right]^{1/2} = 1$$

$$\therefore \boxed{\hat{r} = \sin\phi\cos\theta\,\hat{x} + \sin\phi\sin\theta\,\hat{y} + \cos\phi\,\hat{z}}$$

$$\hat{\theta} = \frac{d\vec{r}}{d\theta} = -r\sin\phi\sin\theta\,\hat{x} + r\sin\phi\cos\theta\,\hat{y} \quad ; \text{ Thus:}$$

$$\left|\frac{d\vec{r}}{d\theta}\right| = \left[r^2\sin^2\phi\sin^2\theta + r^2\sin\phi\cos^2\theta\right]^{1/2}$$

$$= r\sin\phi\sqrt{\sin^2\theta + \cos^2\theta}$$

$$= r\sin\phi$$

$$\boxed{\hat{\theta} = -\sin\theta\,\hat{x} + \cos\theta\,\hat{y}}$$

$$\vec{r} = r\sin\phi\cos\theta\,\hat{x} + r\sin\phi\sin\theta\,\hat{y} + r\cos\phi\,\hat{z}$$

$$\frac{\partial\vec{r}}{\partial\phi} = r\cos\phi\cos\theta\,\hat{x} + r\cos\phi\sin\theta\,\hat{y} - r\sin\phi\,\hat{z}$$

$$\left|\frac{\partial\vec{r}}{\partial\phi}\right| = \sqrt{r^2\cos^2\phi\cos^2\theta + r^2\cos^2\phi\sin^2\theta + r^2\sin^2\phi}$$

$$= r\sqrt{\cos^2\phi\left(\cos^2\theta + \sin^2\theta\right) + \sin^2\theta} \quad = 1$$

$$\therefore \boxed{\hat{\phi} = \cos\phi\cos\theta\,\hat{x} + \cos\phi\sin\theta\,\hat{y} - \sin\phi\,\hat{z}}$$

$$\vec{v} = \frac{d\vec{r}}{dt} = \dot{r}\hat{r} + r\frac{d\hat{r}}{dt} \; ; \qquad \hat{r} = \sin\phi\cos\theta\,\hat{x} + \sin\phi\sin\theta\,\hat{y} + \cos\phi\,\hat{z}$$

$$\frac{d\hat{r}}{dt} = (\dot{\phi}\cos\phi\cos\theta - \dot{\theta}\sin\phi\sin\theta)\,\hat{x} + (\dot{\phi}\cos\phi\sin\theta + \dot{\theta}\sin\phi\cos\theta)\,\hat{y} - \dot{\phi}\sin\phi\,\hat{z}$$

$$\frac{d\hat{r}}{dt} = \dot{\theta}\left(-\sin\phi\sin\theta\,\hat{x} + \sin\phi\cos\theta\,\hat{y}\right) + \dot{\phi}\left(\cos\phi\cos\theta\,\hat{x} + \cos\phi\sin\theta\,\hat{y} - \sin\phi\,\hat{z}\right)$$

$$\frac{d\hat{r}}{dt} = \dot{\theta}\sin\phi\,\hat{\theta} + \dot{\phi}\,\hat{\phi}$$

$$= \frac{d}{dt}\left(\dot{\theta}\sin\phi\,\hat{\theta} + \dot{\phi}\,\hat{\phi}\right)$$

$$= \frac{d}{dt}\left(\dot{\theta}\sin\phi\,\hat{\theta}\right) + \frac{d}{dt}\left(\dot{\phi}\,\hat{\phi}\right)$$

Eq^n A

$$\frac{d\hat{\phi}}{dt} = -\dot{\phi}\sin\phi\cos\theta\,\hat{x} - \dot{\theta}\cos\phi\sin\theta\,\hat{x} - \dot{\phi}\sin\phi\sin\theta\,\hat{y} + \dot{\theta}\cos\phi\cos\theta\,\hat{y} - \dot{\phi}\cos\phi\,\hat{z}$$

$$= \dot{\phi}\left[-\sin\phi\cos\theta\,\hat{x} - \sin\phi\sin\theta\,\hat{y} - \cos\phi\,\hat{z}\right] + \dot{\theta}\left[-\cos\phi\sin\theta\,\hat{x} + \cos\phi\cos\theta\,\hat{y}\right]$$

$$= -\dot{\phi}\,\hat{r} + \dot{\theta}\cos\phi\,\hat{\theta}$$

$$\hat{\theta} = -\sin\theta\,\hat{x} + \cos\theta\,\hat{y}$$

$$\frac{d\hat{\theta}}{dt} = -\dot{\theta}\cos\theta\,\hat{x} - \dot{\theta}\sin\theta\,\hat{y}$$

TRICK: [In terms of ϕ and r]

$$\sin\phi\,\hat{r} = \sin^2\phi\cos\theta\,\hat{x} + \sin^2\phi\sin\theta\,\hat{y} + \sin\phi\cos\phi\,\hat{z}$$

$$+ \quad = \sin^2\phi\,(\cos\theta\,\hat{x} + \sin\theta\,\hat{y}) + \sin\phi\cos\phi\,\hat{z}$$

$$\cos\phi\,\hat{\phi} = \cos^2\phi\,(\cos\theta\,\hat{x} + \sin\theta\,\hat{y}) - \sin\phi\cos\phi\,\hat{z}$$

$$\sin\phi\,\hat{r} + \cos\phi\,\hat{\phi} = (\cos\theta\,\hat{x} + \sin\theta\,\hat{y})(\sin^2\phi + \cos^2\phi) = \cos\theta\,\hat{x} + \sin\theta\,\hat{y}$$

$$\frac{d\hat{\theta}}{dt} = -\dot{\theta}\,(\cos\theta\,\hat{x} + \sin\theta\,\hat{y})$$

$$\therefore \frac{d\hat{\theta}}{dt} = -\dot{\theta}\sin\phi\,\hat{r} - \dot{\theta}\cos\phi\,\hat{\phi}$$ ← Student task to prove this.

$$= \cancel{-\dot{\theta}(\sin\phi\,\hat{r} + \cos\phi\,\hat{\phi})} = -\dot{\theta}(\sin\phi\,\hat{r} + \cos\phi\,\hat{\phi})$$

Hence:

$$\boxed{\begin{aligned} \vec{r} &= r\hat{r} \\ \frac{d\hat{r}}{dt} &= \dot{\theta}\sin\phi\,\hat{\theta} + \dot{\phi}\,\hat{\phi} \\ \frac{d\hat{\phi}}{dt} &= -\dot{\phi}\,\hat{r} + \dot{\theta}\cos\phi\,\hat{\theta} \\ \frac{d\hat{\theta}}{dt} &= -\dot{\theta}(\sin\phi\,\hat{r} + \cos\phi\,\hat{\phi}) \end{aligned}}$$

$$\vec{v} = \frac{d\vec{r}}{dt} = \dot{r}\hat{r} + r\frac{d\hat{r}}{dt} = \boxed{\dot{r}\hat{r} + r\dot{\theta}\sin\phi\,\hat{\theta} + r\dot{\phi}\hat{\phi}}$$

$$\dot{r} = \frac{dr}{dt}$$

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(\dot{r}\hat{r} + r\dot{\theta}\sin\phi\,\hat{\theta} + r\dot{\phi}\hat{\phi}\right)$$

$$\cancel{\ddot{r}\hat{r} + \dot{r}\frac{d\hat{r}}{dt} + \dot{r}\dot{\theta}\sin\phi\,\hat{\theta} + r\ddot{\theta}\sin\theta\,\hat{\theta} + r\dot{\theta}\dot{\phi}\cos\phi\,\hat{\theta} + r\dot{\theta}\sin\phi\left(-\dot{\theta}\sin\phi\,\hat{r} - \dot{\theta}\cos\phi\,\hat{\phi}\right)}$$

$$= \ddot{r}\hat{r} + \dot{r}\dot{\theta}\sin\phi\,\hat{\theta} + \dot{r}\dot{\phi}\hat{\phi} + \dot{r}\dot{\theta}\sin\phi\,\hat{\theta} + r\ddot{\theta}\sin\phi\,\hat{\theta} + r\dot{\theta}\dot{\phi}\cos\phi\,\hat{\theta} + r\dot{\theta}\sin\phi\left(-\sin\phi\,\hat{r} - \dot{\theta}\cos\phi\,\hat{\phi}\right)$$

$$+ \dot{r}\dot{\phi}\hat{\phi} + r\ddot{\phi}\hat{\phi} - r\dot{\phi}\dot{\phi}\hat{r} + r\dot{\phi}\dot{\theta}\cos\phi\,\hat{\theta}$$

$$\therefore \boxed{\vec{a} = \hat{r}\left(\ddot{r} - r\dot{\theta}^2\sin^2\phi - r\dot{\phi}^2\right) + \hat{\theta}\left(2\dot{r}\dot{\theta}\sin\phi + r\ddot{\theta}\sin\phi + 2r\dot{\theta}\dot{\phi}\cos\phi\right) + \hat{\phi}\left(2\dot{r}\dot{\phi} - r\dot{\theta}^2\sin\phi\cos\phi + r\ddot{\phi}\right)}$$

Students TASK.

Newton's Second Law in Various Coordinate Systems

Vector Form

$$\vec{F} = m\ddot{\mathbf{x}}$$

Cartesian (x, y, z)

$$F_x = m\ddot{x}$$

$$F_y = m\ddot{y}$$

$$F_z = m\ddot{z}$$

2D Polar (r, ϕ)

$$F_r = m(\ddot{r} - r\dot{\phi}^2)$$

$$F_\phi = m(r\ddot{\phi} + 2\dot{r}\dot{\phi})$$

Cylindrical Polar (ρ, ϕ, z)

$$F_r = m(\ddot{\rho} - \rho\dot{\phi}^2)$$

$$F_\phi = m(\rho\ddot{\phi} + 2\dot{\rho}\dot{\phi})$$

$$F_z = m\ddot{z}$$

Spherical (r, θ, ϕ)

$$F_r = m(\ddot{r} - r\dot{\theta}^2\sin^2\phi - r\dot{\phi}^2)$$

$$F_\theta = m(2\dot{r}\dot{\theta}\sin\phi + r\ddot{\theta}\sin\phi + 2r\dot{\theta}\dot{\phi}\cos\phi)$$

$$F_\phi = m(2\dot{r}\dot{\phi} - r\dot{\theta}^2\sin\phi\cos\phi + r\ddot{\phi})$$

Example 1.12.1

A bead slices on a wire bent into the form of a helix, the motion of the bead being given in cylindrical coordinate by $R = b$, $\phi = wt$, $z = ct$. find the velocity and acceleration vectors as a functions of time.

In cylindrical coordinate system:

$$\begin{cases} \vec{V} = \dot{R}\hat{R} + R\dot{\phi}\hat{\phi} + \dot{z}\hat{z} \\ \vec{a} = \hat{R}(\ddot{R} - R\dot{\phi}^2) + (2\dot{R}\dot{\phi} + R\ddot{\phi})\hat{\phi} + \ddot{z}\hat{z} \end{cases}$$

Thus:

$$\dot{R} = \ddot{R} = 0, \quad \dot{\phi} = w, \quad \ddot{\phi} = 0, \quad \dot{z} = c, \quad \ddot{z} = 0$$

$$\therefore \vec{V} = bwt\,\hat{\phi} + c\hat{z}$$

$$\vec{a} = -bw^2\hat{R}$$